Problem 1.

Let T be the torus $S^1 \times S^1$ and let T' be T with a small open disk removed. Let X be obtained from T by attaching two copies of T', identifying their boundary circles with the longitude and meridian circles $S^1 \times \{x_0\}$ and $\{x_0\} \times S^1$ in T. Find a presentation for $\pi_1(X)$.

Solution.

We will compute the fundamental group of X by applying van Kampen's theorem. We first begin by verifying the hypotheses of the theorem. Consider an open neighborhood of X, say U_X embedded in $\mathbb{R}^3$ chosen so that it deformation retracts onto X. Because deformation retraction is a homotopy equivalence, $\pi_1(U_X) \cong \pi_1(X)$. If we let X_1 be the embedding of T in U_X and X_2, X_3 be the respective embeddings of T', then the union of open neighborhoods about each of them will cover X. Each of the X_i is path-connected and the intersection of $X_i \cap X_j \simeq S^1$ for $i \neq j$, which is also path-connected. Because the intersection $X_1 \cap X_2 \cap X_3 = \{x_0\}$, where x_0 is the intersection of the boundary circles, the three way intersection is path connected as well. So we have a map

$$\Phi : \pi_1(X_1) * \pi_1(X_2) * \pi_1(X_3) \longrightarrow \pi_1(X)$$

that is surjective. Moreover, we see that $\pi_1(X) \cong *_i X_i / \ker(\Phi)$. So we are left to compute $\pi_(X_i)$ as well as impose relations for each element of $\ker(\Phi)$ and we will be done.

The fundamental group of $X_1 \simeq S^1 \times S^1$ is known, $\mathbb{Z} \times \mathbb{Z}$. To compute the fundamental group $\pi_1(T')$ we again consider the square with a small disk removed. If we look at this as a CW-complex we have one vertex and 5 edges as in the attached diagram (see attached).

We can then read off the relation on the fundamental group from the above by noting that it deformation retracts onto the boundary (via a straight line homotopy and read off the relation $\ell[e_1, e_2]$, where $[e_1, e_2] = e_1 e_2 e_1^{-1} e_2^{-1}$ is the commutator of e_1 and e_2. So we see that

$$\pi_1(T') \cong \{e_1, e_2, e_3 \mid e_3[e_1, e_2]\}$$

Now we note that the space X could be indentified as in the attached picture with the identifications $\ell_1 \simeq a$ and $\ell_2 \simeq b$. Looking at the space X_1, we get the relation $[a, b] = 1$. After the identifications and the observation that $\ell_n = [g_n, g_{n+1}]$ (by the relation in $\pi_1(T')$ this translates to the relations

$$[a, b] = aba^{-1}b^{-1} = [g_1, g_2][g_3, g_4][g_1, g_2]^{-1}[g_3, g_4]^{-1} = 1$$

Note, that these commutators are the elements in $\ker(\Phi)$ because they are all the elements of the form $i_{nm}(\omega) i_{mn}^{-1}(\omega)$, which is precisely $\ker(\Phi)$. Thus, we have the computation

$$\pi_1(X) \cong \{g_1, g_2, g_3, g_4 \mid [g_1, g_2][g_3, g_4][g_1, g_2]^{-1}[g_3, g_4]^{-1}\}$$

Problem 2.

Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpindicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a CW complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is th quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, of order eight.

Solution.

Consider the cube identified as in the attached diagram: We then do the identification via the "screw motion". and get the quotient space X, (in diagram).

To see the CW-complex structure we see there are two 1-cells, one for vertices v_0, v_2, v_5, v_7 another for the vertices v_1, v_3, v_4, v_6. The 2-cells would correspond to the opposite faces of the original cube (th three classes are shown in the attached diagram.. Lastly, the 3-cell would attach along each face and "fill" the space.

Now we will compute the fundamental group, $\pi_1(X)$. Let us look at the 2-cells, these give the relations $abcd, c^{-1}d^{-1}ba, adb^{-1}c^{-1}$ (all of these products are the identity, this will be used in the computations). In addition, we showed that attaching a k-cell for $k > 2$ does not affect the computation of $\pi_1(X)$, the the one 3-cell in the construction is irrelevant.

