Advanced Game Engineering - Mathematical Questions for Games

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1 What is the difference between a point and a vector?

The difference is precisely that between location and displacement.

- Points are locations in space.
- Vectors are displacements in space.

point: A point is a location in a coordinate system, that is a location defined relatively to an origin. If you were to move the origin without moving the point, then the coordinates of the point would change.Point usually refers to topological structure. Eg., point-set topology; a sequence of points converging; a neighborhood of a point; etc.

vector: A vector is a more general object. No matter where you draw a vector

$$\vec{v} \tag{1}$$

on a plane, it is still the same. If you were to move the origin, the components of the vector would not change. You can also think of a vector as a transformation. It can be applied anywhere and have the same effect: displacing a point of a certain distance in a precise direction.Vector usually refers to vector space/normed space/inner product space structure - Eg., adding two vectors; scaling a vector; taking the norm of a vector; etc.

2 What is a right handed coordinate system?

A right-hand coordinate system is modelled by the right hand! Hold out your right hand in a fist with the thumb facing up. Stick up your thumb like your giving a thumbs up. Now extend the index finger, like your pointing at something in from of you. Finally, extend the middle finger side ways so that it's at a right-angle with the thumb and the index finger. The index finger is in the direction of the positive xx-axis, the middle finger in the direction of the positive yy-axis and the thumb the positive zz-axis.

The rule which determines the orientation of the cross product

$$\vec{u} \times \vec{v} \tag{2}$$

. The right-hand rule states that the orientation of the vectors' cross product is determined by placing

$$\vec{u} \tag{3}$$

and

$$\vec{v} \tag{4}$$

tail-to-tail, flattening the right hand, extending it in the direction of

$$\vec{u} \tag{5}$$

, and then curling the fingers in the direction that the angle

$$\vec{v} \tag{6}$$

makes with

$$\vec{u} \tag{7}$$

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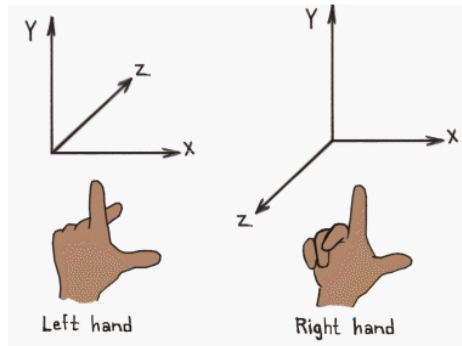

Figure 1: *A three-dimensional coordinate system in which the axes satisfy the right-hand rule.*

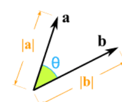

Figure 2: *Dot Product-1.*

. The thumb then points in the direction of

$$\vec{u} \times \vec{v} \tag{8}$$

A three-dimensional coordinate system in which the axes satisfy the right-hand rule is called a right-handed coordinate system, while one that does not is called a left-handed coordinate system.

3 How do you compute the dot product of two vectors? Informally, what does the dot product do? Give an example where we need the dot product for Computer Graphics applications.

Vectors can be multiplied using the "Dot Product". The Dot Product is written using a central dot:

$$\vec{a} \cdot \vec{b} \tag{9}$$

This means the Dot Product of a and b.

We can calculate the Dot Product of two vectors this way:

$$\vec{a} \cdot \vec{b} = |a| \times |b| \times \cos(\theta) \tag{10}$$

$$|a|\ is\ the\ magnitude\ (length)\ of\ vector\ \vec{a}$$

$$|b|\ is\ the\ magnitude\ (length)\ of\ vector\ \vec{b}$$

$$\theta\ is\ the\ angle\ between\ \vec{a}\ and\ \vec{b}$$

we can calculate it this way:

$$\vec{a} \times \vec{b} = a_x \times b_x + a_y \times b_y \tag{11}$$

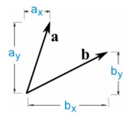

Figure 3: *Dot Product-2.*

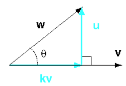

Figure 4: *The Projection.*

Computer Graphics applications using the dot prouct:

- 3D projection
  3D projection is a method of mapping three dimensional points to a two dimensional plane. As most current methods for displaying graphical data are based on planar two dimensional media, the use of this type of projection is widespread, especially in computer graphics, engineering and drafting.
- Ray tracing
  Ray tracing is a technique for generating an image by tracing the path of light through pixels in an image plane. The technique is capable of producing a very high degree of photorealism; usually higher than that of typical scanline rendering methods, but at a greater computational cost.
- Shading
  Shading refers to depicting depth in 3D models or illustrations by varying levels of darkness. It is a process used in drawing for depicting levels of darkness on paper by applying media more densely or with a darker shade for darker areas, and less densely or with a lighter shade for lighter areas. There are various techniques of shading including cross hatching where perpendicular lines of varying closeness are drawn in a grid pattern to shade an area. The closer the lines are together, the darker the area appears. Likewise, the farther apart the lines are, the lighter the area appears. The term has been recently generalized to mean that shaders are applied.

