Equivariant CSM classes of coincident root loci

Balázs Kőműves

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Coincident root loci

Consider a degree m homogeneous binary form $f \in \mathrm{Sym}^m V^*$ on a two dimensional complex vector space $V \cong \mathbb{C}^2$. Taking the roots of the equation $f(z) = 0$ gives us a bijection between $\mathbb{P}\mathrm{Sym}^m V^*$ and the space of unordered multisets of m points in $\mathbb{P}^1 = \mathbb{P}V$.

This space is naturally stratified by specifying the multiplicities of the roots: Given a partition $\mu = (\mu_1, \ldots, \mu_n)$ of m, define $X_\mu \subset \mathbb{P}\mathrm{Sym}^m V^*$ be the set of forms which have n distinct roots, with multiplicities $\mu_1, \mu_2, \ldots, \mu_n$.

$$\mathbb{P}\mathrm{Sym}^m V^* = \coprod_{\mu \vdash m} X_\mu$$

We call the loci X_μ *coincident root loci*.[1]

[1] also called: multiple root loci, pejorative manifolds, discriminant strata, factorization manifolds, λ-Chow varieties, etc.

The goal: Compute $c_{\mathrm{SM}}(X_\mu)$

Our goal here is to compute the GL_2-equivariant Chern-Schwartz-MacPherson classes $c_{\mathrm{SM}}(X_\mu) \in H^*_{\mathsf{GL}_2}(\mathbb{P}\mathsf{Sym}^m V^*)$ of the loci X_μ.

Motivation:

- it is a very natural question (already studied by Hilbert, Schubert)
- it has lots of potential applications in enumerative geometry
- we want to see more worked-out examples anyway

Theorem (Hilbert):

$$\mathsf{deg}(\overline{X_\mu}) = \frac{n!}{\prod_i e_i!} \cdot \prod_i \mu_i$$

where $\mu = (\mu_1, \dots, \mu_n) = (1^{e_1}, 2^{e_2}, \dots, r^{e_r})$.

Previous results

- Schubert (1886): some enumerative consequences for some particular μ-s
- Hilbert (1887): the degree: $\deg(\overline{X_\mu}) \in \mathbb{N}_+$
- Aluffi (1998): the non-equivariant CSM:

$$c_{\mathrm{SM}}(X_\mu) \in H^*(\mathbb{P}^m) = \mathbb{Z}[h] \,/\, h^{m+1}$$

- Fehér, Némethi, Rimányi ($\sim$2003; published in 2006): the equivariant dual via localization

$$[\mathsf{cone}(X_\mu)] \in H^*_{\mathsf{GL}_2}(\mathbb{C}^{m+1}) = \mathbb{Z}[\alpha, \beta]^{S_2} = \mathbb{Z}[c_1, c_2]$$

- Kőműves (2003): the same equivariant dual via restriction equations

Remarks:

- The dual class is the lowest degree part of the CSM class
- The projective ($X_\mu \subset \mathbb{P}^m$) and the affine ($\mathsf{cone}(X_\mu) \subset \mathbb{C}^{m+1}$) versions are equivalent
- The localization and the restriction methods are secretly the same (in this particular case)

Software

There is a software package implementing all computations described here, and also those in previous works. It is available at:

```
http://hackage.haskell.org/package/coincident-root-loci
```

It is written in the Haskell programming language. Installation:

1. install the Haskell Platform (`http://www.haskell.org/platform`)
2. then type:

```
cabal upgrade
cabal install coincident-root-loci
```

Example (in the interactive shell `ghci`):

```
ghci> pretty $ convertGam chernToSchur
             $ umbralClosedCSM $ toPartition [4,3,1,1]

"1694520*s[3,2] + 13548870*s[3,3] + 1006344*s[4,1] + 19483740*s[4,2] +
93239748*s[4,3] + 190659015*s[4,4] + 181440*s[5] + 6904440*s[5,1] +
...
8280*g^7*s[1,1] + 6960*g^7*s[2] + 45*g^8 + 405*g^8*s[1] + 10*g^9"
```

Ambient CSM classes

We are always working in the following situation: $j : X \subset M$ is a possibly singular, G-invariant locally closed subvariety in a smooth ambient variety M.

With some abuse of notation, in this context by $c_{\mathrm{SM}}(X)$ we always mean the Poincaré dual of the pushforward of the CSM class from X to M:

$$\underbrace{c_{\mathrm{SM}}(X \subset M)}_{\text{our version}} := \mathrm{Dual}\big[j_* \underbrace{c_{\mathrm{SM}}(X)}_{\text{standard}}\big] \in H^*_G(M)$$

This seems to be the natural thing to do in our setting, when M is stratified by invariant subvarieties. It also fits better with the applications. Finally, it's much simpler to work in $H^*_G(M)$ which is typically very well understood. (Working in cohomology instead of homology is just personal preference).

Note that Aluffi also came to this conclusion, from different considerations.[2]

[2] P. Aluffi: Characteristic classes of singular varieties; Warsaw lecture notes

Projective vs. affine

We have three different versions of CSM classes here:

- projective, non-equivariant classes: $c_{\mathrm{SM}}(X_\mu \subset \mathbb{P}^m) \in H^*(\mathbb{P}^m)$
- projective, equivariant classes: $c_{\mathrm{SM}}^{\mathsf{equiv}}(X_\mu \subset \mathbb{P}^m) \in H^*_{\mathsf{GL}_2}(\mathbb{P}^m)$
- affine, equivariant classes: $c_{\mathrm{SM}}^{\mathsf{equiv}}(\mathsf{cone}(X_\mu) \subset \mathbb{C}^{m+1}) \in H^*(B\mathsf{GL}_2)$