We need to find the isomorphism between $\pi_1(X)$ and Q_8. We have a familiar presentation for Q_8 (c.f. *Abstract Algebra* by Dummit and Foote) given by

$$Q_8 = \{x, y | x^4, y^{-2}x^2, xyxy^{-1}\}$$

We need to find how the relations above for π_1 are equivalent to these. We begin by finding the elements of order two. Note that

$$abab = adcb = adad$$

For the element of order 4, we compute

$$(ab)^4 = (cd)^{-1}(ab)^2(cd)^{-1} = (d^{-1}c^{-1})adcb(d^{-1}c^{-1}) = d^{-1}bac^{-1} = d^{-1}dcc^{-1}$$

Thus, (ab) has order 4 in $\pi_1(X)$. For the last relation, we will show the equivalent condition $(ad)(ab)(ad)^{-1} = (ab)^{-1}$. Indeed, observe that

$$(ad)(ab)(ad)^{-1} = (ad)abb^{-1}c^{-1} = adac^{-1} = cbac^{-1} = cd = (ab)^{-1}$$

By the relations, we also see that $(cd), (cb)$ are generated by $(ab), (ad)$ and so $(ab), (ad)$ are the desired generators for the presentation of Q_8. So the map $(ab) \mapsto x$, $(ad) \mapsto y$ is the desired isomorphism.

Problem 3.

1. *Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that both compositions gf and fh are homotopic to the identity.*
2. *Let X and Y be path-connected and locally path-connected, and let $\tilde{X} \to X$ and $\tilde{Y} \to Y$ be simply-connected covering spaces. Show that if $X \simeq Y$ then $\tilde{X} \simeq \tilde{Y}$.*

Solution.

1. Suppose that $f : X \to Y$ has both left and right inverses $g, h : Y \to X$, respectively. Because $fh \simeq \mathbb{1}_Y$ and $gf \simeq \mathbb{1}_X$, we see $g \simeq gfh$ and $h \simeq gfh$. Applying f to these relations we get that $fh \simeq \mathbb{1}_Y \simeq fg$. Thus, g is in fact a right-inverse to f so f is a homotopy equivalence.
2. Let $p_1 : \tilde{X} \to X$ and $p_2 : \tilde{Y} \to Y$ be the covering maps. We will construct a map $\tilde{f} : \tilde{X} \to \tilde{Y}$ that is a homotopy equivalence by precomposing the lifts of the homotopy equivalence f, g with their respective covering maps. More precisely, if we have a point $\tilde{\alpha} \in \tilde{X}$, there is a path from $\tilde{\alpha}$ to the desired basepoint in $\tilde{X}$, say $\tilde{x}$. We know that such a path must exist because

of Proposition 1.33. If we let $\tilde{ell}$ be the path, then we can apply p_1 to $\tilde{ell}$ and project down to a usual path in X from the basepoint x to $p\tilde{\alpha} = \alpha$, ℓ. We then apply f to map into y to get a path in Y that lifts $f\ell$. We define our desired map, $\tilde{f}$ as moving $\tilde{alpha}$ to the endpoint of this lifted path. Note that this construction says that $p_2(\tilde{f\ell}) \simeq f\ell$ and so $p_2\tilde{f} \simeq fp_1$. We define the proposed homotopy inverse for $\tilde{h}$, call it $\tilde{g}$ analogously. Namely, in such a way that $p_1(\tilde{g}) \simeq gp_2$.

We need to show that this is indeed a homotopy equivalence. If we consider the construction we have shown that $p_1\tilde{g}\tilde{f} \simeq gp_2\tilde{f} \simeq p_1$ and likewise $p_2\tilde{f}\tilde{g} \simeq fp_1\tilde{g} \simeq p_2$. These equivalences follow from the homotopy lifting property because we can simply restrict the homotopy of fg and gf to the respective identities, and by the uniqueness this lift must be $\tilde{f}\tilde{g}$ and $\tilde{g}\tilde{f}$, respectively. Thus, this is in fact a homotopy equivalence and we are done.

