Problem 3.25.2

(b) First we observe that the function $f(x) = x - x_0$ is a homeomorphism because it preserves the metric. As a result, we have that x, y are in the same path component if and only if $x - y$ and 0 are in the same component, which means the sequence $z = x-y$ is bounded. Consequently, we can assume without loss of generality that $y = 0$. In the forward direction, we assume that $0, x$ are in the same path component, which guarantees the existence of a continuous path

$$\alpha : [0,1] \longrightarrow \mathbb{R}^\omega$$

Such that $\alpha(0) = 0$ and $\alpha(1) = 1$. Furthermore, the image $\alpha([0,1])$ is connected, meaning that $\alpha(1) = x$ is bounded because α is continuous and the subspace of bounded sequences is connected.

In the reverse direction, we suppose that x is bounded. We need to find a path α that begins at 0 and ends at x. the most natural choice for this path is $\alpha(t) = tx$. It is clear that α is bijective, so we need to show that this map is continuous. Using the boundedness of x we can find some k such that $x_n \leq k$ for all n. With this n in mind we apply the definition of the uniform metric to observe that

$$\overline{\rho}(\alpha(u), \alpha(v)) = \sup_n \{\min(|u-v|x_n, 1)\} = |u-v|n$$

whenever $u, v \in [0,1]$. Now fix an $\epsilon > 0$ and not that if we take $\delta < \min(M, \epsilon/M)$ then

$$\overline{\rho}(\alpha(u), \alpha(v)) < \epsilon$$

for $|u - v| < \delta$. This completes the proof.

Problem 3.25.3

The ordered square S is the product of $[0,1] \times [0,1]$ where both have the order topology. Hence, open neighborhoods of any point x are of the form $(x-\delta, x+\delta) \times (x-\epsilon, x+\epsilon)$. Each factor in the Cartesian product is connected, and hence any open neighborhood about a point is connected. This shows that S is locally connected.

To see that S is not locally path connected, we suppose towards a contradiction that it is. This means that there is some path $\alpha : [0,1] \to S$ wholly contained in S such that $\alpha(0) = u$ and $\alpha(1) = v$. Following the outline of Example 24.6 we note that the image of $\alpha([0,1])$ must contain each point of S by the intermediate value theorem. So we see that for every $x \in [0,1]$ we have

$$U_x = \alpha^{-1}(x \times (0,1))$$

is non-empty and open. For this x, we choose a rational $u_x \in U_x$ and note that because the sets U_x are disjoint, the map $x \to u_x$ is injective. But this is impossible because U_x is an interval and hence uncountable. Note that this relies on the fact that for the particular choice of $u = (x, 0)$ we have that any open set containing u contains a point $(x-\epsilon, 0)$ for some $\epsilon > 0$. Hence, S is not locally path connected.

In light of the above we can see that the path components of S are the vertical lines in S. The intuition that led us to this consequence was that the reason that we are not locally connected at $(x, 0)$ is because there is no smooth way to move right, continuously because that would require a discontinuity at $x = 1$. More formally, each of these vertical segments is of the form

$$V = \{[(x, 0), (x, 1)] \mid x \in S\}$$

Each of these is homeomorphic to the unit interval in the order topology, and so they are path connected. And furthermore, there are no continuous paths $[(x, 0), (x, 1)] \to [(y, 0), (y, 1)]$ for $y \neq x$ for the reasons outlined in the proof that S was not locally path-connected.

Problem 3.25.4
Let X be a locally path-connected topological space and let $\mathcal{O} \subset X$ be open and connected. We use Theorem 25.4 to get that each path component of $\mathcal{O}$ is open in X. We then observe that $\mathcal{O}$ is connected and therefore must have only one path component. Therefore, $\mathcal{O}$ is path connected because the one path component is all of $\mathcal{O}$.