4 How do you compute the projection of one vector onto another?

$$In\ the\ diagram\ \vec{w}\ and\ \vec{v}\ are\ any\ two\ vectors.$$

$$We\ want\ a\ vector\ \vec{u}\ that\ is\ orthogonal\ to\ \vec{v}.$$

$$And\ we\ want\ scalar\ \vec{k}\ so\ that:$$

$$\vec{w} = k \times vecv + \vec{u}$$

$$Thenk\vec{v}iscalledtheprojectionof\vec{u}onto\vec{v}.$$

5 What are the implicit and parametric forms of a line? Why are they named implicit and parametric?

Parametric Forms(Parametric Equation): In mathematics, parametric equations of a curve express the coordinates of the points of the curve as functions of a variable, called a parameter.

For example
$$\begin{cases} x = \cos t; \\ y = \sin t; \end{cases} \tag{12}$$

are parametric equations for the unit circle, where t is the parameter. Together, these equations are called a parametric representation of the curve.

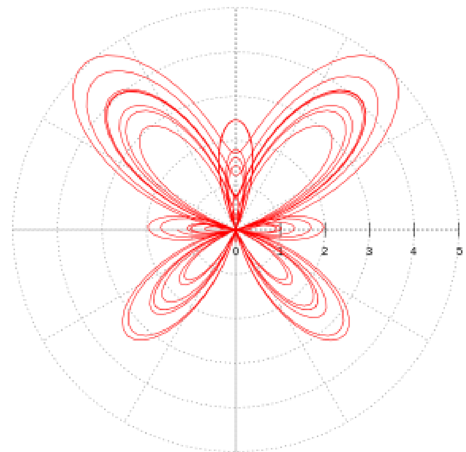

Figure 5: *One example of a sketch defined by parametric equations is the butterfly curve.*

A common example occurs in kinematics, where the trajectory of a point is usually represented by a parametric equation with time as the parameter.

The notion of parametric equation has been generalized to surfaces, manifolds and algebraic varieties of higher dimension, with the number of parameters being equal to the dimension of the manifold or variety, and the number of equations being equal to the dimension of the space in which the manifold or variety is considered (for curves the dimension is one and one parameter is used, for surfaces dimension two and two parameters, etc.).

The parameter typically is designated t because often the parametric equations represent a physical process in time. However, the parameter may represent some other physical quantity such as a geometric variable, or may merely be selected arbitrarily for convenience. Moreover, more than one set of parametric equations may specify the same curve.

Implicit Forms(Implicit Equation): In mathematics, an implicit equation is a relation of the form R(x1,..., xn) = 0, where R is a function of several variables (often a polynomial). For example, the implicit equation of the unit circle is
$$x^2 + y^2 - 1 = 0 \tag{13}$$

An implicit function is a function that is defined implicitly by an implicit equation, by associating one of the variables (the value) with the others (the arguments). Thus, an implicit function for y in the context of the unit circle is defined implicitly by

$$x^2 + [f(x)]^2 - 1 = 0 \tag{14}$$

This implicit equation defines f as a function of x only if -1 x 1 and one considers only non-negative (or non-positive) values for the values of the function.

The implicit function theorem provides conditions under which a relation defines an implicit function.

6 What is the implicit form of a circle centered at (xc, yc) with radius r?

The implicit form of a circle centered at (xc, yc) with radius r
$$\begin{gathered} (xc)^2 + (yc)^2 - r = 0 \\ c \neq 0, r > 0 \end{gathered} \tag{15}$$

7 Write an implicit equation given two points.

Given two points that is

$$(x, y) = (1, 4) \tag{16}$$

$$(x, y) = (2, 3) \tag{17}$$

So the implicit equation of the line is

$$y + x - 5 = 0 \tag{18}$$

8 Write a parametric equation for a line given two points.

Given two points that is

$$(x, y) = (1, 4) \tag{19}$$

$$(x, y) = (2, 3) \tag{20}$$

So the parametric equation of the line is

$$\begin{cases} y = 5 + t; \\ x = 5 - t; \end{cases} \tag{21}$$

9 Suppose you are given a parametric equation for a line. Convert from this representation to an implicit equation for that same line.

The simplest parametric equation of the unit circle is

$$\begin{cases} x = \cos t; \\ y = \sin t; \end{cases} \tag{22}$$

Convert from this representation to an implicit equation

$$x^2 + y^2 - 1 = 0 \tag{23}$$

10 Given an implicit equation for a line, convert it to a parametric representation.

The implicit form of a circle centered at (xc, yc) with radius r is

$$\begin{aligned} (xc)^2 + (yc)^2 - r = 0 \\ c \neq 0, r > 0 \end{aligned} \tag{24}$$

convert it to a parametric representation is

$$\begin{cases} x = \dfrac{\cos(t)}{c} + r; \\ y = \dfrac{\sin(t)}{c} - r; \\ c \neq 0, r > 0 \end{cases} \tag{25}$$

11 Given a slope / intercept equation for a line, convert it to parametric and implicit representations.

In mathematics, the slope or gradient of a line is a number that describes both the direction and the steepness of the line.Slope is often denoted by the letter m. The direction of a line is either increasing, decreasing, horizontal or vertical.

- A line is increasing if it goes up from left to right. The slope is positive, i.e. m¿0.

