1 Section 38: The Stone-Čech Compactification

Exercise 38.4:

We are given a compactification Y of the completely regular space X. Let $c : X \to Y$ be the imbedding of X into Y and let $\beta : X \to \beta(X)$ be the imbedding from X into its Stone-Čech Compactification. By the definition we have that c extends uniquely to a continuous map $g : \beta(X) \to Y$ as seen below

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "\beta"'] & Y \\ \beta(X) \arrow[ur, "g"'] & \end{tikzcd}$$

It is clear that g is the identity on X because c is the identity on X and g extends c. To get that g is a closed map we need the following

Lemma (Exercise 26.4)**.** *A continuous map $f : X \to Y$ from the compact space X to the Hausdorff space Y is closed.*

Proof. Let A be a closed subset of X. Then by Theorem 26.2 (Munkres pg. 165), A is compact. We then apply Theorem 26.5 (Munkres pg. 166) to get that $f(A)$ is compact as well. Then because Y is Hausdorff we can apply Theorem 26.3 (Munkres pg. 165) to get that A is closed as well. Hence, f is a closed map. $\square$

We simply apply the above lemma to g using the fact that X is compact and Y is Hausdorff by construction. Finally, to get surjectivity we note that the image of g is dense in Y because g extends c and we know that $\overline{c(X)} = \overline{X} = Y$ by definition. Because $\overline{c(X)} \subset \beta(X)$ by construction, and g is a closed map that extends c, we see that $g(\overline{X})$ is a closed set containing X and therefore must contain $\overline{X} = Y$. Hence, it contains all of Y and we are done.

Exercise 38.5:

(a) Let $f : S_\Omega \to \mathbb{R}$ be continuous. Following the hint, we fix an $\epsilon > 0$. We need to show that there is some $\alpha \in S_\Omega$ such that $|f(\beta) - f(\alpha)| < \epsilon$ for all $\beta > \alpha$. Suppose not, then we can find an increasing sequence $\{s_n\}_{n=1}^\infty$ in S_Ω such that $|f(s_n) - f(s_{n-1})| \geq \epsilon$. We then note that the sequence $\{s_n\}_{n=1}^\infty$ must converge to its least upper bound in S_Ω, but the sequence $\{f(s_n)\}_{n=1}^\infty$ in $\mathbb{R}$ does not converge by construction. This contradicts the continuity of f and so we must be able to find such an α.

Now we use the preceding to find that f must be constant. Indeed, for each integer n we know there is some element α_n such that $|f(\beta) - f(\alpha_n)| < 1/n$ by the above argument. This forms a sequnce $\{\alpha_n\}_{n=1}^\infty$ in S_Ω and let α be an upper bound. Then we have that $|f(\beta) - f(\alpha)| < 1/n$ for all $\beta > \alpha$ and every n. Hence, $|f(\beta) - f(\alpha)| = 0$ and so f is constant on (α, Ω) as desired.

(b) By the preceeding part, we know that any continuous function $S_\Omega \to \mathbb{R}$ is eventually constant and hence, is bounded. We then have the following

Claim. *Any continuous, bounded function $f : S_\Omega \to \mathbb{R}$ extends uniquely to the one-point compactification $\overline{S_\Omega}$.*

Proof. It is clear that boundedness is necessary because the extension of f will send S_Ω into $\mathbb{R}$ and will therefore be bounded. Let $\bar{f}$ be the extension of f to S_Ω. The condition that $\bar{f}$ must satisfy is that for any sequence $\{s_n\}_{n=1}^{\infty}$ such that $s_n \to s$

$$\lim_{n\to\infty} \bar{f}(s_n) = \bar{f}(s)$$

This is clear when $s \in S_\Omega$ because f is continuous on S_Ω. If $s_n \to \overline{\Omega}$ then we note that $\mathbb{R}$ has the least upper bound property and define $\bar{f}(\overline{\Omega}) = \sup f(s_n)$. This mapping is well-defined because f is continuous and so if $\{t_n\}_{n=1}^{\infty}$ was another sequence converging to $\overline{\Omega}$ then $|f(s_n) - f(t_n)| < \epsilon$ for any $\epsilon > 0$ and $n > N$ for a sufficiently large N. This shows that $\bar{f}$ is well-defined and moreover, unique. □

Because every continuous function on S_Ω extends to the one point compactification we can apply Theorem 38.5 to the one-point compactification and $\beta(S_\Omega)$ to get that the two compactifications are equivalent.

(c) To see that any other compactification Y of S_Ω is equivalent to $\overline{S_\Omega}$ we appeal to the previous exercise and note that Y must be a quotient of $\beta(S_\Omega)$, and hence for each f from S_Ω to $\mathbb{R}$ let the extension $\bar{f}_Y$ be the restriction of $\bar{f}$ to Y. Then $\bar{f}_Y$ is uniquely determined and continuous because it is the composition of continuous functions. We then apply Theorem 38.5 again to see that Y is equivalent to $\overline{S_\Omega}$. Hence, any two compactifications of S_Ω are equivalent.

