Exercise 1.1.2:

Consider the change-of-basepoint homomorphism $\beta_h : f \mapsto [h \cdot f \cdot \bar{h}]$. Suppose that $h' \simeq h$. We need to show that for any f, $\beta_h(f) \simeq \beta_{h'}(f)$. Indeed, we can find a homotopy of paths $H(s,t) = g_t(s)$ from $\beta_h(f)$ to $\beta_{h'}(f)$ and define a new family of paths $\beta H(s,t) = g_t \cdot f \cdot \bar{g}_t$. We need to show that βH is a homotopy between $\beta_h(f)$ and $\beta_{h'}(f)$. It is clear from the construction that $\beta H(s,0) = \beta_h(f)$ and $\beta H(s,1) = \beta_{h'}(f)$. Moreover, continuity of βH follows from the continuity of H.

Exercise 1.1.3:

Let X be path-connected and abelian. Then for any basepoint-change homomorphisms $\beta_f : \pi_1(X, x_1) \to \pi_1(X, x_0)$ and β_k do not depend on the endpoints. If we choose maps g, h so that $[h \cdot \bar{g}] \in \pi_1(X, x_0)$ then for any path $[f]$ we have

$$\beta_f([h \cdot \bar{g}]) = [\bar{f} \cdot h \cdot \bar{g} \cdot f] = [\bar{g} \cdot f \cdot \bar{f} \cdot h] = [\bar{g} \cdot h]$$

Then for any path k from x_0 to x_1 we get

$$[\bar{g} \cdot h] = [\bar{g} \cdot k \cdot \bar{k} \cdot h] = [\bar{k} \cdot h \cdot \bar{g} \cdot k] = \beta_{\bar{k}}([h \cdot \bar{g}])$$

So $\beta_f = \beta_k$ as desired.

Conversely, suppose that every basepoint-change homomorphism depends only on the endpoints of h. Then for any two paths h, j with the same basepoint. Then decompose f into components $h_1 = h([0, 1/2])$ and $h_1 = h([1/2, 1])$. So by assumption we have

$$\beta_{h_1}(j) = \beta_{h_2}(j)$$
$$[h_1^{-1} \cdot j \cdot h_1] = [h_2 \cdot j \cdot h_2^{-1}]$$

We then multiply by h_1 on the left and $[h_2]$ on the right to get

$$[h_1] \cdot [h_1^{-1} \cdot j \cdot h_1] \cdot [h_2] = [h_1] \cdot [h_2 \cdot j \cdot h_2^{-1}] \cdot [h_2]$$
$$[j \cdot h_1 \cdot h_2] = [h_1 \cdot h_2 \cdot j]$$
$$[j] \cdot [h] = [h] \cdot [j]$$

So $\pi_1(X)$ is abelian.

Exercise 1.1.4:

Let $X \subset \mathbb{R}^n$ be locally star-shaped and let $\gamma : I \to X$ be a path in X. To each point $\gamma(t)$ we can find a star-shaped set S_t containing $\gamma(t)$ and so $\gamma(I) \subset \bigcup_{t \in I} S_t$. Because $\gamma(I)$ is the continuous image of a compact set, it is also compact and we can find a finite subcover $S_{t_0}, S_{t_2}, \ldots, S_{t_n}$. Moreover, we have that $S_{t_i} \cap \bigcup_{i \neq j} S_{t_j} \neq \varnothing$ because then the S_{t_i} would not cover $\gamma(I)$. We now construct a piecewise linear path from $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. Begin at $x_0 \in S_{t_0}$. Because S_{t_0} is star-shaped there is a point $c_0 \in S_{t_0}$ such that the line segment between x_0 and c_0 is contained in S_{t_0}. Now choose the smallest i such that $S_{t_0} \cap S_{t_i} \neq \varnothing$. Choose a point x_i in the intersection and a point c_i such that the line segment joining x_i and c_i is wholly contained in S_{t_i}.

