Online Companion for Exploiting Identical Generators in Unit Commitment

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Nomenclature

Indices and Sets

$g \in \mathcal{G}$	Thermal generators
$l \in \mathcal{L}^g$	Piecewise production cost intervals for generator g: $1, \dots, \mathbf{L}_g$.
$s \in \mathcal{S}^g$	Startup categories for generator g, from hottest (1) to coldest ($\mathbf{S}_g$).
$t \in \mathcal{T}$	Hourly time steps: $1, \dots, \mathbf{T}$.
$[t, t') \in \mathcal{X}^g$	Feasible intervals of non-operation for generator g with respect to its minimum downtime, that is, $[t, t') \in \mathcal{T} \times \mathcal{T}$ such that $t' \geq t + \mathbf{DT}^g$, including times (as necessary) before and after the planning period $\mathcal{T}$.
$[t, t') \in \mathcal{Y}^g$	Feasible intervals of operation for generator g with respect to its minimum uptime, that is, $[t, t') \in \mathcal{T} \times \mathcal{T}$ such that $t' \geq t + \mathbf{UT}^g$, including times (as necessary) before and after the planning period $\mathcal{T}$.

Parameters

$\mathbf{c}^{l,g}$	Cost coefficient for piecewise segment l for generator g (\$/MWh).
$\mathbf{c}^{R,g}$	Cost of generator g running and operating at minimum production $\underline{\mathbf{P}}_g$ (\$/h).
$\mathbf{c}^{s,g}$	Startup cost of category s for generator g (\$).
$\mathbf{D}_t$	Load (demand) at time t (MW).
$\mathbf{DT}^g$	Minimum down time for generator g (h).
$\overline{\mathbf{P}}^g$	Maximum power output for generator g (MW).
$\overline{\mathbf{P}}^{l,g}$	Maximum power available for piecewise segment l for generator g (MW) ($\overline{\mathbf{P}}^{0,g} = \underline{\mathbf{P}}^g$).
$\underline{\mathbf{P}}^g$	Minimum power output for generator g (MW).
$\mathbf{R}_t$	Spinning reserve at time t (MW).
$\mathbf{RD}^g$	Ramp-down rate for generator g (MW/h).
$\mathbf{RU}^g$	Ramp-up rate for generator g (MW/h).
$\mathbf{SD}^g$	Shutdown rate for generator g (MW/h).
$\mathbf{SU}^g$	Startup rate for generator g (MW/h).
$\mathbf{TC}^g$	Time down after which generator g goes cold, i.e., enters state S^g.
$\underline{\mathbf{T}}^{s,g}$	Time offline after which the startup category s is available ($\underline{\mathbf{T}}^{1,g} = \mathbf{DT}^g$, $\underline{\mathbf{T}}^{S^g,g} = \mathbf{TC}^g$)
$\overline{\mathbf{T}}^{s,g}$	Time offline after which the startup category s is no longer available ($= \underline{\mathbf{T}}^{s+1,g}$, $\overline{\mathbf{T}}^{S_g,g} = +\infty$)
$\mathbf{UT}^g$	Minimum run time for generator g (h).

Variables

p_t^g	Power output for generator g at time t (MW).
$p_t^{l,g}$	Power from piecewise interval l for generator g at time t (MW).
r_t^g	Spinning reserves provided by generator g at time t (MW), ≥ 0.
u_t^g	Commitment status of generator g at time t, $\in \{0,1\}$.
v_t^g	Startup status of generator g at time t, $\in \{0,1\}$.
w_t^g	Shutdown status of generator g at time t, $\in \{0,1\}$.
$c_t^{SU,g}$	Startup cost for generator g at time t (\$), ≥ 0.
$x_{[t,t')}^g$	Indicator arc for shutdown at time t, startup at time t', uncommitted for $i \in [t,t')$, for generator g, $\in \{0,1\}$, $[t,t') \in \mathcal{X}^g$.
$y_{[t,t')}^g$	Indicator arc for startup at time t, shutdown at time t', committed for $i \in [t,t')$, for generator g, $\in \{0,1\}$, $[t,t') \in \mathcal{Y}^g$.

1 Unit Commitment Formulation

This section lays out the basic unit commitment formulation we consider for the computational tests in [1].

$$\min \sum_{g \in G} \mathbf{c}^g(p^g) \tag{1a}$$

subject to:

$$\sum_{g \in \mathcal{G}} p_t^g = \mathbf{D}_t \qquad \forall t \in \mathcal{T} \tag{1b}$$

$$\sum_{g \in \mathcal{G}} r_t^g \geq \mathbf{R}_t \qquad \forall t \in \mathcal{T} \tag{1c}$$

$$(p^g, r^g) \in \Pi^g \qquad \forall g \in \mathcal{G}, \tag{1d}$$

where $\mathbf{c}^g(p^g)$ is the cost function of generator g producing an output of p^g over the time horizon $\mathcal{T}$, and Π^g represents the set of feasible schedules for generator g. Here $\mathbf{c}^g$ includes possibly piecewise linear convex production costs, as well as time-dependent startup costs. We will see how we can modify the formulations in [1] to allow for these additional modeling features while still maintaining the property that we can disaggregate solutions to aggregated generators.

For completeness we restate the theorems from [1].

Theorem 1. *Consider identical generators $g_1, g_2 \in \mathcal{G}$, and assume their production costs are increasing and convex. Then there exists an optimal solution with $p_t^{g_1} = p_t^{g_2}$ or (inclusive) one of the nominal or startup/shutdown ramping constraints is binding for generator g_1 or g_2 for all times t for which they are both on.*

Theorem 2. *Suppose generator g_1 is turned off at time t. If identical generator g_2 can also be turned off at time t, there exists an optimal solution where the generator that has been on for the least amount of time is turned off.*

Theorem 3. *Suppose generator g_1 is turned on at time t If identical generator g_2 can also be turned on at time t, (and there are no time-dependent startup costs), then there exists an optimal solution where g_2 is turned on at time t.*

While Theorem 3 is useful for when start-up costs are not time-dependent, this is often not a realistic assumption. Therefore, we have the following two theorems which will be of use when start-up costs are time-dependent.

Theorem 4. *Suppose generator g_1 is turned on at time t, and has been off for at least $\mathbf{TC}(= \mathbf{TC}^{g_1} = \mathbf{TC}^{g_2})$ time periods. If identical generator g_2 can also be turned on at time t and has been off for at least $\mathbf{TC}$ time periods, there exists an optimal solution where where g_2 is turned on at time t.*

Proof. Similar to the proof of Theorem 3. □

Theorem 5. *If identical generators $g_1, g_2 \in \mathcal{G}$ both shut down at time t and g_1 starts up at time $t_1 \geq t + \mathbf{DT}$ ($\mathbf{DT} = \mathbf{DT}^{g_1} = \mathbf{DT}^{g_2}$) and g_2 starts up at time $t_2 \geq t + \mathbf{DT}$, then an equally good solution exists where g_1 starts up at time t_2 and g_2 starts up at time t_1.*

Proof. Like in the proof of Theorem 3, we may permute the remainder of each generator's schedule without affecting feasibility or the objective value. □

2 Disaggregating the Extended Formulation

First, we consider the extended formulation, which is more straightforward than disaggregating the 3-bin formulation. We can add reserves and piecewise linear production costs by adding new variables $r_t^{[a,b),g}$ $\forall [a,b) \in \mathcal{Y}^g$, $\forall t \in \mathcal{T}$ and $p^{[a,b),l,g}$ $\forall [a,b) \in \mathcal{Y}^g$, $\forall l \in \mathcal{L}^g$, $\forall t \in \mathcal{T}$. We add time-dependent startup costs by replacing the packing polytope we used in [1] with the shortest path polytope [2]. The resulting formulation is

$$\mathbf{A}^{[a,b)} p^{[a,b)} + \mathbf{A}'^{[a,b)} r^{[a,b)} + \sum_{l \in \mathcal{L}} \mathbf{A}^{[a,b),l} p^{[a,b),l} \leq \mathbf{b}^{[a,b)} y_{[a,b)} \qquad \forall [a,b) \in \mathcal{Y} \tag{2a}$$

$$\sum_{[a,b) \in \mathcal{Y}} p_t^{[a,b)} = p_t \qquad \forall t \in \mathcal{T} \tag{2b}$$

$$\sum_{[a,b) \in \mathcal{Y}} r_t^{[a,b)} = r_t \qquad \forall t \in \mathcal{T} \tag{2c}$$

