Notes and Results for Chapter 7b

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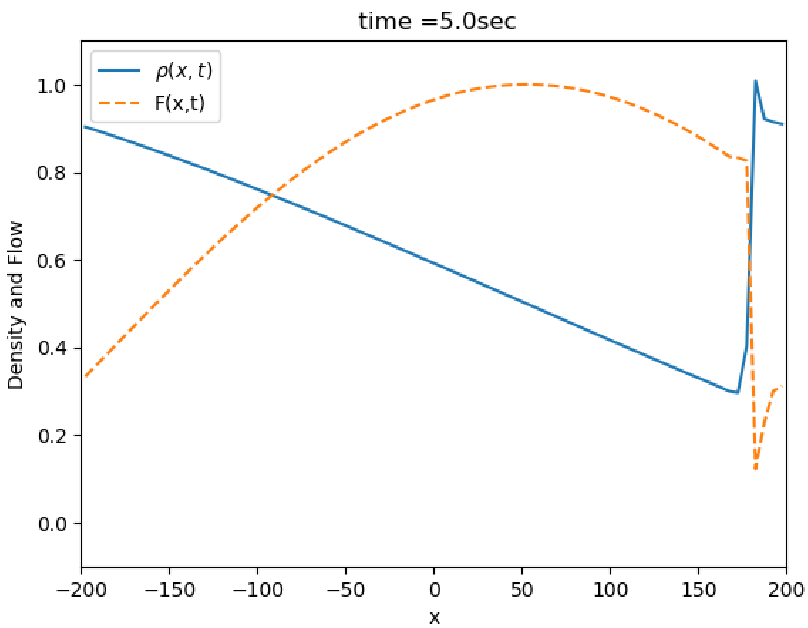
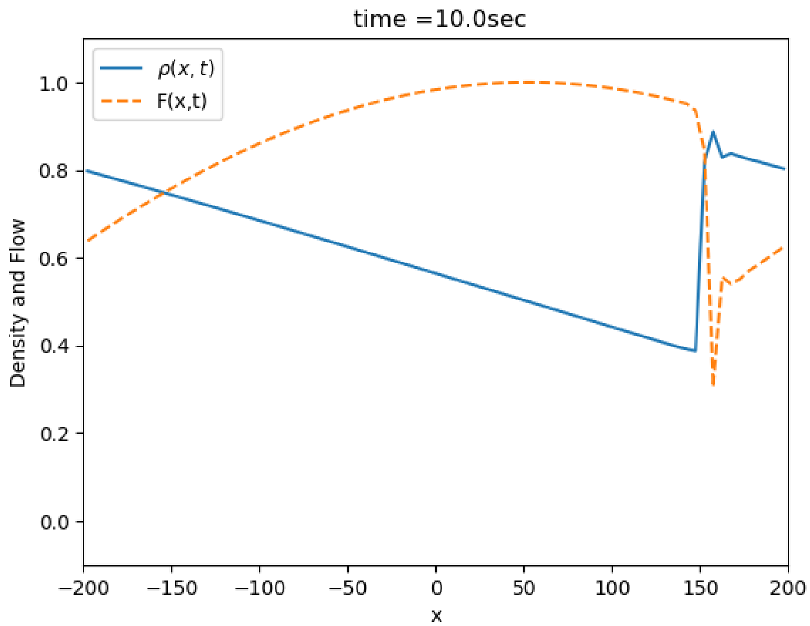
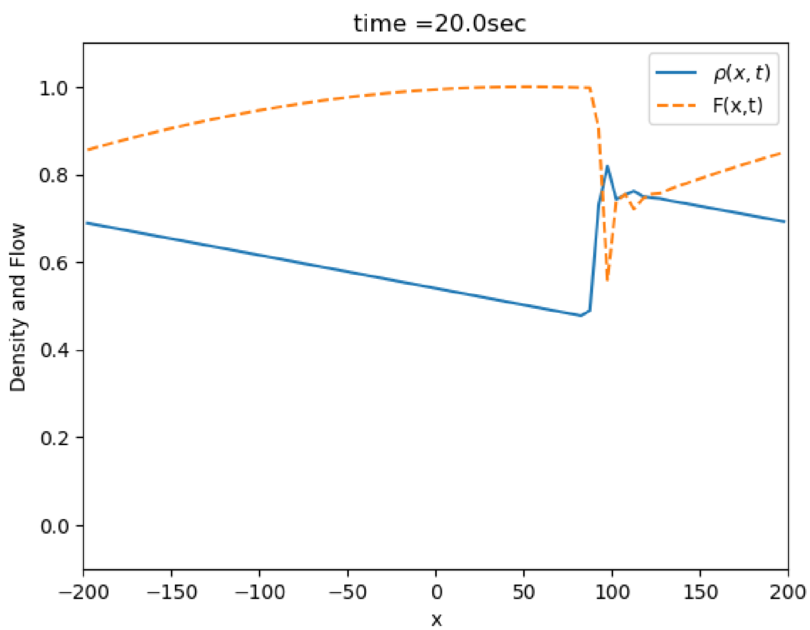
Problem 7.11

