Problem 1.6.6
Let $A = \{1, 2, \ldots, n\}$ and let $X = \{0, 1\}$. Consider the map

$$\begin{aligned} f : \mathcal{P}(A) &\longrightarrow X^n \\ B &\mapsto (x_1, x_2, \ldots, x_n) \end{aligned}$$

Where $m \in B$ implies $x_m = 1$ and otherwise $x_m = 0$. This map is clearly injective because each set $B \subset A$ has less than or equal to n elements, and therefore will get mapped onto the n-vector corresponding to its elements. Furthermore, B_1 and B_2 have the same image under f means that if $f(B_1) = (x_1, \ldots, x_n)$ and $f(B_2) = (y_1, \ldots, y_n)$ then $x_k = y_k$ for all k. Hence, B_1 and B_2 have the same elements and are therefore equal. For surjectivity take $x = (x_1, \ldots, x_n)$ and let so $f^{-1}(x) = \{i : x_i = 1\}$. Hence, each x has an inverse image and f is surjective. Thus, f is bijective.

Problem 1.7.5
(a) The set $A = \{f \mid f : \{0, 1\} \to \mathbb{Z}_+\}$ is countable. Each function in A is uniquely determined by the where it sends 0 and 1. There are only a countable number of possibilities for each one, and so we may think of f as lying in $\mathbb{Z}_+ \times \mathbb{Z}_+$ as a 2-tuple $(f(0), f(1))$. This means that $A \subseteq \mathbb{Z}_+ \times \mathbb{Z}_+$ and because $\mathbb{Z}_+ \times \mathbb{Z}_+$ is a finite product of countable sets this gives that A is a subset of a countable set and hence, countable.
(b) The set $B = \{f \mid f : \{1, 2, \ldots, n\} \to \mathbb{Z}_+\}$ is also countable. The reasoning is the same as in **(a)** except that now we may think of f as lying in $\mathbb{Z}_+^n$ represented by $(f(1), f(2), \ldots, f(n))$, which is also a finite product of countable sets and thus, countable. So B is also a subset of a countable set and therefore countable.
(c) Let $C = \bigcup_{n \in \mathbb{Z}_+} B_n$. If B_n is an at most countable set for each n then C is countable because it is the countable union of countable sets. If any of the B_n is uncountable then so is C because there is a natural injection $B_n \hookrightarrow C$ given by the identity on B and so $|C| \geq |B_n|$.
(d) Let $D = \{f \mid f : \mathbb{Z}_+ \to \mathbb{Z}_+\}$. I claim that D is uncountable. We can think of each f as an element of $\mathbb{Z}^\omega$ represented by $(f(1), f(2), \ldots)$. Note that if we set $X = \{0, 1\}$ then $X^\omega \subset \mathbb{Z}^\omega \cong D$. Recall that X^ω is uncountable and because the identity on X^ω defines an injection into $\mathbb{Z}^\omega$ we have that $|D| = |\mathbb{Z}^\omega| \geq |X^\omega|$ and so D is uncountable. (Note: we could also have just done the diagonalization directly on D)
(e) If we take $E = \{f \mid f : \mathbb{Z}_+ \to \{0, 1\}\}$ and $X = \{0, 1\}$ then we can think of each $f \in E$ as the vector $(f(1), f(2), \ldots) \in X^\omega$. As we saw in the book, X^ω is uncountable and hence so is E.
(f) Let F be the set of eventually zero functions $\mathbb{Z}_+ \to \{0, 1\}$. More precisely, take $F = \{f \mid \exists N > 0, \forall n > N, f(n) = 0\}$. Then F breaks up naturally into pieces $F_n = \{g \mid \forall m > n, g(m) = 0\}$ so that $F = \bigcup_{n=1}^{\infty} F_n$. Now we observe that $F_n \cong \mathbb{Z}_+^n$. To see this observe that we can view F_n as an n-tuple $(f(1), f(2), \ldots, f(n))$. This defines an injection $F_n \hookrightarrow \mathbb{Z}_+^n$ because if f, g map to the same n-tuple, then $f(k) = g(k)$ for each $k \leq n$, and because $f, g \in F_n$ $f(m) = g(m)$ for every $m > n$. So $f(x) = g(x)$ for each x and thus $f = g$. Furthermore, the mapping is surjective because for each $(y_1, \ldots, y_n)$ we can

construct the function given by $f(k) = y_k$. This bijection verifies that $F_n \cong \mathbb{Z}_+^n$. Hence,