Exercise 38.7:

(a) Consider the function $f : X \to \{0, 1\}$ given by

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in X - A \end{cases}$$

This function is continuous when X has the discrete topology. As such, it has an extension $\bar{f} : \beta(X) \to \{0, 1\}$ that is also continuous. Suppose that $\overline{A} \cap \overline{X - A} \neq \emptyset$, where closures are taken in $\beta(X)$. Then we would have some point y which belongs to both. Then any neighborhood V of y intersects both A and $X - A$. This would mean that if $\bar{f}$ must be the constant function because otherwise there would be a discontinuity at y. But this is impossible because $\bar{f}$ extends f and maps A and $X - A$ to different values. Hence, $\overline{A}$ and $\overline{X - A}$ are disjoint.

(b) The first thing to notice is the following

Lemma. *Let A and X be as in the preceding part. Then $\overline{A}$ is both open and closed in $\beta(X)$.*

Proof. We note that the conlusion of the previous problem is the same as saying that $\overline{X - A} \subset \beta(X) - \overline{A}$. We then have the chain of inclusions

$$\begin{aligned} \beta(X) - \overline{A} &= \overline{X} - \overline{A} \\ &\subseteq \overline{X - A} \\ &\subseteq \beta(X) - \overline{A} \end{aligned}$$

So we must have that $\beta(X) - \overline{A} = \overline{X - A}$ and thus $\overline{A}$ is open as well as closed. □

The plan is to use this fact to show that $\overline{U}$ is open (it is clearly closed). Indeed, we verify the following

Claim. *If $U \in \beta(X)$ is open then $\overline{U \cap X} = \overline{U}$.*

Proof. One inclusion is obvious, namely $U \cap X \subset U$ implies that $\overline{U \cap X} \subset \overline{U}$. For the reverse inclusion let u be point of $\overline{U}$ and let V be a neighborhood of u. Then we see that $V \cap U \neq \emptyset$ because $u \in \overline{U}$. Consequently,

$$(V \cap U) \cap X = V \cap (U \cap X) \neq \emptyset$$

because X is dense. So $x \in \overline{U \cap X}$, which gives the reverse inclusion so we are done. $\square$

With the clain in hand we apply the lemma to the set $U \cap X$ to get that $\overline{U \cap X} = \overline{U}$ is both open and closed.

(c) Let T be a subset of βX containing at least two points x and y. We need to show that T is not connected. If we pick any open $U \subset \beta(X)$ such that $x \in U$ and $y \notin U$, and such a choice is possible because $\beta(x)$ is Hausdorff, then we have that

$$T = (T \cap \overline{U}) \cup (T - \overline{U})$$

is a separation of T. Thus, we have that the connected components of βX are singletons, because they are both open and connected and any set larger than a singleton is not connected. Thus, $\beta(X)$ is totally disconnected.

Exercise 38.9:

(a) We will suppose to the contrary and pursue a contradiction. The only natural thing to do is to find an "inclusion-map" test like we did in the previous exercise, and use the lift of that map to find a contradiction. So we need to find suitable sets A, B such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\beta"'] & {[0,1]} \\ \beta(X) \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

$\bar{f}^{-1}(0)$ and $\bar{f}^{-1}(1)$ are disjoint, but map to the same value under f, and then we will have our contradiction.

Indeed, suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in X that converges to $x \in \beta(X) - X$. We need to construct disjoint sets such that $x_n \to x$ in both A and B. One way to do this is to remove the duplicates from x_n. We create the new sequence x'_n from x_n inductively by setting $x'_1 = x_1$ and $x_n = \inf_m\{x_m \mid x_m \neq x'_k, k < n\}$. It is easy to see that $x'_n \to x$ because any neighborhood of x contains all but finitely many of the x_n, and let x_N be the highest index not contained in that neighborhood. Then we have that for sufficiently large M, $x'_m = x_n$ with $m > M, n > N$ by construction. So $x'_n \to x$ as well.

By construction, we have that the sets $A = \{x'_1, x'_3, x'_5, \ldots\}$ and $B = \{x'_2, x'_4, x'_6, \ldots\}$ are disjoint, and have limit point x. We then verify the following

Claim. *We have* $\overline{A} = A \cup \{x\}$ *and* $\overline{B} = B \cup \{x\}$*, where the closures are taken in* $\beta(X)$.

Proof. We have already shown that x is a limit point of A and hence, $x \in \overline{A}$. Since $A \subset A \cup \{x\} \subseteq \overline{A}$ and $\overline{A}$ is the smallest closed set containing A it will suffice to show that $A \cup \{x\}$ is closed.

Consider $C = \beta(X) - (A \cup \{x\})$. If we choose any point $y \in C$ then y is not a limit of the sequence $\{x_{2n+1}\}_{n=1}^{\infty}$ by construction. Because y is not a limit of the sequence we an find a neighborhood V of y that contains only finitely many of the x_{2n+1}. Because y is not in A we can remove these finitley many points from V to get a new neighborhood V' such that $V' \cap (A \cup \{x\}) = \emptyset$. Hence, V' is wholly contained in C and so C is open and as a result $A \cup \{x\}$ is closed.

