1 Economic preliminaries

- This document describes a two sector economy with N^f final good firms and N^c capital goods firms.

- Final good firms use capital goods and labor to produce the consumption good. Capital good firms uses land and labor to produce the capital goods.

- In addition to the two types of firms, a representative Household supply labor to both sectors and consume.

- We first present a competitive equilibrium model with representative agents.

- We then write the economy in strategic game form, which makes it suitable for learning based optimization.

2 Equilibrium model

2.1 Structure of the economy

- There are N^f final good firms indexed by i and N^c capital good firms indexed by j.

- In each period, final good firm i prduces the consumption good Y_i using labor L_i^f and capital good $K_{i,j}$, where j indexes capital goods sold by firm j. The production function is:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} \left(L_i^f \right)^{1-\alpha^f}$$

- Capital goods firm j uses land X_j and labor L_j^f to produced new capital good I_j. The production function is

$$I_j^c = X_j^{\alpha^c} \left(L_j^c \right)^{1-\alpha^C}$$

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- The representative household decides how much to consume of each final good, C_i, and how many hours to work L^h. The household have utility function

$$U(C, L^h) = \frac{1}{1 - \gamma_h} \left(C - \phi \frac{\left(L^h \right)^{1+\eta}}{1 + \eta} \right)^{1-\gamma}$$

where C is a bundle with constant elasticity of substitution:

$$C = \left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

2.2 Final Good firm (superscript f)

- The production function is:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} \left(L_i^f \right)^{1-\alpha^f}$$

where $Z_{i,t}$ is the TFP shock and follows a stochastic process $Z_{i,t+1}^f \sim \mathcal{P}(Z_{i,t}^f)$.

- Profits in each period are

$$\pi_{i,t}^f = p_{i,t}^c Y_{i,t} - w_t L_{i,t}^f - \sum_{j=1}^{N^c} p_{j,t}^k I_{i,j,t}$$

- The firm is risk-adverse and it maximizes the discounted sum of utility over profits $U(\pi_t) = \frac{\pi^{1-\gamma^f}}{1-\gamma^f}$.

- The problem of producers is:

$$\mathcal{V}_t^f \left(\{K_{i,j,t}\}_j, Z_t^f \right) = \max_{\{K_{i,j,t+1}, I_{i,j,t}\}_j, L_{i,t}^f} \frac{\left(\pi_{i,t}^f\right)^{1-\gamma^f}}{1-\gamma^f} + \beta_f E_t \mathcal{V}_{t+1}^f(\{K_{i,jt+1}\}_j, Z_{t+1}^f) \quad \text{s.t.}$$

$$K_{i,j,t+1} \leq (1-\delta)K_{i,j,t} + I_{i,j,t} \quad \text{for} \quad j \in [1, ..., N^c]$$

- The lagrangian of the problem for firm i is:

$$\mathcal{L}_{i,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(\frac{\left(\pi_{i,t+r}^f\right)^{1-\gamma^f}}{1-\gamma^f} + \sum_{j=1}^{N^c} (Q_{i,j,t+r}[(1-\delta)K_{i,j,t+r} + I_{i,j,t+r} - K_{i,j,t+r+1}]) \right)$$

- The first order conditions are:

$$[K_{i,j,t+1}] \qquad Q_{i,j,t} = \beta_f \mathbb{E}_t \left[\left(\pi_{i,t+1}^f \right)^{-\gamma^f} p_{i,t+1}^c \frac{\alpha^F}{N^f} \frac{Y_{i,t+1}}{K_{i,k,t+1}} + Q_{i,j,t+1}(1 - \delta) \right] \qquad (2.1)$$

$$\left[I_{i,j,t}^f \right] \qquad Q_{i,j,t} = \left(\pi_{i,t}^f \right)^{-\gamma^f} p_{j,t}^k \qquad (2.2)$$

$$\left[L_{i,t}^f \right] \qquad w_t = p_{i,t}^c (1 - \alpha^f) \frac{Y_{i,t}}{L_{i,t}^f} \qquad (2.3)$$

