Exercise 2.6.15:

Following the hint, we extend f to all of $\mathbb{C}$ via the function

$$F(z) = \begin{cases} f(z) & |z| \leq 1 \\ \frac{1}{\overline{f(1/\bar{z})}} & |z| > 1 \end{cases}$$

Now we establish that F is continuous in the whole plane. Because f is holomorphic in $\mathbb{D}$, it is also continuous there. In $|z| > 1$, F is also continuous becuase f is non-vanishing in $\mathbb{D}$ and F is the composition of continuous functions. On the boundary, $\partial\,\mathbb{D}$, we consider a point $z \in \partial\,\mathbb{D}$ and the limit of $F(w)$ as $w \to z$ from the complement of the disk. Then $1/\bar{w} \to 1/z = z$ so that

$$F(w) = 1/\overline{1/\bar{w}} \to 1/\overline{f(z)} = f(z) = F(z)$$

So F is continuous in the whole plane.

Now we will show that F is entire. By assumption, we know that F is holomorphic in the interior of $\mathbb{D}$. For the complement of $\mathbb{D}$, consider a contour C in $\mathbb{D}^c$. Consider the image of C under the map $w = 1/z$, C', which will be a curve strictly contained in $\mathbb{D}$ because $|z| > 1$ for every $z \in C$. Moreover, the origin is not in the interior of C' because $|z| < \infty$ for all $z \in C$. Consequently,

$$\int_C F(z)dz = \int_{C'} \frac{-1dw}{w^2\overline{f(\bar{w})}} = 0$$

where the second equality is because $1/w^2\overline{f(\bar{w})}$ is analytic on C' and its interior.

We now need to verify that F is analytic at the boundary $\partial\,\mathbb{D}$. We now argue as in the proof of the Schwartz reflection principle. Let T be a triangle that crosses the boundary of $\mathbb{D}$. We subdivide T into the following types of subtriangles, T_i

1. A vertex of T_i lies on the boundary of $\mathbb{D}$.
2. An edge of T_i is a chord of $\mathbb{D}$.

In the first case, we argue as in the Schwartz reflection principle: perturb the vertex that lies on $\partial\,\mathbb{D}$ by ϵ and note that for each $\epsilon > 0$ thi integral of F around T_i is 0 because it lies either only in the interior or complement of $\mathbb{D}$. In the second case we continue to subdivide T_i. We consider the arc in $\partial\,\mathbb{D}$ that is subtended by the edge in T_i and its midpoint p. Divide T_i into smaller triangles by drawing the triangles e_1pe_2 where e_1 and e_2 are the endpoint of the chord. This strictly decreases the distance between a point in a triangle, and the boundary of the disk. We continue this process until all points are within ϵ distance of $\partial\,\mathbb{D}$. We now apply the same argument as above to see that the integral across the subtriangles, and hence T_i, is zero. Consequently, F is entire.

Now we notice that $f(\mathbb{D})$ is the continuous image of a compact set (that does not include the origin), hence F bounded. So $1/f$ is also bounded on $\mathbb{D}$. F is then a bounded entire function and therefore constant. As a result we see that f must be constant.

Exercise 3.8.2:

We want to evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx$$

We proceed by studying the complex function $f(z) = 1/(1+z^4)$. First, we need to find the poles of f. We can see that f has a singularity at points where $z^4 = -1$. By setting

$z = re^{i\theta}$ and noting that $e^{i\pi} = -1$, we see that the only singularities in the the interval are at $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

We can determine the type of singularity that f has at each of the points above. If we look at the denominator $1 + z^4$ we can use the fact that polynomials are a product of linear factors to see that each pole is a simple pole (no roots are repeated in this case).

Now we consider the only solutions in the upper half plane, $\theta = \pi/4, 3\pi/4$ and integrate f over the following contour

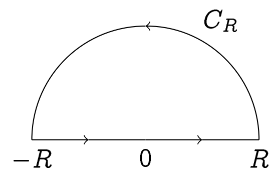

Figure 1: A semicircular contour.

Set $z_0 = e^{i\pi/4}$ and $z_1 = e^{3\pi/4}$ then apply the residue theorem to see that

$$\int_C \frac{1}{1+z^4} dz = 2\pi i(\mathrm{res}_{z_0} f(z) + \mathrm{res}_{z_1} f(z))$$

We begin by calculating the residues at z_0, z_1. Using the factorization for $1 + z^4$ we get

$$\mathrm{res}_{z_0} f(z) = \lim_{z \to z_0} (z - z_0) \cdot \frac{1}{(z - z_0)(z - z_1)(z - e^{5i\pi/4})(z - e^{7\pi i/4})} = \frac{1}{4} e^{5\pi i}$$

and

$$\mathrm{res}_{z_0} f(z) = \lim_{z \to z_1} (z - z_1) \cdot \frac{1}{(z - z_0)(z - z_1)(z - e^{5i\pi/4})(z - e^{7\pi i/4})} = \frac{1}{4} e^{7\pi i}$$

