1 Baseline framework

- N^h households indexed by i choose how much to consume and how much to invest in order to maximize $\sum_{t=0}^{\infty} \beta^t U(C_{i,t})$.

- The final good is produced using N^c capital goods indexed by j according to technology

$$Y_{i,t} = Z^h_{i,t} \left(\sum_{j=1}^{N^c} K_{i,j,t}^{1/N^c} \right)^{\alpha}$$

  where total factor productivity $Z^h_{i,t}$ follows the stochastic process $Z^h_{i,t+1} \sim \mathcal{P}(Z^h_{i,t})$.

- The agent needs to decide how much of the final good to consume and how much to use for investment in new capital goods. The evolution of the stock of capital is

$$K_{i,j,t+1} = (1-\delta)K_{i,j,t} + I^h_{i,j,t}$$

  where $I^h_{i,j,t}$ represents investment in capital good j by household i in period t[1].

- **Adjustment costs:** The cost in term of the final good of producing $I^h_{j,t}$ new units of capital good j is $\frac{\phi}{2}\left(I^h_{j,t}\right)^2$.

- In order to frame the problem, we define the saving rate $s_{j,t}$ as the expenditure on new capital goods j in terms of the final good. Thus, consumption is $C_{i,t} = (1 - \sum_{j=1}^{N^c} s_{i,j,t})Y_{i,t}$

- We will consider two cases:

  - First, we assume that the households itself pays the adjustment cost. In that case, investment is defined by the equation $\frac{\phi}{2}\left(I^h_{i,j,t}\right)^2 = s_{i,j,t}Y_{i,t}$. Thus:

$$I^h_{i,j,t} = \sqrt{\frac{2}{\phi} s_{i,j,t} Y_{i,t}}$$

    .

  - Second, we assume that there is a market for investment goods and we introduce capital good firms that pay the adjustment cost and sell the investment good at price $p^k_{j,t}$. Thus, from the point of view of the household, investment is $I^h_{i,j,t} = s_{i,j,t}/p^k_{j,t}$.

[1] The superscript h is added because we will consider both planner and market solutions to the problem, and in the latter it is useful to keep track of the agent that is choosing the variable I.

1.1 Planner Solution

- The recursive formulation of the problem is:

$$
\begin{aligned}
V\left(\{K_{i,j,t}\}_{i,j}\right) = \max_{\{s_{i,j,t},K_{i,j,t+1}\}_{i,j}} & \sum_{i=1}^{N^h} U(C_{i,t}) + \beta \mathbb{E}_t V(\{K_{i,j,t+1}\}_{i,j}) \qquad \text{s.t.} \\
& C_{i,t} = (1 - \sum_{j=1}^{N^c} s_{i,j,t})Y_{i,t} \quad \text{for } i \in [1, ..., N^h] \\
& Y_{i,t} = Z^h_{i,t} \left(\sum_{j=1}^{N^c} K_{i,j,t}^{1/N^c} \right)^{\alpha} \qquad \text{for } i \in [1, ..., N^h] \\
& K_{i,j,t+1} \leq (1-\delta)K_{i,j,t} + \sqrt{\frac{2}{\phi} s_{i,j,t} Y_{i,t}} \quad \text{for } i \in [1, ..., N^h] \text{ and } j \in [1, ..., N^c]
\end{aligned}
$$

- The lagrangian of the problem for firm i is:

$$
\mathcal{L}_t = \mathbb{E}_t \sum_{r=0}^{\infty} \beta^r \left((1 - s_{t+r})K_{t+r}^{\alpha} + Q_{t+r} \left[(1-\delta)K_{t+r} + \sqrt{2 s_{t+r} K_{t+r}^{\alpha}} - K_{t+r+1} \right] \right)
$$

