Exercise 6.3.5:

We will use the fact that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

And the following

Claim. $\overline{\Gamma(s)} = \Gamma(\overline{s})$

Proof. This is clear from the product formula for $\Gamma(s)$. By Theorem 6.1.7 we see

$$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

Consequently,

$$\Gamma(z) = \frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{s} n$$

Then we note that complex conjugates are preserved under products and take the limit of the comjugates of partial products to get that

$$\overline{\Gamma(s)} = \Gamma(\overline{s})$$

□

We can then compute

$$|\Gamma(1/2+it)|^2 = \Gamma(1/2+it)Gamma(1/2-it) = \frac{\pi}{\sin \pi(1/2+it)}$$

Then note that $\sin(\theta + \pi/2) = \cos(\theta)$ to see that

$$\frac{\pi}{\sin \pi(1/2+it)} = \frac{\pi}{\cos \pi it} = \frac{\pi}{\cosh \pi t} = \frac{2\pi}{e^{\pi t} + e^{-\pi t}}$$

Then

$$|\Gamma(1/2+it)|^2 = \frac{2\pi}{e^{\pi t} + e^{-\pi t}}$$

Taking square roots yields

$$|\Gamma(1/2+it)| = \sqrt{\frac{2\pi}{e^{\pi t} + e^{-\pi t}}}$$

Exercise 6.3.7:

(a) We first observe that

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= \left(\int_0^{\infty} e^{-t} t^{\alpha-1} dt\right)\left(\int_0^{\infty} e^{-s} s^{\beta-1} ds\right) \\ &= \int_0^{\infty}\int_0^{\infty} t^{\alpha-1} s^{\beta-1} e^{-t-s} dt ds\end{aligned}$$

Following the hint, we change variables to $s = ur, t = u(1-r)$ and compute

$$\begin{aligned}\int_0^{\infty}\int_0^{\infty} t^{\alpha-1} s^{\beta-1} e^{-t-s} dt ds &= \int_0^{\infty}\int_0^{1} (u(1-r))^{\alpha-1}(ur)^{\beta-1} e^{-u} u dr du \\ &= \left(\int_0^{\infty} u^{\alpha+\beta-1} e^{-u} du\right)\left(\int_0^1 r^{\beta-1}(1-r)^{\alpha-1} dr\right) \\ &= \Gamma(\alpha+\beta)B(\alpha,\beta)\end{aligned}$$

Dividing on both sides gives

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(b) This also follows by a change of variable. Inspection of the formula suggests that the thing to do set

$$t=\frac{1}{1+u}\qquad 1-t=\frac{u}{1+u}$$

Consequently, we have that

$$dt=\frac{-1}{(1+u)^2}du$$

Hence,

$$\begin{aligned}B(\alpha,\beta)=\int_0^1(2-t)^{\alpha-1}t^{\beta-1}dt&=-\int_0^\infty\left(\frac{u}{1+u}\right)^{\alpha-1}\left(\frac{1}{1+u}\right)^{\beta-1}\left(\frac{-1}{(1+u)^2}\right)du\\&=\int_0^\infty\frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du\end{aligned}$$

as desired.

Exercise 6.3.11:

We begin with the function

$$f(z)=e^{az}e^{-e^z}$$

We then recall the definition

$$\hat{f}(\xi)=\int_{-\infty}^{\infty}f(x)e^{-2\pi ix\xi}dx$$

and then compute

$$\int_{-\infty}^{\infty}f(x)e^{-2\pi ix\xi}dx=\int_{-\infty}^{\infty}e^{ax}e^{-e^x}e^{-2\pi ix\xi}dx=\int_{-\infty}^{\infty}e^{x(a-2\pi i\xi)}e^{-e^x}dx$$

Then make the change of variable $t=e^x$ so that

$$\int_{-\infty}^{\infty}e^{x(a-2\pi i\xi)}e^{-e^x}dx=\int_0^\infty t^{(a-2\pi i\xi)}e^{-t}\frac{dt}{t}=\int_0^\infty t^{(a-2\pi i\xi-1)}e^{-t}dt$$

Then observe that

$$\Gamma(a-2\pi i\xi)=\int_0^\infty t^{(a-2\pi i\xi-1)}e^{-t}dt$$

So that

$$\hat{f}(\xi)=\Gamma(a-2\pi i\xi)$$

Exercise 6.3.13:

We recall from the textbook that

$$\frac{1}{\Gamma(s)}=e^{\gamma s}s\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-s/n}$$

Taking the logarithm yields

$$-\log\Gamma(s)=\log s+\gamma s+\sum_{n=1}^{\infty}\log\left(1+\frac{s}{n}\right)-\frac{s}{n}$$

