Mathematics in Games

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Abstract

Mathematics and games have many features in common. This commonality is mainly reflected in their properties, structures and practice, thus mathematics and game brought up a relationship of which were mutually penetrated and mutually unified. We can found a lot of methods of mathematics were used in designing a game. And we solved some problems in the article to help us further learning the game engineering.

Keywords: mathematics, vector, implicit equation, parametric equation

1 Introduction

When we are playing a game, the mathematics are everywhere, the scores, the amount of money, the remain health of the character. Also we use mathematics to solve the challenging tasks.

When studying a game's core mathematics, arithmetic theory is generally of higher utility than actively playing or observing the game itself. To analyze a game numerically, it is particularly useful to study the rules of the game insofar as they can yield equations or relevant formulas. This is frequently done to determine winning strategies or to distinguish if the game has a solution.

There are knowledge points like vectors, matrix and physics will be put into our game. So learning the mathematics is very useful and challenging.

2 Answer the questions

There are several questions for us to further understand the using of mathematics in designing a game.

✔ What is the difference between a point and a vector?

A vector has direction and magnitude whereas a point doesn't. It is just a direction and it do not have a location. Points are measured relative to the origin. Vectors are intrinsically relative to everything. So a vector can be used to represent a point.

Also the difference is precisely that between location and displacement.

- Points are locations in space.
- Vectors are displacements in space.

It may be specific to explain it in the coordinate system. When we store a vector as (x,y,z), the vector represents the direction of a line beginning at (0,0,0) and ending at the point (x,y,z) with a magnitude of sqrt(x*x+y*y+z*z). The vector from (0,0,0) to (x,y,z) is the same as the vector from (1,1,1) to (x+1, y+1, z+1), because their magnitudes and directions are the same.

✔ What is a right handed coordinate system?

Coordinates are usually right-handed. A three-dimensional coordinate system in which the axes satisfy the right-hand rule.

For Right-handed coordinates use right hand. For right-handed coordinates your right thumb points along the Z axis and the curl of your fingers represents a motion from the first or X axis to the second or Y axis. When viewed from the top or Z axis the system is counter-clockwise.[Watson 1998] We can see an example in Figure1.

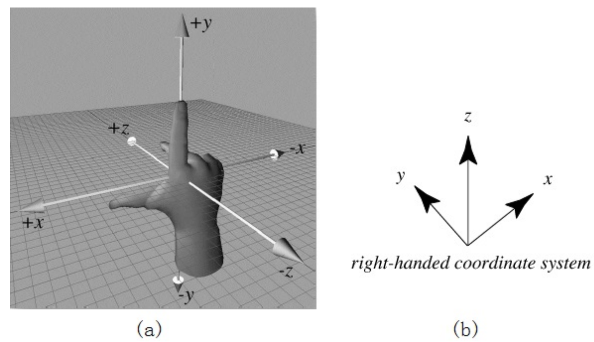

Figure 1: *Right-handed Coordinate System. The (a) shows that each fingers represent the X, Y, Z axis respectively. And Figure (b) is a simplified version of figure (a).*

✔ How do you compute the dot product of two vectors? Informally, what does the dot product do? Give an example where we need the dot product for Computer Graphics applications.

✔ How do you compute the projection of one vector onto another?

We can calculate the Dot Product of two vectors these ways:

- Algebraic definition
  Can take the dot product of two vectors of the same dimension. The result is a scalar. And the equation 1is below:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i \tag{1}$$

- Geometric definition
  The equation 2 is below

$$\mathbf{a} \cdot \mathbf{b} = \|a\| \, \|b\| \, cos\theta \tag{2}$$

  Where: The cross product of a and bis a vector perpendicular to the plane of a and b. The magnitude is related to the magnitude of a and band the angle between a and b. The magnitude is equal to the area of a parallelogram with sides a and b.

When dot product works in 3 or more dimensions, it can measure the end-points of two poles. Also it can shine a light to see where the shadow lies for example. 3D graphics compared to 2D graphics are graphics that use a three-dimensional representation of geometric data. 2D applications may use dot product to achieve effects such as lighting. Dot product is the magnitude of the projection of one vector onto another.

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The article is based on work to support Advanced Game Engineering.
Advanced Game Engineering (2016/17)
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✔ What are the implicit and parametric forms of a line? Why are they named implicit and parametric?

In mathematics, an implicit equation is a relation of the form R(x_1,..., x_n) = 0, where R is a function of several variables (often a polynomial). The equation 3 can be explained to form a line.

$$y = ax + b \tag{3}$$

In mathematics, parametric equations of a curve express the coordinates of the points of the curve as functions of a variable, called a parameter.[Thomas and Finney 1979][Weisstein] For example,4 and 5

$$x = acos\theta \tag{4}$$

$$y = bsin\theta \tag{5}$$

We call implicit equations implicit because although they imply a curve or surface, they cannot explicitly generate the points that comprise it. And parametric equations offer the capability to generate continuous curves and surfaces.

