Exercise 3.4.16:

1. To see that the Cauchy-Riemann operator

$$\partial_{\bar{z}}\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

is elliptic we compute the characteristic polynomial

$$P(i\xi)=\frac{1}{2}i\left(\xi_1+i\xi_2\right)$$

Then the triangle inequality immediately implies that

$$|\xi|\leq|\xi_1|+|i\xi_2|=\frac{1}{2}|P(\xi)|$$

which shows that $\partial_{\bar{z}}$ is elliptic.

2. To see that $1/\pi z$ is a fundamental solution of $\partial_{\bar{z}}$ first note that because $\frac{1}{|z|}\in L^1_{\text{loc}}$ the function $f(z)=1/z$ defines a distribution on $\mathbb{C}\cong\mathbb{R}^2$. Moreover, by Theorem 2.9 we know that $\frac{1}{2\pi}\log|x|$ is a fundamental solution of Δ on $\mathbb{R}^2$. So for any $\varphi\in\mathcal{D}$ we see that

$$\frac{1}{2\pi}\int\log|x|\Delta\varphi dxdy=\delta(\varphi)=\varphi(0)$$

Because $\Delta=4\partial_{\bar{z}}\partial z$ we see that

$$\Delta(\frac{1}{2\pi}\log|x|)(\varphi)=4\partial_{\bar{z}}(\frac{1}{2\pi}\log|x|)(\partial_z\varphi)=4(\partial_z\frac{1}{2\pi}\log|x|)(\partial_{\bar{z}}\varphi)$$

But the above is equivalent to the statement that

$$(\frac{1}{\pi z})(\partial_{\bar{z}}\varphi)=\partial_{\bar{z}}\frac{1}{\pi z}(\varphi)=\delta(\varphi$$

which means that $1/\pi z$ is a fundamental solution of $\partial_{\bar{z}}$.

3. Suppose that f is continuous and $\partial_{\bar{z}}f=0$ in the sense of distributions and let C be a simple, closed curve in the plane. Because $\partial_{\bar{z}}$ is elliptic the distribution $U=\frac{1}{\pi z}*f$ satisfies $\partial_{\bar{z}}U=f$ by the previous part. We then apply Theorem 2.14 to the distribution U to see that U is in act a C^∞ function that satisfies

$$\partial_{\bar{z}}\int_C U\,dz=\partial_{\bar{z}}\int\frac{f(\zeta}{\zeta-z}dz=\int\frac{\partial_{\bar{z}}f(\zeta)}{\zeta-z}dz$$

Then because $\partial_{\bar{z}}f=0$ we have that

$$\int_C f dz=\int_C U\,dz=0$$

So f is analytic by Morera's Theorem.

Exercise 3.4.17:

1. To see that $\log|f(z)|$ is locally integrable we begin by recalling that because f is meromorphic it can be written as the quotient of two entire functions $f(z)=g(z)/h(z)$. Then we have the identity

$$\log|f(z)|=\log\left|\frac{g(z)}{h(z)}\right|=\log|g(z)|-\log|h(z)|$$

Thus, it suffices to show that $\log|g(z)|$ is locally integrable when g is entire.

In the case that g is an entire function then g admits a factorization

$$g(z) = z^m \prod_{i=1}^{\infty} E_n(z/a_n)$$

where the a_n are the roots of g and the E_n are the canonical factors

$$E_n(z) = \log(1-z)\sum_{i=1}^{\infty}\frac{z^k}{k}$$

Thus,

$$\int \log\left|z^m \prod_{n=1}^{\infty} E_n(z/a_n)\right| dz \leq \int m + \sum_{n=1}^{\infty} \log|E_n(z/a_k)|$$

So $\log|g(z)|$ is locally integrable if each of the $\log|E_n(z/a_n)|$ is. To verify this we expand

$$\log|E_n(z/a_n)| = \log|(1-z/a_k)e^{\sum_{i=1}^{\infty} a_k^{-i} z^i/i} \leq \log|1-z_k/a_k| + \sum_{i=1}^{\infty}\frac{z^i}{ia_k^i}$$

