Exercise 1.2.2:

Suppose that $X \subset \mathbb{R}^n$ is the union of convex open sets, $X_1, \dots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for every i, j, k. We will first establish that X is path-connected. To see this, we use the hypothesis that $X_i \cap X_j \cap X_k \neq \emptyset$ and note that this implies that the pairwise intersections $X_i \cap X_j \neq \emptyset$ because otherwise we would have $X_i \cap X_j \cap X_k \subset \emptyset$, but the former is non-empty so this is impossible. With this fact in hand, we choose any two points x, y in X and note that if $x \in X_i$ and $y \in X_j$ then there is a point z in $X_i \cap X_j$ so we can draw the line between x and z and then from z to y, both of which are contained in X because each of the X_j is convex. Hence, X is path-connected.

Now we need to compute the fundamental group of X. First we note that the intersection of two convex sets must be convex because any two points x, y in the intersection $X_i \cap X_j$ is in X_i and X_j and hence the line joining them is also in $X_i \cap X_j$ by convexity. We can then note that because convex sets deformation retract to a point via the nullhomotopy, their fundamental groups are trivial. We then apply the Van-Kampen theorem to see that $\pi_1(X) \cong *_{i,j,k}\pi_1(X_i \cap X_j \cap X_k) = *0 = 0$. So the fundamental group of X is trivial and X is simply connected.

Exercise 1.2.3:

As a base case, we refer to the fact that $\mathbb{R}^n - \{x\} \cong S^{n-1} \times \mathbb{R}$ (c.f. pg 35) and note that $\pi_1(\mathbb{R}^n - \{x\}) \cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R})$. We then use the fact that S^n is simply connected for $n \geq 2$ so that $\pi_1(\mathbb{R} - \{x\}) = 0$ when $n \geq 3$. We then proceed by induction on the number of removed points to prove the result. Indeed we note that

$$\mathbb{R}^n - \{x_1, \dots, x_n\} \cong (\mathbb{R}^n - \{x_1, \dots, x_{n-1}\}) \times (\mathbb{R}^n - \{x_n\}) / \{x_1, \dots, x_n\}^c$$

This quotient identifies all points that are not one of the x_i. Geometrically, it is a superposition of the two versions of $\mathbb{R}^n$, one with $n-1$ points removed, and the other with 1 point removed. Then we note that by the induction hypothesis $\pi_1(\mathbb{R}^n - \{x_1, \dots, x_{n-1}\} = 0$ and $\pi_1(\mathbb{R}^n - \{x_n\}) = 0$ as seen in the base case. Consequently, the inclusion map $\mathbb{R}^n - \{x_1, \dots, x_n\} \hookrightarrow (\mathbb{R}^n - \{x_1, \dots, x_{n-1}\}) \times (\mathbb{R}^n - \{x_n\})$ induces an injective homomorphism from $\pi_1(\mathbb{R}^n - \{x_1, \dots, x_n\} \to 0$. This implies that $\pi_1(\mathbb{R}^n - \{x_1, \dots, x_n\} = 0$ as desired.

Exercise 1.2.4:

Now we consider the space X which is $\bigcup_{j=1}^n \mathcal{L}_j$, where $\mathcal{L}_j$ is a line through the origin. First we note that $\mathbb{R}^3 - X$ deformation retracts onto $S^2 - \{\ell_1, \dots, \ell_{2n}$, where each of the ℓ_j is one of the points where the line $\mathcal{L}_j$ intersects S^2, via the straight line homotopy (note that the origin is deleted so we retain continuity). Then we note that $S^2 - \{x\} \cong \mathbb{R}^2$ via stereographic projection, and this induces a map from $S^2 - \{\ell_1, \dots, \ell_{2n-1}\} \hookrightarrow \mathbb{R}^2$. We then note that $\mathbb{R}^2 - \{\ell_1, \dots, \ell_{2n-1}\}$ deformation retracts onto the loops around each of the deleted points, which is $\bigvee_{i=1}^{2n-1} C_i$. This gives a string of isomorphisms

$$\pi_1(\mathbb{R}^3 - X) \cong \pi_1(S^2 - \{\ell_1, \dots, \ell_{2n}\}) \cong \pi_1(\mathbb{R}^2 - \{\ell_1, \dots, \ell_{2n-1}\}) \cong \pi_1(\bigvee_{i=1}^{2n-1} C_i) \cong *_{i=1}^{2n-1} \mathbb{Z}$$

