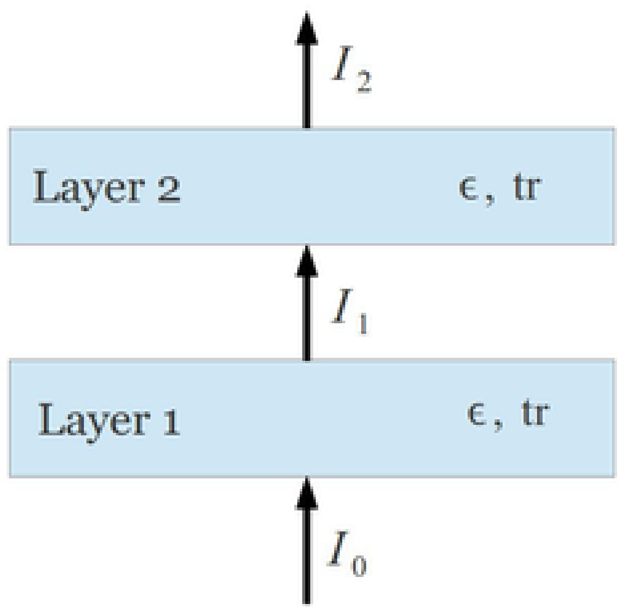
1 Introduction

This reading extends the one and two layer model of the greenhouse effect given in Dessler Section 4.3 to a more realistic atmosphere like Earth's that isn't completely black in the longwave. We will be coming back to these equations quite a few times in the class problems/assignments. For the Day 5 quiz, you only need to read through the first paragraph of section 4 below.

2 Transmitted radiation through two layers

From earlier classes, we are aware of how radiation can be transmitted, reflected or absorbed by matter. We've discussed how a "grey" surface absorbs a fraction of the incident radiation, the rest being either reflected or transmitted. Let's now consider what happens when light is shone on two or more layers with the same optical properties in series. We will take the reflectivity of the material to be zero (α=*0*), which corresponds to greenhouses gasses and longwave radiation, which is transmitted, absorbed, emitted but not reflected (at least in EOSC 340). The situation is illustrated below in Figure 1.



We can use the relationships between incident and reflected/absorbed radiation that we learned about in Day 03 to calculate the intensity of the radiation being transmitted through layers 1 and 2. For starters, we know that the absorptivity, abs, of the material is equal to the emissivity, ϵ. This is just Kirchoff's law. As the material is non-reflective, the transmissivity of the material is tr=1-abs = 1-ϵ. **(From now on be aware that ϵ can represent either emissivity or absorptivity, depending on context).** The radiation transmitted by layer 1 is proportional to the incident radiation so we write

$$I_1 = trI_0 = (1-\epsilon)I_0$$

Likewise, the radiation transmitted by layer 2 is proportional to the incident radiation on it (i.e. *I1*). Hence,

$I_2 = trI_1 = tr\left(tr\, I_0\right) = \mathrm{tr}^2 I_0 = \left(1-\epsilon\right)^2 I_0$

Adding a second layer with identical optical properties is like making the first layer twice as thick but, rather than doubling, the total transmissivity is the square of the individual transmissivities. Total transmissivity is given by

$tr_{total} = (1-\epsilon)^2$

You may note that, as $1-\epsilon < 1$ by definition, $(1-\epsilon)^2 < (1-\epsilon)$ which means that the total transmissivity is always less than the transmissivity of an individual layer, which makes sense because if we look through two sheets of semi-transparent material, objects on the other side are less visible than if we were to look through just a single sheet. Using the total transmissivity relationship, we can also figure out a value for the total absorptivity

