Problem 3.28.2
We will modify the proof that the interval $(0, 1)$ is not compact in the usual topology. Consider the set of points

$$A = \{1 - 1/n\}_{n=1}^{\infty}$$

Then it is clear that $A \subset [0, 1]$ because each point of A is in $[0, 1]$. We begin by arguing that if A had a limit point, then this limit point would be 1. Indeed, suppose that $\ell \in [0, 1)$ is a limit point of A. We can find some interval

$$I_\ell = [1 - 1/k, 1 - 1/(k + 1))$$

That contains ℓ. It is clear from the construction that $I_\ell \cap A = \{1 - 1/k\}$. This means that the interval $I'_\ell = [\ell, 1 - 1/(k + 1))$ intersects A in at most one point. We see that $I'_\ell \cap A = \ell$ when $\ell \in A$ and that $I'_\ell \cap A = \emptyset$ when $x \notin A$. But I'_ℓ is open in the lower limit topology and so there is a neighborhood of ℓ that contains finitely many points of A, so ℓ is not a limit point of A.

Thus we consider the point 1. The set $[1, 1) \subset [0, 1]$ is open in the lower limit topology, and furthermore, it contains no points of A because for all n the point $1 - 1/n \notin [1, 1)$. So 1 cannot be a limit point of A either. As a result we have that A has no limit points in $[0, 1]$ and is not limit point compact.

Problem 3.28.6
We are given a mapping $f : X \to X$ such that

$$d(f(x), f(y)) = d(x, y)$$

The first observation is that f is continuous. We simply note that for any $\epsilon > 0$ if $d(x, y) < \delta$ with $\delta < \epsilon$ then we must have that

$$d(f(x), f(y)) = d(x, y) = \delta < \epsilon$$

It is clear that f must be injective because we have that if $f(x) = f(y)$ then

$$d(f(x), f(y)) = 0 = d(x, y)$$

So $x = y$.

Now we need to show that f is surjective. Following the hint, we proceed by contradiction. If f were not surjective then there is some $y \notin f(X)$. Moreover, we know that $f(X)$ is compact and closed because X is compact and f is continuous. As a result of this we can find an $\epsilon > 0$ such that

$$B_\epsilon(y) \cap f(X) = \emptyset$$

Then we define the following sequence inductively: Let $x_1 = y$ and then let $x_{n+1} = f(x_n)$ for $n > 1$. Because $d(y, f(X)) > \epsilon$ we have that $d(x_i, x_j) > \epsilon$ for $i \neq j$ because f is an isometry. But then no subsequence of $\{x_n\}$ can converge. Because X is compact, it must also be sequentially compact, so this is a contradiction.

We then note that f is a continuous bijection, and hence, a homeomorphism.

Problem 3.28.7
(a) Suppose that $f : X \to X$ is a contraction map from a compact space X to itself. We will show that f has a unique fixed point. We begin by showing that f is necessarily continuous. Indeed, fix $\epsilon > 0$ and then observe that if we set $\delta = \epsilon/\alpha$ then we have that

$$d(f(x), f(y)) < \alpha d(x, y) = \alpha \cdot \frac{\epsilon}{\alpha} = \epsilon$$

So f is continuous. Then define recursively the sequence of functions $\{f_n\}_{n=1}^{\infty}$ by setting $f_1 = f$ and $f_{n+1} = f \circ f_n$. Then consider the sequence of images of these functions, the sets $A_n = f_n(X)$. We can see that each of the sets A_n is compact. Indeed, $A_1 = f_1(X)$, and f_1 is a continuous image of a compact set, hence, compact. Proceeding by induction yields that if $f_n(X)$ is compact, then $f_{n+1} = f(f_n(X))$, is also a continuous image of a compact set, and therefore is compact. We now show that the set

$$A = \bigcap_{n=1}^{\infty} A_n$$

is non-empty. To do this, we observe that $A_n \subset A_{n+1}$ because f is a contraction map. Then we have that $A \neq \emptyset$ because it is the intersection of a nested sequence of compact sets. We now see that $f(A) \subset A$. Suppose not, then there would be some $a \in A$ such that $f(a) \notin A$, so that $a \in A_n - A_{n+1}$ for some n. But then $a \notin A_{n+1}$, a contradiction. Now note that $\mathrm{diam}(X) = sup_{x,y\in X} d(x, y)$. As a result we can see that

$$\mathrm{diam}(A_1) = \sup_{x,y\in A_1} d(x, y) = \sup_{u,v\in X} d(f(u), f(v)) = \alpha \sup_{u,v\in X} d(u, v) = \alpha \mathrm{diam}(X)$$

