Exercise 3.4.3:

Suppose that f is a bounded Lipschitz function with Lipschitz constant C. Then $f \in L^\infty$. To reason about the first order derivatives $\frac{\partial}{\partial x_j} f$, we convolve with an approximate identity as in Proposition 3.1.2 and define the sequence $f_n = f * \phi_n$, where ψ_n is the approximate identity. Then we see that $\frac{\partial}{\partial x_j} f_n = \frac{\partial}{\partial x_j} f * \psi_n \to \frac{\partial}{\partial x_j} f$ uniformly with n. Then we compute the partial derivatives via the difference quotient yielding

$$\int \frac{f(x+y+\epsilon e_j) - f(x+y)}{\epsilon} \psi_n(y)dy \le C \int \psi_n(y)dy$$

This goes to C uniformly in n. So the partial derivatives are uniformly bounded by the Lipshitz constant C.

Exercise 3.4.4:

(a) Because F is a distribution on Ω, we can apply Proposition 3.1.2 to get a sequence of C^∞ functions f_n such taht $f_n \to F$ in the sense of distributions. Now we use the fact that $C^\infty_{Com} \subset C^\infty$ is a dense containment to get a sequence of functions $f_{n,m} \to f_n$ as $m \to \infty$, where the convergence takes place in C^∞. Because this convergence is in the sense of functions, we can conclude that $f_{n,m} \to f_n$ in the sense of distributions as well. Finally, we note that in the sense of distributions

$$|f_{n.n}(\varphi) - F(\varphi)| = |(f_{n.n}(\varphi) - f_n(\varphi)) + (f_n(\varphi - F(\varphi))| \le |(f_{n.n}(\varphi) - f_n(\varphi))| + |(f_n(\varphi - F(\varphi))|$$

Letting $n \to \infty$ gives the result.

(b) Now let C be the support of F. We need to restrict the approximating functions f_n to an ϵ-neighborhood of C. For $0 \le \delta \le \epsilon$ let C_δ be all the points within δ of C/ Fix a $\delta_0 < \epsilon/2$ and choose a function φ supported in the unit ball such that $\int \varphi = 1$. Then we consider the family of scaling functions

$$\varphi_R(x) = R^{-d}\varphi(x/R)$$

which are supported in the balls of radius R about the origin. We then consider the convolutions

$$\varphi_{\delta_0} * \chi_{C_{\delta_0}} = \int_{\mathbb{R}^d} \varphi_{\delta_0}(y)\chi_{C_{\delta_0}}(x-y)dy = \int_{|y|\le\delta_0} \varphi_{\delta_0}(y)\chi_{C_{\delta_0}}(x-y)dy$$

because φ_{δ_0} is supported on the ball of radius δ_0. Now if $x \in C$ and $|y| \le \delta_0$ then $\chi_{C_{\delta_0}}(x-y) = 1$, but if $x \notin C_\epsilon$ and $|y| \le \delta_0$ then $x - y \notin C_\epsilon$. Thus, we see that

$$(\varphi_{\delta_0} * \chi_{C_{\delta_0}})(x) = \begin{cases} 1 & x \in C \\ 0 & \text{outside } C_\epsilon \end{cases}$$

Now we simply note that $\varphi_{\delta_0} * \chi_{C_{\delta_0}} \in \mathbb{C}^\infty_{Com}$ by construction and we then simply take a sequence $f_n \to F$ with the $f_n \in C^\infty$ and define the new sequence $(\varphi_{\delta_0} * \chi_{C_{\delta_0}}) \cdot f_n$ which has the desired properties.

Exercise 3.4.7: In the forward direction, we suppose that F is a tempered distribution. The by Proposition 3.1.4 we can find an N such that $|F(\varphi)| \le C\|\varphi\|_N$ for all $\varphi \in \mathcal{S}$. Now suppose that $\varphi \in \mathcal{D}$ and has supposrt in the ball $|x| \le R$, then we have the bound $|x|^\beta \le c_{n,\beta}|x|^{|\beta|} \le c_{n,N}R^n$. This implies that

$$|F(\varphi)| = \sup_{x\in\mathbb{R}^d} |x^\beta \partial_x^\alpha \varphi| \le Cc_{n,N}R^N \sup_{x\in\mathbb{R}^d} |\partial_x^\alpha \varphi|$$

with $A = Cc_{n,N}$, as desired.