$abs_{total} = \epsilon = 1 - tr_{total} = 1 - (1-\epsilon)^2$

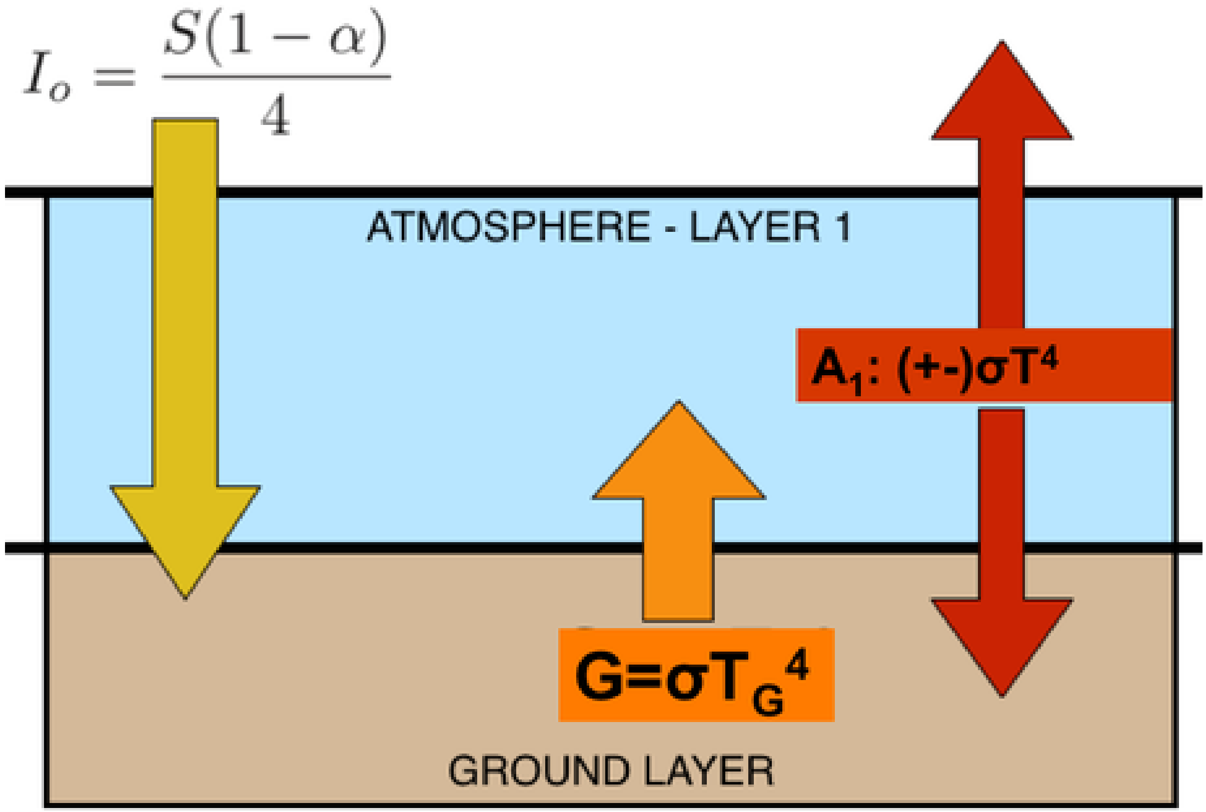
Clearly the same analysis can be applied to more than two layers, and also to layers whose transmissivities differ. We'll visit examples of this in class and at the end of this reading.

3 Stock and flow models for a multi-layered atmosphere

On Day 04 we saw a very simple radiation balance for the Earth: (short wave) solar radiation heating the Earth's surface balanced by cooling through the Planck feedback (i.e. long-wave radiation) which is a stock and flow model with energy as the stock. For equilibrium situations, these two effects balance and we can use the Stefan-Boltzmann equation to estimate the surface temperature of the Earth. This introduces the idea of a **layer model**, which represents the Earth as an infinitely large horizontal plane. We assume the plane has the mean properties of the Earth averaged over several years of time everywhere. This allows us to do calculations on a single representative square metre of the plane's surface, and ignore effects like the curvature of the Earth, the day-night cycle, and day-to-day weather variations.

Next we assume that ϵ, α, and tr can be treated as if they have one set of values in the long wave and another set in the short wave, i.e. ϵ, α, and tr can have different values at different wavenumbers. For example, in the short wave part of the spectrum, the Earth reflects a large fraction of the incident radiation. The rest of the radiation is absorbed by the Earth with none being transmitted. (From previous classes and the Trenberth diagram, α_{sw} = 0.3, $tr_{sw} = 0$, and $\epsilon_{sw} = 0.7$.) On the other hand, in the long-wave portion of the spectrum we assume the Earth is a blackbody, with $\alpha_{lw} = 0$, tr_{lw} ≈ 0, and $\epsilon_{lw} \approx 1$ Furthermore, these values can be different for the stratosphere and the troposphere, and even for the upper and lower regions of the troposphere/stratosphere.

