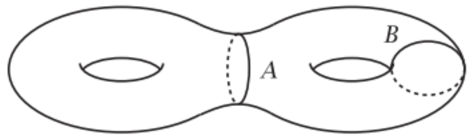
Problem 1.

(a) Compute the groups $H_n(X,A)$ when X is S^2 of $S^1 \times S^1$ and A is a finite set of points.

(b) Compute $H_n(X,A)$ and $H_n(X,B)$ where X is a closed orientable surface of genus 2 and A and B are the circles shown in the figure below.



Solution.

(a) We begin with the former. Let $X = S^2$ and $A = \{a_1, \ldots, a_n\}$ be a finite set of points. If $n = 0$ then trivially $H_n(X,A) = H_n(X)$. for the non-trivial case we observe that (X,A) is a good pair because we can surround each of the points $a_i \in A$ by a small disk in X which deformation retracts onto a_i via the straight line homotopy. We then note that there are no k-cells in X for $k > 2$ and consequently $H_n(X,A) = 0$ for $n > 2$. For $n \leq 2$ we can consider the long exact sequence of the pair

$$\cdots \longrightarrow H_2(A) \longrightarrow H_2(X) \longrightarrow H_2(X,A) \longrightarrow H_1(A) \longrightarrow \cdots$$

Then we note that A is a discrete set of points and consequently $H_n(A) = 0$ for $n > 0$ and moreover $H_2(X) = \mathbb{Z}$. This reduces the above to

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(X,A) \longrightarrow 0$$

Then exactness of the sequence implies that $H_2(X,A) \cong \mathbb{Z}$. Looking further down the sequence we see

$$\cdots \longrightarrow H_1(A) \longrightarrow H_1(X) \longrightarrow H_1(X,A) \longrightarrow H_0(A) \longrightarrow \cdots$$

This time we have $H_1(A) = H_1(X) = 0$, $H_0(A) = \bigoplus_{i=1}^n \mathbb{Z}$ (one generator for each of the a_i) and $H_0(X) = \mathbb{Z}$. Thus, the above becomes

$$0 \longrightarrow 0 \longrightarrow H_1(X,A) \longrightarrow \bigoplus_{i=1}^n \mathbb{Z} \xrightarrow{p} \mathbb{Z}$$

Exactness implies $H_1(X,A) = \ker p$. We know that homomorphisms must fix the identity and consequently must send only one generator of the product to $1 \in \mathbb{Z}$. This means that $\ker p = \bigoplus_{i=1}^{n-1} \mathbb{Z}$ and $H_1(X,A) = \bigoplus_{i=1}^{n-1} \mathbb{Z}$. Lastly, we have that

$$\cdots \longrightarrow H_0(A) \longrightarrow H_0(X) \longrightarrow H_0(X,A) \longrightarrow 0$$

By previous arguments these give

$$\cdots \longrightarrow \bigoplus_{i=1}^n \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow H_0(X,A) \longrightarrow 0$$

In particular the end of the sequence implies that $H_0(X, A) = 0$. This gives

$$H_k(X, A) = \begin{cases} 0 & k = 0 \\ \bigoplus_{i=1}^{n-1} \mathbb{Z} & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & k > 2 \end{cases}$$

When $X = S^1 \times S^1$ we proceed much in the same way as previously. The pair (X, A) is still a good pair for the same reason as before. for $H_2(X, A)$ we note that $H_2(A) = 0$, $H_2(X) = \mathbb{Z}$ and $H_1(A) = 0$ so we get the sequence

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow H_2(X, A) \longrightarrow 0$$

Which implies that $H_2(X, A) = \mathbb{Z}$. Proceeding down the sequence we recall that $H_1(X) = \mathbb{Z} \times \mathbb{Z}$ so the sequence at this level (terminated with a 0) becomes

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(X, A) \longrightarrow \bigoplus_{i=1}^{n} \mathbb{Z} \longrightarrow 0$$

We then apply the Splitting Lemma to conclude that

$$H_1(X, A) \cong (\mathbb{Z} \oplus \mathbb{Z}) \oplus \bigoplus_{i=1}^{n} \mathbb{Z} \cong \bigoplus_{i=1}^{n+1} \mathbb{Z}$$

Finally, for $H_0(X, A)$ we have the sequence

$$0 \longrightarrow \bigoplus_{i=1}^{n} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow H_0(X, A) \longrightarrow 0$$

Which immediately implies $H_0(X, A) = 0$. Aggregating these computations we have

$$H_k(X, A) = \begin{cases} 0 & k = 0 \\ \bigoplus_{i=1}^{n+1} \mathbb{Z} & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & k > 2 \end{cases}$$

Again, we remark that if $A = \emptyset$ then $H_k(X, A) = H_k(X)$.