Conversely, suppose that F is a functional that satisfies

$$|F(\varphi)| \leq AR^N \sup_{x\in\mathbb{R}^d} |\partial_x^\alpha \varphi|$$

for all φ supported in $|x| \leq R$. Then we see that $F \in \mathcal{D}^*$. Choose a partition of unity on $\mathbb{R}^d$, say ψ_j with each of the ψ_j supported in a ball about the origin. TO construct such a partition we will use a function $\eta(x) + \eta(x-1) = 1$ supported in $(-1, 1)$. Then η induces a partition of unity given by $\psi_j(x) = \eta(|x| - j)$ supported in the disk of radius j. For a given $\varphi_j \in \mathcal{D}$ so that

$$\sum_{j=0}^{\infty} \psi_j \varphi = \varphi$$

where the convergence is in $\mathcal{D}$. Thus, we can set

$$|F(\varphi)| \leq |F(\psi_0\varphi)| + \sum_{j=1}^{\infty} |F(\psi_j\varphi)|$$

Becuase ψ_0 is supported in the unit ball, we take $R = 1$ to see that

$$|F(\psi_0\varphi)| \leq C \sum_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} |\partial_x^\alpha(\psi_0\varphi)| \leq C' \sum_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} |\partial_x^\alpha(\varphi)|$$

where C' is a constant depending only on the ψ_j and N. Now we will go through a similar process for each of the $|F(\psi_j\varphi)|$ yielding

$$|F(\psi_j\varphi)| \leq C(j+1)^N \sum_{|\alpha|\leq N} \sup_{x\in B_j} |\partial_x^\alpha \varphi|$$

As before we can then bound this with a new constant C_j' such that

$$\begin{aligned} C(j+1)^N \sum_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} |\partial_x^\alpha \varphi| &\leq C_j'(j+1)^N \sum_{|\alpha|\leq N} \sup_{x\in B_j} |\partial_x^\alpha \varphi| \\ &= C_j'(j+1)^{-2} \left(\frac{j+1}{j}\right)^{N+2} \sum_{|\alpha|\leq N} j^{N+2} |\partial_x^\alpha \varphi| \end{aligned}$$

Now we use the fact that $j \leq 1 + |x|$ for x in the ball of radius j and get a bound

$$\begin{aligned} C_j'(j+1)^{-2} \left(\frac{j+1}{j}\right)^{N+2} \sum_{|\alpha|\leq N} j^{N+2} |\partial_x^\alpha \varphi| &\leq C_j'(j+1)^{-2} 2^{N+2} \sum_{|\alpha|\leq N} (1+|x|)^{N+2} |\partial_x^\alpha \varphi| \\ &\leq C_j'(j+1)^{-2} 2^{N+2} \sum_{|\alpha|\leq N} \|\varphi\|_{N+2} \end{aligned}$$

So we have bounded $F(\varphi)$ in terms of finite sums of $\|\varphi\|_N$ for various N. Hence, F extends to be bounded on $\mathcal{S}$ by density and so we have that $F \in \mathcal{S}^*$.

Exercise 3.4.8:

Suppose that F is a homogeneous distribution of degree λ. Then we know that $|F_a(\varphi)| = a^\lambda F$. Consider a scaling operator $\eta \in \mathcal{D}$ with $\eta(x)$ in $|x| \leq 1$, supported in $|x| \leq 2$ and $\eta_R(x) = \eta(x/R)$. We know because F is a distribution and η has compact support

$$|\eta_1 F(\varphi)| = |f(\eta_1\varphi)| \leq C \sum_{|alpha|\leq N} \sup_{x\leq 1} |\partial_x^\alpha \varphi| \leq C\|\varphi\|_N$$

We can see this because we can approximate F by a sequence of distributions with compact sense $F_n \to F$ in the weak sense as $n \to \infty$ and by the discussion after Proposition 3.1.4, the limit holds for each of the F_n and thus for F as well. Now suppose that we have a distribution $\varphi \in \mathcal{D}$ that is supported in the ball of radius R about the origin with $R \geq 1$. Then we see that $\eta_R\varphi$ is supported in the unit ball and the above estimate gives

$$\begin{aligned}|\eta_R F(\varphi)| &\leq C \sum_{|\alpha|\leq N} \sup_{|x|\leq 1} |\partial_x^\alpha(\eta_R\varphi)| \\ &= C \sum_{|\alpha|\leq N} \sup_{|x|\leq 1} R^{-|\alpha|}|\partial_x^\alpha(\varphi)|\end{aligned}$$

Now because F is homogenous we also know that $F(\eta_R\varphi) = R^{-n-\lambda}|F(\varphi)|$. Combining these facts we see that

$$|F(\eta_R\varphi)| \leq CR^{n+\lambda} \sum_{|\alpha|\leq N} \sup_{|x|\leq R} |\partial_x^\alpha\varphi| \leq CR^{n+\lambda}\|\varphi\|_N$$

We can then apply the previous exercise to see that F must be tempered.