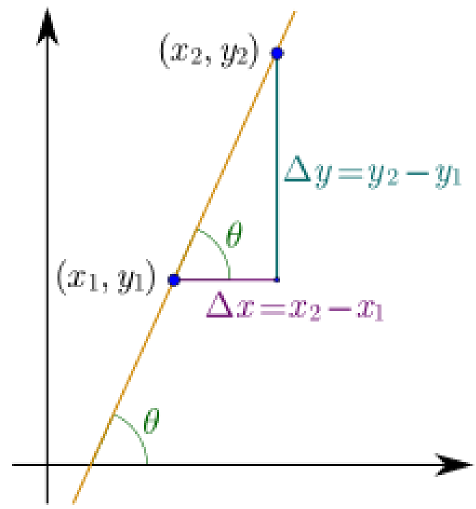

Figure 6: *Slope:*$m = \frac{\Delta y}{\Delta x} = \tan(\theta)$.

- A line is decreasing if it goes down from left to right. The slope is negative, i.e. m¡0.
- If a line is horizontal the slope is zero. This is a constant function.
- If a line is vertical the slope is undefined (see below).

The steepness, incline, or grade of a line is measured by the absolute value of the slope. A slope with a greater absolute value indicates a steeper line

Suppose a line runs through two points:

$$\begin{cases} P = (1, 2); \\ Q = (13, 8); \end{cases} \tag{26}$$

By dividing the difference in y-coordinates by the difference in x-coordinates, one can obtain the slope of the line:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 2}{13 - 1} = \frac{6}{12} = \frac{1}{2} \tag{27}$$

The implicit form of a circle centered at (xc, yc) with radius r is

$$\frac{1}{2}x + y - \frac{3}{2} = 0 \tag{28}$$

convert it to a parametric representation is

$$\begin{cases} x = 2t; \\ y = \dfrac{3}{2} - t; \end{cases} \tag{29}$$

12 Give a situation in a computer graphics application where you would prefer a parametric representation of a line to an implicit representation.

3D examples Parametric equations are convenient for describing curves in higher-dimensional spaces. For example:

$$\begin{cases} x = \dfrac{a}{\cos(t)} \\ y = \dfrac{a}{\sin(t)} \\ z = bt \end{cases} \tag{30}$$

describes a three-dimensional curve, the helix, with a radius of a and rising by 2b units per turn. Note that the equations are identical in

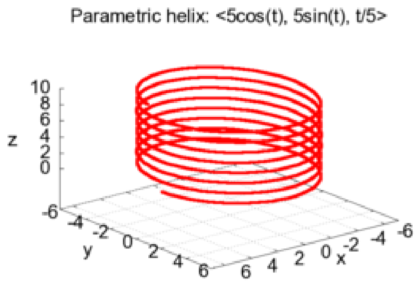

Figure 7: *Parametric helix.*

the plane to those for a circle. Such expressions as the one above are commonly written as

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = (a\cos(t), a\sin(t), bt) \tag{31}$$

13 Give a situation where you would prefer an implicit representation of a line to a parametric one.

Applications in economics

Marginal rate of substitution In economics, when the level set R(x, y) = 0 is an indifference curve for the quantities x and y consumed of two goods, the absolute value of the implicit derivative dy/dx is interpreted as the marginal rate of substitution of the two goods: how much more of y one must receive in order to be indifferent to a loss of one unit of x.

Marginal rate of technical substitution Similarly, sometimes the level set R(L, K) is an isoquant showing various combinations of utilized quantities L of labor and K of physical capital each of which would result in the production of the same given quantity of output of some good. In this case the absolute value of the implicit derivative dK/dL is interpreted as the marginal rate of technical substitution between the two factors of production: how much more capital the firm must use to produce the same amount of output with one less unit of labor.

Optimization Often in economic theory, some function such as a utility function or a profit function is to be maximized with respect to a choice vector x even though the objective function has not been restricted to any specific functional form. The implicit function theorem guarantees that the first-order conditions of the optimization define an implicit function for each element of the optimal vector x * of the choice vector x. When profit is being maximized, typically the resulting implicit functions are the labor demand function and the supply functions of various goods. When utility is being maximized, typically the resulting implicit functions are the labor supply function and the demand functions for various goods.

Moreover, the influence of the problem's parameters on x *łthe partial derivatives of the implicit functionłcan be expressed as total derivatives of the system of first-order conditions found using total differentiation.

14 How do you convert an arbitrary vector to a unit vector?

vector In mathematics, a unit vector in a normed vector space is a vector (often a spatial vector) of length 1. A unit vector is often denoted

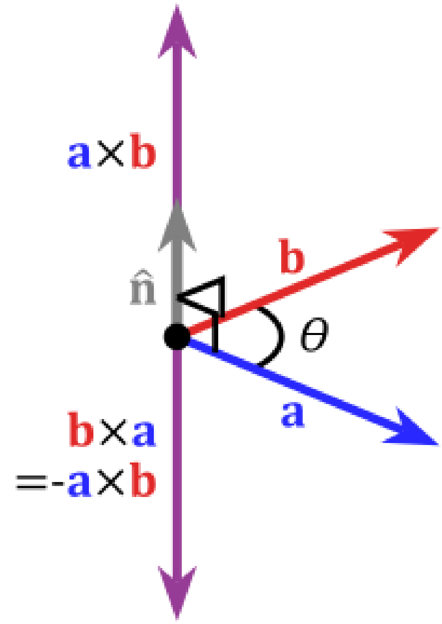

Figure 8: *The cross-product in respect to a right-handed coordinate system.*

by a lowercase letter with a "hat":

$$\vec{i} \tag{32}$$

i-hat (pronounced "i-hat"). The term direction vector is used to describe a unit vector being used to represent spatial direction, and such quantities are commonly denoted as d. Two 2D direction vectors, d1 and d2 are illustrated. 2D spatial directions represented this way are equivalent numerically to points on the unit circle.