To illustrate how layer models work, we show a very simple model in figure 2:



where the atmosphere is represented by just a single layer above the ground. For further simplicity, we take ϵ = 1 for both the ground and the atmosphere. The (long-wave) radiative flux from the ground is labelled G and the flux emitted by the atmospheric layer is A_1 . We will write G as a positive σT^4 to keep the equations simple and place a minus sign in front of it when we need to account for its upward direction. Note that the atmospheric level radiates flux A_1 both up and downwards and so will be both added and subtracted in the equations below. The next step is to write a stock-and-flow balance for the ground for an equilibrium situation. Since no energy is coming up through the bottom of the ground layer, the gain or loss of energy into the ground layer is given by the flux balance at the ground. Calling downward fluxes into the ground layer positive, and upward fluxes out negative:

$$\frac{dE_G}{dt} = I_0 + A_1 - G = 0$$

Now we do the same for the atmospheric Layer 1, again considering fluxes into the layer positive, and fluxes out of the layer negative. The equilibrium energy balance for Layer 1 is then:

$$\frac{dE_G}{dt} = G - 2A_1 = 0$$

Notice that the incoming solar radiation I_0 does not appear in this equation because the shortwave solar radiation does not interact with the atmosphere. When we subtract the net flux at the top layer from the net flux at the bottom, the two I_0 fluxes cancel. In the same way we pick up ($-2A_1$) because we when we subtract top and bottom we have $(-A_1 - +A_1) = -2A_1$. Regardless, I_0 is known (Dessler, Equation 4.2, with $I_0 = E_{in}/area$), so we have two equations: and in two unknowns (G and A). We can therefore solve for either the ground or the atmospheric radiative flux by substitution:

$$G = 2A_1 \tag{1}$$
$$G = I_0 + A_1 \tag{2}$$
$$A_1 = I_0 \tag{3}$$
$$G = 2I_0 \tag{4}$$

Thus the presence of a black atmosphere doubles the energy reaching the surface without changing the energy leaving the top of the atmosphere.

The next step is to split the atmosphere into several layers: the ground is one layer and the atmosphere can be represented by two-or-more atmospheric layers. The reason for doing this is so that each layer can have its own optical properties (troposphere and stratosphere may not have the same values of ϵ, α, and tr) or that increasing concentrations of greenhouse gases, such as CO2 can be represented by adding another layer to that part of the atmosphere.

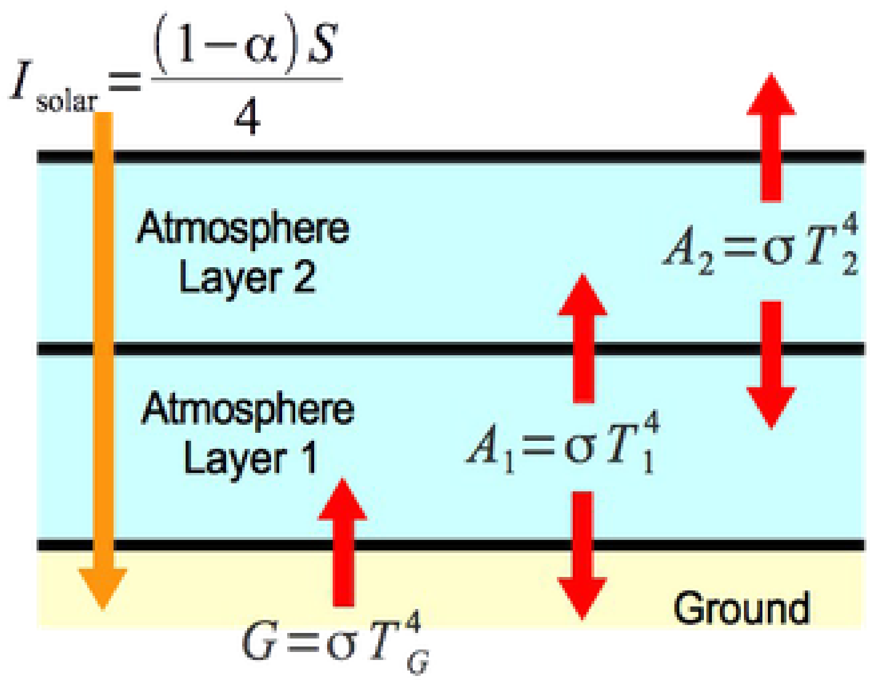


Figure 3: schematic of a two-layer atmosphere model

A schematic of a very simple two-layer atmosphere model is shown in figure 3. In this model, the net solar radiation (i.e. short wave) being absorbed by the ground is represented by an orange arrow and the long-wave radiation emitted by the ground and each of the two atmospheric layers is represented by red arrows. We can use this representation to help us establish stock and flow models for each layer. Then, by substitution and elimination of terms, we can work out the radiative power and therefore the temperature of each layer.

