# Euler Angles Are Not Evil, Just Misunderstood

Michael J. Pfauth

August 23, 2012

**Abstract**

What Euler angles are, how they are computed, different Euler angle conventions and how to convert among them, identifying and handling gimbal lock, viewing geometry. Appendix A contains the complete set of equations for all 12 unique Euler angle sequences.

# 1 Introduction

I was faced with the problem of handling, in a unified way, space vehicle orientations specified in a variety of often arbitrary conventions. I discovered that there is little practical information on this, despite the simplicity of the mathematics. This tutorial will discuss:

- How Euler angles are defined
- How to compute any desired sequence
- How to get any desired sequence given the direction cosine matrix
- How to convert from one Euler angle convention to another
- How to handle gimbal lock
- How to handle different body axis conventions.

In what follows, you need only basic matrix and trigonometry skills.

A good, though theoretical, introduction to Euler angles can be found on Wolfram's web site: http://mathworld.wolfram.com/EulerAngles.html. If you are new to Euler angles, this is a good place to start. Don't worry about the quaternion material, I won't use it here.

# 2 Euler Angle Math

A common problem in working with Euler angles is knowing how the rotations are defined and the order in which they are applied. It is important to understand that there is no standard way of defining Euler angles, though particular fields tend to use one convention.

## 2.1 Primitive Rotations

In this tutorial, I will use a convention common in aeronatical work: yaw ($\psi$) is rotation about the Z axis, pitch ($\theta$) is rotation about the Y axis and roll ($\phi$) is rotation about the X axis. All angles are positive per the right-hand rule and the coordinate systems are all right-handed Cartesian. I occaisionally work with data in a left-handed coordinate system. I find it most convenient to convert to a right handed system. Do this by multiplying the Y coordinate by -1.

Given the above Euler angle conventions, we can define some primitive rotation functions:

$$yaw(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

$$pitch(\theta) = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \tag{2}$$

$$roll(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \tag{3}$$

### 2.2 Composite Rotations and Euler's Theorem

Euler's theorem states (more or less) that any finite sequence of primitive rotations can be expressed as an equivalent sequence of no more than 3 primitive rotations as given in equations 1 to 3. What this means practically is that we can multiply the matrices representing the desired inidividual rotations, then re-interpret the product matrix into a sequence of no more than 3 primitive rotations. These 3 rotations are the "Euler angles" and the matrix is often known as the "direction cosine matrix".

But how do we do this?

### 2.3 Getting the Euler Angles

The first step is to multiply all the individual transformation matrices to obtain the final matrix. This matrix is unique; that is, it represents one and only one orientation of the object in space.

There are exactly 12 ways to multiply the 3 primitive rotations: 6 of the form ABC where the primitive rotations are all different such as YAW-PITCH-ROLL and 6 of the form ABA where the first and last rotations are the same such as YAW-ROLL-YAW.

For example, the YAW-PITCH-ROLL matrix is computed from the primitive rotations as follows:

$$YPR(\psi, \theta, \phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$
$$\begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

The final result:

$$YPR(\psi, \theta, \phi) = \left( \begin{array}{c} \cos(\theta)\cos(\psi) \\ \sin(\phi)\sin(\theta)\cos(\psi) - \cos(\phi)\sin(\psi) \\ \cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi) \end{array} \right.$$

$$\begin{matrix} \cos(\theta)\sin(\psi) & -\sin(\theta) \\ \sin(\phi)\sin(\theta)\sin(\psi)+\cos(\phi)\cos(\psi) & \sin(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta)\sin(\psi)-\sin(\phi)\cos(\psi) & \cos(\phi)\cos(\theta) \end{matrix} \right) \tag{5}$$

The complete set of sequences is at Appendix A.

Note that I use right multiplication of the matrices because I represent vector quantities such as position and velocity as 3 by 1 column matrices. Some authors use 1 by 3 row matrices. In this case, all of my matrices should be transposed and the order of the multiplications reversed. The practical effect is that with right multiplication, the matrix representing the first rotation is written last. Whether you use right or left matrix multiplication is purely a matter of convenience; it has no effect on the final result.

Now that we have the final transformation matrix, we can solve it for any one of the 12 Euler angle sequences. Considering equation 5 (and using M as the matrix), we have:

$$\psi = \operatorname{atan2}(M_{0,1}, M_{0,0}) \tag{6}$$

$$\theta = \sin^{-1}(-M_{0,2}) \tag{7}$$

$$\phi = \operatorname{atan2}(M_{1,2}, M_{2,2}) \tag{8}$$

where atan2 is the four-quadrant inverse tangent function found in the C and FORTRAN math libraries. This function is necessary because the standard $\tan^{-1}$ function only returns an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. It also correctly handles the cases where either, but not both, of its arguments are zero.