They are related by the substitutions:

$$
\begin{aligned}
c_{\mathrm{SM}}(X_\mu) &= c_{\mathrm{SM}}^{\mathsf{equiv}}(X_\mu)\,|_{\{\alpha\mapsto 0,\ \beta\mapsto 0\}} \\
c_{\mathrm{SM}}^{\mathsf{equiv}}(\mathsf{cone}(X_\mu)) &= c_{\mathrm{SM}}^{\mathsf{equiv}}(X_\mu)\,|_{\{\gamma\mapsto 0\}} \\
c_{\mathrm{SM}}^{\mathsf{equiv}}(X_\mu) &= c_{\mathrm{SM}}^{\mathsf{equiv}}(\mathsf{cone}(X_\mu))\,|_{\{\alpha\mapsto\alpha+\gamma/m,\ \beta\mapsto\beta+\gamma/m\}}
\end{aligned}
$$

Note that in the affine case, there is an extra stratum $X_0 = \{0\} \subset \mathbb{C}^{m+1}$.
It's CSM class is:

$$
c_{\mathrm{SM}}(X_0) = \big[\{0\}\big] = \prod_{i=0}^{m} \big(\underbrace{i\alpha + (m-i)\beta}_{w_i}\big)
$$

Segre-SM classes and intersection theory

The Segre-SM classes, while seemingly just a simple variation:

$$s_{\mathrm{SM}}(X \subset M) = \frac{c_{\mathrm{SM}}(X)}{c(TM)}$$

are much more useful for doing intersection theory.

The reason for this is that they behave well with respect to pullback (and as a corollary, also wrt. intersection). In particular:

Theorem (Ohmoto): Given a G-representation W, an invariant subvariety $X \subset W$, a W-bundle $E \to B$ with classifying map $\varphi : B \to BG$, and a section $\sigma : B \to E$ transversal to X, we have

$$\underbrace{s_{\mathrm{SM}}(\sigma^{-1}(X_E \subset E) \subset B)}_{\text{non-equivariant}} = \varphi^* \underbrace{s_{\mathrm{SM}}(X \subset W)}_{\text{equivariant}}$$

Applications to enumerative geometry

The most straightforward application of this idea is the following: Given a generic degree d hypersurface $\mathcal{H} \subset \mathbb{P}^n$, intersecting it with any line $\mathbb{P}^1 \subset \mathbb{P}^n$ gives us d points on that line.

More precisely, if the hypersurface is defined by the equation $F = 0$ with $F \subset \mathrm{Sym}^d(\mathbb{C}^{n+1})^*$, then restricting F to the fibers of the tautological subbundle $K^2 \to \mathrm{Gr}_2\mathbb{C}^{n+1}$ gives us a section $\sigma = F|_K$ of the bundle $\mathrm{Sym}^d K^*$. Then $\sigma^{-1}(X_\mu)$ is the locus of lines in $\mathbb{P}^n$ which meet $\mathcal{H}$ with the prescribed contact multiplicities.

For example $\mu = (2, 1^{d-2})$ gives the set of tangent lines; $\mu = (3, 1^{d-3})$ the flex lines, $\mu = (2, 2, 1^{d-4})$ the bitangent lines, etc. The zero stratum gives the lines lying on $\mathcal{H}$.

Already the equivariant dual allows us to answer questions like: How many $4\times$ tangent lines are to a generic degree d surface in $\mathbb{P}^3$?

The geometric situation

$$
\begin{array}{ccccccc}
\mathcal{U}_n & \subset & \overbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}^{\mathcal{M}^n} & \xrightarrow{\Delta^\mu} & \overbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1}^{\mathcal{M}^m} & \supset & Y_\mu \\
 & & & & \Big\downarrow \pi & & \\
 & & & & \mathbb{P}\mathsf{Sym}^m V^* = \mathbb{P}^m & \supset & X_\mu
\end{array}
$$

Notations:

- n is the number of parts of the partition $(\mu_1, \ldots, \mu_n)$
- $m \geq n$ is the total number of points $m = \mu_1 + \mu_2 + \cdots + \mu_n$
- $\mathcal{M}^k = \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, product of k projective lines
- $\mathcal{U}_k = \{(z_1, \ldots, z_k) \mid z_i \neq z_j\} \subset \mathcal{M}^k$ is the set of distinct points
- Δ^μ is the diagonal map corresponding to μ
- π simply forgets the order of points.

Clearly, we have $X_\mu = \pi(\Delta^\mu(\mathcal{U}_n))$.

Computation strategies

Strategy I:

- Observe that $\mathcal{M}^n$ is a smooth blow-up of $\overline{X_\mu}$. Taking the pushforward of $c(\mathcal{M}^n)$ we get a linear combination of the $c_{\mathrm{SM}}(X_\lambda)$ classes, where $X_\lambda \subset \overline{X_\mu}$ (equivalently, μ is a refinement of λ);
- Since the smallest stratum, $X_{(m)}$ is smooth (it's just the rational normal curve), we know its CSM class, and we can work out the rest recursively.

Strategy II:

- Solve the analogous problem in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ to get $c_{\mathrm{SM}}(\mathcal{U}_n)$;
- Compute the pushforward $c_{\mathrm{SM}}(X_\mu) = \frac{1}{\mathsf{aut}(\mu)} \cdot \pi_* \, \Delta_*^\mu \, c_{\mathrm{SM}}(\mathcal{U}_n)$.

Computation strategies, page 2

Both strategies work.

Unfortunately "Strategy I" requires a Möbius inversion for the poset defined by the closure relation between the strata: $\lambda \prec \mu$ if $X_\lambda \subset \overline{X_\mu}$. Combinatorially, this is the (inverse of the) *refinement poset* of partitions.

