Problem 9
(b) We can partition $\mathbb{Z}$ into two distinct pieces based on the position of x. Namely,

$$X = \{n \in \mathbb{Z} : n < x\} \text{ and } Y = \{n \in \mathbb{Z} : n > x\}$$

Noting that $X, Y \subset \mathbb{Z}$ means that we can apply the Well-Ordering Principle to that X has a largest element and Y a smallest element. If we set n to be the largest element in X then $n < x < n+1$. Furthermore, this n is unique due to the Well-Ordering Principle.
(d) For x, y rational and $y < x$ we can write $y = p/q$ and $x = m/n$. So $y < x$ implies that $np < mq$. We then create the new rational

$$z = \frac{np + mq}{2nq} = \frac{1}{2}(\frac{p}{q} + \frac{m}{n})$$

This shows that $y < z < x$.

Problem 10
(a) Let $x >$ and $0 \leq h < 1$. Then we can see that

$$\begin{aligned}(x+h)^2 &= x^2 + 2hx + h^2 \\ &\leq x^2 + 2hx + h \\ &= x^2 + h(2x+1)\end{aligned}$$

Where the inequality holds because $h^2 < h$ for $h \, in[0, 1)$. Similarly,

$$\begin{aligned}(x-h)^2 &= x^2 - 2hx + h^2 \\ &\geq x^2 - h(2x)\end{aligned}$$

because $h^2 > 0$.
(b) Now let $x > 0$ and fix $a \in \mathbb{R}$. If $x^2 < a$ then we can choose $h < \frac{a-x^2}{2x+1}$ so that

$$(x+h)^2 \leq x^2 + h(2x+1) < a$$

Analogously, if $x^2 > a$ then we can choose $h > \frac{a-x^2}{2x}$ so that

$$(x-h)^2 \geq x^2 - h(2x) > a$$

(c) Fix a real number a and suppose that the set

$$B = \{x \in \mathbb{R} : 0 < x^2 < a\}$$

is unbounded. Then there exists some integer n such that $n < a < n+1$ and for each $m \in Z$ we can find an $x \in B$ such that $m < x < m+1$ because B is unbounded. But then for large enough m

$$a < (n+1) < m^2 < x^2$$

and then $x \notin B$. This contradiction implies that B must be bounded. Furthermore, we can see that $B \neq \emptyset$ by considering the two cases $a < 1$ and $a \geq 1$. We see that in the first case $x^2 < a$ means that $x < 1$ and so $x^2 < x$. We then consider some $0 < h < 1$ to see that

$$x^2 < (x+h)^2 < x(1+2h) + h$$

if we choose $h < a$ then $x = \frac{a-h}{1+2h}$ will satisfy $x \in B$ so B is not empty. If $a > 1$ then we can choose $k < 1$

$$(1+k)^2 = 1 + 2k + k^2 < 1 + 3k$$

Then because $a \geq 1$, $a - 1 > 0$ and we can take $k < \frac{a-1}{3}$ to see that $(1+k) \in B$. Finally, for $a = 1$ we can choose any $x \in (0,1)$ and $x \in B$. So we are done.
Now let $b = \sup B$. We will see that $b^2 = a$. It is clear that $b^2 \leq a$, so it suffices to prove the reverse inequality. Fix any $\epsilon > 0$ and observe that we can choose a sequence $h_k \to 0$ such that for each k

$$b^2 \geq (b - h_k)^2 \geq b^2 - h_k(2b)$$

Then we have that $a - b^2 < \epsilon - h_k(2b)$. Letting $k \to \infty$ gives the desired result.

(d) Let b and c be positive integers with $b^2 = c^2$. Suppose that $b \neq c$ and without loss of generality that $b < c$. Then by assumption

$$\frac{b}{c} = \frac{c}{b}$$

but the left side is less than 1 and the right side is greater than 1 which is impossible. Therefore $b = c$.

Problem 11

(a) Suppose that m is an odd integer. Then we know that $m/2 \notin \mathbb{Z}$ and so we can pick n such that $n < m/2 < n+1$. We multiply this inequality by 2 to see that $2n < m < 2n+2$. Using the fact that $m \in \mathbb{Z}$ we can see that $m = 2n+1$.

(b) Suppose that p and q are add integers. Then we can write $p = 2n+1$ and $q = 2m+1$. So

$$pq = (2n+1)(2m+1) = 4mn + 2(m+n) + 1 = 2k + 1$$

where $k = mn + m + n$. Hence, pq is also odd. Further, we can see p^n is odd by induction on n.

(c) Suppose that $a > 0$ is rational. Consider the set

$$N = \{x \in \mathbb{Z}^+ : ax \in \mathbb{Z}^+\}$$

Then we set $n = \min_{x \in N} x$. We see that $an = m$ for some integer m. If n, m are both even, then they must share a factor of 2 so $a = \frac{m'}{n'}$, where $n' = n/2$

and $m' = m/2$. This contradicts the minimality of n and therefore n, m cannot both be even.

(d)
Theorem. *The $\sqrt{2}$ is irrational.*

Proof. Suppose that $\sqrt{2}$ were rational. Then we would have $\sqrt{2} = m/n$ with n, m not both even. If we square both sides we see that $2n^2 = m^2$. Hence, we have that m^2 is even by part **(b)**. However, then $m = 2k$ and $m^2 = 4k^2$. We divide by 2 to see that $n^2 = 2k^2$ which means that n must also be even. But this contradicts the fact that n, m are not both even. Hence, a cannot be written as m/n and is therefore irrational. $\square$

Problem 4
(d) Consider the map

$$f : X^n \times X^\omega \longrightarrow X^\omega$$
$$((x_1, \ldots, x_n), (y_1, y_2, \ldots)) \mapsto (x_1, \ldots, x_n, y_1, y_2, \ldots)$$

We need to show that f is bijective. The fact that f is injective is obvious from the definition. To see that f is also surjective take any element $z = (z_1, z_2, \ldots) \in X^\omega$. Then we can see that $f^{-1}(z) = ((z_1, \ldots, z_n), (z_{n+1}, \ldots))$. And furthermore, this inverse is unique because if any other element k had $f(k) = z$ then $k_i = f^{-1}(z)_i$ for all i.

Problem 5
(a) Yes. Write $\mathbf{x} \in \mathbb{Z}^\omega$.
(b) Yes. Take $A_i = \{x \in \mathbb{R} : x \geq i\}$ and then write $\mathbf{x} \in \prod_{i=1}^\infty A_i$.
(c) Yes. Take

$$A_i = \begin{cases} \mathbb{R} & if i < 100 \\ \mathbb{Z} & otherwise \end{cases}$$

Then write $\mathbf{x} \in \prod_{i=1}^\infty A_i$.
(d) No. However, this set is isomorphic to a Cartesian product of subsets of $\mathbb{R}$ via a canonical projection.