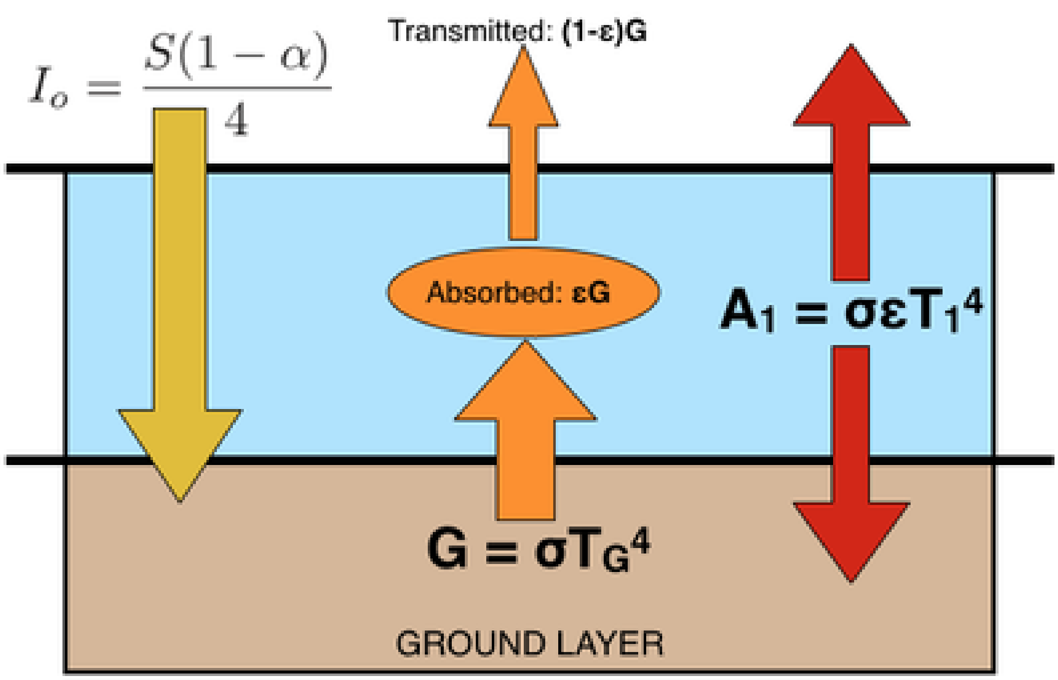
4 Non-equilibrium stock-and-flow layer models

If the stock-and-flow models for each layer are out of equilibrium, the analysis presented above cannot be followed – a more complicated analysis using a system of differential equations must be solved in order to find the radiative fluxes. (Remember that each stock-and-flow model is itself a differential equation.) However, we can still use these equations to determine qualitatively whether a given layer is heating and cooling: the stock is the energy in each layer so if the stock is increasing ($I_{in} + I_{out} > 0$ so $dE/dt > 0$) then the layer is heating and vice versa.

5 The Greenhouse Effect

Figure 4 on below shows a more realistic (single) layer model, with a layer representing the troposphere lying over the surface of the Earth. In this course we define the **greenhouse effect** as the difference between the amount of longwave flux escaping to space (the orange and red arrows at the top of the figure) and the longwave radiation leaving the ground (the arrow labeled G at the surface). We assume the atmosphere is completely transparent to short-wave radiation, but is semi-transparent to long-wave radiation (α = 0 and tr = 1 – ϵ where ϵ is somewhere in the range $0 \rightarrow 1$). The radiation emitted by the ground is $G = \sigma T_G^4$. However, *now only a fraction* ϵ of the Earth's radiation is absorbed by the atmosphere (note that we are using ϵ=abs), which heats the atmosphere so that it emits thermal radiation $A_1 = \epsilon\sigma T_1^4$. How much is kept in the layer? If we subtract the net flux at the top from the bottom, keeping track of signs, we get:

$$top - bottom = \left(I_0 - (1-\varepsilon)\,\sigma T_G^4 - \varepsilon\sigma T_1^4\right) - \left(I_0 - \sigma T_G^4 + \varepsilon\sigma T_1^4\right) = \varepsilon\sigma T_G^4 - 2A_1 = \varepsilon G - 2A_1$$