✔ What is the implicit form of a circle centered at (x_c, y_c) with radius R?

It can be shown that if R(x, y) is given by a smooth submanifold M in R^2, and (a, b) is a point of this submanifold such that the tangent space there is not vertical , then M in some small enough neighbourhood of (a, b) is given by a parametrization (x, f(x)) where f is a smooth function.

In less technical language, implicit functions exist and can be differentiated, unless the tangent to the supposed graph would be vertical. In the standard case where we are given an equation 6

$$R(x, y) = 0 \tag{6}$$

the condition on R can be checked by means of partial derivatives. [Stewart 1998] And we can see an example in Figure2.

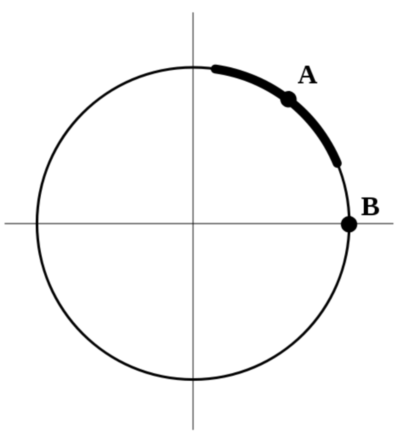

Figure 2: *The unit circle can be defined implicitly as the set of points (x_c, y_c) satisfying ${x_c}^2$+${y_c}^2$=R^2.*

✔ Write an implicit equation for a line given two points.

We use Figure3 for example. The generalized equation of implicit can be equation9

$$y - y_A = m(x - x_A) \tag{7}$$

$$m = \frac{y_A - y_B}{x_A - x_B} \tag{8}$$

$$y - y_A = \frac{y_A - y_B}{x_A - x_B}(x - x_A) \tag{9}$$

✔ Write a parametric equation for a line given two points.

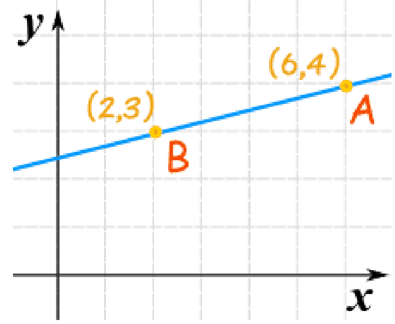

Figure 3: *A line to explain implicit equation*

What we know is that there are two points A(x_A,y_A) and B(x_B,y_B).

$$x = x_A + t; \tag{10}$$

$$y = y_A + \frac{y_B - y_A}{x_B - x_A}t; \tag{11}$$

✔ Suppose you are given a parametric equation for a line. Convert from this representation to an implicit equation for that same line.

✔ Given an implicit equation for a line, convert it to a parametric representation.

✔ Given a slope / intercept equation for a line, convert it to parametric and implicit representations.

Those are similar questions. Once we know one of the equations, we can get one point on this line. And it will be easy to found the results.

✔ Give a situation in a computer graphics application where you would prefer a parametric representation of a line to an implicit representation.

✔ Give a situation where you would prefer an implicit representation of a line to a parametric one.

When the two intersection points of the straight line and the conic curve are not the fixed points, the parametric equation will be used to solve it more easily. It will be larger calculation if we use implicit equation.

When we solve planimetry problems, it may use implicit equations more often.

✔ How do you convert an arbitrary vector to a unit vector?

The normalized vector or versor of a non-zero vector u is the unit vector in the direction of u, i.e.,

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{||\mathbf{u}||} \tag{12}$$

where $||u||$ is the norm (or length) of u. The term normalized vector is sometimes used as a synonym for unit vector.

A vector can be ”scaled” off the unit vector. Unit vectors are often chosen to form the basis of a vector space. Every vector in the space may be written as a linear combination of unit vectors.

✔ How do you compute the cross product of two vectors? Informally, what does the cross product do? Give an example where we need the cross product for Computer Graphics applications.

We can calculate the Cross Product this way:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\, sin\theta\mathbf{n} \tag{13}$$

- $|a|$ is the magnitude (length) of vector a
- $|b|$ is the magnitude (length) of vector b
- θ is the angle between a and b

- n is the unit vector at right angles to both a and b

So the length is: the length of a times the length of b times the sine of the angle between a and b.

Then we multiply by the vector n to make sure it heads in the right direction (at right angles to both a and b).

Or we can calculate it this way:

When a and b start at the origin point (0,0,0), the Cross Product will end at:

$$c_x = a_y b_z - a_z b_y \tag{14}$$

$$c_y = a_z b_x - a_{xz} b_z \tag{15}$$

$$c_z = a_x b_y - a_y b_x \tag{16}$$

The cross product has applications in various contexts: e.g. it is used in computational geometry, physics and engineering.

The cross product appears in the calculation of the distance of two skew lines (lines not in the same plane) from each other in three-dimensional space.

