Discriminants in the relative Grothendieck group of varieties

(work in progress, as always...)

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Introduction

The starting point of this work was the problem of computing the (equivariant) motivic Chern classes of *discriminant strata* of $\mathbb{P}^1(\mathbb{C})$ (also called *coincident root loci*, λ-Chow-varieties, etc).

For a quasi-projective variety X over an algebraically closed field K (most often, $K = \mathbb{C}$), define the discriminant strata X_λ indexed by partitions λ of n to be the locally closed subvariety

$$j_\lambda \ : \ X_\lambda \ \subset \ \mathsf{Sym}^n X$$

whose underlying set is the set of unordered n-tuples of points in X with multiplicities given by λ. Clearly,

$$\mathsf{Sym}^n X = \coprod_{\lambda \vdash n} X_\lambda$$

The name "discriminant strata" is motivated by the classical example of the discriminant which is just the closure of $X_{(2,1,1,\dots)}$.

(For more general fields k the situation is more delicate, but essentially all this should still work in some form or other...)

Motivation / goals

It turns out that there is a relatively simple recursive algorithm (B. K. 2018) computing motivic invariants and motivic characteristic classes of X_λ, where the general skeleton of the algorithm does not depend on the concrete invariant.

Furthermore, with some additional work we can also understand the output of this algorithm, which results in *formulas* for the generating functions of the invariants of X_λ.

Here, I try to explain this in the "universal setting", which needs the language of the (relative) Grothendieck group of varieties.

But we can also do more concrete computations for concrete invariants and characteristic classes - including the original problem of computing the equivariant motivic Chern classes of X_λ for $X = \mathbb{P}^1$; or for example computing the zeta function of E_λ where E is an elliptic curve over $\mathbb{F}_q$.

Takeaways (advertisement)

- the main result is a generic formula for the generating function of the classes $[X_\lambda \subset \mathrm{Sym}^n X]$ in the relative Grothendieck group of varieties, expressed via a relative analogue of the motivic zeta function
- we have generic formulas for the g.f. of χ_y genus, E-polynomial, (equivariant) CSM class, Hirzebruch class of X_λ (because Macdonald-type formulas are available)
- we can compute the Hasse-Weil zeta function of X_λ at least for some varieties $X/\mathbb{F}_q$ (like $X = \mathbb{P}^n$ or elliptic curves)
- for $X = \mathbb{P}^1/\mathbb{C}$ we have formulas for the "umbral" g.f. of the (non-equivariant) CSM, Hirzebruch and MC classes
- and also for the $\mathbb{T}^2$-equivariant fundamental class, CSM and motivic Chern (MC) classes
- these g.f.-s can be used to understand the families $[X_{(\mu,1^n)}]$
- an application is the class of the locus of μ-tangent lines of a degree d projective hypersurface

Plan of this talk

This is the (overly ambitious) plan for today:

1. building up the language
2. unifying the absolute (invariant) and relative (characteristic class) settings
3. the motivic discriminant generating function
4. examples: χ_y genus and Hasse-Weil zeta function
5. generic formulas for the CSM and Hirzebruch classes
6. concrete (umbral) formulas for $X = \mathbb{P}^1$ (GL_2-equivariant CSM and motivic Chern classes)
7. ~~looking at the stabilization~~ $\cancel{n \mapsto [X_{(\mu,1^n)}]}$
8. (closure of the strata and applications)
9. there won't be time for the proof of the main formula, but a sketch is included in these slides

The relative Grothendieck group of varieties

Let k be a field and S a scheme over k. Denote by Var_S the category of S-varieties (separated, reduced schemes of finite type over S).

The relative Grothendieck group $K_0(\mathsf{Var}_S)$ of varieties over S is defined to be the free Abelian group generated by the isomorphism classes $[X]_S$ of S-varieties, modulo the relations

$$[X]_S = [X - Y]_S + [Y]_S$$

whenever $Y \subset X$ is a closed subvariety.

For a map $f : S \to T$ we have the operations:

$$f_! : K_0(\mathsf{Var}_S) \to K_0(\mathsf{Var}_T)$$
$$f^* : K_0(\mathsf{Var}_T) \to K_0(\mathsf{Var}_S)$$

defined by composition and pullback diagram, respectively.

The product in the Grothendieck ring

We also have an exterior product, and via fiber products an internal product:

$$\boxtimes : K_0(\mathsf{Var}_S) \times K_0(\mathsf{Var}_T) \to K_0(\mathsf{Var}_{S\times T})$$
$$\circledast : K_0(\mathsf{Var}_S) \times K_0(\mathsf{Var}_S) \to K_0(\mathsf{Var}_S)$$

satisfying a lot of identities like:

$$\begin{aligned}(f \times g)_!\,(A \boxtimes B) &= f_!\,A \boxtimes g_!\,B \\ (f \times g)^*(A \boxtimes B) &= f^*A \boxtimes g^*B \\ f^*(A \circledast B) &= f^*A \circledast f^*B\end{aligned}$$

The product $\circledast$ and $\mathbb{1} = [S]_S$ makes $K_0(\mathsf{Var}_S)$ a ring; the pullback f^* is a ring homomorphism, and for $f : S \to T$ the pushforward $f_!$ is a $K_0(\mathsf{Var}_T)$-module homomorphism.

The latter fact can be encoded in the projection formula:

$$f_!(f^*x \circledast y) = x \circledast f_!\,y$$

Motivic measures

A map $\mathfrak{m} : \mathsf{Var}_S \to A$ to a ring A is called a *motivic measure* if it satisfies the following properties:

$$\begin{aligned} \mathfrak{m}[S] &= 1 \\ \mathfrak{m}[X] &= \mathfrak{m}[X - Y] + \mathfrak{m}[Y] \\ \mathfrak{m}[X \times_S Y] &= \mathfrak{m}[X] \cdot \mathfrak{m}[Y] \end{aligned}$$

Theorem (see eg. [Motivic Integration]). Any motivic measure $\mathfrak{m} : \mathsf{Var}_S \to A$ factors through a unique ring homomorphism $K_0(\mathsf{Var}_S) \to A$ (which we will also denote by $\mathfrak{m}$).

Setting $\mathfrak{m} = \mathrm{id}$ we get the *universal Euler characteristic* [Bittner] taking values in $K_0(\mathsf{Var}_S)$ itself.

Examples of motivic measures

Some examples of motivic measures:

- over $S = \operatorname{Spec}(\mathbb{F}_q)$, the counting measure $X \mapsto |X(\mathbb{F}_q)| \in \mathbb{Z}$
- more generally, over $\mathbb{F}_q$, the (local) Hasse-Weil zeta function $Z(X;t)$ in the Witt ring $W(\mathbb{Z})$
- over $S = \operatorname{Spec}(\mathbb{C})$: the Euler characteristic; the Hirzebruch χ_y genus; the virtual Poincaré polynomial; and the Hodge-Deligne E-polynomial
- over $\mathbb{C}$, the Hodge characteristic in the Grothendieck group of Hodge structures
- the universal Euler characteristic $[X]_S$

Almost-examples (group morphisms instead of ring morphisms):

- over a variety $X/\mathbb{C}$, the Chern-Schwartz-MacPherson class in $H_{2*}^{BM}(X)$
- over a variety $X/\mathbb{C}$, the motivic Chern class in $K^0(X)[y]$

These are however compatible with the *exterior* product, and also with proper pushforwards!

Symmetric products

For a quasi-projective variety X/k, the symmetric product

$$\mathrm{Sym}^n X = \big(X \times X \times \cdots \times X\big)/S_n$$

exists, and similary for a map $f : Y \to X$ between quasi-projective varieties we have

$$\mathrm{Sym}^n f \ : \ \mathrm{Sym}^n Y \to \mathrm{Sym}^n X.$$

There are also natural maps ("merging" resp. "replicating" points):

$$\begin{aligned} \Psi_{n,m} &: \mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Sym}^{n+m} X \\ \Omega^d &: \mathrm{Sym}^n X \to \mathrm{Sym}^{dn} X \end{aligned}$$

Remark: for arbitrary base fields, the symmetric product is a very subtle thing. But our main example (apart from $\mathbb{C}$) is $\mathbb{F}_q$, where we are just counting points, in which case we can pretend that this problem does not exist.

Motivic zeta function

Kapranov's motivic zeta function for a motivic measure $\mathfrak{m}$ taking values in A, and for a variety X/k is defined by

$$Z_{\mathfrak{m}}(X;t) \ := \ \sum_{n\geq 0} \mathfrak{m}[\mathsf{Sym}^n X]\, t^n \quad \in \quad 1 + t\cdot A[[t]].$$

In particular, we can take the universal measure $\mathfrak{m} = \mathrm{id}$:

$$Z(X;t) \ := \ \sum_{n\geq 0} [\mathsf{Sym}^n X]\, t^n \quad \in \quad 1 + t\cdot K_0(\mathsf{Var}_k)[[t]].$$

In fact, $Z_{\mathfrak{m}}$ is a group homomorphism: $K_0(\mathsf{Var}_k) \to W(A)$. For those measures $\mathfrak{m}$ which are also λ-ring morphisms, $Z_{\mathfrak{m}}$ is also a ring homomorphism, in which case $Z_{\mathfrak{m}}$ itself is again a motivic measure! An example of this is the local Hasse-Weil zeta function over $\mathbb{F}_q$ (see eg. [Ramachandran]).

