

1 Sum from 1 to n

We want to prove that $1 + 2 + \dots + n$ can be calculated with $\frac{n(n+1)}{2}$

1.1 Definitions

We will define a function to let us talk about the sum of numbers from 1 to n .
Let:

$$F(n) = 1 + 2 + \dots + n \quad (1)$$

We will define a predicate to let us talk about the relationship between $F(n)$ and the shortcut calculation. Let:

$$P(n) : F(n) = \frac{n(n+1)}{2} \quad (2)$$

Note that $P(n)$ evaluates to a boolean. It can be true or false for any particular n . It is true for a particular value of n if $F(n)$ does in fact equal $\frac{n(n+1)}{2}$ and it is false if these two things are not equal.

1.2 Goal

Our goal is to prove that $P(n)$ holds (is true) for all values of n greater than 0.
Prove:

$$\forall n \in N : P(n) \quad (3)$$

1.3 Proof by induction

1.3.1 Base case

To show our base case $P(1)$ is true, we will state the base case, then show that the left side does in fact equal the right side. Prove:

$$P(1) : F(1) = \frac{1(1+1)}{2} \quad (4)$$

$$F(1) = 1$$

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

1.3.2 Inductive step

We will prove that **if** $P(k)$ holds (is true) for some $k \in N$, **then** $P(k+1)$ is also true. Prove:

$$P(k) \implies P(k+1), \forall k \in N \quad (5)$$

We start with the *inductive hypothesis*, we assume for the time that $P(k)$ holds.
Assume:

$$P(k) : F(k) = \frac{k(k+1)}{2} \quad (6)$$

Now, assuming that $P(k)$ is true, prove:

$$P(k+1) : F(k+1) = \frac{(k+1)((k+1)+1)}{2} \quad (7)$$

By definition:

$$F(k+1) = 1 + 2 + \dots + k + (k+1)$$

which is by definition:

$$F(k+1) = F(k) + (k+1)$$

which by our inductive hypothesis is:

$$F(k+1) = \frac{k(k+1)}{2} + (k+1)$$

simplifying is:

$$F(k+1) = (k+1)\left(\frac{k}{2} + 1\right)$$

which is equivalent to:

$$F(k+1) = (k+1)\left(\frac{k}{2} + \frac{2}{2}\right)$$

which simplifies to:

$$F(k+1) = \frac{(k+1)(k+2)}{2}$$

which is clearly:

$$F(k+1) = \frac{(k+1)((k+1)+1)}{2}$$

And so we have proved $P(k+1)$ (7) by showing that the left side is equal to the right side (assuming that $P(k)$ is true).

1.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(1)$ and that for all $k \in N$, $P(k)$ being true implies that $P(k+1)$ is also true. Therefore $P(n)$ holds for all $n > 0$ (all natural numbers).

$$P(1) : F(1) = \frac{1(1+1)}{2}$$

$$P(k) \implies P(k+1), \forall k \in N$$

$$\therefore P(n), \forall n \in N$$

2 Making postage with 3 and 5 cent stamps

We want to prove that all postage amounts greater than 7 cents can be made with combinations of 3 and 5 cent stamps

2.1 Definitions

We will define a predicate to talk about whether a particular number can be represented as a summation of a non-negative multiple of 3 and a non-negative multiple of 5.

$$P(n) : n = 3a + 5b \mid a, b \in \mathbb{Z}_{\geq 0} \quad (8)$$

Note that $P(n)$ may be true or false for any given number n . For example, $P(2)$ is false, as 2 cents of postage cannot be made with 3 and 5 cent stamps. However, $P(11)$ is true, because 11 cents of postage can be made with a 5 cent stamp and two 3 cent stamps.

2.2 Goal

Our goal is to prove that $P(n)$ holds for all values of n greater than 7.