$$\sum_{\{[c,d) \in \mathcal{X} \ | \ t=d\}} x_{[c,d)} = \sum_{\{[a,b) \in \mathcal{Y} \ | \ t=a\}} y_{[a,b)} \qquad \forall t \in \mathcal{T} \tag{2d}$$

$$\sum_{\{[a,b) \in \mathcal{Y} \ | \ t=b\}} y_{[a,b)} = \sum_{\{[c,d) \in \mathcal{X} \ | \ t=c\}} x_{[c,d)} \qquad \forall t \in \mathcal{T} \tag{2e}$$

$$\sum_{\{[a,b) \in \mathcal{Y} \ | \ a \leq 0\}} y_{[a,b)} + \sum_{\{[c,d) \in \mathcal{X} \ | \ c \leq 0\}} x_{[c,d)} = 1 \tag{2f}$$

$$\sum_{\{[a,b) \in \mathcal{Y} \ | \ b > \mathbf{T}\}} y_{[a,b)} + \sum_{\{[c,d) \in \mathcal{X} \ | \ d > \mathbf{T}\}} x_{[c,d)} = 1, \tag{2g}$$

where the polytope

$$\{p^{[a,b)}, r^{[a,b)}, p^{[a,b),1}, \ldots, p^{[a,b),\mathbf{L}} \in \mathbb{R}_+ \mid \mathbf{A}^{[a,b)} p^{[a,b)} + \mathbf{A}'^{[a,b)} r^{[a,b)} + \sum_{l \in \mathcal{L}} \mathbf{A}^{[a,b),l} p^{[a,b),l} \leq \mathbf{b}^{[a,b)}\} \tag{3}$$

represents feasible production given that the generator is turned on at time a and turned off at time b. Piecewise production costs can then be handled by placing the appropriate objective coefficient on the $p^{[a,b),l}$ variables and time-dependent startup costs are accounted for by placing the appropriate objective coefficient on the $x_{[c,d)}$ variables.

Similar to the EF presented in [1], we see that the underlying shortest path polytope (2e, 2e, 2f, 2g) with nonnegativity, has a totally unimodular constraint matrix, and thus has the integer decomposition property [3]. Hence if we have k generators have identical parameters, we can replace (2f) and (2g) with

$$\sum_{\{[a,b)\in\mathcal{Y} \ \mid \ a\leq 0\}} Y_{[a,b)} + \sum_{\{[c,d)\in\mathcal{X} \ \mid \ c\leq 0\}} X_{[c,d)} = k \tag{4a}$$

$$\sum_{\{[a,b)\in\mathcal{Y} \ \mid \ b>\mathbf{T}\}} Y_{[a,b)} + \sum_{\{[c,d)\in\mathcal{X} \ \mid \ d>\mathbf{T}\}} X_{[c,d)} = k, \tag{4b}$$

thus pushing k units of flow through the graph. Changing these right-hand-sides doesn't affect the integrality of the full polytope (2a – 2e), (4) [4].

As before, allowing capital variables to represent aggregated variables for identical generators, now $Y_{[a,b)}$ represents *how many* of the generators are on during the interval $[a,b)$ and $X_{[a,b)}$ represents how many generators are off during the interval $[a,b)$. Since there are separate power variables for each on interval $[a,b)$, like before with the EF in [1], Theorem 1 enables us to disaggregate power easily once the status variables are disaggregated.

Letting $\mathcal{K} \subset \mathcal{G}$ be some set of identical generators, consider the extended formulation for these aggregated generators

$$\mathbf{A}^{[a,b)}P^{[a,b)} + \mathbf{A}'^{[a,b)}R^{[a,b)} + \sum_{l\in\mathcal{L}} \mathbf{A}^{[a,b),l}P^{[a,b),l} \leq \mathbf{b}^{[a,b)}Y_{[a,b)} \qquad \forall [a,b)\in\mathcal{Y} \tag{5a}$$

$$\sum_{[a,b)\in\mathcal{Y}} P_t^{[a,b)} = P_t \qquad \forall t\in\mathcal{T} \tag{5b}$$

$$\sum_{[a,b)\in\mathcal{Y}} R_t^{[a,b)} = R_t \qquad \forall t\in\mathcal{T} \tag{5c}$$

$$\sum_{\{[c,d)\in\mathcal{X} \ \mid \ t=d\}} X_{[c,d)} = \sum_{\{[a,b)\in\mathcal{Y} \ \mid \ t=a\}} Y_{[a,b)} \qquad \forall t\in\mathcal{T} \tag{5d}$$

$$\sum_{\{[a,b)\in\mathcal{Y} \ \mid \ t=b\}} Y_{[a,b)} = \sum_{\{[c,d)\in\mathcal{X} \ \mid \ t=c\}} X_{[c,d)} \qquad \forall t\in\mathcal{T} \tag{5e}$$

$$\sum_{\{[a,b)\in\mathcal{Y} \ \mid \ a\leq 0\}} Y_{[a,b)} + \sum_{\{[c,d)\in\mathcal{X} \ \mid \ c\leq 0\}} X_{[c,d)} = |\mathcal{K}| \tag{5f}$$

$$\sum_{\{[a,b)\in\mathcal{Y} \ \mid \ b>\mathbf{T}\}} Y_{[a,b)} + \sum_{\{[c,d)\in\mathcal{X} \ \mid \ d>\mathbf{T}\}} X_{[c,d)} = |\mathcal{K}|. \tag{5g}$$

We can then write down an easy algorithm to decompose solutions to (5).

Algorithm 1 (Peel Off EF) Constructs feasible generator schedules from a solution of (5).

Initialize all $\hat{P}^{[a,b)}$ to $P^{*[a,b)}$ and all $p^{g,[a,b)}$ to 0.
Find a feasible s, t path based on (5d – 5g) and store in $x^g,\ y^g$.
$\hat{X} \leftarrow X^* - x^g$, $\hat{Y} \leftarrow Y^* - y^g$
for $[a, b) \in \mathcal{Y}$ with $y^g_{[a,b)} = 1$ **do**
5:    **for** $t \in [a, b) \cap \mathcal{T}$ **do**
      $p_t^{g,[a,b)} \leftarrow P_t^{*[a,b)}/Y^*_{[a,b)}$; $\hat{P}_t^{[a,b)} \leftarrow P_t^{*[a,b)} - p_t^{g,[a,b)}$
      $r_t^{g,[a,b)} \leftarrow R_t^{*[a,b)}/Y^*_{[a,b)}$; $\hat{R}_t^{[a,b)} \leftarrow R_t^{*[a,b)} - r_t^{g,[a,b)}$
      for $l \in \mathcal{L}$ **do**
         $p_t^{g,[a,b),l} \leftarrow P_t^{*[a,b),l}/Y^*_{[a,b)}$; $\hat{P}_t^{[a,b),l} \leftarrow P_t^{*[a,b),l} - p_t^{g,[a,b),l}$
10: **for** $t \in \mathcal{T}$ **do**
   $\hat{P}_t \leftarrow \sum_{[a,b)\in\mathcal{Y}} \hat{P}_t^{[a,b)}$
   $\hat{R}_t \leftarrow \sum_{[a,b)\in\mathcal{Y}} \hat{R}_t^{[a,b)}$
   $p_t^g \leftarrow \sum_{[a,b)\in\mathcal{Y}} p_t^{g,[a,b)}$
   $r_t^g \leftarrow \sum_{[a,b)\in\mathcal{Y}} r_t^{g,[a,b)}$

After running Algorithm 1 $|\mathcal{K}| - 1$ times we are left with $|\mathcal{K}|$ feasible (and by Theorem 1 optimal) schedules, one for each generator in $\mathcal{K}$. We formalize this in Theorem 6.

First we need a simple lemma regarding the decomposability of polytopes.

Lemma 1. *Let P a polytope such that $P := \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$ and for $k \geq 1$ define $kP := \{x \mid \frac{1}{k}x \in P\} = \{x \in \mathbb{R}^n_+ \mid Ax \leq kb\}$. If $y \in kP$, then $\frac{1}{k}y \in P$ and $\frac{(k-1)}{k}y \in (k-1)P$.*

Proof. It suffices to notice that $\frac{1}{k}y$ is feasible for the system $\{x \in \mathbb{R}^n_+ \mid Ay - (k-1)b \leq Ax \leq b\}$. □

Now we turn to the main result.