Interlude: λ-rings

A *pre-λ-ring* is a pair of a commutative ring A and a group morphism $\lambda_t : (A, +) \to \Lambda(A) := 1 + tA[[t]]$; then the components λ^r of λ_t

$$\lambda_t(a) \;=\; 1 + \sum_{r \geq 1} \lambda^r(a)\, t^r$$

behave like exterior powers: $\lambda^r(a+b) = \sum_i \lambda^{r-i}(a)\lambda^i(b)$. Given such a λ_t, we can define the Adams operations via the logarithmic differential:

$$t \cdot \frac{\partial}{\partial t} \log(\lambda_t(a)) = \sum_{r \geq 1} \psi^r(a)\, t^r$$

(A, λ_t) is a pre-λ ring iff $\psi^r(a+b) = \psi^r(a) + \psi^r(b)$. A pre-$\lambda$-ring is called a λ-ring iff $\psi^r \circ \psi^s = \psi^{rs}$ and $\psi^r(ab) = \psi^r(a)\psi^r(b)$.

Some examples of (pre-)λ-rings:

- $\mathbb{Z}$ with $\lambda_t(n) = (1 \pm t)^{\pm n}$ and $\psi^r(n) = \pm(\pm 1)^r \cdot n$
- the ring of symmetric functions, with $\lambda^r(e_1) = e_r$ and $\psi_r(e_1) = p_r$
- K-theory $K^0(X)$ with the exterior power $\lambda^r([E]) = [\bigwedge^r E]$
- the Grothendieck ring of varieties $K_0(\mathsf{Var}_k)$ with $\lambda^r([X]) = [\mathsf{Sym}^n X]$
- the Witt ring $W(A)$ with $\lambda^{\geq 2}([a]) = 1$

Relative zeta "function"

Motivated by Kapranov's motivic zeta function, define the following *"relative zeta function"* for a map $f : Y \to X$:

$$\mathcal{Z}(f;t) \ := \ \sum_{n \geq 0} [\mathsf{Sym}^n f]\, t^n \ \in \ \Lambda(X;t)$$

This lives in the group (for which I don't have a good name):

$$\Lambda(X;t) \ := \ 1 + \widehat{\bigoplus_{n \geq 1}} t^n \, K_0(\mathsf{Var}/\mathsf{Sym}^n X)$$

(here t is just a formal variable so that it is easier to distinguish the components; sometimes we will omit it, as it's not important).

Note that for $X = \mathrm{Spec}(k)$ we get back Kapranov's zeta function, and the ring multiplication $\odot$ (defined on the next slide) becomes just the usual power series multiplication.

The relative ring $\Lambda(X)$

There is a natural ring structure on

$$\Lambda(X;t) := 1 + \widehat{\bigoplus_{n\geq 1}} t^n \, K_0(\mathsf{Var}/\mathsf{Sym}^n X)$$

given by the multiplication $\odot$ induced by the maps

$$\begin{aligned} K_0(\mathsf{Var}/\mathsf{Sym}^n X) \times K_0(\mathsf{Var}/\mathsf{Sym}^m X) \;&\to\; K_0(\mathsf{Var}/\mathsf{Sym}^{n+m} X) \\ [f] \odot [g] \;&:=\; (\Psi_{n,m})_! [f \times g] = \Psi_! ([f] \boxtimes [g]) \end{aligned}$$

We also have the embedding (mapping $+$ to $\odot$)

$$\mathcal{Z} : K_0(\mathsf{Var}/X) \to \Lambda(X;t)$$

which gives a "relative pre-λ-ring" structure on it.

The maps $\Omega^d_!$ also extend to $\Lambda(X;t)$; for $X = \mathrm{Spec}(k)$ these are simply the substitution $t \mapsto t^d$.

Relative Adams operations

Define the "relative Adams operations" $\mathcal{A}_m$ for $m \geq 1$

$$\mathcal{A}_m : K_0(\mathsf{Var}/X) \to K_0(\mathsf{Var}/\mathsf{Sym}^m X)$$

by the equation

$$\mathcal{Z}[f] = \sum_{n \geq 0} [\mathsf{Sym}^n f]\, t^n = \exp\left[\sum_{m \geq 1} \frac{t^m}{m} \cdot \mathcal{A}_m[f]\right]$$

Again, for $X = \mathrm{Spec}(k)$ we get back the usual Adams operations corresponding to the pre-λ-ring structure on $K_0(\mathsf{Var}_k)$.

$\mathcal{A}_m[\mathrm{id}_X]$ can be expressed as a $\mathbb{Z}$-linear combination of $[\Psi_{k_1,k_2,\ldots}]$ with $m = k_1 + k_2 + \ldots$, and more generally $\mathcal{A}_m[f]$ by the same combination of the pushforwards $(\Psi_{\underline{k}})_![\mathsf{Sym}^{\underline{k}} f]$. In particular:

$$\begin{aligned}
\mathcal{A}_1 &= \quad [\Psi_1] & &= [\mathrm{id}] \\
\mathcal{A}_2 &= 2[\Psi_2] - [\Psi_{1,1}] & &= 2[\mathrm{id}] - [\Psi_{1,1}] \\
\mathcal{A}_3 &= 3[\Psi_3] - 3[\Psi_{2,1}] + [\Psi_{1,1,1}] \\
\mathcal{A}_4 &= 4[\Psi_4] - 4[\Psi_{3,1}] + 4[\Psi_{2,1,1}] - 2[\Psi_{2,2}] - [\Psi_{1,1,1,1}]
\end{aligned}$$

Additivity of the Adams operations

The (relative) zeta function is multiplicative: If $f : Y \to X$ and $Y = Z + U$ with $Z \subset Y$ a closed subvariety, then

$$\mathcal{Z}[f] \;=\; \mathcal{Z}[f|_Z] \odot \mathcal{Z}[f|_{Y-Z}]$$

because

$$\sum_n t^n[\mathsf{Sym}^n(Z+U)] = \sum_n t^n \sum_{k=0}^{n} [\mathsf{Sym}^{n-k} Z] \odot [\mathsf{Sym}^k U] =$$

$$= \left(\sum_n t^n [\mathsf{Sym}^n Z]\right) \odot \left(\sum_m t^m [\mathsf{Sym}^m U]\right).$$

Hence, the (relative) Adams operations are additive:

$$\mathcal{A}_m[f] \;=\; \mathcal{A}_m[f|_Z] + \mathcal{A}_m[f|_{Y-Z}].$$

The (absolute) discriminant generating function

We can collect all the discriminant strata together into a big generating function

$$G(X;\underline{x}) := \sum_{\lambda} [X_\lambda]\, \underline{x}^\lambda \quad \in \quad K_0(\mathsf{Var}_k)[[x_1, x_2, \dots]]$$

where $\underline{x}^\lambda$ is shorthand for $x_1^{e_1} x_2^{e_2} \dots$ with $\lambda = (1^{e_1} 2^{e_2} \dots)$.

Substituting $x_i \mapsto t^i$ we get back the motivic zeta function:

$$G(X; t, t^2, t^3, \dots) = Z(X;t)$$

We can also look at 1-parameter families: For μ with $\mu_i \geq 2$

$$F_\mu(X;t) := \sum_{n=0}^{\infty} [X_{(\mu,1^n)}] \cdot t^{|\mu|+n}$$

Vakil and Wood [VW2015] gave a (different) algorithm which computes F_μ, and in particular proved the following nice formula:

$$F_\emptyset(X;t) = \sum [X_{(1^n)}]\, t^n = \frac{Z(X;t)}{Z(X;t^2)}$$

The relative discriminant generating function

We can also consider this in the relative setting introduced above: For $f : Y \to X$, define

$$\mathcal{G}(f;\underline{x}) \;:=\; \sum_{\lambda} \underbrace{[\mathsf{Sym}^n f \circ j_\lambda]}_{[f_\lambda]} \cdot \underline{x}^\lambda \;\;\in\;\; \Lambda(X;\underline{x})$$

where j_λ denotes the embedding $j_\lambda : Y_\lambda \to \mathsf{Sym}^{|\lambda|}Y$, and

$$\Lambda(X;\underline{x}) \;:=\; \widehat{\bigoplus_{\lambda}}\, \underline{x}^\lambda\, K_0(\mathsf{Var}/\mathsf{Sym}^{|\lambda|}X) \;\subset\; \Lambda(X)[[x_1, x_2, \ldots]]$$

Everything works pretty much the same, and again we get back the absolute case when $X = \mathrm{Spec}(k)$.

However, the relative setting allows us to work with *characteristic classes* (as opposed to the absolute setting, which corresponds to *invariants*).

The main formula

Define the formal power series $\mathcal{L} \in \mathbb{Q}[[\underline{x}, t]]$ and $\mathcal{P} \in \mathbb{Z}[[\underline{x}, t]]$ and their homogeneous component polynomials L_d and P_d by

$$\sum_{d\geq 1} L_d(\underline{x})t^d = \mathcal{L}(\underline{x}; t) := \log\left[1 + \sum_{i\geq 1} x_i t^i\right] = \sum_{m\geq 1} \frac{1}{m}\mathcal{P}(\underline{x}^m; t^m)$$

$$\sum_{d\geq 1} P_d(\underline{x})t^d = \mathcal{P}(\underline{x}; t) := \sum_{m\geq 1} \frac{\mu(m)}{m}\mathcal{L}(\underline{x}^m; t^m)$$

Theorem. The central result of this work is the following formula:

$$\mathcal{G}(f; \underline{x}) = \sum_{\lambda} [f_\lambda]\, \underline{x}^\lambda = \exp\left[\sum_{m\geq 1} \frac{1}{m} \sum_{d\geq 1} P_d(\underline{x}^m) \cdot \Omega_!^d \mathcal{A}_m[f]\right]$$

where $f : Y \to X$ and the exponential is defined by its power series in the ring $\Lambda(X; \underline{x})$.