Apparently, these posets behave badly:[3] We don't even know the *signs* of the Mobius function in general! So "Strategy I" gives us a working algorithm, but it is slow, and gives no insight.

Hence, we will follow "Strategy II" instead. This is exactly the same strategy Aluffi[4] follows, we just work out the equivariant version here (which is rather more intricate).

[3] G. Ziegler: On the poset of partitions of an integer (1986)

[4] P. Aluffi: Char. classes of discriminants and enumerative geometry (1998)

Equivariant cohomology

We have to describe the cohomology rings of the spaces we work in.

First the projective lines:

$$H^*_{\mathsf{GL}_2}(\mathbb{P}^1) = \mathbb{Z}[\alpha,\beta;\xi]^{S_2} \,/\, \big(\ {\scriptstyle (\alpha+\xi)(\beta+\xi)=0}\ \big) = \mathbb{Z}[c_1,c_2;\xi] \,/\, \big(\ {\scriptstyle \xi^2+c_1\xi+c_2=0}\ \big)$$

$$H^*_{\mathsf{GL}_2}(\mathcal{M}^m) = \mathbb{Z}[\alpha,\beta;u_1,\dots,u_m]^{S_2} \,/\, \big(\ {\scriptstyle (\alpha+u_i)(\beta+u_i)=0\ :\ 1\le i\le m}\ \big)$$

where $\xi = -c_1(L)$ and $u_i = -c_1(L_i)$, the Chern classes of the tautological line bundles; c_1, c_2 are the generators of $H^*(B\mathsf{GL}_2)$: $c_i = c_i(K)$ for the tautological bundle $K^2 \to \mathsf{Gr}_2(\mathbb{C}^\infty) = B\mathsf{GL}_2$; and the Chern roots α,β via the splitting principle: $c_1 = \alpha+\beta$ and $c_2 = \alpha\beta$.

More generally, given a representation W (in our case $W = \mathsf{Sym}^m V^*$):

$$H^*_{\mathsf{GL}_2}(\mathbb{P}W) = \mathbb{Z}[\alpha,\beta;\gamma]^{S_2} \,/\, \big(\ {\scriptstyle \prod_i (w_i+\gamma)=0}\ \big)$$

where $w_i \in H^*_{\mathbb{T}^2}(\mathsf{pt}) = \mathbb{Z}[\alpha,\beta]$ are the *weights* of the representation. In our case $w_i = \pm\big((n-i)\alpha + i\beta\big)$ for $0 \le i \le n$.

A warning about signs

There are several sign choices to be made here:

- $\xi = \pm c_1(L^1) \in H^*(\mathbb{P}^m)$
- $\alpha + \beta = c_1 = \pm c_1(K^2) \in H^*(B\mathsf{GL}_2)$
- which representation to use: $\mathsf{Sym}^m V^2$ or $\mathsf{Sym}^m V^{2*}$

For the first two, our choices are $\xi = -c_1(L)$ and $c_1 = +c_1(K)$.

The third one is more confusing, because there are *canonical* isomorphisms between $\mathbb{P}^1 \cong \mathbb{P}^{1*}$ and $\mathbb{P}\mathsf{Sym}^m V^2 \cong \mathbb{P}\mathsf{Sym}^m V^{2*}$.

For brevity, we will abuse the notation and pretend we are working with $\mathsf{Sym}^m V$ instead of $\mathsf{Sym}^m V^*$. This is not important except for the applications (and for positivity), and we can just put back the signs at that point.

The pushforward along the diagonal maps

First, consider the small diagonal $\Delta^k : \mathbb{P}^1 \to \mathcal{M}^k$:

$$\Delta^k(z) = (z, z, \dots, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 = \mathcal{M}^k$$

Lemma:

$$\begin{aligned} \Delta_* 1 &= \sum_{j=0}^{m-1} \sigma_{m-1-j}(\mathbf{u}) \cdot \tau_j(\alpha, \beta) \\ \Delta_* \xi &= -\alpha\beta \cdot \sum_{j=0}^{m} \sigma_{m-j}(\mathbf{u}) \cdot \tau_{j-2}(\alpha, \beta) \end{aligned}$$

where τ_k is defined by

$$\tau_k(\alpha, \beta) = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} = \begin{cases} \sum_{i=0}^{k} \alpha^{k-i}\beta^i, & k \geq 0 \\ 0, & k = -1 \\ -\frac{1}{\alpha\beta}, & k = -2 \end{cases}$$

The pushforward along the diagonal maps, page 2

The general diagonal map Δ^μ is simply assembled from copies of Δ^k with $k = \mu_i$.

Sketch of proof of the Lemma: Clearly we have

$$\xi = -c_1(L) = -c_1(\Delta^* L_i) = \Delta^*(-c_1(L_i)) = \Delta^* u_i,$$

and thus from the adjunction formula:

$$B = \Delta_* \xi = \Delta_*(\xi \cdot 1) = \Delta_*(\Delta^* u_i \cdot 1) = u_i \cdot \Delta_* 1 = u_i \cdot A.$$

The left-hand side is independent of i, and it turns out that there is a unique pair of polynomials A and B (up to a scalar factor) of the right degree satisfying this equation.

Remark: τ_k satisfies the recurrence

$$\tau_k = (\alpha + \beta)\tau_{k-1} - (\alpha\beta)\tau_{k-2} = c_1 \cdot \tau_{k-1} - c_2 \cdot \tau_{k-2}.$$

The space of n-tuples of points in $\mathbb{P}^1$

Consider the set of distinct points

$$\mathcal{U}_n = \{\,(z_1,\dots,z_n) \in \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \ :\ z_i \neq z_j\,\}$$

and more generally, for a *set partition* $\varrho \in \mathcal{P}(n)$ of $\{1,\dots,n\}$,

$$\begin{aligned} Y_\varrho = \{\,(z_1,\dots,z_n) \in \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \ :& \\ :\ z_i = z_j \text{ iff } i,j \in A \text{ for } A \in \varrho\,\}& \end{aligned}$$

Note: $\mathcal{U}_n$ corresponds to the set partition $\{\{1\},\dots,\{n\}\}$.