Theorem 6. *Algorithm 1 returns a feasible solution for* (2) *and a feasible solution for* (5) *for the remaining $\mathcal{K} \setminus \{g\}$ generators. That is, after applying Algorithm 1 $|\mathcal{K}| - 1$ times we have a feasible and optimal solution for every $g \in \mathcal{K}$.*

Proof. Notice the feasible s, t path leaves the equalities (2d – 2g) and (5d – 5g) feasible for $x^g,\ y^g$ and $\hat{X},\ \hat{Y}$ respectively. Further, the solutions constructed for the $p^{g,[a,b)}$, $r^{g,[a,b)}$, and $p^{g,[a,b),l}$ variables (lines 6 – 9) are exactly of the type prescribed by Lemma 1, and so both these and the $\hat{P}^{[a,b)}$, $\hat{R}^{[a,b)}$, and $\hat{P}^{[a,b),l}$ variables are feasible for (2a) and (5a) respectively. Theorem 1 ensures this assignment is optimal as well. The equalities (2b, 2c) and (5b, 5c) follow from lines 11 – 14.

The last statement follows from inducting on the size of $\mathcal{K}$. □

3 Disaggregating the 3-bin polytope

Recalling the traditional 3-bin formulation for fast ramping generators (when $\mathbf{UT}^g \geq 2$) [5,6]:

$$\underline{\mathbf{P}}^g u_t^g \leq p_t^g, \qquad \forall t \in \mathcal{T}, \tag{6a}$$

$$p_t^g + r_t^g \leq \overline{\mathbf{P}}^g u_t^g + (\mathbf{SU}^g - \overline{\mathbf{P}}^g) v_t^g + (\mathbf{SD}^g - \overline{\mathbf{P}}^g) w_{t+1}^g, \qquad \forall t \in \mathcal{T}, \tag{6b}$$

$$u_t^g - u_{t-1}^g = v_t^g - w_t^g, \qquad \forall t \in \mathcal{T}, \tag{6c}$$

$$\sum_{i=t-\mathbf{UT}+1}^{t} v_i^g \leq u_t^g, \qquad \forall t \in [\mathbf{UT}^g, \mathbf{T}], \tag{6d}$$

$$\sum_{i=t-\mathbf{DT}+1}^{t} w_i^g \leq 1 - u_t^g, \qquad \forall t \in [\mathbf{DT}^g, \mathbf{T}], \tag{6e}$$

$$p_t^g,\ r_t^g \in \mathbb{R}_+, \qquad \forall t \in \mathcal{T}, \tag{6f}$$

$$u_t^g,\ v_t^g,\ w_t^g \in \{0, 1\}, \qquad \forall t \in \mathcal{T}. \tag{6g}$$

It has the property that the constraint matrix defined by (6c, 6d, 6e) is totally unimodular [7], and so it too has the integer decomposition property [3]. When $\mathbf{UT}^g = 1$, (6b) is replaced with the following [5,6]:

$$p_t^g + r_t^g \leq \overline{\mathbf{P}}^g u_t^g + (\mathbf{SU}^g - \overline{\mathbf{P}}^g) v_t^g, \qquad \forall t \in \mathcal{T}, \tag{7a}$$

$$p_t^g + r_t^g \leq \overline{\mathbf{P}}^g u_t^g + (\mathbf{SD}^g - \overline{\mathbf{P}}^g) w_{t+1}^g, \qquad \forall t \in \mathcal{T}. \tag{7b}$$

3.1 Generator Production Cost Function

The convex production costs are typically approximated by piecewise linear costs. This is done by partitioning the interval $[\underline{\mathbf{P}}, \overline{\mathbf{P}}]$ into L subintervals with breakpoints $\overline{\mathbf{P}}^l$, with $\overline{\mathbf{P}}^0 = \underline{\mathbf{P}}$, $\overline{\mathbf{P}}^L = \overline{\mathbf{P}}$, and $\overline{\mathbf{P}}^l < \overline{\mathbf{P}}^{l+1}$. Let $\mathbf{c}^l$ be the marginal cost for the segment $[\overline{\mathbf{P}}^{l-1}, \overline{\mathbf{P}}^l]$. The variable c_t, then, represents the production cost for time t given the following constraints:

$$c_t = \sum_{l=1}^{k} \mathbf{c}^l p_t^l \qquad \forall t \in \mathcal{T} \tag{8a}$$

$$p_t = \underline{\mathbf{P}} u_t + \sum_{l=1}^{k} p_t^l \qquad \forall t \in \mathcal{T} \tag{8b}$$

$$0 \leq p_t^l \leq (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1}) u_t \qquad \forall l \in \mathcal{L},\ \forall t \in \mathcal{T} \tag{8c}$$

The cost functions of the generators need to be modified to account for the aggregation. Fortunately, using the intuition behind Theorem 1, the lack of ramping constraints ensure if two identical generators are on in a given time period, they must have the same production, meaning that allowing u to be a general integer is also sufficient. The only exception to this rule is when there is a startup/shutdown rate. Without loss of generality assume $\mathbf{SU} = \mathbf{SD} = \overline{\mathbf{P}}^{l'}$. We substitute (8c) by

$$p_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})u_t \qquad \forall l \in [l'],\ \forall t \in \mathcal{T} \tag{9a}$$
$$p_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})(u_t - v_t - w_{t+1}) \qquad \forall l > l',\ \forall t \in \mathcal{T} \tag{9b}$$
$$p_t^l \ge 0 \qquad \forall l \in \mathcal{L},\ \forall t \in \mathcal{T} \tag{9c}$$

when $\mathbf{UT} \ge 2$ and

$$p_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})u_t \qquad \forall l \in [l'],\ \forall t \in \mathcal{T} \tag{10a}$$
$$p_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})(u_t - v_t) \qquad \forall l > l',\ \forall t \in \mathcal{T} \tag{10b}$$
$$p_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})(u_t - w_{t+1}) \qquad \forall l > l',\ \forall t \in \mathcal{T} \tag{10c}$$
$$p_t^l \ge 0 \qquad \forall l \in \mathcal{L},\ \forall t \in \mathcal{T} \tag{10d}$$

when $\mathbf{UT} = 1$. When the generator has just turned on (about to turn off), then its power output cannot be above $\mathbf{SU}$ ($\mathbf{SD}$). Hence constraints of the form (9b) and (10b, 10c) cut off these solutions in the p^l variables just as (6b) and (7a, 7b) do for the p variables.

3.2 Generator Startup Costs

There are many different proposed formulations for time-dependent startup costs. One of which, [2], provides a perfect (and totally unimodular) formulation. It is:

$$\sum_{\{t' \mid [t,t') \in \mathcal{Y}\}} y_{[t,t')} = v_t \qquad \forall t \in \mathcal{T} \tag{11a}$$
$$\sum_{\{t' \mid [t',t) \in \mathcal{Y}\}} y_{[t',t)} = w_t \qquad \forall t \in \mathcal{T} \tag{11b}$$
$$\sum_{\{t' \mid [t',t) \in \mathcal{X}\}} x_{[t',t)} = v_t \qquad \forall t \in \mathcal{T} \tag{11c}$$
$$\sum_{\{t' \mid [t,t') \in \mathcal{X}\}} x_{[t,t')} = w_t \qquad \forall t \in \mathcal{T} \tag{11d}$$
$$\sum_{\{[\tau,\tau') \in \mathcal{Y} \mid t \in [\tau,\tau')\}} y_{[\tau,\tau')} = u_t \qquad \forall t \in \mathcal{T}. \tag{11e}$$

Like the extended formulation in the generator model, this formulation can be quite large, containing $O(T^2)$ many variables. However, after adding these constraints to 3-bin model for fast-ramping generators, the resulting formulation still satisfies the integer decomposition property, and we can show this aggregation is valid for the generator schedule.

[8] suggest a more compact formulation of startup costs:

$$\sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} x_{[t',t)} \leq v_t \qquad \forall t \in \mathcal{T}, \tag{12a}$$

$$\sum_{t'=t+\mathbf{DT}}^{t+\mathbf{TC}-1} x_{[t,t')} \leq w_t \qquad \forall t \in \mathcal{T}, \tag{12b}$$

(where the sums are understood to be taken over valid t') and the objective function is

$$c_t^{SU} = \mathbf{c}^S v_t + \sum_{s=1}^{S-1} (\mathbf{c}^s - \mathbf{c}^S) \left(\sum_{t'=t-\overline{\mathbf{T}}^s+1}^{t-\underline{\mathbf{T}}^s} x_{[t',t)} \right) \qquad \forall t \in \mathcal{T}, \tag{12c}$$

and dropping the remaining $x_{[t,t')}$ for which $t' \geq t + \mathbf{TC}$. The resulting computational experiments suggest that this formulation dominates (11) computationally, in addition to the other formulations examined. The reason is that while (12) is not an integer polytope, it is integer "in the right direction," that is, any fractional vertex for 3-bin models using (12) will be dominated by an integer solution for all reasonable objective coefficients (the fractionally enters the formulation by allowing cooler startups to be assigned even when the generator is still hot).