Now we observe that

$$\left| \int_{C_R} \frac{1}{1+z^4} dz \right| \leq \int_{C_R} \left| \frac{1}{1+z^4} \right| dz \leq \int_{C_R} \left| \frac{1}{|z|^4 - 1} \right| dz \leq \int_0^{\pi} \frac{|iRe^{i\theta}|}{|Re^{i\theta}| - 1} d\theta \leq \frac{\pi R}{R^4 - 1}$$

Sending $R \to \infty$ gives that

$$\lim_{R \to \infty} \int_{C_R} \frac{1}{1+z^4} dz = 0$$

So that as $R \to \infty$ the integral around C becomes

$$\int_{-\infty}^{\infty} \frac{1}{1+z^4} dz = 2\pi i \left(\frac{1}{4} e^{5\pi i} + \frac{1}{4} e^{7\pi i} \right) = 2\pi i \left(\frac{\sqrt{2} i}{4} \right) = \frac{\pi}{\sqrt{2}}$$

Exercise 3.8.4:

We want to show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}$$

We set

$$f(z) = \frac{ze^{iz}}{z^2 + a^2} = \frac{ze^{iz}}{(z + ia)(z - ia)}$$

so that

$$\text{Im}\,(f(z)) = \frac{x \sin x}{x^2 + a^2}$$

We see that f has simple poles as $z = \pm ia$. If we integrate about the semicicular contour of radius R then the residue formula says that

$$\int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i\, \text{res}_{ia}\, f(z)$$

We compute the residue at $z = ia$ via

$$\text{res}_{ia}\, f(z) = \lim_{z\to ia}(z - ia) \cdot \frac{ze^{iz}}{(z + ia)(z - ia)} = \frac{e^{-a}}{2}$$

Now we observe compute the integral about the arc

$$\left|\int_{C_R} \frac{ze^{iz}}{z^2 + a^2} dz\right| = \left|\int_0^{\pi} \frac{Re^{i\theta}e^{iRe^{i\theta}}}{R^2e^{2i\theta} + a^2} d\theta\right| \le \int_0^{\pi} \left|\frac{Re^{i\theta}e^{iRe^{i\theta}}}{R^2e^{2i\theta} + a^2}\right| d\theta \le \frac{R\pi}{R^2 + a^2}$$

which goes to zero as $R \to \infty$. Consequently, the integral about all of C is given only by the value on the real line as $R \to \infty$ so

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \text{Im}\left(2\pi i \frac{e^{-a}}{2}\right) = \pi e^{-a}$$

Exercise 3.8.5:

We need to verify that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1 + x^2)^2} dx = \frac{\pi}{2}(1 + 2\pi|\xi|)e^{-2\pi|\xi|}$$

Consider the function

$$f(z) = \frac{e^{-2\pi iz\xi}}{(1 + z^2)^2}$$

This function has poles of order 2 at points where $z^2 = -1$, namely, $\pm i$. We apply Theorem 1.4 in the text to compute the residue

$$\begin{aligned}
\text{res}_i\, f &= \lim_{z\to i} \frac{d}{dz}\left((z - i)^2 \cdot \frac{e^{-2\pi iz\xi}}{(1 + z^2)^2}\right) \\
&= \lim_{z\to i} \frac{d}{dz}\left(\frac{e^{-2\pi iz\xi}}{(z + i)^2}\right) \\
&= \lim_{z\to i} \frac{e^{-2\pi iz\xi}(-2\pi i(z + i) - 2)}{(z + i)^3} \\
&= \frac{e^{2\pi\xi}(4\pi\xi - 2)}{8i}
\end{aligned}$$

Again, we will integrate f about a semicircle of radius R in the upper half plane, so that the only pole is at $z = i$. Applying the residue formula we see that

$$\int_C f(z)dz = 2\pi i\left(\frac{e^{2\pi\xi}(4\pi\xi - 2)}{8i}\right)$$

We then look at the values of f along the arc in C. We can estimate the integral by

$$\left|\int_{C_R} \frac{e^{-2\pi iz\xi}}{(1 + z^2)^2} dz\right| \le \int_{C_R} \left|\frac{e^{-2\pi iz\xi}}{(1 + z^2)^2}\right| dz = \int_0^{\pi} \left|\frac{iRe^{i\theta}e^{-2\pi iRe^{i\theta}\xi}}{(1 + R^2e^{2i\theta})^2}\right| d\theta \le \frac{2\pi R}{R^2 + 1}$$

We then let $R \to \infty$ so that the integral about C_R will vanish. So the integral about all of C reduces to

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2} dx = 2\pi i \left(\frac{e^{2\pi\xi}(4\pi\xi - 2)}{8i} \right) = \frac{\pi}{2}(1+2|\xi|)e^{-2\pi|\xi|}$$

Exercise 3.8.6:

