Exercise 3.8.13:

Consider the function $g(z) = (z - z_0)f(z)$. By the assumptions on f we see that

$$\lim_{z\to z_0} |g(z)| \leq \lim_{z\to z_0} A|z - z_0| \cdot |z - z_0|^{-1+\epsilon} = \lim_{z\to z_0} A|z - z_0|^{\epsilon} = 0$$

Hence, g is bounded in $D_r(z_0)$ and so the singularity of g is removable. We gan then extend g to an analytic function on all of $D_r(z_0)$. Hence, because $g(z_0) = 0$ we can write $g(z) = (z - z_0)h(z)$ in a neighborhood of z_0. By definition, we must have that $f(z) = h(z)$ in a deleted neighborhood of z_0. By setting $f(z_0) = h(z_0)$ we extend f to a holomorphic function on all of D_r. Thus, the singularity at z_0 is removable.

Exercise 3.8.14:

Let f be an entire, injective function and consider $g(z) = f(1/z)$. It is clear that g is holomorphic in the punctured plane because f is entire. We will study the type of singularity that g can have at the origin. We see that f cannot have a removable singularity at the origin because that would imply that f is bounded and hence, constant. Thus, f could not be injective.

Moreover, f cannot have an essential singularity at the origin. Consider a punctured neighborhood of the origin $N_R(0)$ and a $z \in N_R(0)$. We then look at a neighborhood of z with radius less than $|z|$, say $N_r(z)$. If we look at the image $g(N_r(z))$, the Open Mapping theorem guarantees that this contains a neighborhood of the point $g(z)$. But the Casorati-Weierstrass theorem says that the image of $N_R(0)$ is dense in $\mathbb{C}$ and therefore has non-empty intersection with the neighborhood of $g(z)$. So f could not have been injective.

Consequently, the singularity of g at the origin must be a pole. We can then write

$$g(z) = \frac{a_m}{z^m} + \frac{a_{m-1}}{z^{m-1}} + \cdots + \frac{a_1}{z} + h(z)$$

Where h is holomorphic at the origin. We then see this implies that

$$f(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + h(1/z)$$

This function cannot be holomorphic unless h is bounded at infinity, and hence constant. So f is a polynomial.

To see that f must be linear, we note that f must be of the form

$$f(z) = a(z - z_0)^m$$

because if f had multiple distinct roots, it would not be injective. To see that $m = 1$ we observe that if not, then we can consider the values of f on a unit disk centered at the root, z_0. We see that $f(z_0 + 1) = f(z_0 + e^{2\pi i/m})$, so that f would not be injective. Hence, $m = 1$ and we see that $f(z) = a(z - z_0)$. Finally, we observe that $a \neq 0$ because otherwise f would be constant. This completes the proof.

Exercise 3.8.15:

(d) Suppose that Re (f) is bounded and so $|$ Re $(f)| < M|$ for some $M \geq 0$. Consider the function $g(z) = e^{f(z)}$. g is entire and $|g| = e^{\mathrm{Re}(f)}$ is bounded by e^M. Because g is bounded and entire, it must be a constant, say k. So that $f = \log k$ everywhere. Although the logarithm is multi-valued we note that the values differ by a discrete amount $2\pi i$, but f is continuous, and so it must be constant.

Exercise 3.8.17:

(a) Suppose that f is holomorphic in an open set containing the closed unit disk and that $|f(z)| = 1$ whenever $|z| = 1$. We will show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$. We will instead show that $f(z) = 0$ has a root, because by Rouché's theorem $f(z)$ and $f(z) - w_0$ have the same number of zeros in the interior of $\mathbb{D}$ (because $w_0 < 1$).

If f is non-zero then $g(z) = 1/f(z)$ is holomorphic in the disk and so $|g(z)| \leq 1$ by the maximum modulus principle. This implies that $|f(z)| \geq 1$ for $z \in \mathbb{D}$. Consider the image of f on the boundary of $\mathbb{D}$, because $|f(z)| = 1$ for $z \in \partial\mathbb{D}$ the Open Mapping theorem implies that the image $f(N_r(z))$ contains an open ball about $f(z)$. Because $|f(z)| = 1$, any neighborhood contains points in the interior of $\mathbb{D}$, so $|w| < 1$. But this contradicts that $|f(z)| \geq 1$ for all $z \in \mathbb{D}$. So $f(z) = 0$ must have a root in $\mathbb{D}$.

