Računanje s približki

Taylorjeva vrsta

$f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x)$
$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \quad \xi \in [x_0, x]$
$\implies R_n \to 0$, ko $n \to \infty$ za $x = konst.$

Asimptotske vrste

Obnašanje tipa $\frac{n!}{x^n}$
$\implies R_n \to 0$, ko $x \to \infty$ za $n = konst.$
$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = 1 - \int_0^x e^{-t^2}\,\mathrm{d}t$
$\mathrm{erfc}(x) \asymp \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2} + \cdots + (-1)^n \frac{(2n-1)!!}{(2x^2)^n}\right)$

Metoda stacionarne faze

$f(\lambda) = \int_a^b g(z) e^{i\lambda h(z)}\,\mathrm{d}z$
h linearna, g pohlevna, λ velik $\implies f(\lambda) = 0$
Drugače razvoj okoli stacionarne točke in dobimo:
$f(\lambda) \approx \left[\frac{2\pi}{|h''(z_0)|\lambda}\right]^{\frac{1}{2}} e^{i\mu\frac{\pi}{4}} e^{i\lambda h(z_0)} g(z_0), \quad \mu = \mathrm{sgn}(h''(z_0))$

Eliptični integrali in funkcije

$z = F(k,\varphi) = \int_0^\varphi \frac{\mathrm{d}\psi}{\sqrt{1-k^2\sin^2\psi}} = \int_0^u \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}} =$
$= \int_v^1 \frac{\mathrm{d}v}{\sqrt{(1-v^2)(1-k^2-k^2v^2)}} = \int_w^1 \frac{\mathrm{d}w}{\sqrt{(1-w^2)(k^2-1+w^2)}}$
$E(k,\varphi) = \int_0^\varphi \sqrt{1-k^2\sin^2\psi}\,\mathrm{d}\psi$
$u = \mathrm{sn}(z,k) \quad v = \mathrm{cn}(z,k) \quad w = \mathrm{dn}(z,k)$
$K(k) = F(k, \frac{\pi}{2}) = \frac{\pi}{2}\sum_{n=0}^\infty \left[\frac{(2n-1)!!}{2n!!}\right]^2 k^{2n}$
$E(k) = E(k, \frac{\pi}{2}) = \frac{\pi}{2}\left(1 - \sum_{n=1}^\infty \left[\frac{(2n-1)!!}{2n!!}\right]^2 \frac{k^{2n}}{n-1}\right)$

Picardova iteracija

Imejmo enačbo $\dot{x}(t) = h(x(t), t)$ in zaporedje $\varphi_{n+1}(t) = x_0 + \int_{t_0}^t h(\varphi_n(t), t)\,\mathrm{d}t$ z danim $\varphi_0(t)$. Tedaj $\lim_{n\to\infty}\varphi_n(t) = x(t)$ (pod določenimi pogoji).
Podobno velja za vektorske funkcije.

Diracova delta funkcija

$\int_D f(x)\delta(x-a)\,\mathrm{d}x = \begin{cases} f(a); & a \in D \\ 0; & \text{sicer} \end{cases}$
$\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}H(t)$
$\delta(ax) = \frac{1}{|a|}\delta(x)$
$\delta(-x) = \delta(x)$

$$\delta(t) = \lim_{\sigma\to 0}\frac{1}{2\sqrt{\pi}\sigma}e^{-\frac{t^2}{2\sigma^2}} = \lim_{\varepsilon\to 0}\frac{1}{\pi}\frac{\varepsilon^2}{t^2+\varepsilon^2} =$$
$$= \lim_{\varepsilon\to 0^+}\frac{\sin(t/\varepsilon)}{\pi t} = \lim_{\sigma\to\infty}\frac{\sigma}{\pi}\frac{1}{1-\sigma^2t^2} =$$
$$= \begin{cases} \lim_{a\to 0}\frac{1}{a}; & a \in [0,a] \\ 0; & \text{sicer} \end{cases}$$
$\delta(h(x)) = \sum_i \frac{\delta(x-x_i)}{|h'(x)|} = \sum_i \frac{\delta(x-x_i)}{|h'(x_i)|}, \quad x_i$ ničla h.

