Notes and Results for Chapter 3 Problems

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1 Problem 2

The following demonstrates that the Euler-Cromer method conserves angular momentum ($\vec{L} = \vec{r} \times m\vec{v}$) when applied to the Kepler problem. Note that for arbitrary vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \times \vec{a} = \vec{0}$ and $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

$$\vec{L}_{n+1} = \vec{r}_{n+1} \times m\vec{v}_{n+1} \quad \text{where}$$
$$\vec{v}_{n+1} = \vec{v}_n - \tfrac{\tau GM}{|\vec{r}|^3}\vec{r}_n \text{ and } \vec{r}_{n+1} = \vec{r}_n + \tau\vec{v}_n - \tfrac{\tau^2 GM}{|\vec{r}|^3}\vec{r}_n$$

$$\vec{L}_{n+1} = (\vec{r}_n + \tau\vec{v}_n - \tfrac{\tau^2 GM}{|\vec{r}|^3}\vec{r}_n) \times (m\vec{v}_n - \tfrac{\tau GMm}{|\vec{r}|^3}\vec{r}_n)$$

$$= (\vec{r}_n \times m\vec{v}_n) - (\vec{r}_n \times \tfrac{\tau GMm}{|\vec{r}|^3}\vec{r}_n) + (\tau\vec{v}_n \times m\vec{v}_n) - (\tau\vec{v}_n \times \tfrac{\tau GMm}{|\vec{r}|^3}\vec{r}_n) - (\tfrac{\tau^2 GM}{|\vec{r}|^3}\vec{r}_n \times m\vec{v}_n) + (\tfrac{\tau^2 GM}{|\vec{r}|^3}\vec{r}_n \times \tfrac{\tau GMm}{|\vec{r}|^3}\vec{r}_n)$$

$$= \vec{r}_n \times m\vec{v}_n - \tfrac{\tau^2 GMm}{|\vec{r}|^3}\vec{r}_n(\vec{v}_n \times \vec{r}_n) - \tfrac{\tau^2 GMm}{|\vec{r}|^3}\vec{r}_n(\vec{r}_n \times \vec{v}_n)$$

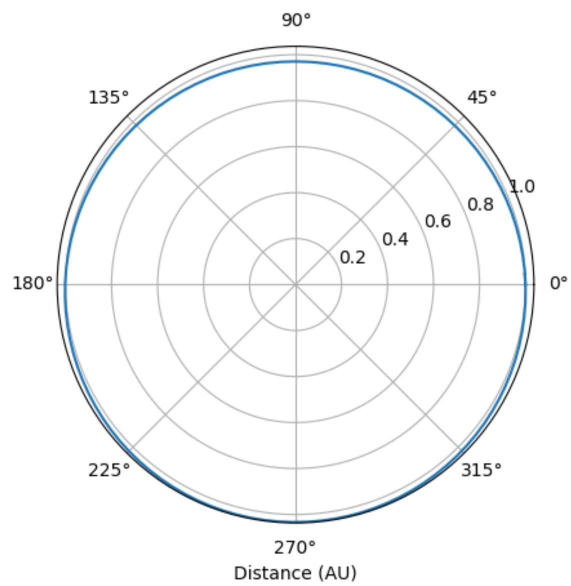
$$= \vec{r}_n \times m\vec{v}_n - \tfrac{\tau^2 GMm}{|\vec{r}|^3}\vec{r}_n(\vec{v}_n \times \vec{r}_n) + \tfrac{\tau^2 GMm}{|\vec{r}|^3}\vec{r}_n(\vec{v}_n \times \vec{r}_n)$$

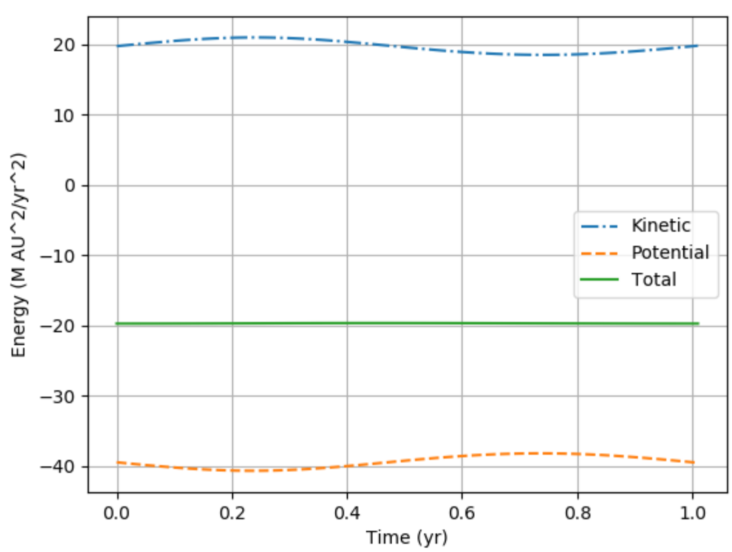
$$\vec{L}_{n+1} = \vec{r}_n \times m\vec{v}_n \text{ and angular momentum is conserved}$$

