1 Theoretical background

The partition function of the system is given,

$$Z = \sum e^{-\beta E[\{\sigma_i\}]} = \sum e^{-\beta(J\sum_{<i,j>}\sigma_i\sigma_j - h\sum_i \sigma_i)}$$

Expectation value of the energy is

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z = \sum E[\{\sigma_i\}]\frac{1}{Z}e^{-\beta E[\{\sigma_i\}]}$$

Specific heat can be obtained by differentiating $\langle E\rangle$ by temperature.

$$\begin{aligned}
c_V &= \frac{1}{N}\frac{\partial}{\partial T}\langle E\rangle = \frac{\partial\beta}{\partial T}\left(-\frac{\partial^2}{\partial\beta^2}\ln Z\right) \\
&= \frac{1}{NT^2}\sum E[\{\sigma_i\}]^2\frac{1}{Z}e^{-\beta E[\{\sigma_i\}]} - \frac{1}{NT^2}\sum E[\{\sigma_i\}]\frac{1}{Z}e^{-\beta E[\{\sigma_i\}]}\sum E[\{\sigma_i\}]\frac{1}{Z}e^{-\beta E[\{\sigma_i\}]} \\
&= \frac{1}{NT^2}\left(\left\langle E^2\right\rangle - \langle E\rangle^2\right) = \frac{N}{T^2}\left(\left\langle (E/N)^2\right\rangle - \langle E/N\rangle^2\right) \\
&= \frac{N}{T^2}\mathrm{Var}(E/N)
\end{aligned}$$

Magnetization and magnetic susceptibility can be calculated in a similar manner.

$$\frac{\partial}{\partial h}\ln Z = \sum \beta\left(\sum_i \sigma_i\right)\frac{1}{Z}e^{-\beta(J\sum_{<i,j>}\sigma_i\sigma_j - h\sum_i\sigma_i)} = \frac{1}{T}\langle M\rangle$$

$$\begin{aligned}
\chi &= \frac{1}{N}\frac{\partial}{\partial h}\langle M\rangle = \frac{\partial}{\partial h}\left(T\frac{\partial}{\partial h}\ln Z\right) \\
&= \frac{1}{NT}\sum \beta^2\left(\sum_i\sigma_i\right)^2\frac{1}{Z}e^{-\beta(J\sum_{<i,j>}\sigma_i\sigma_j - h\sum_i\sigma_i)} \\
&\quad - \frac{1}{NT}\sum\beta\left(\sum_i\sigma_i\right)\frac{1}{Z}e^{-\beta(J\sum_{<i,j>}\sigma_i\sigma_j - h\sum_i\sigma_i)}\sum\beta\left(\sum_i\sigma_i\right)\frac{1}{Z}e^{-\beta(J\sum_{<i,j>}\sigma_i\sigma_j - h\sum_i\sigma_i)} \\
&= \frac{1}{NT}\left(\left\langle M^2\right\rangle - \langle M\rangle^2\right) = \frac{N}{T}\left(\left\langle m^2\right\rangle - \langle m\rangle^2\right) \\
&= \frac{N}{T}\mathrm{Var}(m)
\end{aligned}$$