See below for several density versus position plots for various times. Note that the initial cosine wave becomes more like a sawtooth wave.





time =10.0sec
ρ(x, t)
F(x,t)
Density and Flow
x

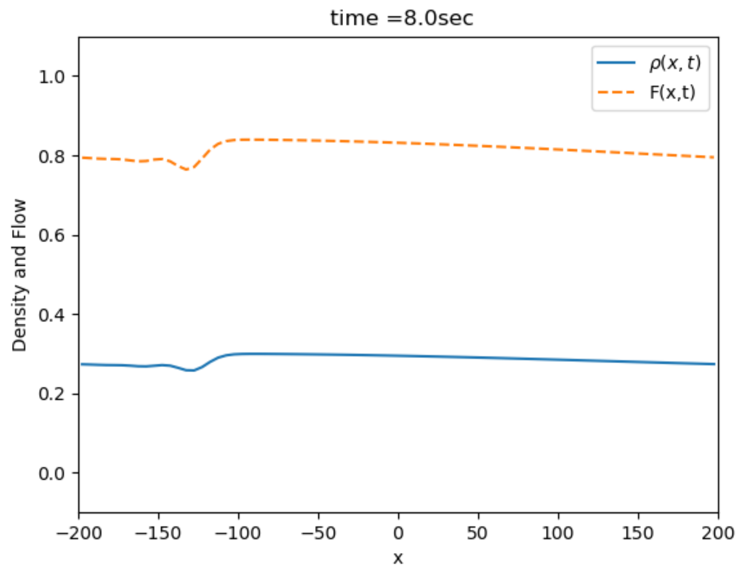


time =20.0sec
ρ(x, t)
F(x,t)
Density and Flow
x

Problem 7.12

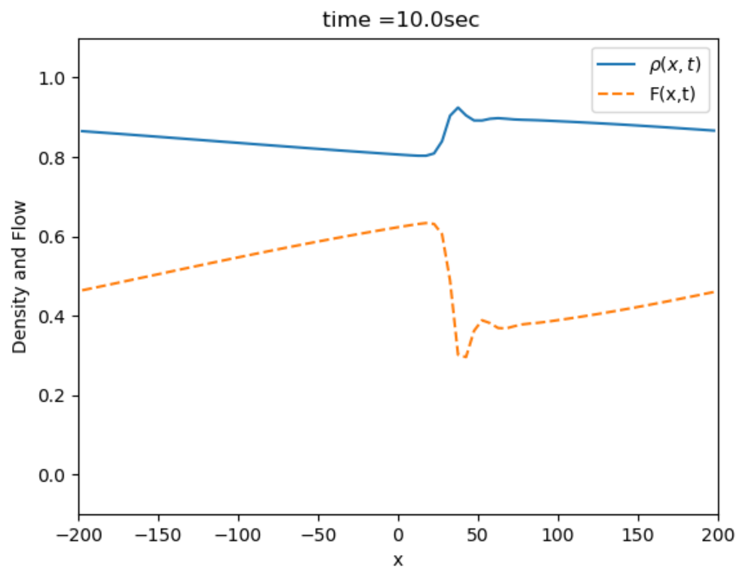
a)

In the plot below you can see that the perturbation (which begins at the far left of the plot) moves forward (e.g. towards the right) for light traffic ($\rho_0 = \rho_m/4$).



