Exercise 1.3.12: (Drawing attached: Looks like 8 S^1's wedged together in a loop)

Let $N \leq \pi_1(S^1 \vee S^1)$ be the smallest normal subgroup generated by $a^2, b^2, (ab)^4$. If we consider the action of N on the universal cover of $S^1 \vee S^1$, we will identify the orbits of each of the actions of each of these three generators. I constructed this Cayley graph iteratively via the following process:

1. Start at e and add a vertex for each of a, b. Add the edges a, b to these vertices and then add the reverse edges due to the relations a^2, b^2 respectively. Proceed to one of the new vertices.
2. Every time we visit a vertex we add new vertices according to the action of a, b on the group element corresponding to that vertex, subject to the restrictions given by the relations which are
   (a) If we just traversed an a edge to get to the current vertex then only consider multiplications by b.
   (b) If we just traversed a b edge to get to the current vertex, then we only consider multiplications by a.
3. Reduce each new product by associating the elements to elements that have already been computed in the construction thus far, if possible.
4. Use the previous step to get additional equalities and add this to the list of relations.

Steps 3. and 4. above happen when we get to the products aba, bab and the new relation forces $(ab)^2$ to get its own vertex that resolves the reductions to pre-computed products.

To see that this is the correct Cayley graph, first note that each of the generators is a loop in X/N. Second observe that each generator of $\pi_1(X)$ is conjugate to an element in X/N (just follow the arrows). This shows that this is indeed the correct Cayley graph.

Exercise 1.3.15:

Let $p : \tilde{X} \to X$ be a simply connected covering space of X and suppose that $A \subset X$ is path-connected and locally path-connected. Suppose that $\tilde{A}$ is a path component in $p^{-1}(A)$. Now let $f : \tilde{A} \to \tilde{X}$ be the inclusion of $\tilde{A}$ in $\tilde{X}$. Let $\varphi : \pi_1(A) \to \pi_1(X)$ be the inclusion map. We are considering the map $q : \tilde{a} \to A$ obtained by restricting p. First we will see that $\ker(\varphi) \subset \ker(f_*)$. Indeed, because $\tilde{X}$ is simply connected, $\pi_1(\tilde{X})$ is the trivial group and so f_* must be the trivial map. As a result $p_* \circ f_*$ is trivial and also $f_* \circ q_*$ is trivial because $id \circ p = p \circ f$. This implies that $q_*(\pi_1(\tilde{A})) \subset \ker f_*$.

For the converse, consider $\alpha \in \ker(\varphi)$. We can lift α to a loop in $\tilde{A}$. Moreover, we must have that $\alpha \in p_*(\pi_1(\tilde{A}))$ because $\tilde{X}$ is simply connected. Hence, $\ker(f_*) \subset \ker(\varphi)$. Then by Proposition 1.40 we have that : $\tilde{A} \to A$ is the covering space corresponding to $\ker(f_*)$.

Exercise 1.3.21:

Let M be the Möbius band and $T \cong S^1 \times S^1$ be the usual torus. We consider the space X obtained from T by attaching ∂M to the circle $S^1 \times \{x_0\}$. We first compute $\pi_1(T_m)$ by applying the van Kampen Theorem to the open cover obtained by taking an ϵ-neighborhood of T and M within X. Clearly, T and M are deformation retracts of such neighborhoods and so

$$\pi_1(X) \cong \pi_1(T) * \pi_1(M) \cong (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$$

We will now construct the universal covering space of X. We know that a universal cover for T is a grid in $\mathbb{R}^2$. We begin with such a space and proceed to "stack" copies of this grid on top of each other and then attach a 2-cell along each vertical line $\mathbb{R} \times \{n\}$. The action of $\pi_1(X)$ on this space is determined by where it sends the generators: $(1,0), (0,1), n$. If we identify each point in $\tilde{X}$ with (x, y, z) where x, y are the points in each grid, and z is the grid number the vertex is on then

$$\begin{aligned}
(1,0) \cdot (x,y,z) &= (x+2, y, z) \\
(0,1) \cdot (x,y,z) &= (x, y+2, z) \\
n \cdot (x,y,z) &= (x, y, z+1)
\end{aligned}$$

The reason that the translation is by 2 rather than one in the first two cases comes from the fact that the action of the generator for $\pi_1(M)$ must act twice before achieving the same translation as the generators for $\pi_1(T)$ because paths in the Möbius band must loop around twice before getting back to the basepoint, where are the loops in the boundary circles must only loop once in the base space. This discrepancy forces the lifts to satisfy the relation above.