The same construct is used to specify spatial directions in 3D. As illustrated, each unique direction is equivalent numerically to a point on the unit sphere.

The normalized vector or versor ? of a non-zero vector u is the unit vector in the direction of u, i.

How do you convert an arbitrary vector to a unit vector? u-hat equals the vector u divided by its length

$$\vec{u} = \frac{\vec{u}}{||u||} \tag{33}$$

15 How do you compute the cross product of two vectors? Informally, what does the cross product do? Give an example where we need the cross product for Computer Graphics applications.

Cross Product In mathematics and vector calculus, the cross product or vector product (occasionally directed area product to emphasize the geometric significance) is a binary operation on two vectors in three-dimensional space (R3) and is denoted by the symbol . Given two linearly independent vectors a and b, the cross product, a b, is a vector that is perpendicular to both and therefore normal to the plane containing them. It has many applications in mathematics, physics, engineering, and computer programming. It should not be confused with dot product (projection product).

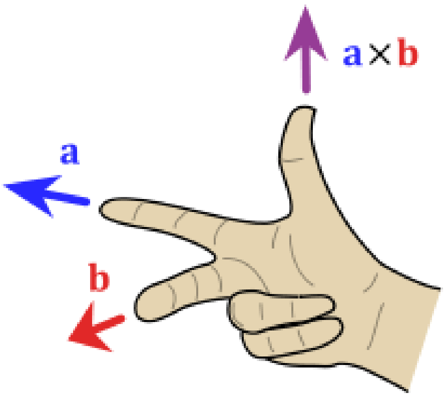

Figure 9: *Finding the direction of the cross product by the right-hand rule.*

If two vectors have the same direction (or have the exact opposite direction from one another, i.e. are not linearly independent) or if either one has zero length, then their cross product is zero. More generally, the magnitude of the product equals the area of a parallelogram with the vectors for sides; in particular, the magnitude of the product of two perpendicular vectors is the product of their lengths. The cross product is anticommutative.

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \tag{34}$$

and is distributive over addition

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \tag{35}$$

The space R3 together with the cross product is an algebra over the real numbers, which is neither commutative nor associative, but is a Lie algebra with the cross product being the Lie bracket.

Like the dot product, it depends on the metric of Euclidean space, but unlike the dot product, it also depends on a choice of orientation or "handedness". The product can be generalized in various ways; it can be made independent of orientation by changing the result to pseudovector, or in arbitrary dimensions the exterior product of vectors can be used with a bivector or two-form result. Also, using the orientation and metric structure just as for the traditional 3-dimensional cross product, one can in n dimensions take the product of n ? 1 vectors to produce a vector perpendicular to all of them. But if the product is limited to non-trivial binary products with vector results, it exists only in three and seven dimensions. If one adds the further requirement that the product be uniquely defined, then only the 3-dimensional cross product qualifies. (See Generalizations, below, for other dimensions.

Compute The cross product of two vectors a and b is defined only in three-dimensional space and is denoted by a b. In physics, sometimes the notation a b is used,[2] though this is avoided in mathematics to avoid confusion with the exterior product.The cross product a b is defined as a vector c that is perpendicular to both a and b, with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span.

The cross product is defined by the formula

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta \; \mathbf{n} \tag{36}$$

The standard basis vectors i, j, and k satisfy the following equalities in

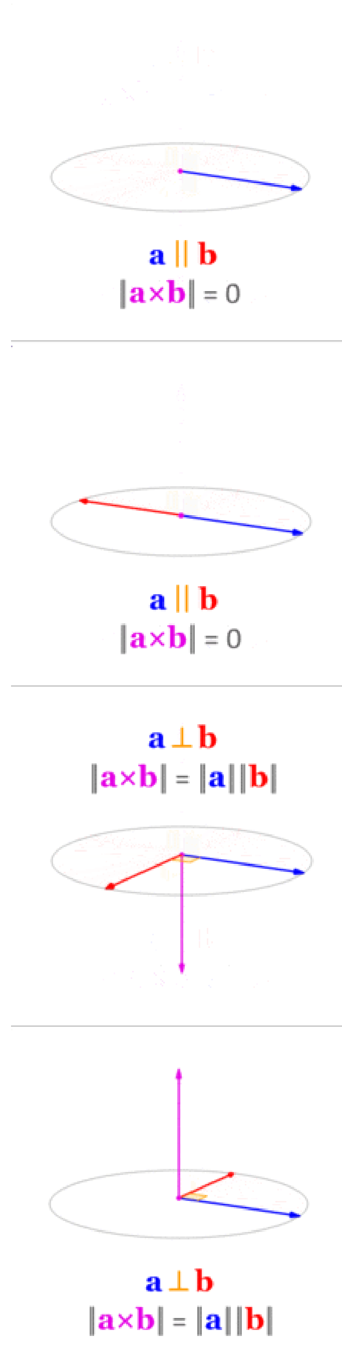

Figure 10: *The cross product a × b (vertical, in purple) changes as the angle between the vectors a (blue) and b (red) changes. The cross product is always perpendicular to both vectors, and has magnitude zero when the vectors are parallel and maximum magnitude* $||a||$ $||b||$ *when they are perpendicular..*