3.3 Disaggregating schedules

In this section we will show how to decompose solutions to the aggregated formulation with various common features, such as time-dependent startup costs, reserves, and piecewise linear production costs. Suppose $\mathcal{K} \subset \mathcal{G}$ such that all generators in $\mathcal{K}$ have identical properties (save initial status). Let $U = \sum_{g \in \mathcal{K}} u^g$, $V = \sum_{g \in \mathcal{K}} v^g$, $W = \sum_{g \in \mathcal{K}} w^g$, and $X = \sum_{g \in \mathcal{K}} x^g$. Let $\mathbf{UT} = \mathbf{UT}^g$, $\mathbf{DT} = \mathbf{DT}^g$, and $\mathbf{TC} = \mathbf{TC}^g$ for some (every) $g \in \mathcal{K}$.

First consider the aggregated 3-bin model for commitment status with startup costs:

$$U_t - U_{t-1} = V_t - W_t \qquad \forall t \in \mathcal{T} \tag{13a}$$

$$\sum_{i=t-\mathbf{UT}+1}^{t} V_i \leq U_t \qquad \forall t \in [\mathbf{UT}, \mathbf{T}] \tag{13b}$$

$$\sum_{i=t-\mathbf{DT}+1}^{t} W_i \leq |\mathcal{K}| - U_t \qquad \forall t \in [\mathbf{DT}, \mathbf{T}] \tag{13c}$$

$$\sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} X_{[t',t)} \leq V_t \qquad \forall t \in \mathcal{T}, \forall g \in \mathcal{G} \tag{13d}$$

$$\sum_{t'=t+\mathbf{DT}}^{t+\mathbf{TC}-1} X_{[t,t')} \leq W_t \qquad \forall t \in \mathcal{T},\ \forall g \in \mathcal{G} \tag{13e}$$

$$U_t, V_t, W_t \in \{0, \dots, |\mathcal{K}|\} \qquad \forall t \in \mathcal{T} \tag{13f}$$

$$X_{[t,t')} \in \{0, \dots, |\mathcal{K}|\} \qquad \forall\ t, t' \in \mathcal{T}^2 \text{ with } \mathbf{DT} \leq t' - t < \mathbf{TC}. \tag{13g}$$

Algorithm 2 demonstrates how to disaggregate a solution to (13).

We establish the correctness of Algorithm 2 with the next theorem.

Theorem 7. *Suppose* (U^*, V^*, W^*, X^*) *is a feasible solution for* (13)*. Then for every* $g \in \mathcal{K}$ *there exist* $(u^{g*}, v^{g*}, w^{g*}, x^{g*})$ *feasible for the minimum up-time/down-time system for* $\mathcal{K}$*:*

$$\begin{aligned}
& u_t^g - u_{t-1}^g = v_t^g - w_t^g && \forall t \in \mathcal{T},\ \forall g \in \mathcal{K} \\
& \sum_{i=t-\mathbf{UT}+1}^{t} v_i^g \leq u_t^g && \forall t \in [\mathbf{UT}, \mathbf{T}],\ \forall g \in \mathcal{K} \\
& \sum_{i=t-\mathbf{DT}+1}^{t} w_i^g \leq 1 - u_t^g && \forall t \in [\mathbf{DT}, \mathbf{T}],\ \forall g \in \mathcal{K} \\
& \sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} x_{[t',t)} \leq v_t^g && \forall t \in \mathcal{T},\ \forall g \in \mathcal{K} \\
& \sum_{t'=t+\mathbf{DT}}^{t+\mathbf{TC}-1} x_{[t,t')} \leq w_t^g && \forall t \in \mathcal{T},\ \forall g \in \mathcal{K} \\
& u_t^g, v_t^g, w_t^g \in \{0, 1\} && \forall t \in \mathcal{T},\ \forall g \in \mathcal{K} \\
& x_{[t,t')} \in \{0, \dots, |\mathcal{K}|\} && \forall\ t, t' \in \mathcal{T}^2 \textit{ with } \mathbf{DT} \leq t' - t < \mathbf{TC}.
\end{aligned}$$

Proof. Clearly this is true when $|\mathcal{K}| = 1$. We will proceed by induction on the size of $\mathcal{K}$, "peeling off" feasible binary vectors and leaving behind a still feasible crushed system. Suppose (U^*, V^*, W^*, X^*) is feasible (13), and $|\mathcal{K}| = I$. We wish to find a feasible solution to the following system:

$$U_t^* = \hat{U}_t + u_t^g,\ V_t^* = \hat{V}_t + v_t^g,\ W_t^* = \hat{W}_t + w_t^g \qquad \forall t \in \mathcal{T} \tag{14a}$$

$$X_{[t,t')}^* = \hat{X}_{[t,t')} + x_{[t,t')}, \qquad \forall\ t, t' \in \mathcal{T}^2 \text{ with } \mathbf{DT} \leq t' - t < \mathbf{TC} \tag{14b}$$

$$u_t^g - u_{t-1}^g = v_t^g - w_t^g \qquad \forall t \in \mathcal{T} \tag{15a}$$

$$\sum_{i=t-\mathbf{UT}+1}^{t} v_i^g \leq u_t^g \qquad \forall t \in [\mathbf{UT}, \mathbf{T}] \tag{15b}$$

Algorithm 2 (PeelOff) Constructs feasible generator schedules from a solution of (13).

Initialize all u^g, v^g, w^g, x^g to 0.
Initialize all $\hat{U}$, $\hat{V}$, $\hat{W}$, $\hat{X}$ to U^*, V^*, W^*, X^*, respectively.
if $U_1^* \geq 1$ **then**
  $u_1^g \leftarrow 1$; $\hat{U}_1 \leftarrow U_1^* - 1$;
  Assign historical startup v_t, for $-\mathbf{UT} < t \leq 0$.
else
  Assign historical shutdown w_t, for $-\mathbf{DT} < t \leq 0$.
$t \leftarrow 2$
while $t \leq \mathbf{T}$ **do**
  if $u_{t-1}^g = 1$ **then** ▷ If on in the previous period
    if $\sum_{i=t-\mathbf{UT}+1}^{t-1} v_i = 0$ **and** $W_t^* \geq 1$ **then** ▷ If we can turn off and a turn off is available
      $w_t^g \leftarrow 1$; $\hat{W}_t \leftarrow W_t^* - 1$; ▷ Turn off
      if $\exists t'$ s.t. $t + TD \leq t' < t + \mathbf{TC}$ **and** $X_{[t,t')}^* \geq 1$ **then** ▷ If there is a hot-start available
        $u_{t'}^g, v_{t'}^g \leftarrow 1$; $x_{[t,t')}^g \leftarrow 1$; ▷ Take it
        $\hat{U}_{t'} \leftarrow U_{t'}^* - 1$; $\hat{V}_{t'} \leftarrow V_{t'}^* - 1$;
        $\hat{X}_{[t,t')} \leftarrow X_{[t,t')}^* - 1$;
        $t \leftarrow t' + 1$;
      else if $\exists t' \geq t + \mathbf{TC}$ s.t. $\sum_{t'=t+\mathbf{DT}}^{t+\mathbf{TC}-1} X_{[t,t')}^* < V_{t'}^*$ **then** ▷ Else take some cold start, if possible
        $u_{t'}^g, v_{t'}^g \leftarrow 1$; $\hat{U}_{t'} \leftarrow U_{t'}^* - 1$; $\hat{V}_{t'} \leftarrow V_{t'}^* - 1$;
        $t \leftarrow t' + 1$;
      else ▷ If not, stay off for the rest of the time horizon
        $t \leftarrow \mathbf{T} + 1$;
    else ▷ If we could not turn off or a turn off was not available
      $u_t \leftarrow 1$; $\hat{U}_t \leftarrow U_t^* - 1$; ▷ Stay on
      $t \leftarrow t + 1$;
  else ▷ $u_{t-1} = 0$, i.e., off previously
    if $\sum_{i=t-\mathbf{DT}+1}^{t-1} w_i = 0$ **and** $V_t^* \geq 1$ **then** ▷ If we can turn on and a turn on is available
      $u_t, v_t \leftarrow 1$; $\hat{U}_t \leftarrow U_t^* - 1$; $\hat{V}_t \leftarrow V_t^* - 1$; ▷ Turn on
      if $\exists t' \in (t - \mathbf{TC}, t - \mathbf{DT}] \cap \mathbb{Z}$ s.t. $X_{[t',t)}^* \geq 1$ **then** ▷ If there is a historical hot-start
        $x_{[t',t)}^g \leftarrow 1$; $\hat{X}_{[t',t)} \leftarrow X_{[t',t)}^* - 1$; ▷ Take it
        $w_{t'}^g \leftarrow 1$; $\hat{W}_{t'} \leftarrow W_{t'}^* - 1$; ▷ Assign historical data for $t \leq -\mathbf{DT}$
      $t \leftarrow t + 1$;
    else ▷ If no turn on feasible or available
      $t \leftarrow t + 1$; ▷ Stay off

