Problem 2.19.10

(a) We want to find a topology $\mathcal{T}$ on A which has the fewest open sets such that each f_α is continuous with respect to $\mathcal{T}$. Let $\mathcal{C}$ be the collection of all such topologies on A, more precisely define

$$\mathcal{C} = \{\overline{\mathcal{T}} \,|\, \forall \alpha, f_\alpha \text{ is continuous relative to } \overline{\mathcal{T}}\}$$

We can see that $\mathcal{C} \neq \emptyset$ because the discrete topology must be in $\mathcal{C}$ because every set is open. We then set

$$\mathcal{T} = \bigcup_{\overline{\mathcal{T}} \in \mathcal{C}} \overline{\mathcal{T}}$$

It is clear that $\mathcal{T}$ is a topology because $\emptyset, A \in \mathcal{T}$ because they are in each $\overline{\mathcal{T}}$. Furthermore, any countable union of open sets in $\mathcal{T}$ is open because this union must be open in each of the $\overline{\mathcal{T}}$. The argument is analogous for finite intersections. Finally, we can see that $\mathcal{T}$ is the coarsest because it is coarser than each of the $\overline{\mathcal{T}}$. The uniqueness of $\mathcal{T}$ follows from the fact that if $\mathcal{T}'$ was another topology as coarse as $\mathcal{T}$, but was not equal to $\mathcal{T}$ then we would have that $\mathcal{T} \cap \mathcal{T}'$ is coarser than both, but this is impossible because $\mathcal{T}$ was the coarsest topology on A.

(b) We define

$$\mathcal{S}_\beta = \{f_\beta^{-1}(U_\beta) \,|\, U_\beta \text{ is open in } X_\beta\}$$

And let $\mathcal{S} = \bigcup \mathcal{S}_\beta$. We want to show that $\mathcal{S}$ is a subbasis for $\mathcal{T}$ as defined in part **(a)**. For each $S \in \mathcal{S}$ we must have that $S \in \mathcal{T}$ because $S = f_\beta^{-1}(U_\beta)$ and f_β is continuous relative to $\mathcal{T}$ for every β. Hence, $\mathcal{T}$ must contain all unions of sets in $\mathcal{S}$, and all finite intersections of sets in $\mathcal{S}$ because it is a topology. The coarsest topology satisfying these properties is $\mathcal{T}$, so $\mathcal{S}$ is a subbasis for $\mathcal{T}$.

(c) We need to show that $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each map $f_\alpha \circ g$ is continuous relative to $\mathcal{T}$. In the forward direction we have that g is continuous. But then $f_\alpha \circ g$ is the composition of continuous maps, and is therefore continusous. In the reverse direction, suppose that $f_\alpha \circ g$ is continuous for each α. then for each open set $U_\alpha \in X_\alpha$ we have that the inverse image $(g^{-1} \circ f_\alpha^{-1})(U_\alpha)$ is open. Observe that because each of the f_α is continuous relative to $\mathcal{T}$ we have that $g^{-1}(f_\alpha^{-1}(U_\alpha))$ is open. So the inverse image of any subbasis element under g is open. But because we can write any open set $\mathcal{O} \subset A$ as the union of finite intersections of subbasis elements such as $\bigcup_\beta \bigcap_{k=1}^n f_{\beta,n}^{-1}(U_{\beta,n})$ we know that the inverse image of any open set in A is open. So g must be continuous.

(d) Choose any open set $U \subset A \in \mathcal{T}$ and some element $x = f(u) \in f(U)$. Because U is open we can find some basis element $B \in \mathcal{T}$ such that $u \in B \subset U$. Moreover, we must have that $B = \bigcap_{k=1}^n f_{\alpha_k}^{-1}(V_{\alpha_k})$, because the sets $f_\beta^{-1}(V_\beta)$ form a subbasis for A and the V_{α_k} are open in X_{α_k}. For all other α we set $V_\alpha = X_\alpha$. Then we have that $V = f^{-1}(\prod_\alpha V_\alpha)$ and therefore $f(B) = f(A) \cap \prod_\alpha V_\alpha$ is open in the subspace topology. We can see this because only finitely many of the $V_\alpha \neq X_\alpha$. Moreover, $x \in f(B) \subset f(U)$ so $f(U)$ is open because x was arbitrary.

Problem 2.20.4
(a) We consider each function separately.

1. For the function $f : t \mapsto (t, 2t, 3t, \ldots)$ we can see that each component is linear, and hence continuous (the inverse image of the interval (a, b) is $(a/k, b/k)$), and so f in continuous in the product topology. In the uniform topology we consider the neighborhood $B_{\overline{\rho}}(\epsilon, 0)$, which is the intersection of a sequence of descending neighborhoods about 0, and hence it is equal to 0, and as a result f is not continuous. In the box topology observe that the open set $\mathcal{O} = \prod_{k=1}^{\infty}(-1/k^2, 1/k^2)$, has preimage $\{0\}$ as well, so f is not continuous relative to the box topology.