$$\forall n \in \mathbb{Z}_{>7} : P(n) \quad (9)$$

2.3 Proof by induction

2.3.1 Base case

To show that the base case $P(8)$ is true, we will state the base case, then show that we can find suitable non-negative integers a and b . Prove:

$$P(8) : 8 = 3a + 5b \quad (10)$$

$$a, b = 1$$

2.3.2 Inductive step

We will prove that **if** $P(k)$ holds (is true) for some $k \in \mathbb{Z}_{>7}$, **then** $P(k + 1)$ is also true. Prove:

$$P(k) \implies P(k + 1), \forall k \in \mathbb{Z}_{>7} \quad (11)$$

We start with the *inductive hypothesis*, we assume for the time that $P(k)$ holds. Assume:

$$P(k) : k = 3a + 5b \mid a, b \in \mathbb{Z}_{\geq 0} \quad (12)$$

Now, assuming that $P(k)$ is true, prove:

$$P(k + 1) : k + 1 = 3c + 5d \mid c, d \in \mathbb{Z}_{\geq 0} \quad (13)$$

If $b > 0$, then $d = b - 1$ and $c = a + 2$ results in:

$$3c + 5d = 3(a + 2) + 5(b - 1) = 3a + 5b + 1 = k + 1$$

By our inductive hypothesis, we assumed that a and b were non-negative integers, and so c and d will be too.

If $b = 0$, then k must be a multiple of 3. The smallest multiple of 3 in $\mathbb{Z}_{>7}$ is 9, which is $3 * 3$. All other multiples of 3 in $\mathbb{Z}_{>7}$ will have a greater or equal

number of 3 in their all-3 representation ($b = 0$). Therefore if $b = 0$, then $a \geq 3$. So $d = b + 2$ and $c = a - 3$ results in:

$$3c + 5d = 3(a - 3) + 5(b + 2) = 3a + 5b + 1 = k + 1$$

By our inductive hypothesis and our reasoning that a must be greater than or equal to 3 in the case where $b = 0$, then c and d will be non-negative integers too.

And so we have proved $P(k + 1)$ (13) by assuming $P(k)$ (12) was true. If $P(k)$, then $P(k + 1)$ (11).

2.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(8)$ and that for all $k \in \mathbb{Z}_{>7}$, $P(k)$ being true implies that $P(k + 1)$ is also true. Therefore $P(n)$ holds for all $n > 7$.

$$P(8) : 8 = 3(1) + 5(1)$$

$$P(k) \implies P(k + 1), \forall k \in \mathbb{Z}_{>7}$$

$$\therefore P(n), \forall n \in \mathbb{Z}_{>7}$$

3 Proof with inequality

We want to prove that for any integer n greater than 6, $n!$ is greater than 3^n .

$$n! > 3^n \mid n > 6$$

3.1 Definitions

We will define a predicate to let us talk about an inequality relationship between $n!$ and 3^n . Let:

$$P(n) : n! > 3^n \tag{14}$$

Note that $P(n)$ may be true or false for any particular choice of n . For example, $P(2)$ is false because $2!$ is not greater than 3^2 . But $P(7)$ is true because $7!$ (5040) is greater than 3^7 (2187).

3.2 Goal

Our goal is to prove that $P(n)$ holds for all values of n greater than 6.

$$\forall n \in \mathbb{Z}_{>6} : P(n) \tag{15}$$

3.3 Proof by induction

3.3.1 Base case

To show that the base case $P(7)$ is true, we will state the base case, then show that the inequality is true. Prove:

$$\begin{aligned} P(7) : 7! &> 3^7 \\ 7! &= 5040 > 2187 = 3^7 \end{aligned} \tag{16}$$

3.3.2 Inductive step

We will prove that **if** $P(k)$ holds (is true) for some $k \in \mathbb{Z}_{>6}$, **then** $P(k+1)$ is also true. Prove:

$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{>6} \tag{17}$$

We start with the *inductive hypothesis*, we assume for the time that $P(k)$ holds. Assume:

$$P(k) : k! > 3^k \mid k \in \mathbb{Z}_{>6} \tag{18}$$

Now, assuming that $P(k)$ is true, prove:

$$P(k+1) : (k+1)! > 3^{k+1} \tag{19}$$

Starting with the left side:

$$(k+1)! = (k+1)k!$$

Using our inductive hypothesis:

$$\begin{aligned} (k+1)! &> (k+1)3^k \\ (k+1)! &> \frac{1}{1}(k+1)3^k \\ (k+1)! &> \frac{3}{3}(k+1)3^k \\ (k+1)! &> \frac{k+1}{3}3 * 3^k \\ (k+1)! &> \frac{k+1}{3}3^1 * 3^k \\ (k+1)! &> \frac{k+1}{3}3^{k+1} \end{aligned}$$

And if $\frac{k+1}{3}$ is greater than 1:

$$(k+1)! > \frac{k+1}{3}3^{k+1} > 3^{k+1}$$

therefore:

$$(k+1)! > 3^{k+1}$$

Since $k \in \mathbb{Z}_{>6}$, we know that $\frac{k+1}{3}$ will always be greater than $\frac{6}{3}$ which is greater than 1. And so we have proved $P(k+1)$ (19) by assuming $P(k)$ (18) was true. If $P(k)$, then $P(k+1)$ (17).