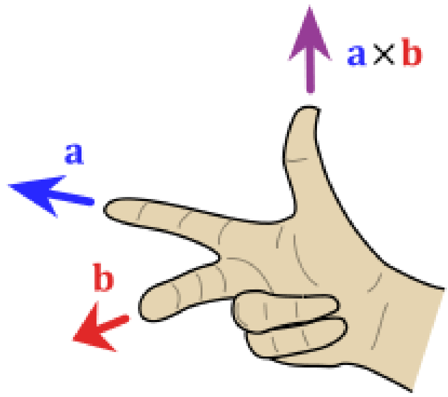

Figure 11: *Finding the direction of the cross product by the right-hand rule.*

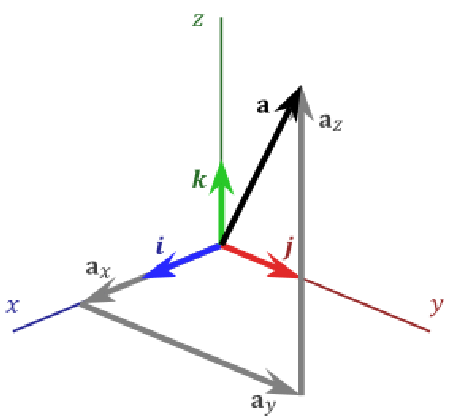

Figure 12: *Standard basis vectors (i, j, k, also denoted e1, e2, e3) and vector components of a (ax, ay, az, also denoted a1, a2, a3).*

a right hand coordinate system:

$$\begin{cases} \mathbf{i} = \mathbf{j} \times \mathbf{k} \\ \mathbf{j} = \mathbf{k} \times \mathbf{i} \\ \mathbf{k} = \mathbf{i} \times \mathbf{j} \end{cases} \tag{37}$$

which imply, by the anticommutativity of the cross product, that

$$\begin{cases} \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} \end{cases} \tag{38}$$

The definition of the cross product also implies that

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \tag{39}$$

These equalities, together with the distributivity and linearity of the cross product (but both do not follow easily from the definition given above), are sufficient to determine the cross product of any two vectors u and v. Each vector can be defined as the sum of three orthogonal components parallel to the standard basis vectors:

$$\begin{cases} \mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \\ \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \end{cases} \tag{40}$$

Their cross product u v can be expanded using distributivity:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1(\mathbf{i} \times \mathbf{i}) + u_1v_2(\mathbf{i} \times \mathbf{j}) + u_1v_3(\mathbf{i} \times \mathbf{k}) + \\ &\quad u_2v_1(\mathbf{j} \times \mathbf{i}) + u_2v_2(\mathbf{j} \times \mathbf{j}) + u_2v_3(\mathbf{j} \times \mathbf{k}) + \\ &\quad u_3v_1(\mathbf{k} \times \mathbf{i}) + u_3v_2(\mathbf{k} \times \mathbf{j}) + u_3v_3(\mathbf{k} \times \mathbf{k}) \end{aligned} \tag{41}$$

This can be interpreted as the decomposition of u v into the sum of nine simpler cross products involving vectors aligned with i, j, or k. Each one of these nine cross products operates on two vectors that are easy to handle as they are either parallel or orthogonal to each other. From this decomposition, by using the above-mentioned equalities and collecting similar terms, we obtain:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} = {} & u_1v_1\mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - \\ & u_2v_1\mathbf{k} - u_2v_2\mathbf{0} + u_2v_3\mathbf{i} + \\ & u_3v_1\mathbf{j} - u_3v_2\mathbf{i} - u_3v_3\mathbf{0} \\ = {} & (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \end{aligned} \tag{42}$$

meaning that the three scalar components of the resulting vector s = s1i + s2j + s3k = u v are

$$\begin{cases} s_1 = u_2v_3 - u_3v_2 \\ s_2 = u_3v_1 - u_1v_3 \\ s_3 = u_1v_2 - u_2v_1 \end{cases} \tag{43}$$

Using column vectors, we can represent the same result as follows:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} \tag{44}$$

Computational geometry The cross product appears in the calculation of the distance of two skew lines (lines not in the same plane) from each other in three-dimensional space.

The cross product can be used to calculate the normal for a triangle or polygon, an operation frequently performed in computer graphics. For example, the winding of a polygon (clockwise or anticlockwise) about a point within the polygon can be calculated by triangulating the polygon (like spoking a wheel) and summing the angles (between the spokes) using the cross product to keep track of the sign of each angle.

In computational geometry of the plane, the cross product is used to determine the sign of the acute angle defined by three points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ and $p_3 = (x_3, y_3)$. It corresponds to the direction of the cross product of the two coplanar vectors defined by the pairs of points p_1, p_2 and p_1, p_3, i.e., by the sign of the expression $P = (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)$. In the "right-handed" coordinate system, if the result is 0, the points are collinear; if it is positive, the three points constitute a positive angle of rotation around p_1 from p_2 to p_3, otherwise a negative angle. From another point of view, the sign of P tells whether p_3 lies to the left or to the right of line p_1, p_2.