2 Problem 3

Below the orbit, energy graph, and console output for eccentricity, period, semi-major axis, etc for both a circular and elliptical orbit.

2.1 Circular Orbit





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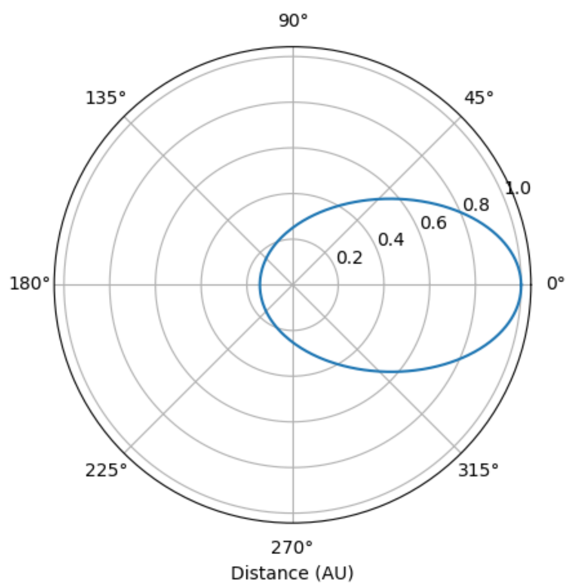
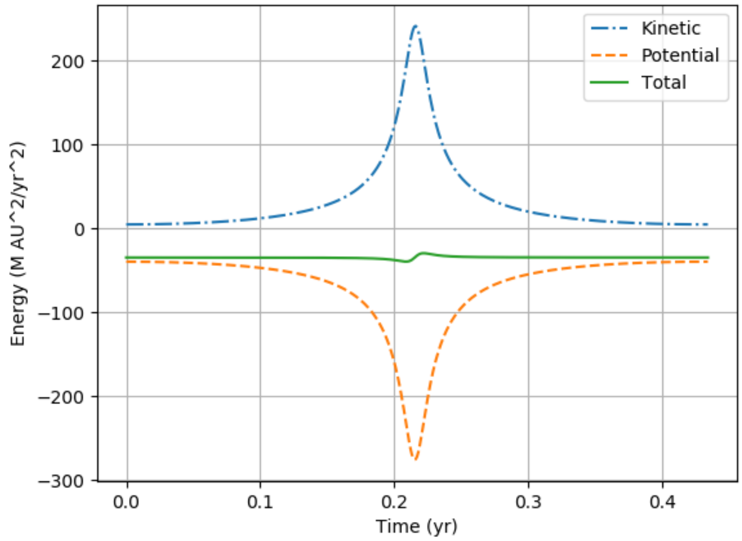
Enter initial radial distance (AU): 1
Enter initial tangential velocity (AU/yr) as a number or multiple of Pi (e.g.,2*pi): 2*pi
Enter number of steps: 250
Enter time step (yr): 0.01
Choose a number for a numerical method:
 1-Euler, 2-Euler-Cromer, 3-Runge-Kutta 4-Adaptive R-K: 2
Period =  1.01 years
Semi-major axis =  1.0000131416645683  AU
Eccentricity =  0.0036251195658040377
Perihelion distance =  0.996387974458659  AU
Analytic solution for eccentricity:  0.0036251195658040377
Check if Kepler's 3rd law is confirmed:
 T^2 =  1.0201
 a^3 =  1.0000394255118172
Time average kinetic energy <K>:  19.70158408844291  J
Time average potential energy <V>:  -39.40259887316827  J
The virial theorem says that <K> = -0.5<V>:  19.70158408844291  =  19.701299436584137
```

2.2 Elliptical Orbit

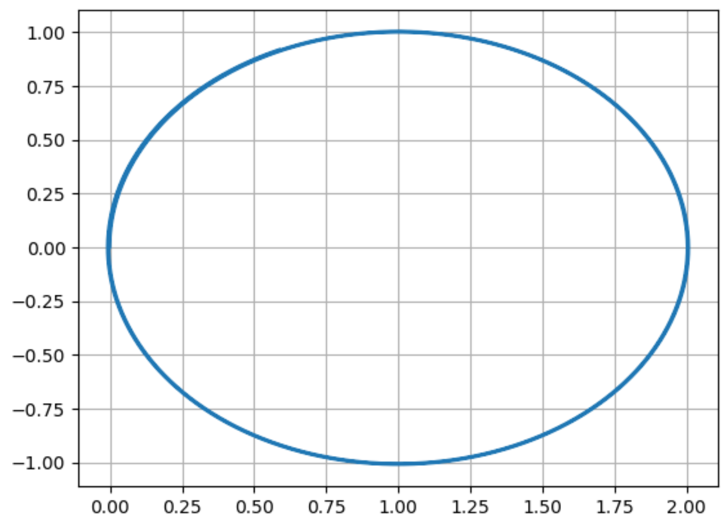




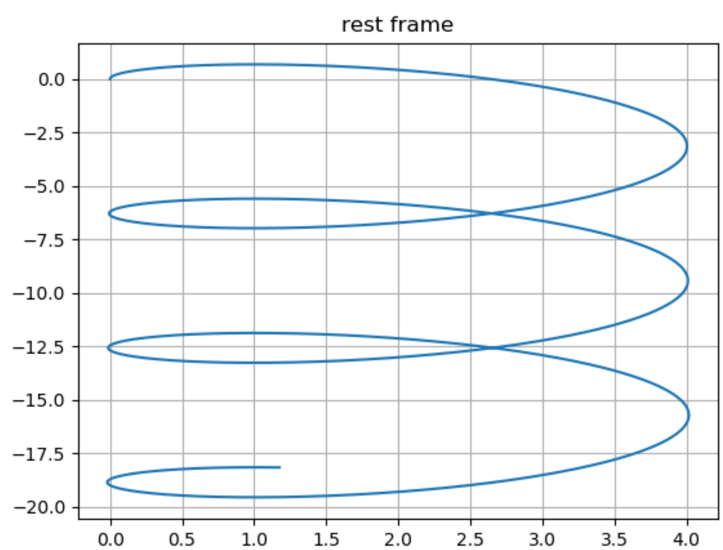

