Problem 3.26.13

(a) Suppose without loss of generality that $AB \neq G$, because otherwise $AB = G$ and is trivially closed. Otherwise, we can find some $x \in (G - AB)$. Therefore we have that $AB \cap \{x\} = 0$ which implies that $A \cap xB^{-1} = \emptyset$. Because B is compact and multiplication and conversion are continuous we have that the set xB^{-1} is compact. We then have the following

Lemma. *Let G be a topological group and $A, B \subset G$ be disjoint sets with A closed and B compact. Then there exists a neighborhood $\mathcal{C}$ of the identity, e, such that $A \cap \mathcal{C}B = \emptyset$*

Proof. We begin with any $x \in B$, so that $x \in G - A$, an open set in G. Hence, we have that $e \in (G - A)x^{-1}$ and that $(G - A)x^{-1}$ is open because inversion and multiplication are continuous. So we find a neighborhood W_x of x such that $W_x W_x \subset (G - A)x^{-1}$. To see this let $I = \text{int}(G - A)x^{-1}$ (the interior). Then $m^{-1}(I)$ (m is multiplication) contains (e, e) and is open. So we find V_1, V_2 such that $(e, e) \in V_1 \times V_2$ and $V_1 \times V_2 \subset (G - A)x^{-1}$. Take $V_3 = V_1 \cap V_2$ so that $V_3 V_3 \subset (G - A)x^{-1}$ and $e \in V_3$. Set $W_x = V_3 \cap V_3^{-1}$. Then W_x is open and satisfies $W_x W_x \subset (G - A)x^{-1}$. Now we observe that

$$B \subset \bigcup_x W_x x$$

We can find a subcover $\bigcup_{i=1}^n W_{x_i}$. Then we take the set

$$\mathcal{C} = \bigcap_{i=1}^n W_{x_i}$$

So for any $x \in B$ we have that $x \in W_{x_i} x_i$ for some i. So

$$\mathcal{C}x \subset W x_i x \subset W_{x_i} W_{x_i} x_i \subset G - A$$

So we have that $A \cap \mathcal{C}x = \emptyset$. As a result we have that $A \cap \mathcal{C}B = \emptyset$ and we are done. $\square$

We then apply the lemma to the sets A and xB^{-1} so that we find a neighborhood U with $A \cap UxB^{-1} = \emptyset$. Consequently, we must have that $AB \cap Ux = \emptyset$. Furthermore, because $e \in U$ we have that $x \in Ux \subset (G - AB)$. So $G - AB$ must be open and so AB must be closed.

(b) Let A be a closed set in G. Then if $xH \notin p(A)$ then x cannot be in AH, which is closed by the previous part (H is assumed compact). So we can find a neighborhood N of x such that $N \cap AH = \emptyset$. Then $p(N)$ is open because the quotient map is open. Moreover, it contains xH and has $p(N) \cap p(A) = \emptyset$. Hence, $p(A)$ is closed.

(c) We know that $p : G \to G/H$ is continuous. By the previous part we deduce that it is also closed. Then we note that $p^{-1}(xH) = xH$ is compact because the group operation is a homeomorphism. Then we know that p is a perfect map (c.f Munkres pg 172) and as a result the preimage G of G/H is compact.

Problem 4.30.13
This is clear. Let $\mathcal{C} = \{U_\alpha\}_{\alpha \in J}$ be a collection of disjoint subsets of X. X has some countable dense subset D and so $\{D \cap U_\alpha\}_{\alpha \in J}$ is also a disjoint collection of subsets of X. We then choose d_α to be an element of $D \cap U_\alpha$. Clearly, $\alpha \neq \beta$ implies that $d_\alpha \neq d_\beta$ because otherwise $d_\beta \in D \cap U_\alpha \subset U_\alpha$, a contradiction. But then we have a bijective correspondence $d_\alpha \leftrightarrow U_\alpha$ and the d_α are countable so the U_α must also be countable.

Problem 4.31.5
We want to impose the condition that $f(x) = g(x)$ after we apply the map to X. Let

$$P = \{x \,|\, f(x) = g(x)\}$$

Consider the map

$$\begin{aligned} h : X &\to Y \times Y \\ x &\mapsto (f(x), g(x)) \end{aligned}$$

Then points of the form $(y, y) \in Y \times Y$ have preimage $h^{-1}(\{(y, y) \,|\, y \in Y\}) = P$, because we have that $y = f(x) = g(x)$. Furthermore, we have that the set of points $\{(y, y) \,|\, y \in Y\}$ is closed in Y because Y is Hausdorff. Because each of the maps f and g are continuous we have that h is continuous and so the preimage of $\{(y, y) \,|\, y \in Y\}$ is closed. This gives precisely that P is closed and we are done.

