Exercise 2.7.3:

Consider inequalities of the form

$$||\hat{f}||_{L^q} \leq A||f||_{L^p}$$

where f is a simple function. We see in Corollary 2.2.6 that if p, q are conjugate then such an inequality is possible. We need to show that this is necessary. Indeed, if we look at the family of functions $f_r(x) = f(rx)$ for $r > 0$. Then we compute

$$\hat{f}_r(\xi) = \int f_r(x)e^{-2\pi ix\cdot\xi}dx = r^{-d}\int f(u)e^{-2\pi iu\cdot\xi/r}du = r^{-d}\hat{f}_r(\xi/r)$$

We then note that because f is simple, it is in both L^p and L^q for all p and q. This implies that the inequality above satisfies

$$||\hat{f}||_{L^q} \leq Ar^d||f||_{L^p}$$

due to translation invariance of the integral. We know by the Hausdorff-Young inequality that the Fourier transform of an L^p function is in L^q if q is conjugate to p. Letting $r \to \infty$ the right hand is unbounded we have a contradiction unless p, q are conjugate.

Exercise 2.7.4:

We are considering estimates of the type

$$\int_{|\xi|\leq 1} |\hat{f}(\xi)|d\xi \leq A||f||_{L^p}$$

Given $f \in L^p$, consider the function

$$f^s(x) = s^{-d/2}e^{-\pi|x|^2/s}$$

where $s = \sigma + it$, $\sigma > 0$. We then compute the Fourier transform $\hat{f^s}(\xi)$ via

$$\hat{f^s}(\xi) = \int_{\mathbb{R}^d} s^{-d/2}e^{-\pi|x|^2/s}e^{-2\pi ix\cdot\xi}dx$$

Then note that because $|x|^2 = x \cdot x$ the inner products in the exponents above allow the integral to admit a factorization into single variable integrals yielding

$$\begin{aligned}
\hat{f^s}(\xi) &= \prod_{j=1}^{d} s^{-d/2}\int_{-\infty}^{\infty} e^{-\pi x_j^2/s}e^{-2\pi ix_j\xi_j}dx_j \\
&= \prod_{j=1}^{d} s^{-d/2}\int_{-\infty}^{\infty} e^{-\frac{\pi}{s}(x_j-is\xi_j)^2}e^{-\pi s\xi_j^2}dx_j \\
&= \prod_{j=1}^{d} s^{-d/2}e^{-\pi s\xi_j^2}\int_{-\infty}^{\infty} e^{-\frac{\pi}{s}(x_j-is\xi_j)^2}dx_j
\end{aligned}$$

Where the third equality above follows by completing the square. We can then simplify each term using contour integration using a reduction to the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\frac{\pi}{s}(x_j - is\xi_j)^2} dx_j = \int_{-\infty - is\xi_j}^{\infty - is\xi_j} e^{-\pi u_j^2/s} du_j = s^{1/2}$$

Pluggin this into the product formula above yields

$$\hat{f}^s = \prod_{j=1}^{d} s^{-d/2} e^{-\pi s \xi_j^2} \int_{-\infty}^{\infty} e^{-\frac{\pi}{s}(x_j - is\xi_j)^2} dx_j = e^{-\pi s|\xi|^2} s^{-d/2} \prod_{j=1}^{d} s^{1/2} = e^{-\pi s|\xi|^2}$$

Now we restrict our attention to the line $\sigma = 1$ so that $s = 1 + it$. We then compute

$$\begin{aligned}
||f^s||_{L^p}^p &= \int \left| s^{-d/2} e^{-\pi|x|^2/s} \right|^p dx \\
&\leq |s^{-d/2}|^p \int \prod_{j=1}^{d} \left| e^{-\pi x_j^2/s} \right|^p dx_1 \dots dx_d \\
&\leq |s^{-d/2}| \prod_{j=1}^{\infty} \int \left| e^{-\pi x_j^2/s} \right|^p dx_j
\end{aligned}$$

Now we use the fact that $s = 1 + it$. Then we see that

$$1/(1+it) = \frac{1}{t^2+1} - \frac{it}{t^2+1}$$

So the exponent in the integral resolves to

$$(-\pi i x_j^2)\left(\frac{1}{t^2+1} - \frac{it}{t^2+1}\right) = \left(\frac{-\pi t x_j^2}{t^2+1} - \frac{\pi i x_j^2}{t^2+1}\right)$$

Now we use the fact that

$$\int e^{-\pi a x^2} dx = \frac{1}{\sqrt{a}}$$

to continue the above computation as follows

$$\begin{aligned}
\int \left| e^{-\pi x_j^2/s} \right|^p dx_j &= \int \left| \mathrm{Re}\, e^{-\pi x_j^2/s} \right|^p dx_j \\
&= \int e^{-\pi(pt/t^2+1)x_j^2} dx_j \\
&= \left(\frac{pt}{t^2+1}\right)^{-1/2} \\
&= O(t^{1/2})
\end{aligned}$$

Plugging this into the integral formula above gives

$$||f^s||_{L^p}^p \leq |(1+it)^{-d/2}|^p O(t^{1/2}) = O(t^{-pd/2}) O(t^{d/2}) = O(t^{pd-d/2})$$

Then taking p-th roots gives $\|f^s\|_{L^p} \leq ct^{d(1/p-1/2)}$. We then let $t \to \infty$ and observe that if $1/p < 1/2$ then the right hand side of the inequality

$$\int_{|\xi|\leq 1} |\hat{f}(\xi)| d\xi \leq A\|f\|_{L^p}$$

will go to zero, but the left hand side will remain positive, a contradiction. Hence, $1/p \geq 1/2$ so we must have that $p \leq 2$.