2. Next we consider $g : t \mapsto (t, t, t \ldots)$. Again, each of the component functions is linear, and hence continuous, so that g is continuous in the product topology. In the uniform topology we choose $\epsilon > 0$ and $x \in \mathbb{R}$. Then if $d(x, y) < \epsilon$ we must have that $d_{\overline{\rho}}(g(x), g(y)) = d(x, y) < \epsilon$, therefore g is continuous. In the box topology, we consider the same $\mathcal{O}$ as above, and note that the preimage of $\mathcal{O}$ under g is $\{0\}$, so g is not continuous in this topology.

3. Finally we consider $h : t \mapsto (t, t/2, t/3, \ldots)$. As before, each of the component functions is continuous and so h is continuous in the product topology. In the uniform topology we note that $d_{\overline{\rho}}(h(x), h(y)) = d(x, y)$ because that is the usual distance in the Euclidean metric in the first component, which is the largest. If we take $d(x, y) < \epsilon$ for any choice of ϵ we have that h is continuous in the uniform topology. Lastly, we note that $h^{-1}(\mathcal{O}) = \{0\}$, again. And therefore h is not continuous in the box topology.

(b)

1. We begin by considering the sequences w_n for $n = 1, 2, 3, \ldots$ In the product topology each of the sequences clearly converge (to $\vec{0}$), because every neighborhood of zero will contain all but finitely many of the w_n. In the uniform topology this sequence does not converge. Consider the ball $B_{\overline{\rho}}(\epsilon, x)$ in the uniform topology. If $\epsilon < 1$ then $B_{\overline{\rho}}(\epsilon, x)$ can contain at most one point of w_n. And it cannot converge in the box topology, because the box topology is finer than the uniform topology.

2. For the sequence x_n we begin with the uniform topology and note that the distance in the uniform metric from $\vec{0}$ goes to zero because it decreases at the rate of the harmonic sequence $1/k$ which goes to 0 as $k \to \infty$. That is, $d_{\overline{\rho}}(x_n, 0) = 1/n$ which goes to 0 as $n \to \infty$. Because x_n converges in the uniform topology, it must also converge in the product topology, because the product topology is coarser. Finally, in the box topology x_n does not converge because the open set $\mathcal{O}$ about 0 does not contain any point of

x_n, and clearly x_n cannot converge elsewhere.

3. The sequence y_n is similar. It converges in the uniform topology because the supremum distance of any coordinate from 0 is $1/n$, which goes to 0 as $n \to \infty$. Likewise, it must also converge in the product topology because the product topology is coarser. Lastly, we look at $\mathcal{O}$ in the box topology, and again note that $\mathcal{O}$ does not contain any points of y_n.

4. The sequence z_n does actually converge in the box topology because any neighborhood of zero all but perhaps the first two coordinates of z_n. However, because the harmonic sequence goes to 0, the first to components of all but finitely many of the z_n are contained in a neighborhood of 0. Hence, z_n converges in the box topology. Because both the product and uniform topologies are coarser than the box topology, we have that z_n must converge in these two topologies as well.

Problem 2.20.5

Intuitively, the closure of the set of sequences that are eventually zero should be the set of sequences that are 0 "at infinity". So we are looking for the limit points of $\mathbb{R}^\infty$, which should be the set of sequences that converge to 0 in the uniform topology. Consider any such limit point $\ell \in \mathbb{R}^\omega$. Because ℓ is a limit point, given any $\epsilon > 0$ the ball $B_{\overline{\rho}}(\epsilon, \ell)$ must contain a point of $\mathbb{R}^\infty$ other than ℓ. which means that for all but finitely many k, $|x_k| < \epsilon$. This is precisely what it means for the sequence $x_k \to 0$. Hence, the set of sequences that converge to 0 is the closure of $\mathbb{R}^\infty$ in the uniform topology.