Exercise 3.4.10:

To see that $\mathcal{D}$ is dense in $\mathcal{S}$ we let $\varphi \in S$ fix an $\eta \in \mathcal{D}$ that has $\eta = 1$ on the unit ball. We then set $\eta_k(x) = \eta(x/k)$ and $\varphi_k = \eta_k\varphi$. Then consider $\|\varphi_k - \varphi\|_N$,

$$\begin{aligned}\|\varphi_k - \varphi\|_N &= \sup_{\substack{x\in\mathbb{R}^d \\ |\alpha|,|\beta|\leq N}} |x^\beta\partial_x^\alpha(\eta_k\varphi - \varphi)| \\ &= \sup_{\substack{x\in\mathbb{R}^d \\ |\alpha|,|\beta|\leq N}} |x^\beta\partial_x^\alpha(\eta_k\varphi) - x^\beta\partial_x^\alpha(\varphi)|\end{aligned}$$

Then we compute

$$\begin{aligned}\partial_x^\alpha(\eta_k\varphi) &= \sum_{|\gamma|\leq|\alpha|} \binom{|\alpha|}{|\gamma|} k^{|\gamma-\alpha|}\partial_x^\gamma\varphi\partial_x^{\gamma-\alpha}\eta_k \\ &= \partial_x^\alpha\varphi + \sum_{\substack{|\gamma|\leq|\alpha| \\ \gamma\neq\alpha}} \binom{\alpha}{\gamma} k^{|\gamma-\alpha|}\partial_x^\gamma\varphi\partial_x^{\gamma-\alpha}\eta_k\end{aligned}$$

So we can estimate the above by

$$\begin{aligned}\sup_{\substack{x\in\mathbb{R}^d \\ |\alpha|,|\beta|\leq N}} |x^\beta\partial_x^\alpha(\eta_k\varphi) - x^\beta\partial_x^\alpha(\varphi)| &\leq \left| x^\beta \sum_{\substack{|\gamma|\leq|\alpha| \\ |\gamma-\alpha|\geq 1}} \binom{|\alpha|}{|\gamma|} k^{|\gamma-\alpha|}\partial_x^\gamma\varphi\partial_x^{\gamma-\alpha}\eta_k \right| \\ &\leq \frac{\|\varphi\|_N}{k} \sum_{\substack{|\gamma|\leq|\alpha| \\ |\gamma-\alpha|\geq 1}} \|\varphi\|_{|\alpha|}\end{aligned}$$

This clearly goes to zero as $k \to \infty$ for all N and so we have that $\varphi_k \to \varphi$ as desired.

Exercise 3.4.11:

1. Suppose that both $\varphi_1, \varphi \in \mathcal{S}$. Then we need to show that

$$\|\varphi_1\varphi_2\|_N = \sup_{\substack{x\in\mathbb{R}^d \\ |\alpha|,|\beta|\leq N}} |x^\beta\partial_x^\alpha(\varphi_1\varphi_2)| < \infty$$

We then compute the product via

$$\partial_x^\alpha(\varphi_1\varphi_2) = \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}(\partial_x^{\alpha-\gamma}\varphi_1)(\partial_x^\gamma\varphi_2)$$

Substituting this formula above gives

$$\begin{aligned}
\sup_{\substack{x\in\mathbb{R}^d\\|\alpha|,|\beta|\leq N}}|x^\beta\partial_x^\alpha(\varphi_1\varphi_2)| &= \sup_{\substack{x\in\mathbb{R}^d\\|\alpha|,|\beta|\leq N}}\left|x^\beta\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}(\partial_x^{\alpha-\gamma}\varphi_1)(\partial_x^\gamma\varphi_2)\right| \\
&\leq \sup_{\substack{x\in\mathbb{R}^d\\|\alpha|,|\beta|\leq N}}\left|\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\left(x^{N-|\gamma|}\partial_x^{\alpha-\gamma}\varphi_1\right)\left(x^{|\gamma|}\partial_x^\gamma\varphi_2\right)\right| \\
&\leq \sup_{\substack{x\in\mathbb{R}^d\\|\alpha|,|\beta|\leq N}}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}||\varphi_1||_{N-|\gamma|}||\varphi_2||_{|\gamma|}
\end{aligned}$$

This is a finite sum and hence, is bounded so that $||\varphi_1\varphi_2||_N \leq \infty$ and so $\varphi_1\varphi_2 \in \mathcal{S}$ as claimed.