1-parameter families (or "stabilization")

Corollary:

$$\mathcal{F}_{\emptyset}(f;t) \;:=\; \sum_{n\geq 0}[f_{(1^n)}]\cdot t^n \;=\; \frac{\mathcal{Z}(f;t)}{\Omega_!^2\,\mathcal{Z}(f;t)}$$

Proof: $\mathcal{P}(x) := \mathcal{P}(x;1) := \mathcal{P}(x,0,0,\ldots;1) = x - x^2$. QED.

More generally, for any μ (it also works with f instead of X):

$$\mathcal{F}_{\mu}(X;t) = \sum_{n\geq 0}[X_{(\mu,1^n)}]\cdot t^{|\mu|+n} = \mathcal{F}_{\emptyset}(X;t)\cdot\mathcal{R}_{\mu}(X;t)$$

where $\mathcal{R}_{\mu}(X;t)$ is something explicitly computable; for example

$$\begin{aligned}
\mathcal{R}_{(a)} &= \phi(a;1) & & a>1\\
\mathcal{R}_{(a,b)} &= \phi(a;1)\cdot\phi(b;1) - \phi(a+b;k+1) & & a>b>1
\end{aligned}$$

with

$$\phi\big(a;f(k)\big) = \sum_{k\geq 0}(-1)^k t^{k+a}\cdot f(k)\cdot\Omega_!^{k+a}[X]$$

Example: Hirzebruch χ_y genus

Let X be a quasi-projective variety over $\mathbb{C}$. We can combine the Borisov-Libgober-Zhou formula for the χ_y genus of symmetric products:

$$\sum_{n\geq 0} \chi_{-y}(\mathsf{Sym}^n X) \cdot t^n = \exp\left[\sum_{m\geq 1} \chi_{-y^m}(X) \frac{t^m}{m}\right]$$

with our main formula (the absolute version) to get

$$\sum_{\lambda} \chi_{-y}(X_\lambda) \cdot \underline{x}^\lambda = \exp\left[\sum_{k\geq 1} \frac{\mathcal{L}(\underline{x}^m)}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) \chi_{-y^d}(X)\right]$$

There is a similar formula for the Hodge-Deligne E-polynomial, too.

Example: Elliptic curves in the Grothendieck ring

Let E be an elliptic curve over a field k. It's known (see eg. [Motivic Integration]) that the motivic zeta function of E is

$$Z(E;t) \;=\; \frac{1+([E]-1-\mathbb{L})t+\mathbb{L}t^2}{(1-t)(1-\mathbb{L}t)} \;\in\; K_0(\mathsf{Var}_k)[[t]]$$

From this, $\mathcal{A}_m[E]$ can be computed:

$$\mathcal{A}_m[E] = 1+\mathbb{L}^m - (-i)^m\mathbb{L}^{m/2}\cdot L_m\left(-i\cdot\frac{[E]-1-\mathbb{L}}{\sqrt{\mathbb{L}}}\right)$$

where $L_m(x)$ is the m-th Lucas polynomial. Then $G(E;\underline{x})$ is

$$\exp\left[\sum_{m\geq 1}\frac{\mathcal{P}(\underline{x}^m)}{m}\cdot\mathcal{A}_m[E]\right] = \exp\left[\sum_{m\geq 1}\frac{\mathcal{L}(\underline{x}^m)}{m}\cdot\sum_{d|m}\mu\left(\frac{m}{d}\right)\mathcal{A}_m[E]\right]$$

When $k=\mathbb{F}_q$, simply substituting $\mathbb{L}\mapsto q$ and $[E]\mapsto \#E(\mathbb{F}_q)$ we get the point counting generating functions over $\mathbb{F}_q$.

Interlude II: the Witt ring

Given a commutative ring A, the group $(W(A), \oplus)$ is isomorphic to $(1 + t\,A[[t]], \times)$, and the multiplication $\otimes$ on $W(A)$ is defined by

$$[a] \otimes [b] := (1-at)^{-1} \otimes (1-bt)^{-1} = (1-abt)^{-1} =: [ab]$$

plus the functoriality of $W(-)$. The elements $[a] := (1-at)^{-1}$ are called *Teichmüller elements*; $a \mapsto [a]$ is a multiplicative group homomorphism. Remark: the Chern character of a tensor product is the Witt product of the Chern characters.

The "ghost map" (isomorphism) $gh : W(A) \to A^{\mathbb{N}}$ defined by

$$\begin{array}{ccccccc} W(A) & \longrightarrow & t\,A[[t]] & = & t\,A[[t]] & \xrightarrow{\cong} & A^{\mathbb{N}} \\ P(t) & \longmapsto & t \cdot (\partial_t \log(P)) & = & \sum_m g_m(P) t^m & \longmapsto & (g_1, g_2, \dots) \end{array}$$

The Hasse-Weil zeta function of varieties over $\mathbb{F}_q$ is a ring homomorphism $K_0(\mathrm{Var}/\mathbb{F}_q) \to W(\mathbb{Z})$.

Hasse-Weil zeta function of elliptic curves over $\mathbb{F}_q$

The Hasse-Weil zeta function of a variety $X/\mathbb{F}_q$

$$Z_{\#}(X;t) \ := \ \exp\left[\sum_{m\geq 1} \#X(\mathbb{F}_{q^m}) \cdot \frac{t^m}{m}\right] \ \in \ W(\mathbb{Z})$$

is itself a motivic measure, regarded as en element of the Witt ring $W(\mathbb{Z})$ [Ramachandran]. Hence we can formulate *its associated motivic zeta function*

$$Z_{Z_{\#}}(X;u) \ := \ \sum_{n\geq 0} Z_{\#}(\mathsf{Sym}^n X;t) \cdot u^n \ \in \ 1 + uW(\mathbb{Z})[[u]]$$

This was computed by Huang; for example take E to be an elliptic curve, let $a_q = 1 + q - \#E(\mathbb{F}_q)$ then

$$Z_{Z_{\#}}(E;u) \ = \ \frac{(1-[\alpha]u)(1-[\beta]u)}{(1-[1]u)(1-[q]u)} \ \in \ W(\mathbb{Z}) \subset W(\overline{\mathbb{Q}})$$

where $\alpha + \beta = a_q$, $\alpha\beta = q$ and $[\alpha] := (1-\alpha t)^{-1}$ are the Teichmüller elements in the Witt ring $W(\overline{\mathbb{Q}})$.

Hasse-Weil zeta function of discriminants of $E/\mathbb{F}_q$

Huang's formula can be rewritten in terms of $\mathcal{A}_m(E)$:

$$Z_{Z_\#}(E;u) \;=\; \exp\left[\sum_{m\geq 1}\Big(\underbrace{[1]+[q^m]-[\alpha^m]-[\beta^m]}_{\mathcal{A}_m}\Big)\frac{u^m}{m}\right] \;\in\; W(\overline{\mathbb{Q}})[[u]]$$

and then we can proceed as usual.

For example we can compute the zeta function of the first nontrivial discriminant $\overline{E_{(2,1)}}$ by

$$Z_\#(\overline{E_{(2,1)}}) = Z_\#(\mathsf{Sym}^3 E) \ominus Z_\#(E_{(1,1,1)}) =$$

$$= [1]+2[q]+[q^2]-2[\alpha]-2[q\alpha]+[\alpha^2]-2[\beta]-2[q\beta]+2[\alpha\beta]+[\beta]^2 =$$

$$= \frac{(1-a_q t+qt^2)^2\cdot(1-a_q qt+q^3t^2)^2}{(1-t)\cdot(1-qt)^4\cdot(1-q^2t)\cdot(1-a_q^2t+2qt+q^2t^2)}$$

Characteristic classes

Similarly as the absolute version works for any motivic measure, the relative version works for the Chern-Schwartz-MacPherson, Hirzbruch and motivic Chern classes. These commute with exterior product and *proper* pushforward, but all the maps appearing in our algorithm are proper (in fact, they are all finite maps), so this causes no problems.

There is also an equivariant version of all this, and as a consequence, everything works for equivariant characteristic classes, too (or if you don't want to believe this, just repeat the same arguments in all the different contexts).

These transformations going from the relative Grothendieck ring to (co)homology or K-theory also forget a lot of information, which means that we have a chance to get nicer formulas than in the universal setting.

Relative Adams operations, revisited

Recall that $\mathcal{A}_m[f]$ is defined by the equation

$$\sum_{n\geq 0}[\mathsf{Sym}^n f]t^n \;=\; \exp\left[\sum_{m\geq 1}\frac{t^m}{m}\mathcal{A}_m[f]\right]$$

It is easy to see from this (just compute the logarithmic differential) that:

$$\mathcal{A}_m[f] = \sum_{\lambda\vdash m}(-1)^{\ell(\lambda)+1}\cdot\frac{m}{\ell(\lambda)}\cdot\binom{\ell(\lambda)}{e_1,e_2,\ldots,e_s}\cdot(\Psi_\lambda)_![\mathsf{Sym}^\lambda f]$$

Observation: $\Psi_\lambda : \mathsf{Sym}^\lambda X \to \mathsf{Sym}^{|\lambda|}X$ is a branched covering. We can subdivide $\mathsf{Sym}^\lambda X$ into "multi-discriminant strata" $X_{\boldsymbol{\mu}}$ (where $\boldsymbol{\mu}\vdash\lambda$ runs over the vector partitions of λ) s.t. Ψ_λ is unramified on each stratum.

This becomes interesting because the CSM class cannot distinguish between covering maps of the same degree:

$$c_{\mathrm{SM}}[\Psi_\lambda|_{X_{\boldsymbol{\mu}}}] \;=\; \mathsf{deg}(\Psi_\lambda|_{X_{\boldsymbol{\mu}}})\cdot c_{\mathrm{SM}}[\Psi_\lambda(X_{\boldsymbol{\mu}})]$$

Hence, in principle we can compute $c_{\mathrm{SM}}(\mathcal{A}_m)$ using only combinatorics.

