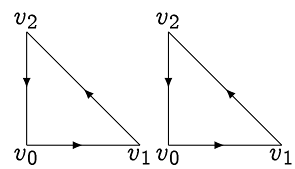
Exercise (2.1.1).

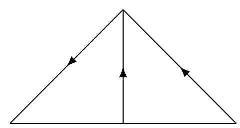
What familiar space is the quotient of a Δ-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1] \sim [v_0, v_1]$, preserving the ordering of the vertices?

Solution.

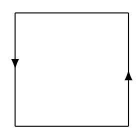
We can construct this space from two copies of the 2-simplex $[v_0, v_1, v_2]$, and then identifying the edges $[x_0, v_1], [v_1, v_2]$. We can construct this as follows:



We then rotate the latter and make the required identifications yielding the space



Unattaching if we break this up along the perpindicular and reattach preserving the edge identifications we get the space



From the last image it is clear that the resulting space X is the Möbius band.

Exercise (2.1.3).

Construct a Δ-complex structure on $\mathbb{R}P^n$ as a quotient of a Δ-complex structure on S^n having two vectors of length 1 along each coordinate axis in $\mathbb{R}^{n+1}$.

Solution.

The Δ-complex structure on S^n is given as follows:

1. For each basis vector e_j in $\mathbb{R}^{n+1}$ we create two vertices $v_{j,0}, v_{j,1}$ corresponding to $\pm e_j$, respsectively.

2. For each n-subset of the $\{v_{j,k}\}$ create an n-simplex and attach it along the set of vertices $\{v_{1,k_1}, \ldots, v_{n,k_n}\}$. Geometrically, this will add an n-simplex for each quadrant of $\mathbb{R}^{n+1}$.

3. At this point we have 2^{n+1} n-simplices, and the resulting space is clearly homeomorphic to S^n by just pushing out the faces.

Now, to get the resulting Δ-complex structure on $\mathbb{R}P^n$, we identify opposite points, so we set

$$(v_{1,j}, v_{2,j}, \ldots, v_{n,j}) \sim (v_{1,j'}, v_{2,j'}, \ldots, v_{n,j'})$$

with $j' = j + 1 \mod 2$. This space is clearly homeomorphic $\mathbb{R}P^n$ because we have identified opposite points in S^n.

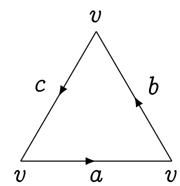
Exercise (2.1.4).

Compute the simplicial homology groups of the triangular patachute obtained from Δ^2

by identifying its three vertices to a single point.

Solution.

After identifying the vertices, we are left with the following Δ-complex:



As we can see, there is one 0-simplex, three 1-simplices, and one 2-simplex. This leads to the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

Where the boundary maps are given by

$$\begin{aligned} \partial_0(v) &= 0 \\ \partial_1(a) = \partial_1(b) = \partial_1(c) &= 0 \\ \partial_2(X) &= a + b - c \end{aligned}$$

So the homology groups become

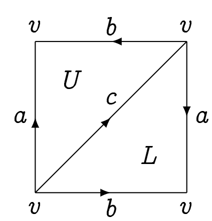
$$\begin{aligned} H_0^\Delta(X) &= \ker \partial_0 / \text{im } \partial_1 = \mathbb{Z} / 0\,\mathbb{Z} \\ H_1^\Delta(X) &= \ker \partial_1 / \text{im } \partial_0 = (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) / \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z} \\ H_2^\Delta(X) &= \ker \partial_2 / \text{im } \partial_1 = 0/0 = 0 \\ H_n^\Delta(X) &= 0, n > 2 \end{aligned}$$

Exercise (2.1.5).

Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.

Solution.

