Deriving the distribution of test statistic for one sample t-test.

Let $x_1, x_2, \ldots, x_n \overset{iid}{\sim} N(\mu, \sigma^2)$. We want to test the hypotheses:

$H_0: \mu = \mu_0$ vs $H_a: \mu \neq \mu_0$

under H_0;

$$T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

By Theorem, $\bar{X} \sim N(\mu, \sigma^2/n)$

Proof

If $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$

Then $Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty$$

$$M_Z(t) = E e^{tZ} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cdot e^{tz} dz$$

$$= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2} dz}_{Z \sim N(t, 1)}$$

$$= e^{\frac{1}{2}t^2}$$

Now $x = \mu + Z\sigma$

$$\Rightarrow M_X(t) = M_{\mu + Z\sigma}(t) = M_\mu(t) \cdot M_Z(\sigma t)$$

$$= e^{\mu t} \cdot e^{\frac{1}{2}\sigma^2 t^2}$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

But $\bar{x} = \frac{1}{n}\Sigma x_i$

$$\Rightarrow M_{\bar{X}}(t) = \left[M_X\left(\frac{t}{n}\right)\right]^n = \left[e^{\frac{\mu}{n}t + \frac{\sigma^2}{2n^2}t^2}\right]^n$$

$$= e^{\mu t + \frac{\sigma^2}{2n}t^2}$$

which is MGF of $N(\mu, \frac{\sigma^2}{n})$

$\therefore \bar{X} \sim N(\mu, \sigma^2/n)$

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Next, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$

$$\Rightarrow \frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sigma^2}$$

But $\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^2} = \frac{\sum(x_i-\bar{x}+\bar{x}-\mu)^2}{\sigma^2} = \frac{\sum(x_i-\bar{x})^2+\sum(\bar{x}-\mu)^2}{\sigma^2} = \frac{\sum(x_i-\bar{x})^2+n(\bar{x}-\mu)^2}{\sigma^2}$

$$\Rightarrow \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^2} = \frac{\sum(x_i-\bar{x})^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2}$$

But $\frac{x-\mu}{\sigma} = z = \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 = \sum_{i=1}^{n} z_i^2 \sim \chi^2_n$

Also, $\frac{n(\bar{x}-\mu)^2}{\sigma^2} = \left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2 = z^2 \sim \chi^2_1$

$$\Rightarrow \chi^2_n = \frac{\sum(x_i-\bar{x})^2}{\sigma^2} + \chi^2_1$$

Let $W = \chi^2_n$, $U = \chi^2_1$ and $V = \frac{(n-1)S^2}{\sigma^2} = \frac{\sum(x_i-\bar{x})^2}{\sigma^2}$

$$\Rightarrow M_W(t) = M_V(t)\cdot M_U(t)$$

$$\Rightarrow M_V(t) = \frac{M_W(t)}{M_U(t)} = \frac{(1-2t)^{-n/2}}{(1-2t)^{-1/2}} = (1-2t)^{-\frac{(n-1)}{2}}$$

which is MGF of $\chi^2_{(n-1)}$

$$\therefore \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

Next, we want to show that $\bar{x}$ and S^2 are independent.
A number of options. Lets use transformation.

WLOG: assume $\mu = 0$ and $\sigma^2 = 1$

Then $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{1}{n-1}\left[x_1-\bar{x}+\sum_{i=2}^{n}(x_i-\bar{x})^2\right]$ since $\sum(x_i-\bar{x})=0$

$$= \frac{1}{n-1}\left[\left[\sum_{i=2}^{n}(x_i-\bar{x})\right]^2+\sum_{i=2}^{n}(x_i-\bar{x})^2\right]$$

$\Rightarrow S^2$ is a function of only $x_i-\bar{x}$ for $i=2,3,\dots$

The joint pdf of $x_1,\dots,x_n$ is

$$f(x_1,\dots,x_n) = \left(\frac{1}{2\pi}\right)^{n/2} e^{-\frac{1}{2}\sum_{i=1}^{n}x_i^2}, \quad -\infty < x_i < \infty$$

We transform $x_1, \dots, x_n$ into $y_1, \dots, y_n$ as :

$$Y_1 = \bar{x} \quad (= \tfrac{1}{n}\sum x_i)$$
$$y_2 = x_2 - \bar{x}$$
$$y_3 = x_3 - \bar{x}$$
$$\vdots$$
$$y_n = x_n - \bar{x}$$

$$\Rightarrow x_1 = y_1 - \sum_{i=2}^{n} y_i \Rightarrow \frac{\partial x_1}{\partial y_1} = 1, \ \frac{\partial x_1}{\partial y_2} = -1, \dots, \frac{\partial x_1}{\partial y_n} = -1$$

$$x_2 = y_1 + y_2 \Rightarrow \frac{\partial x_2}{\partial y_1} = 1, \ \frac{\partial x_2}{\partial y_2} = 1$$
$$\vdots$$
$$x_n = y_1 + y_n \Rightarrow \frac{\partial x_n}{\partial y_1} = 1, \ \frac{\partial x_n}{\partial y_n} = 1$$

So
$$J = \begin{vmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix} = n$$

$$\Rightarrow f(y_1, \dots, y_n) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\left(y_1 - \sum_{i=2}^{n} y_i\right)^2 - \frac{1}{2}\sum_{i=2}^{n}(y_i + y_1)^2} \cdot n$$

$$\left(y_1 - \sum_{i=2}^{n} y_i\right)^2 = y_1^2 - 2y_1\sum_{i=2}^{n} y_i + \left(\sum_{i=2}^{n} y_i\right)^2 \quad \text{—①}$$

$$\sum_{i=2}^{n}(y_i + y_1)^2 = \sum_{i=2}^{n} y_i^2 + 2y_1\sum_{i=2}^{n} y_i + (n-1)y_1^2 \quad \text{—②}$$