- The first order conditions are:

$$
[s_t] \qquad Q_t = \sqrt{2 s_t K_t^{\alpha}} = I_t \tag{1.1}
$$

$$
[K_{t+1}] \qquad Q_t = \beta(1 - s_{t+1})\alpha K_{t+1}^{\alpha-1} + \beta Q_{t+1} \left[(1-\delta) + \frac{s_{t+1}\alpha K_{t+1}^{\alpha-1}}{\sqrt{2 s_{t+1} K_{t+1}^{\alpha}}} \right] \tag{1.2}
$$

1.2 Steady State

- In the steady state, we have that $\sqrt{2sK^{\alpha}} = \delta K$ and so $sK^{\alpha-1} = \delta^2 K/2$.
- By replacing the first F.O.C. in the second F.O. an, getting rid of time subscript we get

$$
\sqrt{2sK^{\alpha}} = \beta\alpha K^{\alpha-1} - \beta\alpha s K^{\alpha-1} + \beta\sqrt{2sK^{\alpha}}(1-\delta) + \beta\alpha s K^{\alpha-1} \tag{1.3}
$$

- Collecting terms and replacing $\sqrt{2sK^{\alpha}} = \delta K$ we get:

$$\delta K = \beta\alpha K^{\alpha-1} + \beta\delta K(1-\delta) \tag{1.4}$$

- Finally, solving for K we get:

$$K = \left[\frac{\beta\alpha}{\delta[1-\beta(1-\delta)]}\right]^{\frac{1}{2-\alpha}} \tag{1.5}$$

1.3 Market Solution

- Now we have agents, the final good agent, which purchases capital goods and consume the final good, and the capital good firm, which produces the capital good subject to a quadratic cost.

1.3.1 Final Good firm

- The production function is:

$$Y = K^{\alpha}$$

where $Z_{i,t}$ is the TFP shock and follows a stochastic process $Z^f_{i,t+1} \sim \mathcal{P}(Z^f_{i,t})$.

- Profits in each period are

$$\pi^f_{i,t} = p^c_{i,t}Y_{i,t} - \sum_{j=1}^{N^c} p^k_{j,t}I_{i,j,t}$$

- The firm is risk-adverse and it maximizes the discounted sum of utility over profits $U(\pi_t) = \frac{\pi^{1-\gamma^f}}{1-\gamma^f}$.
- The problem of producers is:

$$\mathcal{V}^f_t\left(\{K_{i,j,t}\}_j,\, Z^f_t\right) = \max_{\{K_{i,j,t+1},\, I_{i,j,t}\}_j, L^f_{i,t}} \frac{\left(\pi^f_{i,t}\right)^{1-\gamma^f}}{1-\gamma^f} + \beta_f E_t \mathcal{V}^f_{t+1}(\{K_{i,jt+1}\}_j,\, Z^f_{t+1}) \quad \text{s.t.}$$
$$K_{i,j,t+1} \leq (1-\delta)K_{i,j,t} + I_{i,j,t} \quad \text{for} \quad j \in [1, ..., N^c]$$

– The lagrangian of the problem for firm i is:

$$\mathcal{L}_{i,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(\frac{\left(\pi_{i,t+r}^f\right)^{1-\gamma^f}}{1-\gamma^f} + \sum_{j=1}^{N^c} \left(Q_{i,j,t+r}[(1-\delta)K_{i,j,t+r} + I_{i,j,t+r} - K_{i,j,t+r+1}]\right) \right)$$

– The first order conditions are:

$$[K_{i,j,t+1}] \qquad Q_{i,j,t} = \beta_f \mathbb{E}_t \left[\left(\pi_{i,t+1}^f\right)^{-\gamma^f} p_{i,t+1}^c \frac{\alpha^F}{N^f} \frac{Y_{i,t+1}}{K_{i,k,t+1}} + Q_{i,j,t+1}(1-\delta) \right] \tag{1.6}$$

$$\left[I_{i,j,t}^f\right] \qquad Q_{i,j,t} = \left(\pi_{i,t}^f\right)^{-\gamma^f} p_{j,t}^k \tag{1.7}$$

