

1 Economic preliminaries

- This document describes a two sector economy with N^f final good firms and N^c capital goods firms.
- Final good firms use capital goods and labor to produce the consumption good. Capital good firms uses land and labor to produce the capital goods.
- In addition to the two types of firms, a representative Household supply labor to both sectors and consume.
- We first present a competitive equilibrium model with representative agents.
- We then write the economy in strategic game form, which makes it suitable for learning based optimization.

2 Equilibrium model

2.1 Structure of the economy

- There are N^f final good firms indexed by i and N^c capital good firms indexed by j .
- In each period, final good firm i produces the consumption good Y_i using labor L_i^f and capital good $K_{i,j}$, where j indexes capital goods sold by firm j . The production function is:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} (L_i^f)^{1-\alpha^f}$$

- Capital goods firm j uses land X_j and labor L_j^f to produced new capital good I_j . The production function is

$$I_j^c = X_j^{\alpha^c} (L_j^c)^{1-\alpha^c}$$

- The representative household decides how much to consume of each final good, C_i , and how many hours to work L^h . The household have utility function

$$U(C, L^h) = \frac{1}{1-\gamma_h} \left(C - \phi \frac{(L^h)^{1+\eta}}{1+\eta} \right)^{1-\gamma}$$

where C is a bundle with constant elasticity of substitution:

$$C = \left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

2.2 Final Good firm (superscript f)

- The production function is:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} (L_i^f)^{1-\alpha^f}$$

where $Z_{i,t}$ is the TFP shock and follows a stochastic process $Z_{i,t+1}^f \sim \mathcal{P}(Z_{i,t}^f)$.

- Profits in each period are

$$\pi_{i,t}^f = p_{i,t}^c Y_{i,t} - w_t L_{i,t}^f - \sum_{j=1}^{N^c} p_{j,t}^k I_{i,j,t}$$

- The firm is risk-averse and it maximizes the discounted sum of utility over profits $U(\pi_t) = \frac{\pi^{1-\gamma^f}}{1-\gamma^f}$.
- The problem of producers is:

$$\mathcal{V}_t^f \left(\{K_{i,j,t}\}_j, Z_t^f \right) = \max_{\{K_{i,j,t+1}, I_{i,j,t}\}_j, L_{i,t}^f} \frac{(\pi_{i,t}^f)^{1-\gamma^f}}{1-\gamma^f} + \beta_f E_t \mathcal{V}_{t+1}^f(\{K_{i,j,t+1}\}_j, Z_{t+1}^f) \quad \text{s.t.}$$

$$K_{i,j,t+1} \leq (1-\delta)K_{i,j,t} + I_{i,j,t} \quad \text{for } j \in [1, \dots, N^c]$$

- The lagrangian of the problem for firm i is:

$$\mathcal{L}_{i,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(\frac{(\pi_{i,t+r}^f)^{1-\gamma^f}}{1-\gamma^f} + \sum_{j=1}^{N^c} (Q_{i,j,t+r} [(1-\delta)K_{i,j,t+r} + I_{i,j,t+r} - K_{i,j,t+r+1}]) \right)$$

- The first order conditions are:

$$[K_{i,j,t+1}] \quad Q_{i,j,t} = \beta_f \mathbb{E}_t \left[\left(\pi_{i,t+1}^f \right)^{-\gamma^f} p_{i,t+1}^c \frac{\alpha^F}{N^f} \frac{Y_{i,t+1}}{K_{i,k,t+1}} + Q_{i,j,t+1}(1 - \delta) \right] \quad (2.1)$$

$$[I_{i,j,t}^f] \quad Q_{i,j,t} = \left(\pi_{i,t}^f \right)^{-\gamma^f} p_{j,t}^k \quad (2.2)$$

$$[L_{i,t}^f] \quad w_t = p_{i,t}^c (1 - \alpha^f) \frac{Y_{i,t}}{L_{i,t}^f} \quad (2.3)$$

- This sector adds the following equations to the system of equations that describe the economy:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} \left(L_i^f \right)^{1-\alpha^f} \quad (2.4)$$

$$\mathcal{M}_{i,t} = \mathbb{E}_t \left[\beta_f \left(\frac{\pi_{i,t+1}}{\pi_{i,t}} \right)^{-\gamma^f} \right] \quad (2.5)$$

$$K_{i,j,t+1} = (1 - \delta) K_{i,j,t} + I_{i,j,t}^f \quad (2.6)$$

$$p_{j,t}^k = \mathbb{E}_t \left[\mathcal{M}_{i,t} \left(p_{i,t}^c \frac{\alpha^F}{N^f} \frac{Y_{i,t+1}}{K_{i,j,t+1}} + p_{j,t+1}^k (1 - \delta) \right) \right] \quad (2.7)$$

$$w_t = p_{i,t}^c (1 - \alpha^f) \frac{Y_{i,t}}{L_{i,t}^f} \quad (2.8)$$

$$Z_{i,t+1}^f \sim \mathcal{P}(Z_{i,t}^f) \quad (2.9)$$

2.3 Capital good firms (superscript c)

- Capital good firms maximize the discounted sum of profits. Profits in each period are

$$\pi_{j,t}^c = p_{j,t}^k I_{j,t}^c - w_t L_{j,t}^c - p_t^x X_{j,t}$$

- The lagrangian of the problem for firm j is:

$$\mathcal{L}_{j,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(p_{j,t+r}^k I_{j,t+r}^c - w_{t+r} L_{j,t+r}^c - p_{t+r}^x X_{j,t+r} \right)$$

- The first order conditions are:

$$[X_{j,t}] \quad p_t^x = p_{j,t}^k \alpha^c \frac{I_{j,t}}{X_{j,t}} \quad (2.10)$$

$$[L_{j,t}^c] \quad w_t = p_{j,t}^k (1 - \alpha^c) \frac{I_{j,t}}{L_{j,t}^c} \quad (2.11)$$