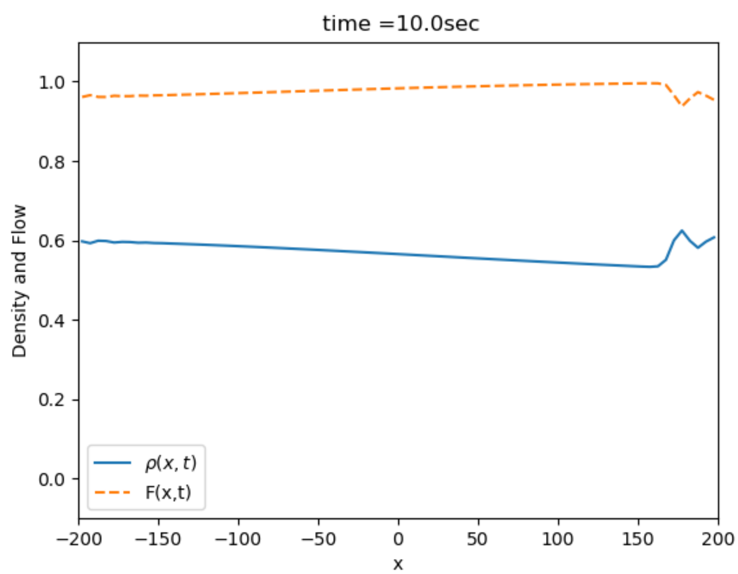
b)

The plot below shows how the perturbation begins at the far right of the plot and then moves backwards for heavy traffic ($\rho_0 = 3\rho_m/2$). Physically, this result means that the backup stays in the same spot. This shows that as a driver approaches the perturbation, she slows down and then speeds back up upon leaving the perturbation.



c)

When the density is half of the maximum density it can be seen in the graph below that the perturbation does not move much at all; after 10 seconds the perturbation had drifted less than 50 feet and becomes slightly distorted.



Problem 7.8

a)

Given that $x_c(t)$ is the position of a given car, we know that

$$\frac{dx_c(t)}{dt} = v(\rho(x_c(t), t))$$

where

$$\rho(x,t) = \begin{cases} \rho_m & x \leq -v_m t \\ \frac{1}{2}\left(1 - \frac{x}{v_m t}\right)\rho_m & -v_m t < x < v_m t \\ 0 & x \geq v_m t \end{cases}$$

and

$$v(\rho) = (1 - \rho/\rho_m)$$

From this we can find that

$$v(\rho(x,t)) = \begin{cases} 0 & x \leq -v_m t \\ \frac{v_m}{2} + \frac{x}{2t} & -v_m t < x < v_m t \\ v_m & x \geq v_m t \end{cases}$$

Upon inspection we can see that when $t < -x/v_m$ the velocity is 0 and so the position of a given car, x_c, will be the initial position $x_c(0)$. For the other solution component we have the equation

$$\frac{dx(t)}{dt} - \frac{x}{2t} = \frac{v_m}{2} \tag{1}$$

and know that an equation of the form $at - b\sqrt{t}$ should be a solution. We can test this by plugging in:

