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Exercises of *Stellar Interiors - Physical Priciples, Structure, and Evolution*
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Interview Questions

Question 1. (Exercises 1.2) We stated, without proof, that the central pressure of the constant density star was a lower limit (§1.4); that is, central pressures must exceed $P_c = 3G\mathcal{M}^2/(8\pi\mathcal{R}^4)$. The proof of that statement requires a bit more work than we wish to attempt now. There is, however, a weaker lower limit on P_c. To get at this consider th function

$$f(r) = P(r) + \frac{G{\mathcal{M}_r}^2}{8\pi r^4}.$$

1) Show that $f(r)$ decreases outward with increasing r. (Hint: Differentiate $f(r)$ with respect to r and use the equation of hydrostatic quilibrium to show $df/dr < 0$.)
2) Assuming zero pressure at $\mathcal{R}$, demonstrate (almost immediately) that

$$P_c > \frac{G\mathcal{M}^2}{8\pi\mathcal{R}^4}$$

which is less stringent than that given by (1.42). Note that you must show ${\mathcal{M}_r}^2/r^4$ goes to zero as $r \to 0$.

1) Take the derivate of $f(r)$ respect to r:

$$\frac{df(r)}{dr} = \frac{dP(r)}{dr} + \frac{d}{dr}\left(\frac{G{\mathcal{M}_r}^2}{8\pi r^4}\right). \tag{1}$$

The equation of hydrostatic equilibrium (1.6) gives:

$$\frac{dP}{dr} = -\frac{G\mathcal{M}_r}{r^2}\rho.$$

We know that $d\mathcal{M}_r = 4\pi r^2\rho(r)dr$, then using the chain rule, the second term in Equation 1 becomes:

$$\begin{aligned}
\frac{d}{dr}\left(\frac{G{\mathcal{M}_r}^2}{8\pi r^4}\right) &= \frac{G}{8\pi}\frac{d}{dr}\left(\frac{{\mathcal{M}_r}^2}{r^4}\right) \\
&= \frac{G}{8\pi}\left[\frac{1}{r^4}\frac{d{\mathcal{M}_r}^2}{dr} + {\mathcal{M}_r}^2\frac{d}{dr}\left(\frac{1}{r^4}\right)\right] \\
&= \frac{G}{8\pi}\left[\frac{1}{r^4}2\mathcal{M}_r\frac{d\mathcal{M}_r}{dr} + {\mathcal{M}_r}^2(-4)\frac{1}{r^5}\right] \\
&= \frac{G}{8\pi}\left[\frac{1}{r^4}2\mathcal{M}_r 4\pi r^2\rho - 4{\mathcal{M}_r}^2\frac{1}{r^5}\right] \\
&= \frac{G\mathcal{M}_r}{r^2}\rho - \frac{G{\mathcal{M}_r}^2}{2\pi r^5}.
\end{aligned}$$

Substitude the results of dP/dr and $d\left(\frac{G{\mathcal{M}_r}^2}{8\pi r^4}\right)/dr$ back to Equation 1:

$$\frac{df(r)}{dr} = -\frac{G\mathcal{M}_r}{r^2}\rho + \frac{G\mathcal{M}_r}{r^2}\rho - \frac{G{\mathcal{M}_r}^2}{2\pi r^5} = -\frac{G{\mathcal{M}_r}^2}{2\pi r^5}. \tag{2}$$

Since $G, \mathcal{M}_r, 2\pi > 0$, and we want $r > 0$ otherwise if $r = 0$ will blow up the equation,

$$\frac{G\mathcal{M}_r{}^2}{2\pi r^5} > 0;$$

thus, $df/dr < 0$.

2) Assuming zero pressure at $\mathcal{R}$: $P(\mathcal{R}) = 0$, then

$$f(\mathcal{R}) = P(\mathcal{R}) + \frac{G\mathcal{M}_r{}^2}{8\pi\mathcal{R}^4} = \frac{G\mathcal{M}_r{}^2}{8\pi\mathcal{R}^4}. \tag{3}$$

Note that, Equation (1.42) in the textbook is the result with the constant-density model. Then, as $r \to 0$, using Equation (1.41) in the textbook, $f(r)$ becomes:

$$\begin{aligned}\lim_{r\to 0} f(r) &= \lim_{r\to 0}\left[P(r) + \frac{G\mathcal{M}_r{}^2}{8\pi r^4}\right] \\ &= \lim_{r\to 0} P_c\left(1 - \frac{r^2}{\mathcal{R}^2}\right) + \frac{G}{8\pi}\lim_{r\to 0}\frac{\mathcal{M}_r{}^2}{r^4},\end{aligned}$$

where the second term, by the constand-density model, $\mathcal{M}_r = r^3\mathcal{M}/\mathcal{R}^3$, is

$$\begin{aligned}\lim_{r\to 0}\frac{\mathcal{M}_r{}^2}{r^4} &= \lim_{r\to 0}\left(\frac{r^3}{\mathcal{R}^3}\mathcal{M}\right)^2\frac{1}{r^4} \\ &= \frac{\mathcal{M}^2}{\mathcal{R}^6}\lim_{r\to 0} r^2 = 0.\end{aligned}$$

Therefore, $\lim_{r\to 0} f(r) = P_c$. Since $df/dr < 0$ as we shown in part 1, that means the function $f(r)$ is decreasing as r increasing. So,

$$\lim_{r\to 0} f(r) > f(\mathcal{R}) \implies P_c > \frac{G\mathcal{M}_r{}^2}{8\pi\mathcal{R}^4}, \tag{4}$$

as stated in the question.