(b) We begin with $H_n(X, A)$. First recall that $H_n(X, A) = \tilde{H}_n(X, A)$ because (X, A) is a good pair. But X/A is a wedge of two tori. If we embed X/A into $\mathbb{R}^3$ in the usual way and consider an open neighborhood that deformation retracts onto each of the $S^1 \times S^1$ factors, respectively, then we can use the Mayer-Vietoris sequence to compute the homology. Indeed, if we let $U \supset S^1 \times S^1$ and $V \supset S^1 \times S^1$ be such neighborhoods then we have

$$\cdots \longrightarrow \tilde{H}_n(U \cap V) \longrightarrow \tilde{H}_n(U) \oplus \tilde{H}_n(V) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_{n-1}(U \cap V) \longrightarrow \cdots$$

We then note that $U \cap V$ deformation retracts to a point and therefore has the homology of a point. This reduces the sequence for each n to

$$0 \longrightarrow \tilde{H}_n(S^1 \times S^1) \oplus \tilde{H}_n(S^1 \times S^1) \longrightarrow \tilde{H}_n(X/A) \longrightarrow 0$$

Exactness then states precisely that

$$\tilde{H}_n(X/A) = \tilde{H}_n(S^1 \times S^1) \oplus \tilde{H}_n(S^1 \times S^1)$$

In particular we see that

$$H_n(X,A) = \tilde{H}_n(X/A) = \begin{cases} 0 & n = 0 \\ \bigoplus_{i=1}^4 \mathbb{Z} & n = 1 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}$$

If we identify X with the octagon (having the usual identifications), we can see that collapsing B to a point removes one of the generators and we see that the quotient space is homeomorphic to $S^1 \times S^1/\{x_0, x_1\}$ where $\{x_0, x_1\}$ result from cancelling the other generator for that loop. We can then apply the previous part of this question to see that

$$H_0(X,B) = \begin{cases} 0 & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 \end{cases}$$

Problem 2.

Show that $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology of the union of n cones CX with their bases identified.

Solution.

For the fist part, we will use the Mayer-Vietoris sequence. Indeed, we take the cover to be $A = p(X \times [0, 1/2 + \epsilon])$ and $B = p(X \times [1/2 - \epsilon, 1]$ where p is the quotient map for the suspension. It is clear that $A \cap B$ deformation retracts onto X, so the Mayer-Vietoris sequence in reduced homology becomes

$$\cdots \longrightarrow \tilde{H}_{n+1}(A \cap B) \longrightarrow \tilde{H}_{n+1}(A) \oplus \tilde{H}_{n+1}(B) \longrightarrow \tilde{H}_{n+1}(SX) \longrightarrow \tilde{H}_n(A \cap B) \longrightarrow \cdots$$

We then note that A and B deformation retract to the cone CX on X, which is homotopy equivalent to a point, and consequently has the homology of a point. Thus, the above sequence becomes

$$0 \oplus 0 \longrightarrow \tilde{H}_{n+1}(SX) \longrightarrow \tilde{H}_n(X) \longrightarrow 0 \oplus 0$$

Exactness of the sequence gives $\tilde{H}_{n+1}(SX) \cong \tilde{H}_n(X)$ as desired.

For the union of n cones we proceed by induction on the number of cones. Let C^nX be the union of n cones on XWe will show that

$$\tilde{H}_{n+1}(C^nX) = \bigoplus_{i=1}^{n-1} \tilde{H}_n(X)$$

We have just done the base case in the previous part. We can choose a cover of C^nX by $C^{n-1}X$ and CX then the Mayer-Vietoris sequence becomes

$$0 \longrightarrow \tilde{H}_{n+1}(CX) \oplus \tilde{H}_{n+1}(C^{n-1}X) \longrightarrow \tilde{H}_{n+1}(C^nX) \longrightarrow \tilde{H}_n(X) \longrightarrow 0$$

By the splitting lemma we have that

$$\tilde{H}_{n+1}(C^nX) \cong \tilde{H}_n(X) \oplus \left(\bigoplus_{i=1}^{n-2} \tilde{H}_n(X)\right) \cong \bigoplus_{i=1}^{n-1} \tilde{H}_n(X)$$

Problem 3.