Problem 4.

1. *Using covering spaces, enumerate all the different subgroups of $\mathbb{Z} * \mathbb{Z}$ of index 3 and determine which of these are normal subgroups. Your answer should include some justification why this list is complete.*

2. *For p prime, find all the index p normal subgroups of $\mathbb{Z} * \mathbb{Z}$ and the corresponding covering spaces of $S^1 \vee S^1$.*

Solution.

1. Let $X = S^1 \vee S^1$ so $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$. We then apply Proposition 1.36 from the book to see that every subgroup of $G \leq \mathbb{Z} * \mathbb{Z}$ has a corresponding covering space $p : \tilde{X}_G \to X$. Because the subgroups we are looking for have index 3, we apply Theorem 1.38 to see that the corresponding covering spaces must be the connected 3-sheeted covering spaces of $S^1 \vee S^1$.

   We know that each covering space must be regular of degree 4 with in-degree 2 and out-degree 2, because the covering space must be locally homeomorphic to the base space. Moreover, because each edge must be the lift of a generating loop in the base space, we see that there must be one "in" edge and one "out" edge for each generator in the base space. In particular, this means that a self-loop in the covering space forces the other edges to go to a different vertex because otherwise the graph would be disconnected. As a result, we can characterize the graph by the number of vertices with self-loops. We consider each case:

   (a) If there are no self loops, then there is only one graph possible (up to isomorphism) because of the relations on the edges. We get the graph attached.

   (b) If there is one self loop then we can see that there is only one possible graph satsifying the restriction on the edges.

   (c) If there are two loops we also have only one possibility.

   (d) Finally, in the case that there are three self loops, there can only be one possible choice.

   We can see that only two of the covering spaces are regular: the one with no self-loops and the one with three self-loops. The intuition is clear: the other two do not have maximal symmetry. Phrasing this in the language of deck transformations, we see that there must be an automorphism of the covering space that takes a given vertex, say v_0, to another vertex, v_1. This is clearly impossible, because any graph automorphism would preserve self loops, so there is no autmorphism that could take a vertex with a self-loop to one without one. So all of the vertices must have the same number of self loops, either one or zero.

2. The critical fact is the subgroups we are looking for are normal. By the same reasoning as the previous part, we are looking for connected p-sheeted covering spaces of $S^1 \vee S^1$. By proposition 1.39 we see that normal covering spaces are in bijective correspondence with normal subroups. Moreover, we knkow that $G \cong \pi_1(X)/H$, where H is the given normal subgroup. Because $G \cong \mathbb{Z} * \mathbb{Z}$ this implies that there is a surjective homomorphism $\varphi : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. Because homomorphisms are determined by their values on the generators, we can specify all possible homomorphisms by considering where it would send the generating loops of the base space.

   Because φ is surjective, we see that it cannot send both of the generating loops to 0. There are then two cases to consider

   (a) If φ sends one generator to zero then it must send the other generator a non-zero element, which has order p. This leads to a covering space that has one p-cycle and every vertex has a self-loop, corresponding to the generator mapped to zero.

   (b) Otherwise φ sends both generators to elements of order p in $\mathbb{Z}/p\mathbb{Z}$. This leads to a graph that has two p cycles, one for each generator.

   This is all of the covering spaces, and therefore subgroups, up to isomorphism. To count them the first case gives us one possible covering space up to isomorphism: the graph with p self-loops. The second case only contributes one addition covering space (up to isomorphism) because we can simply relabel the vertices to have the two p-cycles map to one another (they are conjugates in $\mathbb{Z} * \mathbb{Z}$).

Problem 5.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R}^2 - \{0\}$.

1. *Show this action of $\mathbb{Z}$ on X is a covering space action.*
2. *Show that $X/\mathbb{Z}$ is not a Hausdorff space and describe how it is the union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$, two coming from the complementary components of the x-axis and two coming from the complimentary components of the y-axis.*
3. *Show that $\pi_1(X/\mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$ and construct a map $S^1 \times S^1 \to X/\mathbb{Z}$ that induces an isomorphism on π_1 and all higher homotopy groups π_n.*

Solution.

