Exercise 1.1.12:

If we are given a homomorphism $f : \pi_1(S^1) \to \pi_1(S^1)$. Because $\pi_1(S^1)$ is cyclic f is uniquely determined by its value on the generator ω_1. By definition, f maps loops to loops and therefore there is some k such that $f([\omega_1]) = [\omega_k]$. I claim that f induces a homomorphism $\varphi_k : \theta \mapsto k\theta$ for each $\theta \in S^1$. To see this, note that the loop $\gamma : \theta \mapsto 2\pi\theta$ satisfies $(\varphi_k \circ \gamma)(\theta) = 2\pi k\theta$ and so $(\varphi \circ \gamma)(I) = \omega_k$. We have shown that $(\varphi_k)_*([\omega_1]) = f_*([\omega_1])$ and so $(\varphi_k)_* = f_*$.

Exercise 1.1.16:

(a) Suppose that there were a retraction $r : \mathbb{R}^3 \to S^1$. Then there would be some induced homomorphism that there is some homomorphism $\pi_1(S^1) \to \pi_1(\mathbb{R}^3)$. Because r is a deformation retract this is an injective homomorphism $\mathbb{Z} \hookrightarrow 0$, which is impossible.

(b) Suppose that $r : S^1 \times D^2 \to S^1 \times S^1$ were a deformation retraction. If we consider the fundamental groups of both spaces

$$\pi_1(S^1 \times D^2) \cong \pi_1(S^1) \times \pi_1(D^2) \cong \mathbb{Z}$$
$$\pi_1(S_1 \times S_1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$$

Then r induces an injective homomorphism $\mathbb{Z} \times \mathbb{Z} \hookrightarrow \mathbb{Z}$. But this is impossible because both each generator $(0, 1)$ and $(1, 0)$ must map to the identity, so the map cannot be injective.

(c) If there were a retraction $r : S^1 \times D^2 \to A$ then there would be some induced homomorphism $\varphi_* : \pi_1(A) \to \pi_1(S^1 \times D^2)$. The generator is sent to a loop in X that loops around once, and then retraces back on the S^1 factor. This is homotopic to a constant loop. Hence, the induced map is trivial, not injective, so we have a contradiction.

(d) Again, we proceed by contradiction. The space $X = D^2 \times D^2$ is clearly path-connected and deformation retracts onto a point, hence $\pi_1(X) \cong 0$. Conversely, we know that the fundamental group of the figure-eight is $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$. Hence, the deformation retraction r induces an injective homomorphism $\mathbb{Z} * \mathbb{Z} \hookrightarrow 0$, which is impossible.

(e) The disk with two points on the boundary identified is homotopic to the circle S^1, and therefore has fundamental group $\mathbb{Z}$. However, the boundary torus has fundamental group $\mathbb{Z} * \mathbb{Z}$. So if there were a retraction then we would have some induced homomorphism $\mathbb{Z} * \mathbb{Z} \hookrightarrow \mathbb{Z}$, which should be injective. But this is impossible, so there cannot be such a retraction.

(f) To see that the Möbius strip does not deformation retract onto its boundary circle, we suppose that there were such a retraction. This retraction would induce a homomorphism of the fundamental group, say φ. Consider the generator for the fundamental group of the Möbius strip and its image under φ. We see that the loop must wrap around twice around the circle. So $\varphi : \pi_1(A) \hookrightarrow \pi_1(X)$ is multiplication by 2. This implies that there is a homomorphism $\mathbb{Z} \twoheadrightarrow 2\,\mathbb{Z}$ that restricts to the identity, but this is impossible.

Exercise 1.1.17:

To see the family of non-homotopic retractions $S^1 \vee S^1 \to S^1$ consider the family

$$r_n(\theta, \varphi) = \begin{cases} \varphi & \theta = 0 \\ n\theta & \theta \neq 0 \end{cases}$$

We need to check the following facts

1. Each r_n is a deformation rectraction.

2. $r_n \not\simeq r_m$ unless $n = m$.

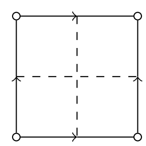
For (1) we need to verify that the map is continuous and restricts to the identity on the circle. The latter is clear from the definition (the identity on the second coordinate). Geometrically, the map is the identity on the "left circle" and loops around the circle n times on the "right circle". The map is continuous on each circle separately, so we only need check the common point of intersection. We see that if we set the common point of intersection to $\theta = \phi = 0$ (reparametrize if not) then the map agrees on the intersection and is therefore continuous.

To see (2) we note that r_n could not be homotopic to r_m unluss $m = n$ because otherwise the induced homomorphisms of these maps could send $[\omega_n] \mapsto [\omega_m]$, which is clearly impossible.

Hence, the family $\{r_n\}_{n\in\mathbb{Z}}$ is an infinite family of retractions $S^1 \vee S^1 \to S^1$.