This is completely analogous to the situation with unordered points, but we have set partitions instead of partitions.

Obs.: $Y_\varrho = \Delta^\varrho(\mathcal{U}_k)$ where $k = \ell(\varrho)$ is the number of parts of ϱ.

Computing $c_{\mathrm{SM}}(\mathcal{U}_n)$

Observe that Y_ϱ stratifies the space $\mathcal{M}^n = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, hence we have

$$c(T\mathcal{M}^n) = \sum_{\varrho} c_{\mathrm{SM}}(Y_\varrho) = c_{\mathrm{SM}}(\mathcal{U}_n) + \sum_{\ell(\varrho)<n} \Delta_*^\varrho(c_{\mathrm{SM}}(\mathcal{U}_{\ell(\varrho)}))$$

From this, we can compute $c_{\mathrm{SM}}(\mathcal{U}_n)$ recursively, since we know that $c(T\mathbb{P}^1) = 1 + \alpha + \beta + 2\xi$.

For $n = 1, 2, 3$, they are:

$$c_{\mathrm{SM}}(\mathcal{U}^1) = 1 + \alpha + \beta + 2u_1$$
$$c_{\mathrm{SM}}(\mathcal{U}^2) = 1 + \alpha + \beta + 2\alpha\beta + (u_1 + u_2)(1 + \alpha + \beta) + 2u_1u_2$$
$$c_{\mathrm{SM}}(\mathcal{U}^3) = 1 - \alpha^2 - \beta^2 + 2\alpha\beta$$

A formula for $c_{\mathrm{SM}}(\mathcal{U}_n)$

Theorem: For $n \geq 1$, we have

$$\boxed{c_{\mathrm{SM}}(\mathcal{U}_n) = q^3 \cdot (q - u_1 - u_2 - \cdots - u_n)^{n-3}}$$

after the "umbral substitution" $q^k \mapsto Q_k$, where Q_k is defined by the recurrence:

$$\begin{aligned} Q_0 &= 1 \\ Q_{k+1} &= \big(1 - (k-1)(\alpha + \beta)\big) \cdot Q_k - k(k-3) \cdot \alpha\beta \cdot Q_{k-1} \end{aligned}$$

This to be understood in the cohomology ring, where $u_i^2 = 0$.

This is an "umbral q-deformation" of Aluffi's formula for the non-equivariant case (which is the same with $q = 1$).

Lemma: The coefficients of Q_k are polynomials in k; thus we can define a "stable" $Q_\infty \in \mathbb{Q}[k][[c_1, c_2]]$

Sketch of proof

Clearly $c_{\mathrm{SM}}(\mathcal{U}_n)$ must be symmetric in $u_1, \ldots, u_n$, hence

$$c_{\mathrm{SM}}(\mathcal{U}_n) = \sum_{i=0}^{n} \sigma_i(\mathbf{u}) \cdot p_{n,i}(\alpha, \beta)$$

for some polynomials $p_{n,k}$.

Consider the projection maps $\vartheta : \mathcal{M}^n \to \mathcal{M}^{n-1}$ which simply forgets the last coordinate. Clearly $\vartheta(\mathcal{U}_n) = \mathcal{U}_{n-1}$, thus

$$\vartheta_* \, c_{\mathrm{SM}}(\mathcal{U}_n) = \chi(\vartheta^{-1}(\mathsf{pt})) \cdot c_{\mathrm{SM}}(\mathcal{U}_{n-1})$$

where the fibrum $\vartheta^{-1}(z_1, \ldots, z_{n-1})$ is $\mathbb{P}^1$ minus those points, having Euler characteristics $\chi = 2 - (n-1) = 3 - n$.

It's easy to show that ϑ_* simply extracts the coefficient of u_n, which shows how $p_{n,i}$ depends on n.

Sketch of proof, page 2

It follows that $c_{\mathrm{SM}}(\mathcal{U}_n)$ has the following form (for $n \geq 3$):

$$c_{\mathrm{SM}}(\mathcal{U}_n) = \sum_{i=0}^{n-3} (-1)^i \cdot \frac{(n-3)!}{(n-3-i)!} \cdot \sigma_i(\mathbf{u}) \cdot Q_{n-i}(\alpha, \beta)$$

for some Q_k (not depending on n).

To understand Q_k, decompose $\mathcal{U}_n \times \mathbb{P}^1$ according which (if any) of the points z_i the new point $z_{n+1} \in \mathbb{P}^1$ coincides with:

$$\mathcal{U}_n \times \mathbb{P}^1 = \mathcal{U}_{n+1} \cup \coprod_{i=1}^{n} \Delta^{(i)}(\mathcal{U}_n)$$

where $\Delta^{(i)}$ duplicates the i-th point, so that $z_i = z_{n+1}$ in the image.

Take the CSM of this equation; some more computation with that results the earlier recurrence.

The pushforward along the order forgetting map

Let $\pi : \mathcal{M}^m \to \mathbb{P}^m$ the order-forgetting map. This is a degree $m!$ finite map.

