Exercise 8.5.5:

First note that $f(z) = -\frac{1}{2}(z + 1/z)$ is holomorphic in the half disk $U = \{z = x + iy : |z| < 1, y > 1\}$ because the origin is excluded. We will first prove that f is surjective from $U \to \mathbb{H}$. Indeed, we note that if $f(z) = w$ then $z^2 + 2wz + 1 = 0$. This equation has two distinct roots in $\mathbb{C}$ if $w \neq \pm 1$, which is true because we exclude the real line in $\mathbb{H}$. Moreover, we see that one of the roots lies in the interior of U because the roots are of the form $z_0 = -w \pm (w^2 - 1)$ and the product of the roots must be 1. So if we note that

$$|z_0| = |-w - (w^2 - 1)| = |w + w^2 + 1| > 1, w \in \mathbb{H}$$

Then the other root must satisfy $|z_1| = |1/w| < 1$ and lies in U. Then $f(z_1) = w$ and f is surjective.

For injectivity, we suppose that there were two distinct $z_0, z_1 \in U$ such that $w = f(z_0) = f(z_1)$. We again use the fact that z_0, z_1 must be the roots of the equation $z^2 + 2wz + 1$ and by the above we see that only one of them can be in U. So f is injective on U.

Finally, we check that f maps into $\mathbb{H}$. This is clear because if $z \in U$ then

$$\text{Im }(-\frac{1}{2}(z + 1/z)) = \frac{-1}{2}(\text{Im }(z) - \frac{1}{|z|}\text{ Im }(z)) = -\frac{1}{2}\left(1 - \frac{1}{|z|}\right)\text{ Im }(z)$$

Which is greater than zero when $|z| < 1$ (i.e. $z \in U$). So f maps U into $\mathbb{H}$.

Exercise 8.5.6:

We will verify this with a direct calculation. We let $u : U \to \mathbb{C}$ be a harmonic function and $F = f(x, y) + ig(x, y)$ be holomorphic from $V \to U$. We then use the fact that

$$\frac{\partial^2}{\partial z\overline{z}} = \frac{1}{4}\Delta$$

to compute

$$\begin{aligned}
\Delta(u \circ F) &= \frac{1}{4}(u \circ F)_{z\overline{z}} \\
&= \frac{1}{4}((u_z \circ F) \cdot F')_{\overline{z}} \\
&= \frac{1}{4}(u_{z\overline{z}} \circ F) \cdot F' \cdot \overline{F'} \\
&= (\Delta u \circ F) \cdot |F'|^2
\end{aligned}$$

And because $\Delta u = 0$ the entire expresion is zero and we see that

$$\Delta(u \circ F) = 0$$

as well. Hence, $u \circ F$ is harmonic.

Exercise 8.5.12:

(a) Suppose that $f : \mathbb{D} \to \mathbb{D}$ is analytic and has two fixed points z_0, z_1. Then we recall the function

$$\psi_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0}z}$$

from the text, whose important property is that $\psi_{z_0}(0) = z_0$ and $\psi_{z_0}(z_0) = 0$. Then the function

$$F = \psi_{z_0}^{-1} \circ f \circ \psi_{z_0}$$

fixes the origin. Now we apply the Schwarz lemma to see that $|F(z)| \leq |z|$ for $|z| \in \mathbb{D}$. Moreover, because equality holds, then $F(z) = e^{i\theta}z$ for some θ. Because $F(b) = b$ we see that $\theta = 0$. As a result

$$F(z) = (\psi_{z_0}^{-1} \circ f \circ \psi_{z_0})(z) = z$$

And hence, $f(z) = z$ for every z and f is the identity.

(b) Following the hint, we look at maps in $\mathbb{H}$. Consider the horizontal translations in the upper half plane of the form $\varphi_n : z \mapsto z + n$ for some integer n. Now take any conformal map $f : \mathbb{D} \to \mathbb{H}$ and consider $F = \varphi_n^{-1} \circ f \circ \varphi_n$. Then $F : \mathbb{D} \to \mathbb{D}$ is holomorphic, but has no fixed points.

Exercise 8.5.13:

(b) We are considering the hyperbolic distance defined as

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|$$

By part **(a)** we have that if $f \in \mathrm{Aut}(\mathbb{D})$ then

$$\begin{aligned} \rho(f(z), f(w)) &\leq \rho(z, w) \\ \left| \frac{f(z) - f(z)}{1 - \overline{f(w)}f(z)} \right| &\leq \left| \frac{z - w}{1 - \overline{w}z} \right| \\ \left| \frac{f(z) - f(z)}{z - w} \right| \cdot \frac{1}{|1 - \overline{f(w)}f(z)|} &\leq \frac{1}{|1 - \overline{w}z|} \end{aligned}$$

Now we let $w \to z$ and get the estimate

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

for all $z \in \mathbb{D}$.

Exercise 8.5.14:

Let $f : \mathbb{H} \to \mathbb{D}$ be a conformal map. Recall from the text the map

$$G(w) = i\frac{1 - w}{1 + w}$$

which is conformal from $\mathbb{D} \to \mathbb{H}$. Then the composition of functions $f \circ G : \mathbb{D} \to \mathbb{D}$ is holomorphic and furthermore it is an automorphism of $\mathbb{D}$. As a result, we can apply Theorem 8.2.2 to see that

$$(f \circ G)(z) = e^{i\theta}\frac{\alpha - z}{2 - \overline{\alpha}z}$$

or some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$. Then we simplify by computing setting $z = G(w)$ and getting

$$f(z) = f\left(i\frac{1 - w}{1 + w}\right) = e^{i\theta}\frac{\alpha - w}{1 - \overline{\alpha}w}$$

Then recall that G has an inverse map

$$F(z) = \frac{i - z}{i + z}$$

and do $w = F(z)$ and we can compute

$$e^{i\theta}\frac{\alpha - w}{1-\overline{\alpha}w} = e^{i\theta}\frac{\alpha - \frac{i-z}{i+z}}{1-\overline{\alpha}\frac{i-z}{i+z}} = e^{i\theta}\frac{(i+z)\alpha-(i-z)}{(i+z)-\overline{\alpha}(i-z)}$$

Rearranging terms gives

$$e^{i\theta}\frac{(i+z)\alpha-(i-z)}{(i+z)-\overline{\alpha}(i-z)} = e^{i\theta}\frac{z(1+\alpha)-i(1-\alpha)}{z(1+\alpha)-i(1-\overline{\alpha})} = e^{i\theta}\frac{z-i\frac{1-\alpha}{1+\alpha}}{z+i\frac{1-\overline{\alpha}}{1+\alpha}}$$

We then make the substitution

$$\beta = i\frac{1-\alpha}{1+\alpha}$$

in the above to get

$$e^{i\theta}\frac{z(1+\alpha)-i(1-\alpha)}{z(1+\alpha)-i(1-\overline{\alpha})} = e^{i\theta}\frac{z-i\frac{1-\alpha}{1+\alpha}}{z+i\frac{1-\overline{\alpha}}{1+\alpha}} = e^{i\theta}\frac{z-\beta}{z-\beta}$$

as desired.