For example, to convert from a YAW-ROLL-YAW sequence to YAW-PITCH-ROLL, we first compute the YAW-ROLL-YAW matrix and then apply equations 6 to 8 above to get the YAW-PITCH-ROLL Euler angles.

The other sequences are derived in the same way.

### 2.4 Gimbal Lock

The observant reader will have noticed a problem with equations 6 to 8. When $\theta$, the middle rotation, is $\frac{\pi}{2}$, the transformation matrix is:

$YPR(\psi, \frac{\pi}{2}, \phi) =$

$$\begin{pmatrix} 0 & 0 & -1 \\ \sin(\phi)\cos(\psi)-\cos(\phi)\sin(\psi) & \sin(\phi)\sin(\psi)+\cos(\phi)\cos(\psi) & 0 \\ \sin(\phi)\sin(\psi)+\cos(\phi)\cos(\psi) & \cos(\phi)\sin(\psi)-\sin(\phi)\cos(\psi) & 0 \end{pmatrix} \tag{9}$$

The arguments for the atan2 functions are all zero; *i.e.* undefined and the function can't be used. This is the "gimbal lock" problem; fortunately the solution is simple.

Recall from trigonometry these formulas:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \tag{10}$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) \tag{11}$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B) \tag{12}$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B) \tag{13}$$

Using these formulas, the transformation matrix (equation 9) becomes:

$$YPR(\psi, \frac{\pi}{2}, \phi) = \begin{pmatrix} 0 & 0 & -1 \\ -\sin(\psi-\phi) & \cos(\psi-\phi) & 0 \\ \cos(\psi-\phi) & \sin(\psi-\phi) & 0 \end{pmatrix} \tag{14}$$

$$YPR(\psi, -\frac{\pi}{2}, \phi) = \begin{pmatrix} 0 & 0 & 1 \\ -\sin(\psi+\phi) & \cos(\psi+\phi) & 0 \\ -\cos(\psi+\phi) & -\sin(\psi+\phi) & 0 \end{pmatrix} \tag{15}$$

What equations 14 and 15 mean is that the final orientation can be achieved using only *two* rotations. For the case of equation 14 the two rotations are a yaw of $\psi - \phi$ followed by a pitch of $\frac{\pi}{2}$. For the case of equation 15 the two rotations are a yaw of $\psi + \phi$ followed by a pitch of $-\frac{\pi}{2}$.

The other Euler angle sequences are handled in the same way except that sequences of the form ABA have gimbal lock at 0 and $\pi$.

### 2.5 Visualizing Euler Angles

You can visualize this by making some “coordinate sticks”. These are nothing more than three small sticks held together at right angles representing the positive coordinate axes. Mine are made of coffee stirrers fastened together with scotch tape and triangular pieces of a file folder. Mark the axes so that you can distinguish them.

Do some small positive rotation about the Z axis (a yaw), then a pitch of 90 degrees and then finish with a small positive rotation about the X axis (a roll). With a little care you can see that the final orientation could have been achieved by only two rotations: this first rotation being the difference between the original yaw and roll and the second being a pitch of 90 degrees.

## 3 Euler Angles in the Real World

I mentioned at the start that I would discuss how to convert from one Euler angle convention to another. There are two related concepts which need to be explained before I give the full method.

### 3.1 Different Conventions

First, we must know how the Euler angles (ie the primitive rotations) are defined. In the equations above, I sued the “areonautical” convention: yaw is about the Z axis, pitch is about the Y axis and roll is about the X axis. Suppose we have a different convention, *e.g* the one used by Open GL Performer:

heading is about the Z axis, pitch is about the X axis and roll is about the Y axis. If we let the areonautical convention be our reference, then the Performer convention is equivalent to yaw then roll then pitch in the reference convention. To convert from aeronautical yaw-pitch-roll to Performer heading-pitch-roll we first compute the direction cosine matirx using the aeronautical convention; ie yaw-pitch-roll. We get the Performer Euler angles by interpreting the direction cosine matrix as yaw-roll-pitch. Performer heading = aeronautical yaw, Performer pitch = aeronautical roll and Performer roll = aeronautical pitch.

Don't let notation confuse you. There are 12 and only 12 unique Euler angle sequences. No matter how the sequence you have is defined, it will be one of the 12. Also remember that the direction cosine matrix is unique: it is independent of any particular Euler angle sequence and represents exactly one orientation in space.