We then complete the argument by noting that $f(z) = w_0$ must have a root for each w_0 by Rouché's theorem and so the image of f contains the unit disk.

(b) Now we know that $|f(z)| \geq 1$ whenever $|z| = 1$ and that there is some point $z_0 \in \mathbb{D}$ such that $|f(z)| < 1$. We apply Rouché's theorem again to see that $f(z)$ and $f(z) - w$ have the same number of roots in the disk for all $w \in \mathbb{D}$. Then note that $f(z) - f(z_0)$ has a root and by the above, this implies that $f(z)$ has a root for all $w \in \mathbb{D}$. So the image of f contains $\mathbb{D}$.

Exercise 3.8.19:

(a) Suppose that u attained a maximum at z_0. Consider the complex function $f(z) = u + iv$, where v is the conjugate harmonic to u. Then f is holomorphic at z_0 and is therefore an open mapping in a neighborhood of z_0. Thus, the image of any neighborhood of z_0 under f contains a neighborhood of $f(z_0)$. But this implies that there are points $f(w)$ in this neighborhood such that $\text{Re}\,(f(w)) > \text{Re}\,(f(z_0))$, which means that $u(w) > u(z_0)$ and so z_0 was not a local maximum.

(b) This is a direct consequence of part **(a)**. Because u is harmonic on Ω it is continuous there and also continuous on its closure, which is assumed compact, thus it must attain its maximum in $\overline{\Omega}$. However, by **(a)** it cannot achieve its maximum on the interior, Ω, and therefore must achieve the maxima on the boundary. This is equivalent to saying that

$$\sup_{z\in\Omega} |u(z)| \leq \sup_{z\in\overline{\Omega}-\Omega} |u(z)|$$

Exercise 3.8.22:

Suppose that f, holomorphic in $\mathbb{D}$, could extend continuously to a function such that $f(z) = 1/z$ for $z \in \partial\mathbb{D}$. Because $\mathbb{D}$ is simply connected we see that $\int_\gamma f(z)dz = 0$ for any closed γ in $\mathbb{D}$. In particular, we should have that $\int_{\partial\mathbb{D}} f(z)dz = 0$ but we know that

$$\int_{\partial\mathbb{D}} f(z)dz = \int_{\partial\mathbb{D}} \frac{1}{z}dz = 2\pi i$$

This contradiction shows that no continuous extension of f could have existed.

Exercise 4.4.3:

We want to verify

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi}dx = e^{-2\pi a|\xi|}$$

We will show this using contour integration. Consider the function

$$f(z) = \frac{a}{a^2 + z^2} e^{-2\pi i z \xi}$$

This function has a simple pole at $z = \pm ia$. We consider the case $\xi > 0$ and integrate over the semicircle of radius R in the upper half-plane, denoted by C_R. Let γ_R be the arced part of the semicircle, and I_R be the interval $[-R, R]$. Then the residue theorem guarantees that

$$\int_{\gamma_R} \frac{a}{a^2 + z^2} e^{-2\pi i z \xi} dz + \int_{I_R} \frac{a}{a^2 + x^2} e^{-2\pi i x \xi} dx = 2\pi i \operatorname{res}(f, ia)$$

We will first show that the integral over $\gamma_R \to 0$ as $R \to \infty$. We note that

$$\begin{aligned} \left| \int_{\gamma_R} \frac{a}{a^2 + z^2} e^{-2\pi i z \xi} dz \right| &= \int_0^\pi \left| \frac{a}{a^2 + R^2 e^{2i\theta}} e^{-2\pi i R e^{i\theta} \xi} i R e^{i\theta} \right| d\theta \\ &\leq \int_0^\pi \left| \frac{aR}{a^2 + R^2} \right| d\theta \\ &\leq \frac{aR\pi}{a^2 + R^2} \end{aligned}$$

Which goes to zero as $R \to \infty$.

Now we compute the residue at $z = ia$

$$\lim_{z \to ia} (z - ia) \cdot \frac{a}{(z + ia)(z - ia)} e^{-2\pi i z \xi} = \frac{1}{2i} e^{2\pi a \xi}$$

This formula gives that

$$\int_{-\infty}^{\infty} \frac{a}{a^2 + x^2} e^{-2\pi i x \xi} dx = 2\pi i \cdot \frac{1}{2i} e^{2\pi a \xi} = \pi e^{2\pi a \xi}$$

as desired. The case for $\xi \leq 0$ is the same, reflecting the contour across the real axis.