The proof for $B \cup \{x\}$ is analogous. □

This shows the first property that we wanted of A and B, namely that $\overline{A} \cap \overline{B} \neq \emptyset$ while $A \cap B = \emptyset$.

Now we make the observation that A and B are both closed in X because we have that

$$\overline{A} = X \cap \overline{A} = X \cap (A \cup \{x\}) = A$$

and similarly for B. Then we can apply the Urysohn lemma, recalling that X is normal and A, B are disjoint closed subsets of X. Hence, A and B can be separated by a continuous function $f : X \to [0,1]$. We then lift f to a function $\bar{f} : \beta(X) \to [0,1]$ that agrees with f on X. We then note that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$ by construction so that $\overline{A} \subset \bar{f}^{-1}(0)$ and $\overline{1} \subset \bar{f}^{-1}(1)$. So $\overline{A}$ and $\overline{B}$ are disjoint.

This gives a contradiction in that $\overline{A} \cap \overline{B}$ is both empty and non-empty and so no point of $\beta(X) - X$ can be a limit of a sequence of points in X.

(b) Suppose that X is completely regular and noncompact. Then X must be properly contained in $\beta(X)$ because $\beta(X)$ is compact and X is not. Because X is completely regular we can apply the previous part of this exercise to see that no point in $\beta(X) - X$ is a limit of a sequence in X. However, because one-point sets are closed in X there is a function that separates every point of $\beta(X) - X$ and X. If we extend such a function to $\beta(X)$ we see that $\beta(X)$ does not satisfy Theorem 21.3 (Munkres pg. 130), and is therefore not metrizable.

Exercise 38.10:

(i) Let 1_X be the identity map of X. Then if we consider the canonical imbedding of X into $\beta(X)$, which is equal to 1_X on X, we can lift this function to a unique continuous function $\bar{f} : \beta(X) \to \beta(X)$ that is the identity on X. Because X is dense in $\beta(X)$ and $\beta(X)$ is compact and Hausdorff we have that $\bar{f}$ determined by its values on X and therefore must be the identity on all of $\beta(X)$.

(ii) This is a repeated application of the universal property of $\beta(X)$. The situation is most readily understood with a diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \\
\beta(X) \arrow[r] \arrow[ur, "\bar{f}"] & \beta(Y) \arrow[ur, "\bar{g}"] &
\end{tikzcd}$$

We simply apply the lifting twice. We first are given a function $f : X \to Y$, which can be uniquely extended to the function $\bar{f} : \beta(X) \to Y$ and so $g \circ \bar{f}$ is an extension of $g \circ f$ from $\beta(X) \to Z$. However, g can be lifted to a map $\bar{g} : \beta(Y) \to Z$ and so $g \circ \bar{f}$ extends to the map $\bar{g} \circ \bar{f}$ and so because all extensions are unique we have

$$\beta(g \circ f) = \beta(g) \circ \beta(f)$$

In other words, the diagram above commutes.

2 Section 43: Complete Metric Spaces

Exercise 43.5 (Banach Contraction Principle)**:**

Consider any point $x_0 \in X$. From x_0 we construct the sequence $x_0, f(x_0), f(f(x_0)), f^3(x_0), \ldots$, call it $\{x_n\}_{n=1}^{\infty}$. We first show that this sequence converges to some $x \in X$, which we hope will be our fixed point. Let $D = d(x_0, x_1)$, so because f is a contraction $d(x_2, x_1) \leq \alpha D$ and in general we can estimate

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m} d(x_k, d_{k+1}) \\ &= \sum_{k=1}^{m} d(x_k, d_{k+1}) - \sum_{k=1}^{n} d(x_k, d_{k+1}) \\ &= \frac{D}{1-\alpha}(\alpha^{n+1} - \alpha^{m+1}) \\ &\leq \alpha^{n+1}\frac{D}{1-\alpha} \end{aligned}$$

This quantity goes to 0 as n gets large because $\alpha < 1$. Consequently, we have that for sufficiently large n, m that $d(x_n, x_m) < \epsilon$ for any $\epsilon > 0$ and so $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Because X is a complete metric space, we have that $\{x_n\}_{n=1}^{\infty}$ converges in X, say to x.

We will now show that x is the desired fixed point. Note that

$$\lim_{n\to\infty} d(f(x_n), x_n) = \lim_{n\to\infty} \alpha^n D = 0$$

However, because d is continuous on X that we can exchange the limiting operation (twice) and get that $d(f(x), x) = 0$ so $x = f(x)$ and x is the desired fixed point.