Iterate this process until all of the S_{t_i} have been exhausted. At this point there are two cases: either $\gamma(1) \in S_{t_n}$ and we add the segments x_n to c_n and c_n to $\gamma(1)$ and we are done, or $\gamma(1) \notin S_{t_{n-1}}$. In this case we have to do some backtracking. We retrace the path until we reach the star that contains $\gamma(1)$ and then do the same steps as in the previous case. So in the end we have a piecewise linear path from $\gamma(0)$ to $\gamma(1)$.

Now we need to verify that the piecewise linear path is homotopic to $\gamma(I)$. Consider the restriction of γ to S_{t_i}. Now we need to construct the valid homotopy. We will construct the homotopy piecewise by looking at its behavior on an individual domain. There are two cases: Either S_{t_i} two segments within it, or it has three due to the"backward" segements out of the center.

In the former case suppose that S_{t_i} and S_{t_j} intersect and the linear path goes through the intersection. Now find the smallest t such that $\gamma(t) \in S_{t_i} \cap S_{t_j}$ and the smallest t such that $\gamma(t) \in S_{t_j} - S_{t_i}$; call them t_i, t_j respectively. Then we construct the homotopy of paths by first composing the straight line homotopy to the center point c_i for each $\gamma(t)$ with $t \leqslant t_i$ and then send each point $\gamma(t)$ from c_i to $(1-t)x_i + tc_i$. Do the same for $t \geqslant t_j$. For the points of γ in the intersection we define a path p_i from $H_s(t_i)$ to $H_s(t_j)$ (the locations of $\gamma(t_i)$ and $\gamma(t_j)$ at time s in the homotopy) that is contained in the stars. For each t in $[t_i, t_j]$ we send $\gamma(t)$ to $p(1/(t_j - t_i)$. Becuase this path is continuous and agrees with H_t on the boundaries, the concatenation of this path with H_{t_i} and H_{t_j} is also continuous. So this is a valid homotopy on these stars.

In the second case we have that there is a "backward" line in S_{t_i}. This modifies the construction only slightly. Instead of sending all of the $\gamma(t)$ to a single segment, we must divide them in half and send half of the $\gamma(t)$ to one segment, and the other half to the other segment. The fact that this is continuous follows from the fact that both γ and the linear path are continuous at c_0 which is the boundary point for this operation. Other than this, the construction is identical.

Hence, every path in X is homotopic to a piecewise linear path in X.

Exercise 1.1.5:

We need to show that the following are equivalent:

(a) Every map $S^1 \to X$ is homotopic to the constant map, with image a point.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

We begin by showing that **(a)** $\Rightarrow$ **(b)**. Indeed, suppose that every map $f : S^1 \to X$ is homotopic to the constant map. Then there is a homotopy $H(t,s)$ connecting f to 1. Now we consider the function

$$\begin{aligned} F : D^2 &\to X \\ (r, \theta) &\mapsto H(r, \theta/2\pi) \end{aligned}$$

To see that F is continuous note that $F = g \circ H$ where g divides the second coordinate by 2π. g is linear and hence continuous so F is the composition of two continuous functions and is continuous as well. Moreover, $F|_{S^1} = f$ by the definition of a homotopy.

For **(b)** $\Rightarrow$ **(c)** we simply do the reverse process of the above. Suppose that every map $f : S^1 \to X$ extends to a map $F : D^2 \to X$. We can create a homotopy as follows: Let $H_f(0,s) = x_0$ for $x_0 = F(0,0)$ and $H_f(1,s) = f(s)$ and then $H_f(t,s) = F(t, 2\pi\theta)$. The continuity of H_f follows immediately from the continuity of F. Now consider any two loops, f and g in X. We can create a homotopy between them by setting

$$C(t,s) = \begin{cases} H_f(1-2t, s) & t \leqslant 1/2 \\ H_g(2t-1, s) & t > 1/2 \end{cases}$$

The continuity is immediate by observing that $C(1/2, s) = x_0$. Thus, there is only one element in $\pi_1(X, x_0)$ for every x_0 in X.