- This sector adds the following equations to the system of equations that describe the economy:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} \left(L_i^f \right)^{1-\alpha^f} \qquad (2.4)$$

$$\mathcal{M}_{i,t} = \mathbb{E}_t \left[\beta_f \left(\frac{\pi_{i,t+1}}{\pi_{i,t}} \right)^{-\gamma^f} \right] \qquad (2.5)$$

$$K_{i,j,t+1} = (1 - \delta) K_{i,j,t} + I_{i,j,t}^f \qquad (2.6)$$

$$p_{j,t}^k = \mathbb{E}_t \left[\mathcal{M}_{i,t} \left(p_{i,t}^c \frac{\alpha^F}{N^f} \frac{Y_{i,t+1}}{K_{i,j,t+1}} + p_{j,t+1}^k (1 - \delta) \right) \right] \qquad (2.7)$$

$$w_t = p_{i,t}^c (1 - \alpha^f) \frac{Y_{i,t}}{L_{i,t}^f} \qquad (2.8)$$

$$Z_{i,t+1}^f \sim \mathcal{P}(Z_{i,t}^f) \qquad (2.9)$$

2.3 Capital good firms (superscript c)

- Capital good firms maximize the discounted sum of profits. Profits in each period are

$$\pi_{j,t}^c = p_{j,t}^k I_{j,t}^c - w_t L_{j,t}^c - p_t^x X_{j,t}$$

- The lagrangian of the problem for firm j is:

$$\mathcal{L}_{j,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(p_{j,t+r}^k I_{j,t+r}^c - w_{t+r} L_{j,t+r}^c - p_{t+r}^x X_{j,t+r} \right)$$

- The first order conditions are:

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$$[X_{j,t}] \qquad\qquad p_t^x = p_{j,t}^k \alpha^c \frac{I_{j,t}}{X_{j,t}} \qquad\qquad (2.10)$$

$$[L_{j,t}^c] \qquad\qquad w_t = p_{j,t}^k(1-\alpha^c)\frac{I_{j,t}}{L_{j,t}^c} \qquad\qquad (2.11)$$

2.4   Household (superscript h)

- Te household consume a bundle:

$$C = \left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

- The household maximize intertemporal utility of consumption and leisure:

$$U(C,L^h) = \frac{1}{1-\gamma_h}\left(C - \phi\frac{\left(L^h\right)^{1+\eta}}{1+\eta} \right)^{1-\gamma}$$

- The lagrangian is

$$\mathcal{L} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_h^r \left\{ \frac{1}{1-\gamma}\left(C_{t+r} - \phi\frac{\left(L_{t+r}^h\right)^{1+\eta}}{1+\eta} \right)^{1-\gamma} + \right.$$

$$\left. \mu_{t+r}\left[w_{t+r}L_{t+r}^h + p_{t+r}^x X_{t+r}^g + \sum_{i=1}^{N^f} \pi_{i,t+r}^f + \sum_{j=1}^{N^c} \pi_{j,t+r}^c - \sum_{i=1}^{N^f} p_{i,t+r}^c C_{i,t+r} \right] \right\}$$

- The first order conditions are

$$[C_{i,t}^S] \qquad \left(C_t - \phi\frac{\left(L_t^h\right)^{1+\eta}}{1+\eta} \right)^{-\gamma} \left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} C_i^{\frac{-1}{\sigma}} - \mu_t p_{i,t}^c = 0 \qquad (2.12)$$

$$[L_t^h] \qquad \left(C_t - \phi\frac{\left(L_t^h\right)^{1+\eta}}{1+\eta} \right)^{-\gamma} \phi\left(L_t^h\right)^{\eta} - \mu_t w_t = 0 \qquad (2.13)$$

- We can summarize the conditions as:

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$$\frac{p_{i,t}^c}{w_t} = \frac{\left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} C_i^{\frac{-1}{\sigma}}}{\phi\left(L_t^h\right)^{\eta}} \quad \text{for} \quad i \in [1, ..., N^f] \tag{2.14}$$

2.5 Market Clearing

- Final good: $Y_{i,t} = C_{i,t}$ for $i \in [1, ..., N^f]$

- Capital Goods: $\sum_{i=1}^{N^f} I_{i,j,t}^f = I_{j,t}^c$ for $j \in [0, ..., N^c]$

- Labor Market: $L_t^s = \sum_{i=1}^{N^f} L_{i,t}^f + \sum_{j=1}^{N^c} L_{j,t}^c$

- Land: $1 = \sum_{j=1}^{N^c} X_{j,t}$

2.6 Deterministic Steady-State

- There are x unkowns: prices w, p^x, p^k, quantities Y, L^f, L^c, I, K.