For uniqueness, we suppose that there were another point y such that $f(y) = y$. If $y \neq x$ then we have that $d(x, y) > 0$ and so we apply f to get that

$$d(f(x), f(y)) \leq \alpha d(x, y) < d(x, y)$$

But then either $f(x) \neq x$ or $f(y) \neq y$ which is a contradiction. So x must be the unique fixed point. $\square$

Exercise 43.9:

(a) To see that $\sim$ is an equivalence relation we need to verify that it is reflexive, symmetric and transitive. Reflexivity is clear because $d(x_n, x_n) = 0$ for all n and so it clearly converges to 0. Symmetry is also evident from the symmetry of the metric. If we have that $\mathbf{x} \sim \mathbf{y}$ then we have that

$$d(x_n, y_n) = d(y_n, x_n) = 0$$

Which gives that $\mathbf{y} \sim \mathbf{x}$. Transitivity follows from the triangle inequality. Fix an $\epsilon > 0$ and suppose that $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{y} \sim \mathbf{z}$. Then we see that

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$$

So that when we let $n \to \infty$ we get that $d(x_n, z_n) \to 0$. So $\sim$ is an equivalence relation.

(b) Let x, y be two elements of X. We need to show that the map h is injective, continuous and preserves the metric. Indeed, we see that h is injective because if $h(x) = h(y)$ then $[(x, x, \ldots)] = [(y, y, \ldots)]$ which clearly implies that $x = y$ because otherwise the two sequences would differ in every term.

To see that f is continuous we fix an $\epsilon > 0$ and note that if we set $\delta = \epsilon$ then $d(h(x), h(y)) < \epsilon$ whenever $d(x, y) < \delta$ and so f is continuous.

To see that h preserves the metric observe that

$$d(x, y) = \lim_{n\to\infty} d(x, y) = D([(x, x, \ldots)], [(y, y, \ldots)])$$

Because the limit of a constant sequence is just that constant.

(c) Now suppose that we are given an $\mathbf{x} = (x_1, x_2 \ldots) \in Y$. We need to show that there is a sequence of points in Y converging to $\mathbf{x}$. Indeed, let $\epsilon > 0$ be given and recall that $x_1, x_2 \ldots$ is a Cauchy sequence in Y because it is in the image of X, which is defined as a set of Cauchy sequences. We then know that there is some N such that $m, n > N$ implies that $d(x_m x_n) < \epsilon/2$. Set $\mathbf{y} = h(x_N)$ then we note that

$$D(\mathbf{x}, \mathbf{y}) = \lim_{n\to\infty} d(x_n, x_N) \leq \epsilon/2$$

So that $D(\mathbf{x}, h(x_n)) \leq D(\mathbf{x}, h(x_N)) + D(h(x_N), h(x_n)) < \epsilon$. So that $h(x_n) \to \mathbf{x}$ and hence X is dense in Y.

(d) We are given that every Cauchy sequence in A converges in (Z, ρ). Suppose that $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in Z. We need to prove that it converges as well. Well, to each z_k we can find an $a_k \in A$ such that $\rho(a_k, z_k) \leq \epsilon/3$ for any choice of $\epsilon > 0$. Then we can see that $\{a_n\}_{n=1}^{\infty}$ is Cauchy because for sufficiently large n, m we have that

$$\rho(a_n, a_m) \leq \rho(a_n, z_n) + \rho(z_n, z_m) + \rho(z_m, a_m) \leq 3(\epsilon/3)$$

Whenever n, m are chosen large enough that $\rho(z_n, z_m) \leq \epsilon/3$. So a_k is a Cauchy sequence as well. As a result, it must converge in Z, say to a. We will see that $z_n \to a$ as well. Indeed we use the triangle inequality to see that

$$\rho(z_n, a) \leq \rho(z_n, a_n) + \rho(a_n, a)$$

We can pick n large enough that $\rho(a_n, a) \leq \epsilon/2$ and the result follows.

(e) By the previous part, it suffices to show that every Cauchy sequence in the dense subspace $h(X)$ converges in Y. Let $\mathbf{y}_k$ be a Cauchy sequence in $h(X)$, where each $\mathbf{y}_k = (y_k, y_k, \ldots)$. Since h is an isometry we see that $d(y_n, y_m) = D(\mathbf{y}_n, \mathbf{y}_m)$ for each pair n, m. Hence, we see that $(y_1, y_2, \ldots)$ is a Cauchy sequence in X. If we set $\mathbf{y}^* = (y_1, y_2, \ldots)$ then we want to show that $\mathbf{y}_n \to \mathbf{y}^*$. Choose $\epsilon > 0$ and note that there is an N such that $d(y_n, y_m) < \epsilon/2$ for any $n, m > N$. So for each $m \geq N$ we have that

$$D(\mathbf{y}_m, \mathbf{y}^*) = \lim_{n\to\infty} d(y_m, y_n) \leq \epsilon/2$$

So we see that $\mathbf{y} \to \mathbf{y}*$ and we are done.