$$\sum_{i=t-\mathbf{DT}+1}^{t} w_i^g \leq 1 - u_t^g \qquad \forall t \in [\mathbf{DT}, \mathbf{T}] \qquad (15c)$$

$$\sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} x_{[t',t)} \leq v_t^g \qquad \forall t \in \mathcal{T} \qquad (15d)$$

$$\sum_{t'=t+\mathbf{DT}}^{t+\mathbf{TC}-1} x_{[t,t')} \leq w_t^g \qquad \forall t \in \mathcal{T} \qquad (15e)$$

$$u_t^g, v_t^g, w_t^g \in \{0,1\} \ \forall t \in \mathcal{T}, \qquad x_{[t,t')}^g \in \{0,1\}, \ \forall \ t, t' \in \mathcal{T}^2 \text{ with } \mathbf{DT} \leq t' - t < \mathbf{TC} \qquad (15f)$$

$$\hat{U}_t - \hat{U}_{t-1} = \hat{V}_t - \hat{W}_t \qquad \forall t \in \mathcal{T} \qquad (16a)$$

$$\sum_{i=t-\mathbf{UT}+1}^{t} \hat{V}_i \leq \hat{U}_t \qquad \forall t \in [\mathbf{UT}, \mathbf{T}] \qquad (16b)$$

$$\sum_{i=t-\mathbf{DT}+1}^{t} \hat{W}_i \leq (I-1) - \hat{U}_t \qquad \forall t \in [\mathbf{DT}, \mathbf{T}] \qquad (16c)$$

$$\sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} \hat{X}_{[t',t)} \leq \hat{V}_t \qquad \forall t \in \mathcal{T} \qquad (16d)$$

$$\sum_{t'=t+\mathbf{DT}}^{t+\mathbf{TC}-1} \hat{X}_{[t,t')} \leq \hat{W}_t \qquad \forall t \in \mathcal{T} \qquad (16e)$$

$$\hat{U}_t, \hat{V}_t, \hat{W}_t \in \{0,\ldots,I-1\} \ \forall t \in \mathcal{T}, \qquad \hat{X}_{[t,t')} \in \{0,\ldots,I-1\} \forall \ t, t' \in \mathcal{T}^2 \text{ with } \mathbf{DT} \leq t' - t < \mathbf{TC}. \qquad (16f)$$

Algorithm 2 constructs a feasible solution to (14, 15, 16) from a solution of (13). To see this, first notice that the solution returned by Algorithm 2 always has (14). Similarly, Algorithm 2 constructs a feasible solution for (u^g, v^g, w^g, x^g), so (15) holds. Further, (13a) and (15a) together imply (16a). Notice that given the bounds on $\hat{U}$ and (16b - 16e), we get the bounds on $\hat{V}$, $\hat{W}$, and $\hat{X}$, and the proof is finished. Therefore we check the bounds on $\hat{U}$ and (16b–16e), proceeding by contraction each time.

$\hat{U}_t \leq I-1$: Let t be the first time period such that $\hat{U}_t > I-1$ (notice $t > 1$ by line 8 in Algorithm 2). Then $\hat{U}_t = I$ and further, $u_t = 0$, $U_t^* = I$. Now by (13c), $\sum_{i=t-\mathbf{DT}+1}^{t} W_i^* \leq I - I = 0$. Therefore $W_i^* = 0$, $\forall i \in [t-\mathbf{DT}+1, t]$, yielding $w_i^g = 0$, $\forall i \in [t-\mathbf{DT}+1, t]$. Since $U_t^* = I$ and $W_t^* = 0$, (13a) gives $I = U_{t-1}^* + V_t^*$. If $V_t^* > 0$, since g is eligible for a turn-on, this contracts line 27 in Algorithm 2. If $V_t^* = 0$, then $U_{t-1}^* > I-1$, contracting the minimality of t.

(16b): Suppose there is t with $\sum_{i=t-\mathbf{UT}+1} \hat{V}_i > \hat{U}_i$. We have the following relation:

$$U_t^* \overset{v_i^g \geq 0}{\geq} U_t^* - \sum_{i=t-\mathbf{UT}+1}^{t} v_i^g \overset{(13b)}{\geq} \sum_{i=t-\mathbf{UT}+1}^{t} (V_i^* - v_i^g) \overset{(14a)}{=} \sum_{i=t-\mathbf{UT}+1}^{t} \hat{V}_t > \hat{U}_t \overset{(14a)}{=} U_t^* - u_t^g \overset{u_t^g \leq 1}{\geq} U_t^* - 1 \tag{17}$$

Since the far left and far right differ by only 1, and all quantities are integer, in order for the strict inequality to hold, all weak inequalities in (17) must be equalities. Therefore we have (a) $u_t^g = 1$, (b) $\sum_{i=t-\mathbf{UT}+1}^{t} v_i = 0$, and (c) $\sum_{i=t-\mathbf{UT}+1}^{t} V_i^* = U_t^*$. Together (a) and (b) imply $u_i^g = 1$ and $U_i^* \geq 1$ for every $i \in \{t - \mathbf{UT}, \ldots, t\}$. Therefore generator g started-up most recently at some time $\hat{i}$ such that $\hat{i} \leq t - \mathbf{UT}$; i.e $v_{\hat{i}} = 1$. Line 11 of Algorithm 2 implies then that $W_i^* = 0$ for every $i \in \{\hat{i} + \mathbf{UT}, \ldots, t\}$, and hence (d) $U_i^* - U_{i-1}^* = V_i^*$ for every $i \in \{\hat{i} + \mathbf{UT}, \ldots, t\}$ by equation (13a). There are but two cases then.

Case 1. Suppose $\hat{i} + \mathbf{UT} \leq t - \mathbf{UT} + 1$. Then $\sum_{i=t-\mathbf{UT}+1}^{t} V_i^* = U_t^* - U_{t-\mathbf{UT}}^*$ by (d). By (c) then $U_{t-\mathbf{UT}}^* = 0$ but (a) and (b) give $U_{t-\mathbf{UT}}^* \geq 1$.

Case 2. Suppose $t - \mathbf{UT} + 1 < \hat{i} + \mathbf{UT}$. We have:

$$U_t^* \overset{(c)}{=} \sum_{i=t-\mathbf{UT}+1}^{t} V_i^* = \sum_{i=t-\mathbf{UT}+1}^{\hat{i}+\mathbf{UT}-1} V_i^* + \sum_{i=\hat{i}+\mathbf{UT}}^{t} V_i^* \overset{(d)}{=} \sum_{i=t-\mathbf{UT}+1}^{\hat{i}+\mathbf{UT}-1} V_i^* + U_t^* - U_{\hat{i}+\mathbf{UT}-1}^* \tag{18}$$

Subtracting U_t^* from both sides we get the relation $0 = \sum_{i=t-\mathbf{UT}+1}^{\hat{i}+\mathbf{UT}-1} V_i^* - U_{\hat{i}+\mathbf{UT}-1}^*$. Finally then, we see

$$0 = \sum_{i=t-\mathbf{UT}+1}^{\hat{i}+\mathbf{UT}-1} V_i^* - U_{\hat{i}+\mathbf{UT}-1}^* \overset{(13b)}{\leq} \sum_{i=t-\mathbf{UT}+1}^{\hat{i}+\mathbf{UT}-1} V_i^* - \sum_{i=\hat{i}}^{\hat{i}+\mathbf{UT}-1} V_i^* = - \sum_{i=\hat{i}}^{t-\mathbf{UT}} V_i^* \overset{v_{\hat{i}}=1}{\leq} -1, \tag{19}$$

yielding the contraction desired.