Recall the Δ-complex structure on the Klein bottle, K, is given by:



So from the above it is clear that we have one 0-simplex, three 1-simplices, and two 2-simplices. So we have the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

We then compute the boundary maps:

$$\begin{aligned} \partial_0(v) &= 0 \\ \partial_1(a) = \partial_1(b) = \partial_1(c) &= 0 \\ \partial_2(U) &= a+b-c \\ \partial_2(L) &= a-b+c \end{aligned}$$

Thus, we see that

$$\text{im } \partial_2 \cong \{a+b-c, a-b+c\} \cong \{2a, a+b-c\} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$$

Using this we see that the homology groups are

$$H_n^{\Delta}(X) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} & n=1 \\ 0 & n \geq 2 \end{cases}$$

Exercise (2.1.9).

Compute the homology groups of the Δ-complex X obtained from Δ^n by identifying all faces of the same dimension. Thus, X has a single k-simplex for each $k \leq n$.

Solution.

We will show that

$$H_n^{\Delta}(X) = \begin{cases} \mathbb{Z} & n = 0, n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Indeed, because there is a unique k-simplex for each $k \leq n$ we have the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_n} \mathbb{Z} \xrightarrow{\partial_{n-1}} \dots \longrightarrow \mathbb{Z} \xrightarrow{\partial_0} 0$$

So we only need to compute the boundary maps. Let σ_k be the unique k-simplex in X then

$$\partial_k(\sigma_k) = \sum_{i=0}^{n} (-1)^i \sigma_{k-1} = \begin{cases} \sigma_{k-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

So that $\ker \partial_k$ is trivial when k is even and is equal to all of $\mathbb{Z}$ when k is odd. Likewise, we see that im $\partial_k \cong \mathbb{Z}$ when k is even and 0 when k is odd.Therefore, we can compute the homology groups as

$$H_k^{\Delta}(X) = \ker \partial_k / \text{im } \partial_{k+1} = \begin{cases} \mathbb{Z} & k=0 \\ 0 & k \leq n \text{ even} \\ 0 & k < n \text{ odd} \\ \mathbb{Z} & k = n \text{ odd} \end{cases}$$

Exercise (2.1.11).

Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Solution.

Because A is a retract of X we have that there is a map $r : X \to X$ such that $r|_A = 1_A$. So we have that maps

$$A \xrightarrow{i} X \xrightarrow{r} A$$

as well as the induced homomorphisms of the homology groups

$$H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{r_*} H_n(A)$$

Then we note that because $(ri)_* = r_* i_*$ that $(ri)_* = (1_{H_n(A)})_*$. The identity function is clearly an isomorphism and so i_* must be injective.

Exercise (2.1.12).

Show that a homotopy of chain maps is an equivalence relation.

Solution.

We will say that $f \sim g$ if there is a chain homotopy between them. We will first show that the relation $\sim$ is reflexive. Indeed, let F be the sequence of trivial homomorphisms (all 0 maps) then we see that for a map $f : X \to Y$ we have

$$f_\# - f_\# = \partial 0 + 0\partial = \partial F + F\partial$$

and so $f \sim f$. Next, suppose that $f \sim g$, we need to show that $g \sim f$. By assumption we have a chain homotopy F between f and g and so $\partial F + F\partial = f_\# - g_\#$. To compute the other direction we multiply this identity by -1 yielding

$$g_\# - f_\# = -\partial F + (-F)\partial = \partial(-F) + (-F)\partial$$

and so $-F$ is the desired chain homotopy between f and g. Finally, wee need to show transitivity. Suppose that $f \sim g$ and $g \sim h$, so that there are chain homotopies F_0, F_1 such that $f_\# - g_\# = \partial F_0 + F_0\partial$ and $g_\# - h_\# = \partial F_1 + F_1\partial$, respectively. Then we can compute

$$\begin{aligned} f_\# - h_\# &= f_\# - g_\# + g_\# - h_\# \\ &= \partial F_0 + F_0\partial + \partial F_1 + F_1\partial \\ &= \partial(F_0 + F_1) + F_0\partial + F_1\partial \\ &= \partial(F_0 + F_1) + (F_0 + F_1)\partial \end{aligned}$$

Where the third equality follows because ∂ is a homomorphism, and the fourth equality follows from the definition of $F_0 + F_1$. Thus, $F_0 + F_1$ is the desired chain homotopy, verifying that $\sim$ is an equivalence relation.