2. To see that $\varphi_1 * \varphi_2 \in \mathcal{S}$ whenever $\varphi_1, \varphi_2 \in \mathcal{S}$ we will invoke the Fourier transform. We saw that the Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ was a homeomorphism and that

$$\mathcal{F}(\varphi_1 * \varphi_2) = \mathcal{F}(\varphi_1)\mathcal{F}(\varphi_2)$$

Because φ_1 and φ_2 are both in $\mathcal{S}$ we conclude that $\mathcal{F}(\varphi_1 * \varphi_2) \in \mathcal{S}$ as well. Then applying Fourier inversion in $\mathcal{S}$ we get that $\varphi_1 * \varphi_2 \in \mathcal{S}$ as desired.

3. To verify that $\varphi_1 * \varphi_2 \in \mathcal{S}$ directly from the definition we first recall that differentiation commutes with the convolution to compute

$$\begin{aligned}
||\varphi_1 * \varphi_2||_N &= \sup_{\substack{x\in\mathbb{R}^d\\|\alpha|,|\beta|\leq N}}|x^\beta\partial_x^\alpha(\varphi_1 * \varphi_2)| \\
&= \sup_{\substack{x\in\mathbb{R}^d\\|\alpha|,|\beta|\leq N}}\left|x^\beta\varphi_1 * \partial_{x-y}^\alpha\varphi_2\right| \\
&\leq \sup_{\substack{x\in\mathbb{R}^d\\|\alpha|,|\beta|\leq N}}\int_{\mathbb{R}^d}|\varphi_1(y)|\cdot|x^\beta\partial_{x-y}^\alpha\varphi(x-y)|dy \\
&\leq \sup_{\substack{x\in\mathbb{R}^d\\|\alpha|,|\beta|\leq N}}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\int_{\mathbb{R}^d}|y^\ell\varphi_1(y)|\cdot|(x-y)^{\beta-\ell}\partial_{x-y}^\alpha\varphi(x-y)|dy \\
&\leq \sup_{\substack{x\in\mathbb{R}^d\\|\alpha|,|\beta|\leq N}}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}||\varphi_2||_N\int_{\mathbb{R}^d}|y^\ell\varphi_1(y)|\frac{1+y^2}{1+y^2}dy \\
&\leq \sup_{\substack{x\in\mathbb{R}^d\\|\alpha|,|\beta|\leq N}} C\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}||\varphi_2||_N(||\varphi_1||_\gamma + ||\varphi||_{|\gamma|+2})
\end{aligned}$$

The last term is bounded because it is a finite sum of bounded terms. Hence $\varphi_1 * \varphi_2 \in \mathcal{S}$

Exercise 3.4.12: Let F be a distribution with compact support and let $\varphi \in \mathcal{S}$. We will first show that for each N we can find a constant C_N such that

$$||\varphi_y^\sim||_N \leq C_N(1+|y|)^N||\varphi||_N$$

Indeed, we first note that

$$\|\varphi_y^{\sim}\|_N = \sup_{\substack{y\in\mathbb{R}^d \\ |\alpha|,|\beta|\leq N}} |y^\beta \partial_y^\alpha \varphi(x-y) \leq \sup_{\substack{y\in\mathbb{R}^d \\ |\alpha|,|\beta|\leq N}} |(1+|y|)^\beta \partial_y^\alpha \varphi(y-y)$$

Then we note that $(1+|y|) \leq (1+|y|)(1+|x-y|)$ to conclude that

$$\begin{aligned}
\sup_{\substack{y\in\mathbb{R}^d \\ |\alpha|,|\beta|\leq N}} |(1+|y|)^\beta \partial_y^\alpha \varphi(x-y)| &\leq \sup_{\substack{y\in\mathbb{R}^d \\ |\alpha|,|\beta|\leq N}} |(1+|y|)^\beta (1+|x-y|)^\beta \partial_y^\alpha \varphi(x-y)| \\
&\leq C_N(1+|y|)^N \sup_{\substack{y\in\mathbb{R}^d \\ |\alpha|,|\beta|\leq N}} ||x-y|^\beta \partial_y^\alpha (x-y)| \\
&\leq C_N(1+|y|)^N \|\varphi\|_N
\end{aligned}$$

Now we note that because F has compact support it must also be tempered and therefore we can apply Proposition 3.1.4 to see that the function $F * \varphi = F(\varphi_y^{\sim})$ must satisfy

$$|F(\varphi_x^{\sim})| \leq c\|\varphi_y^{\sim}\|_N \leq C'(1+|x|)^N \|\varphi\|_N$$

Thus, we see that

$$\|F(\varphi_x^{\sim}\|_N = \sup_{\substack{x\in\mathbb{R}^d \\ |\alpha|,|\beta|\leq N}} |x^\beta \partial_x^\alpha$$