2.4 Household (superscript h)

- The household consume a bundle:

$$C = \left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

- The household maximize intertemporal utility of consumption and leisure:

$$U(C, L^h) = \frac{1}{1 - \gamma_h} \left(C - \phi \frac{(L^h)^{1+\eta}}{1 + \eta} \right)^{1-\gamma}$$

- The lagrangian is

$$\mathcal{L} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_h^r \left\{ \frac{1}{1 - \gamma} \left(C_{t+r} - \phi \frac{(L_{t+r}^h)^{1+\eta}}{1 + \eta} \right)^{1-\gamma} + \right. \\ \left. \mu_{t+r} \left[w_{t+r} L_{t+r}^h + p_{t+r}^x X_{t+r}^g + \sum_{i=1}^{N^f} \pi_{i,t+r}^f + \sum_{j=1}^{N^c} \pi_{j,t+r}^c - \sum_{i=1}^{N^f} p_{i,t+r}^c C_{i,t+r} \right] \right\}$$

- The first order conditions are

$$[C_{i,t}^S] \quad \left(C_t - \phi \frac{(L_t^h)^{1+\eta}}{1 + \eta} \right)^{-\gamma} \left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} C_i^{\frac{-1}{\sigma}} - \mu_t p_{i,t}^c = 0 \quad (2.12)$$

$$[L_t^h] \quad \left(C_t - \phi \frac{(L_t^h)^{1+\eta}}{1 + \eta} \right)^{-\gamma} \phi (L_t^h)^\eta - \mu_t w_t = 0 \quad (2.13)$$

- We can summarize the conditions as:

$$\frac{p_{i,t}^c}{w_t} = \frac{\left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} C_i^{\frac{-1}{\sigma}}}{\phi(L_t^h)^\eta} \quad \text{for } i \in [1, \dots, N^f] \quad (2.14)$$

2.5 Market Clearing

- Final good: $Y_{i,t} = C_{i,t}$ for $i \in [1, \dots, N^f]$
- Capital Goods: $\sum_{i=1}^{N^f} I_{i,j,t}^f = I_{j,t}^c$ for $j \in [0, \dots, N^c]$
- Labor Market: $L_t^s = \sum_{i=1}^{N^f} L_{i,t}^f + \sum_{j=1}^{N^c} L_{j,t}^c$
- Land: $1 = \sum_{j=1}^{N^c} X_{j,t}$

2.6 Deterministic Steady-State

- There are x unknowns: prices w, p^x, p^k , quantities Y, L^f, L^c, I, K .
- Given these X variables we have the following conditions:

$$\begin{aligned} Y_i &= \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c}\right)^{\alpha^f} (L_i^f)^{1-\alpha^f} \quad \text{for } i \in [1, \dots, N^f] \\ I_{i,j}^f &= \delta K_{i,j} \quad \text{for } i \in [1, \dots, N^f] \text{ and } j \in [1, \dots, N^c] \\ I_j^c &= X_j^{\alpha^c} (L_j^c)^{1-\alpha^c} \quad \text{for } j \in [1, \dots, N^c] \\ p_j^k &= \beta_f \left(p_i^c \frac{\alpha^f}{N^f} \frac{Y_i}{K_{i,j}} + p_j^k (1 - \delta) \right) \quad \text{for } i \in [1, \dots, N^f] \text{ and } j \in [1, \dots, N^c] \\ w &= p_i^c (1 - \alpha^f) \frac{Y_i}{L_i^f} \quad \text{for } i \in [1, \dots, N^f] \\ p^x &= p_j^k \alpha^c \frac{I_j^c}{X_j} \quad \text{for } j \in [1, \dots, N^c] \\ w &= p_j^k (1 - \alpha^c) \frac{I_j^c}{L_j^c} \quad \text{for } j \in [1, \dots, N^c] \\ \frac{p_i^c}{w} &= \frac{\left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} C_i^{\frac{-1}{\sigma}}}{\phi(L_t^h)^\eta} \quad \text{for } i \in [1, \dots, N^f] \\ \sum_{i=1}^{N^f} p_i^c C_i &= w L^h + p^x \end{aligned}$$

2.7 The $N^f = 1$ and $N^c = 1$ case

- We impose
- We can take out many variables from the steady state

$$\begin{aligned}
 Y &= K^{\alpha^f} \left(L^f \right)^{1-\alpha^f} \\
 w &= \phi \left(L^f + L^c \right)^\eta \\
 p^x &= p^k \alpha^c \delta K \\
 I &= \delta K \\
 I &= (L^c)^{1-\alpha^c} \\
 p^k K &= \frac{\beta_f}{1 - (1 - \delta)\beta_f} \alpha^f K^{\alpha^f} \left(L^f \right)^{1-\alpha^f}
 \end{aligned}$$

- The steady state is

$$\begin{aligned}
 \phi \left(L^f + L^c \right)^\eta L^f &= (1 - \alpha^f) \left(\delta^{-1} (L^c)^{1-\alpha^c} \right)^{\alpha^f} \left(L^f \right)^{1-\alpha^f} \\
 \phi \left(L^f + L^c \right)^\eta L^c &= (1 - \alpha^c) \delta \frac{\beta_f}{1 - (1 - \delta)\beta_f} \alpha^f \left(\delta^{-1} (L^c)^{1-\alpha^c} \right)^{\alpha^f} \left(L^f \right)^{1-\alpha^f}
 \end{aligned}$$

- We can confirm our solution by confirming that the goods market clear:

$$Y = w \left(L^f + L^c \right) + p^x$$