Exercise (2.1.15).

For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and the map $D \to E$ is injective. Hence, for a pair of space (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.

Solution.

First we will give each of the maps a name. Suppose that

$$A \xrightarrow{\alpha_0} B \xrightarrow{\alpha_1} C \xrightarrow{\alpha_2} D \xrightarrow{\alpha_3} E$$

is our exact sequence.In the forward direction we suppose that $C = 0$. Then we decompose the exact sequence above into two short eact sequences

$$A \xrightarrow{\alpha_0} B \xrightarrow{\alpha_1} 0$$

$$0 \xrightarrow{\alpha_2} D \xrightarrow{\alpha_3} E$$

The first says that α_0 must be surjective and the latter says that $D \to E$ is injective (c.f. page 114) as desired.

In the reverse direction we suppose that the sequence is exact and that $\alpha_0 : A \to B$ and $\alpha_3 : D \to E$ are surjective and injective, respectively. Because α_3 is an injective homomorphism, it must have trivial kernel. Moreover, because the sequence is exact we have that $\ker \alpha_3 = \text{im } \alpha_2 = 0$. Because the image of α_2 is trivial, it must be the trivial homomorphism and so $\ker \alpha_2 = C$. Now we note that because α_0 is surjective and $\text{im } \alpha_0 = \ker \alpha_1 = B$. Because $\ker \alpha_1 = B$ we have that $\text{im } \alpha_1 = 0$. Combining these two facts gives that $C = 0$.

Thus, for a pair of spaces (X, A) there long exact sequence of homology groups

$$\cdots H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \longrightarrow \cdots$$

Now because the inclusion maps are always injective, if i_* is an isomorphism i.e. surjective then we apply the above to see that $H_n(X, A) = 0$ for all n, and conversely.

Exercise (2.1.16)**.**

1. *Show that $H_0(X, A) = 0$ iff A meets each path-component of X.*
2. *Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.*

Solution.

1. We begin by showing that there is only one homology class in $H_0(X)$ for each path-component of X. Indeed, suppose that x_0, x_1 are points in the same path-component of X. Then we can find a path α whose endpoints are x_0, x_1, respectively, and so α is a singular 1-simplex such that $\partial\alpha = x_1 - x_0$. Hence, x_0 and x_1 are homologous in $H_0(X)$.

   Proceeding with the forward direction of the problem, suppose that $H_0(X, A) = 0$. If we look at the long exact sequence of the pair we see

   $$\cdots \longrightarrow H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \longrightarrow 0$$

   Our assumption that $H_0(X, A) = 0$ along with exactness of the sequence implies that i_* is surjective. Thus the inclusion $A \to X$ induces an isomorphism on H_0 and by the previous discussion, A must intersect each path-component of X.

   Conversely, suppose that A intersects each path-component of X. For each path component of X we can find an $x \in X$ that is homologous in $H_0(X)$ to an $a \in A$. Thus the induced map $i_*(a) = [a] = [x]$ where $[x]$ is the homology class of X in $H_0(X)$. So i_* is surjective. Then if we look at

   $$H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \longrightarrow 0$$

   Then exactness of the short exact sequence above (a subsequence of the long exact sequence of the pair) combined with the surjectivity of i_* imply that $\ker j_* = H_0(X)$, and consequently im $j_* = 0$. Using exactness again on the zero map at the end of the sequence gives that

$$\ker 0 = H_0(X, A) = 0 = \text{im } j_*$$

as desired.

2. We proceed in a similar manner as the previous part. If we look at the long exact sequence of the pair we get

$$\cdots \longrightarrow H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\partial} H_0(A) \longrightarrow \cdots$$

If we assume that $H_1(X, A) = 0$ then we see that $\ker j_* = H_1(X)$. By exactness this says that im $i_* = H_1(X)$ and so i_* is injective. Moreover, we still have that

$$H_1(X, A) = 0 \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \longrightarrow \cdots$$

Which says that i_* is injective. Applying the claim at the beginning of the previous part, this says that distict path components of A map into distinct path components in X. Thus, each path component in X contains at most one path-component of A.