Problem 4.31.6
Let $\mathcal{C} \subset Y$ be closed and suppose that V is open and $\mathcal{C} \subset V$. We define $A = p^{-1}(\mathcal{C})$ and $U = p^{-1}(V)$, and note that $A \subset U$ because p is continuous and $\mathcal{C} \subset V$. Find some open set $W \supset A$ such that $\overline{W} \subset U$. Then we see that $p(X - W)$ is closed and

$$\mathcal{C} \subset Y - p(X - W) \subset p(W) \subset V$$

Because p is surjective we see that $y \in \mathcal{C}$ implies the existence of an $a \in A$ such that $y = p(a)$. However, for no $b \in X - A$ do we have $y \notin p(X - A) \supset p(X - W)$. Furthermore, $y \notin p(X - W)$ implies that $y \in p(W)$ by surjectivity. So set $W' = Y - p(X - W)$ and observe that $\mathcal{C} \subset W' \subset V$ is open. So we have that $p(W) \subset V$ is closed and contains W' giving that $\overline{W}' \subset V$. Then for any two $A, B \subset Y$ closed and disjoint use the sets W'_A and W'_B to give a separation into disjoint open sets.

Problem 4.32.6
In the forward direction we suppose that A and B are separated in X. Following the hint, we consider the space $Y = X - (\overline{A} \cap \overline{B})$. Clearly, Y is open because it is the complement of a closed set. Moreover, we have that $A, B \subset Y$. So we

have that

$$\overline{A}_Y \cap \overline{B}_Y = Y \cap \overline{A} \cap \overline{B} = \emptyset$$

We then apply the normality property to get that $\overline{A}$ and $\overline{B}$ can be separated by disjoint open neighborhoods in X, say U and V, respectively. Then we have that $Y \cap U$ and $Y \cap V$ give a separation in Y and we are done.

Conversely, take $Y \subset X$ and $A, B \subset Y$ to be disjoint closed sets in Y. We see that

$$\overline{A}_X \cap B = \overline{A}_X \cap Y \cap B \subset \overline{A}_Y \cap B = \emptyset$$

Analogously, we have that $\overline{B} \cap A = \emptyset$. Thus, A and B can be separated by neighborhoods in X, U and V, respectively, such that $Y \cap U$ and $Y \cap V$. This completes the proof of the claim.

Problem 4.33.5

Most of the work for this theorem actually gets done in the preceding problem, which we will do for the sake of completeness. We first establish the following

Theorem. *Let X be normal. There exists a continuous function $f : X \to [0,1]$ defined by*

$$f(x) = \begin{cases} 0 & x \in A \\ y > 0 & \textit{otherwise} \end{cases}$$

if and only if A is a closed G_δ set in X.

Proof. In the forward direction we are given f. We simply note that we can write the set $\{0\}$ as the intersection

$$\{0\} = \bigcap_n [0, 1/n)$$

Because f is continuous so that we can write

$$f^{-1}(\bigcap_n [0, 1/n) = \bigcap f^{-1}([0, 1/n)$$

Which is a G_δ set in X.

In the reverse direction we have that if A is G_δ then the set $\cap_n V_n = A$. Consider the sets $X - V_n$ for each n. We apply the Urysohn lemma to each of these sets because $A \cap V_n^c = \emptyset$ by construction and both are closed. As a result of this fact, we get a family of functions f_n such that $f_n(A) = \{0\}$ and $f_n(V_n) = 1$. Then we define

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x)$$

Then we have that $f(A) = \{0\}$ because each of the f_n is 0 on A. Furthermore, it is strictly greater than 0 on A^c because $f_n(V_n^c) = 1$ for some n so at least one term in the sum is greater than 0. Finally it is clear that f is continuous because the sum converges absolutely (monotone increasing and bounded by geometric series). Therefore it converges uniformly, and the uniform limit of continuous functions is continuous. □

With this result in hand, the following is more straightforward.

Theorem. *Let X be a normal space. there is a continuous function $f : X \to [0, 1]$ such that*

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \\ 0 < y < 1 & \textit{otherwise} \end{cases}$$

if and only if A and B are disjoint closed G_δ sets in X.

Proof. In the forward direction, suppose that we are given our function f. Then we apply the previous theorem twice on f, A and $(1 - f), B$. As a result this gives that both A and B are G_δ.

For the reverse we apply the proceeding theorem to get the existence of g and h such that $g^{-1}(\{0\}) = A$ and $h^{-1}(\{0\}) = B$. Then we simply define

$$f(x) = \frac{g(x)}{g(x) + h(x)}$$

The claim is that f is the desired function. Clearly, if $x \in A$ then $g(x) = 0$ and so $f(x) = 0$. Next we check the case that $x \in B$, in which case $h(x) = 0$ and we are left with $f(x) = g(x)/g(x) = 1$, as desired. Lastly, we see that otherwise we have a positive real bounded above by 1 and below by 0 as the value of $f(x)$. It is clear that the function is continuous because it is a non-vanishing quotient of continuous functions. This completes the proof. □