Exercise 43.10:

Theorem (Uniqueness of the Completion)**.** *Let $h : X \to Y$ and $h' : X \to Y'$ be isometric imbeddings of the complete metric space (X, d) in the complete metric spaces (Y, D) and (Y', D'), respectively. Then there is an isometry of $(\overline{(h(X)}, D)$ with$(\overline{(h'(X)}, D')$ that equals $h'h^{-1}$ on the subspace $h(X)$.*

Proof. This is another situation that is made clear with a diagram

$$\begin{tikzcd}
 & X \arrow[dl, "h"'] \arrow[dr, "h'"] & \\
Y \arrow[rr, dotted, leftrightarrow] & & Y'
\end{tikzcd}$$

We need to "connect the dots", so to speak. We can construct an isometry from the image of $h(X) \subset Y$ to the image $h'(X) \subset Y'$ by composing the maps above, namely, $h' \circ h^{-1}$ is the desired isometry. It is clear that this is an isometry because h, h' are isometries and hence, continuous, injective, and distance preserving, which are all properties preserved under composition.

Then we use the fact that $h(X)$ and $h'(X)$ are dense in Y and Y', respectively to extend $h' \circ h^{-1}$ to some continuous map $\varphi : Y \to Y'$ that agrees with $h' \circ h^{-1}$ on $h(X)$ by defining $\varphi(x) = \lim_{n\to\infty} \varphi(x_n)$ whenever $x_n \to x$ in Y. This sequence converges in Y' because it is Cauchy and Y' is complete. Because limits are unique in metric spaces, the map φ is well-defined.

We now need to verify that φ is an isometry. To see that φ is surjective we note that if $y \in Y'$ if we let u_n be a sequence in $h'(X)$ with limit y then we consider the sequence of preimages $\varphi^{-1}(u_n)$ in Y. Because the space Y' is Hausdorff we must have that

$$\lim_{n\to\infty} \varphi(\varphi^{-1}(u_n)) = \lim_{n\to\infty} u_n = u$$

as desired.

For injectivity suppose that $\lim_{n\to\infty} \varphi(u_n) = \lim_{n\to\infty} \varphi(v_n)$ and $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. Fix an $\epsilon > 0$ and choose M large enough that $D'(\varphi(u_n), \varphi(v_n)) < \epsilon$ for all $n > M$. Then we have that

$$D(u_n, v_n) = D'(\varphi(u_n), \varphi(v_n)) \leq \epsilon$$

since x is Hausdorff we conclude that $u = v$ and so φ is injective.

Finally, we note that the map $\varphi : \overline{h(X)} \to \overline{h'(X)}$ is continuous, because it is the continuous extension of $h' \circ h^{-1}$ and moreover it is bijective, and isometric, and thus is the desired function. $\square$

3 Section 45: Compactness in Metric Spaces

Exercise 45.3:

Theorem (Arzela's Theorem)**.** *Let X be compact; let $f_n \in \mathcal{C}(X, \mathbb{R}^k)$. If the collection $\{f_n\}$ is pointwise bounded and equicontinuous, then the sequence f_n has a uniformly convergent subsequence.*

Proof. We begin with the following

Lemma (Exercise 31.4)**.** *Every compact metrizable space is separable.*

Proof. For each integer n consider a covering of the compact space X by a ball of radius $1/n$ centered at each point of X. Because X is compact, we can extract a finite subcover. If we take the union of all such subcovers we are left with a countable set of balls that we claim is a countable base for X. Indeed, given an open set U in X that contains x there is a $\delta > 0$ such that $B_\delta(x) \subset U$. If we choose n such that $1/n < \delta/2$ then we can find some i such that $x \in B_{1/n}(x_i) \subset B_\delta(x)$ because we chose $1/n < \delta/2$ we have that $B_{1/n}(x_i)$ is an element that contains x and is contained in U. So we have found our countable base.

Constructing the countable dense subset from this point is clear, we simply choose one point from each of the balls in out countable base and call this set E. I claim that E is our countable dense subset. E is clearly countable, so we need only verify that it is dense. We will show that $\overline{E}^c = \emptyset$. We have that $\overline{E}^c$ is open because its complement is closed. As a result, we use our base of balls to find some base element that is contained in $\overline{E}^c$, say $B_\delta(x_i)$. This implies that $x_n \in \overline{E}^c$. But this is impossible because $x_n \in E$ means that $x_n \in \overline{E}$. This contradiction shows that $\overline{E}^c$ is empty and so E is a countable dense subset of X. □

We can now apply the lemma to the space X to get a countable dense subset $E = \{x_1, x_2, \ldots\}$ in X. Our next goal is to capture the convergence of a subsequence of the f_n by keeping track of which indices correspond to the sequence of convergent functions. More precisely, we have the following

Claim. *There exists a decreasing chain of infinite sets $C_0 \supset C_1 \supset C_2 \supset \cdots$ with the property that $\lim_{n\to\infty} f_n(x_j)$ exists for $j \le k$ if $n \to \infty$ within S_k.*

Proof. The idea is that each of the C_k captures the indices of a subsequence of the f_n that converges. With this in mind, we set $C_0 = \mathbb{N}$ and consider the set of points $\{f_n(x_k) : n \in C_0\}$ is a bounded sequence in $\mathbb{R}^k$ and therefore has a convergent subsequence. Let C_1 be the indices of the elements of the convergent subsequence. then $C_1 \subset C_0$ and C_1 is infinite. We continue inductively defining C_k by considering the set $\{f_n(x_k) : n \in C_{k-1}\}$. As before, this is a bounded sequence in $\mathbb{R}^k$ and has a convergent subsequence and we set C_k to be the set of indicees of that convergent subsequence. So we have that we have that $f_n(x_k)$ exists as $n \to \infty$ within C_k. This completes the construction. □

Let i_k be the k^{th} term of C_k in the preceeding lemma and set

$$C = \{i_1, i_2, \ldots\}$$

By construction, there are at most $k-1$ terms of C that are not in C_k and so $\lim_{n\to\infty} f_n(x)$ exists for every $x \in E$ as $n \to \infty$ within C.