- Given these X variables we have the following conditions:

$$Y_i = \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c}\right)^{\alpha^f} \left(L_i^f\right)^{1-\alpha^f} \quad \text{for } i \in [1, ..., N^f]$$

$$I_{i,j}^f = \delta K_{i,j} \quad \text{for } i \in [1, ..., N^f] \text{ and } j \in [1, ..., N^c]$$

$$I_j^c = X_j^{\alpha^c} \left(L_j^c\right)^{1-\alpha^c} \quad \text{for } j \in [1, ..., N^c]$$

$$p_j^k = \beta_f \left(p_i^c \frac{\alpha^F}{N^f} \frac{Y_i}{K_{i,j}} + p_j^k(1-\delta)\right) \quad \text{for } i \in [1, ..., N^f] \text{ and } j \in [1, ..., N^c]$$

$$w = p_i^c(1-\alpha^f)\frac{Y_i}{L_i^f} \quad \text{for } i \in [1, ..., N^f]$$

$$p^x = p_j^k \alpha^c \frac{I_j^c}{X_j} \quad \text{for } j \in [1, ..., N^c]$$

$$w = p_j^k(1-\alpha^c)\frac{I_j^c}{L_j^c} \quad \text{for } j \in [1, ..., N^j]$$

$$\frac{p_i^c}{w} = \frac{\left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} C_i^{\frac{-1}{\sigma}}}{\phi\left(L_t^h\right)^{\eta}} \quad \text{for} \quad i \in [1, ..., N^f]$$

$$\sum_{i=1}^{N^f} p_i^c C_i = wL^h + p^x$$

2.7 The $N^f = 1$ and $N^c = 1$ case

- We assume that land is fixed. The first order condition of the capital goods sector now are:

$$[X_{j,t}] \qquad\qquad p_t^x = p_{j,t}^k \alpha^c \frac{I_{j,t}}{X_{j,t}} \qquad\qquad (2.15)$$

$$\left[L_{j,t}^c\right] \qquad\qquad w_t = p_{j,t}^k (1 - \alpha^c) \frac{I_{j,t}}{L_{j,t}^c} \qquad\qquad (2.16)$$

- We can take out many variables from the steady state

$$Y = K^{\alpha^f} \left(L^f\right)^{1-\alpha^f}$$

$$w = \phi \left(L^f + L^c\right)^{\eta}$$

$$p^x = p^k \alpha^c \delta K$$

$$I = \delta K$$

$$I = (L^c)^{1-\alpha^c}$$

$$p^k K = \frac{\beta_f}{1 - (1 - \delta)\beta_f} \alpha^f K^{\alpha^f} \left(L^f\right)^{1-\alpha^f}$$

- The steady state is

$$\phi \left(L^f + L^c\right)^{\eta} L^f = (1 - \alpha^f) \left(\delta^{-1} (L^c)^{1-\alpha^c}\right)^{\alpha^f} \left(L^f\right)^{1-\alpha^f}$$

$$\phi \left(L^f + L^c\right)^{\eta} L^c = (1 - \alpha^c)\delta \frac{\beta_f}{1 - (1 - \delta)\beta_f} \alpha^f \left(\delta^{-1} (L^c)^{1-\alpha^c}\right)^{\alpha^f} \left(L^f\right)^{1-\alpha^f}$$

- We can confirm our solution by confirming that the goods market clear:

$$Y = w \left(L^f + L^c\right) + pi^c + pi^f + p^x$$

3 Model with no labor and exogenous cost curve

3.1 Final Good firm (superscript f)

- The production function is:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f}$$

where $Z_{i,t}$ is the TFP shock and follows a stochastic process $Z_{i,t+1}^f \sim \mathcal{P}(Z_{i,t}^f)$.