Now we consider the space Y obtained by attaching the Möbius band to a circle in $\mathbb{R}P^2$ that lifts to the equator in the covering space $p : S^2 \to \mathbb{R}P^2$. We proceed as before, we create an open cover of Y by taking open neighborhoods of both M and $\mathbb{R}P^2$ and apply van Kampen's theorem to conclude that

$$\pi_1(Y) \cong (\pi_1(\mathbb{R}P^2) * \pi_1(M))/\pi_1(S^1) \cong \{a, b \mid a, a^{-1}b^2\} \cong \mathbb{Z}_4$$

To see the action of $\pi_1(Y)$ on $\tilde{Y}$ we look at where it sends the generators. The generator of order two must follow the edge in a loop (right translation by 2 vertices). The generator of infinite order corresponding to the generator for $\pi_1(M)$ acts by vertical translation.

Exercise 1.3.30:

Because $\mathbb{Z} * \mathbb{Z}/2$ has two generators, the Cayley graph must be 4-regular. Moreover, each edge corresponding to multiplication by part of a loop due to the relation b^2. This leads to the following picture:

Exercise 1.3.31:

The forward direction is clear from the definition: Given a presentation for a group G with two generators we construct the space X_G as the wedge of 2 circles as well as 2-cells attached along the relations in the presentation of G. This space has $S^1 \vee S^1$ as a deformation retract. If we let $\tilde{X}_G$ be the Cayley graph of G then $\tilde{X}/G = X_G$ and when we compose the covering map with the retraction, we retain the covering map, so $\tilde{X}$ is a covering space of $S^1 \vee S^1$. Moreover, proposition 1.26 guarantees that the covering space is normal.

In the other direction we are given a normal covering space of $S^1 \vee S^1$ and must show that this is a Cayley graph of some group on two generators. If $\tilde{X}$ is the normal covering space then we know by Proposition 1.39 that the group of deck transformations $G(X) \cong \pi_1(S^1 \vee S^1)/p_*(\pi_1(\tilde{X}))$. We then construct the Cayley graph of $p_*(\pi_1(\tilde{X}))$ and note that the construction (c.f. discussion at the top of pg 77) guarantees that the graph X_G satisfies

$$X_G \cong \pi_1(S^1 \vee S^1)/p_*(\pi_1(\tilde{X})) \cong G(X)$$

as desired. The proof of the general case is exactly the same.

Exercise A1:

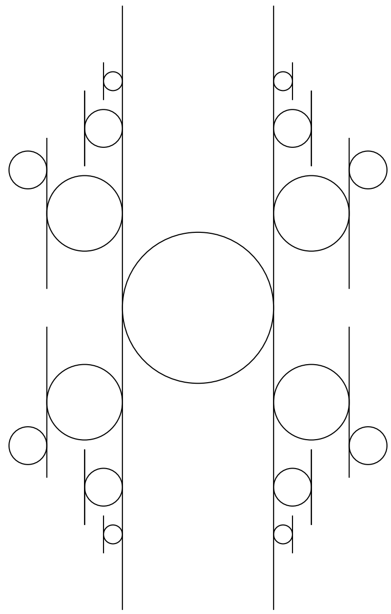

Figure 1: The Cayley graph of $\mathbb{Z} * \mathbb{Z}/2$.

If X is Hausdorff and has a compact covering space $p : \tilde{X} \to X$. Construct an open neighborhood U_x about each $x \in X$. If we lift each of these into $\tilde{X}$ we get an open cover for $\tilde{X}$. Because $p^{-1}(U_x)$ is the disjoint union of copies of U_x we can cover $\tilde{X}$ by a collection of copies of $U_x, x \in X$. By compactness we can extract a finite subcover, $\{U_{x_i,k}\}$ and we can take each of these sets to be disjoint because X is Hausdorff. Because the fibers above each point have the discrete topology, the finite disjoint cover implies that the fiber above each point must also be finite. Then because a loop in $\pi_1(X, x_0)$ lifts to a unique path in $\tilde{X}$ rooted at some point in $p^{-1}(x_0)$ and the number of such paths is finite we see that $\pi_1(X, x_0)$ must be finite as well. Because X is path connected, this does not depend on the choice of x_0 and we are done.

Exercise A2:

For the first part, consider the binary expansion of each real number. We construct a family of group presentations as follows:

1. To each real $r = d_0 d_1 \ldots$ assign the sequence of groups

$$\{a, b \mid a^{d_0}\}$$
$$\{a, b \mid a^{d_0} b^{d_1}\}$$
$$\{a, b \mid a^{d_0} b^{d_1} a^{d_2}\}$$
$$\vdots$$

2. To construct a bijection from the reals to groups on two generators, let G_r be the first group in the sequence for r that is not isomorphic to G_ℓ for any $\ell < r$. We know this is possible because the decimal expansion of each distinct reals is distinct.

3. Let $\tilde{X}_{G_r}$ be the Cayley graph of G_r. We showed in a previous problem that each of the X_{G_r} are normal covering spaces for $S^1 \vee S^1$.