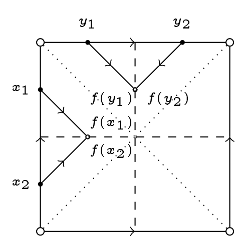
Exercise 0.1:

Consider the torus $S^1 \times S^1$ identified with the square $I \times I$ in the usual way. So

$S^1 \times S^1 \simeq$ 

Consequently, the four corner points of the square are identified as the same. We will show that the torus with this point removed has the union of two circles (the dashed lines in the figure above) as a deformation retract.

Indeed, we can visualize the map as follows: in each quadrant of the square, draw the line connecting the center of the square to the corner in that quadrant, and move each point along the line parallel to this line until you meet one of the inner lines, defining the two circles.



Then we can define the retraction f explicity by defining it piecewise on each wedge of the square

$$f(x,y) = \begin{cases} (x+y,0) & x \le 0, y \ge 0, |y| \le |x| \\ (0,x+y) & x \le 0, y \ge 0, |y| \ge |x| \\ (0,y-x) & x \ge 0, y \ge 0, |y| \ge |x| \\ (x-y,0) & x \ge 0, y \ge 0, |y| \le |x| \\ (y-x,0) & x \ge 0, y \le 0, |y| \ge |x| \\ (0,x+y) & x \ge 0, y \le 0, |y| \le |x| \\ (0,y-x) & x \le 0, y \le 0, |y| \le |x| \\ (y-x,0) & x \le 0, y \le 0, |y| \ge |x| \end{cases}$$

To see that f is continuous, we note that it is a straight line on each of the interiors, so we need only verify continuity on the boundary. We see that on each of the diagonals, we follow a straight line to the center of the square, so we agree. Finally, we see that if either $x, y = 0$ then the map is the identity (as it should be because it is a retraction. Hence, f is continuous.

Exercise 0.3:

(a) Let $f : X \to Y$ and $g : Y \to Z$ be homotopy equivalences. This guarantees the existence of maps $f' : Y \to X$ and $g' : Z \to Y$ such that $ff' = f'f = 1$ and $gg' = g'g = 1$. So

$$(g \circ f)(f' \circ g') = g \circ (f' \circ f) \circ g = g \circ g' = 1$$

Similarly,

$$(f' \circ g')(g \circ f) = f' \circ (g' \circ g) \circ f = f' \circ f = 1$$

So, $f \circ g$ is a homotopy equivalence as well.

This establishes that homotopy equivalence is an equivalence relation because we have relfexivity (use the identity map), symmetry (because $fg = gf = 1$), and the above established transitivity.

(c) Now suppose that h is homotopic to a homotopy equivalence f. Because f is a homotopy equivalence we have a map g such that $fg = gf = 1$.

Exercise 0.4:

Suppose that X deformation retracts to A in the weak sense and consider the inclusion map $i : A \hookrightarrow X$. We must have some homotopy $H(t, s)$ such that $H_0 = 1$, $H_1(X) \subset A$ and $H_t(A) \subset A$ for each t. We will see that H_1 is a two-sided inverse for i. So we need to show first that $H_1 \circ i = 1$. Consider the homotopy given by $F_t = H_{1-t}|_A$. So that $F_0 = H_1 \circ i$ and $F_1 = H_0 \circ i = 1$. Now we need to see that $i \circ H_1 = 1$. We then consider the homotopy $F_t = H_{1-t}$. We see that $H_1 = i \circ H_1$ because $H_1(X) \subset A$. Moreover, $H_0 = 1$ so we are done.

Exercise 0.5:

Suppose that X deformation retracts onto a point $x \in X$. Let $H(t, s)$ be a homotopy for this retraction. Let U be a neighborhood of x and consider its set of preimages under H

$$H^{-1}(U) = \{(t, x) : H_t(x) \in U\}$$

Now observe that $I \times \{x\} \subset U$ by the definition of a retraction. For each t we construct a neighborhood U_t containing x and a neighborhood I_t of t such that $U_t \times J_t \subset H^{-1}(U)$ because $H^{-1}(U)$ is open. Then we note that

$$I \times \{x\} \subset \bigcup_{t \in I} U_t \times I_t$$

Because $I \times \{x\}$ is compact, we can find a finite subcover such that

$$I \times \{x\} \subset \bigcup_{k=1}^{n} U_{t_k} \times I_{t_k}$$

We can then extract our deired neighborhood V by setting

$$V = \bigcap_{k=1}^{n} U_{t_k}$$

So V is an open neighborhood of x and $H_t(V) \subset U$ for each t.

If we then look at the inclusion map $i : V \hookrightarrow U$ then we see that the map $H_t \circ i$ gives a homotopy $G(t, s)$ between $H_0 = i$ and H_1, the constant map to x. So i is nullhomotopic.

Exercise 0.6:

(a) To see that X deformation retracts onto $\{x_0\} \times \{0\}$, we construct the retraction. We begin by retracting each interval $\{r\} \times [0, 1 - r]$ to $\{r\} \times \{0\}$ via the straight-line retraction. We then compose this with the retraction of $\{0\} \times [0, 1]$ onto the point $\{0\}$. Finally, because the interval is contractible, we retract onto the desired point $\{x_0\} \times \{0\}$ (see Exercise 0.5).