$\delta^{(3)}(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$
$\delta^{(3)}(\mathbf{H}(\mathbf{z})) = \sum_i \frac{\delta^{(3)}(\mathbf{z}-\mathbf{z}_i)}{|J\mathbf{H}(\mathbf{z})|} = \sum_i \frac{\delta^{(3)}(\mathbf{z}-\mathbf{z}_i)}{|J\mathbf{H}(\mathbf{z}_i)|}, \quad \mathbf{z}_i$ ničla $\mathbf{H}$.

$\int_{-\infty}^\infty \varphi(x)\frac{\mathrm{d}^n\delta(x-a)}{\mathrm{d}x^n}\,\mathrm{d}x = (-1)^n\frac{\mathrm{d}^n\varphi(a)}{\mathrm{d}x^n}$

Vektorji

Polarni in aksialni vektorji

Skalar: $f(-\mathbf{r}) = f(\mathbf{r})$
Psevdo-skalar: $f(-\mathbf{r}) = -f(\mathbf{r})$
Vektor (oz. polarni vektor): $\mathbf{F}(-\mathbf{r}) = -\mathbf{F}(\mathbf{r})$
Psevdo-vektor (oz. aksialni vektor): $\mathbf{F}(-\mathbf{r}) = \mathbf{F}(\mathbf{r})$

$M \in \mathbb{R}^{n\times n}$ ortogonalna:
Polarni vektor: $\mathbf{a}' = M\mathbf{a}$
Aksialni vektor: $\mathbf{b}' = (\det M)M\mathbf{b}$

$\mathbf{c} = \mathbf{a}\times\mathbf{b}$:
$\mathbf{a}, \mathbf{b}$ polarna $\implies \mathbf{c}$ aksialen
$\mathbf{a}, \mathbf{b}$ aksialna $\implies \mathbf{c}$ aksialen
$\mathbf{a}, \mathbf{b}$ en polaren, en aksialen $\implies \mathbf{c}$ polaren

Rotacije

$$A(\hat{\mathbf{i}},\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \quad A(\hat{\mathbf{j}},\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$
$$A(\hat{\mathbf{k}},\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad L_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$L_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad L_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{L} = (L_x, L_y, L_z)$$
$[L_x, L_y] = L_z \quad [L_y, L_z] = L_x \quad [L_z, L_x] = L_y$
$A(\hat{\mathbf{e}}_\alpha, \varphi) = e^{L_\alpha\varphi} \quad A(\hat{\mathbf{n}}, \varphi) = e^{(\mathbf{L}\cdot\hat{\mathbf{n}})\varphi}$

Pasivne rotacije dobimo s transponiranjem aktivnih. Če smo bazne vektorje zavrteli z aktivno rotacijo, potem vektor v novo bazo zavrtimo s pasivno rotacijo.

Vektorske baze

$\hat{\mathbf{e}}_i = \frac{1}{\left|\frac{\partial\mathbf{r}}{\partial u_i}\right|}\frac{\partial\mathbf{r}}{\partial u_i} = \frac{1}{h_i}\frac{\partial\mathbf{r}}{\partial u_i}$
$\mathrm{d}\mathbf{r} = \sum_i \frac{\partial\mathbf{r}}{\partial u_i}\mathrm{d}u_i = \sum_i h_i\hat{\mathbf{e}}_i\mathrm{d}u_i$
$\mathrm{d}s^2 = \sum_i h_i^2\mathrm{d}u_i^2$
$\mathrm{d}V = h_1h_2h_3\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3$
$\mathrm{d}S_1 = \left|\frac{\partial\mathbf{r}}{u_2}\mathrm{d}u_2 \times \frac{\partial\mathbf{r}}{u_3}\mathrm{d}u_3\right| = h_2h_3\,\mathrm{d}u_2\,\mathrm{d}u_3\,|-\hat{\mathbf{e}}_1|$

Tangentna (kontravariantna) baza - $\mathbf{r} = \mathbf{r}(u^1, u^2, u^3)$
$\frac{\partial\mathbf{r}}{\partial u^i} = \alpha_i = h_i\hat{\mathbf{e}}_i$
$\mathrm{d}\mathbf{r} = \sum_i \alpha_i\,\mathrm{d}u^i$
$\mathbf{A} = \sum_i \hat{A}^i\hat{\mathbf{e}}_i = \sum_i A^i\alpha_i$

