Problem 9.59.1
Choose two copies of S^2, say A and B and let $p_1 \in A$ and $p_2 \in B$ be the points such that if we let $\sim$ be the relation that identifies p_1 with p_2 then we have that $(A \cup B)/\sim \ \approx X$. Because S^2 is a manifold we can find neighborhoods U_1 of p_1 and U_2 of p_2 in A and B respectively that is homeomorphic to a disk in $\mathbb{R}^2$. We then define $V_A = A \cup U_2$ and $V_B = B \cup U_1$. Then V_1, V_2 are open and their union is X. Furthermore, their intersection is $U_1 \cup U_2$, which is non-empty and path connected. We want to show that V_A is simply connected. We see that U_2 deformation retracts to the point p_1 because we can take a homeomorphism of the straight line homotopy. Hence, U has the same homotopy type as A, and is therefore simply connected. The same argument shows that B is simply connected. This means that the fundamental group of X is the trivial group.

Problem 9.59.2
The problem with the proof is that it may be impossible to pick a point not in the image of f. It is possible to have continuous surjections $f : I \to S^2$. An example of such a function is the space-filling curve $s : I \to I^2$ composed with a homeomorphism $t : I^2 \to S^2 f$ so that $t \circ s$ is a continuous surjection $I \to S^2$.

Problem 9.59.3
(a) If we have that $h : \mathbb{R}^1 \to \mathbb{R}^n$ is a homeomorphism then h has a restriction to a homeomorphism $h^* : \mathbb{R}^1 - \{0\} \to \mathbb{R}^n - \{h(0)\}$. But, $\mathbb{R}^1 - \{0\}$ is disconnected and $\mathbb{R}^n - \{x\}$ is connected for any $x \in \mathbb{R}^n$ and $n > 1$.

(b) If $h : \mathbb{R}^2 \to \mathbb{R}^n$ was a homeomorphism, then h has a restriction to a homeomorphism $h^* : \mathbb{R}^2 - \{0\} \to \mathbb{R}^n - \{h(0)\}$. We then note that $\mathbb{R}^2 - \{0\}$ is homotopy equivalent to the circle S^1 and therefore is simply connected. However, $\mathbb{R}^n - \{x\}$ is homotopy equivalent to S^{n-1} and therefore is simply connected when $n > 2$. Hence, these two spaces cannot be homeomorphic.

Problem 9.59.4
(a) If we are given that j_* is the trivial homomorphism then the fundamental group of X must be generated by the image of i_* alone because it is generated by the identity and the image of i_*, which is the same as just the image of i_* because that image is a subgroup and therefore contains the identity.

If i_* and j_* are both trivial homomorphisms then the fundamental group of X must be trivial as well because it is the subgroup generated by the identity, which is just the identity again.

(b) Again we take two copies of S^2, call them A and B. Then choose four distinct points p_1, p_2, p_3, p_4 and define

$$A' = A - \{p_1, p_2\} \text{ and } B' = B - \{p_3, p_4\}$$

Then both A' and B' contain a copy of S^1 as a deformation retract, and consequently have the fundamental group $\mathbb{Z}$. However, we note that if $X = A' \cup B'$, then $X \cong S^2$ which has the trivial fundamental group. So each of the i_* and j_* must send every loop class to the identity and are therefore trivial homomorphisms.

Problem 9.60.5
Let E be the covering space $(A_0 \cup A_1 \cup B_0 \cup B_1$ as denoted in the diagram on pg. 375. Let e_1 be the point where A_1 intersects B_0 and let e_2 be the point where B_1 intersects A_0. Let $\bar{\alpha}$ be a path in A_1 from e_0 to e_1 and $\bar{\beta}$ be a path in B_1 from e_0 to e_2. Then we set $\alpha = p \circ \bar{\alpha}$ and $\beta = p \circ \bar{\beta}$, which are loops in X at x_0. Let α_ℓ be the lift of α at e_2 and let β_ℓ be the lift of β at e_2. The image of α_ℓ is contained in $A_0 \cup A_1$ and therefore in the component of A_0 of the set that contains its initial point. Thus, α_ℓ is a loop at e_2. Analogously, we have that β_ℓ is a loop at e_1. We now compute the lift of $f * g$ at e_0 to be $\bar{\alpha} * \beta_\ell$ and similarly

that of $g * f$ is $\bar{\beta} * \alpha_\ell$. So if we use the lifting correspondence

$$\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$$

which is determined by

$$\phi([f * g]) = e_1 \text{ and } \phi([g * f]) = e_2$$

we see that $[f] * [g] \neq [g] * [f]$ and so $\pi_1(X, x_0$ is not abelian.

Problem 13.79.1
We have that S^n is simply connected for $n > 1$. Consequently, we can apply Lemma 9.79.1 to see that any continuous map $f : S^n \to S^1$ lifts to a map $\tilde{f} : S^n \to \mathbb{R}$. The map is clearly nullhomotopic by a straight-line homotopy and so $f = p \circ \tilde{f}$ is nullhomotopic as well.

Problem 13.79.2
(a) Let $f : P^2 \to S^1$ be a continuous map. Consider the induced map f_* on the fundamental groups. We know that

$$\pi_1(P^2, x) \cong \mathbb{Z}_2$$
$$\pi_1(S^1, y) \cong \mathbb{Z}$$

From algebra we know that the only homomorphism $\varphi : \mathbb{Z}_2 \to \mathbb{Z}$ is the trivial homomorphism. Hence, we have that $f_*(\pi_1(P^2, x)) = 0$. So we can lift f to a map $\tilde{f} : P^2 \to \mathbb{R}$ as in the previous problem. As a result, we again get that f is nullhomotopic.

Problem 13.79.3

Problem 13.81.2
(a) Clearly, this is a 2-sheeted cover from the definition. Hence , we have that $H_0 = p_*(\pi_1(E, e_0))$ has index 2 in $\pi_1(X, x_0)$. Then we recall that any subgroup of index 2 is normal and so $H_0 \trianglelefteq \pi_1(X)$.

(b) We follow the argument given in example 13.81.2. So if $p : (E, e_0) \to (B_, b_0)$ is a covering map and h is a covering transformations, then any loop at b_0 lifts to a loop at e_0 iff it lifts to a loop at $h(e_0)$. But we can see that a loop that wraps B once lifts to a loop at e_1 but not a loop at e_0. Because $h(e_0)$ is one of e_0, e_1, e_2 we see that $h(e_0)$ is not e_1. Analogously, we see that $h(e_0)$ is not equal to e_2 by the same argument for loops wrapping around A. So we have that $h(e_0) = e_0$ and so h is the identity and therefore the covering is not regular.

(c)

(d)