Problem 5.37.3
(a) We will follow the outline of the proof given for the Tychonoff theorem. Let $\{L_\alpha\}_{\alpha\in J}$ be a collection of Lindelöf spaces indexed by J and define $L = \prod_{\alpha\in J} L_\alpha$. Pick a collection $\mathcal{A}$ with the countable intersection property. By assumption, we can extract a subcollection $\mathcal{D}$ of $\mathcal{A}$ that is maximal with respect to the countable intersection property and still covers $\mathcal{A}$. Consider the canonical projections of the elements of $\mathcal{D}$ onto their α coordinate, namely $\mathcal{D}_\alpha = \{p_\alpha(D) \,|\, D \in \mathcal{D}\}$. Then we must have that $\mathcal{D}_\alpha$ must also have the countable intersection property. Because each of the L_α is Lindelöf we have that given any α we can find some $\ell_\alpha \in L_\alpha$ such that $\ell_\alpha \in \bigcap_D \overline{p_\alpha(D)}$. From each of these ℓ_α we can construct an element $\ell = (\ell_\alpha)_{\alpha\in J} \in L$. Consider the inverse image of any open neighborhood U_α about ℓ_α. Then we must have that $p_\alpha^{-1}(U_\alpha)$ must intersect every element of1 $\mathcal{D}$ by construction. Then we observe that every basis element containing ℓ must intersect $\mathcal{D}$ because of hypothesis (ii) and the preceding argument (the projections of the basis elements must contain ℓ_α). Consequently, we have that $\ell \in \overline{D}$ for each D and therefore $\ell \in \bigcap_{D\in\mathcal{D}} \overline{D}$.

(b) Let $\{E_n\}$ be a countable collection of elements of $\mathcal{D}$ and set $E = \bigcap_n E_n$. We then define a new collection $\mathcal{E} = \mathcal{D} \cup \{E\}$ and take a countable subcollection of $\mathcal{E}$. We now consider cases for the constituents of this subcollection. If none of the elements are in E then we must have that the intersection is non-empty because then all it is a countable subcollection of $\mathcal{D}$, which has the countable intersection property. For each element that is in E, we can see that this element must be $E \cap \bigcap_n D_n$ for some of the countable collection D_n. Because E is a countable intersection of elements of $\mathcal{D}$, we have that this too is a countable intersection of elements of $\mathcal{D}$, and is therefore non-empty. Becuase we assumed that $\mathcal{E}$ was maximal, it cannot be a proper subset of $E \cup \mathcal{D}$. Hence, we must have that $\mathcal{E} = E \cup \mathcal{D}$ and so $E \in \mathcal{D}$. Now suppose that in addition that $Y \subset X$ intersects each element in $\mathcal{D}$. We then set $\mathcal{E} = \mathcal{D} \cup \{Y\}$. Like before, we suppose that if we extract countably many elements from this collection, that none of them are in Y, so their intersection must be nonempty. On the other hand if their intersection foes contain Y, then it must be of the form $Y \cap \bigcap_n D_n$. Clearly, the right term side is non-empty and Y must intersect everything in the right term by assumption. Hence, this collection must also have the countable intersection property, meaning that $Y \in \mathbb{D}$ and we are done.

(c) The part of the prof that breaks down is the assumption that the collection $\mathcal{D}$ exists. In a general Lindelöf space we may not be guaranteed that such a set exists and therefore the proof given by Munkres fails at this point.

Problem 5.37.4
(a) Let $\mathfrak{B} = \bigcap_{B\in\mathcal{B}} B$. We will see that $\mathfrak{B}$ is closed in X and that x and y lie in the same quasicomponent of $\mathcal{A}$. It is clear that $\mathfrak{B}$ is closed because it is the intersection of closed sets and that $x, y \in \mathfrak{B}$ because x, y are in each of the elements of $\mathcal{B}$. Suppose that there was a separation into $C, D \subset \mathfrak{B}$ disjoint. Then we must have that C and D are closed in $\mathfrak{B}$, which implies further that C, D are closed in X. Because X is a compact Hausdorff space we can find open and disjoint open C_o, D_o such that $C \subset C_o$ and $D \subset D_o$. For any $B \in \mathcal{B}$ that $B - (C_o \cup D_o) \neq \emptyset$ because $B \in \mathcal{A}$. Now we consider the ordering of $\mathcal{B}$ to see that the set $\mathcal{U} = \{B - (C_o \cup D_o) \,|\, B \in \mathcal{B}\}$ must have the finite intersection property because we are ordered by inclusion. Because $\mathcal{U}$ contains only closed sets in the compact space X, we know that the intersection must be non-empty. However, the intersection of the elements of $\mathcal{U}$ is given by $\mathfrak{B} - (C_o \cup D_o) = \emptyset$ because $C \cup D \subset C_o \cup D_o$. Hence, there could not have been such a separation and so we have that x, y must belong to the same quasicomponent of $\mathcal{A}$.