```
Enter initial radial distance (AU): 1
Enter initial tangential velocity (AU/yr) as a number or multiple of Pi (e.g.,2*pi): 1*pi
Enter number of steps: 1800
Enter time step (yr): 0.001
Choose a number for a numerical method:
 1-Euler, 2-Euler-Cromer, 3-Runge-Kutta 4-Adaptive R-K: 2
Period =  0.434 years
Semi-major axis =  0.5713990645928156  AU
Eccentricity =  0.7499849382902803
Perihelion distance =  0.1428583723950489  AU
Analytic solution for eccentricity:  0.7499849382902803
Check if Kepler's 3rd law is confirmed:
 T^2 =  0.188356
 a^3 =  0.1865600181198875
Time average kinetic energy <K>:  34.323615149017584  J
Time average potential energy <V>:  -68.85487356269662  J
The virial theorem says that <K> = -0.5<V>:  34.323615149017584  =  34.42743678134831
```

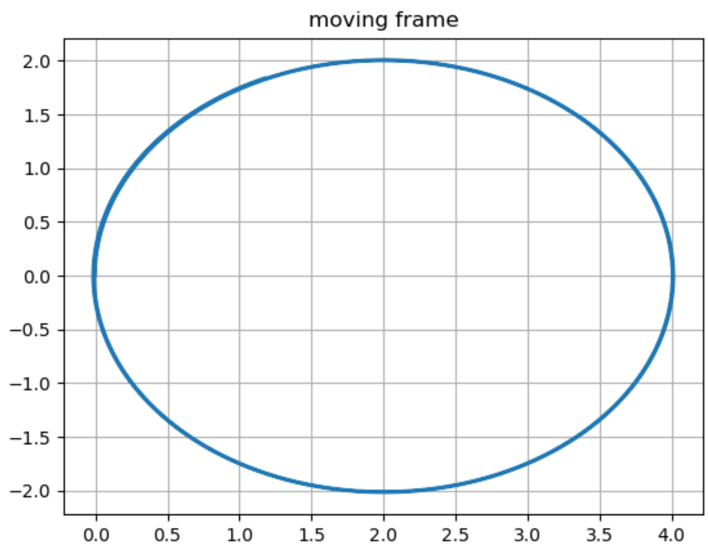
3 Problem 7

Throughout the problem I used a time step $\tau = 0.001$ s.
Trajectory for $\hat{E} = 0$, $\hat{B} = \hat{z}$:

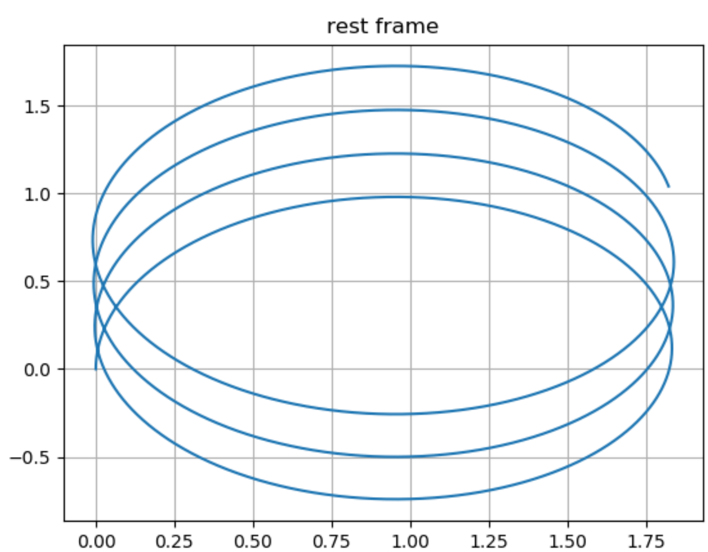


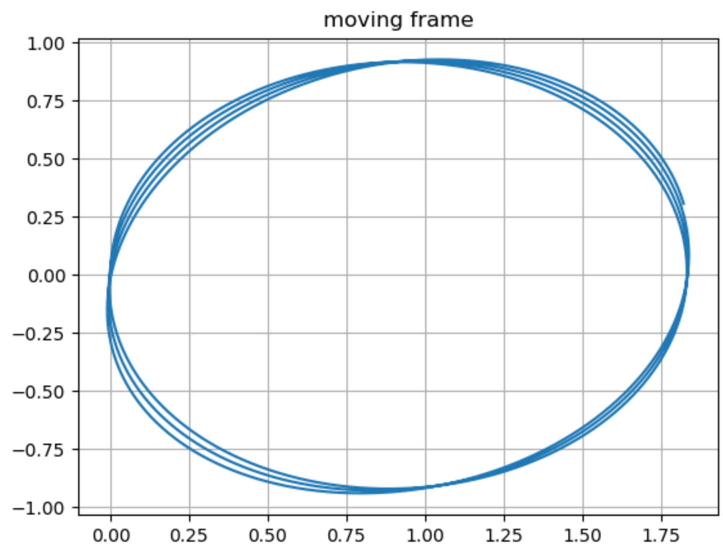
Trajectory for $\hat{E} = \hat{x}$, $\hat{B} = \hat{z}$ in both rest frame and frame moving with the $E \times B$ velocity:





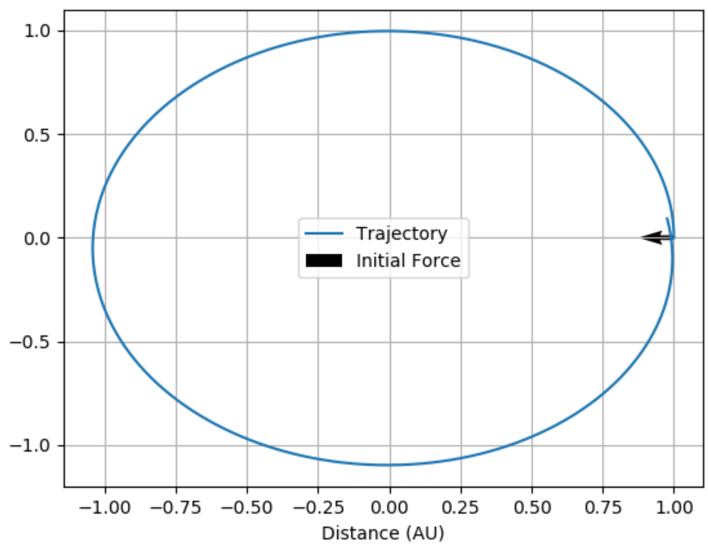
Note that when the frame is moving the trajectory does not precess as it does when the frame is at rest.