In the reverse direction, suppose that $i_* : H_1(A) \to H_1(X)$ is surjective and that each path component of X contains at most path-component of A. We again consider the segment of the sequence

$$H_1(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \longrightarrow \cdots$$

As before, we apply the claim at the beginning of the previous part to conclude that i_* is injective and so $\ker i_* = 0$. Using exactness of the long exact sequence we see that im $\partial = 0$. We assumed that $i_* : H_1(A) \to H_1(X)$ was surjective and so im $i_* = H_1(X)$ and by exactness we see that $\ker j_* = H_1(X)$. Because j_* is a homomorphism we see that im $j_* = 0$. We then compile these facts to see that im $\partial = \ker \partial = 0$ and so $H_1(X, A) = 0$.

Exercise (2.1.18).

Show that for the subspace $\mathbb{Q} \subset \mathbb{R}$ the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian and find a basis.

Solution.

Because $\mathbb{Q} \subset \mathbb{R}$ we can find a long exact sequence in the homology groups

$$\cdots \to H_n(\mathbb{Q}) \xrightarrow{i_*} H_n(\mathbb{R}) \xrightarrow{j_*} H_n(\mathbb{R}, \mathbb{Q}) \xrightarrow{\partial} H_{n-1}(\mathbb{Q}) \xrightarrow{i_*} H_{n-1}(\mathbb{Q}) \to \cdots$$

Restricting this sequence and terminating with 0 we get a short exact sequence

$$0 \longrightarrow H_1(\mathbb{R}, \mathbb{Q}) \xrightarrow{\partial} H_0(\mathbb{Q}) \xrightarrow{i_*} H_0(\mathbb{R}) \longrightarrow 0$$

Now we apply Proposition 2.6 to compute the homology groups $H_0(\mathbb{Q})$ and $H_0(\mathbb{R})$ by making a generator for each path component. Consequently, we have

$$H_0(\mathbb{Q}) = \bigoplus_{q \in \mathbb{Q}} \mathbb{Z}, H_0(\mathbb{R}) = \mathbb{Z}$$

So the above resolves to the exact sequence

$$0 \longrightarrow H_1(\mathbb{R}, \mathbb{Q}) \xrightarrow{\partial} \bigoplus_{q \in \mathbb{Q}} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \longrightarrow 0$$

Now we apply exactness of the sequence to conclude that $\ker \partial = 0$ and ∂ is injective. Thus, $H_1(\mathbb{R}, \mathbb{Q})$ is a subgroup of a free group and therefore, free.

Now we need to compute a basis. The elements of $C_1(\mathbb{R}, \mathbb{Q})$ consists of cycles in $\mathbb{R}$ whose boundaries lie in $\mathbb{Q}$. Therefore, the boundary map sends them to some interval $[r_1, r_2]$ with $r_2 - r_1 \in \mathbb{Q}$ or to some point p (i.e. $r_1 = r_2$). The boundaries are such that if we have a map $\sigma \in C_2(\mathbb{R}, \mathbb{Q})$ then its boundary is some cycle $\varphi = \partial_2 \sigma$. So if $\varphi \neq 0$ then we must have that $\partial\sigma : \partial\Delta^2 \to \mathbb{R}$ maps to an an interval $[r_1, r_2]$ such that $r_2 - r_1 \in \mathbb{Q}$ or it maps to a point that is not in $\mathbb{Q}$.

By the above discussion we see that im ∂_2 differs from $\ker \partial_1$ only on maps $\varphi : \Delta^1 \to \mathbb{R}$ that map to a point. Thus, we see that

$$H_1(\mathbb{R}, \mathbb{Q}) = \{\varphi : \Delta^1 \to \mathbb{R} \mid \text{im } \varphi \in \mathbb{Q}\} \cong \mathbb{Q}$$

Where the last identification just identifies a $q \in \mathbb{Q}$ with the singular simplex φ that maps onto $\mathbb{Q}$. Thus, any basis for $\mathbb{Q}$ as an additive group will be a basis for $H_1(\mathbb{R}, \mathbb{Q})$ for example, the elements $1, 1/2, 1/3, \ldots$.