(b) For this part we appeal to Zorn's Lemma. We saw that every totally ordered subset of the collection $\mathcal{A}$ must have a lower bound given by $\mathfrak{B}$ above. So by Zorn's lemma we can find the minimal element in $\mathfrak{B}$, say, D.

(c) As usual, we suppose that it is not connected. So we can find some separation U, V such that $U \cap V = \emptyset$ and $D = U \cup V$. We take our minimal element D and note that x and

y must both lie in the same quasicomponent, say U. Because U is closed in D it must also be closed in X and by hypothesis we have that $V \neq \emptyset$ so that U lies properly in D. As a result, $U \notin \mathcal{A}$ because D is the minimal element. As a result of this fact we can find disjoint open O_1, O_2 such that $x \in O_1$ and $y \in O_2$. However, this is impossible because then we get that $O_1 \cup V$ and O_2 are disjoint open sets that separate x and y in D, which is impossible. Hence, there could not have been such a separation. So we are done.

Problem 9.51.3
(a) Consider the map $F(x, y) = xy$. Clearly, this map is continuous for each $(x, y) \in \mathbb{R}^2$. If we restrict y to the interval $[0, 1]$ then F is a homotopy between the identity and the constant map to zero for both I and $\mathbb{R}$.

(b) If X is a contractible space with nullhomotopy $F : X \times I \to X$ and let $x \in X$ be the image of the constant map $F(z, 0)$. If y is another point in X then $F(y, t)$ is a path from y to x. This precisely means that all the points are in the same path component as x.

(c) Let $F : Y \times I \to Y$ be a nullhomotopy for Y with y the image of the particular constant map. We construct a homotopy between $f : X \to Y$ and the constant map from X to y. This homotopy is the function $H : X \times I \to Y$ defined by $(x, r) \mapsto F(f(x), r)$.

(d) If we refer to result of Exercise 2(b) (c.f pg 330) then we see that all constant maps into a path-connected space are homotopic. Our goal is to show that each $f : X \to Y$ is homotopic to a constant map. Let $F : X \times I \to X$ be a nullhomotopy of X and let x be the image of he constant map $F(z, 0)$. Then we define $H : X \times I \to Y$ by $(z, r) \mapsto f(F(z, r))$. Then we see that G must be a homotopy between f and the constant map from X to $f(x)$ and the proof is complete.

Problem 9.52.3
This is a straightforward computation. We apply Theorem 5.51.2 to get

$$[f] * [\alpha] = [f] * [\alpha] * [e_{x_1}] = [f] * [\alpha] * [\bar{\beta}] * [\beta]$$

Then we apply the group operation to get

$$[f] * [\alpha] * [\bar{\beta}] * [\beta] = [f] * [\alpha * \bar{\beta}] * [\beta]$$

We then note that $\pi_1(X, x_0)$ is abelian so

$$[f] * [\alpha * \bar{\beta}] * [\beta] = [\alpha * \bar{\beta}] * [f] * [\beta] = [\alpha] * [\bar{\beta}] * [f] * [\beta]$$

Then we compute using Theorem 5.51.2 again after multiplying on the left to get

$$\begin{aligned}
[\bar{\alpha}] * [f] * [\alpha] &= [\bar{\alpha}] * [\alpha] * \bar{\beta}] * [f] * [\beta] \\
[\bar{\alpha}] * [f] * [\alpha] &= [e_{x_1}] * [\bar{\beta}] * [f] * [\beta] \\
[\bar{\alpha}] * [f] * [\alpha] &= [\bar{\beta}] * [f] * [\beta] \\
\hat{\alpha}([f]) &= \hat{\beta}([f])
\end{aligned}$$

and we are done.

Problem 9.52.6
Again, we compute

$$\begin{aligned}
(h_{x_1}) \circ \hat{\alpha}([f]) &= (h_{x_1}) * ([\bar{\alpha} * f * \alpha]) \\
&= [h \circ (\bar{\alpha} * f * \alpha)] \\
&= [(h \circ \bar{\alpha}) * (h \circ f) * (h \circ \alpha)] \\
&= [\bar{\beta} * (h \circ f) * \beta] \\
&= \hat{\beta}([h \circ f]) \\
&= \hat{\beta} \circ (h_{x_0}) * ([f])
\end{aligned}$$

And we are done.

Problem 9.52.7
(a)

(b)

(c)

(d)