To see that

$$\int_{-\infty}^{\infty} e^{-2\pi a |\xi|} e^{2\pi i \xi x} d\xi = \frac{1}{\pi} \frac{a}{a^2 + x^2}$$

We apply the previous part and break up the integral to see

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-2\pi a |\xi|} e^{2\pi i \xi x} d\xi &= \int_{-\infty}^{0} e^{-2\pi a(-\xi) + 2\pi i (-\xi) x} d\xi + \int_0^\infty e^{-2\pi a \xi + 2\pi i \xi x} d\xi \\ &= \int_{-\infty}^{0} e^{2\pi \xi (a + ix)} d\xi + \int_0^\infty e^{2\pi i \xi (-a + ix)} d\xi \\ &= \frac{1}{2\pi} \frac{1}{a + ix} + \frac{1}{2\pi} \frac{1}{a - ix} \\ &= \frac{1}{\pi} \frac{a}{a^2 + x^2} \end{aligned}$$

Exercise 4.4.4:

We want to encapsulate of the roots of Q inside a disk so that the poles of the function

$$f(z) = \frac{e^- 2\pi i z \xi}{Q(z)}$$

will occur at precisely the roots of Q. The nature of each pole depends on the factorization of Q and what multiplicity the root is at that point. We will be integrating over a disk of radius R in the plane, C_R. We estimate the value of the integral via

$$\left|\int_{C_R} \frac{e^-2\pi iz\xi}{Q(z)} dz\right| \leq \int_0^{2\pi} \left|\frac{e^-2\pi iRe^{i\theta}\xi}{Q(Re^{\theta})} iRe^{i\theta}\right| d\theta \leq O(1/R^{\deg(Q)})$$

Which goes to 0 as $R \to \infty$. We then see that

$$\sum_i \operatorname{res}(f, z_i) = 0$$

The exact form of each residue depends on the order of the pole at each root z_i of Q.

Exercise 4.4.6:

This follows from Exercise 4.4.3. We showed that if

$$f(x) = \frac{1}{\pi} \frac{a}{a^2 + x^2}$$

then its Fourier transform is

$$\hat{f}(\xi) = e^{-2\pi a|\xi|}$$

Applying the Poisson summation forumula immediately shows that

$$\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{a}{a^2 + n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|}$$

We then observe that because the sum is symmetric we note that

$$\sum_{n=-\infty}^{\infty} e^{-2\pi a|n|} = 2\sum_{n=0}^{\infty} e^{-2\pi a|n|} - 1 = \frac{2}{1 - e^{-2\pi a}} - 1 = \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}} = \frac{e^{\pi a} + e^{-\pi a}}{e^{\pi a} - e^{-\pi a}} = \coth \pi a$$

Exercise 4.4.7:

(a) First we compute the Fourier transform of

$$f(x) = (\tau + z)^{-k}$$

We fix a $\xi \leq 0$ and use contour integration (around a semicircle of radius R, C_R, in the lower half-plane). Note that if $k \geq 2$ the function $f(z)e^{-2\pi iz\xi}$ is analytic in C_R

$$\begin{aligned} 0 &= \int_{C_R} f(z)e^{-2\pi iz\xi} dz \\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi} dx + \int_{\gamma_R} f(z)e^{-2\pi iz\xi} dz \end{aligned}$$

We then compute

$$\begin{aligned} \left|\int_{\gamma_R} f(z)e^{-2\pi iz\xi} dz\right| &= \left|\int_0^{\pi} f(Re^{i\theta})e^{-2\pi iRe^{\theta}\xi} iRe^{i\theta} d\theta\right| \\ &\leq \int_0^{\pi} \left|\frac{iRe^{i\theta}e^{-2\pi iRe^{\theta}\xi}}{(Re^{i\theta} + \tau)^k}\right| d\theta \\ &\leq \frac{R\pi}{(|R| + |\tau|)^k} \end{aligned}$$

Letting $R \to \infty$ shows that this integral goes to zero. This implies that

$$\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx = 0$$

If $\xi > 0$ then $f(z)e^{-2\pi i z\xi}$ has a pole at $-\tau$ of order k so that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(\tau + x)^k}dx = 2\pi i \operatorname{res}(f, -\tau)$$