As on page 136 of the book, the local homology groups of a space X at a point $x \in X$ are defined to be the relative homology groups $H_n(X, X - \{x\})$. Let X be the cone on the 1-skeleton of Δ^3, the union of all the line segments joining the six edges of Δ^3 to the barycenter of Δ^3

(a) Compute the local homology groups $H_n(X, X - \{x\})$ for all $x \in X$.

(b) Define ∂X to be the subset of X consisting of points x such that the local homology groups $H_n(X, X - \{x\})$ are zero for all n. Compute the local homology groups $H_n(\partial X, \partial X - \{x\})$ for all $x \in \partial X$.

(c) Use these calculuations to determine which subsets $A \subset X$ have the property that $f(A) \subset A$ for all homeomorphisms $f : X \to X$.

Solution.

1. We begin by noting that there are five cases depending on where x lies inside of $C\Delta^3$. We handle them in sequence:

   (a) Suppose that x is a vertex in Δ^3. We see that locally we have a star-shaped nieghborhood U corresponding to the three faces and the edge leading to the barycenter. This neighborhood deformation retracts to a point (first project down to the plane so we have the wedge of four line segments, and then retract each of the segments). Consequently, each (sufficiently) small neighborhood of x has the homotopy type of a point. We then recall that homology is a homotopy invariant and so $\tilde{H}_n(U, U - \{x\}) = 0$.

   (b) The next case is the "distinguished vertex" that is the barycenter. Note that if we remove the barycenter, the resulting space $C\Delta^3 - \{x\}$ deformation retracts to Δ^3_1, the 1-skeleton of Δ^3. We can then compute the homology groups based on the k-skeletons of Δ^3. Indeed we have the chain complex

   $$\cdots \longrightarrow 0 \longrightarrow \bigoplus_{i=1}^{6} \mathbb{Z} \xrightarrow{\partial_1} \bigoplus_{i=1}^{4} \mathbb{Z} \xrightarrow{\partial_0} 0$$

   It is clear from this that $\tilde{H}_1(X) = \bigoplus_{i=1}^{4} \mathbb{Z}$ and $\tilde{H}_n(X) = 0$ otherwise. This implies that

   $$H_n(U, U - \{x\}) \cong \tilde{H}_{n-1}(\Delta^3_1) = \begin{cases} \bigoplus_{i=1}^{3} \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

   (c) Now we look at the interior points of $C\Delta^3$. Suppose that x lies on an edge in Δ^3. Then a neighborhood U of x will be contractible and thus we have $\tilde{H}_n(U - \{x\}) = 0$ and consequently $H_n(U, U - \{x\}) = 0$.

   (d) If x lies on an edge connecting to the barycenter, then the situation is a bit different.

We see that x will intersect three faces in this case.

(e) The last case is clear. If x is in the interior of the space then any neighborhood of x is homeomorphic to a disk in $\mathbb{R}^2$ and so the homology is 0 for all n. Thus, we have a situation very similar to the barycenter. After removing x we have a space that deformation retracts onto tetrahedron and so the homology must be the same as the barycenter for this reason. Thus we have that

$$H_n(U, U - \{x\}) \cong \tilde{H}_{n-1}(\Delta_1^3) = \begin{cases} \bigoplus_{i=1}^3 \mathbb{Z} & n = 2 \\ 0 & \text{else} \end{cases}$$

again.

2. In this case, we simply have the exterior of Δ^3 (if we aggregate the results from the previous part.) There are only a few changes when we compute the local homology in ∂X. The change occurs when we consider the vertices of Δ^3. In this case we note that a vertex on an edge must have a neighborhood that deformation retracts onto $S^2 \wedge S^2$ so that

$$H_n(X, X - \{x\} \cong \tilde{H}_n(S^2 \wedge S^2 \cong \tilde{H}_2(S^2) \oplus \tilde{H}_2(S^2)$$

The others are unchanged.