## 3.2 Using Real Geometry

The next concept is more subtle. So far, we have not tied the X, Y and Z axes to an object. Obviously this is important if we want the object to be correctly drawn. In our reference convention, we add the stipulation (imagine this applied to an airplane) that the X axis is out the nose of the airplane, the Y axis is out the starboard (right) side (as seen by the pilot looking forward) and the Z axis (to complete the right-hand system) is down. If the geometry we want to draw has its body axes defined this way and we know how the Euler angles are defined, we can proceed as described in the paragraph above; nothing more needs to be done.

But suppose the geometry has different body axis conventions? For example: the body X axis is out the port (left) side of the airplane, body Y axis points up and the body Z axis is out the nose of the airplane. For simplicity, the Euler angle sequence for this geometry is the areonautical convention we have been using. Without correcting for the different body axis definitions, the airplane will be drawn incorrectly.

We perform the correction as follows. Perform the given set of Euler angle rotations then perform any additioanl rotations needed to bring the given body axes into alignment with our desired convention. In this case, the additional rottions would be a yaw of 90 degrees followed by a pitch of -90 degrees. (Use your coordinate sticks and see for yourself.) The complete sequence for this case is: yaw(given yaw angle) pitch(given pitch angle) roll(given roll angle) yaw(90) pitch(-90). If we were using Open GL Performer to draw the airplane, we would take the matrix just computed, interpret it is Performer heading-pitch-roll as described above and use these angles as the HPR values of the airplane's DCS node.

The complete conversion process goes like this:

1. Examine the given Euler angle sequence and decide to which sequence in your preferred convention it is equivalent.

2. Examine how the body axes are defined and determine any additional rotations needed to bring them into alignment with your preferred convention.

3. Perform the given Euler angle rotations (using equivalent rotations in your prefered convention if necessary).

4. Perform the additional alignment rotations, if any.

5. Interpret the direction cosine matrix into your preferred convention (or any of the 12 sequences).

## 3.3 Pointing a Camera

Pointing a virtual camera is a special case. Since it has no geometry, there is no need to determine extra alignment rotations and you only need to do steps 1, 3 and 5 above. As long as the axis representing the camera lens is pointed at the object of interest, the object will be visible.

# A Euler Angle Equations

## A.1 Yaw - Pitch - Roll

The direction cosine matrix $M(\psi,\theta,\phi) = \text{roll}(\phi)\text{pitch}(\theta)\text{yaw}(\psi)$:

$$M(\psi,\theta,\phi) = \begin{pmatrix} \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ \sin(\phi)\sin(\theta)\cos(\psi) - \cos(\phi)\sin(\psi) & \sin(\phi)\sin(\theta)\sin(\psi) + \cos(\phi)\cos(\psi) & \sin(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi) & \cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi) & \cos(\phi)\cos(\theta) \end{pmatrix} \tag{16}$$

The Euler angles $\psi,\theta,\phi$:

$$\begin{array}{lllll} |\theta| \neq \frac{\pi}{2} & |M_{0,2}| \neq 1 & \psi = \text{atan2}(M_{0,1}, M_{0,0}) & \theta = \sin^{-1}(-M_{0,2}) & \phi = \text{atan2}(M_{1,2}, M_{2,2}) \\ \theta = \frac{\pi}{2} & M_{0,2} = -1 & \psi = \text{atan2}(M_{2,1}, M_{1,1}) & \theta = \frac{\pi}{2} & \phi = 0 \\ \theta = -\frac{\pi}{2} & M_{0,2} = 1 & \psi = \text{atan2}(M_{1,0}, M_{2,0}) & \theta = -\frac{\pi}{2} & \phi = 0 \end{array} \tag{17}$$

## A.2 Yaw - Roll - Pitch

The direction cosine matrix $M(\psi,\phi,\theta) = \text{pitch}(\theta)\text{roll}(\phi)\text{yaw}(\psi)$:

$$M(\psi,\phi,\theta) = \begin{pmatrix} \cos(\theta)\cos(\psi) - \sin(\theta)\sin(\phi)\sin(\psi) & \cos(\theta)\sin(\psi) + \sin(\theta)\sin(\phi)\cos(\psi) & -\sin(\theta)\cos(\phi) \\ -\cos(\phi)\sin(\psi) & \cos(\phi)\cos(\psi) & \sin(\phi) \\ \sin(\theta)\cos(\psi) + \cos(\theta)\sin(\phi)\sin(\psi) & \sin(\theta)\sin(\psi) - \cos(\theta)\sin(\phi)\cos(\psi) & \cos(\theta)\cos(\phi) \end{pmatrix} \tag{18}$$