Normalna (kovariantna) baza - $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$
$\nabla u^i = \beta^i = H_i\hat{\mathbf{e}}^i$
$\mathbf{A} = \sum_i \hat{A}_i\hat{\mathbf{e}}^i = \sum_i A_i\beta^i$

$\alpha_i\cdot\beta^j = \delta_{ij}$
$g_{ij} = \alpha_i\cdot\alpha_j \quad g^{ij} = \beta^i\cdot\beta^j$ (metrični tenzor)
$A^2 = \sum_i\sum_j g_{ij}A^iA^j = \sum_i\sum_j g^{ij}A_iA_j = \sum_i A^iA_i$

Diferencialni operatorji

$\frac{\partial\psi}{\partial\hat{a}} = \hat{a}\cdot\nabla\psi$
$\mathrm{d}\psi = \nabla\psi\cdot\mathrm{d}\mathbf{r}$
$f(\mathbf{r}+\boldsymbol{\Delta}) \approx f(\mathbf{r}) + (\nabla f(\mathbf{r}))\cdot\boldsymbol{\Delta}$

Divergenca je uporabna pri ohranitvenih zakonih, recimo:
$\oint_{\partial V}\mathbf{j}_Q\cdot\mathrm{d}\mathbf{S} = -\frac{\mathrm{d}H}{\mathrm{d}t} \quad \implies \quad \nabla\cdot\mathbf{j}_Q = -\frac{\mathrm{d}h}{\mathrm{d}t}$
Če izvor polja (recimo e polja $\mathbf{D}$) ni zvezno porazdeljen (vsebuje δ funkcije), je divergenca polja vedno 0. Rečemo, da se izvor skriva v singularnostih polja.

Odvod vektorja v smeri $\mathbf{a}$: $(\mathbf{a}\cdot\nabla)\mathbf{v}$
$\mathbf{v}(\mathbf{r}+\mathbf{a}) \approx \mathbf{v}(\mathbf{r}) + (\mathbf{a}\cdot\nabla)\mathbf{v}(\mathbf{r})$
$\mathbf{F}_e = (\mathbf{p}_e\cdot\nabla)\mathbf{E}, \quad \mathbf{F}_m = (\mathbf{p}_m\cdot\nabla)\mathbf{B}$
$\mathbf{f}(\mathbf{r}(t), t) \quad \implies \quad \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} = \frac{\partial\mathbf{f}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{f}$ (substancialni odvod)

Maxwellove enačbe

$\oint_{\partial S}\mathbf{H}\cdot\mathrm{d}\mathbf{r} = I + \frac{\mathrm{d}\Phi_e}{\mathrm{d}t} = \int_S \mathbf{j}_e\cdot\mathrm{d}\mathbf{S} + \frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{D}\cdot\mathrm{d}\mathbf{S}$
$\nabla\times\mathbf{H} = \mathbf{j}_e + \frac{\partial\mathbf{D}}{\partial t}$
$\oint_{\partial S}\mathbf{E}\cdot\mathrm{d}\mathbf{r} = -\frac{\mathrm{d}\Phi_m}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B}\cdot\mathrm{d}\mathbf{S}$
$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
$\oint_{\partial V}\mathbf{B}\cdot\mathrm{d}\mathbf{S} = 0$
$\nabla\cdot\mathbf{B} = 0$
$\oint_{\partial V}\mathbf{D}\cdot\mathrm{d}\mathbf{S} = e = \int_V \rho_e\,\mathrm{d}V$
$\nabla\cdot\mathbf{D} = \rho_e$
$\oint_{\partial V}\mathbf{j}_e\cdot\mathrm{d}\mathbf{S} = -\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_v \rho_e\,\mathrm{d}V$
$\nabla\cdot\mathbf{j}_e = -\frac{\partial\rho_e}{\partial t}$

Helmholtzov izrek

Vsako polje $\mathbf{v}$ lahko razstavimo v brezvrtinčni (skalarni potencial ∇U) in brezizvirni (vektorski potencial $\nabla\times\mathbf{A}$) prispevek.

$\mathbf{B} = \nabla\times\mathbf{A}$
$\mathbf{E} = -\nabla U - \frac{\partial A}{\partial t}$
Lorentzova umeritev: $\nabla\cdot\mathbf{A} = -\frac{1}{c^2}\frac{\partial U}{\partial t}$

Tenzorji

Efektivna vrednost tenzorja v smeri $\hat{n}$: $\hat{n}\cdot\underline{T}\hat{n}$.