3. Suppose that f is a homeomorphism of X. Then we must have that

$$H_n(X, X - \{x\}) \cong H_n(X, X - \{f(x)\})$$

for each n. Thus, we can restrict f to a homeomorphism on Δ_1^3. In particular, see from part (b) that f must map vertices to vertices, and so it must permute the vertices. Also, we see that f must fix the barycenter of X. Finally, we notice that f must also fix the edges within X and so if we let $v_0, \ldots, v_3$ denote the vertices b the barycenter $e_1, \ldots, e_n$ the edges then we have that the invariant sets under any homeomorphism must be obtained from the following:

(a) $\{v_0, \ldots, v_3$

(b) $\{b\}$

(c) $\{\Delta_1^3\}$

(d) $\{e_1, \ldots, e_n$

(e) $X = C\Delta^3$

under the set theoretic operations of union, intersection and set subtraction.

Problem 4.

On the first exam,there was a problem asking for the calculation of the fundamental froup of the CW complex obtained from a cube I^3 by identifying opposite faces by a one-quarter twist. Now compute the homology groups of this complex, without using the general fact that H_1 is the abelianization of π_1.

Solution.

See the attached diagram.

Problem 5.

Two problems to solve using Mayer-Vietoris sequences:

(a) *Show that a closed nonorientable surface, or more generally any finite CW complex X for which $H_1(X)$ contains torsion (an element of finite order), cannot be embedded as a subspace fo $\mathbb{R}^3$ in such a way that it has a neighborhood which is a mapping cylinder of a map from a closed orientable surface to X.*

(b) *The closed orientable surgace M_g of genus g embedded in $\mathbb{R}^3$ in the standard way bounds a compact region R. Two copies of R glued together by the identity map between their boundaries form a closed 3-manifold X. Compute the homology groups of X.*

Solution.

1. Suppose towards a contradiction that X could be embedded in $\mathbb{R}^3$. Then the mapping cylinder neighborhood M of $f(X)$ deformation retracts onto $\underline{f(X)}$. We then consider the Mayer-Vietoris sequence for the pair $U = \overline{M}$ and $V = \overline{\mathbb{R}^3 - M}$. Then observe that the intersection $U \cap B = \partial M$. The sequence in reduced homology becomes

$$\tilde{H}_2(\mathbb{R}^3) \longrightarrow \tilde{H}_1(\partial M) \longrightarrow \tilde{H}_1(U) \oplus \tilde{H}_1(V) \longrightarrow \tilde{H}_1(\mathbb{R}^3)$$

We then note that $\partial M \cong M_g$ the surface of genus g and as a result we see that the above actually gives

$$0 \longrightarrow \mathbb{Z}^{2g} \longrightarrow \tilde{H}_1(f(X)) \oplus \tilde{H}_1(V) \longrightarrow \tilde{H}_1(\mathbb{R}^3)$$

and by exactness we see that $\mathbb{Z}^{2g} \cong \tilde{H}_1(f(X)) \oplus \tilde{H}_1(V)$. But we assumed that $H_1(X)$ had torsion which is a contradiction because $\mathbb{Z}^{2g}$ is torsion-free.

2. the Mayer-Vietoris sequence says that

$$\cdots \longrightarrow H_n(M_g) \longrightarrow H_n(R) \oplus H_n(R) \longrightarrow H_n(X) \longrightarrow H_{n-1}(M_g) \longrightarrow \cdots$$

We then note that R deformation retracts onto $\vee_{i=1}^{g} S^1$ in the usual way. Thus, $H_0(R) \cong \bigoplus_{i=1}^{g} \mathbb{Z}$. As a result we see that

$$0 \longrightarrow H_n(X) \longrightarrow H_{n-1}(M_g) \longrightarrow 0$$

Thus we get that

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ \bigoplus_{i=1}^{g} \mathbb{Z} & n = 1 \\ \bigoplus_{i=1}^{g} \mathbb{Z} \cong H_1(M_g) & n = 2 \\ \mathbb{Z} \cong H_2(M_g) & n = 3 \\ 0 & n \geq 4 \end{cases}$$

where the $n = 1$ case follows because the image of M_g in the induced map is the diagonal in $H_1(R) \oplus H_1(R)$.

Problem 6.