$$F = \bigcup_{n=1}^{\infty} F_n \cong \bigcup_{n=1}^{\infty} \mathbb{Z}_+^n$$

F is a countable union of countable sets and thus, countable.
(g) In the same spirit as the previous part we set $G = \{f \mid \exists N > 0, \forall n > N, f(n) = 1\}$. In the same way as before we can decompose G into pieces G_n such that $G_n = \{g \mid \forall m > n, g(m) = 1\}$. The identification with the n-tuple $(g(1), g(2), \ldots, g(n))$ is a bijection (for the same reasons as in **(f)**) and so we have that $G_n \cong \mathbb{Z}_+^n$. So we see that

$$G = \bigcup_{n=1}^{\infty} G_n \cong \bigcup_{n=1}^{\infty} \mathbb{Z}_+^n$$

(h) It is clear from the preceding to parts that the set of functions which are eventually k is countable. So if we take

$$H = \{f : \mathbb{Z}_+ \to \mathbb{Z}_+ \mid \exists N, k \text{ such that } \forall n > N, f(n) = k\}$$

We now will decompose H into a union as we did before. We define

$$H_{n,k} = \{g \,|\, \forall m > n g(m) = k\}$$

then it is clear that

$$H = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} H_{n,k}$$

For each fixed n, k, $H_{n,k}$ is countable as seen through identification with an n-tuple and so for a fixed n, $\bigcup_{k=1}^{\infty} H_{n,k}$ is a countable union of countable sets and therefore countable. Then we have that H is a countable union of countable sets, and it too, is countable.
(i) The set I of two element subsets of $\mathbb{Z}_+$ is also countable. We can identify with each set $\{a, b\}$ with $a \leq b$ the 2-tuple $(a, b) \in T \subset \mathbb{Z}_+ \times \mathbb{Z}_+$ where $T = \{(x, y) \,|\, x < y\}$. To see that this map a bijection we note that the map is an injection because if $f(\{a, b\}) = f(\{c, d\}) = (x, y)$ then we must have that $\{a, b\} = \{c, d\}$. This is because if we order the sets in increasing order then $a = c = x$ and $b = d = y$, which means that the two sets must have been the same (up to a possible re-ordering of their elements). To see that it is surjective take any 2-tuple (x, y) and observe that the set $\{x, y\}$ maps onto it. So this map is a bijection onto $T \subset \mathbb{Z}_+ \times \mathbb{Z}_+$ and so I is (equivalent to) a subset of a countable set and therefore countable.
(j) The set J of all finite subsets of all finite subsets of $\mathbb{Z}_+$ is also countable. By reasoning analogous to part **I** we have that the set J_n of all subsets of size n is countable (this time we order the sets in increasing order and map to an n-tuple). Let J_n be this set. Then we can write

$$J = \bigcup_{n=1}^{\infty} J_n$$

This gives that J is a countable union of countable sets and therefore countable.

Problem 1.7.7
Let F be the set of functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ and G be the set of functions $g : \mathbb{Z}_+ \to \{0,1\}$. We want to show that $|F| = |G|$. First note that there is a natural injection $p : G \hookrightarrow F$ where

$$(p \circ g)(x) = g(x) + 1$$

then the image of each $g \in G$ is $p \circ g$ which maps into $\{1,2\} \subset \mathbb{Z}^+$. Now we will construct an injection in the reverse direction from $F \hookrightarrow G$. The first step is to represent for each n, the quantity $f(n)$ as a vector in $\mathbb{Z}_2^\omega$. To do this we note that each integer has a unique binary expansion

$$f(n) = \sum_{k=1}^{\infty} \alpha_k 2^{k-1}$$

where $\alpha_k \in \{0,1\}$ for all k, and only finitely many α_k are non-zero. So we associate to each f a matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & \cdots \\ \hline f(1) & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \cdots \\ f(2) & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \cdots \\ f(3) & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Where $\alpha_{n,k}$ is the k^{th} digit in the binary expansion of $f(n)$. We then associate to f the vector $\alpha = (\alpha_{1,1}, \alpha_{2,1}, \alpha_{1,2}, \ldots)$, which is obtained via the usual diagonal traversal, and define a new function $g : \mathbb{Z} \to \{0,1\}$ such that $g(n)$ is the n^{th} component of α. I claim that the map $F : f \mapsto g$ is injective. Suppose it were not, then we would have two functions f_1, f_2 such that $F(f_1) = F(f_2) = g$. If we look at the definition of g this means that if $f_1(n) = \sum_{k=1}^{\infty} \alpha_{n,k} 2^{k-1}$ and $f_2(n) = \sum_{k=1}^{\infty} \beta_{n,k} 2^{k-1}$ then $\alpha_{n,k} = \beta_{n,k}$ for every n, k. But the binary representation of a number is unique, and so $f_1(n) = f_2(n)$ for each n. This means that $f_1 = f_2$ and so the map F is injective.