The multi-discriminant strata $X_{\boldsymbol{\mu}}$

Given a dimension vector $\underline{n} = (n_1, n_2, \dots, n_s)$, we associate to each vector partition $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k) \vdash \underline{n}$ a locally closed variety $X_{\boldsymbol{\mu}} \subset \mathrm{Sym}^{\underline{n}} X$ consisting of k disjoint points with multiplicity vectors $\boldsymbol{\mu}_i \in \mathbb{N}^s$. Let $\psi(\boldsymbol{\mu}) = (\sum \boldsymbol{\mu}_1, \sum \boldsymbol{\mu}_2, \dots)$ denote the partition of $\sum n_i$ we get by collapsing the vectors. Clearly $\Psi_{\underline{n}}(X_{\boldsymbol{\mu}}) = X_{\psi(\boldsymbol{\mu})}$.

As an example consider $\underline{n} = (4, 5, 2)$ and the following $\boldsymbol{\mu}$ with $\psi(\boldsymbol{\mu}) = (4, 4, 2, 1)$:

$\underline{n}$	4	5	2	
$\boldsymbol{\mu}_1$	1	3	0	4
$\boldsymbol{\mu}_2$	2	1	1	4
$\boldsymbol{\mu}_3$	1	0	1	2
$\boldsymbol{\mu}_4$	0	1	0	1

Lemma (trivial): $\Psi_{\underline{n}}$ is unramified on $X_{\boldsymbol{\mu}}$ and its degree is

$$\deg(\Psi_{\underline{n}}|_{X_{\boldsymbol{\mu}}}) = \frac{\mathrm{aut}(\psi(\boldsymbol{\mu}))}{\mathrm{aut}(\boldsymbol{\mu})}$$

where $\mathrm{aut}(\lambda) = e_1! \cdot e_2! \cdots e_r!$ is the number of automorphisms of the (possibly vector) partition λ.

Relative Adams operations for the CSM class

Theorem [Ohmoto]: $c_{\mathrm{SM}}[\mathcal{A}_m(X)] = \Omega_!^m c_{\mathrm{SM}}(X)$.

Proof. Based on the above, this is equivalent to the following purely combinatorial statement:

$$\sum_{\lambda\vdash m} (-1)^{\ell(\lambda)+1} \frac{m}{\ell(\lambda)} \binom{\ell(\lambda)}{\underline{e}} \sum_{\substack{\boldsymbol{\mu}\vdash\lambda \\ \psi(\boldsymbol{\mu})=\nu}} \frac{\mathsf{aut}(\nu)}{\mathsf{aut}(\boldsymbol{\mu})} = \begin{cases} 1, & \nu = (n) \\ 0, & \nu \neq (n) \end{cases}$$

This can be further rephrased as counting nonnegative integral $N \times k$ matrices A_{ij} with fixed row sums ν, strictly positive column sums (corresponding to λ), and weights $(-1)^{k+1}k^{-1}\Big(\sum_{i,j} A_{ij}\Big)$.

Which can be then further rewritten using g.f.-s as the equation

$$\frac{\partial}{\partial t} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(-1 + \prod_{i=1}^{\infty} \frac{1}{1-y_i t}\right)^k \stackrel{?}{=} \sum_{i=1}^{\infty} \frac{y_i}{1-y_i t}$$

the proof of which is a simple computation. QED.

Chern-Schwartz-MacPherson classes of X_λ

Let X be a quasi-projective variety. We have just seen Ohmoto's Macdonald-type formula for the (rational) CSM classes of the symmetric products of X:

$$\sum_{n\geq 0} c_{\mathrm{SM}}(\mathsf{Sym}^n X)\cdot t^n = \exp\left[\sum_{m\geq 1}\frac{t^m}{m}\cdot \Omega_!^m c_{\mathrm{SM}}(X)\right]$$

That is, $\mathcal{A}_m(X) = \Omega_!^m c_{\mathrm{SM}}(X)$. Hence we have:

$$\sum_{\lambda} c_{\mathrm{SM}}(X_\lambda)\underline{x}^\lambda = \exp\left[\sum_{m\geq 1}\frac{1}{m}\mathcal{P}(\underline{x}^m;\Omega_!^m)c_{\mathrm{SM}}(X)\right] = \exp\left[\mathcal{L}(\underline{x};\Omega_!)c_{\mathrm{SM}}(X)\right]$$

Note: As before, the $\Omega_!$ substitutions are understood in the umbral sense, and the exponentiation is in the ring

$$\Lambda_H(X;\underline{x})\otimes\mathbb{Q} \;:=\; \widehat{\bigoplus_{\lambda}}\, \underline{x}^\lambda\cdot H_{2*}^{\mathrm{BM}}(\mathsf{Sym}^{|\lambda|}X;\mathbb{Q})$$

Hirzebruch classes

The (motivic) Hirzebruch class T_y is the homology version of the motivic Chern class mc_y. There are two variations, the normalized $T_y := \widetilde{\mathrm{td}}_y$ and the unnormalized $U_y := \mathrm{td}_y$ Hirzebruch class. For a vector bundle E with Chern roots α_i these are defined by

$$T_y(E) = \prod_i \left(\frac{\alpha_i(1+y)}{1 - e^{-\alpha_i(1+y)}} - \alpha_i y \right) \qquad U_y(E) = \prod_i \frac{\alpha_i(1+ye^{-\alpha_i})}{1 - e^{-\alpha_i}}$$

and for smooth varieties by the class of their tangent bundle.

These are related by

$$\begin{aligned} T_y[Y \subset X] &= \psi_{(1+y)} U_y[Y \subset X] \\ U_y[Y \subset X] &= \mathrm{ch}(\mathrm{mc}_y[Y \subset X]) \cdot \mathrm{td}(X) \end{aligned}$$

where ch is the Chern character, $\mathrm{td} = U_0 = T_0$ is the Todd class, and ψ_r acts on $H_{2k}^{\mathrm{BM}}(X)$ via multiplication by r^{-k}.

Hirzebruch classes of X_λ

In [CMSSY] the authors prove the following generalization of Ohmoto's CSM formula:

$$\sum_{n\geq 0} U_{-y}(\mathsf{Sym}^n[f]) \cdot t^n = \exp\left[\sum_{m\geq 1} \frac{t^m}{m} \cdot \psi_m\big(\Omega_!^m\, U_{-y^m}[f]\big)\right]$$

where ψ_m is what they call the m-th *homology Adams operation*, acting on $H_{2k}^{\mathrm{BM}}(X)$ by multiplication by m^{-k} (these should commute with $\Omega_!$, which does not change the homology degree).

From, this, we can conclude that the g.f. of the unnormalized motivic Hirzebruch classes $U_{-y}(X_\lambda)$ is

$$\sum_{\lambda} U_{-y}[X_\lambda] \cdot \underline{x}^\lambda = \exp\left[\sum_{m\geq 1} \frac{1}{m} \sum_{d\geq 1} P_d(\underline{x}^m) \cdot \Omega_!^{dm}\, \psi_m\, U_{-y^m}[X]\right]$$

Question: Is there an analogous formula for the mc_y classes?

Chern-Schwartz-MacPherson classes for $X = \mathbb{P}^1$

Consider $\mathbb{T}^2$-equivariant cohomology (GL_2 is the same), and equivariant CSM classes. We will deal with the ring $\Lambda_H(\mathbb{P}^1; t)$ by mapping it to a power series ring: Introduce the "umbral basis"

$$Q_{n,k} := 1^{\odot k} \odot u^{\odot(n-k)} = (\Psi_{1,1,\ldots,1})_! \, (1 \otimes \cdots \otimes 1 \otimes u \otimes \cdots \otimes u)$$

where $u = -c_1(L)$ and $L \to \mathbb{P}^1$ is the tautological line bundle. There is a simple a recursive formula for $Q_{n,k}$. Now we encode a class $\xi \in H^*_{\mathbb{T}^2}(\mathsf{Sym}^n\mathbb{P}^1)$ as:

$$\xi = \sum_{k=0}^{n} a_k Q_{n,k} \quad \longleftrightarrow \quad \hat{\xi} = h^n \sum_{k=0}^{n} a_k z^k$$

Clearly

$$Q_{n,k} \odot Q_{m,l} = Q_{n+m,k+l}$$

hence with this encoding, the ring multiplication of $\Lambda_H(\mathbb{P}^1; t)$ becomes simply the usual power series multiplication.