Because of symmetry reasons, π_* is fully determined by the polynomials $P_k(m)$ for $0 \leq k \leq m$:

$$P_k(m) := \pi_*(u_1 u_2 \cdots u_k) = \pi_*(u_{\sigma(1)} \cdots u_{\sigma(k)}) \in \mathbb{Z}[\alpha, \beta; \gamma]^{S_2}$$

These can be computed recursively by considering subspaces of the form

$$Z_{k,l} = \underbrace{\{0\} \times \cdots \times \{0\}}_{k \text{ times}} \times \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{m-k-l \text{ times}} \times \underbrace{\{\infty\} \times \cdots \times \{\infty\}}_{l \text{ times}} \subset \mathcal{M}^m$$

Lemma: $P_k(m) = (m-k)! \cdot \widehat{P}_k(m)$ where $\widehat{P}_k$ satisfies the recurrence

$$\begin{aligned} \widehat{P}_0(m) &= 1 \\ \widehat{P}_{k+1}(m) &= \big(\gamma + k(\alpha+\beta)\big) \cdot \widehat{P}_k(m) + k(m-k+1) \cdot \alpha\beta \cdot \widehat{P}_{k-1}(m) \end{aligned}$$

Observation: $\widehat{P}_k$ is a homogeneous degree k polynomial in α, β, γ; furthermore, the coefficients of $\widehat{P}_k(m)$ are polynomials in m.

The umbral formula for $c_{\mathrm{SM}}(X_\mu)$

Theorem: Define the polynomial $\Theta(k)$ by the formula:

$$\Theta(k) = \frac{(\beta+q)(\alpha+t)^k - (\alpha+q)(\beta+t)^k}{(\alpha-\beta)} \in \mathbb{Z}[\alpha,\beta;t,q]$$

then

$$\boxed{c_{\mathrm{SM}}(X_\mu) = \frac{1}{\mathsf{aut}(\mu)} \prod_{i=1}^{n} \Theta(\mu_i)}$$

after the umbral substitution

$$\begin{aligned} t^j &\longmapsto P_j(m) = (m-j)! \cdot \widehat{P}_j(m) \\ q^k &\longmapsto Q_k \cdot \underbrace{(n-3)(n-4)\cdots(k-4)}_{\text{falling factorial } (n-3)_{(n-k)}} \end{aligned}$$

Here $\mathsf{aut}(\mu) = e_1! \cdot e_2! \cdots e_r!$ where $\mu = (1^{e_1}, 2^{e_2}, \ldots, r^{e_r})$.

Stability

It's a natural question, and also important for applications, to consider the family of partitions $(\mu, 1^d)$ for $d \in \mathbb{N}$. Note that codim$(X_{\mu,1^d})$ does not depend on d.

Theorem: Assuming that $n_0 = \ell(\mu) \geq 3$, the coefficients of $c_{\mathrm{SM}}(\text{cone}(X_{\mu,1^d}))$ are polynomials in d (in any of the three $\mathbb{Z}$-module bases $\alpha^i\beta^j$, $c_1^e c_2^f$ or $s_{a,b}$).

Furthermore the degrees of these polynomials are bounded by:

- deg$(p_{e,f}(d)) \leq 2e + 3f$ for the coefficient of $c_1^e c_2^f$
- deg$(p_{i,j}(d)) \leq 2(i + j)$ for the coefficient of $\alpha^i\beta^j$
- deg$(p_{a,b}(d)) \leq 2(a + b)$ for the coefficient of $s_{a,b}$

Hence, we can interpolate the coefficient polynomials from the first few values (which we can compute with software).

Stability, sketch of proof, page 1

Step 1: The coeffs of Q_k are polynomials in k, with the same degree bounds.

Denoting the coeff. of $c_1^i c_2^j$ in Q_k by $q_{ij}(k)$, we can rewrite the recurrence as:

$$\underbrace{q_{ij}(k+1) - q_{ij}(k)}_{\Delta_{ij}(k)} = -(k-1)\cdot q_{i-1,j}(k) - k(k-3)\cdot q_{i,j-1}(k-1)$$

from which the statement follows by induction on i, j:

$$\begin{aligned} q_{ij}(k) &= q_{ij}(0) + \sum_{r=0}^{k-1} \Delta_{ij}(r) \\ &= q_{ij}(0) - \sum_{r=0}^{k-1} \underbrace{(r-1)\cdot q_{i-1,j}(r) + r(r-3)\cdot q_{i,j-1}(r-1)}_{\text{polynomial in } r} \end{aligned}$$

The degree bound follows (again by induction) from:

$$\deg(q_{ij}) = 1 + \max\Big\{ \underbrace{2(i-1)+3j}_{\deg(q_{i-1,j})} + 1 \,,\, \underbrace{2i + 3(j-1)}_{\deg(q_{i,j-1})} + 2 \Big\} = 2i + 3j$$

Stability, sketch of proof, page 2

Step 2: Observe that $\Theta(1) = q - t$, hence (assuming $1 \notin \mu$):

$$c_{\mathrm{SM}}(X_{\mu,1^d}) = \frac{1}{d!} \cdot c_{\mathrm{SM}}(X_\mu) \cdot (q-t)^d$$

Considering a single term $c_1^e c_2^f t^a q^b$ in $c_{\mathrm{SM}}(X_\mu)$, that will become

$$\frac{1}{d!} \cdot c1^e c2^f \cdot t^a q^b \cdot (q-t)^d = \frac{1}{d!} \cdot c1^e c2^f \cdot \sum_{i=0}^{d} (-1)^i \binom{d}{i} t^{i+a} q^{d-i+b}$$

After the substitution $q^k \mapsto (n-3)_{(n-k)} \cdot Q_k$ and $t^j \mapsto (m-j)! \cdot \widehat{P}_j(m)$:

$$c1^e c2^f \cdot \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \widehat{P}_i(m_0+d) \cdot \frac{(m_0-a+d-i)!}{(d-i)!} \cdot Q_{d-i+b} \cdot (n_0-3+d)_{(n_0-b+i)}$$

To finish the proof, stare at this formula for a long time, and also consider very carefully what happens when $i > d$...

