Notes and Results for Chapter 6 Problems

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1 Problem 6

(a)

Given that the Gaussian solution to the diffusion equation is

$$T_G(x,t) = \frac{1}{\sigma(t)\sqrt{2\pi}} \exp\left(-\frac{(x-x_0)^2}{2\sigma^2(t)}\right)$$

where $\sigma(t) = \sqrt{2\kappa t}$, we can use the method of images to find the solution $T(x,t)$ for the initial condition $T(x,0) = \delta(x)$ with the Neumann boundary conditions

$$\left.\frac{\partial T}{\partial x}\right|_{x=-\frac{L}{2}} = \left.\frac{\partial T}{\partial x}\right|_{x=\frac{L}{2}} = 0$$

Because Gaussians are symmetric and we want the derivative of our solution at $\pm\frac{L}{2}$ to be 0, we can place shifted versions of the Gaussian with maxima at $\pm L$ around our initial Gaussian; at each boundary the initial Gaussian and the appropriate image are symmetric about that boundary, so any central difference used to approximate the derivative will be 0 and satisfy our boundary condition. Therefore

$$T(x,t) = \sum_{n=-\infty}^{\infty} T_G(x+nL,t)$$

which is very similar to the solution for the Dirichlet boundary condition excluding the sign flipping.

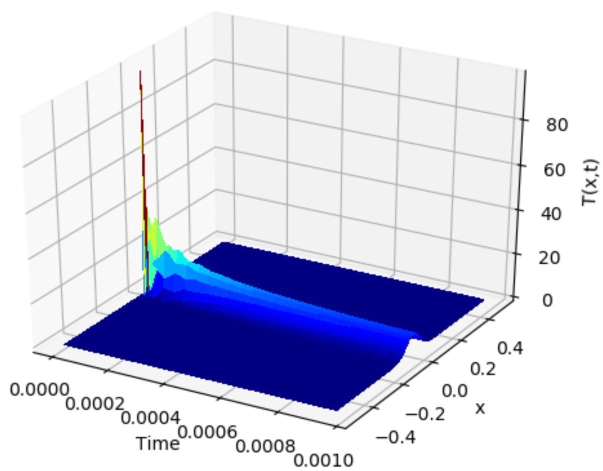
(b)

For this problem we used a spatial discretization of

$$x_i = \left(i - \frac{3}{2}\right)h - \frac{L}{2}$$

This is used instead of the "normal" discretization from the notes because by using Neumann boundary conditions we are dealing with derivatives; using this

discretization allows for points an equal distance on either side of the boundary, which allows us to easily take a central difference as an accurate approximation of the derivative (i.e. our condition requiring that the flux through the boundary is 0 which reduces to the points on either side of the boundary being of equal value).
Below is the output for the program when run with 101 grid points, a time step of 4×10^{-5} s, and a total time of 0.001 s:

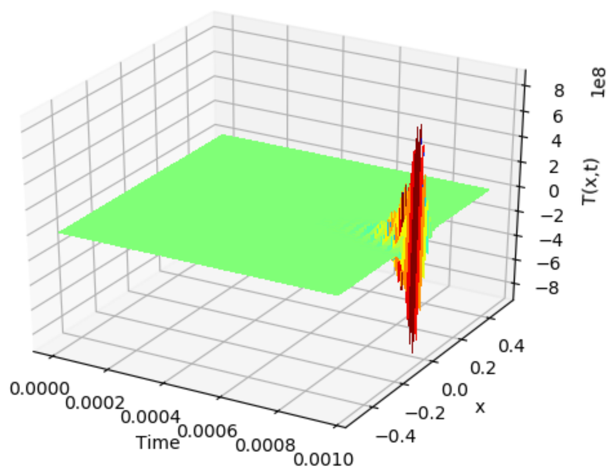


Note that these results compare favorably with the results from part (a) above, as the graph is at 0 for $x = \pm\frac{1}{2}$, which is half of the given length of the bar in our code.

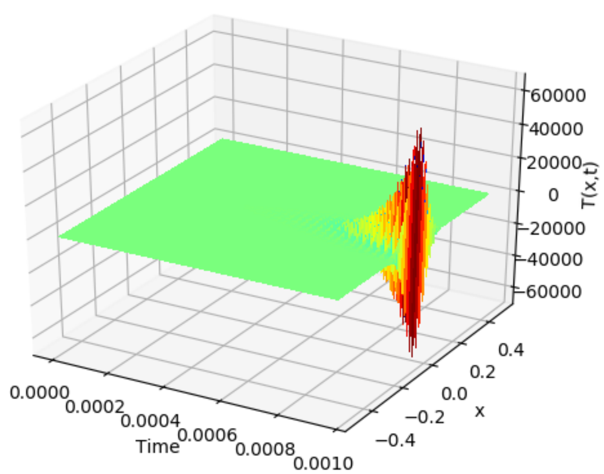
2 Problem 8

(a)

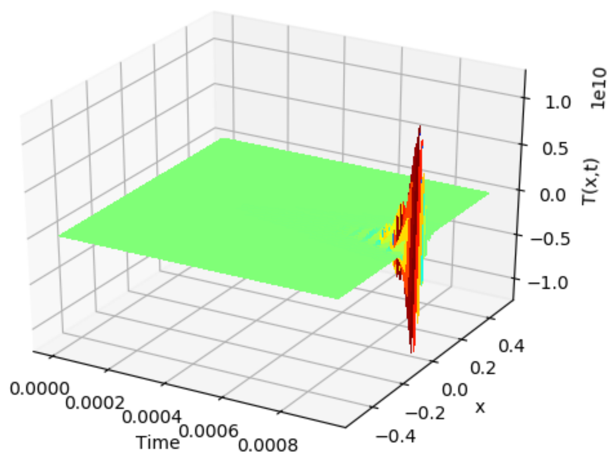
Below are several plots of the Richardson scheme, each with a different timestep below the threshold for stability for the FTCS scheme. Note that all are unstable. Each plot was made from 101 grid points with a total time of 0.001 s.



1e8
8
6
4
2
0
−2
−4
−6
−8
T(x,t)
0.4
0.2
0.0
−0.2
−0.4
x
0.0000
0.0002
0.0004
0.0006
0.0008
0.0010
Time



60000
40000
20000
0
−20000
−40000
−60000
T(x,t)
0.4
0.2
0.0
−0.2
−0.4
x
0.0000
0.0002
0.0004
0.0006
0.0008
0.0010
Time



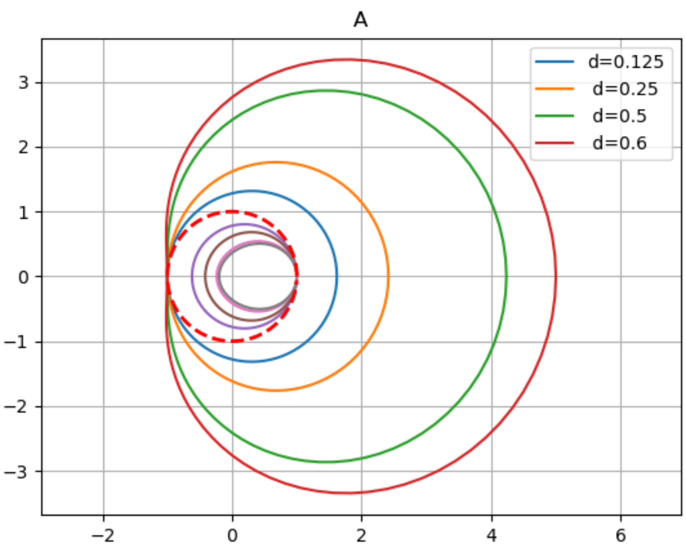
(b)

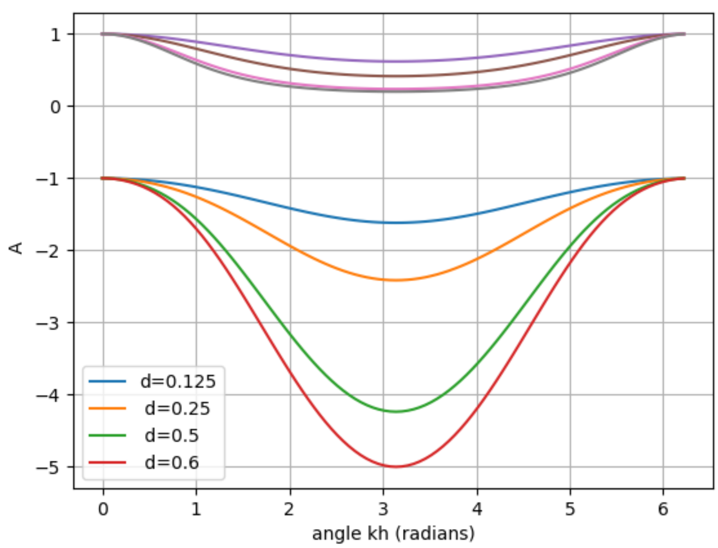
Performing a von Neumann stability analysis of the Richardson scheme (replacing T_j^n with $T^n e^{ikhj}$) and simplying, we obtain a quadratic in the gain, A:

$$A^2 + 8d\sin^2\left(\frac{kh}{2}\right)A - 1 = 0$$

where $d = \frac{2\tau\kappa}{h^2}$ and $h = \frac{L}{N-1}$ as the grid spacing. For stability the gains $|A_1|$ and $|A_2| \leq 1$. However, since a quadratic produces an equation of the form $(A - A_1)(A + A_2) = A^2 + bA + c$ with $b = -(A_1 + A_2)$ and $c = A_1 A_2$ and $|c| = 1$ but $|b| \neq 2$ it must be that one gain is larger than 1 while the other is smaller than 1, meaning that the scheme is always unstable since one gain must always be greater than 1.

Below are plots of the amplification factors A as a function of the phase angle kh. Note that there is always a value with a magnitude greater than 1, showing that the scheme is always unstable.


A
d=0.125
d=0.25
d=0.5
d=0.6


d=0.125
d=0.25
d=0.5
d=0.6
A
angle kh (radians)

3 Problem 12

(a)

I edited the solution for the system to satisfy the Neumann boundary conditions $\left.\frac{\partial n}{\partial x}\right|_{x=-\frac{L}{2}} = \left.\frac{\partial n}{\partial x}\right|_{x=\frac{L}{2}} = 0$:

$$\sum_{j=0}^{\infty} a_j \cos\left(\frac{j\pi}{L}(x + \frac{L}{2})\right)$$

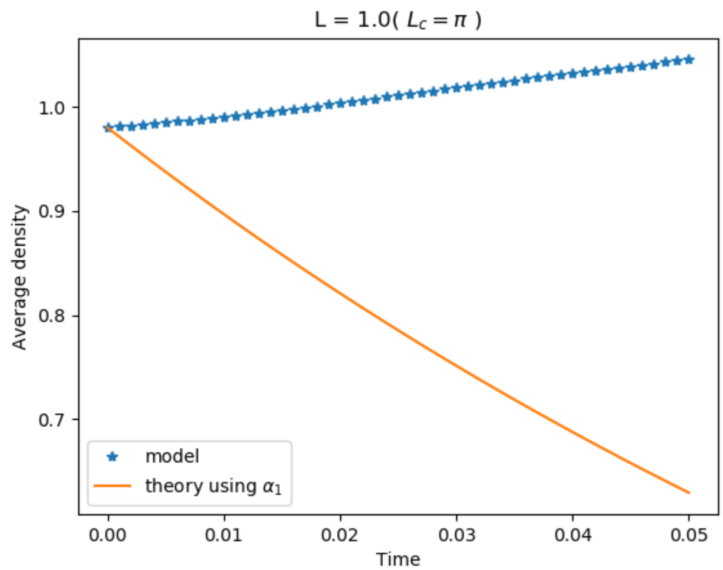
Plugging this into the original equation $\frac{d^2X}{dx^2} = \frac{\alpha - C}{D}X$, I got

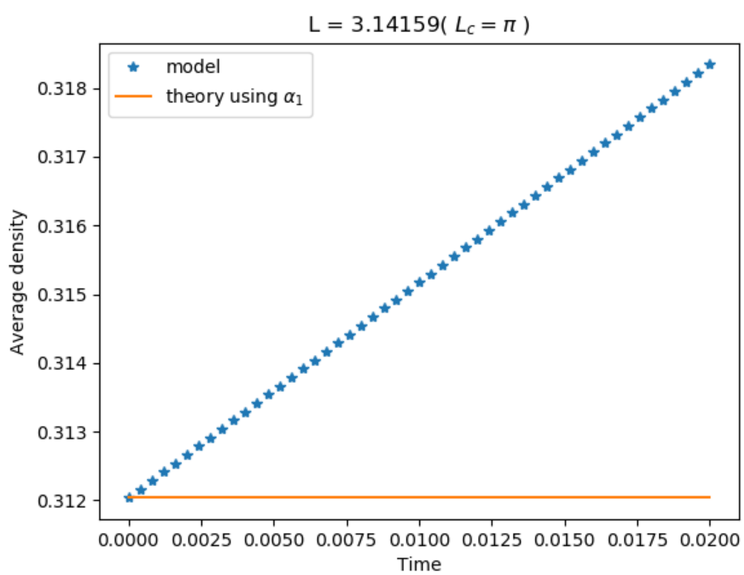
$$-\left(\frac{j\pi}{L}\right)^2 = \frac{\alpha - C}{D}$$
$$\Rightarrow\ C - D\left(\frac{j\pi}{L}\right)^2 = \alpha$$

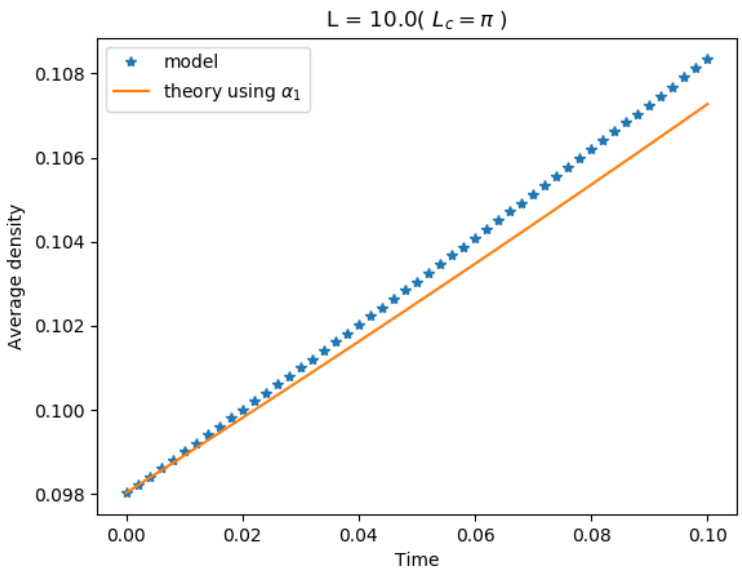
At $j = 0$, the above equation reduces to $C = \alpha$, which is strictly greater than 0. Because of this, there is always a component of the solution that is greater than zero which means that the system is always supercritical.

(b)

Below are plots for various system lengths; note that even when the system length is below the critical length the system becomes supercritical. This matches with part (a) above, which shows using separation of variables that the system should always be supercritical.


L = 1.0($L_c = \pi$)
1.0
0.9
0.8
0.7
Average density
model
theory using α_1
0.00
0.01
0.02
0.03
0.04
0.05
Time


L = 3.14159($L_c = \pi$)
model
theory using α_1
0.318
0.317
0.316
0.315
0.314
0.313
0.312
Average density
0.0000
0.0025
0.0050
0.0075
0.0100
0.0125
0.0150
0.0175
0.0200
Time



L = 10.0($L_c = \pi$)
model
theory using α_1
Average density
Time
0.098
0.100
0.102
0.104
0.106
0.108
0.00
0.02
0.04
0.06
0.08
0.10