How do we solve for these values? We need to write two equations for the ground and the atmosphere and solve for two unknowns. As before, the ground is heated by fluxes from the sun and the atmosphere, I_0 and A_1, and cools by emitting flux G. The energy balance for the ground is as before

$\frac{dE_G}{dt} = I_0 - G + A_1$

The atmosphere, on the other hand, absorbs radiation *G* from the ground below but emits radiation A_1 both upward and downward. Unlike previously, in this scenario not all the radiation incident from the ground below is absorbed in the atmosphere; only the fraction ϵ*G* is absorbed (since ϵ represents the absorptivity) and the remaining fraction *(1-ϵ)G* flows out the top of the atmosphere. Using our result, we get the following energy stock and flow equation for the atmospheric layer:

$\frac{\mathrm{dE_1}}{\mathrm{dt}} = \varepsilon G - 2A_1$

Now if we again assume the climate is in equilibrium, so that $dE/dt = 0$ for the ground and the atmosphere, we get two equations, $\epsilon G = 2A_1$ and $I_0 = G - A_1$. Substituting the first equation into the second gives $A_1 = \epsilon I_0/(2-\epsilon)$,* and substituting this back into the first equation gives

$G = 2I_0/(2-\epsilon)$.

Now we can substitute $G = \sigma T_G^4$ and $A_1 = \epsilon\sigma T_1^4$ and we get:

$$\frac{\epsilon I_0}{(2-\epsilon)} = \epsilon\sigma T_1^4$$

$$\frac{2I_0}{(2-\epsilon)} = \sigma T_G^4$$

and solving for T_1 and T_G gives:

$$T_1 = \sqrt[4]{\frac{I_0}{(2-\epsilon)\sigma}} \quad (5)$$

$$T_G = \sqrt[4]{\frac{2I_0}{(2-\epsilon)\sigma}} \quad (6)$$

We can check our result by setting ϵ = 0 in (6); if the atmosphere is transparent, we should get the same answer for the ground temperature as we did for the Earth without an atmosphere. We find:

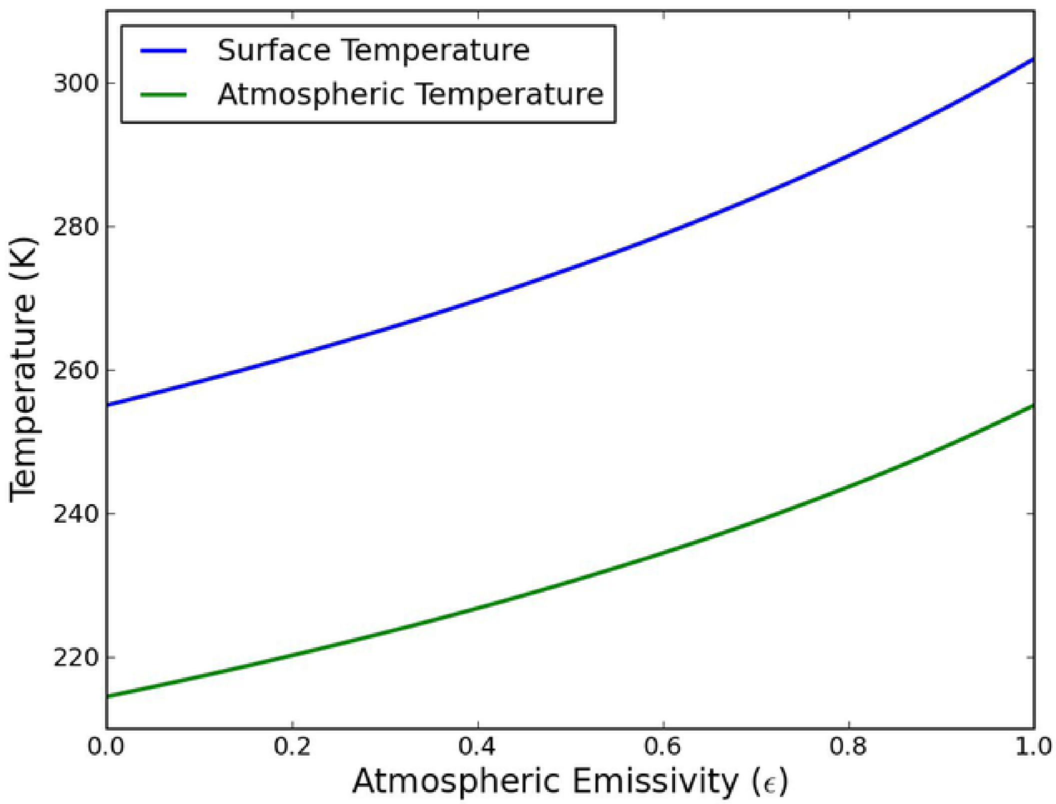
$$T_G = \sqrt[4]{\frac{I_0}{\sigma}}$$

which is indeed just the Planck function, rearranged to solve for the temperature!

