# 1 Sum from 1 to n

We want to prove that $1 + 2 + ... + n$ can be calculated with $\frac{n(n+1)}{2}$

## 1.1 Definitions

We will define a function to let us talk about the sum of numbers from 1 to $n$. Let:

$$F(n) = 1 + 2 + ... + n \tag{1}$$

We will define a predicate to let us talk about the relationship between $F(n)$ and the shortcut calculation. Let:

$$P(n) : F(n) = \frac{n(n+1)}{2} \tag{2}$$

Note that $P(n)$ evaluates to a boolean. It can be true or false for any particular $n$. It is true for a particular value of $n$ if $F(n)$ does in fact equal $\frac{n(n+1)}{2}$ and it is false if these two things are not equal.

## 1.2 Goal

Our goal is to prove that $P(n)$ holds (is true) for all values of $n$ greater than 0. Prove:

$$\forall n \in N : P(n) \tag{3}$$

## 1.3 Proof by induction

### 1.3.1 Base case

To show our base case $P(1)$ is true, we will state the base case, then show that the left side does in fact equal the right side. Prove:

$$P(1) : F(1) = \frac{1(1+1)}{2} \tag{4}$$

$$F(1) = 1$$

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

### 1.3.2 Inductive step

We will prove that **if** $P(k)$ holds (is true) for some $k \in N$, **then** $P(k+1)$ is also true. Prove:

$$P(k) \implies P(k+1) : \forall k \in N \tag{5}$$

We start with the *inductive hypothesis*, we assume for the time that $P(k)$ holds. Assume:

$$P(k) : F(k) = \frac{k(k+1)}{2} \tag{6}$$

Now, assuming that $P(k)$ is true, prove:

$$P(k+1) : F(k+1) = \frac{(k+1)((k+1)+1)}{2} \tag{7}$$

By definition:

$$F(k+1) = 1 + 2 + ... + k + (k+1)$$

which is by definition:

$$F(k+1) = F(k) + (k+1)$$

which by our inductive hypothesis is:

$$F(k+1) = \frac{k(k+1)}{2} + (k+1)$$

simplifying is:

$$F(k+1) = (k+1)(\frac{k}{2} + 1)$$

which is equivalent to:

$$F(k+1) = (k+1)(\frac{k}{2} + \frac{2}{2})$$

which simplifies to:

$$F(k+1) = \frac{(k+1)(k+2)}{2}$$

which is clearly:

$$F(k+1) = \frac{(k+1)((k+1)+1)}{2}$$

And so we have proved $P(k+1)$ (7) by showing that the left side is equal to the right side (assuming that $P(k)$ is true).

### 1.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(1)$ and that for all $k \in N$, $P(k)$ being true implies that $P(k+1)$ is also true. Therefore $P(n)$ holds for all $n > 0$ (all natural numbers).

$$P(1) : F(1) = \frac{1(1+1)}{2}$$
$$P(k) \implies P(k+1) : \forall k \in N$$
$$\therefore P(n) : \forall n \in N$$

## 2 Making postage with 3 and 5 cent stamps

We want to prove that all postage amounts greater than or equal to 8 cents can be made with combinations of 3 and 5 cent stamps

## 3 Another summation

We want to prove that $1 + 4 + 7 + ... + (3n - 2)$ can be calculated with $\frac{n(3n-1)}{2}$

## 4 Proof with inequality

We want to prove that for any number $n$ greater than or equal to 7, $n!$ is greater than $3^n$.

$$n! > 3^n : n \geq 7$$