– This sector adds the following equations to the system of equations that describe the economy:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} \tag{1.8}$$

$$\mathcal{M}_{i,t} = \mathbb{E}_t \left[\beta_f \left(\frac{\pi_{i,t+1}}{\pi_{i,t}} \right)^{-\gamma^f} \right] \tag{1.9}$$

$$K_{i,j,t+1} = (1-\delta)K_{i,j,t} + I_{i,j,t}^f \tag{1.10}$$

$$p_{j,t}^k = \mathbb{E}_t \left[\mathcal{M}_{i,t} \left(p_{i,t}^c \frac{\alpha^F}{N^f} \frac{Y_{i,t+1}}{K_{i,j,t+1}} + p_{j,t+1}^k (1-\delta) \right) \right] \tag{1.11}$$

$$Z_{i,t+1}^f \sim \mathcal{P}(Z_{i,t}^f) \tag{1.12}$$

1.3.2 Capital good firms (superscript c)

– Capital good firms maximize the discounted sum of profits. Profits in each period are

$$\pi_{j,t}^c = p_{j,t}^k I_{j,t}^c - \frac{1}{2}\left(I_{j,t}^c\right)^2$$

– The lagrangian of the problem for firm j is:

$$\mathcal{L}_{j,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(p_{j,t+r}^k I_{j,t+r}^c - \frac{1}{2} I_{j,t+r}^2 \right)$$

– The first order conditions is:

$$[i_{j,t}] \qquad p_t^K = I_{j,t} \tag{1.13}$$

- For $N^f = 1$, $N^c = 1$, The steady state is:

$$\begin{aligned}
Y &= K^{\alpha^f} \\
I &= \delta K \\
I &= p^k \\
p^k &= \frac{\beta_f}{1-(1-\delta)\beta_f}\alpha^f K^{\alpha^f - 1}
\end{aligned}$$

- Solution:

$$\begin{aligned}
Y &= K^{\alpha^f} \\
I &= \delta K \\
I &= p^k \\
K &= \left(\frac{\beta_f}{1-(1-\delta)\beta_f}\frac{\alpha^f}{\delta}\right)^{\frac{1}{2-\alpha^f}}
\end{aligned}$$

2 Two sector model

- This document describes a two sector economy with N^f final good firms and N^c capital goods firms.

- Final good firms use capital goods and labor to produce the consumption good. Capital good firms uses land and labor to produce the capital goods.

- In addition to the two types of firms, a representative Household supply labor to both sectors and consume.

- We first present a competitive equilibrium model with representative agents.

- We then write the economy in strategic game form, which makes it suitable for learning based optimization.

2.1 Structure of the economy

- There are N^f final good firms indexed by i and N^c capital good firms indexed by j.

- In each period, final good firm i prduces the consumption good Y_i using labor L_i^f and capital good $K_{i,j}$, where j indexes capital goods sold by firm j. The production function is:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} \left(L_i^f \right)^{1-\alpha^f}$$

- Capital goods firm j uses land X_j and labor L_j^f to produced new capital good I_j. The production function is

$$I_j^c = X_j^{\alpha^c} \left(L_j^c \right)^{1-\alpha^C}$$

.

- The representative household decides how much to consume of each final good, C_i, and how many hours to work L^h. The household have utility function

$$U(C, L^h) = \frac{1}{1-\gamma_h} \left(C - \phi \frac{\left(L^h\right)^{1+\eta}}{1+\eta} \right)^{1-\gamma}$$

where C is a bundle with constant elasticity of substitution:

$$C = \left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

2.2 Final Good firm (superscript f)

- The production function is:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} \left(L_i^f \right)^{1-\alpha^f}$$

where $Z_{i,t}$ is the TFP shock and follows a stochastic process $Z_{i,t+1}^f \sim \mathcal{P}(Z_{i,t}^f)$.