The cross product can be used to calculate the normal for a triangle or polygon, an operation frequently performed in computer graphics. For example, the winding of a polygon (clockwise or anticlockwise) about a point within the polygon can be calculated by triangulating the polygon (like spoking a wheel) and summing the angles (between the spokes) using the cross product to keep track of the sign of each angle.

In computational geometry of the plane, the cross product is used to determine the sign of the acute angle defined by three points p_1=(x_1,y_1), p_2=(x_2,y_2) and p_3=(x_3,y_3). It corresponds to the direction of the cross product of the two coplanar vectors defined by the pairs of points p_1, p_2 and p_1, p_3, i.e., by the sign of the expression P =(x_2-x_1)(y_3-y_1)-(y_2-y_1)(x_3-x_1). In the "right-handed" coordinate system, if the result is 0, the points are collinear; if it is positive, the three points constitute a positive angle of rotation around p_1 from p_2 to p_3, otherwise a negative angle. From another point of view, the sign of P tells whether p_3 lies to the left or to the right of line p_1,p_2.

The cross product is used in calculating the volume of a polyhedron such as a tetrahedron or parallelepiped.[wik]

✔ What are implicit and parametric expressions for a plane?

Example: A plane is flying along, pointing North, but there is a wind coming from the North-West. We could see in the Figure4[pla]

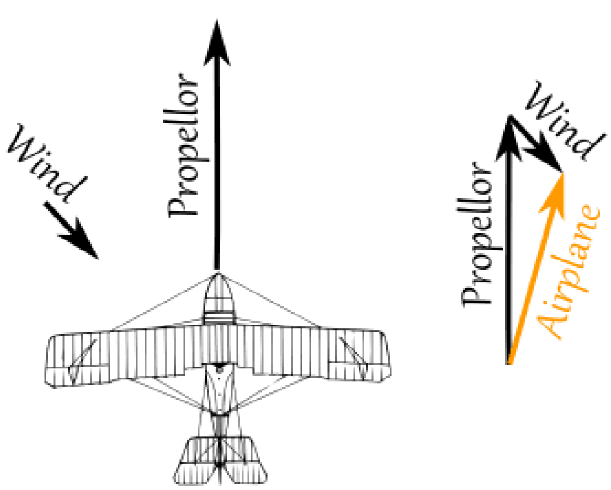

Figure 4: *A plane is flying along, pointing North, but there is a wind coming from the North-West.*

The location of the plane can be implicit expression, and the relative position of the wind or ground can be parametric expression.

✔ Write an implicit equation for a plane given three points.

What we know is that there are three points A(x_1,y_1,z_1), B(x_2,y_{21},z_2), C(x_3,y_3,z_3).

$$a\,(x - x_1) + b\,(y - y_1) + c(z - z_1) = 0 \tag{17}$$

$$a\,(x - x_2) + b\,(y - y_2) + c(z - z_2) = 0 \tag{18}$$

$$a\,(x - x_3) + b\,(y - y_3) + c(z - z_3) = 0 \tag{19}$$

✔ Write a parametric equation for a plane given three points.

What we know is that there are three points A(x_1,y_1,z_1), B(x_2,y_{21},z_2), C(x_3,y_3,z_3).

$$x = x_1 + u\,(x_2 - x_1) + v\,(x_3 - x_1) \tag{20}$$

$$y = y_1 + u\,(y_1 - y_1) + v\,(y_3 - y_1) \tag{21}$$

$$z = z_1 + u\,(z_2 - z_1) + v\,(z_3 - z_1) \tag{22}$$

✔ Compute the intersection between two arbitrary lines. What is the best choice for representing each line: a parametric or implicit expression? Justify your answer.

Implicit expression.

For linear problems, using implicit equations is more efficient. It is unconditionally stable and people can use a large time step.

✔ Compute the intersection between a line and a plane. What is the best choice for representing the line: a parametric or an implicit expression? What is the best choice for representing the plane: a parametric or an implicit expression? Justify your answers.

Parametric equations offer the capability to generate continuous curves and surfaces.

The parameters for these equations are scalars that range over a continuous(possibly infinite) interval. And varying the parameters over their intervals smoothly generates every point on the curve or surface.

✔ Compute the intersection between a line and an arbitrary sphere. Which representation did you choose for representing the line and the sphere (implicit or parametric)? Justify your choice. What are the conditions for this intersection to exist?

Parametric. It is the same reason as the last questions. The parameters for these equations are scalars that range over a continuous(possibly infinite) interval. And varying the parameters over their intervals smoothly generates every point on the curve or surface. For an sphere, using parametric to identify each points will be more efficient.

✔ Be prepared to multiply two arbitrary matrices together. What conditions must be satisfied for this operation to be possible?

For point multiplication, the ranks of the two matrices should be equal. For matrix multiplication, the number of columns of the first matrix should be equal to the number of rows of the second matrix.

3 Conclusion

We found it is useful to learning the mathematics and it can be used in our game designing. So our game will be well structured, more feasible and based on the reality.

To analyze a game numerically, it is particularly useful to study the rules of the game insofar as they can yield equations or relevant formulas.

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