Continuing in this way we see that

$$\mathrm{diam}(A_n) = \alpha^n \mathrm{diam}(X)$$

As a result of this, $\mathrm{diam}(A_n) \to 0$ as $n \to \infty$ because $\alpha < 1$. So A contains exactly one point, x, that satisfies $f(x) = x$ because we already showed that $f(A) \subset A$.

(b) We proceed in the same way as in the previous part. Define f_n and A_n in the same way, and let $A = \bigcap_n A_n$. We begin by showing that if f has at most one fixed point. We argue again by contradiction. If there were two fixed point u and v then we would have that $f(u) = u$ and $f(v) = v$. So we compute

$$d(u, v) = d(f(u), f(v)) < d(u, v)$$

This is impossible, so f has at most one fixed point. Now we need to figure out how to find. Also, we have $f(A) \subset A$ by the same reasoning as before. Now suppose that $a \in A$. Then for each n, $a \in A_n$, and as a result we can find a sequence of points $\{y_n\}_{n=1}^{\infty}$ such that $f(y_n) = a$. We also know that X is compact, and as a result, it is sequentially compact. This means that the sequence $\{y_n\}$ has a convergent subsequence $\{y_{n_k}\}$. Suppose that $y_{n_k} \to y$ as $n_k \to \infty$. Then we have that y is a limit point of each of the A_n and because each A_n is closed, $y \in A_n$ for all a. Then we know that $y \in A$.

Now if A consists of only one point then we are done, and we have that y is the fixed point of f. To see this, we note that $d : A \times A \to \mathbb{R}$ is a continuous function on a compact set, and therefore attains its maximum, say at (u, v). Observe that $\operatorname{diam}(A) = \sup_{(u,v)\in A\times A} d(u, v)$. Therefore is we show that $d(u, v) = 0$, then A contains only one point and we are done. Indeed, we have that $u = f(a)$ and $v = f(b)$ and that $(a, b) \in A$. But then

$$d(a, b) = d(u, v) < d(a, b)$$

A contradiction. So A contains exactly one point, y, as determined above and $f(y) = y$.

(c) Take $X = [0, 1]$ and let

$$f(x) = x - \frac{x^2}{2}$$

Furthermore, f is strictly increasing on $[0, 1]$ (its derivative is strictly positive) and attains its maximum at $x = 1$ with $f(x) = 1/2$. Now observe that

$$\begin{aligned} f(y) - f(x) &= (y - y^2/2) - (x - x^2/2) \\ &= (y - x) - \frac{x^2 - y^2}{2} \\ &= (y - x)(1 - \frac{(y + x)}{2}) \end{aligned}$$

Taking the absolute value gives

$$|f(x) - f(y)| = \left|(y - x)\left(1 - \frac{(y + x)}{2}\right)\right| \leq |x - y||1 - \frac{x + y}{2}| < |x - y|$$

This shows that f is a shrinking map (the fixed point is 0). To see that f is not a contraction observe that if $x, y < 1 - \alpha$ for any $\alpha \in [0, 1]$ then we would have that

$$|f(y) - f(x)| = \left|(y - x)\left(1 - \frac{(y + x)}{2}\right)\right| > |(y - x)|\left|\left(1 - \frac{2(1 - \alpha)}{2}\right)\right| = |x - y|\alpha$$

Which does not satisfy the definition of a contraction.

