Exercise 1.4.7:

(a) We want to show that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1$$

whenever $w, z \in \mathbb{C}$ satisfy $\overline{z}w \neq 1$. Note that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = |w-z|\left|\frac{1}{1-\overline{w}z}\right| = |w-z||1-\overline{w}z|^{-1}$$

So it suffices to show that

$$|w-z|^2 < |1-\overline{w}z|^2$$

Indeed, we see that

$$\begin{aligned}
|w-z|^2 &< |1-\overline{w}z|^2 \\
(w-z)\overline{(w-z)} &< (1-\overline{w}z)\overline{(1-\overline{w}z)} \\
w\overline{w} - w\overline{z} - z\overline{w} + z\overline{z} &< 1 - \overline{z}w - \overline{w}z + \overline{z}z\overline{w}w \\
|w|^2 + |z|^2 &< 1 + |z|^2|w|^2 \\
0 &< 1 - |w|^2 - |z|^2 + (|z||w|)^2 \\
0 &< (1-|z|^2)(1-|w|^2)
\end{aligned}$$

And the last equality must hold if $|z| < 1$ and $|w| < 1$. Moreover, we see that if $|z| = 1$ or $|w| = 1$, then we must have equality because one of the factors will be zero.

(b) Now we consider the map

$$F : z \mapsto \left|\frac{w-z}{1-\overline{w}z}\right|$$

We will show the following

(i) *F maps the unit disk to itself and is holomorphic.*

It is clear from part **(a)** that $F : \mathbb{D} \to \mathbb{D}$ because if $|z| \leq 1$ then $|F(z)| \leq 1$. It is also easy to see that F is holomorphic in the whole disk because it is the quotient of two holomorphic functions and the denominator is non-zero everywhere $\overline{z}w \neq 1$.

(ii) *F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.*

Observe that,

$$F(0) = \frac{w}{1-0} = w \qquad \text{and} \qquad F(w) = \frac{w-w}{1-|w|^2} = 0$$

(iii) *$|F(z)| = 1$ if $|z| = 1$.*

This also follows from **(a)**. We saw that if $|z| = 1$ then

$$|F(z)| = \left|\frac{w-z}{1-\overline{w}z}\right| = 1$$

as desired.

(iv) *$F : \mathbb{D} \to \mathbb{D}$ is bijective.*

Following the hint we observe that

$$\begin{aligned}(F\circ F)(z) &= \frac{w-\left(\frac{w-z}{1-\overline{w}z}\right)}{1-\overline{w}\left(\frac{w-z}{1-\overline{w}z}\right)}\\ &= \frac{w(1-\overline{w}z)-(w-z)}{(1-\overline{w}z)-\overline{w}(w-z)}\\ &= \frac{w-z|w|^2-w+z}{1-\overline{w}z-|w|^2+\overline{w}z}\\ &= \frac{z(1-|w|^2)}{1-|w|^2}\\ &= z\end{aligned}$$

So $F=F^{-1}$ and the inverse is defined for each $z\in\mathbb{D}$ and hence F is bijective.

Exercise 1.4.9:

In polar coordinates we have the familiar relations $x=r\cos\theta$ and $y=r\sin\theta$. If we consider the function $f(z)=u+iv$ and the associated map $F(x,y)=u(x,y)+iv(x,y)$ we see that

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}$$

Similarly, we have

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}=-r\sin\theta\frac{\partial u}{\partial x}+r\cos\theta\frac{\partial u}{\partial y}$$

Analogously,

$$\frac{\partial v}{\partial r}=\cos\theta\frac{\partial v}{\partial x}+\sin\theta\frac{\partial v}{\partial y}\qquad\text{and}\qquad\frac{\partial v}{\partial \theta}=-r\sin\theta\frac{\partial v}{\partial x}+r\cos\theta\frac{\partial v}{\partial y}$$

Now we use the rectangular Cauchy-Riemann equations to see that

$$\frac{\partial u}{\partial r}=\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}=\cos\theta\frac{\partial v}{\partial y}-\sin\theta\frac{\partial v}{\partial x}=\frac{1}{r}\frac{\partial v}{\partial \theta}$$

And

$$\frac{\partial u}{\partial \theta}=-r\sin\theta\frac{\partial u}{\partial x}+r\cos\theta\frac{\partial u}{\partial y}=-r\sin\theta\frac{\partial v}{\partial y}-r\cos\theta\frac{\partial v}{\partial x}=-r\frac{\partial v}{\partial r}$$

So we have

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}\qquad\text{and}\qquad\frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial v}{\partial \theta}$$

Exercise 1.4.10:

Recall that

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right)\qquad\text{and}\qquad\frac{\partial}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y}\right)$$

Then we see that

$$\begin{aligned}\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} &= \frac{1}{4}\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right)\\ &= \frac{1}{4}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial x}\frac{\partial}{\partial y}+\frac{1}{i}\frac{\partial}{\partial x}\frac{\partial}{\partial y}+\frac{\partial}{\partial y}\frac{\partial}{\partial y}\right)\\ &= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\end{aligned}$$

An identical computation shows that

$$\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

If we recall that the Laplacian is $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ then we see that

$$4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = \triangle$$

as desired.

