Exercise 5.6.4:

(a) We are interested in

$$F(z) = \prod_{n=1}^{\infty}(1 - e^{-2\pi nt}e^{2\pi iz})$$

Following the hint, we consider the functions

$$F_1(z) = \prod_{n=1}^{N}(1 - e^{-2\pi nt}e^{2\pi iz})$$

and

$$F_2(z) = \prod_{n=N+1}^{\infty}(1 - e^{-2\pi nt}e^{2\pi iz})$$

Where N is chosen such that $N \approx c|z|$ for a choice of c we will determine later. We then look at the behavior of F_2. We note that

$$\prod_{n=N+1}^{\infty}(1 - e^{-2\pi nt}e^{2\pi iz}) = \prod_{n=N+1}^{\infty} e^{\log(1-e^{-2\pi nt}e^{2\pi iz})} = e^{S_{N+1}}$$

where

$$S_{N+1} = \sum_{n=N+1}^{\infty} \log(1 - e^{-2\pi nt}e^{2\pi iz})$$

Now we will choose c such that S_{N+1} will be bounded. Indeed, consider $c = (1/t + \epsilon)$ for any $\epsilon > 0$. We then observe that because of our choice of c and for sufficiently large z

$$e^{-2\pi(N+1)t}e^{2\pi|z|} = e^{-2\pi((N+1)t-|z|)} \leq e^{-2\pi t(\epsilon|z|+1)} \leq 1/2$$

We then note that $|\log(1+x)| \leq 2|x|$ when $|x| \leq 1/2$ so

$$|S_{N+1}| = \left|\sum_{N+1}^{\infty} \log(1 - e^{-2\pi nt}e^{2\pi iz})\right| \leq \sum_{N+1}^{\infty} |\log(1 - e^{-2\pi nt}e^{2\pi iz})|$$

Applying the inequality from our choice of c gives us

$$\sum_{N+1}^{\infty} |\log(1 - e^{-2\pi nt}e^{2\pi iz})| \leq \sum_{N+1}^{\infty} 2|e^{-2\pi nt}e^{2\pi iz}| \leq 2\sum_{N+1}^{\infty} e^{-2\pi nt}e^{2\pi|z|}$$

Changing indices yields

$$2\sum_{N+1}^{\infty} e^{-2\pi nt}e^{2\pi|z|} = 2e^{-2\pi((N+1)t-|z|)}\sum_{m=0}^{\infty} e^{-2\pi mt}$$

We note that the factor outside the sum is less than 1 and that the series converges by the geometric series test. These two facts imply that $|F_2(z)| \leq A$ for some constant A.

We now turn our attention for $F_1(z)$. Again, we know that for any n

$$|1 - e^{-2\pi nt}e^{2\pi iz}| \leq 1 + e^{2\pi|z|} \leq 2e^{2\pi|z|}$$

As a result

$$|F_1(z)| = \left|\prod_{n=1}^{N}(1 - e^{-2\pi nt}e^{2\pi iz})\right| \leq \prod_{n=1}^{N}|1 - e^{-2\pi nt}e^{2\pi iz}| \leq \prod_{n=1}^{N} 2e^{2\pi|z|} = 2^N e^{2\pi N|z|}$$

Then because $N = O(|z|)$ we can modify the constants so that

$$|F_1(z)| \leq e^{a|z|^2}$$

We then combine the result from F_1 and F_2 to see that

$$F(z) \leq Ae^{a|z|^2}$$

And so F is of order 2.

(b) We apply Proposition 5.3.1 to verify that F is zero exactly when $z = -int + m$ for $n \geq 1$, $n, m \in \mathbb{Z}$. Indeed, by the proposition F is zero precisely when one of its factors is zero. So we need solutions of the equation

$$e^{-2\pi nt}e^{2\pi iz} = 1$$

This happens when $nt - iz = 0$ so $z = -int + m$ for some integer m.

Let z_n be an enumeration of the zeros. To show that $\sum 1/|z_n|^2 = \infty$ we note that by construction

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^2} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{|-int+m|^2} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{m^2+(nt)^2}$$

Then recall the identity

$$\pi \cot \pi a = \sum_{n=-\infty}^{\infty} \frac{a}{a^2+n^2}$$

Substituting this in our sum gives

$$\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{m^2+(nt)^2} = \frac{1}{nt}\sum_{m=-\infty}^{\infty} \frac{nt}{(nt)^2+m^2} = \sum_{n=1}^{\infty} \frac{\pi \coth \pi nt}{nt}$$

The last sum diverges by the comparison test. Hence, $\sum 1/|z_n|^2 = \infty$.