We then note that we can differentiate the sum term by term because the convergence is uniform on compact sets that do not contain a singularity. We then compute for $s > 0$

$$\frac{d^2}{ds^2}(-\log\Gamma(s)) = \frac{-1}{s^2} + \sum_{n=1}^{\infty}\frac{-1}{(n+s)^2} = \sum_{n=0}^{\infty}\frac{-1}{(n+s)^2}$$

The right hand side is an analytic function whenever s is not a negative integer because the sum converges uniformly in this region. If we consider the left hand side to be $(\Gamma'/\Gamma)'$ then we observe that both functions are analytic, and Γ is non-vanishing and the formula must hold by analytic continuation.

Exercise 6.3.15:

We suppose that $\text{Re}\,(s) > 1$ and need to show that

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x - 1}dx$$

Following the hint, we note that

$$\frac{1}{e^x - 1} = \sum_{n=1}^{\infty} e^{-nx}$$

We then see that

$$\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx = \int_0^{\infty} x^{s-1}\sum_{n=1}^{\infty} e^{-nx}dx = \int_0^{\infty}\sum_{n=1}^{\infty} x^{s-1}e^{-nx}dx$$

We then note that on any interval $(\epsilon, 1/\epsilon)$ the convergence of the sum is uniform and we can exchange the sum and the integral. So we then compute

$$\begin{aligned}\int_{\epsilon}^{1/\epsilon}\sum_{n=1}^{\infty} x^{s-1}e^{-nx}dx &= \sum_{n=1}^{\infty}\int_{\epsilon}^{1/\epsilon} x^{s-1}e^{-nx}dx \\ &= \sum_{n=1}^{\infty}\int_{n\epsilon}^{n/\epsilon}\left(\frac{t}{n}\right)^{s-1}e^{-t}dt\end{aligned}$$

Factoring gives

$$\sum_{n=1}^{\infty}\int_{n\epsilon}^{n/\epsilon}\left(\frac{t}{n}\right)^{s-1}e^{-t}dt = \sum_{n=1}^{\infty}\frac{1}{n^s}\int_{n\epsilon}^{n/\epsilon} t^{s-1}e^{-t}dt$$

We then note that because this holds for every $\epsilon > 0$ we can apply dominated convergence to let $\epsilon \to 0$ and yield

$$\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx = \sum_{n=1}^{\infty}\frac{1}{n^s}\int_0^{\infty} t^{s-1}e^{-t}dt = \zeta(s)\Gamma(s)$$

Dividing both sides by $\Gamma(s)$ gives

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx$$

Exercise 6.3.16:

By the previous exercise we have that

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx$$

We then use linearity to see that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^1 \frac{x^{s-1}}{e^x - 1} dx + \frac{1}{\Gamma(s)} \int_1^\infty \frac{x^{s-1}}{e^x - 1} dx$$

We see that the second term is an entire function becuase $1/\Gamma$ is entire. For the first term we must consider the behavior of

$$\int_0^1 \frac{x^{s-1}}{e^x - 1} dx = \int_0^1 x^{s-2} \frac{x}{e^x - 1} dx$$

We then recall the definition of the Bernoulli numbers

$$\frac{x}{e^x - 1} = \sum_{m=0}^\infty \frac{B_m}{m!} x^m$$

This gives

$$\int_0^1 x^{s-2} \frac{x}{e^x - 1} dx = \int_0^1 x^{s-2} \sum_{m=0}^\infty \frac{B_m}{m!} x^m dx$$

We then note that because the convergence is uniform (it is the Taylor series) in the disk $|z| < 2\pi$ we can exchange the sum and the integral to get

$$\int_0^1 x^{s-2} \sum_{m=0}^\infty \frac{B_m}{m!} x^m dx = \sum_{m=0}^\infty \frac{B_m}{m!} \int_0^1 x^{m+s-2} dx = \sum_{m=0}^\infty \frac{B_m}{m!(m+s-1)}$$

Rewriting this gives

$$\int_0^1 \frac{x^{s-1}}{e^x - 1} dx = \sum_{m=0}^\infty \frac{B_m}{m!(m+s-1)}$$

Hence, the function is clearly analytic except possibly when $s = 1$. We then observe that we have an analytic continuation

$$\frac{\Gamma(m+s-1)}{\Gamma(m+s)} = \frac{1}{m+s-1}$$

When $s = 1$ we have a simple pole at $s = 1$ and nowhere else. Thus, we have defined a function that is continuable, using the above extension, to the whole plane with a simple pole at $s = 1$.