(d) We now consider the map $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{x + (x^2 + 1)^{1/2}}{2}$$

We will show that f is a shrinking map that is not a contraction. To see that f is a shrinking map we observe that

$$f(y) - f(x) = \frac{1}{2}(y-x)\left(1 + \frac{(y+x)}{\sqrt{y^2+1}+\sqrt{x^2+1}}\right)$$

Taking norms we get

$$|f(y)-f(x)| = \left|\frac{1}{2}(y-x)\left(1 + \frac{(y+x)}{\sqrt{y^2+1}+\sqrt{x^2+1}}\right)\right| \leq |x-y|\left|\left(1 + \frac{(y+x)}{\sqrt{y^2+1}+\sqrt{x^2+1}}\right)\right|$$

We then note that

$$\left|\left(1 + \frac{(y+x)}{\sqrt{y^2+1}+\sqrt{x^2+1}}\right)\right| < 1$$

to see that f is shrinking map. to see that f is not a contraction note that

$$\left|\frac{f(x)-f(y)}{x-y}\right| = \frac{1}{2}\left|\left(1 + \frac{(y+x)}{\sqrt{y^2+1}+\sqrt{x^2+1}}\right)\right|$$

So if we let $x, y \to \infty$ then $\left|\frac{f(x)-f(y)}{x-y}\right| \to 1$. As a result, f cannot be a shrinking map because any $\alpha < 1$ with

$$|f(x) - f(y)| < \alpha|x-y|$$

would imply

$$\left|\frac{f(x)-f(y)}{x-y}\right| < \alpha$$

Which contradicts $\left|\frac{f(x)-f(y)}{x-y}\right| \to 1$. So f is not a contraction.

We can see that f is strictly increasing by examining the derivative

$$f'(x) = \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}} + 1\right)$$

Which is always strictly positive. We then note the inequality

$$x \leq \frac{x+|x|}{2} < f(x)$$

As a result of this, there is can be no fixed point of f.

Problem 3.29.7
We will get the result by applying Theorem 3.29.1 to the spaces S_Ω and $\overline{S}_\Omega$ in the order topology. We need to check the following three conditions:

(i) S_Ω is a subspace of $\overline{S}_\Omega$.

(ii) The set $\overline{S}_\Omega - S_\Omega$ consists of a single point.

(iii) $\overline{S}_\Omega$ is a compact Hausdorff space.

We know that (i) is clear by the definition of closure (a set together with its limit points). Furthermore, (ii) is clear because the only limit point of S_Ω which is not contained in S_Ω is Ω itself. To see this we appeal to the definition (c.f Munkres pg. 66) we have that $\overline{S}_\Omega = S_\Omega \cup \{\Omega\}$ so that $\overline{S}_\Omega - S_\Omega = \{\Omega\}$, a single point. So we only need to show that $\overline{S}_\Omega$. So we need only check compactness. Note that any open covering must contain an open interval containing Ω. These intervals are of the form (x, ∞), leaving the rest of the space $[x_0, x]$. But $[x_0, x]$ is compact in the order topology because the order topology has the least upper bound property. Hence, any open cover contains a finite subcover by appending the interval (x, ∞) to the finite subcover of $[x_0, x]$. Consequently, $\overline{S}_\Omega$ is compact. We then apply the theorem to see that $S_\Omega \approx \overline{S}_\Omega$.

Problem 4.30.1
(a) Let X be a first-countable T_1 space. Then for every x we can find a countable basis $B_1, B_2, \ldots$ at x. If we pick some $y \neq x$ then we know that there is some neighborhood of x, say U, such that $y \notin U$. So this means that there is some B_n such that $U \not\subset B_n$. This means that $\bigcap_n B_n = \{x\}$. As a result, we have that $\{x\}$ is a G_δ set in X.