The cross product is used in calculating the volume of a polyhedron such as a tetrahedron or parallelepiped.

16 What are implicit and parametric expressions for a plane?

Parametric Forms(Parametric Equation): In mathematics, parametric equations of a curve express the coordinates of the points of the curve as functions of a variable, called a parameter.

For example

$$\begin{cases} x = \cos t; \\ y = \sin t; \end{cases} \tag{45}$$

are parametric equations for the unit circle, where t is the parameter. Together, these equations are called a parametric representation of the curve.

A common example occurs in kinematics, where the trajectory of a point is usually represented by a parametric equation with time as the parameter.

The notion of parametric equation has been generalized to surfaces, manifolds and algebraic varieties of higher dimension, with the number of parameters being equal to the dimension of the manifold or variety, and the number of equations being equal to the dimension of the space in which the manifold or variety is considered (for curves the dimension is one and one parameter is used, for surfaces dimension two and two parameters, etc.).

The parameter typically is designated t because often the parametric equations represent a physical process in time. However, the parameter may represent some other physical quantity such as a geometric variable, or may merely be selected arbitrarily for convenience. Moreover, more than one set of parametric equations may specify the same curve.

As for a plane

$$\begin{cases} x = \dfrac{1}{a} + \dfrac{t}{2a}; \\ y = \dfrac{1}{b} + \dfrac{t}{2b}; \\ z = \dfrac{1}{c} - \dfrac{t}{c}; \\ a, b, c \neq 0 \end{cases} \tag{46}$$

Implicit Forms(Implicit Equation): In mathematics, an implicit equation is a relation of the form R(x1,..., xn) = 0, where R is a function of several variables (often a polynomial). For example, the implicit equation of the unit circle is

$$x^2 + y^2 - 1 = 0 \tag{47}$$

An implicit function is a function that is defined implicitly by an implicit equation, by associating one of the variables (the value) with the others (the arguments). Thus, an implicit function for y in the context of the unit circle is defined implicitly by

$$x^2 + [f(x)]^2 - 1 = 0 \tag{48}$$

This implicit equation defines f as a function of x only if -1 x 1 and one considers only non-negative (or non-positive) values for the values of the function.

The implicit function theorem provides conditions under which a relation defines an implicit function.

As for a plane

$$\begin{cases} ax + by + cz + d = 0; \\ a, b, c \neq 0 \end{cases} \tag{49}$$

17 Write an implicit equation for a plane given three points.

Giving three points :

$$\begin{cases} p_1 = \left\{1, 2, \dfrac{21}{8}\right\} \\ p_2 = \left\{6, 8, \dfrac{15}{2}\right\} \\ p_3 = \left\{-3, 2, \dfrac{9}{8}\right\} \end{cases} \tag{50}$$

The implicit equation for the plane is:

$$\left\{ 3x + 4y + (-8z) + 10 = 0; \right. \tag{51}$$

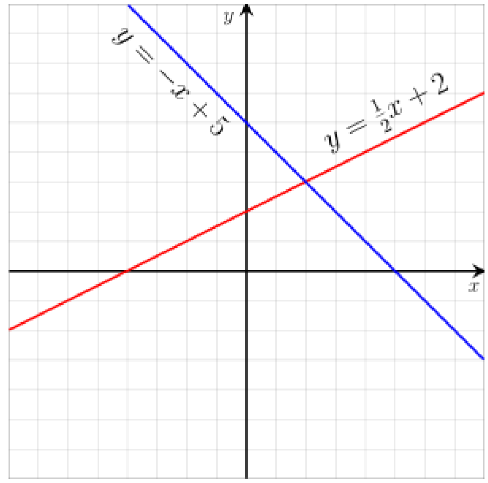

Figure 13: *Graph sample of linear equations.*

18 Write a parametric equation for a plane given three points.

Giving three points :

$$\begin{cases} p_1 = \left\{1, 2, \dfrac{21}{8}\right\} \\ p_2 = \left\{6, 8, \dfrac{15}{2}\right\} \\ p_3 = \left\{-3, 2, \dfrac{9}{8}\right\} \end{cases} \tag{52}$$

The implicit equation for the plane is:

$$\begin{cases} x = \dfrac{1}{3} + \dfrac{8}{3}; \\ y = \dfrac{1}{4} + \dfrac{1}{4}; \\ z = -\dfrac{1}{8} + \dfrac{1}{8} \end{cases} \tag{53}$$

19 Compute the intersection between two arbitrary lines. What is the best choice for representing each line: a parametric or implicit expression? Justify your answer.

I choose the parametric expression because the can make up a Linear equation.

A linear equation is an algebraic equation in which each term is either a constant or the product of a constant and (the first power of) a single variable. The constants may be numbers, parameters, or even non-linear functions of parameters, and the distinction between variables and parameters may depend on the problem (for an example, see linear regression).

Linear equations can have one or more variables. Linear equations occur abundantly in most subareas of mathematics and especially in applied mathematics. While they arise quite naturally when modeling many phenomena, they are particularly useful since many non-linear equations may be reduced to linear equations by assuming that quantities of interest vary to only a small extent from some "background" state. Linear equations do not include exponents.