To see that $\sum 1/|z_n|^{2+\epsilon}$ converges. We note that by part **(a)** F has order 2 so for any $\epsilon > 0$ we satisfy the hypotheses for Theorem 5.2.1 which says precisely that

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{2+\epsilon}} < \infty$$

Exercise 5.6.5:

We are looking at

$$F_\alpha(z) = \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi izt}dt$$

for $\alpha > 1$. We will take the hint and begin by showing that

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leq c|z|^{\alpha/(\alpha-1)}$$

First we consider the case that $|z| \leq (1/4\pi)|t|^{\alpha-1}$. This implies that $2\pi|z||t| \leq |t|^\alpha/2$ and so we have that

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leq 0 \leq c|z|^{\alpha/(\alpha-1)}$$

Alternatively if $|z| > (1/4\pi)|t|^{\alpha-1}$ then we can set

$$c = \frac{2\pi}{(4\pi)^{(\alpha-1)^{-1}}}$$

In which case we have that $2\pi|z||t| \le c|z|^{\alpha/(\alpha-1)}$. This gives that

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \le -\frac{|t|^\alpha}{2} + c|z|^{\alpha/(\alpha-1)} \le c|z|^{\alpha/(\alpha-1)}$$

as desired.

We then apply the above inequality to see

$$|F_\alpha(z)| \le \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi|z||t|}dt = \int_{-\infty}^{\infty} e^{-|t|^\alpha/2} e^{2\pi|z||t|-|t|^\alpha/2}dt \le e^{c|z|^{\alpha/(\alpha-1)}} \int_{-\infty}^{\infty} e^{-|t|^\alpha/2}dt$$

We then note that

$$\int_{-\infty}^{\infty} e^{-|t|^\alpha/2}dt = \sqrt{2\pi}$$

So that

$$|F_\alpha(z)| \le \sqrt{2\pi}e^{c|z|^{\alpha/(\alpha-1)}}$$

So that F is of growth order $\alpha/(\alpha-1)$ as desired.

Exercise 5.6.6:

We will evaluate the product formula for $\sin z$

$$\frac{\sin \pi z}{\pi} = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

Changing variables to $w = \pi z$ we get

$$\frac{\sin w}{w} = \prod_{n=1}^{\infty}\left(1 - \frac{w^2}{\pi^2 n^2}\right)$$

Following the hint we evaluate at $w = \pi/2$ to get

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right)$$

Inverting both sides yields

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\left(\frac{4n^2}{4n^2-1}\right) = \prod_{n=1}^{\infty}\frac{2n\cdot 2n}{(2n-1)(2n+1)}$$

as desired.

Exercise 5.6.8:

First let

$$F(z) = \prod_{k=1}^{\infty}\cos(z/2^k)$$

We will first address the issue of convergence. Consider the function

$$G(z) = \prod_{n=1}^{\infty}(1 - \cos(z/2^n))$$

We will show that G converges, and then apply Proposition 5.3.1 to get that F must converge as well. If we look the formula

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

Then when $|z| \approx 0$ we have that $|1 - \cos z| \le |z|^2$. As a result, for sufficiently large n we have that

$$|1 - \cos(z/2^n)| \le (z/2^n)^2 = z^2/4^k$$

Thus, G converges because $\sum_{n>N} |z^2|/4^n$ converges (geometric series test) and by Proposition 5.3.1 F must converge as well.

Now we follow the hint and repeatedly apply the identity

$$\sin 2z = 2 \sin z \cos z$$

This will yield the formula

$$\begin{aligned}
\sin z &= 2 \sin(z/2) \cos(z/2) \\
&= 4 \sin(z/4) \cos(z/2) \cos(z/4) \\
&= 8 \sin(z/8) \cos(z/2) \cos(z/4) \cos(z/8) \\
&\vdots \\
&= 2^n \sin(z/2^n) \prod_{k=1}^{n} \cos(z/2^k)
\end{aligned}$$

Because $\sin|z| \approx |z|$ when $|z|$ is small, we see that

$$\sin z = \lim_{n\to\infty} 2^n \sin(z/2^n) \prod_{k=1}^{n} \cos(z/2^k) = zF(z)$$

Which precisely means that

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty} \cos(z/2^k)$$

Exercise 5.6.9:

We first define

$$F(z) = \prod_{k=0}^{\infty} (1 + z^{2^k})$$

Adressing the issue of convergence, we note that $\sum_{k=0}^{\infty} z^{2^k}$ converges in the disk $|z| < 1$ by comparison with the geometric series. We apply Proposition 5.3.1 to conclude that F converges as well.