The mapping cylinders of a sequence of maps

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \longrightarrow \cdots$$

can be joined together to form the mapping telescope T *which is the quotient space of* $\coprod_{n=1}^{\infty}(X_n \times I)$ *under the identifications* $(x_n, 1) \sim (f(x_n), 0)$ *for all* $x_n \in X_n$*,* $n = 1, 2, \cdots$.

(a) Let each X_n *be* S^1 *and let* $f_n : S^1 \to S^1$ *be a basepoint-preserving map of degree* n*. Compute* $\pi_1(T)$ *and* $H_i(T)$ *for all* i*. Also, show that* $\pi_i(T) = 0$ *for all* $i > 1$ *using a compactness argument and the fact that part of a mapping telescope* T *between* X_1 *and* X_n *deformation retracts onto* X_n*.*

(b) For $k > 1$ *let each* X_n *be* S^k *and let* $f_n : S^k \to S^k$ *be a basepoint preserving map of degree* n*. Compute* $\pi_1(T)$ *and* $H_i(T)$ *for all* i*.*

Solution.

(a) We will compute $\pi_1(T)$ by applying van Kampen's theorem. Each of the open sets in the cover will be a neighborhood A_n of one of the $S^1 \times I$ factors in T. Then each of the A_n is path-connected and open. Moreover, each of the $A_n \cap A_m \cap A_\ell = \emptyset$. Consequently, we have a surjective homomorphism $\Phi : *_{k=1}^{\infty} \mathbb{Z} \to \pi_1(T)$. To compute the elements in $\ker \Phi$ we note that f_n has degree n for each n and therefore is n times the generator for $\pi_1(S^1)$. This yields the following presentation for the fundamental group

$$\pi_1(T) \cong \{\sum_{n=1}^{\infty} a_n \mid n a_n = (n+1)a_{n+1}\}$$

We then remark that this implies that each of the a_n can be written in terms of a_1. This gives an isomorphism $\pi_1(T) \to \bigoplus_{n=1}^{\infty} \mathbb{Z}/n\,\mathbb{Z}$ by sending $a_n \to n a_1$. The reason that this group is free abelian is that each of the generators commute with one another because the union is disjoint. Consequently, $H_1(T)$ as the abelianization of $\pi_1(T)$ is also $\bigoplus_{n=1}^{\infty} \mathbb{Z}/n\,\mathbb{Z}$.

(b) In this case van Kampen's theorem gives us that $\pi_1(T) = 0$. The reason is that we can set the $A_n = S^k \times I$ which is simply connected and therefore there is a surjective homomorphism from $0 \to \pi_1(T)$. For the homology groups, we will do a limiting process. Let T_n be the product $\coprod_{n=1}^{\infty}(S^k \times I)$ modulo the same relations as T. In this case we consider the Mayer-Vietoris sequence (T_n, S^k) so that we have

$$\cdots \longrightarrow H_n(S^k \cap T^{n-1}) \longrightarrow H_n(S^k) \oplus H_n(T^{n-1}) \longrightarrow H_n(T^n) \longrightarrow H_{n-1}(S^k \cap T^{n-1} \longrightarrow \cdots$$

We will show by induction that

$$H_n(T^n) = \begin{cases} \bigoplus_{i=1}^{n} \mathbb{Z}/n\,\mathbb{Z} & n = k \\ 0 & \text{otherwise} \end{cases}$$

Indeed, note that the sequence becomes

$$H_n(S^k) \longrightarrow H_n(S^k) \oplus H_n(T^{n-1}) \longrightarrow H_n(T^n) \longrightarrow H_{n-1}(S^k \cap T^{n-1})$$

When $n = k$ we get

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \left(\bigoplus_{i=1}^{n-1} \mathbb{Z}/n\,\mathbb{Z}\right) \longrightarrow H_n(T^n) \longrightarrow 0$$

then we use degree to see that the induced map sends the generator of $\mathbb{Z}$ to $n\,\mathbb{Z}$ implying that the induced map forces

$$H_n(T^n) \cong \mathbb{Z}\,/n\,\mathbb{Z} \oplus \left(\bigoplus_{i=1}^{n-1} \mathbb{Z}\,/n\,\mathbb{Z} \right) \cong \bigoplus_{i=1}^{n} \mathbb{Z}\,/n\,\mathbb{Z}$$

By the induction hypothesis. If $n \neq k$ then $H_n(S^k) = 0$ implies that $H_n(T^n) = 0$ as well.