Trajectory for $\hat{E} = \hat{x}$, $\hat{B} = (1 + \alpha x)\hat{z}$, $\alpha = 0.1$ in both rest frame and frame moving with the ∇B velocity:

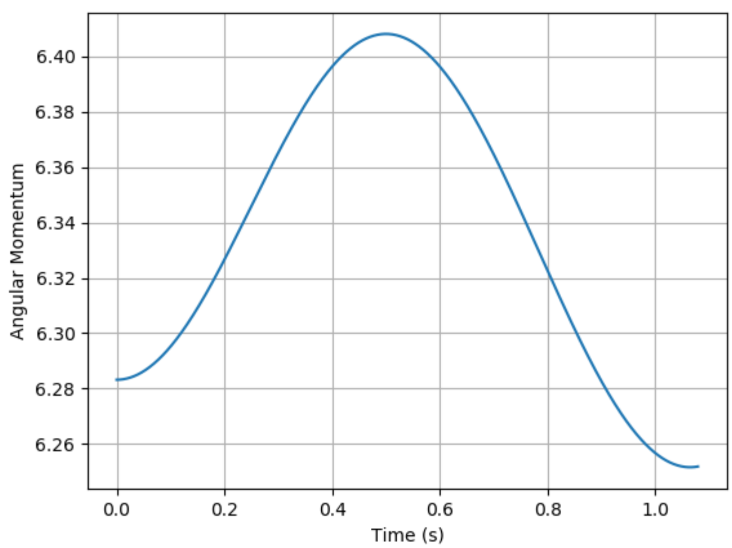




4 Problem 12

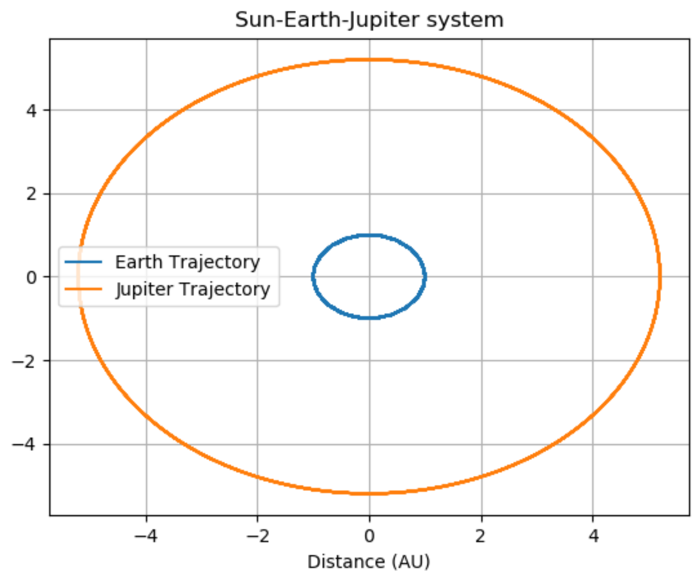
Below are graphs of the orbit and angular momentum of a comet with a perturbing force of 1% initial gravitational force. To prove the semimajor axis is perpendicular to the perturbing force the code outputs the minimum value of both plotting vectors; it can be seen the y vector has a slightly larger value so the semimajor axis is indeed perpendicular to the perturbing force, which is in the -x direction.