Stability for the Segre-SM classes

Recall that

$$c(\mathsf{Sym}^m\mathbb{C}^2) = \prod_{i=0}^{m} \big(1 + \underbrace{i\alpha + (m-i)\beta}_{w_i}\big)$$

Lemma: The coefficients of $c(\mathsf{Sym}^m\mathbb{C}^2)$ are polynomials in m, with the usual degree bounds: $2e+3f$ for $c_1^e c_2^f$ and $2(i+j)$ for $\alpha^i\beta^j$ or $s_{i,j}$.

Remark: as $c(\mathsf{Sym}^m\mathbb{C}^2) = c_{\mathrm{SM}}(\mathsf{cone}(\overline{X_{1^m}}))$, this is not too surprising.

Lemma: The same is true for the inverse $\frac{1}{c(\mathsf{Sym}^m\mathbb{C}^2)}$.

Remark: This is again not too surprising, as we have the duality:

$$c(\mathsf{Sym}^m\mathbb{C}^2) = \sum_{k=0}^{m+1} e_k(\mathbf{w}) \qquad\qquad \frac{1}{c(\mathsf{Sym}^m\mathbb{C}^2)} = \sum_{k=0}^{\infty} (-1)^k h_k(\mathbf{w})$$

where e_k and h_k are the elementary resp. complete symmetric polynomials. It also follows from a direct power series inversion argument.

Corollary: The same is also true for the Segre-SM classes $s_{\mathrm{SM}}(X_\mu)$.

Positivity of Segre-SM classes

Conjecture: Depending on sign conventions, the Schur-coefficients of the Segre-SM classes $s_{\mathrm{SM}}(X_\mu)$ of the open strata (for $m \geq 2$), have either:

- alternating signs, starting with a positive sign at degree $\operatorname{codim}(X_\mu)$;
- are fully positive or fully negative, depending on the parity of $\operatorname{codim}(X_\mu)$.

Remark: Obviously they cannot be just simply positive, as we have

$$1 = s_{\mathrm{SM}}(\mathbb{P}^m) = \sum_{\mu \vdash m} s_{\mathrm{SM}}(X_\mu)$$

Conjecture: For $m \geq 2$, the Segre-SM classes are also alternating linear combinations of the CSM classes $c_{\mathrm{SM}}(\mathbb{S}_{ij}^\circ)$ of Schubert cells $\mathbb{S}_{ij}^\circ \subset \mathrm{Gr}_2(\mathbb{C}^N)$.

It is known that $c_{\mathrm{SM}}(\mathbb{S}_{ij}^\circ)$ are Schur-positive[5]. Conjecture: The Schur polynomials s_{ij} can be written as alternating linear combinations of the $c_{\mathrm{SM}}(\mathbb{S}_{ij}^\circ)$ classes.

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[5] P. Aluffi, C. Mihalcea: Chern classes of Schubert cells and varieties
J. Huh: Positivity of Chern classes of Schubert cells and varieties

Intersection theory

The best kept secret of CSM classes: For $A, B \subset M$ intersecting transversally, we should have

$$c_{\mathrm{SM}}(A \cap B) = \frac{c_{\mathrm{SM}}(A) \cdot c_{\mathrm{SM}}(B)}{c(M)}$$

"Proof": $\quad s_{\mathrm{SM}}(A \cap B) = s_{\mathrm{SM}}(\Delta^{-1}(A \times B)) = \Delta^* s_{\mathrm{SM}}(A \times B)$

Corollary (Aluffi[6]): The non-equivariant CSM class of $X \subset \mathbb{P}^m$ contains the same information as the numbers $\chi(X \cap H_1 \cap \cdots \cap H_k)$ for $k \geq 0$, where $H_i \subset \mathbb{P}^m$ are generic hyperplanes.

Proof: $c_{\mathrm{SM}}(X \cap \mathbb{P}^{m-k}) = c_{\mathrm{SM}}(X) \cdot s_{\mathrm{SM}}(\mathbb{P}^{m-k})$. It's easy to show that $s_{\mathrm{SM}}(H_i) = \frac{h}{1+h}$, hence

$$s_{\mathrm{SM}}(\mathbb{P}^{m-k} \subset \mathbb{P}^m) = \frac{h^k}{(1+h)^k} = h^k \cdot \sum_{i=0}^{\infty} (-1)^i \cdot h^i \cdot \binom{i+k-1}{k-1}$$

[6] P. Aluffi: Euler chars. of general linear sections and poly. Chern classes

Closure of the strata

For the applications, we usually want the CSM or Segre-SM classes of the *closure* $\overline{X_\mu}$ of the strata X_μ. For any *concrete* partition μ, this is easy to compute:

$$c_{\mathrm{SM}}(\overline{X_\mu}) = \sum_{\lambda \prec \mu} c_{\mathrm{SM}}(X_\mu)$$

Unfortunately, we don't have a nice general *formula* for it (and we don't really expect one).

For the applications, we need $c_{\mathrm{SM}}(\overline{X_{\mu,1^d}})$, and that's a problem. However, we can express at least some simple cases. Introduce the shorthand $X_{[\mu]}$ for $X_{\mu,1,1,\dots}$; then we have (set theoretically):

$$\begin{aligned}
\overline{X_{[1]}} &= \mathbb{P}^m \\
\overline{X_{[2]}} &= \overline{X_{[1]}} - X_{[1]} \\
\overline{X_{[2,2]}} &= \overline{X_{[2]}} - X_{[2]} - X_{[3]} \\
\overline{X_{[2,2,2]}} &= \overline{X_{[2,2]}} - X_{[4]} - X_{[2,2]} - X_{[3,2]} - X_{[3,3]} - X_{[5,1,1]} \\
\overline{X_{[3]}} &= \overline{X_{[2]}} - X_{[2,2]} - X_{[2,2,2]} - X_{[2,2,2,2]} - \cdots = \overline{X_{[2]}} - \coprod_{k\geq 1} X_{[2^k]}
\end{aligned}$$