Now choose an $\epsilon > 0$ and apply the equicontinuity of the f_n to find a $\delta > 0$ such that $d(p, q) < \delta$ implies that $|f_n(p) - f_n(q)| < \epsilon/3$ for all n. Cover X with a set of balls of radius $\delta/2$ centered at each point. Because X is compact we can find a finite subcover $B_1, B_2, \ldots, B_M$ of open balls. Because E is dense in X there are points $p_i \in B_i \cap E$ for $i \le M$. Moreover, we see that $p_i \in E$ implies that $\lim_{n\to\infty} f_n(p_i)$ exists within C. Hence, we can find an integer N large enough that $|f_m(p_i) - f_n(p_i)| < \epsilon/3$ for $i \le M$ and $n, m > N$ and $n, m \in C$.

Now we just use the triangle equality to get the uniform convergence. Pick an $x \in X$. then $x \in B_i$ for some i and $d(x, p_i) < \delta$. If we choose δ and N as above we have that

$$\begin{aligned}
|f_m(x) - f_n(x)| &\le |f_m(x) - f_m(p_i)| + |f_m(p_i) - f_n(p_i)| + |f_n(p_i) - f_m(x)| \\
&\le \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&\le \epsilon
\end{aligned}$$

whenever $m, n > N$ and $m, n \in S$. This is precisely what it means for this subsequence to converge uniformly. □

Exercise 43.5:

Theorem. *Let X be a locally compact Hausdorff space and let $\mathcal{C}_0(X, \mathbb{R})$ be the space of continuous functions on X that vanish at infinity with the uniform topology. A subset $\mathcal{F}$ of $\mathcal{C}_0(X, \mathbb{R})$ has compact closure if and only if it is pointwise bounded, equicontinuous, and vanishes uniformly at infinity.*

Proof. Let X be a locally compact Hausdorff space. Then X admits a one-point compactification $Y = X \cup \{\Omega\}$. We then define a map

$$\begin{aligned} F : \mathcal{C}_0(X, \mathbb{R}) &\to \mathcal{C}(Y, \mathbb{R}) \\ f &\mapsto \bar{f} \end{aligned}$$

Where $\bar{f}$ is defined by

$$\bar{f}(x) = \begin{cases} f(x) & x \in X \\ 0 & x = \Omega \end{cases}$$

Then we first establish the following

Claim. *The function $\bar{f}$ as defined above is continuous under the hypotheses of the theorem.*

Proof. It is clear that $\bar{f}$ is continuous on X because $\bar{f}$ restricts to f on X and f is continuous. So we only need to check that $\bar{f}$ is continuous at Ω. Let U be a neighborhood of 0 in $\mathbb{R}$, then we have that $g^{-1}(U)$ contains Ω. We can find some interval $(-\delta, \delta) \subset U$, and use the fact that $\bar{f}$ vanishes at infinity to get some compact K in X such that $f(X - K) \subset (-\delta, \delta)$ and so $Y - K \subset \bar{f}^{-1}(U)$ and $Y - K$ is a neighborhood of Ω in Y. Hence, $\bar{f}$ is continuous at Ω. □

Now we introduce the metric structure on $\mathcal{C}_0(X, \mathbb{R})$ and $\mathcal{C}(Y, \mathbb{R})$. Let d_0, d be the metrics induced by the sup-norm on $\mathcal{C}_0(X, \mathbb{R})$ and $\mathcal{C}(Y, \mathbb{R})$. Note that these metrics are well-defined because continuous functions on X are necessarily bounded. We want to show that E is an isometry of $\mathcal{C}_0(X, \mathbb{R})$ with a closed subspace of $\mathcal{C}(Y, \mathbb{R})$. Indeed, pick $f, g \in \mathcal{C}_0(X, \mathbb{R})$ and observe that

$$\begin{aligned} d_0(f, g) &= \sup_{x \in X} |f(x) - g(x)| \\ &= \sup_{x \in Y} |\bar{f}(x) - \bar{g}(x)| \\ &= d(\bar{f}, \bar{g}) \end{aligned}$$

Note that equality holds moving from the first line above to the second because the point at infinity only adds 0 to a set of non-negative numbers, and therefore will not be the supremum unless f is identically zero on X and equality holds nonetheless. E is also continuous because we simply take $\epsilon = \delta$. Hence, E is a continuous, bijective and distance preserving map, and therefore an isometry (and, consequently, a homeomorphism).