We now need to show that $F(z) = 1/(1-z)$. We recall that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^k$$

for $|z| < 1$. We will proceed to show that the product formula for $F(z)$ is equivalent to this summation formula. More specifically we will show that

$$\prod_{k=0}^{n} (1 + z^{2^k}) = \sum_{\ell=0}^{2^{n+1}-1} z^{\ell}$$

We proceed by induction. The base case $n = 0$ is clear. For the inductive step observe that

$$\prod_{k=0}^{n+1}(1+z^{2^k}) = (1+z^{2^{n+1}})\prod_{k=0}^{n}(1+z^{2^k}) = (1+z^{2^{n+1}})\left(\sum_{\ell=0}^{2^{n+1}-1} z^\ell\right)$$

The last equality follows from the induction hypothesis. We then distribute to see

$$(1+z^{2^{n+1}})\left(\sum_{\ell=0}^{2^{n+1}-1} z^\ell\right) = \left(\sum_{\ell=0}^{2^{n+1}-1} z^\ell\right) + z^{2^{n+1}}\left(\sum_{\ell=0}^{2^{n+1}-1} z^\ell\right) = \left(\sum_{\ell=0}^{2^{n+1}-1} z^\ell\right) + \left(\sum_{\ell=2^{n+1}}^{2^{n+2}-1} z^\ell\right)$$

We then combine the two sums two conclude that

$$\prod_{k=0}^{n+1}(1+z^{2^k}) = \sum_{\ell=0}^{2^{n+2}-1} z^\ell$$

and so equality holds for all n by induction.

We then observe that

$$F(z) = \lim_{n\to\infty}\prod_{k=0}^{n}(1+z^{2^k}) = \lim_{n\to\infty}\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$$

Exercise 5.6.15:

Let f be a meromorphic function with poles at $\{z_n\}$, counted with multiplicity. Then Theorem 5.4.1 guarantees the existence of an entire function g that vanishes precisely at the $\{z_n\}$ and nowhere else. Then the product $h = fg$ is an entire function and we can write $f = h/g$, so f is the quotient of two entire functions.

Now let $\{a_n\}$ and $\{b_n\}$ be sequences that have no finite limit point in the plane. Applying Theorem 5.4.1 to both of these sequences yields two functions f and g with zeros precisely at the $\{a_n\}$ and $\{b_n\}$, respectively. Then the quotient f/g has zeroes at the a_n and poles at the b_n.

Exercise 5.6.17:

(a) This part is the complex version Lagrange Interpolation, which I have seen before. Indeed, we have our two sets of complex numbers $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ with the a_n all distinct. We will look to have some polynomial of the form

$$P(z) = \sum_{k=1}^{n} b_k p_k(z)$$

where $p_k(z)$ is a polynomial with the property that

$$p_k(z) = \begin{cases} 1 & z = a_k \\ 0 & z = a_\ell, \ell \neq j \end{cases}$$

This will give P the desired property that $P(a_k) = b_k$. So the construction of P has been reduced to the construction of the p_k.

We will then define a product formula for $p_k(z)$ via

$$p_k(z) = \prod_{\substack{m\leq n \\ m\neq k}} \frac{z-a_m}{a_k-a_m}$$

To see that p_k is the desired function we note that for $j \neq k$

$$p_k(a_j) = \prod_{\substack{m \leq n \\ m \neq k}} \frac{a_j - a_m}{a_k - a_m} = \left(\frac{a_j - a_j}{a_k - a_j}\right) \prod_{\substack{m \leq n \\ m \neq j,k}} \frac{z - a_m}{a_k - a_m} = 0$$

Moreover,

$$p_k(a_k) = \prod_{\substack{m \leq n \\ m \neq k}} \frac{a_k - a_m}{a_k - a_m} = 1$$

as desired.

(b)