Chern-Schwartz-MacPherson classes for $\mathbb{P}^1$, page 2

With this encoding, we can write the (CSM of) $\mathcal{A}_m(\mathbb{P}^1)$ as

$$\mathcal{A}_m(\mathbb{P}^1) = h^m \cdot \left\{ \underbrace{(1+z\alpha)^m}_{\mathcal{A}_m([0:1])} + \underbrace{(1+z\beta)^m}_{\mathcal{A}_m([1:0])} + \underbrace{\frac{(1+z\alpha)^m - (1+z\beta)^m}{\alpha-\beta}}_{\mathcal{A}_m(\mathbb{C}^\times)} \right\}$$

It's easy to see that for $k \cdot \left[\mathsf{pt}_{[0:1]}\right] + l \cdot \left[\mathsf{pt}_{[1:0]}\right]$ we have

$$\Omega_!^d \left[h^{k+l}(1+z\alpha)^k(1+z\beta)^l\right] = h^{d(k+l)}(1+z\alpha)^{dk}(1+z\beta)^{dl}$$

In particular, $\Omega_!^d \mathcal{A}_m(\mathbb{P}^1) = \Omega_!^{dm}\, c_{\mathrm{SM}}[\mathbb{P}^1]$. As a consequence,

$$\mathcal{Z}(\mathbb{P}^1;h) := \sum_n [\mathsf{Sym}^n \mathbb{P}^1]\, h^n = \Big(1-(1+z\alpha)h\Big)^{-1-\frac{1}{\alpha-\beta}} \Big(1-(1+z\beta)h\Big)^{-1+\frac{1}{\alpha-\beta}}$$

$$\mathcal{C}(\mathbb{P}^1;h) := \sum_n [X_{(1^n)}]\, h^n = \Big(1+(1+z\alpha)h\Big)^{+1+\frac{1}{\alpha-\beta}} \Big(1+(1+z\beta)h\Big)^{+1-\frac{1}{\alpha-\beta}}$$

Chern-Schwartz-MacPherson classes for $\mathbb{P}^1$, page 3

Finally we can write down the gen. function of the classes $c_{\mathrm{SM}}[X_\lambda]$:

$$\mathcal{G}(\mathbb{P}^1;\underline{x}) := \sum_{\lambda} c_{\mathrm{SM}}[X_\lambda]\cdot \underline{x}^{\lambda} = \exp\left[\sum_{m\geq 1}\frac{1}{m}\sum_{d\geq 1} P_d(\underline{x}^m)\mathcal{A}_{dm}\right] =$$

$$= \left(1+\sum_{n=1}^{\infty}(1+z\alpha)^n x_n h^n\right)^{1+\frac{1}{\alpha-\beta}} \left(1+\sum_{n=1}^{\infty}(1+z\beta)^n x_n h^n\right)^{1+\frac{1}{\beta-\alpha}}$$

By differentiating and substituting zeros, we can get generating functions for $c_{\mathrm{SM}}[X_{(\mu,1^n)}]$. For example:

$$\sum_n h^{n+a} c_{\mathrm{SM}}[X_{(a,1^n)}] = \mathcal{C}(\mathbb{P}^1;h)\cdot \mathcal{R}_{(a)}(\mathbb{P}^1)$$

$$\mathcal{R}_{(a)}(\mathbb{P}^1) = \frac{h^a}{\alpha-\beta}\left\{\frac{(1+\alpha-\beta)\cdot(1+z\alpha)^a}{1+h(1+z\alpha)} - \frac{(1+\beta-\alpha)\cdot(1+z\beta)^a}{1+h(1+z\beta)}\right\}$$

Chern-Schwartz-MacPherson classes for $\mathbb{P}^1$, page 4

Remark: The $\mathcal{A}_m$ classes have nice formulas in the standard basis too: $\mathcal{A}_m(\mathbb{P}^1) = \mathcal{A}_m(\mathsf{pt}_{[1:0]}) + \mathcal{A}_m(\mathsf{pt}_{[0:1]}) + \mathcal{A}_m(\mathbb{C}^\times)$ where

$$\begin{aligned}
\mathcal{A}_m(\mathsf{pt}_{[1:0]}) &= \prod_{i=1}^{m}(u+(m-i)\alpha+i\beta) \\
\mathcal{A}_m(\mathsf{pt}_{[0:1]}) &= \prod_{i=0}^{m-1}(u+(m-i)\alpha+i\beta) \\
\mathcal{A}_m(\mathbb{C}^\times) &= m\cdot\prod_{i=1}^{m-1}(u+(m-i)\alpha+i\beta)
\end{aligned}$$

Compare this with the pushforward formula for $\Omega^d : \mathbb{P}^n \to \mathbb{P}^{nd}$:

$$\Omega^d_!\, f(u) = d^n \cdot f(u/d) \cdot \prod_{i\in I_{n,d}} (u+(dn-i)\alpha+i\beta)$$

$$I_{n,d} = \{\ i \ :\ 0 \le i \le nd \ \text{ and } \ d \nmid i\ \}$$

The "umbral basis" in the equiv. cohomology of $\mathrm{Sym}^n\mathbb{P}^1$

We can directly compute an ugly formula for the $Q_{n,k}$ classes:

Let $\mathrm{pt} = [0:1]$, then $A_j = \mathrm{pt}^{\times j} \times (\mathbb{P}^1)^{\times(n-j)} \subset (\mathbb{P}^1)^{\times n}$, and finally $B_j = \{[0,\dots,0,*,\dots,*]\} \subset \mathbb{P}^n$ be the codimension j hyperplane where the first j coordinates are zero; then for the fundamental classes we have $\pi_![A_j]_H = (n-j)! \cdot [B_j]_H$, thus

$$Q_{n,n-k} := \pi_!(u^{\otimes k} \otimes 1^{\otimes n-k}) = \pi_!\big[((u+\alpha)-\alpha)^{\otimes k} \otimes 1^{\otimes n-k}\big] =$$

$$= \sum_{j=0}^{k}(-\alpha)^{k-j}\,\pi_!\,\sigma_j(u_1+\alpha,\dots,u_k+\alpha) = \sum_{j=0}^{k}(-\alpha)^{k-j}\binom{k}{j}\pi_![A_j]_H =$$

$$= \sum_{j=0}^{k}(-\alpha)^{k-j}\binom{k}{j}(n-j)! \cdot \underbrace{\prod_{i=0}^{j-1}(u+(n-i)\alpha+i\beta)}_{[B_j]_H}$$

Unfortunately it's not even clear at first if this is symmetric in α, β!

The umbral basis in the equiv. cohomology, page 2

Lemma: $\widehat{P}_k(n) := Q_{n,n-k}/(n-k)!$ satisfies the recursion $\widehat{P}_{-1} = 0$,

$$\widehat{P}_0(n) = 1$$
$$\widehat{P}_{k+1}(n) = (u + k(\alpha+\beta)) \cdot \widehat{P}_k(n) + k(n-k+1)(\alpha\beta) \cdot \widehat{P}_{k-1}(n)$$

Proof. Rewrite the above as the (finite) hypergeometric sum

$$\widehat{P}_k(n) = (-\alpha)^k \frac{n!}{(n-k)!} \cdot {}_2F_1\left[\frac{u+n\alpha}{\beta-\alpha}\,, -k\,; -n\,; 1-\frac{\beta}{\alpha}\right]$$

The α, β symmetry can be seen by applying the Pfaff transformation

$${}_2F_1\left[a\,, b\,; c\,; z\right] = (1-z)^{-b} \cdot {}_2F_1\left[c-a\,, b\,; c\,; \frac{z}{z-1}\right]$$

and the recursion follows from the hypergeometric recurrence identity

$$\begin{aligned}(b-c) \cdot {}_2F_1\left[a\,, b-1\,; c\,; z\right] &= (2b-c+(a-b)z) \cdot {}_2F_1\left[a\,, b\,; c\,; z\right] \\ &+ \quad b(z-1) \cdot {}_2F_1\left[a\,, b+1\,; c\,; z\right]\end{aligned}$$

applied to $a = (u+n\alpha)/(\beta-\alpha)$, $b = -k$, $c = -n$ and $z = 1-\beta/\alpha$.

Fundamental classes of coincident root loci for $X = \mathbb{P}^1$

The CSM class contains the fundamental class as its lowest degree term; we can extract this information from the above g.f. to derive a g.f. for the (equivariant) fundamental classes $[X_\lambda]_H$.

For this, we have to note that the $Q_{n,k}$ classes are homogeneous of degree $n-k$. Hence, applying the substitution

$$h \mapsto qh \qquad z \mapsto q^{-1}z \qquad \alpha \mapsto q\alpha \qquad \beta \mapsto q\beta$$

the exponent of the new variable q will measure the cohomological degree. Next we can shift the lowest degree term, with

$$\mathrm{degree} \;=\; \mathsf{codim}(X_\lambda \subset \mathbb{P}^n) \;=\; |\lambda| - \ell(\lambda)$$

to degree zero by substituting $x_k \mapsto q^{1-k}x_k$; finally taking the limit $q \to 0$ we get the "umbral g.f. of fundamental classes"

$$\sum_\lambda [X_\lambda]_H \cdot h^{|\lambda|}\,\underline{x}^\lambda = \exp\left(\sum_{m=1}^{\infty} h^k x_k \,\frac{(1+z\alpha)^m - (1+z\beta)^m}{\alpha - \beta}\right)$$

Motivic Chern classes of coincident root loci for $\mathbb{P}^1$

Now consider the motivic Chern classes in $\mathbb{T}^2$-equivariant K-theory:

$$K^0_{\mathbb{T}^2}(\mathbb{P}^n) = \mathbb{Z}[L; X, X^{-1}, Y, Y^{-1}] \;/\; \prod_{i=0}^{n}(1 - X^{n-i}Y^iL)$$

Conventions: L is the class of the tautological line bundle over $\mathbb{P}^1$, and $c_1(X) = -\alpha$, $c_1(Y) = -\beta$.

Again, we want to introduce an “umbral basis” in $\Lambda_K(\mathbb{P}^1; t)$

$$h^n z^k \quad \longleftrightarrow \quad Q_{n,k} := L^{\odot k} \odot (1 - L)^{\odot(n-k)}$$

There is an algorithm which can recursively compute the $Q_{n,k}$ K-classes (as polynomials in L, X, Y) with reasonable efficiency.

Note: we chose a different looking basis than in the cohomology case. However, this choice is not very important; these particular choices are motivated by the fact that the resulting formulas are a bit nicer.