Let the action of $\mathbb{Z}$ on X be given by $n \cdot v = T^n(v)$.

1. We need to find a neighborhood about a point (x, y) such that $T^n(U) \cap T^m(U) = \emptyset$ when $m \neq n$. Consider first positive values of m and n and suppose that $m > n$. Then the radius r must satisfy the relation
$$|2^m(x - r) - 2^n(x + r)| > 0$$
   Solving for r we see that $r < (2^m - 2^n)x/(2^m + 2^n)$. Letting $m, n \to \infty$ sends r to zero, so the largest difference possible is when $n = 1, m = 2$ and we get $r < x/3$. For negative values of n, m we apply the same inequality in the y-coordinate to see that $r < y/3$. If n and m have different signs then the sets $T^m(U)$ and $T^n(U)$ are clearly disjoint by applying the inequality with $-n$ replaced for n. Thus, a ball of radius $\min\{x/3, y/3\}$ is the desired neighborhood.

2. First we will show that the orbit space $X / \mathbb{Z}$ is not Hausdorff. Choose two points $(x_0, y_0), (x_1, y_1)$ in X and two open neighborhoods U_0, U_1 about each point respectively. we must show that $U_1 \cap U_2 \neq \emptyset$ in the orbit space. Because U_0, U_1 are bounded we see that eventually $T^n(U_i)$ will intersect the x-axis for large enough values of N (because $y/2^n \to 0$ for any y as $n \to \infty$. The intersections will be intervals on the axis, both of which are bounded. Suppose that the first interval is $I_0 = \alpha_0 \pm \epsilon_0$ and the second is $I_1 = \alpha_1 \pm \epsilon_1$. We need to find a point that is in the orbit of both of these intervals. Suppose that I_0 is bounded by 2^{N_0} and I_1 is bounded by 2^{N_1} and that $N_0 \leq N_1$. Then a point in the intersection will exist if we can satisfy

$$2^{-N_0}(\alpha_0 + \epsilon_0) \geq 2^{-N_1}(\alpha_1 - \epsilon_1)$$

Or equivalently

$$\frac{\alpha_0 + \epsilon_0}{\alpha_1 - \epsilon_1} \geq 2^{N_0 - N_1}$$

This is clearly possible for m sufficiently large. Thus, any two neighborhoods have non-empty intersection in the quotient space. So the space $X / \mathbb{Z}$ is not Hausdorff.

Now let's see how $X / \mathbb{Z}$ is homeomorphic to the union of four cylinders. Consider the lines $y = \pm x$ in the plane. The orbit space of point in X lies on a unique hyperbola and we can index the hyperbola's by the coordinate that the intersect $y = x$. In a similar manner to the construction from the projective plane (the space of lines in the plane), we will construct the space $X / \mathbb{Z}$. Note that the analog of the stereographic projection is given by the x-coordinate of the intersection point with $y = x$ and the hyperbolic angle u. There are only 4 points at infinity in this construction, two for each axis, and as a result, each hyperbola in X wraps around to a loops in $X / \mathbb{Z}$. Thus, we see a cylinder for each complementary piece of the axis. if we let x_v be the x-coordinate of the point where the hyperbola containing v in $\mathbb{R}^2$ intersects $y = \pm x$ then there is a natural mapping $v \mapsto (x_v, u)$ in $X / \mathbb{Z}$.

3. Now we need to show that $\pi_1(X / \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$. We will do this via a map $\varphi : S^1 \times S^1 \to X / \mathbb{Z}$ that induces an isomorphism. Let the map be given by $(\theta, \psi) \mapsto (\cosh\theta, \sinh\psi)$. The induced map will map the generating circles in the torus onto the image of the deleted disk in X and then the second angle will measure which point on the circle corresponding to the image of a hyperbola under the quotient map. In particular, this map has trivial kernel and therefore this induces an isomorphism on pi_1. To get the isomorphism on the higher homotopy groups because it is a covering space projection and so we can apply Proposition 4.1 to get that it induces an isomorphism on the higher homotopy groups π_n as well.