- Profits in each period are

$$\pi_{i,t}^f = p_{i,t}^c Y_{i,t} - w_t L_{i,t}^f - \sum_{j=1}^{N^c} p_{j,t}^k I_{i,j,t}$$

- The firm is risk-adverse and it maximizes the discounted sum of utility over profits $U(\pi_t) = \frac{\pi^{1-\gamma^f}}{1-\gamma^f}$.
- The problem of producers is:

$$\mathcal{V}_t^f\left(\{K_{i,j,t}\}_j,\, Z_t^f\right) = \max_{\{K_{i,j,t+1},\, I_{i,j,t}\}_j, L_{i,t}^f} \frac{\left(\pi_{i,t}^f\right)^{1-\gamma^f}}{1-\gamma^f} + \beta_f E_t \mathcal{V}_{t+1}^f(\{K_{i,jt+1}\}_j,\, Z_{t+1}^f) \quad \text{s.t.}$$
$$K_{i,j,t+1} \leq (1-\delta)K_{i,j,t} + I_{i,j,t} \quad \text{for} \quad j \in [1, ..., N^c]$$

- The lagrangian of the problem for firm i is:

$$\mathcal{L}_{i,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(\frac{\left(\pi_{i,t+r}^f\right)^{1-\gamma^f}}{1-\gamma^f} + \sum_{j=1}^{N^c} (Q_{i,j,t+r}[(1-\delta)K_{i,j,t+r} + I_{i,j,t+r} - K_{i,j,t+r+1}]) \right)$$

- The first order conditions are:

$$[K_{i,j,t+1}] \qquad Q_{i,j,t} = \beta_f \mathbb{E}_t \left[\left(\pi_{i,t+1}^f\right)^{-\gamma^f} p_{i,t+1}^c \frac{\alpha^F}{N^f} \frac{Y_{i,t+1}}{K_{i,k,t+1}} + Q_{i,j,t+1}(1-\delta) \right] \tag{2.1}$$
$$\left[I_{i,j,t}^f\right] \qquad Q_{i,j,t} = \left(\pi_{i,t}^f\right)^{-\gamma^f} p_{j,t}^k \tag{2.2}$$
$$\left[L_{i,t}^f\right] \qquad w_t = p_{i,t}^c(1-\alpha^f)\frac{Y_{i,t}}{L_{i,t}^f} \tag{2.3}$$

- This sector adds the following equations to the system of equations that describe the economy:

$$Y_i = Z_{i,t}^f \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} \left(L_i^f\right)^{1-\alpha^f} \tag{2.4}$$
$$\mathcal{M}_{i,t} = \mathbb{E}_t \left[\beta_f \left(\frac{\pi_{i,t+1}}{\pi_{i,t}} \right)^{-\gamma^f} \right] \tag{2.5}$$
$$K_{i,j,t+1} = (1-\delta)K_{i,j,t} + I_{i,j,t}^f \tag{2.6}$$
$$p_{j,t}^k = \mathbb{E}_t \left[\mathcal{M}_{i,t} \left(p_{i,t}^c \frac{\alpha^F}{N^f} \frac{Y_{i,t+1}}{K_{i,j,t+1}} + p_{j,t+1}^k(1-\delta) \right) \right] \tag{2.7}$$
$$w_t = p_{i,t}^c(1-\alpha^f)\frac{Y_{i,t}}{L_{i,t}^f} \tag{2.8}$$
$$Z_{i,t+1}^f \sim \mathcal{P}(Z_{i,t}^f) \tag{2.9}$$

2.3 Capital good firms (superscript c)

- Capital good firms maximize the discounted sum of profits. Profits in each period are

$$\pi^c_{j,t} = p^k_{j,t} I^c_{j,t} - w_t L^c_{j,t} - p^x_t X_{j,t}$$

- The lagrangian of the problem for firm j is:

$$\mathcal{L}_{j,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(p^k_{j,t+r} I^c_{j,t+r} - w_{t+r} L^c_{j,t+r} - p^x_{t+r} X_{j,t+r} \right)$$