So we have an injection from $\mathbb{R}$ into the covering spaces of $S^1 \vee S^1$. Because each of these Cayley graphs corresponds to a normal subgroup of $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$ we see that the free group on two generators has uncountabley many subgroups. This is not true for the free abelian group on two generators. This is an immediate consequence of the structure theorem for finitely generated abelian groups (there are only countably many subgroups of $\mathbb{Z}$ and taking products doesn't increase cardinality for infinite sets).

Exercise A3:

Given a finitely generated group G and a subgroup $H \leq G$ of finite index we consider the space X_G which is the two dimensional cell complex corresponding to G such that $\pi_1(X_G) \cong G$. Because G is finitely generated, it's 1-skeleton must be finite. Moreover, because H has finite index in G we apply Proposition 1.36 to get a covering space $\tilde{X}_G$ of X_G with a finite 1-skeleton. Moreover, the proposition says $pi_1(\tilde{X}_G) \cong H$. Because X_G has a finite one skeleton and the orbit space of H is finite H must also have a fninite 1-skeleton which means it too is finitely generated.

In the case that G is a finitely presented group and $H \leq G$ has finite index we follow a similar construction. This time we construct X_G by attaching a 2-cell along each relation in the presentation for G and note that if the number of relations is finite, then G has both a finite 1-skeleton and 2-skeleton. Then the construction above translatees exactly: H induces a covering space that must have a finite 1 and 2 skeleton and therefore is finitely presented.

For the last part, suppose that G is a finitely generated group with n generators. In our initial construction of the space X_G we begin with attaching 1-cells to make loops for each relation and wedging the loops together. Then the Cayley subgraph of the group H in X_G has x vertices and nx edges if $[H : G] = x$. But then we can construct a spanning tree of theis graph with $m-1$ edges and so by Proposition 1A.2 H can have at most $nm-(m-1)$ generators. This gives that the number or generators for H is at most $m(n-1)+1$.

Exercise A4:

(a) We know that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$, which has two subgroups, and so there are two connected covering spaces of $\mathbb{R}P^2$. We know that $\mathbb{R}P^2$ itself is a covering space with the identity map as the cover, and so the other space must be the two sheeted cover $S^2 \to \mathbb{R}P^2$. Consequently, any covering space of $\mathbb{R}P^2$ is a disjoint union of some number these spaces.

Now if we look at the covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ we see that they must contain copies of covering spaces of $\mathbb{R}P^2$ and furthermore, must be they union of covering spaces because if we project onto the first coordinate, we are left with a covering space of $\mathbb{R}P^2$.

Let v be the basepoint in $\mathbb{R}P^2 \vee \mathbb{R}P^2$ and now consider a neighborhood of v. Because this neighborhood must lift to a disjoint union of neighborhoods, each homeomorphic to the original, we can brute-force the ways that $\mathbb{R}P^2$ and S^2 can intersect and satisfy this property:

(a) We can intersect the poles of the two spheres.

(b) We can intersect to $\mathbb{R}P^2$'s at their basepoints.

(c) We can intersect an S^2 and an $\mathbb{R}P^2$ at the pole of the former and the basepoint of the latter.

Because of the fact that the two spaces correspond to different summands in the base space, we can see that they must correspond to different edges in the covering space. So this means that we can construct a covering space as a wedge of spheres, possibly terminated by an $\mathbb{R}P^2$ at the end of any chain. These are all of the possibilities.

(b) We apply the previous part and van Kampen's theorem as follows: because $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$ we see that

$$\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2) \cong \pi_1(\mathbb{R}P^2) * \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2 * \mathbb{Z}/2$$

We know that each subgroup corresponds to a connected covering space of $\mathbb{Z}/2 * \mathbb{Z}/2$ and so each of the covering spaces corresponds to the following:

$$\begin{aligned} \{1\} &\mapsto \text{ the universal cover} \\ \{a, b \mid a\} &\mapsto \mathbb{R}P^2 \\ \{a, b \mid (ab)^n\} &\mapsto \text{ the wedge of spheres} \\ \{a, b \mid a, (ab)^n\} &\mapsto \text{ the wedge of spheres with } \mathbb{R}P^2 \text{ attached} \end{aligned}$$

The other constructions are very similar. We can take the space X_3 to be a circle corresponding to the generator of $\mathbb{Z}/3$ with a 2-cell attached along the relation x^3 so that $\pi_1(X_3) \cong \mathbb{Z}/3$. By the van Kampen, we see that

$$\begin{aligned} \pi_1(\mathbb{R}P^2 \vee X_3) &\cong \mathbb{Z}/2 * \mathbb{Z}/3 \\ \pi_1(X_3 \vee X_3) &\cong \mathbb{Z}/3 * \mathbb{Z}/3 \end{aligned}$$

By the same reasoning as the case of $\mathbb{Z}/2 * \mathbb{Z}/2$ we see that there are 6 isomorphism classes of $\mathbb{Z}/2 * \mathbb{Z}/3$ and 9 of $\mathbb{Z}/3 * \mathbb{Z}/3$.