Electromagnetic Waves Math

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Maxwell's Equations

Gauss's Law for $\vec{\mathbf{E}}$

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\varepsilon_0}$$

Electric flux through a closed surfaceis proportional to the charged enclosed

Gauss's Law for $\vec{\mathbf{B}}$

$$\oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

The total magnetic flux through a closed surface is zero

Faraday's Law

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$

Changing magnetic flux produces an electric field

Ampere-Maxwell Law

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

Electric current and changing electric flux produces a magnetic field

Maxwell's Equations in Empty Space

When we assume $Q = 0$ and $I = 0$, as is the case for travelling electromagnetic waves, these get a bit simpler:

Gauss's Law for $\vec{\mathbf{E}}$

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 0$$

Gauss's Law for $\vec{\mathbf{B}}$

$$\oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

Faraday's Law

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$

Ampere-Maxwell Law

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

EM Wave Basics

A key fact for electromagnetic waves is that they are transverse—both the $\vec{E}$ and $\vec{B}$ fields are perpendicular to the direction of propagation. The fields are also perpendicular to each other, so where $\vec{\mathbf{p}}$ is the direction of propagation:

$$\vec{\mathbf{E}} \times \vec{\mathbf{B}} = \vec{\mathbf{p}}$$

The Wave Equation for EM waves

Our goal is to find the wave equation for the EM wave—something of the form:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi(x,t) = 0$$

Where $\psi(x,t)$ is the wave function itself and v is the wave velocity. To do this, we will apply Faraday's Equation and the Ampere-Maxwell law to an electromagnetic wave.

The Wave Equation for EM waves

Setting our coordinate system such that the $\vec{E}$ field is in the xy plane and the $\vec{B}$ field is in the xz plane, the wave equation for an electromagnetic wave can be found using multivariable calcalculus.

Using the Faraday Equation, we can integrate over a loop in the xy plane to find:

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

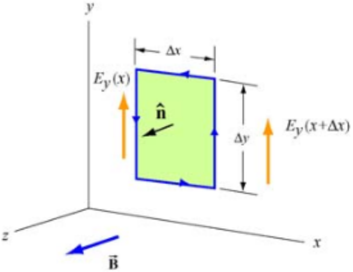

Figure: Applying Faraday's Law to a loop in the xy plane.

The Wave Equation for EM waves

Then, following the same steps with Ampere-Maxwell and a loop in the xz plane, we reach:

$$-\frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t}$$

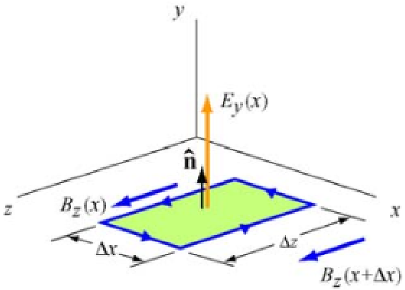

Figure: Applying Ampere-Maxwell to a loop in the xz plane.

Velocity of an EM Wave

The equation describing any wave is given in a general form of:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi(x,t) = 0$$

Where $\psi(x,t)$ is the wave function itself and v is the wave velocity. More manipulation of the two equations from the last sides yields us something in a similar format:

$$\left(\frac{\partial^2}{\partial x^2} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)\left\{\begin{array}{c} E(x,t) \\ B(x,t) \end{array}\right\} = 0$$

We can then clearly find the velocity of an electromagnetic wave:

$$\frac{1}{v^2} = \mu_0\varepsilon_0 \longrightarrow v = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Proving that light is an EM wave

We can use this fact to show that light is an example of an electromagnetic wave.

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\left(4\pi \times 10^{-7} T \cdot m/A\right)\left(8.85 \times 10^{-12} C^2/N \cdot m^2\right)}}$$

$$v = 2.997 \times 10^8 m/s = c$$

More EM Wave Properties

Had we done the full derivation of the above fact, we would also be able to show that:

$$\frac{E}{B} = c$$

This means that in an electromagnetic wave, the $\vec{E}$ field is much larger than the $\vec{B}$ field.

Summary: The Important Slide

That was probably confusing. So to summarize what we know:

- The wave is transverse because $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation $\vec{p}$, which is given by $\vec{p} = \vec{E} \times \vec{B}$.
- The E and B fields are perpendicular to each other, meaning $\vec{E} \cdot \vec{B} = 0$.
- $\frac{E}{B} = \frac{E_0}{B_0} = c$
- The wave's speed of propagation equals $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$
- The superposition principle applies to the waves—two overlapping waves can be added to give the resulting wave.