Exercise 2.7.5:

Let S be the strip $0 <$Re $(z) < 1$. Let $\psi(z) = e^i \pi z$ and let $\Phi(z) = e^{-i\psi(z)}$. First we will check that Φ is continuous on the closure of S. Indeed, we note that $\varphi(z) = e^{iz}$ is clearly continuous on the closed upper half-plane. So we only need to verify that $\psi : S \to \mathbb{H}$ is continuous on the closure of S. Suppose that $\alpha \in \partial S$, then because ψ is a conformal map, $\psi(z)$ must be on the boundary of $\mathbb{H}$. Suppose that ψ is not continuous, then we can find two distinct sequences $\{z_i\}_{i\in\mathbb{N}}, \{w_i\}_{i\in\mathbb{N}}$ converging to α such that $\lim_{i\to\infty} \psi(z_i) \neq \lim_{i\to\infty} \psi(w_i)$. These limits must exist in $\mathbb{H}$ because the z_i, w_i are bounded and convergent. If we set z, w as the respective limits then we know that their are open sets U_z, U_w such that $d(U_z, U_w) > \delta$ for some $\delta > 0$. But then the continuity of ψ on the interior of S guarantees that if $d(w_i, z_j) < \delta_0$ then $d(\psi(w_i), \psi(z_j)) \leq \delta$. Because infinitely many of the z_i, w_i are contained in a δ_0-neighborhood of α we have a contradiction. Thus, ψ is continuous on the boundary.

Now we need to see that $|\Phi(z)| = 1$ on the boundary of S. This follows from the fact that angles are preserved under conformal maps, we see that ψ maps the boundary of S to the boundary of $\mathbb{H}$. This is $\mathbb{R}$ lying in $\mathbb{C}$ which is taken onto the circle S^1 under the map e^{iz}, so that $|\Phi(z) = 1$. Finally, to see that Phi is unbounded on the interior of S we note that it is non-constant and holomorphic and therefore cannot be bounded.

Exercise 2.7.7:

We have that f is bounded with compact support and therefore must be in $L^2(\mathbb{R})$. Then we must have that

$$H(f)(x) = \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{|t|\geq\epsilon} f(x-t)\frac{dt}{t}$$

Moreover, we see that

$$\int_{-\infty}^{\infty} H(f)(x)dx = \int_{-\infty}^{\infty} \left(\lim_{\epsilon\to 0} \frac{1}{\pi} \int_{|t|\geq\epsilon} f(x-t)\frac{dt}{t}\right) dx = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{\pi} \int_{|t|\geq\epsilon} \frac{f(t)}{x-t} dt$$

Note that the change of variable is ok because f is bounded with compact support. Then we integrate by parts to see that if $a = \int f dx$ then $Hf(x) = a/\pi x + O(1/x^2)$. This clearly shows that $H(f)$ cannot be L^1 unless $a = 0$, which precisely says that $\int f dx = 0$.

Exercise 2.7.10:

(a) Suppose $1 \leq p \leq \infty$ and recall that

$$(f * \mathcal{P}_y) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} e^{-2\pi y|\xi|} d\xi$$

Then for $f \in L^p(\mathbb{R})$ we need to compute

$$||f * \mathcal{P}_y||_{L^p}^p = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} e^{-2\pi y|\xi|} d\xi \right|^p dx$$

Indeed, for the inner integral we observe that

$$\left| \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} e^{-2\pi y|\xi|} d\xi \right|^p = \left| \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i (x + i\text{sign}(\xi) y)\xi} d\xi \right|^p$$

Moving the absolute value inside the integral and applying the Fourier inversion formula to the aboslute value gives

$$\left| \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i (x + i\text{sign}(\xi) y)\xi} d\xi \right|^p \leq \left(\int_{-\infty}^{\infty} \left| \hat{f}(\xi) e^{2\pi i x \xi} \right| d\xi \right)^p = |f(x)|^p$$

Note that the imaginary part was eliminated when passing to the absolute value above. Substitution into the formula for the norm gives

$$||f * \mathcal{P}_y||_{L^p}^p \leq \int_{-\infty}^{\infty} |f(x)|^p dx = ||f||_{L^p}^p$$

Taking p^{th} roots gives the result.

(b) We first recall that

$$\int_{-\infty}^{\infty} \mathcal{P}_y(x) dx = 1$$

for each $y > 0$. As a result of this we see that

$$(f * \mathcal{P}_y)(x) - f(x) = \int_{-\infty}^{\infty} [f(x-t) - f(x)] \mathcal{P}_y(t) dt$$

Consequently,

$$|(f * \mathcal{P}_y)(x) - f(x)| \leq \int_{-\infty}^{\infty} |f(x-t) - f(x)|^p \mathcal{P}_y(t) dt$$

We then set $f_t(x) = f(x-t)$ integrate integrate with respect to t and apply Fubini's theorem to get

$$||f * \mathcal{P}_y)(x) - f(x)||_{L^p}^p \leq \int_{-\infty}^{\infty} ||f_t - f||_{L^p}^p \mathcal{P}_y(t) dt$$

The function $g(t) = ||f_t - f||_{L^p}^p$ is bounded and continuous because $t \mapsto f_t$ is uniformly continuous from $\mathbb{R} \to L^p(\mathbb{R})$ and the fact that $g(0) = 0$. Thus, the right hand side of the above goes to zero as $y \to 0$.