From ① and ② ① + ② gives .

$$= ny_1^2 + \left(\sum_{i=2}^{n} y_i\right)^2 + \sum_{i=2}^{n} y_i^2$$

$$\Rightarrow f(y_1, \dots, y_n) = \frac{n}{(2\pi)^{n/2}} e^{-\frac{1}{2}\left[ny_1^2 + \left(\sum_{i=2}^{n} y_i\right)^2 + \sum_{i=2}^{n} y_i^2\right]}$$

$$= \left(\frac{n}{2\pi}\right)^{1/2} e^{-\frac{1}{2}ny_1^2} \cdot \frac{n^{1/2}}{(2\pi)^{n-1/2}} e^{-\frac{1}{2}\left[\left(\sum_{i=2}^{n} y_i\right)^2 + \sum_{i=2}^{n} y_i^2\right]}, \quad -\infty < y_i < \infty$$

Since the joint pdf of $y_1, y_2, \dots, y_n$ factors, y_1 and $y_2, y_3, \dots, y_n$ are independent and hence $\bar{x}$ and s^2 are independent.

So
$$\frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{\sqrt{n}(\bar{x} - \mu)}{s} = \frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} \Big/ s/\sigma$$

$$= \frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} \Big/ \sqrt{\frac{(n-1)s^2}{\sigma^2} / (n-1)}$$

$$\frac{\bar{x}-\mu}{s/\sqrt{n}} = \frac{\sqrt{n}(\bar{x}-\mu)/\sigma}{\sqrt{\frac{(n-1)s^2}{\sigma^2}/(n-1)}} = \frac{z}{\sqrt{\chi^2_{(n-1)}/(n-1)}} = \frac{z}{\sqrt{u/v}}$$

Proof

Let $u = \chi^2_v$ and $z = \frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}$

Then $f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ and $f_u(u) = \frac{u^{v/2-1}}{2^{v/2}\Gamma(v/2)} e^{-u/2}$

Since z and u are independent,

$$f(z,u) = f(z)\cdot f(u)$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cdot \frac{u^{v/2-1}}{2^{v/2}\Gamma(v/2)} e^{-u/2}$$

Let $T = \frac{z}{\sqrt{u/v}}$ and $w = u$

$\Rightarrow u = w$ and $z = T\sqrt{u/v} = T\sqrt{w/v}$

$\Rightarrow \frac{\partial u}{\partial T} = 0, \ \frac{\partial u}{\partial w} = 1$ and $\frac{\partial z}{\partial T} = \sqrt{w/v}, \ \frac{\partial z}{\partial w} = \frac{1}{2}T/\sqrt{wv}$

$$\Rightarrow |J| = \begin{vmatrix} 0 & 1 \\ \sqrt{w/v} & \frac{1}{2}T/\sqrt{wv} \end{vmatrix} = \sqrt{w/v}$$

$$\Rightarrow f(T,w) = \sqrt{w/v}\cdot\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(T\sqrt{w/v}\right)^2} \cdot \frac{w^{v/2-1}}{2^{v/2}\Gamma(v/2)} e^{-w/2}$$
$$= \left(\frac{w}{2\pi v}\right)^{1/2} \cdot e^{-\frac{1}{2}\frac{T^2 w}{v}} \cdot \frac{w^{v/2-1}}{2^{v/2}\Gamma v/2} e^{-w/2}, \quad w>0, \ -\infty < T < \infty$$

Now $f_T(t) = \int_0^\infty f(T,w)\,dw$

$$= \frac{1}{\sqrt{2\pi v}\cdot 2^{v/2}\Gamma v/2} \int_0^\infty e^{-\frac{1}{2}\left(\frac{T^2}{v}+1\right)w} \cdot w^{\frac{v+1}{2}-1}\,dw$$

$$= \frac{1}{\sqrt{2\pi\nu}\, 2^{\nu/2}\, \Gamma_{\nu/2}} \int_0^\infty \underbrace{w^{\frac{\nu+1}{2}-1} e^{\frac{-w}{\left(\frac{2\nu}{T^2+\nu}\right)}}}_{\text{kernel of GAM}\left(\frac{2\nu}{T^2+\nu},\, \frac{\nu+1}{2}\right)} dw$$

$$= \frac{\left(\frac{\cancel{2}\nu}{T^2+\nu}\right)^{\frac{\nu+1}{2}} \cdot \Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\cancel{2}\pi\nu}\, \cancel{2}^{\nu/2}\, \Gamma_{\nu/2}} \underbrace{\int_0^\infty \frac{w^{\frac{\nu+1}{2}-1}}{\left(\frac{2\nu}{T^2+\nu}\right)^{\frac{\nu+1}{2}} \Gamma\left(\frac{\nu+1}{2}\right)} e^{-\frac{w}{\left(\frac{2\nu}{T^2+\nu}\right)}} dw}_{1}$$

$$\Rightarrow f_T(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu/2)} \frac{1}{\sqrt{2\pi\nu}} \left(\frac{\nu + T^2}{\nu}\right)^{\nu/2}$$

$$= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma_{\nu/2}} \cdot \frac{1}{\sqrt{\pi\nu}} \left(1 + \frac{T^2}{\nu}\right)^{\nu/2}$$

$$\therefore f_T(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu/2)} \cdot \frac{1}{\sqrt{\pi\nu}} \left(1 + \frac{T^2}{\nu}\right)^{\nu/2}, \quad -\infty < t < \infty$$

$$\therefore T = \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_\nu, \quad \text{where } \nu = n-1 \qquad \square$$