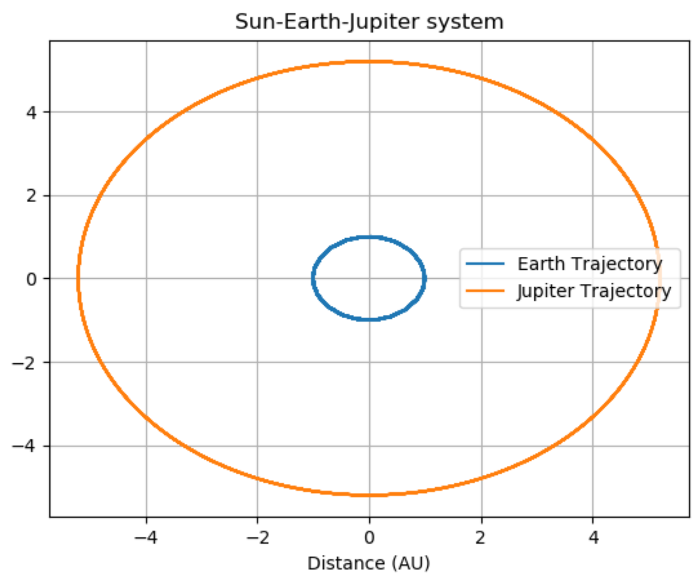


5 Problem A

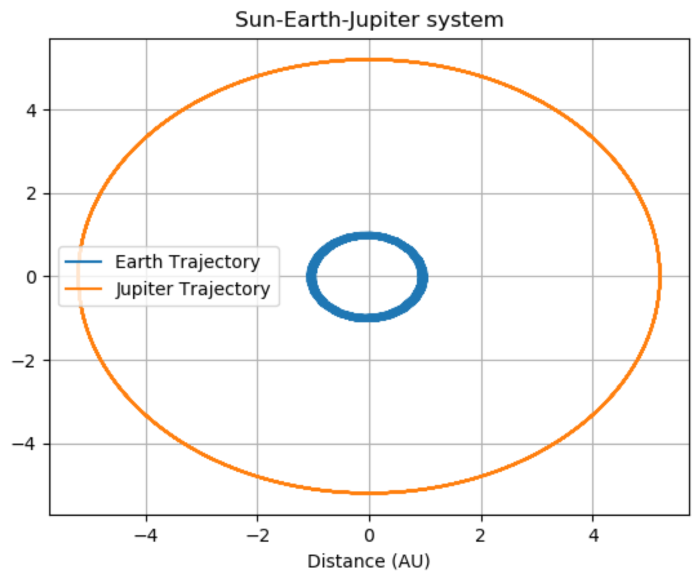
Sun-Earth-Jupiter system, 1 M_J:



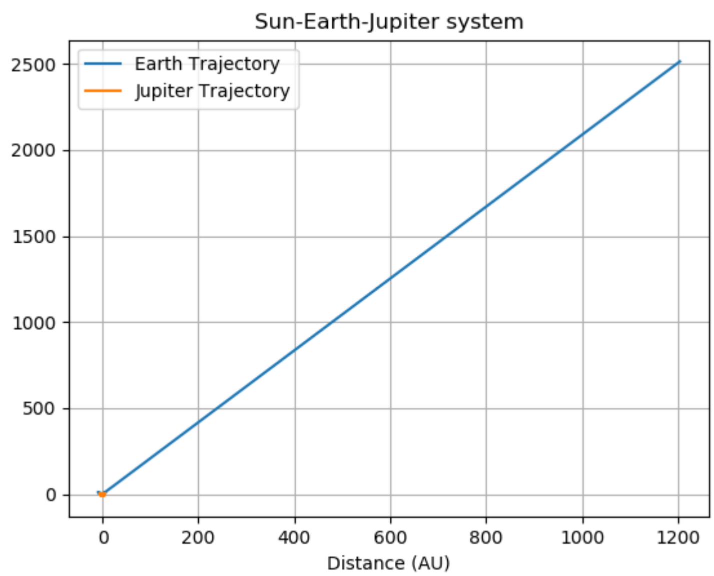
Sun-Earth-Jupiter system, 10 M_J:



Sun-Earth-Jupiter system, 100 M_J. Note that Earth's orbit is becoming unstable, as illustrated by the thickened blue line on the plot:

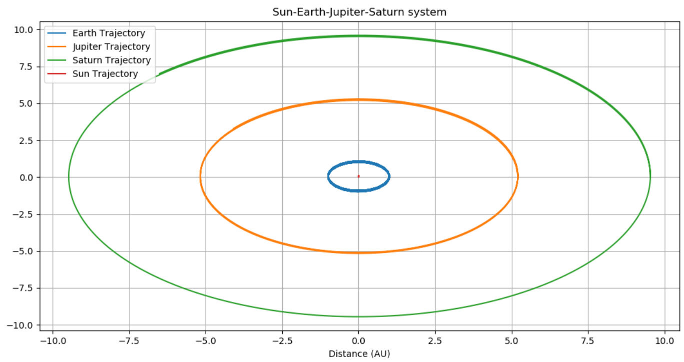


Sun-Earth-Jupiter system, 1000 M_J:



6 Problem B

Introducing Saturn and allowing the Sun to move produces the following plot:



Sun-Earth-Jupiter-Saturn system
Earth Trajectory
Jupiter Trajectory
Saturn Trajectory
Sun Trajectory
Distance (AU)