(16c): Very similar to the proof for (16b).

(16d): Supposing there is t with $\sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} \hat{X}_{[t',t)} > \hat{V}_t$ we can use the same technique from the proof of (16b) to get the following string of inequalities:

$$\begin{aligned} V_t^* \geq V_t^* - \sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} x_{[t',t)}^g &\overset{(13d)}{\geq} \sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} X_{[t',t)}^* - \sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} x_{[t',t)}^g \\ &\overset{(14b)}{=} \sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} \hat{X}_{[t',t)} > \hat{V}_t \overset{(14a)}{=} V_t^* - v_t^g \geq V_t^* - 1 \end{aligned} \tag{20}$$

Hence we may conclude (a) $v_t^g = 1$, (b) $\sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} x_{[t',t)}^g = 0$, and (c) $\sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} X_{[t',t)}^* = V_t^*$. We have (a) implies that $V_t^* \geq 1$, so $\sum_{t'=t-\mathbf{TC}+1}^{t-\mathbf{DT}} X^*_{[t',t)} \geq 1$, so $\exists t' \in \{t - \mathbf{DT}+1, \ldots, t-\mathbf{TC}\}$ such that $X^*_{[t',t)} \geq 1$. But line 14 in Algorithm 2 then sets $x^g_{[t',t)} = 1$, a contraction.

(16e): This is the same as (16d).

Hence Algorithm 2 constructs a feasible solution for (14, 15, 16). We can then proceed in this manner until $I = 1$, proving the theorem. □

Notice that Algorithm 2 constructs solutions exactly of the type in Theorems 2 and 4. Theorem 5 justifies the arbitrary choice in lines 13 – 20 for which shutdown/startup path the generator takes. Hence we do not loose anything in optimality or feasibility by aggregating a fast-ramping generator's status variables, even in the presence of time-dependent startup costs. Next we will see how to disaggregate the power and reserve variables.

3.4 Disaggregating Power and Reserves

3.4.1 Disaggregating Power when $\mathbf{UT} \geq 2$

Consider the power output for an aggregated set of identical fast-ramping generators. First we will consider the case when $\mathbf{UT} \geq 2$. Along with (13), we the aggregated power $P = \sum_{g\in\mathcal{K}} p^g$ and reserves $R = \sum_{g\in\mathcal{K}} r^g$ with the constraints:

$$\underline{\mathbf{P}} U_t \leq P_t \qquad \forall t \in \mathcal{T}, \tag{21a}$$

$$P_t + R_t \leq \overline{\mathbf{P}} U_t + (\mathbf{SU} - \overline{\mathbf{P}}) V_t + (\mathbf{SD} - \overline{\mathbf{P}}) W_{t+1}, \qquad \forall t \in \mathcal{T}. \tag{21b}$$

Since (21) is just a sum of constraints, it is clearly valid.

If $\mathbf{SU} \geq \mathbf{SD}$, then Algorithm 3a demonstrates how to disaggregate power. On the other hand, when $\mathbf{SD} \geq \mathbf{SU}$ disaggregation can be done in an analogous fashion, as shown in Algorithm 3b. The essential logic of Algorithms 3a and 3b is that of Theorem 1. That is, either the power outputs of all generators are equal, or either the startup- and/or shutdown-ramping constraints are active. When $\mathbf{SU} = \mathbf{SD}$ then Algorithms 3a and 3b give the same result.

3.4.2 Disaggregating Power when $\mathbf{UT} = 1$

When $\mathbf{UT} = 1$, we need consider a modified version of the aggregated generator's production constraint, as is the case for a single generator [5, 6]. Again using the aggregated variables from before, consider the

Algorithm 3a (PEEL OFF POWER) Constructs feasible generator schedule from a solution of (21) when $\mathbf{SU} \geq \mathbf{SD}$.

for $g \in \mathcal{K},\ t \in \mathcal{T}$ **do**
  if $P_t^*/U_t^* \leq \mathbf{SD}$ **then** ▷ If the average power is less than **SD**
    if $u_t^g = 1$ **then**
      $p_t^g \leftarrow P_t^*/U_t^*$ ▷ Give all generators on average power
    else
      $p_t^g \leftarrow 0$
  else if $(P_t^* - \mathbf{SD} \cdot W_{t+1}^*)/U_t^* \leq \mathbf{SU}$ **then** ▷ If not, check if remaining average power $\leq$ **SU**
    if $w_{t+1}^g = 1$ **then**
      $p_t^g \leftarrow \mathbf{SD}$ ▷ Give generators shutting down **SD**
    else if $u_t^g = 1$ **then**
      $p_t^g \leftarrow (P_t^* - \mathbf{SD} \cdot W_{t+1}^*)/U_t^*$ ▷ Give all others on remaining average power
    else
      $p_t^g \leftarrow 0$
  else ▷ $(P_t^* - \mathbf{SD} \cdot W_{t+1}^*)/U_t^* > \mathbf{SU}$, so we need separate out generators starting
    if $w_{t+1}^g = 1$ **then**
      $p_t^g \leftarrow \mathbf{SD}$ ▷ Give generators shutting down **SD**
    else if $v_t^g = 1$ **then**
      $p_t^g \leftarrow \mathbf{SU}$ ▷ Give (remaining) generators starting up **SU**
    else if $u_t^g = 1$ **then**
      $p_t^g \leftarrow (P_t^* - \mathbf{SU} \cdot V_t^* - \mathbf{SD} \cdot W_{t+1}^*)/U_t^*$ ▷ all others on get remaining average power
    else
      $p_t^g \leftarrow 0$

Algorithm 3b (Peel Off Power) Constructs feasible generator schedule from a solution of (21) when $\mathbf{SD} \geq \mathbf{SU}$.

for $g \in \mathcal{K},\ t \in \mathcal{T}$ **do**
  if $P_t^*/U_t^* \leq \mathbf{SU}$ **then** ▷ If the average power is less than **SU**
    if $u_t^g = 1$ **then**
      $p_t^g \leftarrow P_t^*/U_t^*$ ▷ Give all generators on average power
    else
      $p_t^g \leftarrow 0$
  else if $(P_t^* - \mathbf{SU} \cdot V_t^*)/U_t^* \leq \mathbf{SD}$ **then** ▷ If not, check if remaining average power $\leq$ **SD**
    if $v_t^g = 1$ **then**
      $p_t^g \leftarrow \mathbf{SU}$ ▷ Give generators starting up **SU**
    else if $u_t^g = 1$ **then**
      $p_t^g \leftarrow (P_t^* - \mathbf{SU} \cdot V_t^*)/U_t^*$ ▷ Give all others on remaining average power
    else
      $p_t^g \leftarrow 0$
  else ▷ $(P_t^* - \mathbf{SU} \cdot V_t^*)/U_t^* > \mathbf{SD}$, so we need separate out generators stopping
    if $v_t^g = 1$ **then**
      $p_t^g \leftarrow \mathbf{SU}$ ▷ Give generators starting up **SU**
    else if $w_{t+1}^g = 1$ **then**
      $p_t^g \leftarrow \mathbf{SD}$ ▷ Give (remaining) generators shutting down **SD**
    else if $u_t^g = 1$ **then**
      $p_t^g \leftarrow (P_t^* - \mathbf{SU} \cdot V_t^* - \mathbf{SD} \cdot W_{t+1}^*)/U_t^*$ ▷ all others on get remaining average power
    else
      $p_t^g \leftarrow 0$

aggregated production constraints

$$\underline{\mathbf{P}}U_t \le P_t \qquad \forall t \in \mathcal{T} \tag{22a}$$

$$P_t + R_t \le \overline{\mathbf{P}}U_t + (\mathbf{SU} - \overline{\mathbf{P}})V_t, \qquad \forall t \in \mathcal{T} \tag{22b}$$

$$P_t + R_t \le \overline{\mathbf{P}}U_t + (\mathbf{SD} - \overline{\mathbf{P}})W_{t+1}, \qquad \forall t \in \mathcal{T}. \tag{22c}$$

When $\mathbf{SU} \ge \mathbf{SD}$, we can again use Algorithm 3a, except we need modify line 20 to

$$p_t^g \leftarrow (P_t^* - \mathbf{SU} \cdot \min\{V_t^* - W_{t+1}^*,\ 0\} - \mathbf{SD} \cdot W_{t+1}^*)/U_t^*. \tag{23}$$

The correctness of Algorithm 3a the modification above to line 20 follows from Theorem 1 and Algorithm 2. That is, $\min\{V_t^*,\ W_{t+1}^*\}$ generators turn on at time t and turn off at time $t+1$, and hence $\min\{V_t^* - W_{t+1}^*,\ 0\}$ will get $\mathbf{SU}$ power at line 18.

