Exercise (2.2.1).

Prove the Brouwer fixed point theorem for maps $f : D^n \to D^n$ by applying degree theory to the map $S^n \to S^n$ that sends both the northern and southern hemispheres of S^n to the southern hemisphere via f.

Solution.

Suppose that there were such a map $f : D^n \to D^n$ such that $f(x) \neq x$ for every $x \in D^n$. Then there is a homotopy

$$H(x,t) = \frac{f(x)(1-t) + x(t)}{\|f(x)(1-t) + x(t)\|}$$

which is well-defined because f fixes no point. This homotopy in particular defines a retraction $r : D^n \to S^{n-1}$ of the disk onto its boundary sphere. Let i be the inclusion $S^{n-1} \to D^n$ so that $r \circ i = \mathbb{1}$. This implies that the induced homomorphism $r_* i_* = \mathbb{1}$ in the (reduced) homology. However, we know that $\tilde{H}_{n-1}(S^{n-1}) \cong Z$ because and that $\tilde{H}_{n-1}(D^n) \cong 0$. This implies an isomorphism $\varphi : 0 \to \mathbb{Z}$ which is clearly impossible. This contradiction establishes that no such f exists.

Exercise (2.2.2).

Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.

Solution.

For the first claim, suppose that f and $-f$ both have no fixed-points. This implies that $\deg(f) = (-1)^{2n+1} = -1$. But $\deg(-f) = (-1)^{2n+1} = -1$ and so $\deg(f) = 1$, a contradiction. Thus, either f or $-f$ has a fixed point and so there is a point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$.

Now let $g : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$. Then g lifts to a map $\tilde{g} : S^{2n} \to S^{2n}$ and so we can find a point $\tilde{x}$ such that $\tilde{g}(\tilde{x}) = \pm x$. Then passing back down to the quotient via that projection p, we see that $p(x)$ must be a fixed point for g.

For the last part of the problem, consider a non-identity automorphism $T \in \mathrm{Aut}(\mathbb{R}^{2n})$. Then T has no fixed points. We then construct a map $f = p \circ T \circ p^{-1} : \mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ with no fixed points by lifting to S^{2n}, apply T and then composing with the projection map. This has no fixed points because otherwise we would have $(p \circ T \circ p^{-1})(x) = x$ which implies that $T(p^{-1}(x)) = p^{-1}(x)$. This is impossible becuase T has no fixed points.

Exercise (2.2.3).

Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.

Solution.

Suppose towards a contradiction that $f(x) \neq x$ for each $x \in S^n$. Then $\deg(f) = (-1)^{n+1}$. Likewise, if $f(y) \neq y$ for each $y \in S^n$, then the line from y to $f(y)$ does not pass through the origin. Given this fact, consider the homotopy

$$H : (y, t) \mapsto \frac{(1-t)f(y) + ty}{\|(1-t)f(y) + ty\|}$$

This is well-defined because the denominator is non-zero by assumption. Moreover, we see that $H(y, 0) = f(y)$ and $H(y, 1) = \mathbb{1}$. So then $\deg(f) = \deg(\mathbb{1}) = 1$, which is non-zero. Therefore, f must have a fixed point. Likewise, $-f$ must also have a fixed point. The contradiction establishes the first part.

Now let F be a non-zero vector field on D^n. First, define the map $\alpha : D^n \to S^{n-1}$ given by $\alpha(x) = F(z)/\|F(z)\|$. Then consider the restriction of α to ∂D^n given by $\alpha\mid_{\partial D^n}: S^{n-1} \to S^{n-1}$. We see that the restriction $\alpha\mid_{\partial D^n}$ factors through D^n, which is contractible, and therefore must be nullhomotopic. Then $\deg(\alpha\mid_{\partial D^n}) = 0$ and so by the above we have points $x, y \in \partial D^n$ such that $\alpha\mid_{\partial D^n}(x) = x$ and $\alpha\mid_{\partial D^n}(y) = -y$. Then by the definition of α we see that $F(x) = x\|F(x)\|$ and $F(y) = -y\|F(y)\|$, which are vectors pointing radially outward and inward, respectively.

Exercise (2.2.4).

Construct a surjective map $S^n \to S^n$ of degree zero, for each $n \geq 1$.

Solution.