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$$\begin{aligned}
\overline{X_{[1]}} &= \mathbb{P}^m \\
\overline{X_{[2]}} &= \overline{X_{[1]}} - X_{[1]} \\
\overline{X_{[2,2]}} &= \overline{X_{[2]}} - X_{[2]} - X_{[3]} \\
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\overline{X_{[3]}} &= \overline{X_{[2]}} - X_{[2,2]} - X_{[2,2,2]} - X_{[2,2,2,2]} - \cdots = \overline{X_{[2]}} - \coprod_{k\geq 1} X_{[2^k]} \\
\overline{X_{[3,2]}} &= \overline{X_{[3]}} - X_{[3]} - X_{[4]} \\
\overline{X_{[k,2]}} &= \overline{X_{[k]}} - X_{[k]} - X_{[k+1]} \\
\overline{X_{[4]}} &= \overline{X_{[3]}} - \coprod_{k\geq 1}\coprod_{j\geq 0} X_{[3^k,2^j]} \\
\overline{X_{[3,3]}} &= \overline{X_{[3,2]}} - X_{[5]} - \coprod_{j\geq 1}\left(X_{[3,2^j]} \cup X_{[4,2^j]} \cup X_{[5,2^j]}\right) \\
\overline{X_{[5]}} &= \overline{X_{[4]}} - \coprod_{k\geq 1}\coprod_{j\geq 0}\coprod_{i\geq 0} X_{[4^k,3^j,2^i]}
\end{aligned}$$

Even though there are potentially infinite sums appearing here, in each codimension the sums are finite. Since the codimension is a lower bound for the degree of terms in the CSM, it follows that the stability is also true for these classes.

The dual curve of a generic plane curve

A simple application is $\mu = (2, 1^d - 2)$; in this case $\overline{X_\mu}$ gives the variety of lines tangent to generic hypersurface in $\mathbb{P}^n$. For $n = 2$ we get the dual curve $\check{C}$ of a generic plane curve C.

Calculation: For a generic plane curve C of degree $d \geq 2$

$$c_{\mathrm{SM}}(\check{C}) = c(\mathbb{P}^2) \cdot \varphi^* s_{\mathrm{SM}}(\bar{X}_{2,1^{d-2}}) =$$

$$= \underbrace{d(d-1)}_{\deg(\check{C})} \cdot s_1 \; + \; \underbrace{\frac{1}{2}(d-3)d(4-d-d^2)}_{\chi(\check{C})} \cdot s_{1,1}$$

The degree is of course well known, and the Euler characteristic can be also checked using classical methods. For $d \geq 2$:

$$\chi(\check{C}_d) = 2, 0, -32, -130, -342, -728, -1360, -2322 \dots$$

The locus of hyperflex lines to a generic surface

Exactly the same way, can consider any tangency condition in any dimension.

For example consider the locus $\Phi_4 \subset \mathsf{Gr}_2(\mathbb{C}^4)$ of hyperflex lines to a generic surface $S \subset \mathbb{P}^3$ (meeting the surface at a point of order at least 4). This is a curve, and its CSM class is:

Calculation: For a generic surface $S \subset \mathbb{P}^3$ of degree $d \geq 4$

$$c_{\mathrm{SM}}(\Phi_4) = c(\mathbb{P}^3) \cdot \varphi^* s_{\mathrm{SM}}(\bar{X}_{4,1^{d-4}}) =$$

$$= \underbrace{2(d-3)d(3d-2)}_{[\Phi_4] \text{ or ``degree''}} \cdot s_{2,1} \; + \; \underbrace{2d(158d-186-31d^2)}_{\chi(\Phi_4)} \cdot s_{2,2}$$

The locus of bitangent lines to a generic surface

Another example is the locus $\Phi_{2,2} \subset \mathrm{Gr}_2(\mathbb{C}^4)$ of bitangent lines to a generic surface $S \subset \mathbb{P}^3$. This is itself a surface in $\mathrm{Gr}_2(\mathbb{C}^4)$, and its CSM class is:

Calculation: For a generic surface $S \subset \mathbb{P}^3$ of degree $d \geq 4$, we have

$$c_{\mathrm{SM}}(\Phi_{2,2}) = c(\mathbb{P}^3) \cdot \varphi^* s_{\mathrm{SM}}(\bar{X}_{2,2,1^{d-4}}) =$$

$$= \underbrace{\frac{1}{2}(d-3)(d-2)d(d+3) \cdot s_{1,1} + \frac{1}{2}(d-3)(d-2)(d-1)d \cdot s_2}_{[\Phi_{2,2}] \text{ or "bidegree"}} +$$

$$+ \underbrace{\frac{1}{3}(d-3)d(2-63d+18d^2+6d^3-2d^4)}_{\text{no direct interpretation (?)}} \cdot s_{2,1}$$

$$+ \underbrace{\frac{1}{12}d(-6144+8096d-1872d^2-909d^3+396d^4+10d^5-24d^6+3d^7)}_{\chi(\Phi_{2,2})} \cdot s_{2,2}$$

The (bi)degree of this and some similar loci was computed in:
E. Arrondo, M. Bertolini, C. Turrini: A focus on focal surfaces (2000)

The locus of flex lines to a generic surface

Similarly we can consider the locus $\Phi_3 \subset \mathrm{Gr}_2(\mathbb{C}^4)$ of flex lines to a generic surface $S \subset \mathbb{P}^3$. This is again a surface in $\mathrm{Gr}_2(\mathbb{C}^4)$, and its CSM class is:

Calculation: For a generic surface $S \subset \mathbb{P}^3$ of degree[7] $d \geq 4$, we have

$$\begin{aligned} c_{\mathrm{SM}}(\Phi_3) &= c(\mathbb{P}^3) \cdot \varphi^* s_{\mathrm{SM}}(\bar{X}_{3,1^{d-3}}) = \\ &= \underbrace{3d(d-2) \cdot s_{1,1} + d(d-1)(d-2) \cdot s_2}_{[\Phi_3] \text{ or “bidegree”}} + \\ &+ \underbrace{(-d)(3d-8)(3d-4)}_{(???)} \cdot s_{2,1} \\ &+ \underbrace{\frac{1}{2}(d+1)(392 - 749d + 265d^2 - 8d^3 + d^4 - d^5)}_{\chi(\Phi_3)} \cdot s_{2,2} \end{aligned}$$

[7]the coefficientof $s_{2,2}$ fails to be a polynomial for $d = 3$

The number of $4\times$ tangent lines to a generic surface

Question: How many $4\times$ tangent lines are to a generic degree $d \geq 8$ surface in $\mathbb{P}^3$? This was first computed by Schubert.

Note that we simply want to count a zero dimensional locus, so we don't actually need the full power of CSM classes; the only thing we need is the equivariant dual of the locus $\bar{X}_{2^4,1^{d-8}}$.

Calculation: For a generic surace S of degree $d \geq 8$

$$c_{\mathrm{SM}}(4\times) = \underbrace{\frac{1}{12}n \cdot \frac{(n-4)!}{(n-8)!} \cdot (n^3 + 6n^2 + 7n - 30)}_{\text{number of } 4\times \text{ tangent lines}} \cdot s_{2,2}$$

For $d \geq 8$ these numbers are:

$$14752,\ 112320,\ 492000,\ 1620080,\ 4445280,\ 10719072 \ldots$$

The number of maximally hyperflex lines

Question: Given a generic degree $(2d+1)$ hypersurface $\mathcal{H}$ in $\mathbb{P}^{d+1}$, how many lines are in $\mathbb{P}^{d+1}$ which meet $\mathcal{H}$ at a single point with a contact of order $(2d+1)$?

Again, we don't need the power of CSM classes, simply the equivariant dual of $X_{(2d+1)}$ (which is a rational normal curve).

Calculation: The locus of maximally hyperflex lines $Z_{2d+1} \subset \mathrm{Gr}_2(\mathbb{C}^{d+2})$ has CSM class

$$c_{\mathrm{SM}}(Z_{2d+1}) = s_{d,d} \cdot \underbrace{\sum_{j=0}^{d} \frac{(2d+1)!}{d-j+1} \cdot \binom{2d-2j}{d-j} \cdot \sigma_j(\Gamma_d)}_{\text{number of max. hyperflex lines}}$$

$$\Gamma_d = \left\{ \tfrac{(2d+1-2i)^2}{i(2d+1-i)} \;\middle|\; i \in \{1,2,\ldots,d\} \right\}$$

For $d \geq 1$ the numbers are:

$$9,\ 575,\ 99715,\ 33899229,\ 19134579541,\ 16213602794675\ldots$$

Linear systems of hypersurfaces

A less trivial application is to consider pencils, nets or higher dimensional linear systems of degree d hyperfaces $\mathcal{H}_y \subset \mathbb{P}^n$ parametrized by $y \in \mathbb{P}^s$.

Such a linear system is encoded by a linear map

$$\mathcal{F} \in \mathsf{Hom}\big[\mathbb{C}^{s+1}, \mathsf{Sym}^d(\mathbb{C}^{n+1})^*\big] = (\mathbb{C}^{s+1})^* \otimes \mathsf{Sym}^d(\mathbb{C}^{n+1})^*$$

Given a tangency condition μ, we can define the incidence variety

$$\mathcal{J}_\mu = \Big\{ (y, K) \in \mathbb{P}^s \times \mathsf{Gr}_2(\mathbb{C}^{n+1}) \;\Big|\; \mathbb{P}K \text{ has contact type } \mu \text{ with } \mathcal{H}_y = \{\mathcal{F}_y = 0\} \Big\} \subset \mathbb{P}^s \times \mathsf{Gr}_2(\mathbb{C}^{n+1})$$

Observation: $\mathcal{J}_\mu = \sigma^{-1}(X_\mu)$ where the section σ of $L^* \otimes \mathsf{Sym}^d K^*$ is defined by restricting $\mathcal{F}$ to $\mathsf{pr}_1^* L \otimes \mathsf{pr}_2^* K$.

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Observation: We can compute $c_{\mathrm{SM}}(\mathcal{J}_\mu)$ using the same "twisting trick" which gives the correspondance between the affine and the projective CSM classes. Unfortunately, when projecting down to the second component, while $(\mathrm{pr}_2)_* \, c_{\mathrm{SM}}(\mathcal{J}_\mu)$ is easy to compute, it does not normally agree with $c_{\mathrm{SM}}(\mathrm{pr}_2(\mathcal{J}_\mu))$...

We can still do some counting though (but again, we don't need the full CSM class for that):

Calculation: Given a generic pencil of degree $d \geq 4$ plane curves, the number of hyperflexes (contact of order ≥ 4) to the members of the family is $6(d-3)(3d-2)$:

$$c_{\mathrm{SM}}(\bar{\mathcal{J}}_{(4,1^{d-4})}) = \underbrace{6(d-3)(3d-2)}_{\#\text{ hyperflex}} \cdot s_{1,1} \cdot \xi \in H^*(\mathbb{P}^1 \times \mathrm{Gr}_2(\mathbb{C}^3))$$

It's easy to show that $(\mathrm{pr}_2)_*$ simply extracts the coefficient of ξ.