Similarly when $\mathbf{SD} \ge \mathbf{SU}$, we can use Algorithm 3b, modifying line 20 to

$$p_t^g \leftarrow (P_t^* - \mathbf{SU} \cdot V_t^* - \mathbf{SD} \cdot \min\{W_{t+1}^* - V_t^*,\ 0\})/U_t^*. \tag{24}$$

The correctness of Algorithm 3b with the modification to line 20 is exactly analogous to that in the $\mathbf{SU} \ge \mathbf{SD}$ case above.

3.4.3 Disaggregating Reserves

Having disaggregated power, reserves r^g can be disaggregated by considering $(p_t^g + r_t^g)$ and $(P_t^* + R_t^*)$ in Algorithm 3a or 3b (or their modified analogs when $\mathbf{UT} = 1$) above in place of p_t^g and P_t^* respectively.

3.5 Disaggregating Piecewise Linear Production Costs

3.5.1 $\mathbf{UT} \ge 2$

In the case of piecewise production costs we can modify (9) by considering the aggregated piecewise production variables $P^l = \sum_{g \in \mathcal{K}} p^{l,g}$. We consider the sum of constraints of the form (9), recalling l' is such that $\mathbf{SU} = \mathbf{SD} = \overline{\mathbf{P}}^{l'}$

$$P_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})U_t \qquad \forall l \in [l'],\ \forall t \in \mathcal{T} \tag{25a}$$

$$P_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})(U_t - V_t - W_{t+1}) \qquad \forall l > l',\ \forall t \in \mathcal{T} \tag{25b}$$

$$P_t^l \ge 0 \qquad \forall l \in \mathcal{L},\ \forall t \in \mathcal{T} \tag{25c}$$

If the generator is on and did not just turn on nor is about to turn off ($u_t^g = 1,\ v_t^g = 0,\ w_{t+1}^g = 0$), then $p_t^{l,g} = P_t^{*l}/U_t^*$. If the generator just turned on ($u_t^g = 1,\ v_t^g = 1,\ w_{t+1}^g = 0$), then

$$p_t^{l,g} = P_t^{*l}/U_t^* \qquad \forall l \in [l'],\ \forall t \in \mathcal{T} \tag{26a}$$

$$p_t^{l,g} = 0 \qquad \forall l > l',\ \forall t \in \mathcal{T}, \tag{26b}$$

and similarly if the generator is just about to turn off ($u_t^g = 1,\ v_t^g = 0,\ w_{t+1}^g = 1$) Notice that with the aggregated linking constraints for piecewise production

$$P_t = \underline{\mathbf{P}}U_t + \sum_{l\in\mathcal{L}} P_t^l \qquad \forall t \in \mathcal{T},$$

the assignment given by (26) is compatible with Algorithms 3a and 3b. $p_t^{l,g}$ is 0 for all l, t when the generator is off ($u_t^g = 0$).

Consider the case when $\mathbf{SU} > \mathbf{SD}$. Letting $\overline{\mathbf{P}}^{l^{\mathbf{SU}}} = \mathbf{SU}$ and $\overline{\mathbf{P}}^{l^{\mathbf{SD}}} = \mathbf{SD}$, ($l^{\mathbf{SU}} > l^{\mathbf{SD}}$) we may use a modified version of (25)

$$P_t^l \leq (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})U_t \qquad \forall l \in [l^{\mathbf{SD}}],\ \forall t \in \mathcal{T} \tag{27a}$$

$$P_t^l \leq (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})(U_t - W_{t+1}) \qquad \forall l \in (l^{\mathbf{SD}}, l^{\mathbf{SU}}],\ \forall t \in \mathcal{T} \tag{27b}$$

$$P_t^l \leq (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})(U_t - V_t - W_{t+1}) \qquad \forall l > l^{\mathbf{SU}},\ \forall t \in \mathcal{T} \tag{27c}$$

$$P_t^l \geq 0 \qquad \forall l \in \mathcal{L},\ \forall t \in \mathcal{T}. \tag{27d}$$

Like before, if the generator did not just turn on nor is about to turn off ($u_t^g = 1,\ v_t^g = 0,\ w_{t+1}^g = 0$), then $p^{l,g} = P_t^{*l}/U_t^*$. If the generator just turned on ($u_t^g = 1,\ v_t^g = 1,\ w_{t+1}^g = 0$), then

$$p_t^{l,g} = P_t^{*l}/U_t^* \qquad \forall l \in [l^{\mathbf{SU}}],\ \forall t \in \mathcal{T} \tag{28a}$$

$$p_t^{l,g} = 0 \qquad \forall l > l^{\mathbf{SU}},\ \forall t \in \mathcal{T}. \tag{28b}$$

If the generator is just about to turn off ($u_t^g = 1,\ v_t^g = 0,\ w_{t+1}^g = 1$), then

$$p_t^{l,g} = P_t^{*l}/U_t^* \qquad \forall l \in [l^{\mathbf{SD}}],\ \forall t \in \mathcal{T} \tag{29a}$$

$$p_t^{l,g} = 0 \qquad \forall l > l^{\mathbf{SD}},\ \forall t \in \mathcal{T}. \tag{29b}$$

Finally, $p_t^{l,g} = 0$ for all l, t when the generator g is off ($u_t^g = 0$).

The case when $\mathbf{SD} > \mathbf{SU}$ can be handled similarly.

3.5.2 $\mathbf{UT} = 1$

When $\mathbf{UT} = 1$, the aggregated piecewise power constraints need to be modified as well. Under the assumption $\overline{\mathbf{P}}^{l'} = \mathbf{SU} = \mathbf{SD}$, consider the aggregated version of (10)

$$P_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})U_t \qquad \forall l \in [l'],\ \forall t \in \mathcal{T} \tag{30a}$$
$$P_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})(U_t - V_t) \qquad \forall l > l',\ \forall t \in \mathcal{T} \tag{30b}$$
$$P_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})(U_t - W_{t+1}) \qquad \forall l > l',\ \forall t \in \mathcal{T} \tag{30c}$$
$$P_t^l \ge 0 \qquad \forall l \in \mathcal{L},\ \forall t \in \mathcal{T}. \tag{30d}$$

Without loss of generality, we can assign the piecewise power making the same modifications that were necessary for the power production. In particular, we have that $p_t^{l,g} = P_t^{*l}/U_t^*$ if $u_t^g = 1,\ v_t^g = w_{t+1}^g = 0$. If is a startup or shutdown (or both), ($u_t^g = 1,\ v_t^g$ and/or $w_{t+1}^g = 1$), then we can use (26).

Assuming $\mathbf{SU} \ne \mathbf{SD}$, we can introduce $l^{\mathbf{SU}}$ and $l^{\mathbf{SD}}$ as before for (27)

$$P_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})U_t \qquad \forall l \in [\max\{l^{\mathbf{SU}}, l^{\mathbf{SD}}\}],\ \forall t \in \mathcal{T} \tag{31a}$$
$$P_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})(U_t - V_t) \qquad \forall l > l^{\mathbf{SU}},\ \forall t \in \mathcal{T} \tag{31b}$$
$$P_t^l \le (\overline{\mathbf{P}}^l - \overline{\mathbf{P}}^{l-1})(U_t - W_{t+1}) \qquad \forall l > l^{\mathbf{SD}},\ \forall t \in \mathcal{T} \tag{31c}$$
$$P_t^l \ge 0 \qquad \forall l \in \mathcal{L},\ \forall t \in \mathcal{T}. \tag{31d}$$

If we have $u_t^g = 1,\ v_t^g = 0,\ w_{t+1}^g = 0$, then $p_t^{l,g} = P_t^{*l}/U_t^{*l}$. Then there are three other cases to consider. Suppose $\mathbf{SU} > \mathbf{SD}$. If the generator is just starting and does not shutdown ($u_t^g = 1, v_t^g = 1, w_{t+1}^g = 0$), then (28) applies, and if the generator is shutting down ($u_t^g = 1,\ v_t^g = 0$ or $1,\ w_{t+1}^g = 1$), then (29) applies.

