Exercise 1.3.5:

We will use the tube lemma to produce such a neighborhood of the left edge. Let $p : \tilde{X} \to X$ be a covering space. We must find points that are "close-enough" in the lift to the inverse image of the left most line. More precisely, if we consider the function $pd : \tilde{X} \times \tilde{X} \to X$ given by $(x, y) \mapsto d(p(x), p(y))$, then because p is locally injective (locally a homeomorphism for a good enough cover by the definition), we can find an open neighborhood of the point $(x, x) \in \tilde{X} \times \tilde{X}$ for x in the inverse image of the left line $U_x \times U_x$ such that

$$(U_x \times U_x) \cap pd^{-1}(0)^c \subset \{(y, y) : y \in \tilde{X}\}$$

Then we use the compactness of the left line $\ell \subset X$ to see that its inverse image ℓ^{-1}, must also be compact and the set

$$U = \bigcup_{y=pd^{-}1x} (U_d \times U_d)$$

is an open cover. So we see that $\ell^{-1} \times \ell^{-1}$ is contained in $U \cup pd^{-}1(0)$ by the above. Now we can apply the tube lemma to find open V_1, V_2 such that

$$\ell^{-1} \times \ell^{-1} \subset V_1 \times V_2 \subset U \cup pd^{-}1(0)$$

Then clearly $V = V_1 \cap V_2$ is a desired enighborhood by construction.

To see that X has no simply connected covering space, we observe that any open neighborhood of the left line necessarily intersects finitely many of the lines $\{1/n\} \times I$. And so is homeomorphic to a disjoint union of intervals. Consequently, X is not locally path-connected. Then the classification theorem for covering spaces is violated and so X cannot have a simply connected covering space.

Exercise 1.3.6:

We are initially given the following covering space $\tilde{X}$ of the shrinking wedge of circles:

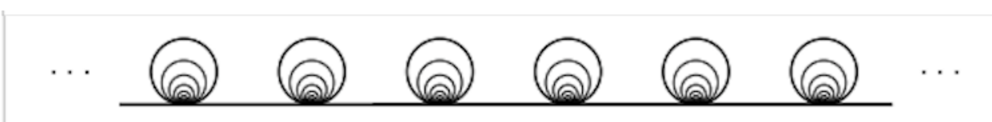

Figure 1: The initial covering space, $\tilde{X}$

We create the covering map as follows: the bottom line is identified with $\mathbb{R}$ and becomes the usual covering space of the outer circle in an individual covering of the shrinking wedge. Each of the circles, of radius $1/n$ is identified with the circle of radius $1/(n+1)$ in the base space. This is clearly a covering space map as the inverse of every open set is a disjoint union of copies in the covering space.

Now we will create a 2-sheeted cover, Y of the space $\tilde{Y}$. By gluing two identical copies together in the following way: We then define the map in the following way:

1. Send one of the base lines (say the bottom one) in Y to the base line in $\tilde{X}$.
2. Send the edge (v_n, v_{n+1}) in the other base line to the outermost loop of the n^{th} wedge in $\tilde{X}$, where v_n is the common vertex to all of the shrinking wedges in the copy of X.
3. Send each pair of shrinking wedges in Y to the corresponding wedge in $\tilde{X}$ (map the aligned copies to the top one in the diagram) in such a way that the loop of radius $1/n$ is sent to a loop of radius $1/(n+1)$

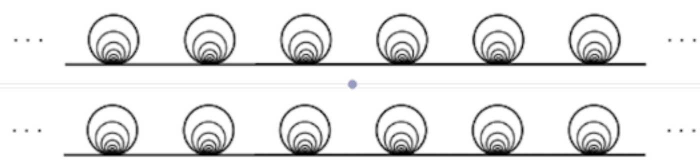

Figure 2: The covering space of $\tilde{X}$, Y

To see that this is a covering map we note that any open set in X lifts to two open sets in Y. If the open set is on the base line, it lifts to a copy of itself in one base line of Y as well as an outer circle of one of the wedges in Y. Otherwise, it is in a wedge and is sent to the corresponding copies of itself in Y but one loop inward as described above. This is a two sheeted cover.

To see that the composition of maps $q : Y \to \tilde{X}$ and $p : \tilde{X} \to X$ is not a covering space consider the common vertex of loops in X, call it x. Then we can lift any neighborhood U of x via p^{-1} to a set of neighborhoods in $\tilde{X}$ about the set of basepoints of each copy of X. We then lift each of these neighborhoods into Y. Because the points to adjacent to $(p^{-1}(x))_n$ will map to different wedges, depending on which side they were on (rule 2 described above),

Exercise 1.3.7:

Exercise 1.3.9:

Let X be a path connected, locally path-connected space such that $\pi_1(X)$ is finite. If we look at the induced homomorphism $f_* : \pi_1(X) \to \pi_1(S^1)$ we see that $f_*(\pi_1(X))$ is a finite subgroup of $\pi_1(S^2) \cong \mathbb{Z}$, which means it is the trivial group and f_* is the trivial homomorphism.

Because X is path-connected and locally path connected any map $f : X \to S^1$ lifts to map $\tilde{f} : \mathbb{R} \to S^1$ such that $f = p\tilde{f}$. Moreover, $\mathbb{R}$ is contractible, so the identity map in $\mathbb{R}$ is homotopic to a constant map. But, $f_*(\pi_1(X)) \leq p_*(\pi_1(\mathbb{R}))$ and f_* is the identity, so it must be homotopic to a constant map and so any map in X is homotopic to a constant.

Exercise 1.3.11:

Consider the covering space consisting of two vertices v_0, v_1 with a self-edge b from v_0 to itself and directed edges (v_0, v_1)

Exercise 1.3.23:

Because the action is properly discontinuous for each x we can find an open set U containing x such that $g(U) \cap U$ is non-empty for only finitely many g. Suppose that $g_1, g_2, \ldots, g_n$ are the elements such that $g(U) \cap U = \emptyset$. Now we use the Hausdorff property to find an open set $V_0 \subset U$ such that $g_i(U) \cap V_0 = \emptyset$ for each i. We iterate this process for each of the g_i (the Hausdorff property guarantees we can do this finitely many times by induction) to find sets $V_j \subset g_j(U)$ containing x that are all disjoint. Then we set

$$W = \bigcap_{j=0}^{n} g_j^{-1}(V_j \cap g_j(V_0))$$

Then W contains X and the $g(W)$ are pariwise disjoint for all $g \in G$.

Exercise A1:

(a)

(b)

Exercise A2:

(a)

(b)