- The first order conditions are:

$$[X_{j,t}] \qquad p^x_t = p^k_{j,t} \alpha^c \frac{I_{j,t}}{X_{j,t}} \tag{2.10}$$

$$\left[L^c_{j,t}\right] \qquad w_t = p^k_{j,t}(1-\alpha^c)\frac{I_{j,t}}{L^c_{j,t}} \tag{2.11}$$

2.4 Household (superscript h)

- Te household consume a bundle:

$$C = \left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

- The household maximize intertemporal utility of consumption and leisure:

$$U(C, L^h) = \frac{1}{1-\gamma_h} \left(C - \phi \frac{\left(L^h\right)^{1+\eta}}{1+\eta} \right)^{1-\gamma}$$

- The lagrangian is

$$\mathcal{L} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_h^r \left\{ \frac{1}{1-\gamma} \left(C_{t+r} - \phi \frac{\left(L^h_{t+r}\right)^{1+\eta}}{1+\eta} \right)^{1-\gamma} + \right.$$
$$\left. \mu_{t+r} \left[w_{t+r} L^h_{t+r} + p^x_{t+r} X^g_{t+r} + \sum_{i=1}^{N^f} \pi^f_{i,t+r} + \sum_{j=1}^{N^c} \pi^c_{j,t+r} - \sum_{i=1}^{N^f} p^c_{i,t+r} C_{i,t+r} \right] \right\}$$

- The first order conditions are

$$
\begin{aligned}
&\left[C_{i,t}^{S}\right] && \left(C_t-\phi\frac{\left(L_t^h\right)^{1+\eta}}{1+\eta}\right)^{-\gamma}\left(\sum_{i=1}^{N^f}C_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}C_i^{\frac{-1}{\sigma}}-\mu_t p_{i,t}^c=0 && (2.12)\\
&\left[L_t^h\right] && \left(C_t-\phi\frac{\left(L_t^h\right)^{1+\eta}}{1+\eta}\right)^{-\gamma}\phi\left(L_t^h\right)^{\eta}-\mu_t w_t=0 && (2.13)
\end{aligned}
$$

- We can summarize the conditions as:

$$
\frac{p_{i,t}^c}{w_t}=\frac{\left(\sum_{i=1}^{N^f}C_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}C_i^{\frac{-1}{\sigma}}}{\phi\left(L_t^h\right)^{\eta}}\quad\text{for}\quad i\in[1,...,N^f] \tag{2.14}
$$

2.5 Market Clearing

- Final good: $Y_{i,t}=C_{i,t}$ for $i\in[1,...,N^f]$
- Capital Goods: $\sum_{i=1}^{N^f}I_{i,j,t}^f=I_{j,t}^c$ for $j\in[0,...,N^c]$
- Labor Market: $L_t^s=\sum_{i=1}^{N^f}L_{i,t}^f+\sum_{j=1}^{N^c}L_{j,t}^c$
- Land: $1=\sum_{j=1}^{N^c}X_{j,t}$

2.6 Deterministic Steady-State

- There are x unkowns: prices w,p^x,p^k, quantities Y,L^f,L^c,I,K.
- Given these X variables we have the following conditions:

$$
\begin{aligned}
Y_i &= \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c}\right)^{\alpha^f} \left(L_i^f\right)^{1-\alpha^f} \quad \text{for } i \in [1, ..., N^f] \\
I_{i,j}^f &= \delta K_{i,j} \quad \text{for } i \in [1, ..., N^f] \text{ and } j \in [1, ..., N^c] \\
I_j^c &= X_j^{\alpha^c} \left(L_j^c\right)^{1-\alpha^c} \quad \text{for } j \in [1, ..., N^c] \\
p_j^k &= \beta_f \left(p_i^c \frac{\alpha^F}{N^f} \frac{Y_i}{K_{i,j}} + p_j^k(1-\delta)\right) \quad \text{for } i \in [1, ..., N^f] \text{ and } j \in [1, ..., N^c] \\
w &= p_i^c(1-\alpha^f) \frac{Y_i}{L_i^f} \quad \text{for } i \in [1, ..., N^f] \\
p^x &= p_j^k \alpha^c \frac{I_j^c}{X_j} \quad \text{for } j \in [1, ..., N^c] \\
w &= p_j^k(1-\alpha^c) \frac{I_j^c}{L_j^c} \quad \text{for } j \in [1, ..., N^j] \\
\frac{p_i^c}{w} &= \frac{\left(\sum_{i=1}^{N^f} C_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} C_i^{\frac{-1}{\sigma}}}{\phi \left(L_t^h\right)^{\eta}} \quad \text{for} \quad i \in [1, ..., N^f] \\
\sum_{i=1}^{N^f} p_i^c C_i &= w L^h + p^x
\end{aligned}
$$

2.7 The $N^f = 1$ and $N^c = 1$ case

- We assume that land is fixed. The first order condition of the capital goods sector now are:

$$
[X_{j,t}] \qquad p_t^x = p_{j,t}^k \alpha^c \frac{I_{j,t}}{X_{j,t}} \tag{2.15}
$$

$$
\left[L_{j,t}^c\right] \qquad w_t = p_{j,t}^k (1-\alpha^c) \frac{I_{j,t}}{L_{j,t}^c} \tag{2.16}
$$

- We can take out many variables from the steady state

$$
\begin{aligned}
Y &= K^{\alpha^f}\left(L^f\right)^{1-\alpha^f} \\
w &= \phi\left(L^f + L^c\right)^{\eta} \\
p^x &= p^k \alpha^c \delta K \\
I &= \delta K \\
I &= \left(L^c\right)^{1-\alpha^c} \\
p^k K &= \frac{\beta_f}{1-(1-\delta)\beta_f}\alpha^f K^{\alpha^f}\left(L^f\right)^{1-\alpha^f}
\end{aligned}
$$

- The steady state is

$$
\begin{aligned}
\phi\left(L^f + L^c\right)^{\eta} L^f &= (1-\alpha^f)\left(\delta^{-1}\left(L^c\right)^{1-\alpha^c}\right)^{\alpha^f}\left(L^f\right)^{1-\alpha^f} \\
\phi\left(L^f + L^c\right)^{\eta} L^c &= (1-\alpha^c)\delta\frac{\beta_f}{1-(1-\delta)\beta_f}\alpha^f\left(\delta^{-1}\left(L^c\right)^{1-\alpha^c}\right)^{\alpha^f}\left(L^f\right)^{1-\alpha^f}
\end{aligned}
$$

- We can confirm our solution by confirming that the goods market clear:

$$
Y = w\left(L^f + L^c\right) + \pi^c + \pi^f + p^x
$$

- The results are have the following directory:
- Figures:

3 Model with no labor and exogenous cost curve

3.1 Final Good firm (superscript f)

- The production function is:

$$Y_i = Z^f_{i,t}\left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c}\right)^{\alpha^f}$$

where $Z_{i,t}$ is the TFP shock and follows a stochastic process $Z^f_{i,t+1} \sim \mathcal{P}(Z^f_{i,t})$.