Einsteinova notacija

$\mathbf{a} = a_i$
$a_ib_i = \sum_i a_ib_i$
$\partial_i r_j = \delta_{ij}$

Levi-Civitajev tenzor $\underline{\varepsilon}$

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$ in 0 za ostale
$\boldsymbol{\omega}\times\mathbf{r} = -(\underline{\varepsilon}\cdot\omega)\cdot\mathbf{r} = \varepsilon_{ijk}\omega_jr_k$
$\det A = \frac{1}{3!}\varepsilon_{ijk}\varepsilon_{lmn}a_{il}a_{jm}a_{kn} = \varepsilon_{ijk}a_{i1}a_{j2}a_{k3}$

$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Diadni produkt
$\underline{A} = \mathbf{a}\otimes\mathbf{b} \quad \Longrightarrow \quad A_{ik} = a_i b_k$
$\mathbf{v} = \mathbf{a}(\mathbf{b}\cdot\mathbf{u}) = (\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{u}$
$(\mathbf{a}\otimes\mathbf{b})^T = \mathbf{b}\otimes\mathbf{a}$

Diferencialne operacije

$(\nabla U)_i = \partial_i U$
$\nabla\cdot\mathbf{B} = \partial_i B_i$
$(\nabla\times\mathbf{B})_i = \varepsilon_{ijk}\partial_j B_k$
$(\mathbf{p}_e\cdot\nabla)\mathbf{E} = \mathbf{p}_e(\nabla\otimes\mathbf{E}) = p_j\partial_j E_i$

$f(\mathbf{r}+\mathbf{a}) = \exp(\mathbf{a}\cdot\nabla)f(\mathbf{r}) = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathbf{a}\cdot\nabla)^n f(\mathbf{r})$

$\partial_i T_{jl} = F_{ijl} = \nabla\otimes\underline{T}$ (gradient tenzorja)
$\partial_i T_{ij} = F_l$ (divergenca tenzorja)

$\int_{S=\partial V} a_{ijk\ldots q} n_j \,\mathrm{d}S = \int_V \partial_j a_{ijk\ldots q}\,\mathrm{d}V$

Ireducibilne komponente

$\underline{\lambda} = \underline{S} + \underline{A} = \frac{1}{2}\left(\underline{\lambda}+\underline{\lambda}^T\right) + \frac{1}{2}\left(\underline{\lambda}-\underline{\lambda}^T\right)$

$\underline{S} = \frac{1}{3}\mathrm{tr}(\underline{S})\underline{I} + \left[\underline{S} - \frac{1}{3}\mathrm{tr}(\underline{S})\underline{I}\right] = \frac{1}{3}\mathrm{tr}(\underline{S})\underline{I} + \overset{\sqcap}{\underline{S}}$

$\Longrightarrow\ \underline{\lambda} = \frac{1}{3}\mathrm{tr}(\underline{S})\underline{I} + \overset{\sqcap}{\underline{S}} + \underline{A}$
Te tri komponente se pri ortogonalnih transformacijah obnašajo neodvisno.

Deformacije

$\mathrm{d}\mathbf{u}(\mathbf{r}) = \mathbf{u}(\mathbf{r}+\mathrm{d}\mathbf{r}) - \mathbf{u}(\mathbf{r}) = (\mathrm{d}\mathbf{r}\cdot\nabla)\mathbf{u} = \mathrm{d}\mathbf{r}(\nabla\otimes\mathbf{u})$
$\mathrm{d}u_i = \frac{1}{2}(\partial_j u_i + \partial_i u_j)\,\mathrm{d}r_j + \frac{1}{2}(\partial_j u_i - \partial_i u_j)\,\mathrm{d}r_j = E_{ij}\,\mathrm{d}r_j + \Lambda_{ij}\,\mathrm{d}r_j$

$\underline{\Lambda}\mathrm{d}\mathbf{r} = \frac{1}{2}(\nabla\times\mathbf{u})\times\mathrm{d}\mathbf{r} = \mathrm{d}\vartheta\times\mathrm{d}\mathbf{r}$ (rotacijski oz. vrtinčni tenzor)