Motivic Chern classes for $\mathbb{P}^1$, page 2

The mc_y-Adams operations on $\mathbb{P}^1$ can be written as:

$$\mathcal{A}_m(\mathbb{P}^1) \;=\; \underbrace{(1+(1-X)z)^m}_{\mathcal{A}_m([0:1])} + \underbrace{(1+(1-Y)z)^m}_{\mathcal{A}_m([1:0])} +$$

$$+ \underbrace{(1-(-y)^m)\frac{-X^m(1+(1-X)z)^m + Y^m(1+(1-Y)z)^m}{X^m - Y^m}}_{\mathcal{A}_m(\mathbb{C}^\times)}$$

Remark: Unlike in the CSM case, $\Omega^d_! \mathcal{A}_m \neq \mathcal{A}_{dm}$.

However, the pushforwards along Ω^d of the linear combinations of the two fixpoints are simple, and with this, we can express $\Omega^d_! \mathcal{A}_m$ too:

$$\Omega^d_! \left[h^n \cdot (1+(1-X)z)^k \cdot (1+(1-Y)z)^{n-k} \right] \;=$$

$$=\; h^{dn} \cdot (1+(1-X)z)^{dk} \cdot (1+(1-Y)z)^{d(n-k)}$$

Motivic Chern classes for $\mathbb{P}^1$, page 3

Then the "relative zeta function" is

$$\mathcal{Z}(\mathbb{P}^1; t) = \sum_{n\geq 0} \mathrm{mc}_y(\mathsf{Sym}^n \mathbb{P}^1) \cdot t^n = \exp\left[\sum_{m\geq 1} \frac{t^m}{m} \mathcal{A}_m(\mathbb{P}^1)\right]$$

with the above $\mathcal{A}_m$, but I don't know a nice closed formula for this. In any case, we have

$$\mathcal{G}(\mathbb{P}^1\ \underline{x}) := \sum_{n=0}^{\infty} \sum_{\lambda\vdash n} \underline{x}^\lambda \mathrm{mc}_y[X_\lambda] = \exp\left[\sum_{m\geq 1}\sum_{d\geq 1} \frac{P_d(\underline{x}^m)}{m} \Omega_!^d \mathcal{A}_m(X)\right]$$

where we have an explicit (umbral) formula for $\Omega_!^d \mathcal{A}_m$.

Observation: Unlike in the CSM case, $\mathcal{A}_m \neq \Omega_!^m \mathcal{A}_1$. However, at least for $\mathbb{P}^1$, there is a "fudge factor" s.t. $\mathcal{A}_m = \Omega_!^m(\mathcal{A}_1 + \mathrm{fudge})|_{y\mapsto -(-y)^m}$:

$$\mathcal{A}_m(\mathbb{P}^1) \ = \ \Omega_!^m\left(\mathrm{mc}_{-(-y)^m}(\mathbb{P}^1) \ - \ (1-(-y)^m)\cdot L \cdot \frac{X^m Y - Y^m X}{X^m - Y^m}\right)$$

But I don't know what what does this means...

Closures of the strata

For the applications, we need the classes of the closures $\overline{X}_{(\mu,1^n)}$. Fortunately, this can be computed too, at least for any fixed μ.

Let $\mathcal{F}_{\mu|a}$ denote the g.f. of strata corresponding to (μ,ν) where $\mu_i > a$ and $\nu_i \leq a$:

$$\mathcal{F}_{\mu|a}(X;t) \;:=\; \sum_{\nu\leq a}[X_{(\mu,\nu)}]\cdot t^{|(\mu,\nu)|}.$$

Theorem (cf. [Vakil-Wood]): For any λ with $\lambda_i \geq 2$ there exists a finite set S_λ of pairs (μ,a) such that

$$\sum_{n\geq 0}\left[\overline{X}_{(\lambda,1^n)}\right]\cdot t^{|\lambda|+n} \;=\; \mathcal{Z}(X;t) - \sum_{(\mu,a)\in S_\lambda}\mathcal{F}_{\mu|a}(X;t)$$

Proof: Somewhat involved, but very elementary combinatorics.

These can be computed, too; the first step is noticing that:

$$\mathcal{F}_{\emptyset|a}(X) \;=\; \mathcal{Z}(X)\,/\,\Omega_!^{a+1}\mathcal{Z}(X).$$

Stability

In general it is an interesting question to ask what happens with $[X_{(\lambda,1^n)}]$ as $n \to \infty$.

There are different kinds of answers we can think of, for example:

- in [Vakil-Wood], they consider the limit of $\mathbb{L}^{-d(|\lambda|+n)}[X_{(\lambda,1^n)}]$ as $n \to \infty$ in the ring $\widehat{\mathcal{M}}_{\mathbb{L}} = \widehat{K_0(\mathsf{Var}_k)[\mathbb{L}^{-1}]}$, where $d = \dim(X)$;
- as before, we can consider the generating function $\mathcal{F}_\lambda(X;t) = \sum [X_{(\lambda,1^n)}] \cdot t^{|\lambda|+n}$;
- we can look at the "coefficients" of $[X_{(\lambda,1^n)}]$ expressed in a suitable basis, and look at their behaviour.

All this can be, in principle, handled in our framework (at least for some X-es).

Stability of coefficients of the CSM classes for $X = \mathbb{P}^1$

We claim that the coeffs. of the equivariant classes $c_{\mathrm{SM}}[X_{(\lambda,1^d)}]$ are polynomials of d (at least above a threshold $d_0(\lambda)$ depending on λ), which can be also computed algorithmically.

Recall the g.f. $\mathcal{G}(\mathbb{P}^1;\underline{x}) = \sum_\lambda c_{\mathrm{SM}}[X_\lambda]\underline{x}^\lambda$ for theses classes

$$\mathcal{G}(\mathbb{P}^1) = \left(1 + \sum_{n=1}^{\infty}(w+\alpha)^n x_n h^n\right)^{1+\frac{1}{\alpha-\beta}} \left(1 + \sum_{n=1}^{\infty}(w+\beta)^n x_n h^n\right)^{1+\frac{1}{\beta-\alpha}}$$

which is to be understood with the umbral subsitution

$$h^n w^k \longmapsto Q_{n,n-k} := 1^{\odot(n-k)} \odot u^{\odot k} = (n-k)! \cdot \widehat{P}_k(n)$$

where $\widehat{P}_k(n)$ are degree k homogenous polynomials in u, α, β, and also degree $\lfloor k/2 \rfloor$ polynomials in n, defined by the recursion $\widehat{P}_{-1} = 0$, $\widehat{P}_0 = 1$, and

$$\widehat{P}_{k+1} = (u + k(\alpha+\beta))\widehat{P}_k + k(n-k+1)(\alpha\beta)\widehat{P}_{k-1}$$

Stability of coefficients of the CSM classes for $\mathbb{P}^1$, page 2

The "stabil g.f." $\mathcal{F}_\lambda(\mathbb{P}^1;h)$ can be obtained by differentating:

$$\mathcal{F}_\lambda(\mathbb{P}^1;h) = \frac{1}{e_2!e_3!\cdots e_r!}\left[\frac{\partial^{e_2}}{\partial x_2^{e_2}}\frac{\partial^{e_3}}{\partial x_3^{e_3}}\cdots\frac{\partial^{e_r}}{\partial x_r^{e_r}}\,\mathcal{G}(\mathbb{P}^1;\underline{x};h)\right]_{x_1\mapsto 1,\, x_{\geq 2}\mapsto 0}$$

It shouldn't be hard to see that

$$\mathcal{F}_\lambda(\mathbb{P}^1;h) = \mathcal{F}_\emptyset(\mathbb{P}^1;h)\cdot\underbrace{\frac{h^{|\lambda|}\cdot\mathcal{R}_\lambda(h,w;\alpha,\beta)}{\big(1+h(w+\alpha)\big)^{\ell(\lambda)}\cdot\big(1+h(w+\beta)\big)^{\ell(\lambda)}}}_{\mathcal{Q}_\lambda(\mathbb{P}^1;h)}$$

where $\mathcal{R}_\lambda$ is a polynomial of h,w,α,β.

The idea is that this has a regular enough structure so that the coefficient of h^d in $\mathcal{F}_\lambda(h)$ (which is our target, as a function of d) can be analysed in enough details. However, this is a convoluted and not exactly an illuminating process... which I skip for now.

Application: classes of loci of μ-tangent lines

Consider a generic degree d hypersurface $\{f=0\} \subset \mathbb{P}^n = \mathbb{P}W^{n+1}$. For any μ with $\mu_i \geq 2$, we can consider the locus of μ-tangent lines $Z_\mu(f) \subset \mathsf{Gr}_2(W)$: for $\mu = (2)$ this is just the tangent lines; for $\mu = (3)$ the flex lines; for $\mu = (2,2)$ the bitangent lines; and so on.

Restricting $f \in \mathsf{Sym}^d W^*$ to the fibers of the tautological subbundle $K^2 \to \mathsf{Gr}_2(W)$ we get a section $\sigma : \mathsf{Gr}_2(W) \to \mathsf{Sym}^d K^*$, and the locus $Z_\mu(f)$ is the inverse image $\sigma^{-1}(\mathsf{cone}(X_{(\mu,1^{d-|\mu|})}))$.

By a theorem of Ohmoto, we have (assuming transversality of σ)

$$s_{\mathrm{SM}}\big[Z_\mu(f) \subset \mathsf{Gr}_2 W\big] = \sigma^* \, s_{\mathrm{SM}}\big[\mathsf{cone}(X_{(\mu,1^\bullet)}) \subset \mathsf{Sym}^d K^*\big]$$

where $s_{\mathrm{SM}}[Y \subset X] := c_{\mathrm{SM}}[Y \subset X] / c_{\mathrm{SM}}[X]$ are the Segre-SM classes.

Hence we can in principle compute the classes $c_{\mathrm{SM}}[Z_\mu(f)]$; these turn out to not depend on f (for generic enough f-s), and be polynomials in d (except for very small d-s).