Exercise (2.1.22).

Prove by induction on dimension the following facts about the homology of a finite dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n spheres:

1. *If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.*
2. *$H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.*
3. *If X has k n-cells, then $H_n(X)$ is generated by at most k elements.*

Solution.

1. Following the hint, we proceed by induction. For the case $n = 0$ note that if there are k 0-cells in X then $H_0(X) = \bigoplus_{i=1}^{k} \mathbb{Z}$, the free abelian group of rank k. Also, we see that $H_k(X) = 0$ for $k > 0$ because $C_k(X) = 0$ for $k > 0$.

   For the inductive step suppose that any $(n-1)$-dimensional CW-complex A satisfies the proposition, so $H_k(A) = 0$ for $k > n-1$ and $H_{n-1}(A)$ is free. For the first part of the claim we observe that there are no k-cells for $k > n$ and so $C_k(X) = 0$ for $k > n$ and as a result $H_k(X) = 0$. For the second part of the claim we look at the pair (X, A) for an $(n-1)$-dimensional subcomplex A. There is a long exact sequence of the pair

   $$\cdots \to H_{n+1}(A) \xrightarrow{i_*} H_{n+1}(X) \xrightarrow{j_*} H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \to \cdots$$

   Then we note that the space X/A is a wedge of n-spheres and therefore has a Δ-complex structure on it and is a good pair. Thus, we can apply Proposition 2.22 to get that $H_n(X, A) \cong H_n(X/A)$. This fact along with the inductive hypothesis yields the exact sequence

   $$0 \longrightarrow H_{n+1}(X/A) \longrightarrow 0 \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X/A) \xrightarrow{\partial} F$$

Where F is a free group. Now we use exactness, noting that im $i_* = \ker j_* = 0$. So we can use the first isomorphism theorem to see that

$$\text{im } j_* \cong H_n(X)/\ker j_* = H_n(X)$$

Now we use exactness again to conclude that im $j_* \cong \ker \partial$ and so $H_n(X) \cong \ker \partial$. However, we note that because $\ker \partial$ is the kernel of a homomorphism it must be isomorphic to a subgroup of F, a free group. Because subgroups of free groups of free (Theorem 1A.4) we see that $H_n(X)$ must be free as claimed. This completes the induction.

2. Suppose that there are no cells in X of dimension $(n-1)$ or $(n+1)$. Moreover, suppose X has exactly k n-cells so that the chain complex

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_n(X) \xrightarrow{\partial_n} \cdots$$

resolves to

$$\cdots \longrightarrow 0 \xrightarrow{\partial_{n+1}} \bigoplus_{i=0}^{k} \mathbb{Z} \xrightarrow{\partial_n} 0 \xrightarrow{\partial_n} \cdots$$

because there are no generators for C_{n+1} and C_{n-1} but k generators for C_n. Then we see that im $\partial_{n+1} = 0$ and $\ker \partial_n = \bigoplus_k \mathbb{Z}$. So $H_n(X) = \bigoplus_k \mathbb{Z}$ is free with k basis elements and so they can be put in bijective correspondence with the k n-cells that make up the n-skeleton of X.

3. Again we will proceed by induction. For the base case $n = 0$ suppose that X has k 0-cells. Then we see that $H_0(X) \cong \bigoplus_k \mathbb{Z}$ because there are k components. For the inductive step suppose that $H_i(X)$ is generated by at most k elements if there are k i-cells in X. By the discussion in the first part of this problem we have a long exact sequence

$$\cdots \to H_n(X^{n-1}) \xrightarrow{i_*} H_n(X^n) \xrightarrow{j_*} H_n(X^n/X^{n-1}) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

where X^n is the n-skeleton of X. Applying the first part of this exercise and the inductive hypothesis, we get an exact sequence

$$\cdots \to 0 \xrightarrow{i_*} H_n(X^n) \xrightarrow{j_*} \bigoplus_k \mathbb{Z} \longrightarrow \cdots$$

We know that im $i_* = 0$ and so $H_n \cong \ker j_*$. Then notice that $\ker j_*$ is a subgroup of a free group with k generators. Thus, it is free with at most k generators. Because X is n-dimensional we see that $H_n(X) \cong H_n(X^n) \cong \bigoplus_\ell \mathbb{Z}$ where $\ell \leq k$ as desired.