Exercise 1.2.7:

Let X be the space $S^2/(n \sim s)$ where n, s are the north and sout poles, respectively. Following the hint, we will create a cell-structure for X. We begin with 0-cells that are simply the points n, s. We then attach a 1-cell as a path connecting n to s. Finally, we add a 2-cell along the path between n and s as well as its inverse path. Passing to the quotient space we are left with a cell structure for X that has the generating loop as the path from n to s, which are now identified. Also, the two cell is attached along the path and its inverse, so the presentation for the group is

$$\pi_1(X) \cong \langle a \mid aa^{-1} = 1 \rangle \cong \mathbb{Z}$$

So the fundamental group of X is $\mathbb{Z}$.

Exercise 1.2.8:

We begin with two separate tori $X = (S^1 \times S^1) \times (S^1 \times S^1)$. Let a, b and c, d be the generators for their respective fundamental groups. Because

$$\pi_1(X) \cong \pi_1(S^1 \times S^1) \times \pi_1(S^1 \times S^1)$$

we conclude that a and b must both commute with c and d, respectively. Then, under the identification $S^1 \times \{x_0\} \sim S^1 \times \{x_1\}$ where each is a subspace of the distinct factors of X, we note that this corresponds to identifying one of the generators. Without loss of generality, suppose that we identify b, c. Let Y denote this quotient space, then have a presentation for the group

$$\pi_1(Y) = \langle a, b, d \mid ad = da \rangle$$

Exercise 1.2.9:

First we want to show that M_h' does not retract onto its boundary circle, C, and that consequently M_g does not deformation retract onto C. Suppose to the contrary that there were such a retraction $r : M_h' \to C$ and let $i : C \hookrightarrow M_h'$ be the inclusion map. Because r is a retraction the induced homomorphism $i_* : \pi_1(C) \hookrightarrow \pi_1(M_h')$ is injective. Consider the abelianizations of $\pi_1(C)$ and $\pi_1(M_h')$ and the map

$$i_*^{ab} : \pi_1(C) \hookrightarrow \pi_1^{ab}(M_h')$$

This map is well defined because the homomorphism must send the commutator subgroup into the commutator subgroup. Also, because $\pi_1(C) \cong \mathbb{Z}$ is already abelian, taking the abelianization does not change it. Consider the identification of M_h' with a regular $4g$-gon in the usual way (c.f. pg 5). Then if we delete a point in the center of the polygon the loop around the boundary, which consists only of commutators, will be abelianized away. Thus, i_*^{ab} is the trivial map, which is impossible. Thus, there could be no such retraction.

Now we need to show that M_g does retract onto the non-separating circle C'. We see that C' can be identified with one of the generators of $\pi_1(M_g)$, say c. If we identity M_g with its $4g$-gon, then we can see that the polygon is homeomorphic to a rectangle with sides $a, b, a^{-1}, [M_g, M_g]cb^{-1}$, where a, b are generators for $\pi_1(M_g)$ and $[M_g, M_g]$ is the commutator subgroup of $\pi_1(M_g)$. The retract on this rectangle is the canonical projection onto the sides labeled a.

Exercise 1.2.10:

Consider the complement of the loops $\alpha \cup \beta$ as seen in the figure on pg 53, denote this space by X.. We note that the ambient space $D^2 \times I$ has the boundary disks $S^1 \times \{0\}$ and $S^1 \times \{1\}$ as contractible subspaces. We apply the contraction and note that the image of α and β is two interlocked loops rooted at the points that we contracted onto at the boundary. Then we note that the resulting figure, which is the union of two cones, is homotopy equivalent to S^2 with interlocking loops. It is shown in example 1.23 (c.f. pg 46) that the fundamental group of the resulting space Y is $\pi_1(Y) \cong \mathbb{Z} \times \mathbb{Z}$. Thus, the retractions would induce injective homomorphisms into $\pi_1(Y) \cong \mathbb{Z} \times \mathbb{Z} \hookrightarrow \pi_1(X)$. But because γ is a non-trivial loop in Y, it cannot be sent to a trivial loop in X. So γ is not nullhomotopic in X.