If we increases ϵ from 0 → 1, we see that both the atmosphere and the ground get warmer (Fig. 5). At ϵ = 1, the ground temperature becomes:

$$T_G = \sqrt[4]{\frac{2I_0}{\sigma}}$$

indicating that the amount of radiation absorbed by the Earth has *doubled* due to the presence of an opaque atmosphere overhead, which agrees with our single-layer black-atmosphere model from Section 2.



In all of the above examples, the presence of an atmosphere increases the outflow of energy from the Earth's surface and its surface temperature. **Recall the definition of the greenhouse effect above: it is s defined as the difference between the outgoing longwave radiation at the top of the atmosphere minus the outgoing longwave radiation at the earth's surface.** Note that defined this way, the greenhouse effect (Wm^{-2}) will be a positive number. Without the atmosphere providing a greenhouse effect, the temperature of the Earth's surface would be below the freezing point of water, and probably would be unable to support life. To illustrate this point, let us consider the Earth's moon which has no atmosphere yet receives the same amount of solar radiation. The surface temperatures range from -153 °C on the dark side to 107 °C on the side facing the sun: that's an average temperature of about -23 °C!

6 More than two layers

If we've got a computer, then we're free to add more layers, which gives us more a more detailed picture of the vertical temperature structure of the atmosphere. For example, suppose we want to construct a 10 layer atmosphere that has a total emissivity of ϵ total =0.8? That means that tr_{total} = 1 - 0.8 = 0.2. If each of the layers has the same transmissivity tr layer , then $tr_{total} = (tr_{layer})^{10} = 0.2$. Therefore

$$tr_{layer} = 0.1^{1/10} = 0.2^{0.1} = 0.85 \tag{7}$$
$$\epsilon_{layer} = 1 - 0.85 = 0.15 \tag{8}$$

With 10 layers, each with $\epsilon = 0.15$, we need to solve 11 equations (including the surface) for the 11 unknown equilibrium temperatures of the layer + surface.

7 Combining two gasses

Suppose an atmospheric layer has two gasses that absorb and emit in the 9-10 μm wavelength range. One has a transmissivity tr 1 =0.8 and the other has a transmissivity tr 2 =0.4. What is the total emissivity of the layer due to the two gasses? As with distinct layers, you can multiply the two transmissivities to find a total transmissivity, and use this to get the total emissivity:

$tr_{tot} = tr_1 \times tr_2 = 0.8x0.4 = 0.32$
$\epsilon_{total} = 1 - tr_{total} = 1 - 0.32 = 0.68$

8 Summary

- The total transmissivity of two layers of the same material is the square of the transmissivities of the individual layers.

$$tr_{total} = (1 - \epsilon)^2$$

- The total absorptivity is then given by

$$abs_{total} = 1 - (1 - \epsilon)^2$$

- Simple layer models can be used to calculate the radiative flues from the ground and atmosphere by setting the dE/dt for each layer to zero (equilibrium). If the layers equation is out of balance, then the layer is either heating or cooling.
- The temperature of the climate is controlled by the energy flow in and out of the climate system, just like a stock and flow problem. The only flows of energy in and out of the climate system as a whole happen due to electromagnetic radiation.
- Short-wave radiation is emitted by the Sun, and is composed of ultraviolet, visible, and near infra-red light. Long-wave radiation is emitted by the Earth's surface and atmosphere, and is composed of far infra-red light.

- The atmosphere increases the temperature of the Earth's surface by absorbing the Earth's long-wave radiation and emitting its own long-wave radiation back to the earth. This extra radiation flow to the surface from the atmosphere causes the greenhouse effect.
- The greenhouse effect is defined as the difference between the long-wave radiation emitted by the surface of the Earth and the long-wave radiation leaving the top of the Earth's atmosphere.