Exercise (2.1.26).

Show that $H_1(X, A)$ is not isomorphic to $\tilde{H}_1(X/A)$ if $X = [0, 1]$ and A is the sequence $1, 1/2, 1/3, \ldots$ together with its limit 0.

Solution.

First we will show that the relative homology group $H_1(X, A)$ is countable. Indeed, consider the long exact sequence of homology groups given by the inclusion:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$

Restricting this sequence to a short exact sequence terminating in zeros we get a short exact sequence like

$$0 \longrightarrow H_1(X,A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \longrightarrow 0$$

We then apply Proposition 2.6 to see that $H_0(A) = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$ and $H_0(X) = \mathbb{Z}$. Consequently, the above becomes the sequence

$$0 \longrightarrow H_1(X,A) \xrightarrow{\partial} \bigoplus_{n\in\mathbb{N}} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \longrightarrow 0$$

Then exactness implies that ∂ is injective and so

$$H_1(X,A) \leq \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$$

In particular, this implies that $H_1(X,A)$ is countable.

Now we will compute the reduced homology group $\tilde{H}_1(X/A)$. We see that the space X/A is the familiar Hawaiian earring. Following the notation in Example 1.25 we let C_n be the circle of radius $1/n$ and center $(0,1/n)$. Each of the retractions $r_n : X/A \to C_n$ induces a homomorphism on the homology $(r_n)_* : \tilde{H}_1(X/A) \to \tilde{H}_1(C_n)$. Now recall that the circle C_n only has one path-component and therefore $\tilde{H}_1(C_n) \cong \mathbb{Z}$. Now we will see that this homomorphism is surjective. If we consider a singular 1-simplex $\sigma : \Delta^1 \to X/A$, then it is non-zero because $\sigma(0) \neq \sigma(1)$ $(\sigma(0) = 1/n, \sigma(1) = 1/(n+1)$, so it must be mapped to a generator of $\tilde{H}_1(C_n)$.

Now consider the product of each of the $(r_n)_*$ which is given by $\rho : \tilde{H}_1(X/A) \to \prod_{n=1}^{\infty} \mathbb{Z}$. This is a direct product and not a direct sum because every loop in X/A corresponds to a distinct sequence of integers. This map must be surjective because because there is a singular 1-simplex $\sigma_k : \Delta^1 \to C_n$ any given number of times. So

$$\prod_{i=1}^{\infty} \mathbb{Z} \leq \tilde{H}_1(X/A)$$

Because $\prod_{i=1}^{\infty} \mathbb{Z}$ is uncountable, we see that $\tilde{H}_1(X/A)$ is uncountable. Thus, we see that $H_1(X,A) \ncong \tilde{H}_1(X/A)$.

Exercise (2.1.29).

Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Solution.

We know that the homology groups of $S^1 \times S^1$ are

$$H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0, 2 \\ 0 & n \geq 3 \end{cases}$$

Now, we apply Corollary 2.25 to see that the inclusions $S_1 \to (S^1 \vee S^1 \vee S^2)$ induce an isomorphism

$$\varphi : H_n(S^1) \oplus H_n(S^1) \oplus H_n(S^2) \longrightarrow H_n(S^1 \vee S^1 \vee S^2)$$

Then we use the fact that

$$H_k(S^n) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & n = k = 0 \\ \mathbb{Z} & k = 0 \\ 0 & 0 < k < n \\ \mathbb{Z} & k = n \end{cases}$$

So that

$$H_k(S^1 \vee S^1 \vee S^2) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & k = 0 \\ \mathbb{Z} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