(b) The familiar space referred to is $\mathbb{R}^\omega$ in the box topology. The intuition leading to this is that if we consider the set $\mathbb{R}^\omega$ in the metric product topology, then we can use the previous part to get that each one point set is the intersection of balls of radius $1/n$ and so each singleton is a G_δ. So if we take a finer topology we should retain this property, but perhaps lose the first countability. We then switch to the box topology on $\mathbb{R}^\omega$ and proceed via a diagonalization argument to see that there is no countable basis at a single point. Indeed, suppose that $B_1, B_2, \ldots$ was a countable basis at the point x. We then note that each of the sets B_k is a product of intervals so

$$B_k = \prod_{k=1}^{\infty} (a_{k_1}, b_{k_1})$$

Then we can diagonalize these sets by noting that $(a_{k_k}/2, b_{k_k}/2) \not\subset (a_{k_k}, b_{k_k})$. So the neighborhood

$$N = \prod_{k=1}^{\infty} (a_{k_k}/2, b_{k_k}/2)$$

Does not contain any of the B_k because it differs from each of them in the k^{th} coordinate, a contradiction. Then R^ω cannot be first-countable in the box topology.

Problem 4.30.3
This is clear. We proceed by contradiction (as usual). Suppose that the number of limit points in A were countable and let L be the set of limit points of A. We

look at the points in $A-L$, and observe that because X has a countable basis we can find a basis neighborhood about each of these points, a, that is disjoint from $A-\{a\}$. Each of these neighborhoods is distinct by construction, and there can only be countably many of them because X has a countable basis. But $A-L$ is uncountable because we assumed L countable, this is a contradiction. Thus we must have uncountably many limit points of A.

Problem 4.30.5

(a) The intuition for this problem is straightforward. If X has a countable dense subset D, then each point of x is close to one of the points in D, so the union of a countable basis centered at each point of D should suffice as a countable basis for X. To formalize this, we let X be a metrizable space with countable dense subset D. To each $u \in D$ we assign the collection of open balls

$$B_u = \{B_{1/n}(u) \,|\, n \in \mathbb{N}\}$$

Then we set

$$B = \bigcup_{u\in D} \bigcup_{n\in\mathbb{N}} B_{1/n}(u)$$

and claim that B is a countable basis for X. First note that B is countable because it is the countable union of countably many sets. Now choose a point $x \in X$ and an open neighborhood $\mathcal{O}$ about x. Then we can find some $r > 0$ such that $B_r(x) \subset \mathcal{O}$. Observe that if we choose $2/r < n$ and find a point $y \in D \cap B_{1/n}(x)$, which is always possible because D is dense, then we have

$$x \in B_{1/n}(y) \subset B_r(x) \subset \mathcal{O}$$

If we do this for each point $p \in \mathcal{O}$, we see that we can write $\mathcal{O}$ as a union of basis elements. So every open set can be written as a union of elements of B, and so B is the desired countable basis for X.

(b) We follow the same general strategy as the previous part. If X is a metrizable Lindelöf space then we know that the covering of X by $1/n$ balls forms a covering of X. Let

$$B_n = \{B_{1/n}(x) \,|\, x \in X\}$$

Then $X \subset B_n$ for every n. Because X is Lindelöf, we can extract a sequence of countable coverings $\{C_n\}$, where C_n is the countable subcovering of B_n. We then propose

$$C = \bigcup_n C_n$$

as a countable basis of X. C is clearly countable because it is the countable union of countably many sets. Now take some open set $\mathcal{O} \subset X$ and let $x \in \mathcal{O}$. Then there is some $r > 0$ such that $B_r(x) \subset \mathcal{O}$. As a result of this we have that for $n > 1/r$, that $x \in B \in C_n$. So we conclude (as in the last part) that we can write $\mathcal{O}$ as a union of elements of C, and so C is a countable basis for X.

Problem 4.30.6
To see that R_ℓ is not metrizable observe that it is separable but not second-countable (c.f. Munkres pg 192), and as a result of this, not metrizable. The ordered square I_o^2 is compact but not separable. To see this, note that any countable dense subset of I_o^2 must contain all elements of the form $\{t\} \times (0,1)$, which are all open and disjoint. So there are at least as many of these sets as there are $t \in (0,1)$, but $(0,1)$ is uncountable, so there can be no countable basis for I_o^2. We appeal to the results of Exercise 4.30.5 (pg 193) to see that this is sufficient for a set to be non-metrizable.