Finally, we need to show that **(c)** $\Rightarrow$ **(a)**. Consider a loop $\varphi : I \to X$. We can identify each loop with a map $\phi : S^1 \to X$ by composing with the quotient map $p : I \to S^1$ that identifies $p(0) = p(1)$. This implies that the space of loops in X is the set of homotopy classes of maps $S^1 \to X$. Because $\pi_1(X, x_0) = 0$ we see that the space of maps $S^1 \to X$ must contain only one homotopy class and is therefore homotopic to the constant map.

Exercise 1.1.6:

Consider the set $[S^1, X]$ of homotopy classes of maps $S^1 \to X$. We can define the map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ by $[f, x_0] \to [f]$, so we ignore the basepoints. We must show the following

(a) If X is path-connected then Φ is surjective.

(b) If X is path-connected then $\Phi([f]) = \Phi([f])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

To see that Φ is surjective suppose that we have a path $[f] \in [S^1, X]$. Then for each $\theta \in [0, 1\pi)$ we have a path g_θ between $f(\theta)$ and x_0 because X is path-connected, and $g_\theta \cdot f \cdot \bar{g}_{\theta'}$ is a loop based at x_0. Moreover, $\Phi(g_\theta \cdot f \cdot \bar{g}_{\theta'}) = [f]$. So we need to take sure that this map is independent of the choice of representative for $[f]$. Suppose that $f, \tilde{f} \in [f]$ and that $F(t, s)$ is a homotopy connecting them. Then let $H(t, s) = g_t$ be the homotopy defined by the g_θ above, so that $H_t \cdot F_t \cdot \bar{H}_t$ (product taken in $\pi_1(X, x_0)$ is always a loop at x_0. This means that $\Phi(f)$ is homotopic to $\Phi(\tilde{f})$ and so Φ is well-defined on $[S^1, X]$ and we are done.

For **(b)** we begin with $\Rightarrow$. Suppose that $\Phi([f]) = Phi([g])$. So f is homotopic to g by a homotopy H, when looked at as paths. We need to verify that this is indeed a homotopy of loops. Suppose that $x_0 = f(\theta) = g(\theta')$ and let h be a path between x_0 and $f(\theta')$. Then $h \cdot f \cdot \bar{h}$ is a loop at x_0 that is loop homotopic to g.

Conversely, for $\Leftarrow$ we suppose that there is an $[h] \in \pi_1(X, x_0)$ such that $[h] \cdot [f] \cdot [\bar{h}] = [g]$. We need to show that f is homotopic to g as maps in $[S^1, X]$. We note that the loop h is homotopic as a map to the constant map via the homotopy $H : I \times I \to X$ given by $H(t, s) = h(ts)$. So then $[h] \cdot [f] \cdot [\bar{h}]$ is homotopic as a map to $[f]$ via the homotopy $H \cdot f \cdot \bar{H}$. So $\Phi([f]) = \Phi([g])$.

Exercise 1.1.10:

Let f be a loop in X based at x_0 and g be a loop in y based at y_0. Consider the concatenation of f and g in $X \times Y$ given by $h = [f \times \{y_0\}] \cdot [\{x_0\} \times g]$ traversed as f first, followed by g. Let h' be the loops traversed in the reverse order. Consider the new family of loops $\varphi_t : [-1, 2] \to X \times Y$ given by

$$\varphi_t(s) = \begin{cases} (x_0, g(t(s-1))) & s \in [-1, 0] \\ (f(s), g(t)) & s \in [0, 1] \\ (x_0, g((1-t)(s-2)+1)) & s \in [1, 2] \end{cases}$$

Geometrically, φ_t follows g from (x_0, y_0) to $(x_0, g(t))$, then traverses the loop $(f, g(t))$ in $X \times \{g(t)\}$ and then follows the remainder of g from $(x_0, g(t))$ back to (x_0, y_0).

To see that φ_t is a homotopy between h and h' we observe from the definition that $\varphi_0 = h$ and $\varphi_1 = g$. Moreover, the continuity of the homotopy follows from the continuity of f

and g as well as the fact that the pieces defining γ_t agree on the endpoints of the intervals. Hence, γ_t is a valid homotopy.