First consider the canonical injection $i : S^n \to D^n$ by sending S^n to the boundary of D^n. Because D^n is contractible, this map is nullhomotopic. We then observe that S^n is the quotient D^n/S^{n-1} and if we let p be the quotient map, we see that $p \circ f$ is surjective, and homotopic to the constant map, and therefore $\deg(p \circ f) = 0$, as desired.

Exercise (2.2.6).

Show that every map $S^n \to S^n$ can be homotoped to have a fixed point if $n > 0$.

Solution.

Consider a map $f : S^n \to S^n$ and a point $x \in f(S^n)$. Let $y \in f^{-1}(x)$ let T be the rotation of the sphere such that $T(x) = y$. Because S^n is path-connected define a homotopy $H(x, t)$ connecting the identity map $\mathbb{1}$ to T by moving each point along a path between x_0 and $T(x_0)$. Then $H(x, 0) = x$ and $H(x, 1) = T(x)$. We then compose f with such a homotopy and observe that

$$(f \circ H)(x, 0) = f(x)$$
$$(f \circ H)(x, 1) = (f \circ T)(x)$$

But then we see that $(f \circ T)(x) = f(y) = x$. So f can be homotoped to a map with a fixed point.

Exercise (2.2.7).

For an invertible linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ show that the induced map on $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \approx \tilde{H}_{n-1}(\mathbb{R}^n - \{0\} \approx \mathbb{Z}$ us $\mathbb{1}$ or $-\mathbb{1}$ according to whether the determinant of f is positive or negative.

Solution.

Following the hint, we begin by noting that we can perform Gaussian elimination via a sequence of multiplcation by elementary matrices on the matrix representation for the map f. This will define a homotopy from A_f, the matrix corresponding to f, to a diagonal matrix D with ± 1 on the diagonals. Because det is a homomorphism, multiplications by suitable elementary doesn't change the sign of the determinant, and so $\det A = \det D$. Now we note that S^{n-1} is a deformation retract of $\mathbb{R}^n - \{0\}$ and so because the degree of any map $S^{n-1} \to S^{n-1}$ that is a negation of one of the coordinates has degree -1 and so the same must hold in $\mathbb{R}^n - \{0\}$. By the long exact sequence of the pair we see that the same must hold in $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}$. So D_* will have degree $(-1)^\ell$ where ℓ is the number of -1's on the diagonal. Becuase $A \sim D$ A_* will have the same degree, which is precisely ± 1 depending on the determinant of D.

Exercise (2.2.8).

A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \to S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

Solution.

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. By the fundamental theorem of algebra f admits a factorization

$$f(z) = \prod_{j=1}^{n} (z - z_j)^{\alpha_j}$$

Now note that $p : S^2 \to \mathbb{C}$ is a covering space and so f lifts to a map $\tilde{f}$. Now for each root z_j we can find a neighborhood U_j containing $p^{-1}(z_j)$ and we can choose the U_j to be disjoint in S^2. Now let U_j is mapped to a neighborhood of 0, say V_j. This induces a map on the relative homology groups

$$\tilde{f}_* : H_2(U_j, U_j - \{\tilde{z}_j\}) \to H_2(V_j, V_j - \{0\})$$

By the commutative diagram at the top of page 136, we see that each of the groups can be indentified with $H_2(S^2) \cong \mathbb{Z}$ so that $\tilde{f}_*$ is multiplication by an integer which is $\deg \tilde{f} \mid z_j$. Now because the neighborhoods U_j are disjoint, we see that $\tilde{z_j}$ is the only root of $\tilde{f}$ in U_j. So if we restrict f to U_j we get a map looking like

$$f\mid_{U_j}(z) = (z - z_j)^{\alpha_j \log(z - z_j)} \prod_{i \neq j} (z - z_i)^{\alpha_i}$$

This map is clearly α_j-to-one onto V_j because the exponential wraps around the circle at least α_j times in a sufficiently small deleted-neighborhood of $\tilde{z}_j$. Thus, a generator for $H_2(U_j, U_j - \{\tilde{z}_j\})$ maps to α_j times a generator for $H_2(V_j, V_j - \{0\})$. This implies that the local degree of $\tilde{f}$ at z_j is α_j. By Proposition 2.30

$$\deg(f) = \sum_j \deg(f) \mid z_j = \sum_j \alpha_j$$

Moreover, by the fundamental theorem of algebra we see that the degree of f (as a polynomial) is $\sum_j \alpha_j$ as desired.