Problem 1.9.2
(a) We appeal to the well ordering principle of $\mathbb{Z}_+$. To each element of $\mathcal{P}(\mathbb{Z}_+)$ we assign its minimal element, which exists because $\mathbb{Z}$ is well ordered.
(b) Similarly to **(a)** we define to each set in $\mathbb{Z}$ its minimal element, which exists and is well-defined because of well-ordering.
(c) This one is slightly more complicated. The set $\mathbb{Q}$ is countable, and therefore we can give an enumeration of them $r_1, r_2, r_3, \ldots$. Then to each set of rationals $\{r_i, r_j, \ldots\}$ we let $m = \min\{i, j, \ldots\}$, which is unambiguous because the integers are well ordered. Then we define the choice function to be $c : \{r_i, r_j, \ldots\} \mapsto r_m$.
(d) In this case it is not possible to define a choice function without the axiom of choice. To see this note that $X^\omega = \prod_{n \in Z_+} X$. This an infinite product of

sets, and so any choice function would have to make countable infinite choices of elements in X. It is seen in the book that this is equivalent to the axiom of choice.

Problem 1.9.3
Let $\{f_n\}_{n\in\mathbb{Z}_+}$ be a family of injective functions

$$f_n : \{0, 1, \ldots, n\} \longrightarrow A$$

We will see that A must be an infinite set. Suppose to the contrary that A is finite. Then $|A| = N$ for some $N > 0$. But then for f_m, $m > N$ the pigeonhole principle says that there must be two elements in $\{0, 1, \ldots, m\}$ that are mapped onto the same element of A. So f_m cannot be injective. This contradiction shows that A must be an infinite set.

To determine a choice function without the axiom of choice we appeal to the principle of recursive definition. We set

$$m = \min\{x \mid x \notin \{f(1), f(2), \ldots, f(i-1)\}\}$$

and take $f(n) = f_n(m)$. this map is well-defined because the f_n are injective by hypothesis.

Problem 1.9.6
(a) Suppose that $\mathcal{A}$ were a set that contained all other sets. Then we would have that $\mathcal{P}(\mathcal{A}) \in \mathcal{A}$ because $\mathcal{P}(\mathcal{A})$ is a set. Hence we can construct a surjection $S : \mathcal{A} \twoheadrightarrow \mathcal{P}(\mathcal{A})$, but this implies that $|\mathcal{A}| \leq |\mathcal{P}(\mathcal{A})|$, which is impossible (proved in the book).
(b) The point of the paradox is that there is no valid choice for where to include $\mathcal{B}$. If $\mathcal{B} \in \mathcal{B}$, then $\mathcal{B}$ is not an element of itself, which is a contradiction. But if $\mathcal{B} \notin \mathcal{B}$ then $\mathcal{B} \in \mathcal{B}$, which is also impossible.

Problem 1.9.8
As before we will appeal to the uniqueness of binary expansions for real numbers. First we will construct a bijection from subsets of Z_+, i.e elements of $\mathcal{P}(\mathbb{Z}_+)$, to $X = \{0, 1\}^\omega$. To each set $S \in \mathcal{P}(\mathbb{Z}_+)$ we associate the vector $x = (x_1, x_2, \ldots) \in X$ such that

$$x_k = \begin{cases} 1 & k \in S \\ 0 & \text{otherwise} \end{cases}$$

Call this function $g : S \mapsto x$. Recall (from previous problems) that this function is a bijection. Now we use the fact that each real number has a unique binary decimal representation that does note end in an infinite string of 1's. Let $f : \mathbb{R} \hookrightarrow X$ be the injection that sends each real number to its binary expansion. Then using the fact that f is injective and g bijective (and hence injective as well) we have that $g^{-1} \circ f$ is an injection from $\mathbb{R} \hookrightarrow \mathcal{P}(\mathbb{Z}_+)$. Now we construct an injection $h : \mathcal{P}(\mathbb{Z}_+) \hookrightarrow \mathbb{R}$. Suppose that $S \in \mathcal{P}(\mathcal{Z}_+)$, then S must be countable. Let $s_1, s_2, \ldots$ be an enumeration of the elements of S. We then define $b : \mathcal{P}(\mathbb{Z}_+) \hookrightarrow X$ via $S \mapsto (0, s_1, 0, s_2, \ldots, 0, s_{n/2}, \ldots)$. Then take $h = f \circ b$, which is an injection. We then apply the Schroeder-Bernstein Lemma to see that $|\mathcal{P}(\mathbb{Z}_+)| = |\,\mathbb{R}\,|$.