Next, we need to find how $E(\mathcal{C}_0(X, \mathbb{R}))$ sits inside of $\mathcal{C}(Y, \mathbb{R})$. Because each of the $f \in \mathcal{C}_0(X, \mathbb{R})$ vanishes at infinity, the corresponding $\bar{f} \in \mathcal{C}(X, \mathbb{R})$ must satisfy $g(\Omega) = 0$. This follows from the fact that $\bar{f}$ is continuous and bounded above by ϵ outside of an arbitrarily large compact set. Because we can find such a set for each ϵ, we can let $\epsilon \to 0$ and get that $f(x) \to 0$ as $x \to \Omega$. Then continuity guarantees that we can exchange limiting operation and function application so

$$\lim_{x \to \Omega} f(x) = f(\lim x \to \Omega) = 0$$

Now we will see that $E(\mathcal{C}_0(X, \mathbb{R}))$ is closed. Well, we recall that each of the functions in $E(\mathcal{C}_0(X, \mathbb{R}))$ are equal to 0 when evaluated at Ω. Hence, if we consider the evaluation map given by

$$\begin{aligned} A : Y \times \mathcal{C}(Y, \mathbb{R}) &\to \mathbb{R} \\ x \times f &\to f(x) \end{aligned}$$

Well $E(\mathcal{C}_0(X,\mathbb{R})) = A^{-1}(0)$, which is the inverse image of a closed set (singletons are closed in $\mathbb{R}$) under a continuous function, and is thus closed.

Now we are at a point where we can apply the Ascoli theorem. We have seen that $E(\overline{\mathcal{F}})$ is a homemorphism. Then we apply Ascoli's theorem to $E(\mathcal{F})$ and get that it has compact closure if and only if it is equicontinuous and pointwise bounded. Above, we saw that this happens if and only if $\mathcal{F}$ is pointwise bounded, equicontinuous, and vanishes uniformly at infinity. We then apply the fact that E is a isometric homeomorphism so that $E(\overline{\mathcal{F}}) = \overline{E(\mathcal{F})}$. And so $\mathcal{F}$ as compact closure if and only if it is equicontinuous, pointwise bounded, and vanishes uniformly at infinity. $\square$

4 Section 46: Pointwise and Compact Convergence

Exercise 46.7:

We begin with a map

$$\Phi : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \longrightarrow \mathcal{C}(X,Z)$$
$$f \times g \longrightarrow g \circ f$$

which is the composition operator on the two function spaces. To show that Φ is continuous, we will prove that the inverse image of each open sub-basis element is open. Indeed, consider the set open (K,U) where K is compact in X and U is open in Z. Suppose that $(f,g) \in \Phi^{-1}(K,U)$ so that (following the hint) $f(K) \subset g^{-1}(U)$. Because $f(K)$ is compact, $g^{-1}(U)$ is open in Y, and X is locally compact and regular we can (APPLY A THEOREM TO) see that there is some compact subset L such that $f(K) \subset \text{int}(L)$ and $L \subset g^{-1}(U)$. The set $(K, \text{int}(L)) \times (L,U)$ is an open set in $\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z)$. Furthermore, it contains the point (f,g).

So if we pick any $h \in (K, \text{int}(L)$ and $i \in (L,U)$ then we have

$$(i \circ h)(K) \subset i(\text{int}(K)) \subset i(L) \subset U$$

This implies that $(h,i) \in (K,U)$. We have shown that

$$(K, \text{int}(L)) \times (L,U) \subset \Phi^{-1}(K,U)$$

So $\Phi^{-1}(K,U)$ is open, and so Φ is continuous.

5 Section 48: Baire Spaces

Exercise 48.7:

Theorem. *If D is a countable dense subset of $\mathbb{R}$, there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous precisely at the points of D.*

Proof.

(a) Following the hint for the problem we begin by noting the following

Lemma. *Given a function $f : \mathbb{R} \to \mathbb{R}$ we have the following*

(i) *f is continuous if and only if*

$$D_f(x) \lim_{\substack{x \in U \\ diam(U) \to 0}} diam(f(U)) = 0$$

(ii) *The set*

$$E_n = \{x \in \mathbb{R} : D_f(x) < 1/n\}$$

is open.

Proof. For the first claim, this is just the definition of continuity in $\mathbb{R}$ phrased in the language of diameters. If f is continuous then we have for any $\epsilon > 0$ we can find $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. If you give an epsilon and the limit holds, then we simply choose the first U such that the diameter is less than ϵ and set $\delta = \operatorname{diam}(()U)$. Conversely, if f is continuous, then we simply say that for each U containing x we have that $\operatorname{diam}(()U) \to 0$ implies that $|x - y| \to 0$ for $y \in U$. This immediately gives the result.