Appendix: Sketch of proof of the main formula

The outline of the proof:

1. describe an algorithm calculating the classes of discriminant strata;
2. show that the generating function of the outputs of this algorithm satisfy some equations;
3. guess the formula for the generating function by actually running the algorithm and looking at the results;
4. show that the conjectured generating function satisfy the equations;
5. then we know that this is the unique solution, because the equations encode the algorithm.

I think that this is a fine example of *experimental mathematics*!

Notations

Fix some notations:

$$
\begin{aligned}
f \ :\ & Y \to X \\
\lambda &= (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0) = (1^{e_1}\, 2^{e_2}\, 3^{e_3} \ldots\, r^{e_r}) \\
\underline{e} &= (e_1, e_2, e_3, \ldots, e_r) \\
\underline{e}\sharp &= (1e_1, 2e_2, 3e_3, \ldots, re_r) \\
n &= 1e_1 + 2e_2 + 3e_3 + \cdots + re_r \\
\underline{x}^{\lambda} &= x_1^{e_1} x_2^{e_2} x_3^{e_3} \cdots x_r^{e_r} \\
\Omega^{123\ldots} &= \Omega^1 \times \Omega^2 \times \Omega^3 \times \cdots \times \Omega^r \\
\Omega^{\lambda} &= \Psi_{\underline{e}\sharp} \circ \Omega^{123\ldots} \\
\mathsf{Sym}^{\underline{e}} X &= \mathsf{Sym}^{e_1} X \times \mathsf{Sym}^{e_2} X \times \cdots \times \mathsf{Sym}^{e_r} X \\
D_X(\underline{e}) \subset \mathsf{Sym}^{\underline{e}} X &= \text{the set of } e_1 + e_2 + \cdots + e_r \text{ points,} \\
& \qquad\qquad \text{all disjoint, ``colored'' with } r \text{ colors}
\end{aligned}
$$

A (commutative) diagram of the maps appearing

$$
\begin{tikzcd}
 & \mathsf{Sym}^{\underline{e}} Y \arrow[rr, "\mathsf{Sym}^{\underline{e}} f"] \arrow[dd] & & \mathsf{Sym}^{\underline{e}} X \arrow[dd, "\Omega^{123\ldots}"'] \arrow[dddd, bend left, "\Omega^{\lambda}"] \\
D_Y(\underline{e}) \arrow[ur, "k_{\underline{e}}"] \arrow[urrr, "D_f(\underline{e})"] \arrow[rr, "\mathsf{Sym}^{\underline{e}} f|_{D_Y}"'] \arrow[dd, "\Omega^{123\ldots}"] \arrow[dddd, bend right, "\cong"'] & & D_X(\underline{e}) \arrow[ur, "k_{\underline{e}}"'] \arrow[dd] & \\
 & \mathsf{Sym}^{\underline{e}\sharp} Y \arrow[rr, "\mathsf{Sym}^{\underline{e}\sharp} f"] \arrow[dd] & & \mathsf{Sym}^{\underline{e}\sharp} X \arrow[dd, "\Psi"'] \\
\Omega^{123}(D_Y) \arrow[ur] \arrow[rr, "\mathsf{Sym}^{\underline{e}\sharp} f|_{\Omega^{123}(D_Y)}"] \arrow[dd, "\Psi"] & & \Omega^{123}(D_X) \arrow[ur] \arrow[dd] & \\
 & \mathsf{Sym}^{n} Y \arrow[rr, "\mathsf{Sym}^{n} f"] & & \mathsf{Sym}^{n} X \\
Y_\lambda \arrow[ur, "j_\lambda"] \arrow[urrr, "f_\lambda"] \arrow[rr, "\mathsf{Sym}^{n} f|_{Y_\lambda}"'] & & X_\lambda \arrow[ur, "j_\lambda"'] &
\end{tikzcd}
$$

The algorithm

The idea of the algorithm is to “cut-and-paste” (that is, to use the scissor relations) to recursively expand the classes of $[X_\lambda]$, or more generally $[f_\lambda]$.

Applying the following rewrite rules until confluence happens:

$$[f_\lambda] \rightsquigarrow \Omega^\lambda_![D_f(\underline{e})] \tag{1}$$

$$[D_f(p)] \rightsquigarrow [\mathsf{Sym}^p f] - \sum_{\lambda \neq (1^p)} [f_\lambda] \tag{2}$$

$$[D_f(p, \underline{n})] \rightsquigarrow [D_f(p) \times D_f(\underline{n})] - \sum_{k, \underline{d} \neq p, \underline{n}} (\Theta_{k,\underline{d},\underline{e}})_![D_f(k, \underline{d}, \underline{e})] \tag{3}$$

we can express any $[f_\lambda]$ as a linear combination of (various pushforwards of) $[\mathsf{Sym}^n f]$.

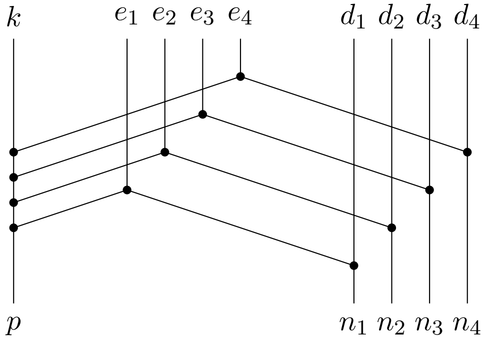
Here, in (2) the sum runs over partitions λ of p; and in (3) the sum runs over triples $(k, \underline{d}, \underline{e})$ such that $k + \sum e_i = p$ and $d_i + e_i = n_i$.

The circuit diagram of $\Theta_{k,\underline{d},\underline{e}} = \Psi \circ \Delta$

This picture illustrates (the pushforward along) the composite map

$$\Theta_{k,\underline{d},\underline{e}} = \Psi \circ \Delta \ : \ D(k, \underline{d}, \underline{e}) \to D(p, \underline{n}).$$

Read from top to bottom, splits (respectively merges) represent individual Δ-s (resp. Ψ-s). The numbers are the orders of the inidividual symmetric powers (number of points).



Another generating function

The proof uses a slightly different generating function:

$$\mathcal{H}(f;\underline{x}) \;:=\; \sum_{\underline{n}} [D_f(\underline{n})]\,\underline{x}^{\underline{n}} \;\in\; \Xi(X;\underline{x}) := \widehat{\bigoplus_{n\geq 1}}\,\underline{x}^{\underline{n}}\,K_0(\mathsf{Var}/\mathsf{Sym}^{\underline{n}}X).$$

The generating function of the coincident root loci can be recovered by

$$\mathcal{G}(f;\underline{x}) = \sum_{\lambda}[f_\lambda]\,\underline{x}^{\lambda} = \Psi_!\,\Omega_!^{123\dots}\,\mathcal{H}(f;\underline{x}).$$

Proposition: $\mathcal{H}_f(\underline{x}) := \mathcal{H}(f;\underline{x})$ satisfies the equations:

$$\mathcal{H}_f(t)\boxtimes\mathcal{H}_f(\underline{x}) = (\Theta_!\mathcal{H}_f)(t,\underline{x}) \qquad (*)$$

$$\mathcal{Z}_f(t) = (\Psi_!\Omega_!^{123\dots}\mathcal{H}_f)(t) \qquad (**)$$

and these have a unique solution.

What do these equations mean?

Equation (**) corresponds to the rewrite rule (2) in the algorithm, and (*) corrsponds to (3). The last remaining rule (1) simply connects the two generating functions $\mathcal{H}$ and $\mathcal{G}$.

The part which can cause confusion is $(\Theta_! \mathcal{H}_f)(t, \underline{x})$ on the RHS of (*). To understand this, consider first the individual maps $\Theta_{k,\underline{d},\underline{e}}$:

$$\Theta_{k,\underline{d},\underline{e}} \;:\; \mathsf{Sym}^k X \times \mathsf{Sym}^{\underline{d}} X \times \mathsf{Sym}^{\underline{e}} X \;\to\; \mathsf{Sym}^p X \times \mathsf{Sym}^{\underline{n}} X$$

where $p = k + \sum_i e_i$ and $n_i = d_i + e_i$. These individual maps can be collected into a big map

$$\Theta_! \;:\; \Xi(X; q, \underline{u}, \underline{v}) \to \Xi(X; t, \underline{x})$$

by linearly extending the components given by:

$$q^k\, \underline{u}^{\underline{d}}\, \underline{v}^{\underline{e}} \cdot [g] \;\longmapsto\; t^{k+\sum e_i}\, \underline{x}^{\underline{d}+\underline{e}} \cdot \Theta_{k,\underline{d},\underline{e}}[g] = t^k \underline{x}^{\underline{d}} (t\underline{x})^{\underline{e}} \cdot \Theta_{k,\underline{d},\underline{e}}[g]$$

Note that $\mathcal{H}_f(\underline{x})$ is *symmetric* in the variables x_i, hence it does not matter how we distribute the variables $q, \underline{u}, \underline{v}$!