Problem 2.20.8

(a) We will break this part down into two claims. The first one being

Claim. *The ℓ^2-topology is coarser than the box topology on $\mathbb{R}^\omega$, i.e. box topology $\supset$ ℓ^2-topology.*

Proof. Fix $\epsilon > 0$ and consider the collection of balls $\mathcal{B} = \{B_{\ell^2}(\epsilon, x) \,|\, x \in X\}$. For each $B_{\ell^2}(\epsilon, x) \in \mathcal{B}$ we need to find an open set in the box topology that contains $B_{\ell^2}(\epsilon, x)$. Consider the set

$$\mathcal{D} = \prod_{n=1}^{\infty} (x - \sqrt{\epsilon}/2, x + \sqrt{\epsilon}/2)$$

It is clear that $\mathcal{D}$ is open in the box topology. Furthermore for any point in $\mathcal{D}$ we can see that the distance in the ℓ^2 metric is less than ϵ (differences are geometric series). Hence, the ℓ^2-topology is coarser than the box topology. $\square$

We then need to show the following

Claim. *The ℓ^2-topology is finer than the uniform topology on $\mathbb{R}^\omega$, i.e. ℓ^2-topology $\supset$ uniform topology.*

Proof. Begin again by fixing an $\epsilon > 0$ and consider all balls of the form $B_{\overline{\rho}}(\epsilon, x)$ where $x \in X$. We will show that for each such ball, the ball $B_{\ell^2}(\epsilon, x) \subset B_{\overline{\rho}}(\epsilon, x)$, which verifies the claim. This is clear if we consider any point $y \in B_{\ell^2}(\epsilon, x)$, then we must have that

$$\left(\sum_{i=1}^{\infty}(x_i - y_i)^2\right)^{1/2} < \epsilon$$

But in order for this to be true we must also have that $\sup_i\{(x_i - y_i)\} < \epsilon$ because otherwise the above inequality would not hold. But this means that $y \in B_{\overline{\rho}}(\epsilon, x)$, and we are done. $\square$

(b) We will go up the chain of increasing fine-ness. To see that the product topologies and the uniform topologies are different we consider any product neighborhood of $\vec{0} = (0, 0, \ldots)$. Any such neighborhood must contain the point $\vec{0} + \epsilon_i = (0, 0, \ldots, , \epsilon, 0, \ldots)$, which has ϵ in the i^{th} coordinate, for sufficiently large i. However, the ball $B_{\overline{\rho}}(\epsilon, 0)$ does not, so the topologies must differ. Next, we compare the uniform and the ℓ^2 topologies. Pick an $\epsilon > 0$ and look at the ball $B_{\ell^2}(\epsilon, 0)$. With this ϵ in mind, choose a $\delta > 0$ corresponding to the ball $B_{\overline{\rho}}(\delta, 0)$. We then want to compare the two balls. Find an integer k such that $k\delta/2 > \epsilon$ and consider the point

$$z = (\overbrace{\delta/2, \delta/2, \ldots,}^{\text{first } n^2 \text{ components}} \; 0, 0, 0, \ldots)$$

Then $z \in B_{\overline{\rho}}(\delta, 0)$ but $z \notin B_{\ell^2}(\epsilon, 0)$. We proceed to the comparison of the ℓ^2 and the box topologies. We consider the usual set $\mathcal{O} = \prod_{k=1}^{\infty}(-1/k^2, 1/k^2)$ about 0 in the box topology and the ball $B_{\ell^2}(\epsilon, 0)$ for every $\epsilon > 0$. Then we have that the vector $0 + (\epsilon/2)_i$ is contained in $B_{\ell^2}(\epsilon, 0)$ for sufficiently large i, but for these same large i this vector is not contained in $\mathcal{O}$.

(c) We are considering the Hilbert cube

$$H = \prod_{n \in \mathbb{Z}^+} [0, 1/n]$$

and the four topologies that it inherits as a subspace of X. We will show that there are only two distinct topologies on the cube by proceeding in steps. The first is the following

Claim. *The ℓ^2 and box topologies on H are distinct.*

Proof. We begin by considering our usual open set $\mathcal{O} = \prod_{k=1}^{\infty}(-1/k^2, 1/k^2)$ in the box topology. Unfortunately, $\mathcal{O}$ does not lie in H, but we can construct a similar set $\mathcal{O}_H = \prod_{n \in \mathbb{Z}^+}[0, 1/2^k)$ as the neighborhodd of zero that we look at in the box topology. Fix any $\epsilon > 0$ and observe that the ball $B_{\ell^2}(\epsilon, 0)$ contains the point

$$p = (\frac{\sqrt{6}}{2\pi}\epsilon, \frac{\sqrt{6}}{4\pi}\epsilon, \dots, \frac{\sqrt{6}}{2k\pi}\epsilon, \dots)$$

This choice of p is motivated by the fact that in the ℓ^2 metric $d(p, 0) = \epsilon/2$, which means that p is not contained in the neighborhood $\mathcal{O}_H$ because the sequence $1/2^k$ decreases too rapidly. Hence, the two topologies on H are different. $\square$