We compute the residue using Theorem 3.1.4 to see that

$$\operatorname{res}(f, -\tau) = \frac{(2\pi i)^k}{(k-1)!}\xi^{k-1}e^{2\pi i z\xi}$$

So that

$$\hat{f}(\xi) = \frac{(2\pi i)^k}{(k-1)!}\xi^{k-1}e^{2\pi i x\xi}$$

We then apply the Poisson summation formula to see that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1}e^{2\pi i m\tau}$$

(b) We set $k = 2$ in the result of part **(a)** to see that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} &= (2\pi i)^2 \sum_{m=1}^{\infty} m e^{2\pi i m\tau} \\ &= -4\pi^2 \sum_{m=1}^{\infty} m e^{-2\pi i m\tau} \end{aligned}$$

Consider the series for $|x| < 1$

$$\sum_{m=1}^{\infty} m x^{m-1} = x\frac{d}{dx}\left(\sum_{m=1}^{\infty} x^m\right) = x\frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{x}{(1-x)^2}$$

We then note that if $\mathrm{Im}\,(\tau) > 0$ then $|e^{\pm 2\pi i m\tau}| < 1$ so we can apply the above to see that

$$-4\pi^2\sum_{m=1}^{\infty} m e^{-2\pi i m\tau} = -4\pi^2\frac{e^{2\pi i\tau}}{(1-e^{2\pi i\tau})^2} = \frac{-4\pi^2}{e^{\pi i\tau} - e^{-\pi i\tau}} = \frac{\pi^2}{\sin^2(\pi\tau)}$$

(c) This is true, you can extend this identity to non-integer complex u. To see this, we take the hint given in Exercise 3.8.12, which considers the function

$$f(z) = \frac{\pi\cot(\pi z)}{(u+z)^2}$$

If the poles of this function occur at the integers, which are simple poles, as well as a pole of order 2 at $z = -u$. We compute the residues at each type of pole by evaluating

$$\operatorname{res}(f, -u) = \lim_{z\to -u}\frac{d}{dz}\pi\cot(\pi z) = \frac{-\pi^2}{\sin^2(\pi u)}$$

At each integral pole we see that

$$\begin{aligned}\operatorname{res}(f,n) &= \lim_{z\to n}(z-n)\frac{\pi\cot(\pi z)}{(u+z)^2}\\ &= \lim_{z\to n}\frac{\pi(z-n)}{\sin(\pi(z-n))}\cdot\frac{\cot(\pi(z-n))}{(u+z)^2}\\ &= \frac{1}{(u+n)^2}\end{aligned}$$

Following the hint, we let C_N be the circle of radius $N+1/2$ and we want to verify that

$$\int_{C_N} f(z)dz \to 0 \text{ as } N\to\infty$$

We then note that for sufficiently large choices of N the conditions $z\in C_N$ and $|\operatorname{Im} z|\leq 1$ imply that $|\operatorname{Re} z-n|>\epsilon$ for some $\epsilon>0$, so that z is bounded away from the poles. We then note that the bound on this set is uniform and so we see that when $z=x+iy$ such that $y\geq 1$ we see

$$|\cot(\pi z) = \left|i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}\right| \leq \frac{1+e^{-2y}}{1-e^{-2y}} \leq \frac{1+e^{-2}}{1-e^{-2}}$$

In the other case when $y\leq -1$ we see that

$$|\cot(\pi z)| \leq \frac{1+e^{2y}}{1-e^{2y}} \leq \frac{1+e^{-2}}{1-e^{-2}}$$

Hence, $cot(\pi z)$ is bounded for large N, say $|\cot(\pi z)|\leq M$ for some $M>0$. We can then bound the integral via

$$\left|\int_{C_N} f(z)dz\right| \leq 2\pi(N+1/2)\frac{M}{(N+1/2)^2-|u|^2}$$

For large enough N. Consequently,

$$\left|\int_{C_N} f(z)dz\right| \to 0$$

as $N\to\infty$.

Now we apply the residue theorem to see that

$$\int_{C_N} f(z)dz = \operatorname{res}(f,-u) + \sum_{n=-\infty}^{\infty}\operatorname{res}(f,n) = 0$$

Plugging in the computed values for the residues gives that

$$\sum_{n=-\infty}^{\infty}\frac{1}{(u+n)^2} = \frac{\pi^2}{\sin^2(\pi u)}$$

As claimed.