- Profits in each period are

$$\pi^f_{i,t} = p^c_{i,t}Y_{i,t} - \sum_{j=1}^{N^c} p^k_{j,t} I_{i,j,t}$$

- The firm is risk-adverse and it maximizes the discounted sum of utility over profits $U(\pi_t) = \frac{\pi^{1-\gamma^f}}{1-\gamma^f}$.
- The problem of producers is:

$$\mathcal{V}^f_t\left(\{K_{i,j,t}\}_j,\, Z^f_t\right) = \max_{\{K_{i,j,t+1}, I_{i,j,t}\}_j, L^f_{i,t}} \frac{\left(\pi^f_{i,t}\right)^{1-\gamma^f}}{1-\gamma^f} + \beta_f E_t \mathcal{V}^f_{t+1}(\{K_{i,jt+1}\}_j,\, Z^f_{t+1}) \quad \text{s.t.}$$
$$K_{i,j,t+1} \leq (1-\delta)K_{i,j,t} + I_{i,j,t} \quad \text{for} \quad j \in [1, ..., N^c]$$

- The lagrangian of the problem for firm i is:

$$\mathcal{L}_{i,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(\frac{\left(\pi^f_{i,t+r}\right)^{1-\gamma^f}}{1-\gamma^f} + \sum_{j=1}^{N^c}\left(Q_{i,j,t+r}[(1-\delta)K_{i,j,t+r} + I_{i,j,t+r} - K_{i,j,t+r+1}]\right)\right)$$

- The first order conditions are:

$$[K_{i,j,t+1}] \qquad Q_{i,j,t} = \beta_f \mathbb{E}_t\left[\left(\pi^f_{i,t+1}\right)^{-\gamma^f} p^c_{i,t+1}\frac{\alpha^F}{N^f}\frac{Y_{i,t+1}}{K_{i,k,t+1}} + Q_{i,j,t+1}(1-\delta)\right] \tag{3.1}$$

$$\left[I^f_{i,j,t}\right] \qquad Q_{i,j,t} = \left(\pi^f_{i,t}\right)^{-\gamma^f} p^k_{j,t} \tag{3.2}$$

- This sector adds the following equations to the system of equations that describe the economy:

$$Y_i = Z^f_{i,t} \left(\sum_{j=1}^{N^c} K_{i,j}^{1/N^c} \right)^{\alpha^f} \tag{3.3}$$

$$\mathcal{M}_{i,t} = \mathbb{E}_t \left[\beta_f \left(\frac{\pi_{i,t+1}}{\pi_{i,t}} \right)^{-\gamma^f} \right] \tag{3.4}$$

$$K_{i,j,t+1} = (1-\delta) K_{i,j,t} + I^f_{i,j,t} \tag{3.5}$$

$$p^k_{j,t} = \mathbb{E}_t \left[\mathcal{M}_{i,t} \left(p^c_{i,t} \frac{\alpha^F}{N^f} \frac{Y_{i,t+1}}{K_{i,j,t+1}} + p^k_{j,t+1}(1-\delta) \right) \right] \tag{3.6}$$

$$Z^f_{i,t+1} \sim \mathcal{P}(Z^f_{i,t}) \tag{3.7}$$

3.2 Capital good firms (superscript c)

- Capital good firms maximize the discounted sum of profits. Profits in each period are

$$\pi^c_{j,t} = p^k_{j,t} I^c_{j,t} - \frac{1}{2} \left(I^c_{j,t} \right)^2$$

- The lagrangian of the problem for firm j is:

$$\mathcal{L}_{j,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(p^k_{j,t+r} I^c_{j,t+r} - \frac{1}{2} I^2_{j,t+r} \right)$$

- The first order conditions is:

$$[i_{j,t}] \qquad\qquad p^K_t = I_{j,t} \tag{3.8}$$

- For $N^f = 1$, $N^c = 1$, The steady state is:

$$\begin{aligned} Y &= K^{\alpha^f} \\ I &= \delta K \\ I &= p^k \\ p^k &= \frac{\beta_f}{1-(1-\delta)\beta_f} \alpha^f K^{\alpha^f - 1} \end{aligned}$$

- Solution:

$$
\begin{aligned}
Y &= K^{\alpha^f} \\
I &= \delta K \\
I &= p^k \\
K &= \left(\frac{\beta_f}{1-(1-\delta)\beta_f} \frac{\alpha^f}{\delta} \right)^{\frac{1}{2-\alpha^f}}
\end{aligned}
$$