A somewhat more surprising fact is the following

Claim. *The product, uniform, and ℓ^2 topologies are equivalent on H.*

Proof. We begin by noting that the usual ordering of these topologies by inclusion is

$$\ell^2 \subset \text{uniform} \subset \text{product}$$

Hence, if we can show that product $\subset \ell^2$ we will have the equivalence of the three topologies on H. To do this we begin by picking an $\epsilon > 0$ and a point $x \in H$. With ϵ in mind we find a $\delta > 0$ such that $\delta < \epsilon^2$ and observe that because the series $\sum_{k=1}^{\infty} 1/k^2$ converges we can use the Cauchy criterion to find an N large enough that

$$\sum_{k=N}^{\infty} 1/k^2 < \delta$$

We then pick $\alpha < \epsilon^2 - \delta$ and observe that the neighborhood

$$\mathcal{N} = \left(\prod_{i=1}^{N-1} (x - \frac{\sqrt{\alpha}}{\sqrt{N-1}}, x + \frac{\sqrt{\alpha}}{\sqrt{N-1}})\right) \times \left(\prod_{i=N}^{\infty} [0, 1/n]\right)$$

is in the product topology and contains the point x. By our choices of ϵ, δ and α we have that for any point $y \in \mathcal{N}$ that $d_{\ell^2}(x, y) < \epsilon$, and so our product neighborhood is contained inside the ℓ^2 ball $B_{\ell^2}(\epsilon, 0)$. As a result, we have that the product topology is finer than the ℓ^2 topology and we are done. $\square$

Problem 2.21.3

(a) We have the function ρ, and need to verify that the axioms for a metric are satisfied. Before we proceed we note that ρ is well-defined because the product is finite. To see that $\rho(x, y) \geq 0$ we observe that $\rho(x, y) \geq d_k(x, y) \geq 0$ because d_k is a valid metric for each k. We can see that equality holds iff $x = y$ because equality holds for each d_k iff $x = y$. To observe that $\rho(x, y) = \rho(y, x)$ observe that $\rho = d_k$ for some k and that $d_k(x, y) = d_k(y, x)$. Finally, the triangle inequality follows from the triangle inequality from each d_k because we have that $\rho(x, z) = \max_k\{d_k(x, z)\} \leq \max_k\{d_k(x, y) + d_k(y, z)\}$ and so we are done.

(b) We appeal to Theorem 20.1 to see that $\overline{d}_i$ is a metric on X_i for each i. Then we can see that $\overline{d}_i/i$ is a metric on x_i by seeing that $\overline{d}_i/i$ is clearly non-negative and that $\overline{d}_i(x,y)/i = 0$ implies that $\overline{d}_i(x,y) = 0$ which only happens when $x = y$. To see symmetry observe that $\overline{d}_i(x,y)/i = \overline{d}_i(y,x)/i$ because $\overline{d}_i(x,y) = \overline{d}_i(y,x)$. For the triangle inequality we see that

$$\overline{d}_i(x,z)/i \leq 1/i(\overline{d}_i(x,y) + \overline{d}_i(y,z)) = \overline{d}_i(x,y)/i + \overline{d}_i(y,z)/i$$

Then the metric $D(x,y) = sup\{\overline{d}_i(x,y)/i\}$ is well-defined and enjoys the properties of each individual metric (analogous to the last problem), and is therefore a valid metric on the product space.

Problem 2.22.2
(a) It is clear that f serves a right inverse for p, and so we know that p is surjective. Because it is also continuous, we need to see that the open saturated sets in X are sent onto open sets in Y. Choose a set $\mathcal{O}_X = p^{-1}(\mathcal{O}_Y)$. Then we have that

$$f^{-1}(\mathcal{O}_X) = f^{-1}(p^{-1}(\mathcal{O}_Y)) = \mathcal{O}_Y$$

So $\mathcal{O}_X$ is open because f is continuous.

(b) The canonical inclusion mapping is a continuous right inverse for the r, which we know is continuous by definition. So r must be a quotient map.

Problem 2.22.4
(a) Following the hint, we consider the map $g : (x \times y) \mapsto x + y^2$. Note that $g : X^* \to \mathbb{R}$. Note that g is a quotient map because the set U is open in $\mathbb{R}$ if and only if g^{-1} is open in $\mathbb{R}^2$. Geometrically, g^{-1} is a collection of left-facing parabolas. We then apply Corollary 22.3 to see that X^* is homeomorphic to $\mathbb{R}$. Alternatively, one could look at g as mapping X^* to the plane so that $(i \circ g)(x \times y) = (x + y^2, 0)$. Then g is a retraction and therefore a quotient map. And we again see that X^* is homeomorphic to $\mathbb{R}$.

(b) We proceed similarly to **(a)** using the map $g(x \times y) = x^2 + y^2$. In this case the retraction view of g is more illuminating because we can wee that $(x^2 + y^2, 0)$ is homeomorphic to the positive reals under the projection onto the first coordinate.

Problem