Exercise 1.2.17:

Let $X = \mathbb{R}^2 - \mathbb{Q}^2$. We will first show that X is path-connected, and so the computation of $\pi_1(X)$ is independent of the choice of basepoint. Consider $(x, y), (z, w) \in X$. By construction we know that one of x and y is irrational, and likewise for z and w. We proceed by case analysis. Suppose that both x and w are irrational then the line $t(x, y) + (1 - t)(x, w)$ is a traverses from $(x, y) \to (x, w)$. Then we traverse the line $t(x, w) + (1 - t)(z, w)$ is a valid path. If we see that z is irrational, then we can find some point (z, w') with w' irrational and traverse this line. We are then in the previous case, and construct the path as we did there. The case that y is irrational is analogous, and so X is path connected.

Let p be some irrational number and for each irrational ξ consider the strips $([-\xi, \xi] \times \{p\}), ([-\xi, \xi] \times \{-p\}), (\{p\} \times [-\xi, \xi]), (\{-p\} \times [-\xi, \xi])$. None of these lines contain any rational points, so they form a square in X. We then consider the basepoint $(0, p)$ and consider the loops that traverse each of the "squares" above in the usual orientation. We need to show that for distinct ξ, the loops are not homotopic. Let S_{ξ_1}, S_{ξ_2} be distinct squares above and consider the path which first traverses around S_{ξ_1} and then S_{ξ_2} in the reverse direction. We need to show that this is not the constant loop.

Because $\mathbb{Q}$ is dense in $\mathbb{R}$ we know that there is a rational r between ξ_1 and ξ_2 so the inclusion map $\mathbb{R}^2 - \mathbb{Q}^2 \hookrightarrow R^2 - (0, p)$, which induces a homomorphism $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2) \to \pi_1(R^2 - (0, p))$. The plane with a point removed is homotopic to S^1 via the straight line homotopy, but the image of the loop $S_{\xi_1} S_{\xi_2}^{-1}$ is a non-trivial loop in S^1. Hence, $S_{\xi_1} S_{\xi_2}^{-1}$ cannot be nullhomotopic in $\mathbb{R}^2 - \mathbb{Q}^2$ because otherwise it would be sent to the trivial element under the homomorphism, which we have just seen is not the case. As a result, there is a homotopy class in $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$ for each irrational, and therefore $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$ is uncountable.

Exercise A1: Let D denote the disk, A denote the annulus and M denote the Möbius band.

(a) First consider the case of the disk. We have seen that if the point removed from the disk is not on the boundary, then $\pi_1(X) = \mathbb{Z} * \mathbb{Z} \ncong 0 = \pi_1(D)$.

In the case of the annulus, we note that if the point removed is not on the boundary then the annulus deformation retracts onto the two generating loops, one about each hole. Hence, there is an isomorphism $\pi_1(A - \{x\}) \cong \mathbb{Z} * \mathbb{Z}$. Also, we see that A retracts onto the circle so that $\pi_1(A) \cong \mathbb{Z}$. However, the induced homomorphism would be injective from $\mathbb{Z} * \mathbb{Z} \hookrightarrow \mathbb{Z}$, which is impossible. However, if the deleted point is on the boundary then the resulting space still deformation retracts onto the inner circle and so the induced homomorphism is an isomorphism $\mathbb{Z} \to \mathbb{Z}$.

Now consider the Möbius band. If we delete a point that is not on the boundary, then M deformation retracts onto a figure-8 and so there should be an induced homomorphism $\pi_1(M - \{x\}) = \mathbb{Z} * \mathbb{Z} \to \mathbb{Z} = \pi_1(M)$, which is impossible. However, if the deleted point is on the boundary, then M deformation retracts onto the remaining boundary circle and so has $\pi_1(M - \{b\}) \cong \mathbb{Z}$, so the induced homomorphism is an isomorphism.