- Profits in each period are

$$\pi_{i,t}^f = p_{i,t}^c Y_{i,t} - \sum_{j=1}^{N^c} p_{j,t}^k I_{i,j,t}$$

- The firm is risk-adverse and it maximizes the discounted sum of utility over profits $U(\pi_t) = \frac{\pi^{1-\gamma^f}}{1-\gamma^f}$.

- The problem of producers is:

$$\mathcal{V}_t^f \left(\{K_{i,j,t}\}_j, Z_t^f \right) = \max_{\{K_{i,j,t+1}, I_{i,j,t}\}_j, L_{i,t}^f} \frac{\left(\pi_{i,t}^f\right)^{1-\gamma^f}}{1-\gamma^f} + \beta_f E_t \mathcal{V}_{t+1}^f(\{K_{i,jt+1}\}_j, Z_{t+1}^f) \quad \text{s.t.}$$

$$K_{i,j,t+1} \leq (1-\delta)K_{i,j,t} + I_{i,j,t} \quad \text{for} \quad j \in [1, ..., N^c]$$

- The lagrangian of the problem for firm i is:

$$\mathcal{L}_{i,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(\frac{\left(\pi_{i,t+r}^f\right)^{1-\gamma^f}}{1-\gamma^f} + \sum_{j=1}^{N^c} (Q_{i,j,t+r}[(1-\delta)K_{i,j,t+r} + I_{i,j,t+r} - K_{i,j,t+r+1}]) \right)$$

- The first order conditions are:

$$[K_{i,j,t+1}] \qquad Q_{i,j,t} = \beta_f \mathbb{E}_t \left[\left(\pi_{i,t+1}^f\right)^{-\gamma^f} p_{i,t+1}^c \frac{\alpha^F}{N^f} \frac{Y_{i,t+1}}{K_{i,k,t+1}} + Q_{i,j,t+1}(1-\delta) \right] \qquad (3.1)$$

$$\left[I_{i,j,t}^f\right] \qquad Q_{i,j,t} = \left(\pi_{i,t}^f\right)^{-\gamma^f} p_{j,t}^k \qquad (3.2)$$

- This sector adds the following equations to the system of equations that describe the economy:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} \tag{3.3}$$

$$\mathcal{M}_{i,t} = \mathbb{E}_t \left[\beta_f \left(\frac{\pi_{i,t+1}}{\pi_{i,t}} \right)^{-\gamma^f} \right] \tag{3.4}$$

$$K_{i,j,t+1} = (1 - \delta) K_{i,j,t} + I_{i,j,t}^f \tag{3.5}$$

$$p_{j,t}^k = \mathbb{E}_t \left[\mathcal{M}_{i,t} \left(p_{i,t}^c \frac{\alpha^F}{Nf} \frac{Y_{i,t+1}}{K_{i,j,t+1}} + p_{j,t+1}^k (1 - \delta) \right) \right] \tag{3.6}$$

$$Z_{i,t+1}^f \sim \mathcal{P}(Z_{i,t}^f) \tag{3.7}$$

3.2 Capital good firms (superscript c)

- Capital good firms maximize the discounted sum of profits. Profits in each period are

$$\pi_{j,t}^c = p_{j,t}^k I_{j,t}^c - \frac{1}{2} \left(I_{j,t}^c \right)^2$$

- The lagrangian of the problem for firm j is:

$$\mathcal{L}_{j,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(p_{j,t+r}^k I_{j,t+r}^c - \frac{1}{2} I_{j,t+r}^2 \right)$$

- The first order conditions is:

$$[i_{j,t}] \qquad\qquad p_t^K = I_{j,t} \tag{3.8}$$

- For $N^f = 1$, $N^c = 1$, The steady state is:

$$Y = K^{\alpha^f}$$
$$I = \delta K$$
$$I = p^k$$
$$p^k = \frac{\beta_f}{1 - (1 - \delta)\beta_f} \alpha^f K^{\alpha^f - 1}$$

- Solution:

$$Y = K^{\alpha^f}$$

$$I = \delta K$$

$$I = p^k$$

$$K = \left(\frac{\beta_f}{1 - (1 - \delta)\beta_f} \frac{\alpha^f}{\delta} \right)^{\frac{1}{2 - \alpha^f}}$$