The difficulty is seeing why a similar construction does not hold elsewhere in X. To see the difference, consider some other point $x = \{r\} \times \{h\}$ in X. If there were some deformation retract f_t then f_t must be homotopic to a constant map on some neighborhood U of x. Since X is path-connected, but $X \setminus ([0, 1] \times \{0\})$ is not, we see that f_t must restrict to the identity on $[0, 1] \times \{0\}$, unless it contracts to a point on $[0, 1] \times \{0\}$. But this contradicts the fact that f_t restricts to the identity on $V \subset U$.

(b) We first observe that Y is contractible. To see this use the straight line homotopy on each copy of the X_i to project onto the bold line (see figure on pg. 18). This line is homotopy equivalent to $\mathbb{R}$ and therefore is contractible. Hence, we can send the identity to a constant map by composing the projection maps with the contraction of the center line. This shows that Y is contractible.

Now we need to show that Y does not deformation retract onto any point. Suppose that there were such a point $y \in Y$ such that Y deformation retracts onto y. Then any open neighborhood U of y would have to contain a neighborhood V of y such that the inclusion map $V \hookrightarrow U$ is nullhomotopic (see Exercise 0.5). However, this is impossible because any V must intersect non-path connected parts of U (due to density).

(c) To see that there is a deformation in the weak sense of Y onto Z we construct it piecewise by looking at the "union" of the retractions on the individual copies of X. More precisely, we note that $Y = \bigcup_{n \in \mathbb{Z}} X_i$ where each X_i is a copy of X and so we consider the homotopy $f_t^{(i)}$ as described in **(a)**. We then define the deformation retraction piecewise on each copy of X. Because these agree on the boundary of each of the X_i we see that this is a valid retraction in the weak sense.

However, Z is homeomorphic to $\mathbb{R}$. Consequently, we see that $\mathbb{Z}$ must retract onto a point, necause $\mathbb{R}$ retracts onto a point. Hence, if there were a true retraction $Y \to Z$, then it would imply that Y deformation retracts onto a point, but we showed in **(b)** that this is impossible. As a result, there is no true deformation retraction.

Exercise 0.9:

Suppose that X is a contractible space and that A is a retract of X. Let f be the retraction $X \twoheadrightarrow A$ and suppose that $H(t, s)$ is a homotopy connecting the identity on X to the constant map. If we compose with the canonical inclusion map $i : A \hookrightarrow X$ then we get a homotopy

$$H' : A \times I \to X \times I \to X \to A$$

This is a homotopy between the identity on A and the constant map.

Exercise 0.10:

We need to prove the following

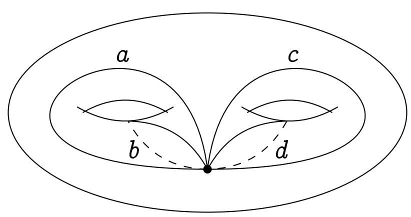
1. A space X is contractible iff every map $f : X \to Y$ for arbitrary Y, is nullhomotopic.
2. X is contractible iff every map $f : Y \to X$ is nullhomotopic.

For (1) we suppose that X is contractible. Then we can find some point $x \in X$ and homotopy $H : X \times I \to X$ such that $H_0 = 1$ and $H_1 = x$. Then the composition with $f : X \to Y$ gives $f \circ H : X \times Y \to Y$, which is a homotopy from f to the constant map at $f(x)$. So f is nullhomotopic. Conversely, suppose that every map $f : X \to Y$ for all Y is nullhomotopic. Then in particular, if we set $Y = X$ and $f = 1$ then the identity is nullhomotopic and X is contractible.

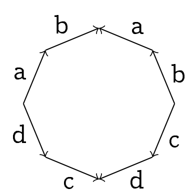
For (2) we again begin with X contractible. If we look at the map $f : Y \to X$, then the map $F : Y \to I \to X$ given by $(y, t) \mapsto H(f(y), t)$ is a homotopy from f to the constant map at a. So f is nullhomotopic. In the reverse direction, we suppose that every map $f : Y \to X$ is nullhomotopic. Then we have that the identity map $X \to X$ is nullhomotopic. Thus, X is contractible.

Exercise A1:

Consider the closed orientable surface of genus 2, $\mathbb{T}_2$. We label the four generators for $\pi_1(\mathbb{T}_2)$ as follows,



We can identify the above figure with the following,



If we look at the fundamental group of an individual torus, $\pi_1(\mathbb{T}) = \mathbb{Z} \times \mathbb{Z}$, we see that if we map each generator of $\pi_1(T)$ to a set of loops, a, b and c, d, respectively. Then we can generate the fundamental group of $\mathbb{T}_2$ by joining all of these at a point. To get the homomorphism onto the free abelian group of rank four we notice that the path around the boundary is given by $aba^{-1}b^{-1}cdc^{-1}d^{-1}$, which is an element of the free product of the groups generated by each loop. When written in group notation this is the product of the commutators $[a, b] \cdot [c, d]$. If C is the commutator subgroup of $\pi_1(\mathbb{T}_2)$ then $\pi_1(G)/C$ is the abelianization of the free product $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. The homomorphism is given explicitly by $x \mapsto xC$.