(b) Let $f : X \to Y$ be a homeomorphism, where X and Y are either both disks, annuli, or Möbius bands. Suppose that f does not restrict to a homeomorphism on the boundary so that $f_R = f\mid_{\partial X}$ is not a homeomorphism. Consider the spaces $X - \{x\}$ and $Y - \{f(x)\}$. We then apply the previous part to the induced isomorphism $\pi_1(X - \{x\}) \cong \pi_1(Y - \{f(x)\})$ such that $x \in \partial X$ and $f(x) \notin \partial Y$ to reach a contradiction. These groups are only isomorphic if both points are contained in the boundary. Thus f must restrict to a homeomorpshism on the boundary.

(c) Finally, to see that $M \not\cong A$ we suppose to the contrary that there were a homeomorphism $f : XtoY$. By the previous part, f would have to restrict to a homeomorphism on the boundary. However, this is impossible because if X is a Möbius band and Y is an annulus then Y deformation retracts onto its boundary circles. However, this implies that $f^1 \circ r$ is a deformation retract of M. However, we showed in last week's assignment that M does not deformation retract onto it's boundary circle. Thus, we have a contradiction and so $M \not\cong A$.

Exercise A2:

Suppose that $X \subset \mathbb{R}^n$ is an open, path-connected set and set $Y = X - \{p\}$ Let Z be a ball about the deleted point p that is contained in X. We can find such a neighborhood because x is assumed open. Observe that $X = Y \cup Z$ and note that the intersection $Y \cap Z$ is just Z, a ball centered at p with the center deleted. We know that when $n \geq 3$ Z is homeomorphic to $\mathbb{R}^{n-1}$, which is path-connected. Hence, the intersection $Y \cap Z \cong Z$ is path-connected and we can apply the van Kampen theorem to see that the map $\pi_1(Y) * \pi_1(Z) \to \pi_1(X)$ is surjective. However, the ball Z is contractible so $\pi_1(Z) = 0$ and so in fact have that $\pi_1(Y) \to \pi_1(X)$ is surjective. We then recall that the inclusion map also induces an injective homomorphism $\pi_1(Y) \to \pi_1(X)$ and so the homomorphism is in fact an isomorphism.

Exercise A3:

Let $X \subset \mathbb{R}^n$ be open and path-connected. Following the hint, we first show that any path in X with rational endpoints is homotopic to a path consisting of a finite number of line segments with rational endpoints. Indeed, let γ be a path in X with endpoints p and q, which have rational endpoints. Cover each point in γ with a ball contained in X, which is always possible because X is open. Because γ is the continuous image of the unit interval, it is comapct, and so only finitely many of these balls are needed to cover γ.

Let $B_1, B_2, \ldots, B_n$ be finite subcover and without loss of generality let $p \in B_1$ and $q \in B_n$. Arrange a sequence $B'_1 B'_2 B'_3 \cdots B'_m$ where each B'_j is one of the B_i such that $B'_{j-1} \cap B_j \neq 0$, $B'_1 = B_1$ and $B'_m = B_n$ (This sequence may contain duplicates, but will be finite). Because balls are convex, we can construct a path from p to q by connecting p to a rational point in the intersection $B'_1 \cap B'_2$ which is possible because the intersection is open and the rationals are dense. This gives us a finite path from p to q consisting only of line segements with rational endpoints.

To see the homotopy between γ and our piecewise linear path we proceed by induction on the number of balls in the subcover. If there were only one ball, then the ball is convex so the straight line homotopy will suffice. For the inductive step, suppose that we have a homotopy that we have n balls that cover γ. We know that there is a homotopy for the first $n - 1$ of them by the inductive hypothesis, and we need only connect them through the intersection of B_{n-1} and B_n. Because the sets are convex, a piecewise homotopy also works to connect the final two balls. (This is a shortened version of a previous exercise).

With the hint shown, we note that any path with irrational endpoints is homotopic to a path with rational endpoints because any ball about the endpoints will contain a rational point by density. Because the ball is path connected, the translation from the irrational point to the rational one does not change homotopy classes. Thus, each path in X is homotopic to a path indexed by $\bigcup_{n=1}^{\infty}(\mathbb{Q}^n \times \mathbb{Q}^n)$ which is the union over all finite length paths of all rational endpoint line segments. This is the countable union of countable sets. Hence there is an injection $\pi_1(X) \hookrightarrow \mathbb{Z}$ so $\pi_1(X)$ is countable.