1 Math Facts

$$\sum_{k=0}^{n-1} ar^k = \left(\frac{1-r^n}{1-r}\right) \qquad \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

2 Properties of Signals

2.1 Complex Exponentials

We can represent signals as complex exponentials to make them a lot easier in certain cases (multiplication).

$$e^{jx} = \cos(x) + j\sin(x)$$

$$cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$sin(x) = \frac{e^{jx} - e^{-jx}}{2j} = -\frac{j}{2}\left(e^{jx} - e^{-jx}\right)$$

2.2 Sampling

Because multiple frequencies can alias to the same frequency when sampled, we use the *Baseband* to describe the frequency represented by a set of samples. The baseband is the range of frequencies $0 \le \Omega \le \pi$.

If there are frequencies in the CT signal greater than the Nyquist frequency $f_N = \frac{f_s}{2}$, they will alias to frequencies in the base band. To avoid distortion, remove these frequencies before sampling (anti-aliasing).

3 Fourier Series

3.1 Continuous Time Fourier Series

Split a signal up into harmonically related sinusoids, which each have a frequency that is an integer multiple of the *fundamental frequency* $\omega_0 = \frac{2\pi}{T}$ of the signal. The fundamental frequency has the same period as the *fundamental period* of the signal, or the smallest value T for which $x(t) = x(t+T)$ holds.

- Represents periodic signals
- Can't perfectly represent discontinuities—results in Gibb's Phenomenon (overshoot)
  - as number of terms increases, width of overshoot gets smaller, but magnitude does not (around 9%).

$$f(t) = \sum_{k=0}^{\infty}(c_k\cos(k\omega_0 t) + d_k\sin(k\omega_0 t))$$

$$c_0 = \frac{1}{T}\int_T f(t)dt \qquad \text{"average"}$$

$$c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_o t)dt$$

$$d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_o t)dt$$

Using exponential form, we get a simpler set of equations for CTFS:

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi kt}{T}}$$

$$X[k] = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi kt}{T}}\,dt$$

3.1.1 Properties

- **Real-Valued Periodic Signal:** $F[k] = F^*[-k]$ for real-valued signals.
- **Symmetric and Antisymmetric Parts**: The real part of $F[k]$ comes from the symmetric part of the signal, the imaginary part comes from the antisymmetric part of the signal. In trig form, symmetric in c_k, antisymmetric in d_k.

property	$\mathbf{y(t)}$	$\mathbf{Y[k]}$
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1[k] + BX_2[k]$
Time Flip	$x(-t)$	$X[-k]$
Time Shift	$x(t-t_0)$	$e^{-j\frac{2\pi k t_0}{T}}X[k]$
Time Derivative	$\frac{d}{dt}x(t)$	$Y[k] = j\frac{2\pi}{T}X[k]$

3.2 DTFS

In CTFS, there can be an infinite number of harmonics, as resolution is infinite. In a DT signal with period N, there can only be N harmonics–the others will alias to a harmonic within those N. We have a similar fundamental frequency $\Omega_0 = \frac{2\pi}{N}$.

$$x[n] = x[n+N] = \sum_{k=k_0}^{k_0+N-1} X[k]e^{j\frac{2\pi}{N}kn}$$

$$X[k] = X[k+N] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

3.2.1 Properties

property	$\mathbf{y[t]}$	$\mathbf{Y[k]}$
Linearity	$Ax_1[n] + Bx_2[n]$	$AX_1[k] + BX_2[k]$
Time Flip	$x[-n]$	$X[-k]$
Time Shift	$x[n-m]$	$e^{-j\frac{2\pi km}{N}}X[k]$
Sym. Part	$\frac{1}{2}(x[n] + x[-n])$	$Re(X[k])$
Antisym. Part	$\frac{1}{2}(x[n] - x[-n])$	$j \cdot Im(X[k])$