For the second part, note that if $x \in E_n$ then we can find some U such that $\operatorname{diam}(U) < \delta$ so that $\operatorname{diam}(f(U)) < 1/n$. So if $y \in B_{\delta/2}(x)$ then $y \in E_n$ because

$$\sup_{p,q \in B_{\delta/2}(y)} |f(p) - f(q)| \leq \sup_{p,qinB_{\delta(x)}} |f(p_- f(q)| < 1/n$$

So $B_{\delta/2}(x)$ is an open ball containing x that is wholly contained in E_n, hence E_n is open. □

This lemma shows that we can write

$$C = \bigcap_{n=1}^{\infty} E_n$$

where C are the points of continuity of f.The first part of the lemma says the points int the intersection are precisely the continuity points of f and the second part of the lemma verifies that this set is a G_δ, as desired.

(b) The result is an immediate consequence of the following

Lemma. *A dense G_δ set in $\mathbb{R}$ is generic.*

Proof. Suppose that Y is a dense G_δ set. Then we can write $Y = \bigcap_{n=1}^{\infty} U_n$, where each of the U_n is open and dense (each must be dense because otherwise the intersection could not be). Suppose to the contrary that Y is of the first category and we could also write $Y = \bigcup_{n=1}^{\infty}$, where each one of the W_n is nowhere dense. We then note that each of the U_n^c is nowhere dense and rewrite

$$\mathbb{R} = Y \cup Y^c = \bigcup_{n=1}^{\infty} W_n \cup \bigcup_{n=1}^{\infty} U_n^c$$

Then $\mathbb{R}$ is the countable union of nowhere dense sets, and therefore is first category in itself. which contradicts the Baire Category theorem. □

Applying this lemma gives immediately that if D is a countable dense set, then D cannot be G_δ, and therefore cannot be the set of continuity points of any function because by the previous part, those must be G_δ.

□

Exercise 48.8:

Theorem (Uniform Boundedness Principle)**.** *Let X be a complete metric space and let $\mathcal{F}$ be a subset of $\mathcal{C}(X, \mathbb{R})$ such that for each $a \in X$ the set*

$$\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$$

is bounded. Then there is a nonempty open set U of X on which the functions in $\mathcal{F}$ are uniformly bounded, that is, there is a number M such that $|f(x)| \leq M$ for all $x \in U$ and $f \in \mathcal{F}$.

Proof. We will begin by proving the result for sets $Y \subseteq X$ of the second category and the theorem will follow because X is a compact metric space, and therefore a Baire space.

Following the hint, we begin by assigning to each integer N the set

$$A_n = \{x : \forall f \in \mathcal{F}, |f(x)| \leq N\}$$

Then, the fact that each of the $\mathcal{F}_a$ is bounded allows us to write

$$A = \bigcup_{N=1}^{\infty} A_N$$

We then note that each the A_N must be closed because we can write

$$A_N = \bigcap_{f \in \mathcal{F}} A_{N,f}$$

And each of the $A_{N,f} = \{f : |f(x)| \leq N\}$ must be closed because f is assumed continuous.

Because A is second category there is A_N for some sufficiently large N that must have a non-empty interior. So we can find an $x_0 \in X$ and an $r > 0$ such that the ball $B_r(x_0)$ is contained in A_N. For each of the $f \in \mathcal{F}$, we get $x \in B_r(x_0)$ implies that $|f(x)| \leq N$. This is the open ball that we were looking for by construction. $\square$

6 Section 55: Retractions and Fixed Points

Exercise 55.3:

Consider the intersection of the sphere with the first octant (all non-negative coordinates) in $\mathbb{R}^3$. Graphically, this is one "slice" of S^2 in 3-space. We know how to parameterize such a space via polar coordinates, as (r, θ, ϕ). All we do need to do is make sure that the map from our slice to the whole sphere is a homeomorphism. this is clear if we simply use the map $(r, \theta, \phi) \mapsto (r, 4\theta, 4\phi)$. Each of these component maps is linear and hence continuous and bijective, so this is a homeomorphism.

Because A has no singularities and each of the components of B are non-negative. We can then map the $F : B \to B$. Because Ax is all non-negative We have that this map is well defined as $F(x) = Ax/\|Ax\|$. Because B is homeomorphic to itself we can apply Brower's fixed point theoerm and then see immediatly that

$$Ax_0/\|Ax_o\| \text{ implies } Ax_0 = \|Ax_0\|x_0$$

So again, this is the same as

$$Ax_0 = \|Ax_0\|x_0$$

The right hand side is a non-negative constant and the formula immediately gives that x_0 is indeed an eigenvector.

7 Section 57: The Borsuk-Ulam Theorem

Exercise 57.2:

Suppose that g were not surjective and let p be a point in the complement of the image of g. Then define a map $h : S^2 - \{p\} \to \mathbb{R}^2$ to be a homeomorphism (following the hint). So we can define a continuous function $f : S^2 \to \mathbb{R}^2$ by $f = h \circ g$. We then apply the Borsuk-Ulam theorem to find a pair of points such that $f(x) = f(-x)$, and because h was a homeomorphism we see that this must be a contradiction. This shows that g must be surjective.