Exercise 1.4.13:

We need to show that if f is holomorphic in an open set Ω then in each of the following cases f must be constant:

(a) Re(f) *is constant.*

This follows immediately from the Cauchy-Riemann equations. If $f = u + iv$ and Re(f) is constant then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

Applying the Cauchy-Riemann equation shows

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

So that Im$(f) = v$ is constant as well. Hence, $f = u + iv$ is constant.

(b) Im(f) *is constant.*

This is equivalent to part **(a)**. This time we have that $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ because Im(f) is constant. Consequently, have that $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ and so u must be constant and thus f as well.

(c) $|f|$ *is constant.*

If $|f|$ is constant then so is $|f|^2 = u^2 + v^2 = M$. Now we differentiate with respect to x and y to get the identities

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$
$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

Eliminating the factor of 2 these imply that

$$u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} - u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial x} = 0$$
$$u(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}) + (v-u)\frac{\partial v}{\partial x}$$

and

$$u\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} - u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial y} = 0$$
$$u(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + (v-u)\frac{\partial v}{\partial y}$$

Adding these identities gives

$$2u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)+(v-u)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

We then make the substitution $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ in the second term and then note that because the derivative is unique

$$f'(z)=\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)$$

The above resolves to

$$\begin{aligned} 2uf'(z)+(v-u)f'(z)&=0\\ (u+v)f'(z)&=0 \end{aligned}$$

This equation is true for all u and v and thus we must have that $f'(z)=0$. We then apply Corollary 3.4 from the text to conclude that f is constant.

Exercise 1.4.18:

Suppose that $f(z)=\sum_{n=0}^{\infty}a_nz^n$ is a power series with radius of convergence R. Choose a z_0 such that $|z_0|<R$. Then we write $z=z_0+(z-z_0)$ and apply the binomial theorem to see that

$$z^n=\sum_{k=0}^{n}\binom{n}{k}z_0^{n-k}(z-z_0)^k$$

If we choose z in the disk $|z|<R-|z_0|$ then we see that

$$\begin{aligned} \sum_{k=0}^{\infty}\left(\sum_{n=k}^{n}|a_n|\binom{n}{k}|z_0^{n-k}|\right)|(z-z_0)^k| &= \sum_{k=0}^{\infty}|a_n|\left|\sum_{n=k}^{n}\binom{n}{k}z_0^{n-k}(z-z_0)^k\right|\\ &=\sum_{n=1}^{\infty}|a_n|(|z_0|+|z-z_0|)^n<\infty \end{aligned}$$

So the series converges absolutely in the disk $|z|<R-|z_0|$. As a result, we can rearrange terms in the series yielding

$$\sum_{n=0}^{\infty}a_nz^n=\sum_{n=0}^{\infty}a_n\left(\sum_{k=1}^{n}\binom{n}{k}z_0^{n-k}(z-z_0)^k\right)=\sum_{k=0}^{\infty}\left(\sum_{n=k}^{n}a_n\binom{n}{k}z_0^{n-k}\right)(z-z_0)^k$$

So because $f(z)$ converges in $|z|<R$ we have that

$$\left|\sum_{k=0}^{\infty}\left(\sum_{n=k}^{n}a_n\binom{n}{k}z_0^{n-k}\right)(z-z_0)^k\right|=\left|\sum_{n=0}^{\infty}a_nz^n\right|<\infty$$

converges for any $|z_0|<R$ and $|z|<R-|z_0|$. And so $f(z)$ has a power series expansion for any z_0 in its disk of convergence.

Exercise 1.4.23:

Consider

$$f(x)=\begin{cases}0 & x\le 0\\ e^{-1/x^2} & x>0\end{cases}$$

We are interested in the behavior of f and its derivatives at the origin. Consider $f(\frac{1}{x}), x>0$ so that $\lim_{x\to 0^+}f(x)=\lim_{x\to\infty}f(\frac{1}{x})$. We see that $f(\frac{1}{x})=e^{-x^2}$. If we write this in terms of power series we get

$$f\left(\frac{1}{x}\right)=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{n!}$$

We then observe that

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |1/n!|^{1/n} < \limsup_{n\to\infty} |1/n^n|^{1/n} = \limsup_{n\to\infty} 1/n = 0$$

And so $f(\frac{1}{x})$ converges on all of $(0,\infty)$. Now we note that

$$\lim_{x\to\infty} f\left(\frac{1}{x}\right) = 0$$

Hence, $\lim_{x\to 0^+} f(x) = 0$. Now we apply Corollary 2.7 from the text to get that $f(\frac{1}{x})$ is infinitely differentiable on $(0,\infty)$ and the derivatives converge on the whole interval as well.