The Euler angles $\psi,\phi,\theta$:

$$\begin{array}{lllll} |\phi| \neq \frac{\pi}{2} & |M_{1,2}| \neq 1 & \psi = \text{atan2}(-M_{1,0}, M_{1,1}) & \phi = \sin^{-1}(M_{1,2}) & \theta = \text{atan2}(-M_{0,2}, M_{2,2}) \\ \phi = \frac{\pi}{2} & M_{1,2} = 1 & \psi = \text{atan2}(M_{0,1}, M_{0,0}) & \phi = \frac{\pi}{2} & \theta = 0 \\ \phi = -\frac{\pi}{2} & M_{1,2} = -1 & \psi = \text{atan2}(M_{0,1}, M_{0,0}) & \phi = -\frac{\pi}{2} & \theta = 0 \end{array} \tag{19}$$

## A.3 Pitch - Yaw - Roll

The direction cosine matrix $M(\theta, \psi, \phi)$:

$$M(\theta,\psi,\phi) = \begin{pmatrix} \cos(\psi)\cos(\theta) & \sin(\psi) & -\cos(\psi)\sin(\theta) \\ -\cos(\phi)\sin(\psi)\cos(\theta)+\sin(\phi)\sin(\theta) & \cos(\phi)\cos(\psi) & \cos(\phi)\sin(\psi)\sin(\theta)+\sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\psi)\cos(\theta)+\cos(\phi)\sin(\theta) & -\sin(\phi)\cos(\psi) & -\sin(\phi)\sin(\psi)\sin(\theta)+\cos(\phi)\cos(\theta) \end{pmatrix} \tag{20}$$

The Euler angles $\theta, \psi, \phi$:

$$\begin{array}{lllll} |\psi| \neq \frac{\pi}{2} & |M_{0,1}| \neq 1 & \theta = \operatorname{atan2}(-M_{0,2}, M_{0,0}) & \psi = \sin^{-1}(M_{0,1}) & \phi = \operatorname{atan2}(-M_{2,1}, M_{1,1}) \\ \psi = \frac{\pi}{2} & M_{0,1} = 1 & \theta = \operatorname{atan2}(M_{2,0}, M_{2,2}) & \psi = \frac{\pi}{2} & \phi = 0 \\ \psi = -\frac{\pi}{2} & M_{0,1} = -1 & \theta = \operatorname{atan2}(M_{2,0}, M_{1,0}) & \psi = -\frac{\pi}{2} & \phi = 0 \end{array} \tag{21}$$

## A.4 Pitch - Roll - Yaw

The direction cosine matrix $M(\theta, \phi, \psi)$:

$$M(\theta,\phi,\psi) = \begin{pmatrix} \cos(\psi)\cos(\theta)+\sin(\psi)\sin(\phi)\sin(\theta) & \sin(\psi)\cos(\phi) & -\cos(\psi)\sin(\theta)+\sin(\psi)\sin(\phi)\cos(\theta) \\ -\sin(\psi)\cos(\theta)+\cos(\psi)\sin(\phi)\sin(\theta) & \cos(\psi)\cos(\phi) & \sin(\psi)\sin(\theta)+\cos(\psi)\sin(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta) & -\sin(\phi) & \cos(\phi)\cos(\theta) \end{pmatrix} \tag{22}$$

The Euler angles $\theta, \phi, \psi$:

$$\begin{array}{lllll} |\phi| \neq \frac{\pi}{2} & |M_{2,1}| \neq 1 & \theta = \operatorname{atan2}(M_{2,0}, M_{2,2}) & \phi = \sin^{-1}(-M_{2,1}) & \psi = \operatorname{atan2}(M_{0,1}, M_{1,1}) \\ \phi = \frac{\pi}{2} & M_{2,1} = -1 & \theta = \operatorname{atan2}(M_{1,0}, M_{0,0}) & \phi = \frac{\pi}{2} & \psi = 0 \\ \phi = -\frac{\pi}{2} & M_{2,1} = 1 & \theta = \operatorname{atan2}(-M_{1,0}, M_{0,0}) & \phi = -\frac{\pi}{2} & \psi = 0 \end{array} \tag{23}$$

## A.5 Roll - Pitch - Yaw

The direction cosine matrix $M(\phi, \theta, \psi)$:

$$M(\phi,\theta,\psi) = \begin{pmatrix} \cos(\psi)\cos(\theta) & \sin(\psi)\cos(\phi) + \cos(\psi)\sin(\theta)\sin(\phi) & \sin(\psi)\sin(\phi) - \cos(\psi)\sin(\theta)\cos(\phi) \\ -\sin(psi)\cos(\theta) & \cos(\psi)\cos(\theta) - \sin(\psi)\sin(\theta)\sin(\phi) & \cos(\psi)\sin(\phi) + \sin(\psi)\sin(\theta)\cos(\phi) \\ \sin(\theta) & -\cos(\theta)\sin(\phi) & \cos(\theta)\cos(\phi) \end{pmatrix} \tag{24}$$