$\underline{E} = \frac{1}{3}\mathrm{tr}(\underline{E})\underline{I} + \overset{\sqcap}{\underline{E}}$ (deformacijski tenzor)
Prvi člen $\underline{E}$ predstavlja spremembo volumna, drugi čisti strig.
$\frac{\Delta V}{V} = \mathrm{tr}\underline{E}$

$\Longrightarrow\ \mathrm{d}\mathbf{u} = \frac{1}{3}(\nabla\cdot\mathbf{u})\mathrm{d}\mathbf{r} + (\nabla\otimes\mathbf{u})^{\sqcap}\mathrm{d}\mathbf{r} + \frac{1}{2}(\nabla\times\mathbf{u})\times\mathrm{d}\mathbf{r}$
sprememba volumna strig rotacija

Hookov zakon

$\sigma_{ij} = \frac{\mathrm{d}F_i}{\mathrm{d}S_j} = -p_{ij}$
$\sigma_{ij} = C_{ijkl}E_{kl}$ (Hookov zakon)

Če je snov izotropna:
$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$\sigma_{ij} = \lambda E_{ll}\delta_{ij} + 2\mu E_{ij}$
$E_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{ll}\delta_{ij}$

Alternativna parametrizacija:

$\sigma_{ij} = KE_{ll}\delta_{ij} + 2G\overset{\sqcap}{E}_{ij}$
$G = \mu$
$K - \frac{2}{3}G = \lambda$
$3K = \frac{Y}{1-2\nu}$ (Y Youngov modul, ν Poissonovo število)
$2G = \frac{Y}{1+\nu}$

Gibalna enačba

$\rho\mathbf{a} = \mathbf{f} = \mathbf{f}_{\mathrm{pr}} - \nabla\cdot\underline{p}\ \Longrightarrow$

$\rho\mathbf{a} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \mathbf{f}_{\mathrm{pr}} + G\nabla^2\mathbf{u} + \left(K + \frac{1}{3}G\right)\nabla(\nabla\cdot\mathbf{u})$

V hidrodinamiki $\mathbf{u}\to\mathbf{v}$, $G\to\eta$ in $K\to\eta_V$. Upoštevamo še hidrostatični tlak p_0. Dobimo Navier-Stokesovo enačbo:
$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{f}_{\mathrm{pr}} - \nabla p_0 + \eta\nabla^2\mathbf{v} + \left(\eta_V + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$

Diferencialne enačbe

Za sistem oblike $\dot{\mathbf{y}} = \underline{A}(t)\mathbf{y}$ glej Dysonovo in Magnusovo metodo.

Parametrični oscilator: $\ddot{x} + 2\beta(t)\dot{x} + \omega(t)^2 x = 0$
Van der Polov oscilator: $\ddot{x} - \varepsilon(1-x^2)\dot{x} + x = 0$
Duffingova enačba $\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t)$

Kaos

Atraktor - množica točk v faznem prostoru, h katerim se sistem razvije po dolgem času.
Čudni (kaotični) atraktor - poljubno majhna variacija začetne točke povzroči evolucijo točk poljubno daleč narazen.

Za pojav determinističnega kaosa potrebujemo nelinearen sistem DE in vsaj 3 spremenljivke (Poincare-Bendixson).

Dodatne enačbe

Biot-Savartov zakon: $\mathrm{d}\mathbf{H} = \frac{I}{4\pi}\frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$

$\mathbf{M} = \mathbf{p}_m\times\mathbf{B} \qquad \mathbf{M} = \mathbf{p}_e\times\mathbf{E}$
$\mathbf{D} = \underline{\varepsilon}\varepsilon_0\mathbf{E}$
$\mathbf{B} = \underline{\mu}\mu_0\mathbf{H}$
$\mathbf{P} = \frac{\mathrm{d}\mathbf{p}_e}{\mathrm{d}V} = \underline{\chi}\varepsilon_0\mathbf{E} = (\underline{\varepsilon} - \underline{I})\varepsilon_0\mathbf{E}$
$\mathbf{M} = \frac{\mathrm{d}\mathbf{p}_m}{\mathrm{d}V} = \underline{\chi}\mathbf{H} = (\underline{\mu} - \underline{I})\mathbf{H}$
$\mathbf{E} = \underline{\zeta}\mathbf{j}$
$\mathbf{j} = -\underline{\lambda}\nabla T$