Another picture of the map $\Theta_{k,\underline{d},\underline{e}} = \Psi \circ \Delta$

The key map $\Theta = \Psi \circ \Delta$, more precisely:

$$\Theta_{k,\underline{d},\underline{e}} \;:\; \mathrm{Sym}^{k,\underline{d},\underline{e}}X \;\to\; \mathrm{Sym}^{p,\underline{n}}X$$

(where $p = k + \sum_i e_i$ and $n_i = d_i + e_i$) is defined as the composition which first duplicates the $\mathrm{Sym}^{\underline{e}}X$ part, then merges one copy with the $\mathrm{Sym}^k X$ part and the other copy with the $\mathrm{Sym}^{\underline{d}}X$ part:

$$\begin{tikzcd}
\mathrm{Sym}^{k,\underline{d},\underline{e}}X \arrow[ddd, "\Theta_{k,\underline{d},\underline{e}}"'] & = & \mathrm{Sym}^k X \arrow[d, equal] & \times & & \mathrm{Sym}^{\underline{e}}X \arrow[d, "\Delta^2"] & & \times & \mathrm{Sym}^{\underline{d}}X \arrow[d, equal] \\
& & \mathrm{Sym}^k X \arrow[d, equal] & \times & (\mathrm{Sym}^{\underline{e}}X \arrow[d, equal] & \times & \mathrm{Sym}^{\underline{e}}X) \arrow[d, equal] & \times & \mathrm{Sym}^{\underline{d}}X \arrow[d, equal] \\
& & (\mathrm{Sym}^k X & \times \arrow[d, "\Psi_{k,\underline{e}}"'] & \mathrm{Sym}^{\underline{e}}X) & \times & (\mathrm{Sym}^{\underline{e}}X & \times \arrow[d, "\Psi_{e_i,d_i}"] & \mathrm{Sym}^{\underline{d}}X) \\
\mathrm{Sym}^{p,\underline{n}}X & = & & \mathrm{Sym}^{k+\sum e_i}X & & \times & & \mathrm{Sym}^{\underline{d}+\underline{e}}X &
\end{tikzcd}$$

The solution

Theorem:

$$\mathcal{H}(f;\underline{x}) = \exp\left[\sum_{m\geq 1}\frac{1}{m}\mathcal{P}(\underline{x}^m\Omega_!;1)(\Delta_!\mathcal{A}_m[f])\right]$$

Corollary:

$$\mathcal{G}(f;\underline{x}) = \Psi_!\,\Omega_!^{123}\,\mathcal{H}(f;\underline{x}) = \exp\left[\sum_{m\geq 1}\frac{1}{m}\mathcal{P}(\underline{x}^m;\Omega_!)(\mathcal{A}_m[f])\right]$$

Where the $\Omega_!$ are "umbral subsitutions": If

$$\mathcal{P}(\underline{x};h) = \sum_{\underline{n},d} a_{\underline{n},d}\cdot \underline{x}^{\underline{n}}h^d$$

then (remark: we always have $d = \sum i\cdot n_i$)

$$\begin{aligned}\mathcal{P}(\underline{x}^m\Omega_!;1)(\Delta_!\mathcal{A}_m[f]) &:= \sum a_{\underline{n},d}\cdot \underline{x}^{m\cdot\underline{n}}\cdot\Omega_!^{\underline{n}}\,\Delta_!\,\mathcal{A}_m[f]\\ \mathcal{P}(\underline{x}^m;\Omega_!)(\mathcal{A}_m[f]) &:= \sum a_{\underline{n},d}\cdot \underline{x}^{m\cdot\underline{n}}\cdot\Omega_!^{d}\mathcal{A}_m[f]\end{aligned}$$

A picture of these maps for $\lambda = (1^4, 2^1, 3^2)$

Here $\underline{n} = (4, 1, 2)$ and $d = \sum i \cdot n_i = 1 \cdot 4 + 2 \cdot 1 + 3 \cdot 2 = 12$:

$$
\begin{tikzcd}
\mathcal{A}_m \arrow[d, mapsto] \arrow[dddd, bend right=60, "\Omega_!^{12}"'] \\
\Delta_! \mathcal{A}_m \arrow[d, mapsto] \\
\Omega_!^{4,1,2} \Delta_! \mathcal{A}_m \arrow[d, mapsto, "\Omega_!^{123}"] \\
\Omega_!^{4,2,6} \Delta_! \mathcal{A}_m \arrow[d, mapsto, "\Psi_!"] \\
\Omega_!^{12} \mathcal{A}_m
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{Sym}^m X \arrow[dl] \arrow[d] \arrow[dr, "\Delta"] & \\
\mathsf{Sym}^m X \arrow[d, "\Omega^4"] & \mathsf{Sym}^m X \arrow[d, "\Omega^1 = \mathrm{id}"] & \mathsf{Sym}^m X \arrow[d, "\Omega^2"] \\
\mathsf{Sym}^{4m} X \arrow[d, "\Omega^1"] & \mathsf{Sym}^m X \arrow[d, "\Omega^2"] & \mathsf{Sym}^{2m} X \arrow[d, "\Omega^3"] \\
\mathsf{Sym}^{4m} X \arrow[dr] & \mathsf{Sym}^{2m} X \arrow[d] & \mathsf{Sym}^{6m} X \arrow[dl, "\Psi"] \\
 & \mathsf{Sym}^{12m} X &
\end{tikzcd}
$$

The key lemma

Recall the definitions $\mathcal{L}(\underline{x}) = \log(1 + \sum_i x_i)$ and

$$\mathcal{P}(\underline{x}) = \sum_{m \geq 1} \frac{\mu(m)}{m} \mathcal{L}(\underline{x}^m).$$

Lemma:

$$\mathcal{L}(t) + \mathcal{L}(\underline{x}) = \mathcal{L}(t, \underline{x}, t\underline{x}) \tag{1}$$

$$\mathcal{P}(t) + \mathcal{P}(\underline{x}) = \mathcal{P}(t, \underline{x}, t\underline{x}) \tag{2}$$

$$\mathcal{L}(t, t^2, t^3, \ldots, t^k) = \log(1 - t^{k+1}) - \log(1 - t) \tag{3}$$

$$\mathcal{P}(t, t^2, t^3, \ldots, t^k) = t - t^{k+1} \tag{4}$$

Proof. Exponentiating both sides of (1), we get the trivial identity:

$$(1+t)(1+x_1+x_2+\ldots) = 1+t+(x_1+x_2+\ldots)+(tx_1+tx_2+\ldots)$$

Then (2) follows from (1) by the definition of $\mathcal{P}$. (3) follows directly from the definition of $\mathcal{L}$; and (4) is a small computation from (3). Note: (3) and (4) are also valid for $k = \infty$.

Finishing the proof

It's enough to show that $\mathcal{H}$ below satisfies equations (*) and (**):

$$\mathcal{H}(f;\underline{x}) = \exp\left[\sum_{m\geq 1}\frac{1}{m}\mathcal{P}(\underline{x}^m\Omega_!;1)(\Delta_!\mathcal{A}_m[f])\right]$$

(**) follows easily from (4) with $k=\infty$; and (*) follows from (2):

First, we can take the logarithm. For this, observe that $\Theta_! = (\Psi\circ\Delta)_!$ commutes $\odot$ and hence with $\exp_\odot$ (because $\odot$ is "vertical" while Δ and Ψ are "horizontal"). Also, it's enough to consider only the m-th term of the sum (eq. (*) is "additive"):

$$\log(\mathcal{H}_m(t)) + \log(\mathcal{H}_m(\underline{x})) \stackrel{?}{=} \Theta_!\log(\mathcal{H}_m(q,\underline{u},\underline{v}))$$

By the definition of the candidate $\mathcal{H}$ this means:

$$\mathcal{P}(t^m\Omega_!)\mathcal{A}_m + \mathcal{P}(\underline{x}^m\Omega_!)(\Delta_!\mathcal{A}_m) \stackrel{?}{=} \Theta_!\Big\{\mathcal{P}\big((q,\underline{u},\underline{v})^m\,\Omega_!\big)\Big\}$$

Finishing the proof, page 2

Expanding further, what it really means is:

$$\mathcal{P}(t^m\Omega_!)\mathcal{A}_m = \sum_n a_n \cdot t^{mn} \cdot \Omega_!^n \mathcal{A}_m$$

$$\mathcal{P}(\underline{x}^m\Omega_!)(\Delta_!\mathcal{A}_m) = \sum_{\underline{n}} a_{\underline{n}} \cdot \underline{x}^{m\underline{n}} \cdot \Omega_!^{\underline{n}}(\Delta_!\mathcal{A}_m)$$

and finally, the interesting part:

$$\begin{aligned}
&\Theta_!\Big\{\mathcal{P}\big((q^m,\underline{u}^m,\underline{v}^m)\,\Omega_!\big)\Big\} = \\
&\quad = \sum_{k,\underline{d},\underline{e}} a_{k,\underline{d},\underline{e}} \cdot (\Theta_{k,\underline{d},\underline{e}})_!\Big\{(q^k\underline{u}^{\underline{d}}\underline{v}^{\underline{e}})^m)\Omega_!^{k,\underline{d},\underline{e}}(\Delta_!\mathcal{A}_m)\Big\} = \\
&\quad = \sum_{k,\underline{d},\underline{e}} a_{k,\underline{d},\underline{e}} \cdot \big(t^k\underline{x}^{\underline{d}}(t\underline{x})^{\underline{e}}\big)^m \cdot (\Theta_{k,\underline{d},\underline{e}})_!\,\Omega_!^{k,\underline{d},\underline{e}}\,(\Delta_!\mathcal{A}_m) = \\
&\quad = \sum_{k,\underline{d},\underline{e}} a_{k,\underline{d},\underline{e}} \cdot \big(t^k\underline{x}^{\underline{d}}(t\underline{x})^{\underline{e}}\big)^m \cdot \Omega_!^{k+\sum e_i,\underline{d}+\underline{e}}\,(\Delta_!\mathcal{A}_m) = \\
&\quad = \mathcal{P}\big((t,\underline{x},t\underline{x})^m\Omega_!\big)\mathcal{A}_m
\end{aligned}$$

using the simple fact that $\Theta_{k,\underline{d},\underline{e}} \circ \Omega^{k,\underline{d},\underline{e}} \circ \Delta = \Omega^{k+\sum e_i,\underline{d}+\underline{e}} \circ \Delta$.

Now you can apply (2) of the lemma. QED.

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