The case when $\mathbf{SD} > \mathbf{SU}$ is handled analogously. That is, if the generator is shutting down and did just startup ($u_t^g = 1,\ v_t^g = 0,\ w_{t+1}^g = 1$), then (29) is used, and if the generator is starting up ($u_t^g = 1,\ v_t^g = 1, w_t^g = 0$ or 1) then (28) applies.

4 Additional Computational Tests

In this section we present some additional computational results to complement those in [1].

4.1 Symmetry Breaking Inequalities

In addition to the formulations from [1], we consider the addition of the "S3" variables and inequalities from [9] to the base 3-bin UC formulation. Lima and Novais [9] propose introducing new variables yon^g

which indicate if generator g ever turned on during the time horizon, along with inequalities to enforce this

$$\sum_{t \in \mathcal{T}} u_t^g \geq yon^g \qquad \forall g \in \mathcal{G} \tag{32}$$

$$u_t^g \leq yon^g \qquad \forall t \in \mathcal{T},\ \forall g \in \mathcal{G}. \tag{33}$$

They then propose the following two symmetry-breaking inequalities for each set of identical generators $\mathcal{K}$

$$yon^g \geq yon^{g+1} \qquad \forall g, g{+}1 \in \mathcal{K} \tag{34}$$

$$\sum_{t \in \mathcal{T}} u_t^g \geq \sum_{t \in \mathcal{T}} u_t^{g+1} \qquad \forall g, g{+}1 \in \mathcal{K}, \tag{35}$$

where $g, g{+}1 \in \mathcal{K}$ is understood to be two consecutive generators in $\mathcal{K}$. (34) enforces that if generator $g{+}1$ ever turns on then generator g does as well, and (35) enforces that generator g is scheduled in at least as many time periods as generator $g{+}1$. As pointed out in [9], this eliminates many, but not every, source of symmetry in UC.

4.2 Computational Results

Here we present computational results for the instances tested in [1] with the addition of the "S3" variables and inequalities from [9], which we label as "3-bin+SBC". The computational platform is as described in [1].

4.2.1 CAISO Instances

In Table 1 we report the computational results for the CAISO instances described in [1]. As we can see, in almost all cases the symmetry-breaking constraints are unhelpful. This is to be expected since these instances have a relatively tight root optimality gap, and the extra constraints serve to slow effective cut generation and heuristic search at the root node. Overall they serve to slow the solver down over 3-bin, though in one instance the 3-bin+SBC variant finds a high-quality solution fastest and with fewest nodes.

4.2.2 Ostrowski Instances

As in [1], we set a time limit of 900 seconds for the Ostrowski instances and report the terminating MIP gap in parentheses when the solver terminates at the time limit. In Table 2 we report the computational results for the Ostrowski instances from [1]. As reported in [9], the symmetry-breaking constraints are helpful overall for the smaller Ostrowski instances (1–10), but they perform worse than 3-bin on the larger instances (11–20). In comparison, EF/3-bin+A has a relatively flat performance profile across all 20 instances, suggesting that, when handled properly, identical generators can be leveraged to significantly reduce the computational burden of UC.

Table 1: Computational Results for CAISO UC Instances

	Time (s)			Nodes		
Instance	**3-bin**	**3-bin+SBC**	**3-bin+A**	**3-bin**	**3-bin+SBC**	**3-bin+A**
2014-09-01 0%	31.35	34.07	**14.25**	0	0	0
2014-12-01 0%	25.77	29.36	**12.38**	0	0	0
2015-03-01 0%	24.08	34.47	**14.27**	0	0	0
2015-06-01 0%	13.11	14.25	**8.50**	0	0	0
Scenario400 0%	27.29	35.99	**23.63**	0	0	0
2014-09-01 1%	20.52	29.38	**16.44**	0	0	0
2014-12-01 1%	38.48	86.41	**24.69**	95	566	**0**
2015-03-01 1%	21.75	35.76	**19.11**	0	0	0
2015-06-01 1%	39.87	30.42	**15.59**	47	**0**	**0**
Scenario400 1%	47.54	57.85	**44.63**	**0**	154	1438
2014-09-01 3%	81.47	92.60	**38.27**	**7**	696	122
2014-12-01 3%	65.01	120.03	**36.53**	1292	**95**	125
2015-03-01 3%	50.79	77.96	**25.04**	**0**	3	**0**
2015-06-01 3%	87.25	147.05	**41.23**	**0**	79	115
Scenario400 3%	131.28	147.61	**69.45**	2055	**140**	880
2014-09-01 5%	47.07	69.51	**30.95**	95	176	**7**
2014-12-01 5%	83.87	132.97	**66.90**	1203	**79**	3978
2015-03-01 5%	80.57	72.70	**21.65**	923	**0**	**0**
2015-06-01 5%	**26.99**	96.56	43.79	**0**	162	402
Scenario400 5%	115.53	**95.50**	118.51	3867	**31**	4225
Geometric Mean:	43.85	59.69	**27.55**			

References

[1] B. Knueven, J. Ostrowski, and J.-P. Watson, "Exploiting identical generators in unit commitment," *Power Systems, IEEE Transactions on*, 2017, to appear.

[2] Y. Pochet and L. A. Wolsey, *Production planning by mixed integer programming.* Springer Science & Business Media, 2006.

[3] S. Baum and L. E. Trotter Jr, "Integer rounding and polyhedral decomposition for totally unimodular systems," in *Optimization and Operations Research.* Springer, 1978, pp. 15–23.

[4] B. Knueven, J. Ostrowski, and J. Wang, "The ramping polytope and cut generation for the unit commitment problem," *INFORMS Journal on Computing*, 2017, to appear.

[5] G. Morales-España, J. M. Latorre, and A. Ramos, "Tight and compact MILP formulation for the thermal unit commitment problem," *Power Systems, IEEE Transactions on*, vol. 28, no. 4, pp. 4897–4908, 2013.

[6] C. Gentile, G. Morales-Espana, and A. Ramos, "A tight MIP formulation of the unit commitment problem with start-up and shut-down constraints," *EURO Journal on Computational Optimization*, pp. 1–25.

Table 2: Computational Results for Ostrowski UC Instances

	Time (s)			Nodes		
Instance	**3-bin**	**3-bin+SBC**	**EF/3-bin+A**	**3-bin**	**3-bin+SBC**	**EF/3-bin+A**
1	**8.44**	54.82	14.02	1509	13700	**68**
2	154.75	44.73	**21.07**	48129	10370	**157**
3	703.94	127.23	**100.33**	316704	25802	**4464**
4	**14.84**	109.15	17.28	8532	26297	**60**
5	143.18	101.73	**57.22**	131320	17784	**4350**
6	95.41	45.03	**28.00**	62394	7160	**72**
7	(0.0238%)	270.34	**119.22**	535361*	75097	**4854**
8	(0.0107%)	**57.36**	71.00	1378310*	29362	**9267**
9	(0.0169%)	167.56	**125.63**	819798*	49318	**12217**
10	(0.0327%)	287.18	**82.89**	751319*	133909	**11549**
11	(0.0186%)	(0.0220%)	**18.76**	73976*	20838*	**1155**
12	(0.0240%)	(0.0265%)	**22.91**	42729*	7505*	**460**
13	(0.0266%)	(0.0264%)	**74.43**	41325*	13420*	**6464**
14	(0.0144%)	(0.0203%)	**19.75**	41469*	4127*	**15**
15	780.76	(0.0105%)	**39.63**	120599	35111*	**3091**
16	(0.0162%)	(0.0223%)	**90.31**	42102*	8770*	**2597**
17	154.37	237.90	**27.88**	2114	2185	**1059**
18	(0.0121%)	(0.0214%)	**22.36**	60651*	3972*	**151**
19	(0.0195%)	(0.0250%)	**21.30**	42683*	6025*	**2436**
20	106.44	628.20	**18.46**	527	4288	**0**
Geometric Mean:	>349.77	>277.82	**38.03**			

[7] P. Malkin, "Minimum runtime and stoptime polyhedra," *Report, CORE, Université catholique de Louvain*, 2003.

[8] B. Knueven, J. Ostrowski, and J.-P. Watson, "A novel matching formulation for startup costs in unit commitment," 2017. [Online]. Available: http://www.optimization-online.org/DB_FILE/2017/03/5897.pdf

[9] R. M. Lima and A. Q. Novais, "Symmetry breaking in MILP formulations for unit commitment problems," *Computers & Chemical Engineering*, vol. 85, pp. 162–176, 2016.