2021 AMC 10A Fall #16

bkf2020

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LICENSE

Commentary

Originally, I got C as the answer. This was because I thought $f(x) = f(-x)$ since I incorrectly assumed $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$. This is false for integers.

The correct solution:

First, assume $x > 0$ and x is not an integer:

$$\begin{aligned} f(x) &= |\lfloor x \rfloor| - |\lfloor 1 - x \rfloor| \\ &= |\lfloor x \rfloor| - |1 + \lfloor -x \rfloor| \\ &= |\lfloor x \rfloor| - |1 - \lfloor x \rfloor - 1| \\ &= |\lfloor x \rfloor| - |-\lfloor x \rfloor| \\ &= 0. \end{aligned}$$

Then, assume $x < 0$ and x is not an integer. To deal with this, we set $y = -x$ and we find $f(-y)$, where $y > 0$ and y is not an integer

$$\begin{aligned} f(-y) &= |\lfloor -y \rfloor| - |\lfloor 1 + y \rfloor| \\ &= |-\lfloor y \rfloor - 1| - |\lfloor y \rfloor| + 1 \\ &= 0. \end{aligned}$$

So when x is not an integer $f(x) = 0$.

Now, when x is an integer, we have

$$f(x) = |\lfloor x \rfloor| - |1 + \lfloor -x \rfloor| = |x| - |1 - x| = |x| - |x - 1|.$$

If $x > 0 \Longrightarrow x \geq 1, f(x) = 1$ and if $x \leq 0, f(x) = -1$.

So finally, we find that at $(1/2, 0)$ has symmetry because $f(1/2+a)+f(1/2-a) = 0$ where $1/2+a$ and $1/2-a$ are integers.

Examples: $a = 1/2 \implies f(1) + f(0) = 0, a = 3/2 \implies f(2) + f(-1) = 0.$