4 Fourier Transforms

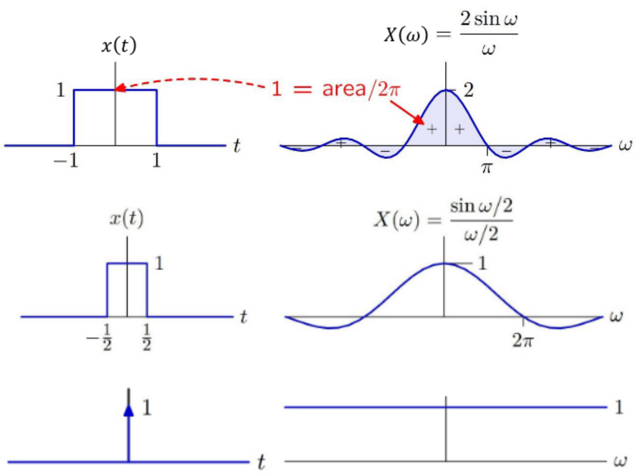
4.1 CT Fourier Transform

Essentially, a fourier series as the period approaches infinity.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}d\omega \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}dt$$

4.1.1 Useful Signals

Stretching in time compresses in frequency, and vice versa.



4.1.2 Duality

If taking the CTFT of $x(t)$ gives us $X(\omega)$, then if we interpret those coefficients as another signal and take the CTFT again, we will get $2\pi x(-\omega)$. Need clarity here.

4.1.3 Properties

property	$\mathbf{y(t)}$	$\mathbf{Y}(\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Time Reversal	$x(-t)$	$X(-\omega)$
Time Delay	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Scaling Time	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Time Derivative	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Freq. Derivative	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$

4.2 Discrete Time Fourier Transform

Same idea, we make a periodic version of $x[n]$ by summing shifted copies and take the DTFS of that, but do so as the distance between copies approaches infinity.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\cdot e^{j\Omega n}d\Omega$$

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\Omega n}$$

4.2.1 Properties

Note that the coefficients are periodic in this case

property	$\mathbf{y(t)}$	$\mathbf{Y}(\omega)$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(\Omega) + bX_2(\Omega)$
Time Reversal	$x[-n]$	$X(-\Omega)$
Time Delay	$x[n-n_0]$	$e^{-j\Omega n_0}X(\Omega)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Freq. Derivative	$nx[n]$	$j\frac{d}{d\Omega}X(\Omega)$

5 Discrete Fourier Transform

Everything so far has had either infinite sums or a continuous domain—difficult for a computer. To solve this, we take an N-sample window of the signal, pretend the signal is periodic in N, and generate the DTFS of this fake-periodic signal.

- Increasing N increases frequency resolution
- If the signal is not periodic in N, we get spectral blurring where frequencies do not line up exactly with the positions of the coefficients.

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n} \qquad X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cdot e^{-j\frac{2\pi k}{N}n}$$

5.0.1 Properties

property	$\mathbf{y(t)}$	$\mathbf{Y[k]}$
Linearity	$Ax_1[n] + Bx_2[n]$	$AX_1[k] + BX_2[k]$
Time Flip	$x_p[N-n]$	$X[-k]$
Time Shift	$x[n-n_0]$	$e^{-j\frac{2\pi k}{N}n_0}X[k]$
Freq. Shift	$e^{-j\frac{2\pi k_0}{N}n}\cdot x[n]$	$X[k-k_0]$
Conjugation	$x^*[n]$	$X^*[-k]$

6 Systems Abstraction

Many applications can be modeled as systems that convert an input signal to an output signal. We focus on systems that are **linear** (you can add them together) and **time invariant** (delaying input delays the output by the same amount).

Systems can be represented with a **difference equation** ($y[n] = \frac{x[n]+x[n-1]}{2}$ or in terms of **convolution** (using its unit-sample response). With convolution, we can use linearity to then construct the output of any signal by just adding and shifting.

6.1 Convolution

Given a unit-sample response for a system, we can compute the output signal by shifting, multiplying, and adding, or integrating for a CT system:

$$y[n] = (x*h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y(t) = (x*h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Convolution is **commutative** (can change the order), **associative** (which pair you convolve first doesn't matter) and **distributive over addition**.