Again we observe that $\lim_{x\to 0^+} f^{(k)}(x) = \lim_{x\to\infty} f^{(k)}(\frac{1}{x})$. Differentiating term term gives

$$f^{(k)}\left(\frac{1}{x}\right) = \sum_{n\geq k} \frac{(-1)^n(2n)!x^{2n-k}}{(2n-k)!n!}$$

Let a_n be the coefficients of $f(\frac{1}{z})$ and let b_n be the coefficients of $f^{(k)}(\frac{1}{x})$ then choose N large enough such that $|f(\frac{1}{x}) - \sum_{n=0}^N a_n x^{2n}| < \epsilon/2$ and $|f^{(k)}(\frac{1}{x}) - \sum_{n=k}^N b_n x^{2n-k}| < \epsilon/2$. Observe that $x > 2n$ implies $x^{2n} > (2n)!x^{2n-k}/(2n-k)!$ (compare factor by factor) then compute

$$\lim_{x\to\infty} \left| f^{(k)}\left(\frac{1}{x}\right)\right| \leq \lim_{x\to\infty} \sum_{n=k}^N b_n x^{2n-k} + \epsilon/2 \leq \sum_{n=0}^N a_n x^{2n} + \epsilon/2 \leq \lim_{x\to\infty} f\left(\frac{1}{x}\right) + \epsilon/2 + \epsilon/2 = \epsilon$$

This shows that $\lim_{x\to\infty} f^{(k)}\frac{1}{x} = 0$ and consequently we have $\lim_{x\to 0^+} f^{(k)}(x) = 0$ as well.

Finally, we can conclude that f is infinitely differentiable on $\mathbb{R}$ because the left and right hand limits of the derivative agree and we have $f^{(n)}(0) = 0$. Consequently, we apply Taylor's formula to see that if f had a power series expansion about the origin it would be of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0$$

But f is non-zero in any interval $(0,\delta)$ and so the power series does not converge.

Exercise 1.4.25: We need to evaluate the following:

(a) $\int_\gamma z^n dz$ *where* γ *is a circle centered at the origin.*

Parametrize γ by setting $z = re^{i\theta}$ where $\theta \in [0,2\pi)$. Then we see that $dz = ire^{i\theta}d\theta$. So we can evaluate

$$\int_\gamma z^n dz = \int_0^{2\pi} (r^n e^{in\theta})(ire^{i\theta})d\theta = ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}d\theta = \frac{ir^{n+1}}{n+1}(e^{2\pi i(n+1)} - e^0) = 0$$

(b) $\int_\gamma z^n dz$ *where* γ *is a circle not enclosing the origin.*

Suppose that γ is centered at z_0. We then use the parametrization $z = z_0 + re^{i\theta}$ where $r < |z_0|$. As before, we have $dz = ire^{i\theta}d\theta$. Consequently,

$$\begin{aligned}
\int_\gamma z^n dz &= \int_0^{2\pi} (z_0 + re^{i\theta})(ire^{i\theta})d\theta \\
&= \int_0^{2\pi} \sum_{k=0}^{n} i\binom{n}{k} z_0^{n-k} r^{k+1} e^{i\theta(k+1)}d\theta \\
&= \sum_{k=1}^{n} i\binom{n}{k} z_0^{n-k} r^{k+1} \int_0^{2\pi} e^{i\theta(k+1)}d\theta
\end{aligned}$$

As in part **(a)** we see that each of the integral terms is zero and hence the sum is zero as well. So

$$\int_\gamma z^n dz = 0$$

(c) *If* $|a| < r < |b|$ *then*

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$$

where γ *denotes the circle centered at the origin, of radius* r*, with the positive orientation.*

Let $f(z) = (z-a)^{-1}(z-b)^{-1}$. We will proceed via integration by parts. We compute

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{1}{a-b}\int_\gamma \frac{dz}{z-a} + \frac{1}{b-a}\int_\gamma \frac{dz}{z-b}$$

The first thing to note is that f has a singularity at a. We evaluate this integral via a change of variable $w = z - a$ and so

$$\int_\gamma \frac{dz}{z-a} = \int_{\gamma - a} \frac{dw}{w} = 2\pi i$$

Now we need to consider $\int_\gamma \frac{dz}{z-b}$. This function has no singularities inside the disk $|z| < r$. We can then note that

$$\frac{1}{z-b} = \frac{-1}{b} \cdot \frac{1}{1-(z/b)} = \frac{-1}{b} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{b^n}$$

The series on the right converges uniformly and absolutely inside the disk $|z| < |b|$ and in particular $|z| < r$. Thus, we can integrate term by term and see

$$\int_\gamma \frac{dz}{z-b} = \frac{-1}{b}\int_\gamma \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{b^n} dz = \sum_{n=0}^{\infty} \frac{(-1)^n}{b^n} \int_\gamma z^n dz$$

Each term in the series evaluates to 0 by part **(a)** and so we have that

$$\int_\gamma \frac{dz}{z-b} = 0$$

Substituting in our original expression gives

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$$

Exercise 1.4.26:

Let f be continuous in the region Ω and suppose that F and G are two primitives of f. Consider $H = F - G$. We see that

$$\frac{d}{dz}H = \frac{d}{dz}F - \frac{d}{dz}G = f - f = 0$$

We then apply Corollary 3.4 to see that $H = F - G$ is constant. So any two primitives of f must differ by a constant.