Exercise 2.6.2:

We want to show

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Consider the integral

$$\frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{i \sin x}{x} dx + \int_{-\infty}^{\infty} \frac{\cos x}{x}$$

Because $\cos x$ is even and $\sin x$ is odd we see

$$\frac{1}{2i}\int_{-\infty}^{\infty} \frac{i \sin x}{x} dx + \frac{1}{2i}\int_{-\infty}^{\infty} \frac{\cos x}{x} = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{i \sin x}{x} = \int_0^\infty \frac{\sin x}{x}$$

We will evaluate this on the contour

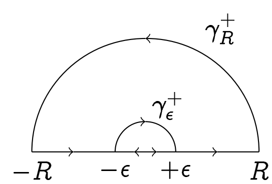

Figure 1: Contour of integration

We apply Cauchy's theorem to see that

$$\int_{\gamma_R^+} \frac{e^{iz}}{z} dz + \int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx + \int_{\gamma_R^+} \frac{e^{iz}}{z} dz + \int_{\epsilon}^{R} \frac{e^{ix}}{x} dx = 0$$

Then we observe

$$\int_{\gamma_R^+} \frac{e^{iz}-1}{z} dz = \int_{\gamma_R^+} \frac{e^{iz}}{z} dz \leq \left| \int_{\gamma_R^+} \frac{e^{iz}}{z} dz \right|$$

We estimate the last term

$$\left| \int_{\gamma_R^+} \frac{e^{iz}}{z} dz \right| = \left| \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta} d\theta \right| \leq \left| i\int_0^\pi e^{iR\cos\theta} e^{-R\sin\theta} d\theta \right|$$

Now estimate

$$\left| i\int_0^\pi e^{iR\cos\theta} e^{-R\sin\theta} d\theta \right| \leq \left| i\int_0^\pi e^{-R\sin\theta} d\theta \right|$$

Note that if $\theta \in (0,\pi)$ then $e^{\sin\theta} > 1$, which can be seen by observing that $\sin\theta > 0$ in $(0,\pi)$ and then applying the esponential function to both sides of the inequality. So

$$\left| i\int_0^\pi e^{-R\sin\theta} d\theta \right| \leq i\pi \sup_{\theta\in(0,\pi)} \left|(e^{\sin\theta})^{-R}\right| \leq i\pi(1+\epsilon)^{-R}$$

Letting $R \to \infty$ shows that the integral goes to 0. This means

$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x} dx = -\int_{\gamma_\epsilon^+} \frac{e^{iz}}{z} dz$$

Now observe that

$$\frac{e^{iz}}{z} = \frac{\sum_{n=1}^{\infty} \frac{(iz)^n}{n!}}{z} = \frac{1+O(z)}{z} = \frac{1}{z} + O(1)$$

So that we have

$$-\int_{\gamma_\epsilon^+} \frac{e^{iz}}{z} dz \approx -\int_{\gamma_\epsilon^+} \frac{1}{z} dz = -\int_\pi^0 \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}} d\theta = \int_0^\pi i d\theta = \pi i$$

Letting $\epsilon \to 0$ and $R \to \infty$ gives that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi i$$

We then apply the original identity (dividing by $2i$) to see that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Note: I think the hint in the book is a typo because the extra -1 in the numerator interferes with the limiting computation.

Exercise 2.6.6:

There are two cases

1. w is in the interior of T.
2. w is on the boundary of T.

We treat both individually. Suppose that w is an interior point of T. Then we will integrate around a "keyhole" toy contour as outlined on page 42 of the text. In this case we let the width of the "corridor" be ϵ for any choice of $\epsilon > 0$. Consequently, we can apply Cauchy's theorem to see that for any $\epsilon > 0$ the integral about the contour is zero. So we only need to deal with the small disk that forms around w. Because f is bounded near w we can make the estimate

$$\left| \int_{\gamma_\epsilon} f(z) dz \right| \leq \int_{\gamma_\epsilon} |f(z)| dz \leq \int_0^{2\pi} M\epsilon e^{i\theta} d\theta \leq \epsilon 2M\pi$$

So the integral around the inner loop goes to zero as $\epsilon \to 0$.

If w is on the boundary of T. Then we choose a slightly different contour: instead of a "corridor" we just use a semicircle around w. This is analogous to the the construction on page 44 of the text, except the contour is cut out of a triangle instead of a larger semicircle. Again, we apply Cauchy's theorem to see that the integral around the whole contour is zero as $\epsilon \to 0$ by estimating again. We make the same estimate as before

$$\left| \int_{\gamma_\epsilon} f(z) dz \right| \leq \int_{\gamma_\epsilon} |f(z)| dz \leq \int_0^{\pi} M\epsilon e^{i\theta} d\theta \leq \epsilon M\pi$$

Which goes to zero. Hence, the integral around the entire contour is zero.

So the claim holds in both cases, and therefore is w is any element of T and f is bounded in a neighbor of w then

$$\int_T f(z) dz = 0$$

Exercise 2.6.11:

(a) We need to show that when $0 < R < R_0$ and $|z| < R$ that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \operatorname{Re}\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z} \right) d\varphi$$

We begin with the right hand side, multiplying by the conjugate to separate and get

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\ \mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi &= \frac{1}{4\pi}\int_0^{2\pi} f(Re^{i\varphi})\ \mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}+\frac{Re^{-i\varphi}+\bar{z}}{Re^{-i\varphi}-\bar{z}}\right)d\varphi \\ &= \frac{1}{4\pi}\int_0^{2\pi} f(Re^{i\varphi})\ \mathrm{Re}\left(2\frac{Re^{i\varphi}}{Re^{i\varphi}-z}-\frac{2\bar{z}}{\bar{z}-Re^{-i\varphi}}\right)d\varphi\end{aligned}$$

Now we distribute and break up the computation as follows

$$\begin{aligned}&\frac{1}{4\pi}\int_0^{2\pi} f(Re^{i\varphi})\ \mathrm{Re}\left(2\frac{Re^{i\varphi}}{Re^{i\varphi}-z}-\frac{2\bar{z}}{\bar{z}-Re^{-i\varphi}}\right)d\varphi \\ &= \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\varphi})\frac{iRe^{i\varphi}}{Re^{i\varphi}-z}d\varphi + \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{iRe^{i\varphi}}{Re^{i\varphi}-R^2\bar{z}^{-1}}d\varphi\end{aligned}$$

Now observe that we can apply the Cauchy integral formula to see

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\varphi})\frac{iRe^{i\varphi}}{Re^{i\varphi}-z}d\varphi$$

Then we note that

$$\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{iRe^{i\varphi}}{Re^{i\varphi}-R^2\bar{z}^{-1}}d\varphi = 0$$

is an analytic function on the interior of $|z| < R$

(b) We begin by computing

$$\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r} = \frac{R\cos\gamma + iR\sin\gamma + r}{R\cos\gamma + iR\sin\gamma - r}$$

Now rationalize the denominator to get

$$\frac{R\cos\gamma + iR\sin\gamma + r}{R\cos\gamma + iR\sin\gamma - r} = \frac{(R\cos\gamma + iR\sin\gamma + r)(R\cos\gamma + iR\sin\gamma - r)}{(R\cos\gamma - r)^2 + R^2\sin^2\gamma}$$

We do the multiplication to see

$$\frac{(R\cos\gamma + iR\sin\gamma + r)(R\cos\gamma + iR\sin\gamma - r)}{(R\cos\gamma - r)^2 + R^2\sin^2\gamma} = \frac{(R^2\cos^2\gamma + R^2\sin^2\gamma - r^2) - i(2Rr\sin\gamma)}{R^2 - 2Rr\cos\gamma + r^2}$$

Taking the real part confirms that

$$\mathrm{Re}\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}$$

Exercise 2.6.12:

(a) Following the hint, we recall that $f'(z) = 2\partial u/\partial z$. Consider $g(z) = 2\partial uz$. We want to show that g is holomorphic. Indeed, observe that

$$\frac{\partial g}{\partial \bar{z}} = 2\frac{\partial u}{\partial \bar{z}}\frac{\partial u}{\partial z} = 0$$

by the equality of mixed partials. Hence, g is holomorphic.

Now we apply Theorem 2.2.1 to see that g must have a primitive, say f. Then we compute

$$\frac{\partial\ \mathrm{Re}\ (f)}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial z} = \frac{1}{2}g = \frac{\partial u}{\partial z}$$

As a result, Re (f) differs from u by some real constant k. Hence, $f(z)$ is the desired function.

(b) We apply the previous exercise (11) to the function

$$f(z) = u(z) + iv(z)$$

So we get

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} (u(e^{i\theta} + iv(e^{i\theta}))\ \mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) d\theta$$

Now we apply **(a)** to separate the real part and see

$$\begin{aligned} u(re^{i\varphi}) &= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\ \mathrm{Re}\left(\frac{e^{i\theta}+re^{i\varphi}}{e^{i\theta}-re^{i\varphi}}\right) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\ \mathrm{Re}\left(\frac{e^{i(\theta-\varphi)}+r}{e^{i(\theta-\varphi}-r}\right) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} P_r(\varphi-\theta)u(e^{i\theta)})d\theta \end{aligned}$$

as desired.