Now we compute the homology of the universal covering spaces. We know that the universal cover of $S^1 \times S^1$ is $\mathbb{R}\times\mathbb{R}$. And in particular we see that $H_2(\mathbb{R}\times\mathbb{R}) = 0$ because $\mathbb{R}$ is contractible. Suppose that $p : \tilde{X} \to S^1 \vee S^1 \vee S^2$ is the universal cover and consider the inclusion $i : S^2 \to \tilde{X}$. The inclusion must factor through the universal cover. Then the induced map $i_* : H_2(S^2) \to H_2(S^1 \vee S^1 \vee S^2)$ is injective and factors through a map in the homology group of the universal cover leading to the diagram

$$\begin{tikzcd} & H_2(\tilde{X}) \arrow[d] \\ H_2(S^2) \cong \mathbb{Z} \arrow[ur] \arrow[r, "i_*"] & H_2(S^1 \vee S^1 \vee S^2) \end{tikzcd}$$

Therefore $H_2(\tilde{X})$ is non-zero and we are done.

Exercise (2.1.30).

In each of the following commutative diagrams assume that all maps but one are isomorphisms. Show that the remaining map must be an isomorphism as well.

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \\ & C \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C \arrow[r] & D \arrow[u] \end{tikzcd}$$

Solution.

For the first diagram, we give the functions names to make this more clear. We have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"] & B \\ & C \arrow[u, "h"] \end{tikzcd}$$

Commutativity implies that $f = hg$. There are three cases

1. If h, g are isomorphisms, then the commutativity of the diagram forces f to be an isomorphism because the composition of isomorphisms is an isomorphism.

2. If f, h are isomorphisms then we apply h^{-1} to both sides to get the equality $g = h^{-1}f$. Thus, g is a composition of isomorphisms and therefore and isomorphisms.

3. In the final case we have that f, g are isomorphisms. Then we precompose both sides with g^{-1} yielding $h = fg^{-1}$ and so h is an isomorphism.

For the second diagram we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"] & D \end{tikzcd}$$

Commutativity gives the equation $g'f = f'g$. Much in the same manner as the last problem we will compose with the correct inverse in order to obtain the excluded map as a composition of three isomorphisms, and thus, it must be an insomorphism as well.

1. If we are looking for f, we note that $f = (g')^{-1}f'g$.
2. For g we see that $g = (f')^{-1}g'f$.
3. Likewise for f' we see that $f' = g'fg^{-1}$.
4. Finally for g' we see that $g' = f'gf^{-1}$.

For the last diagram, we do exactly the same thing. We start with

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C \arrow[r, "f'"] & D \arrow[u, "g'"'] \end{tikzcd}$$

Yielding the equation $(g')^{-1}f = f'g$. So we have

1. If we are looking for f, we note that $f = g'f'g$.
2. For g we see that $g = (f')^{-1}(g')^{-1}f$.
3. Likewise for f' we see that $f' = (g')^{-1}fg^{-1}$.
4. Finally for g' we see that $(g')^{-1} = f'gf^{-1}$. So $g' = fg^{-1}(f')^{-1}$.

Exercise (2.1.31).

using the notation of the five-lemma, give an example where the maps $\alpha, \beta, \delta, \epsilon$ are zero but γ is non-zero. This can be done with short exact sequences in which all the groups are either $\mathbb{Z}$ or 0.

Solution.

Following the hint, we will find a commutative diagram as in the Five-Lemma using only $\mathbb{Z}$ and 0 as the groups. Indeed, consider the diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "0"] \arrow[d, "0"] & 0 \arrow[r, "i"] \arrow[d, "0"] & \mathbb{Z} \arrow[r, "\approx"] \arrow[d, "\approx"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "0"] & 0 \arrow[d, "0"] \\ 0 \arrow[r, "i"] & \mathbb{Z} \arrow[r, "\approx"] & \mathbb{Z} \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \end{tikzcd}$$

where $\approx$ signifies an isomorphism. Each of the rows is easily checked to be exact and moreover, the diagram commutes. Thus, this is an example of such a diagram.