Problem 3.26.5

Suppose that we are given two sets A, B that are disjoint compact subspaces of the Hausdorff space X. Because we assumed that A and B are disjoint we know that any $a \in A$ is not in B. This means that we now satisfy the hypotheses for Lemma 26.4, which we apply to find two open sets U_a and V_a such that $a \in U$ and $B \subset Y$. We then note that

$$A \subset \bigcup_{a \in A} U_a$$

and

$$B \subset \bigcap_{a \in A} V_a$$

Because each of the U_a is open we must also have that their union is open, and hence an open cover of A. We then apply compactness of A to extract a finite subcover $\{U_j\}_{j=1}^n$. Then we note that

$$A \subset \bigcup_{j=1}^{n} U_j$$

and

$$B \subset \bigcap_{j=1}^{n} V_j$$

It is clear that $\bigcup_{j=1}^n U_j$ and $\bigcap_{j=1}^n U_j$ are both open because they are the finite union and intersection of open sets, respectively. These collections are disjoint by construction and so we are done.

Problem 3.26.11

We need to prove the following

Theorem. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of C that is simply ordered by proper inclusion. Then*

$$Y = \bigcap_{A \in \mathcal{A}} A$$

is connected.

Proof. Suppose to the contrary that Y is not connected. Then we must have some separation of Y into open sets C and D with $Y = C \cup D$. Then each of C and D is closed because they are each the intersection of nested (due to the ordering) compact sets in a Hausdorff space. As a reult of this we can apply the result from Exercise 5 to find to disjoint open sets U and V such that $C \subset U$ and $D \subset V$. As a result of this fact we must that for any $A \in \mathcal{A}$ that $A-(U \cup V)$ is closed because $U \cup V$ is open and A is closed. Furthermore, we have that the sets $\{A-(U\cup V)\}$ are a nested collection of closed sets because $\{A\}$ is a nested collection of sets. As a result we see that

$$B = \bigcap_{A\in\mathcal{A}} (A-(U\cup V))$$

must be nonempty because it is the intersection of nested closed sets in a Hausdorff space, and therefore non-empty. This means we can find $b \in B$ that is in neither U nor V. But this is impossible because

$$B \subset Y = (C\cup D) \subset (U\cup V)$$

This contradiction establishes the result. □

Problem 3.26.12
We begin with the following

Lemma. *Let $p : X \to Y$ be a perfect map and y an element of Y. Then if U is an open set such that $p^{-1}(\{y\}) \subset U$ there must be a neighborhood W of y such that $p^{-1}(W) \subset U$*

Proof. Let U be the open set described above. Because U is open we know U^c is closed. As a result of this we see that $p(U^c)$ must also be closed because p is a perfect, hence closed, map. We then set

$$W = (p(U^c))^c$$

Which means W is an open set in Y containing y. Then we note that p is continuous and so $p^{-1}(W)$ is open in X and must be disjoint from U^c by construction. This means $W \subset (U^c)^c = U$ and so we are done. □

With this lemma in hand we now proceed to show that Y is compact then X must also be compact. We begin with some open cover $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{A}}$ of X. We will show that this cover has a finite subcover. Let $\mathcal{C} \subset Y$ be compact. Consider its inverse image $p^{-1}(\mathcal{C}) \subset X$. For every $y \in \mathcal{C}$ the compact space $p^{-1}(y)$ is contained in a finite collection of the $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{A}(y)}$. There is a neighborhood W_y of y such that $p^{-1}(W_y)$ is contained in this union. By the compactness of $\mathcal{C}$ only finitely many of these W_{y_j} cover Y. This means

$$p^{-1}(\mathcal{C}) \subset \bigcup_j \bigcup_{\alpha\in\mathcal{A}(y)} \mathcal{O}_\alpha$$

is a finite cover of $p^{-1}(\mathcal{C})$ and so we are done.

Problem 3.27.5
Let $\mathcal{O}$ be a nonempty subspace of X. We no that $\mathcal{O} \not\subset A_1$ because A_1 has no

interior. So the set $\mathcal{O} - A_1$ is open and non-empty. We then apply Lemma 31.1 to find a nonempty open set $\mathcal{O}_1$ such that

$$\mathcal{O}_1 \subset \overline{\mathcal{O}} \subset (\mathcal{O} - A_1) \subset \mathcal{O}$$

We can continue this process indefinitely because each of the A_n has no interior and each of the $\mathcal{O}_{n-1} - A_n$ will be open. This will lead us to a decreasing sequence of non-empty sets $\mathcal{O}_n \subset \mathcal{O}_{n-1} \subset \cdots$ for every n. Because we know that X is compact we have that $M = \bigcap \mathcal{O}_n = \bigcap \overline{\mathcal{O}_n} \neq \emptyset$. Moreover, we have that

$$\mathcal{O} \cap \bigcap (X - A_n) = \mathcal{O} - \bigcup A_n$$

This completes the proof.

Problem 3.27.6

(a) It is apparent from the definition that the set A_n is the disjoint union of 2^n closed intervals of length 3^{-n}. Pick two points $x, y \in C$ with $x < y$ and choose n such that $|x - y| > 3^{-n}$. Then because $\mathbb{R}$ is Hausdorff with the standard topology we can find a point r such that $x < r < y$ such that $r \notin A_n$, and therefore $r \notin C$. This means that any subspace of C containing x and y has a separation (c.f. pg 149).

(b) It is clear that C is closed because it is defined as the complement of open intervals. We appeal to Theorem 26.2 to see that because C is a closed subset of the compact space $[0, 1]$ that it must be compact.

(c) We proceed by induction. The base case is clear because $A_0 = [0, 1]$ has length $3^0 = 1$. for the inductive step we suppose that A_{n-1} is the union of finitely many disjoint closed intervals of length 3^{-n}. For each of these intervals we remove finitely many disjoint open intervals of length $3^{-(n+1)}$ from them. The remainder is a finite set of disjoint closed intervals of length $3^{-(n+1)}$. Iterating this process finitely many times yields a finite collection of disjoint closed intervals in A_n. This completes the induction. TO see that the the endpoints of each of these intervals is in C is clear, because the endpoints are the boundary points of the intervals, and from the definition we only ever remove interior points from any interval. Hence, the endpoints of every A_n are contained in C.

(d) It is clear from the construction of C that it removes only the interior points of the A_n. Therefore, the boundary of A_n is contained in C for every n. As a result any interval of length $3^{-(n+1)}$ around any point of A_n contains a boundary point of A_{n+1}, and therefore a point of C. So C has no isolated points.

(e) C is a nonempty compact Hausdorff space with no isolated points. Therefore it meets the criteria of Theorem 27.7 and is compact.