Exercise 1.1.13:

We begin with $\Rightarrow$. Suppose that the natural inclusion $\pi_1(A, x_0) \hookrightarrow \pi_1(X, x_0)$ is surjective. Let $f : I \to X$ be a path in X with endpoints in A. Then because A is path-connected we can find a path in A that connects $f(0)$ to x_0 and likewise for $f(1)$; denote these paths by p_0 and p_1, respectively. Then f is homotopic to a loop in $\pi_1(X, x_0)$ given by $p_0 \cdot f \cdot p_1$ via the homotopy concatenating $(p_0)_t \cdot f \cdot (p_1)_{1-t}$ where $(p_i)_t$ is the restriction of p_i to the interval $[t, 1]$. Because the inclusion map is surjective we can find a loop in $f_A \in \pi_1(A, x_0)$ that is homotopic to f. Now we create a path in A homotopic to f by considering $p_0 \cdot f_A \cdot p_1$, which is wholly contained in A.

$\Leftarrow$ is more straightforward. If every path in X with endpoints in A is homotopic to a path in A, then in particular every loop in X with endpoints in A is homotopic to a loop in A. This is exactly what it means for the inclusion map to be surjective.

Exercise 4.1.1:

Let $+'$ denote addition of maps in the k^{th} coordinate defined by

$$(f +' g)(x_1, \ldots, x_n) = \begin{cases} f(x_1, \ldots, 2x_k, \ldots, x_n) & x \leqslant 1/2 \\ g(x_1, \ldots, 2x_k - 1, \ldots, x_n) & x \geqslant 1/2 \end{cases}$$

We want to show that $f + g \simeq f +' g$. Indeed, consider the homotopy

$$H(\bar{x}, t) = \begin{cases} f(2x_1, x_2, \ldots, (1+t)x_k, \ldots, x_n) & x_1 \leqslant 1/2, x_k \leqslant 1/(1+t) \\ g(2x_1 - 1, \ldots, (1+t)x_k - t, \ldots, x_n) & x_1 \geqslant 1/2, x_k \geqslant t/(1+t) \end{cases}$$

So that $H(\bar{x}, 0) = (f + g)(\bar{x})$. Then we set

$$H(\bar{x}, t) = \begin{cases} f((2-t)x_1, x_2, \ldots, 2x_k, \ldots, x_n) & x_1 \leqslant 1/(2-t), x_k \leqslant 1/2 \\ g(2x_1 - 1, \ldots, (1+t)x_k - t, \ldots, x_n) & x_1 \geqslant (1-t)/2, x_k \geqslant 1/2 \end{cases}$$

The continuity of H and K is clear because they are continuous in each of their coordinates. Then we observe that by the construction $K(\bar{x}, 0) = H(\bar{x}, 1)$ and $K(\bar{x}, 1) = (f +' g)(\bar{x})$. Hence the composition of H and K is a homotopy connecting $f+g$ and $f+'g$ so $f+g \simeq f+'g$.

Exercise A1:

Let x_0 and x_1 be points in the same path component of a space X. Consider the set of paths in X from x_0 to x_1. Let $[f]$ be a fixed homotopy class of paths. We can turn f into a loop centered at x_0 by concatenating with a path $\bar{g}$ which moves from x_1 to x_0. Clearly, each homotopy class of $\bar{g}$ determines a homotopy class of loops in $\pi_1(X, x_0)$ because loops $f \cdot g'$ are not homotopic unless $g \simeq \bar{g}$ by definition. Moreover, this map will give all of $\pi_1(X, x_0)$ because X is path-connected so the endpoint change homomorphism is an isomorphism (just rename x_1 to the point you want), so the set of loops generated is independent of the endpoint. In addition we see that changing the homotopy class of f will not change the set of loops because $f \cdot g$ is homotopic to $f' \cdot f$ for some $[f'] = [g]$. So this means that these loops are in the same conjugacy class in $pi_1(X, x_0)$ (solve for the conjugacy class of f). We already have a map that is bijective from $[I, X]$ to conjugacy classes in $\pi_1(X, x_0)$ (Exercise 6) and so this completes the proof.