We know that $\log|w|$ is locally integrable, so we only need to consider the integrability of the sum term. Only finitely many of the series above satisfy $|z^i/ia_k^i| > 1/2$ because the otherwise the sequence of roots would have a limit point and the function would be identically zero, which is clearly integrable. So for sufficiently large k we have that

$$\int |E_n(z/a_k)| \leq \int \log|1-z_k/a_k| + \int \sum_{i=k}^{\infty} 2^{-n}$$

which is the sum of locally integrable functions and hence locally integrable. For large values of z/a_k we will use the analytic continuation of the usual power series representation for $\log(1+z)$ yielding

$$\log(1+z/a_n) + \sum \frac{z^k}{ka_n^k} = -\sum(-1)^n\frac{z^k}{ka_n^k} + \sum\frac{z^k}{ka_n^k} \leq \log|1+z/a_n|$$

Which shows that

$$\int |E_n(z/a_n)| \leq \int \log|1-z/a_n| < \infty$$

for small roots a_k and so each of the E_n is locally integrable and we are done.

2.

Exercise 3.4.18:

The forward direction follows from the homogeneity condition because we note that if we differentiate the identity $F_a(\varphi) = a^\lambda f(\varphi)$ with respect to a we get

$$\frac{\partial}{\partial a}F_a(\varphi) = \frac{\partial}{\partial a}a^\lambda f(\varphi)$$

$$F(\frac{\partial}{\partial a}\varphi^a) = \lambda a^{\lambda-1}F(\varphi)$$

$$F\left(\sum_{i=1}^{n} x_i\frac{\partial\varphi}{\partial x_i}\right) = \lambda a^{\lambda-1}F(\varphi)$$

Then we observe that Linearity reduces the left side to

$$F\left(\sum_{i=1}^{n} x_i \frac{\partial \varphi}{\partial x_i}\right) = \sum_{i=1}^{n} x_i F\left(\frac{\partial \varphi}{\partial x_i}\right) = \left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_j} F\right)(\varphi)$$

Setting $a = 1$ gives

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_j} F = \lambda F$$

as desired.

Exercise 3.4.22:

First we will assume that $f \in L^1_{\text{loc}}$ and define $u(x,t) = f(x-t)$

Exercise 3.4.25:

In the forward direction, we suppose that F is a positive distribution. Then F in particular is a positive linear functional on $\mathbb{R}^d$. We then see that F must in fact be a continuous function because each φ is supported on a compact set in Ω and so F is bounded hence, it is continuous. Then because $\mathcal{D}$ is dense in the space of continuous functions with compact support, C_0, we apply the Hahn-Banach theorem to extend it to a continuous function on C_c.

Now we need to show that the extension of F, call it $\tilde{F}$ is positive. Consider a test function φ convolved with a sequence of mollifiers η_n. If we set $\varphi_n = \varphi * \eta_n$ then each of the φ_n has compact support (because φ does). Because each each of the above have compact support we see that $\varphi_n \to \varphi$ uniformly and moreover that

$$\tilde{F}\varphi = \tilde{F}(\lim_{n\to\infty} \varphi_n) = \lim_{n\to\infty} \tilde{F}\varphi_n = \lim_{n\to\infty} F\varphi_n \geq 0$$

Where the second-to-last equality follows from density. So $\tilde{F}$ is positive and by the Riesz representation theorem we can find a locally finite Borel measure μ such that

$$\tilde{F}(\varphi) = \int \varphi(x) d\mu$$

And in particular we see that

$$F(\varphi) = \tilde{F}(\varphi) = \int \varphi(x) d\mu$$

as desired.

For the converse we simply note that a locally finite Borel measure μ is a continuous function in Ω because it is bounded. Then we use the fact that the test function φ is non-negative and so the distribution associated to the function μ is positive.

Exercise 3.4.27:

1.

2.

3.