Exercise 7.9.1:

To see that f must be identically 0 in the polydisc $P_r(z^0)$ we observe that f has a power series representation

$$f(z) = \sum_{|\alpha| \le d} a_\alpha (z - z^0)^\alpha$$

Moreover, we have a formula for the a_α given by Proposition 7.1.1 which states that

$$a_\alpha = \frac{1}{(2\pi i)^n} \int_{C_r(z^0)} f(\zeta) \prod_{k=1}^{d} \frac{d\zeta_k}{(\zeta_k - z_k^0)^{\alpha_k+1}}$$

Then we use the fact that f vanishes the polydisc $\mathbb{P}_r(z^0)$ to conclude that f must vanish in the disk $|\zeta_1 - z_1^0| = r_1$ to see that

$$a_\alpha = \frac{1}{(2\pi i)^n} \left(\int_{|\zeta_1 - z_1^0| = r_1} \frac{f(\zeta_1)}{(\zeta_1 - z_1^0)} \right) \left(\int_{C_r(z^0)} \prod_{k=2}^{d} \frac{d\zeta_k}{(\zeta_k - z_k^0)^{\alpha_k+1}} \right) = 0$$

because the second term in the product is zero. Hence, f is zero on some neighborhood in $\mathbb{P}_r(z^0)$. Now we apply Proposition 7.1.2 to the functions f and 0 to conclude that they must agree on all of $\mathbb{P}_r(z^0)$.

Exercise 7.9.2:

1.

Exercise 7.9.4:

Exercise 7.9.5:

1.

2.

Exercise 7.9.6:

Let $D_\delta(z) \subset \Omega$ be a disc centered at $z \in \Omega$. We will apply Green's theorem (in the complex sense) to the differential form

$$\frac{F(\zeta) d\zeta}{\zeta - z}$$

in the region $\Omega_{(\delta,z)} = \Omega \setminus D_\delta(z)$. Indeed, Green's theorem states that

$$\int_{\partial\Omega} \frac{F(\zeta) d\zeta}{\zeta - z} - \int_{\partial D_\delta(z)} \frac{F(\zeta) d\zeta}{\zeta - z} = \int_{\Omega_{(\delta,z)}} (\partial + \bar{\partial}) \frac{F(\zeta) d\zeta}{\zeta - z}$$

We then use linearity to expand the right hand side to

$$\int_{\Omega_{(\delta,z)}} \partial \left(\frac{F(\zeta) d\zeta}{\zeta - z} \right) + \int_{\Omega_{(\delta,z)}} \bar{\partial} \left(\frac{F(\zeta) d\zeta}{\zeta - z} \right)$$

Then we note that

$$\partial \left(\frac{F(\zeta) d\zeta}{\zeta - z} \right) = \frac{\partial}{\partial \zeta} \left(\frac{F(\zeta)}{\zeta - z} \right) d\zeta \wedge d\zeta = 0$$

Moreover,

$$\bar{\partial} \left(\frac{F(\zeta) d\zeta}{\zeta - z} \right) = \frac{\partial F}{\partial \bar{\zeta}} \left(\frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} \right) + f \frac{\partial}{\partial \bar{\zeta}} \left(\frac{1}{\zeta - z} \right) d\bar{\zeta} \wedge d\zeta$$

We then use the fact that

$$\frac{\partial}{\partial \bar{\zeta}}\left(\frac{1}{\zeta - z}\right) = 0$$

To conclude that

$$\bar{\partial}\left(\frac{F(\zeta)d\zeta}{\zeta - z}\right) = \frac{\partial F}{\partial \bar{\zeta}}\left(\frac{d\bar{\zeta}\wedge d\zeta}{\zeta - z}\right)$$

Substituting this into the integral formula we see that

$$\int_{D_\delta(z)} (\partial + \bar{\partial})\frac{F(\zeta)d\zeta}{\zeta - z} = \int_{D_\delta(z)} \frac{\partial F}{\partial \bar{\zeta}}\left(\frac{d\bar{\zeta}\wedge d\zeta}{\zeta - z}\right)$$

Now we will simplify the integral

$$\int_{\partial D_\delta(z)} \frac{F(\zeta)d\zeta}{\zeta - z} = \int_{\partial D_\delta(z)} \frac{F(\zeta) - f(z)}{\zeta - z}d\zeta + \int_{\partial D_\delta(z)} \frac{F(z)}{\zeta - z}d\zeta$$

Now because F is C^1 we can find a constant M such that $|F(\zeta) - F(z)| \leq M|\zeta - z|$ on $\partial D_\delta(z)$. This gives the following estimate

$$\left|\int_{\partial D_\delta(z)} \frac{F(\zeta) - f(z)}{\zeta - z}d\zeta\right| \leq M\int_{\partial D_\delta(z)} \left|\frac{\zeta - z}{\zeta - z}\right| |d\bar{\zeta}| = 2\pi\delta M$$

Letting $\delta \to 0$ forces the integral to 0 as well. Finally, we recall that

$$\int_{\partial D_\delta(z)} \frac{F(z)}{\zeta - z}d\zeta = 2\pi i F(z)$$

This implies that

$$\int_{\partial\Omega} \frac{F(\zeta)}{\zeta - z}d\zeta - 2\pi i F(z) = \int_{\Omega_{(\delta,z)}} \frac{\partial F}{\partial \bar{\zeta}}\left(\frac{d\bar{\zeta}\wedge d\zeta}{\zeta - z}\right) + O(\delta)$$

Letting $\delta \to 0$ and rearranging the terms yields

$$F(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{F(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\int_{\Omega} \frac{(\partial F/\partial\bar{\zeta})(\zeta)}{\zeta - z}dm(\zeta)$$

Exercise 7.9.7: