Non-Dimensionalizing a Single-Mode Electric Field QD Micropillar Laser Model which Utilizes Electric Field, QD Occupation Probability, and Reservoir Carrier Density Equations.

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1 Introduction

The goal is to take the dynamic equations for one slowly varying electric field amplitude given in [RLH+16] and make them non-dimensional. Note: This model assumes $E_w(t) = 0$. ρ is already non-dimensional, but we need to remove the dimensions for time t, electric field $E(t)$, and reservoir carrier density $n_r(t)$. An SI unit analysis reveals the units of these values before we non-dimensionalize.

$$
\begin{aligned}
[E(t)] &= V/m \\
[\rho(t)] &= 1 \\
[n_r(t)] &= 1/m^2 \\
[t] &= s
\end{aligned}
$$

2 Redefine variables

We will make the following substitutions to change variables from dimensional variables on the left, to dimensionless variables multiplied by placeholder constants on the right. We will use θ, λ, and ϕ for our placeholder constants. These placeholder constants contain the dimension information which we are trying to remove from our system. Notice ρ has no placeholder constant since it is already a dimensionless value.

$$
\begin{aligned}
E(t) &= \theta \bar{E}(q) \\
\rho(t) &= \rho(q) \\
n_r(t) &= \phi \bar{n}_r(q) \\
t &= \lambda q
\end{aligned}
$$

3 Equations

We are going to be doing bifurcation analysis on this system so I will select equation 4 in [RLH+16], aka the "deterministic representation of spontaneous emission." Starting with equations 5-7 in [RLH+16] we simplify by assuming $E_w(t) = 0$. Since there is only one relevant electric field, $E_s(t)$, I will set $E_s(t) = E(t)$. Similarly for anything with a $j = s$ subscript, the subscript is simply removed.

In addition, I have added in a feedback term called k which is constant and $k \geq 0$. Feedback phase is called C, and feedback time is called τ_{fb}.

This yields the following equations:

$$\frac{d}{dt}E(t) = \frac{\hbar\omega}{\epsilon_0\epsilon_{bg}}\frac{2Z^{QD}}{V}g[2\rho(t)-1]E(t) - \kappa[E(t) - e^{-iC}kE(t-\tau_{fb})] + \frac{\partial}{\partial t}E|_{sp}^{det} \tag{1}$$

$$\frac{d}{dt}\rho(t) = -g[2\rho(t)-1]\,|E(t)|^2 - \frac{\rho(t)}{\tau_{sp}} + S^{in}n_r(t)[1-\rho(t)] \tag{2}$$

$$\frac{d}{dt}n_r(t) = \frac{\eta}{e_0 A}(J-J_p) - S^{in}n_r(t)\frac{2Z^{QD}}{A}[1-\rho(t)] - \frac{n_r(t)}{\tau_r} \tag{3}$$

$$with\ g = \frac{|\mu|^2 T_2}{2\hbar^2}(1+\epsilon_{ss}\tilde{\epsilon}\,|E(t)|^2)^{-1} \tag{4}$$

$$and\ \frac{\partial}{\partial t}E|_{sp}^{det} = \beta\frac{\hbar\omega}{\epsilon_0\epsilon_{bg}}\frac{2Z^{QD}}{V}\frac{\rho}{\tau_{sp}}\frac{E(t)}{|E(t)|^2} \tag{5}$$

4 Substitution

This is a reminder of our dimensionless variable multiplied by placeholder constants. Since we are dealing with delay we need to also scale our τ_{fb} into the proper dimensionless value.

$$E(t) = \theta\bar{E}(q)$$
$$\rho(t) = \rho(q)$$
$$n_r(t) = \phi\bar{n}_r(q)$$
$$t = \lambda q$$
$$\tau_{fb} = \lambda\bar{\tau}_{fb}$$

Notice that $\frac{\partial}{\partial t}E|_{sp}^{det}$ is merely a substitution, so we don't need to scale the partial derivative of time by λ. Once we substitute our dimensionless variables multiplied by placeholder constants we have:

$$\begin{aligned}\frac{\theta}{\lambda}\frac{d}{dq}\bar{E}(q) = \theta\frac{\hbar\omega}{\epsilon_0\epsilon_{bg}}\frac{2Z^{QD}}{V}g[2\rho(q)-1]\bar{E}(q) - \theta\kappa[\bar{E}(q) - e^{-iC}k\bar{E}(q-\bar{\tau}_{fb})] \\ + \frac{\theta}{|\theta|^2}\beta\frac{\hbar\omega}{\epsilon_0\epsilon_{bg}}\frac{2Z^{QD}}{V}\frac{\rho}{\tau_{sp}}\frac{\bar{E}(q)}{|\bar{E}(q)|^2}\end{aligned} \tag{6}$$

$$\frac{1}{\lambda}\frac{d}{dq}\rho(q) = -|\theta|^2 g[2\rho(q)-1]\,|\bar{E}(q)|^2 - \frac{\rho(q)}{\tau_{sp}} + \phi S^{in}\bar{n}_r(q)[1-\rho(q)] \tag{7}$$

$$\frac{\phi}{\lambda}\frac{d}{dq}\bar{n}_r(q) = \frac{\eta}{e_0 A}(J-J_p) - \phi S^{in}\bar{n}_r(q)\frac{2Z^{QD}}{A}[1-\rho(q)] - \phi\frac{\bar{n}_r(q)}{\tau_r} \tag{8}$$

$$with\ g = \frac{|\mu|^2 T_2}{2\hbar^2}(1+\epsilon_{ss}\tilde{\epsilon}\,|\theta|^2\,|\bar{E}(q)|^2)^{-1} \tag{9}$$