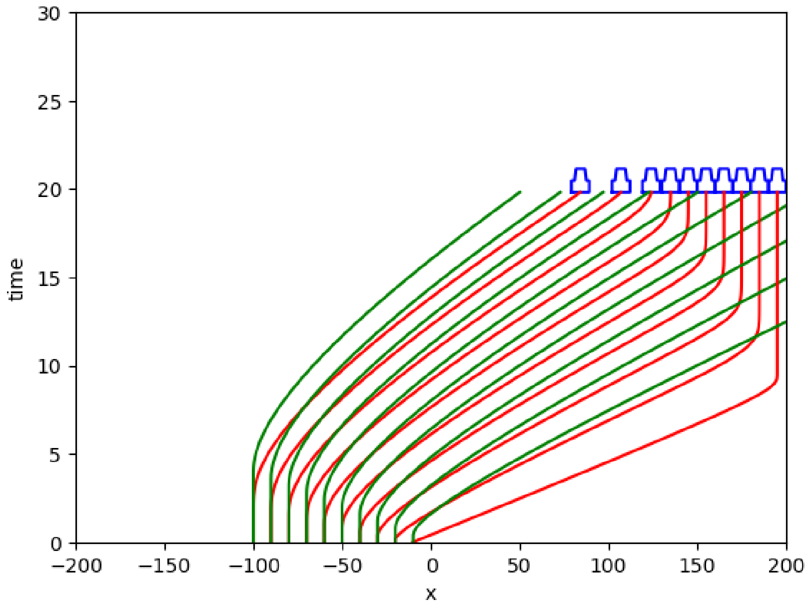
$$\begin{aligned} & a - \frac{b}{2\sqrt{t}} - \frac{1}{2t}\left(at - b\sqrt{t}\right) \\ &= a - \frac{b}{2\sqrt{t}} - \left(\frac{a}{2} - \frac{b}{2\sqrt{t}}\right) \\ &= \frac{a}{2} \end{aligned}$$

This aligns with Equation 1, which also equals a constant over 2; this shows that

$$x_c(t) = \begin{cases} x_c(0) & t < -x_c(0)/v_m \\ v_m t - 2\sqrt{-x_c(0) v_m t} & t > -x_c(0)/v_m \end{cases}$$

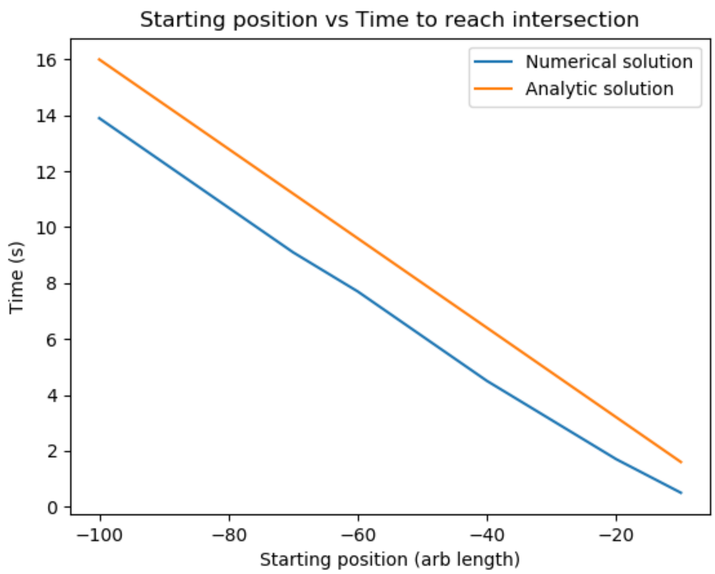
b)

See below for the plot with both numerical solution (red) and analytic solution (green). Note that the analytic solution appears "offset" from the numerical solution.



c)

See below for a graph of the time taken to reach the intersection as a function of starting position as well as the derived analytic solution, $t = \frac{-4x_c(0)}{v_m}$. Note that the analytic solution produces higher values than the numeric solution.


Starting position vs Time to reach intersection
Numerical solution
Analytic solution
Time (s)
Starting position (arb length)
0
2
4
6
8
10
12
14
16
−100
−80
−60
−40
−20