3.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(7)$ and that for all $k \in \mathbb{Z}_{>6}$, $P(k)$ being true implies that $P(k+1)$ is also true. Therefore $P(n)$ holds for all $n > 6$.

$$\begin{aligned} P(7) : 7! &> 3^7 \\ P(k) &\implies P(k+1), \forall k \in \mathbb{Z}_{>6} \\ \therefore P(n), \forall n &\in \mathbb{Z}_{>6} \end{aligned}$$

4 Proof of divisibility

This is related to our stamp problem. We want to prove that for any integer n greater than or equal to 0, $3^{2n} - 1$ is divisible by 8.

$$3^{2n} - 1 \equiv 0 \pmod{8}$$

4.1 Definitions

We will define a predicate to let us talk about the divisibility of our expression. Let:

$$P(n) : 3^{2n} - 1 \equiv 0 \pmod{8} \quad (20)$$

Note that $P(n)$ may be true or false for any particular choice of n . For example, $P(0)$ is true because $3^0 - 1 = 0$ is divisible by 8. But $P(-1)$ is false because $3^{-2} - 1 = -\frac{8}{9}$ is not divisible by 8.

4.2 Goal

Our goal is to prove that $P(n)$ holds for all values of n greater than or equal to 0.

$$\forall n \in \mathbb{Z}_{\geq 0} : P(n) \quad (21)$$

4.3 Proof by induction

4.3.1 Base case

To show that the base case $P(0)$ is true, we will state the base case, then show it is true. Prove:

$$\begin{aligned} P(0) : 3^{2(0)} - 1 &\equiv 0 \pmod{8} \\ 3^0 - 1 &= 0 \end{aligned} \quad (22)$$

4.3.2 Inductive step

We will prove that **if** $P(k)$ holds (is true) for some $k \in \mathbb{Z}_{\geq 0}$, **then** $P(k+1)$ is also true. Prove:

$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{\geq 0} \quad (23)$$

We start with the *inductive hypothesis*, we assume for the time that $P(k)$ holds. Assume:

$$P(k) : 3^{2k} - 1 \equiv 0 \pmod{8} \mid k \in \mathbb{Z}_{\geq 0} \quad (24)$$

Now, assuming that $P(k)$ is true, prove:

$$P(k+1) : 3^{2(k+1)} - 1 \equiv 0 \pmod{8} \quad (25)$$

Starting with the left side:

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1$$

$$3^{2(k+1)} - 1 = 3^{2k} * 9 - 1$$

$$3^{2(k+1)} - 1 = 3^{2k} * (8 + 1) - 1$$

$$3^{2(k+1)} - 1 = 3^{2k} * 8 + 3^{2k} - 1$$

The left expression, $(3^{2k} * 8)$ is clearly divisible by 8 because any whole number multiplied by 8 is divisible by 8. (We know it is a whole number because k is non-negative). And the right expression $(3^{2k} - 1)$ is, by our *inductive hypothesis*, divisible by 8. So

$$3^{2(k+1)} - 1 = 8a + 8b \mid a, b \in \mathbb{Z}$$

$$3^{2(k+1)} - 1 = 8(a + b) \mid a, b \in \mathbb{Z}$$

Therefore $3^{2(k+1)} - 1$ is divisible by 8, because any whole number times 8 is divisible by 8. And so we have proved $P(k+1)$ (25) by assuming $P(k)$ (24) was true. If $P(k)$, then $P(k+1)$ (23).

4.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(0)$ and that for all $k \in \mathbb{Z}_{\geq 0}$, $P(k)$ being true implies that $P(k+1)$ is also true. Therefore $P(n)$ holds for all $n \geq 0$.

$$P(0) : 3^{2(0)} - 1 \equiv 0 \pmod{8}$$

$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{\geq 0}$$

$$\therefore P(n), \forall n \in \mathbb{Z}_{\geq 0}$$

5 Another summation

We want to prove that $1 + 4 + 7 + \dots + (3n - 2)$ can be calculated with $\frac{n(3n-1)}{2}$