General (or standard) form. In the general (or standard) form the linear equation is written as:

$$Ax + By = C \tag{54}$$

where A and B are not both equal to zero. The equation is usually written so that A 0, by convention. The graph of the equation is a straight line, and every straight line can be represented by an equation in the above form. If A is nonzero, then the x-intercept, that is, the x-coordinate of the point where the graph crosses the x-axis (where, y is zero), is C/A. If B is nonzero, then the y-intercept, that is the y-coordinate of the point where the graph crosses the y-axis (where x is zero), is C/B, and the slope of the line is ?A/B. The general form is sometimes written as:

$$ax + by + c = 0 \tag{55}$$

where a and b are not both equal to zero. The two versions can be converted from one to the other by moving the constant term to the other side of the equal sign.

SlopeCintercept form.

$$y = mx + b \tag{56}$$

where m is the slope of the line and b is the y intercept, which is the y coordinate of the location where the line crosses the y axis. This can be seen by letting x = 0, which immediately gives y = b. It may be helpful to think about this in terms of y = b + mx; where the line passes through the point (0, b) and extends to the left and right at a slope of m. Vertical lines, having undefined slope, cannot be represented by this form.

PointCslope form.

$$y - y_1 = m(x - x_1) \tag{57}$$

where m is the slope of the line and (x1,y1) is any point on the line.

The point-slope form expresses the fact that the difference in the y coordinate between two points on a line (that is, y ? y1) is proportional to the difference in the x coordinate (that is, x ? x1). The proportionality constant is m (the slope of the line).

Two-point form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{58}$$

where (x1, y1) and (x2, y2) are two points on the line with x2 x1. This is equivalent to the point-slope form above, where the slope is explicitly given as (y2 ? y1)/(x2 ? x1).

Multiplying both sides of this equation by (x2 ? x1) yields a form of the line generally referred to as the

20 Compute the intersection between a line and a plane. What is the best choice for representing the line: a parametric or an implicit expression? What is the best choice for representing the plane: a parametric or an implicit expression? Justify your answers.

I prefer the line is the implicit expression and the plane is the parametric expression

The implicit expression of the line is

$$\begin{cases} x = m_1 + v_1 \times t \\ y = m_2 + v_2 \times t \\ z = m_3 + v_3 \times t \end{cases} \tag{59}$$

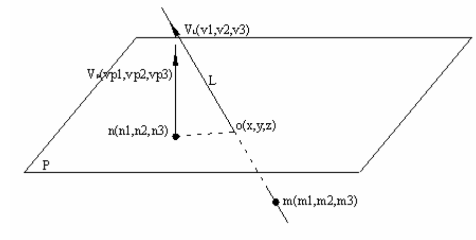

Figure 14: *Compute the intersection between a line and a plane.*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Figure 15: *The quadratic formula for the roots of the general quadratic equation.*

the plane's parametric expression is

$$\{\, vp_1 \times (xCn_1) + vp_2 \times (yCn_2) + vp_3 \times (zCn_3) = 0 \tag{60}$$

The intersection point of the line and the plane must satisfy the formula (59) and (60), simultaneous two, get:

$$t = \frac{((n_1Cm_1) \times vp_1 + (n_2Cm_2) \times vp_2 + (n_3Cm_3) \times vp_3)}{(vp_1 \times v_1 + vp_2 \times v_2 + vp_3 \times v_3)} \tag{61}$$

If the denominator of (61)$(vp_1 \times v_1 + vp_2 \times v_2 + vp_3 \times v_3$ is equal to 0, and is said the line is parallel to the plane so that plane and line and have no intersection.

Otherwise, calculate t, and then put t into equation (59) coordinates can be obtained intersection O (x, y, z).

21 Compute the intersection between a line and an arbitrary sphere. Which representation did you choose for representing the line and the sphere (implicit or parametric)? Justify your choice. What are the conditions for this intersection to exist?

I prefer the line is the implicit expression and the arbitrary sphere is the parametric expression

the arbitrary sphere's parametric expression is

$$\begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - r^2 = 0 \\ (x_0, y_0, z_0)\ is\ the\ center\ of\ the\ arbitrary\ sphere; \end{cases} \tag{62}$$

The implicit expression of the line is

$$\begin{cases} x = m_1 + v_1 \times t \\ y = m_2 + v_2 \times t \\ z = m_3 + v_3 \times t \end{cases} \tag{63}$$

The intersection point of the line and the plane must satisfy the formula (62) and (63), simultaneous two, get a quadratic equation

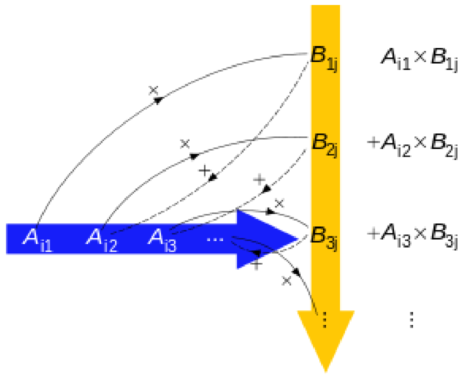

Figure 16: *Arithmetic process of multiplying numbers (solid lines) in row i in matrix A and column j in matrix B, then adding the terms (dashed lines) to obtain entry ij in the final matrix.*