We want to show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi$$

for $n \geq 1$. Indeed, we consider the function $f(z) = 1/(1+z^2)^{n+1}$. This function has poles of order $n+1$ at $z = \pm i$. As before, we will restrict our attention to f in a semicircle contained in the upper half plane. We compute the residue at i by applying Theorem 1.4 to see

$$\begin{aligned}
\operatorname{res}_i f &= \lim_{z \to i} \frac{1}{n!} \frac{d^n}{dz^n} \left((z-i)^{n+1} \cdot \frac{1}{(1+z^2)^{n+1}} \right) \\
&= \lim_{z \to i} \frac{1}{n!} \frac{d^n}{dz^n} \frac{1}{(z+i)^{n+1}} \\
&= \lim_{z \to i} \frac{(-1)^n (2n)!}{(n!)^2} \frac{1}{(z+i)^{2n+1}} \\
&= \frac{(2n)!}{(2^n n!)^2} \cdot \frac{1}{2i} \\
&= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{1}{2i}
\end{aligned}$$

We now estimate the integral of f over the arc C_R by

$$\left| \int_{C_R} \frac{1}{(1+z^2)^{n+1}} dz \right| \leq \int_0^{\pi} \left| \frac{iRe^{i\theta}}{1 + R^{2(n+1)} e^{2(n+1)i\theta}} \right| d\theta \leq \frac{R\pi}{R^{2(n+1)} + 1}$$

Letting $R \to \infty$ forces the integral of f over C_R to 0. As a result the residue formula gives

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}} dx = 2\pi i \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{1}{2i} \right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi$$

Exercise 3.8.10:

We need to see that when $a > 0$ that

$$\int_{-\infty}^{\infty} \frac{\log x}{(x^2+a^2)} dx = \frac{\pi}{2a} \log a$$

In this case we will consider the function

$$f(z) = \frac{\log z}{z^2 + a^2}$$

Where we look at a branch of the logarithm corresponding to $-\pi/2 \leq \theta \leq 3\pi/2$. Then we integrate over the "indented semicircle" contour as seen below.

We begin by esitmating the integral over the arc C_R

$$\left| \int_{C_R} \frac{\log z}{z^2 + a^2} dz \right| \leq \int_0^{\pi} \left| \frac{iRe^{i\theta} \log(Re^{i\theta})}{R^2 e^{2i\theta} + a^2} \right| d\theta \leq \frac{R\pi \log R}{R^2 + a^2}$$

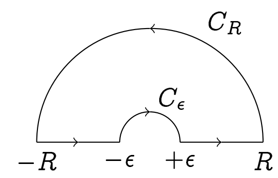

Figure 2: The "indented semicircle".

Letting $R \to \infty$ send the integral to zero.

Along the other curve, C_ϵ, we see that

$$\left| \int_{C_\epsilon} \frac{\log z}{z^2 + a^2} dz \right| \leq \int_0^\pi \left| \frac{i\epsilon e^{i\theta} \log(\epsilon e^{i\theta})}{\epsilon^2 e^{2i\theta} + a^2} \right| d\theta \leq \frac{\epsilon\pi \log \epsilon}{\epsilon^2 + a^2}$$

We then note that

$$\lim_{\epsilon \to 0} \frac{\epsilon\pi \log \epsilon}{\epsilon^2 + a^2} = 0$$

by L'Hôpital's rule.

Integrating along the real parts of the contour gives

$$\begin{aligned} \int_{-R}^{-\epsilon} \frac{\log x}{x^2 + a^2} dx + \int_{\epsilon}^{R} \frac{\log x}{x^2 + a^2} dx &= \int_{\epsilon}^{R} \frac{\log x + \log -x}{x^2 + a^2} dx \\ &= 2 \int_{\epsilon}^{R} \frac{\log x}{x^2 + a^2} dx + i\pi \int_{\epsilon}^{R} \frac{dx}{x^2 + a^2} \end{aligned}$$

Now we compute the residues of f in the upper half plane to apply the residue formula. The only poles are at $z = \pm ia$ and the only one in the upper half plane is $z = ia$. We compute the residue by evaluating

$$\lim_{z \to ia} (z - ia) \frac{\log z}{(z - ia)(z + ia)} = \lim_{z \to ia} \frac{\log z}{(z + ia)} = \frac{\log ia}{2ia} = \frac{\pi}{4a} + \frac{\log a}{2ia}$$

Recall from calculus that

$$\int_0^\infty \frac{dx}{x^2 + a^2} = \left. \frac{\tan^{-1}(x/a)}{a} \right|_0^\infty = \frac{2\pi}{a}$$

We then send $R \to \infty$ and $\epsilon \to 0$ then use the above in conjunction with the residue formula to see that

$$\int_0^\infty \frac{\log x}{x^2 + a^2} = \frac{1}{2} \left(2\pi i \left(\frac{\pi}{4a} + \frac{\log a}{2ia} \right) - \frac{i\pi^2}{2a} \right) = \frac{\pi \log a}{2a}$$