Problem 3.25.5
(a) We take the space T as described in the problem to be the set of all lines connecting $p = (0, 1)$ to rational points of the form $(q, 0)$ with $q \in \mathbb{Q} \cap [0, 1]$. It is clear that T is path connected because given two point $u, v \in T$ we can construct a path α that starts at u, follows the line to p, and then traverses the line to v. All of the points on these lines are contained in T by definition and so α is a valid path. More rigorously take

$$\alpha(t) = \begin{cases} (1 - 2t)u + 2tp & t \leq 1/2 \\ 2tp - (1 - 2t)v & t > 1/2 \end{cases}$$

Which is continuous and traverses the two lines as stated above. It is clear that T is locally path connected at p and so every open set containing p is path connected and hence, connected. This immediately gives that T is locally connected at p. At any other point, an open neighborhood consists of disjoint line segments which are a disjoint union of path connected components, which means that the neighborhood is not connected.

(b) We extend the example in the previous part to construct a space that is path connected but not locally connected anywhere. The intuition is to force there to be a dense set of lines in a neighborhood of p, and preserve the structure of the rest of T. To do this, we simply do the same construction but push the lines up near p. Formally, Take the new space T_p to be the set of all lines connecting $(0, 0)$ to rational points of the form $(-q, 1)$ where $q \in [0, 1]$. Geometrically we reflected T across the y axis and then flipped it upside down. We take our example space $T' = T \cup T_p$. Path connected follows from above because we take u, v and note that the only non-trivial case (as in different from the previous part) is when $u \in T$ and $v \in T_p$. In this case just traverse from u to p, then p to $(0,0)$, then $0,0$ to our destination point, v. So T' is path connected. For local connectedness note that the only points to check are $(0,0)$ and p, all others follow from the preceding part. We will check p, and $(0,0)$ will follow by symmetry. Now any neighborhood of p will contain a dense set of disjoint line segments (on the "left") and because they are disjoint this neighborhood cannot be connected. So T' is not locally connected at p. This completes the proof.

Problem 3.25.9
Following the hint we observe that if $x \in G$ then xC is the component of G containing x. Furthermore, we have that the cosets of C are given by the homeomorphisms $f_\alpha(x) = \alpha \cdot x$ and $g_\alpha(x) = x \cdot \alpha$. Then we see that C is the component of e then we have that xC and Cx are also components of G containing e. This means that $C = xC = Cx$ or equivalently that C is normal in G.

Problem 3.25.10
(b) First we need to show that each component of X is contained in a quasi-component of X. Pick a component Y in X and observe that $x, y \in Y$ we must have that $x \sim y$ because otherwise there would be a separation $A \cup B$ of Y with $x \in A$ and $y \in B$, but this contradicts the fact that Y is a connected component of X. Hence, Y is in contained in some equivalence class of the relation $\sim$.
If we have in addition that X is a locally connected space then we let C be a quasicomponent of X and Y some component of C. We then take U to be the remaining components of Y, not including C. We have that X is locally connected and U, C are open in X and so we can appeal to Theorem 25.3 to see that they are also open in Y. Then we use the same trick as before, if U was non-empty, then we would have a separation on $Y = C \cup U$, which is impossible, and so C is contained in Y. This completes the proof.

Problem 3.26.4
Let X be a metric space and C a compact subspace. It is clear that C is closed in X because X is metric and therefore hausdorff, and we apply Theorem 26.3 to get that C must be closed. Now we need to show that C is bounded in the metric of X. Suppose not, then would see that for any ball $B_r(x)$ there would be some y such that $y \notin B_r(x)$. We then restrict our attention to those balls that have $x \in C$ and note that

$$C \subset \bigcup_{x \in C} \bigcup_{r > 0} B_r(x)$$

But if we try to extract a finite subcover, we can see that we will have a point y such that y is not contained in any finite union of balls, this is because there are finitely many balls and infinitely many points outside of them such as $y \in C$ with $d(y, x) > r$, which is infinite because C is unbounded. But then C is not compact. This contradiction means that C must have been bounded.
For a metric space where not every closed and bounded subspace is compact take any infinite space with the discrete metric. This is clear because a discrete

space is compact if and only if it is finite (the cover of the infinite space by $x_1, x_2, ...$ has no finite subcover).