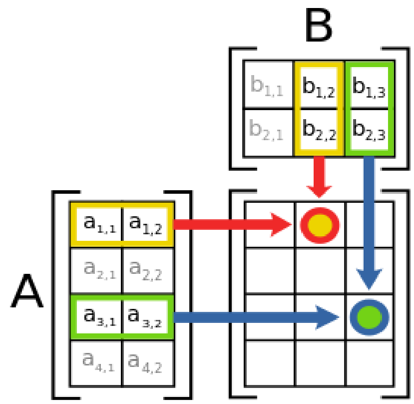

Figure 17: *Matrix2.*

The conditions for this intersection to exist is that the $b^2 - 4ac \succeq 0$

if the $b^2 - 4ac = 0$ there is only one intersection between the line and the arbitrary sphere.

if the $b^2 - 4ac \neq 0$ there are tow intersections between the line and the arbitrary sphere only.

22 Be prepared to multiply two arbitrary matrices together. What conditions must be satisfied for this operation to be possible?

Matrix product Assume two matrices are to be multiplied (the generalization to any number is discussed below).

General definition of the matrix product If A is an n m matrix and B is an m p matrix,

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ B_{21} & B_{22} & \cdots & B_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mp} \end{pmatrix} \tag{64}$$

the matrix product AB (denoted without multiplication signs or dots) is defined to be the n p matrix

$$\mathbf{AB} = \begin{pmatrix} (\mathbf{AB})_{11} & (\mathbf{AB})_{12} & \cdots & (\mathbf{AB})_{1p} \\ (\mathbf{AB})_{21} & (\mathbf{AB})_{22} & \cdots & (\mathbf{AB})_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ (\mathbf{AB})_{n1} & (\mathbf{AB})_{n2} & \cdots & (\mathbf{AB})_{np} \end{pmatrix} \tag{65}$$

where each i, j entry is given by multiplying the entries Aik (across row i of A) by the entries Bkj (down column j of B), for k = 1, 2, ..., m, and summing the results over k:

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj} \tag{66}$$

Thus the product AB is defined only if the number of columns in A is equal to the number of rows in B, in this case m. Each entry may be computed one at a time. Sometimes, the summation convention is used as it is understood to sum over the repeated index k. To prevent any ambiguity, this convention will not be used in the article.

Usually the entries are numbers or expressions, but can even be matrices themselves (see block matrix). The matrix product can still be calculated exactly the same way. See below for details on how the matrix product can be calculated in terms of blocks taking the forms of rows and columns.

Row vector and column vector If

$$\mathbf{A} = \begin{pmatrix} a & b & c \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{67}$$

their matrix products are:

$$\mathbf{AB} = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz \tag{68}$$

and

$$\mathbf{BA} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} xa & xb & xc \\ ya & yb & yc \\ za & zb & zc \end{pmatrix} \tag{69}$$

Note AB and BA are two different matrices: the first is a 1 1 matrix while the second is a 3 3 matrix. Such expressions occur for real-valued Euclidean vectors in Cartesian coordinates, displayed as row and column matrices, in which case AB is the matrix form of their dot product, while BA the matrix form of their dyadic or tensor product.

Square matrix and column vector If

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ p & q & r \\ u & v & w \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{70}$$

their matrix product is:

$$\mathbf{AB} = \begin{pmatrix} a & b & c \\ p & q & r \\ u & v & w \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax + by + cz \\ px + qy + rz \\ ux + vy + wz \end{pmatrix} \tag{71}$$

however BA is not defined.

The product of a square matrix multiplied by a column matrix arises naturally in linear algebra; for solving linear equations and representing linear transformations. By choosing a, b, c, p, q, r, u, v, w in A appropriately, A can represent a variety of transformations such as rotations, scaling and reflections, shears, of a geometric shape in space.

Square matrices If

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ p & q & r \\ u & v & w \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{pmatrix} \tag{72}$$

$$\mathbf{AB} = \begin{pmatrix} a & b & c \\ p & q & r \\ u & v & w \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{pmatrix} = \begin{pmatrix} a\alpha + b\lambda + c\rho & a\beta + b\mu + c\sigma & a\gamma + b\nu + c\tau \\ p\alpha + q\lambda + r\rho & p\beta + q\mu + r\sigma & p\gamma + q\nu + r\tau \\ u\alpha + v\lambda + w\rho & u\beta + v\mu + w\sigma & u\gamma + v\nu + w\tau \end{pmatrix} \tag{73}$$

and

$$\mathbf{BA} = \begin{pmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{pmatrix} \begin{pmatrix} a & b & c \\ p & q & r \\ u & v & w \end{pmatrix} = \begin{pmatrix} \alpha a + \beta p + \gamma u & \alpha b + \beta q + \gamma v & \alpha c + \beta r + \gamma w \\ \lambda a + \mu p + \nu u & \lambda b + \mu q + \nu v & \lambda c + \mu r + \nu w \\ \rho a + \sigma p + \tau u & \rho b + \sigma q + \tau v & \rho c + \sigma r + \tau w \end{pmatrix} \tag{74}$$

Acknowledgements

I would like to thank my teacher.

References