After some algebra

$$\begin{aligned}\frac{d}{dq}\bar{E}(q) = \lambda\frac{\hbar\omega}{\epsilon_0\epsilon_{bg}}\frac{2Z^{QD}}{V}g[2\rho(q)-1]\bar{E}(q) - \lambda\kappa[\bar{E}(q) - e^{-iC}k\bar{E}(q-\bar{\tau}_{fb})] \\ + \frac{\lambda}{|\theta|^2}\beta\frac{\hbar\omega}{\epsilon_0\epsilon_{bg}}\frac{2Z^{QD}}{V}\frac{\rho}{\tau_{sp}}\frac{\bar{E}(q)}{|\bar{E}(q)|^2}\end{aligned} \tag{10}$$

$$\frac{d}{dq}\rho(q) = -\lambda|\theta|^2 g[2\rho(q)-1]\,|\bar{E}(q)|^2 - \lambda\frac{\rho(q)}{\tau_{sp}} + \lambda\phi S^{in}\bar{n}_r(q)[1-\rho(q)] \tag{11}$$

$$\frac{d}{dq}\bar{n}_r(q) = \frac{\lambda}{\phi}\frac{\eta}{e_0 A}(J-J_p) - \lambda S^{in}\bar{n}_r(q)\frac{2Z^{QD}}{A}[1-\rho(q)] - \lambda\frac{\bar{n}_r(q)}{\tau_r} \tag{12}$$

$$with\ g = \frac{|\mu|^2 T_2}{2\hbar^2}(1+\epsilon_{ss}\tilde{\epsilon}\,|\theta|^2\,|\bar{E}(q)|^2)^{-1} \tag{13}$$

4.1 Set constants based on equations and dynamics

It's already clear the sensible value for θ is the reciprocal of the constants in front of the absolute value of the electric field in g.

$$E(t) = \theta\bar{E}(q) = \sqrt{\frac{1}{\epsilon_{ss}\tilde{\epsilon}}}\bar{E}(q)$$

Set λ to the QD lifetime which is 1 ns. This comes up in the parameters τ_{sp} since $\tau_{sp} = 1$ns. This choice is motivated because the relevant dynamics usually occur on the nanosecond timescale. It is not obvious from the equations alone.

$$t = \lambda q = \tau_{sp} q$$
$$\tau_{fb} = \lambda\bar{\tau}_{fb} = \tau_{sp}\bar{\tau}_{fb}$$

After making those substitutions choosing ϕ is somewhat arbitrary. No matter what we choose, many terms will have some kind of multiplicative constants in front of them.

$$n_r(t) = \phi\bar{n}_r(q) = \frac{1}{S^{in}\tau_{sp}}\bar{n}_r(q)$$

4.2 Final Form of Equations

Finally we're looking at something that is non-dimensional. I left 1s in the places where the substitution canceled terms. After completing the substitution for the placeholder constants we have:

$$\begin{aligned}\frac{d}{dq}\bar{E}(q) = \tau_{sp}\frac{\hbar\omega}{\epsilon_0\epsilon_{bg}}\frac{2Z^{QD}}{V}g[2\rho(q)-1]\bar{E}(q) - \tau_{sp}\kappa[\bar{E}(q) - e^{-iC}k\bar{E}(q-\bar{\tau}_{fb})] \\ + \epsilon_{ss}\tilde{\epsilon}\beta\frac{\hbar\omega}{\epsilon_0\epsilon_{bg}}\frac{2Z^{QD}}{V}\frac{\rho}{1}\frac{\bar{E}(q)}{\left|\bar{E}(q)\right|^2}\end{aligned} \tag{14}$$

$$\frac{d}{dq}\rho(q) = -\frac{\tau_{sp}}{\epsilon_{ss}\tilde{\epsilon}}g[2\rho(q)-1]\left|\bar{E}(q)\right|^2 - \frac{\rho(q)}{1} + (1)\bar{n}_r(q)[1-\rho(q)] \tag{15}$$

$$\frac{d}{dq}\bar{n}_r(q) = \tau_{sp}^2 S^{in}\frac{\eta}{e_0 A}(J - J_p) - \tau_{sp}S^{in}\frac{2Z^{QD}}{A}\bar{n}_r(q)[1-\rho(q)] - \tau_{sp}\frac{\bar{n}_r(q)}{\tau_r} \tag{16}$$

$$with\ g = \frac{|\mu|^2 T_2}{2\hbar^2}(1 + \left|\bar{E}(q)\right|^2)^{-1} \tag{17}$$

References

[RLH+16] Christoph Redlich, Benjamin Lingnau, Steffen Holzinger, Elisabeth Schlottmann, Sören Kreinberg, Christian Schneider, Martin Kamp, Sven Höfling, Janik Wolters, Stephan Reitzenstein, and Kathy Lüdge. Mode-switching induced super-thermal bunching in quantum-dot microlasers. *New J. Phys*, 18(063011), 2016.