Question 2. (Exercises 1.8) In the paragraph follwoing the expression of the virial theorem (1.18) we stated that an extra term $-3P_SV_S$ should appear on the righthand side if we had chosen to consider only that part of the spherical star interior to $r = r_S$ having a volume V_S and a surface pressure P_S at r_S.

1) Prove this for the case of hydrostatic equilibrium; that is, show that the correct expression is

$$2K + \Omega - 3P_SV_S = 0.$$

Hint: Integrate (1.22) by parts using the equation of hydrostatic equilibrium (1.5 or 1.16) and the mass equation, and remember to only go out to r_S in that integration and the one for Ω (1.7).

2) Show explicitly that this amended version works for the constant density sphere.

1) Following by the hint, but using Equation (1.21) instead of (1.22), we have

$$\begin{aligned}2K = 3\int_{r=0}^{r=r_S} P\,dV &= 3\left[(P_SV_S - P_0V_0) - \int_P V\,dP\right] \\ &= 3P_SV_S - 3\int_P V\,dP.\end{aligned} \tag{5}$$

Using Equation (1.16), the second term in Equation 5 becomes

$$3\int_P V dP = 3\int_{\mathcal{M}} V\left(-\frac{G\mathcal{M}_r}{4\pi r^4}d\mathcal{M}_r\right)$$
$$= -3\int_{\mathcal{M}}\left(\frac{4}{3}\pi r^3\right)\left(\frac{G\mathcal{M}_r}{4\pi r^4}\right)d\mathcal{M}_r$$
$$= -\int_{\mathcal{M}}\frac{G\mathcal{M}_r}{r}d\mathcal{M}_r = \Omega. \tag{6}$$

Then,

$$2K = 3P_S V_S - \Omega \implies 2K + \Omega - 3P_S V_S = 0,$$

as required by the question.

2) With the assumption of constant density sphere, we can directly calculate $2K$ and Ω. Since the density is unchanged in r dimension, we have $\mathcal{M}_r$:

$$d\mathcal{M}_r = 4\pi r^2 \rho dr \implies \mathcal{M}_r = \frac{4}{3}\pi\rho r^3.$$

We can find $P(r)$ from the equation of hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{G\mathcal{M}_r}{r^2}\rho = -\frac{4\pi}{3}G\rho^2 r. \tag{7}$$

Take the integration on both side, we have

$$P(r) = -\frac{2\pi}{3}G\rho^2 r^2 + \text{const.}, \tag{8}$$

with the assumption that $P(r_S) = P_S$,

$$P(r_S) = P_S = -\frac{2\pi}{3}G\rho^2 {r_S}^2 + \text{const.}$$
$$\implies \text{const.} = P_S + \frac{2\pi}{3}G\rho^2 {r_S}^2.$$

Thus, Equation 8 is

$$P(r) = P_s + \frac{2\pi}{3}G\rho^2({r_S}^2 - r^2). \tag{9}$$

Note that $dV = d(4\pi r^3/3)$, then $2K$ is

$$2K = \int_V P dV = 3\int_0^{r_S}\left[\frac{2\pi}{3}G\rho^2\left({r_S}^2 - r^2\right) + P_S\right] d\left(\frac{4}{3}\pi r^3\right)$$
$$= 3\int_0^{r_S}\left[\frac{2\pi}{3}G\rho^2\left({r_S}^2 - r^2\right) + P_S\right] 4\pi r^2 dr$$
$$= 3\left[\int_0^{r_S}\frac{2\pi}{3}G\rho^2\left({r_S}^2 - r^2\right)4\pi r^2 dr + P_S\int_0^{r_S}4\pi r^2 dr\right]$$
$$= \frac{16\pi^2}{15}G\rho^2 {r_S}^5 + 3P_S V_S. \tag{10}$$

The Ω, by Equation 6, in this situation, is

$$\begin{aligned}\Omega = 3\int_P V dP &= 3\int_0^{r_S} \left(\frac{4}{3}\pi r^3\right)\left(-\frac{4\pi}{3}G\rho^2 r dr\right) \\ &= -3\int_0^{r_S} \frac{16}{9}\pi^2 G\rho^2 r^4 dr \\ &= -\frac{16\pi^2}{15}G\rho^2 {r_S}^5. \end{aligned} \tag{11}$$

Now,

$$2K + \Omega - 3P_S V_S = \frac{16\pi^2}{15}G\rho^2 {r_S}^5 + 3P_S V_S - \frac{16\pi^2}{15}G\rho^2 {r_S}^5 - 3P_S V_S = 0, \tag{12}$$

which the equation still holds for the constant density sphere.

End of the Assignment