The Euler angles $\phi, \theta, \psi$:

$$\begin{array}{lllll} |\theta| \neq \frac{\pi}{2} & |M_{2,0}| \neq 1 & \phi = \mathrm{atan2}(-M_{2,1}, M_{2,2}) & \theta = \sin^{-1}(M_{2,0}) & \psi = \mathrm{atan2}(-M_{1,0}, M_{0,0}) \\ \theta = \frac{\pi}{2} & M_{2,0} = 1 & \phi = \mathrm{atan2}(M_{0,1}, M_{1,1}) & \theta = \frac{\pi}{2} & \psi = 0 \\ \theta = -\frac{\pi}{2} & M_{2,0} = -1 & \phi = \mathrm{atan2}(M_{1,2}, M_{0,2}) & \theta = -\frac{\pi}{2} & \psi = 0 \end{array} \tag{25}$$

## A.6 Roll - Yaw - Pitch

The direction cosine matrix $M(\phi, \psi, \theta)$:

$$M(\phi,\psi,\theta) = \begin{pmatrix} \cos(\psi)\cos(\theta) & \sin(\psi)\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) & \sin(\psi)\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi) \\ -\sin(\psi) & \cos(\psi)\cos(\phi) & \cos(\psi)\sin(\phi) \\ \cos(\psi)\sin(\theta) & \sin(\psi)\sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi) & \sin(\psi)\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\phi) \end{pmatrix} \tag{26}$$

The Euler angles $\phi, \psi, \theta$:

$$\begin{array}{lllll} |\psi| \neq \frac{\pi}{2} & |M_{1,0}| \neq 1 & \phi = \mathrm{atan2}(M_{1,2}, M_{1,1}) & \psi = \sin^{-1}(-M_{1,0}) & \theta = \mathrm{atan2}(M_{2,0}, M_{0,0}) \\ \psi = \frac{\pi}{2} & M_{1,0} = -1 & \phi = \mathrm{atan2}(M_{0,2}, M_{0,1}) & \psi = \frac{\pi}{2} & \theta = 0 \\ \psi = -\frac{\pi}{2} & M_{1,0} = 1 & \phi = \mathrm{atan2}(M_{2,1}, M_{0,1}) & \psi = -\frac{\pi}{2} & \theta = 0 \end{array} \tag{27}$$

## A.7 Yaw - Pitch - Yaw

The direction cosine matrix $M(\psi_1, \theta, \psi_2)$:

$$M(\psi_1, \theta, \psi_2) =$$

$$\begin{pmatrix} \cos(\psi_2)\cos(\theta)\cos(\psi_1) - \sin(\psi_2)\sin(\psi_1) & \cos(\psi_2)\cos(\theta)\sin(\psi_1) + \sin(\psi_2)\cos(\psi_1) & -\cos(\psi_2)\sin(\theta) \\ -\sin(\psi_2)\cos(\theta)\cos(\psi_1) - \cos(\psi_2)\sin(\psi_1) & -\sin(\psi_2)\cos(\theta)\sin(\psi_1) + \cos(\psi_2)\cos(\psi_1) & \sin(\psi_2)\sin(\theta) \\ \sin(\theta)\cos(\psi_1) & \sin(\theta)\sin(\psi_1) & \cos(\theta) \end{pmatrix} \tag{28}$$

The Euler angles $\psi_1, \theta, \psi_2$:

$$\begin{array}{lllll} \theta \neq 0, \pi & |M_{2,2}| \neq 1 & \psi_1 = \text{atan2}(M_{2,1}, M_{2,0}) & \theta = \cos^{-1}(M_{2,2}) & \psi_2 = \text{atan2}(M_{1,2}, -M_{0,2}) \\ \theta = 0 & M_{2,2} = 1 & \psi_1 = \text{atan2}(M_{0,1}, M_{0,0}) & \theta = 0 & \psi_2 = 0 \\ \theta = \pi & M_{2,2} = -1 & \psi_1 = \text{atan2}(M_{1,0}, M_{0,0}) & \theta = \pi & \psi_2 = 0 \end{array} \tag{29}$$

## A.8 Yaw - Roll - Yaw

The direction cosine matrix $M(\psi_1, \phi, \psi_2)$:

$$M(\psi_1, \phi, \psi_2) =$$

$$\begin{pmatrix} \cos(\psi_2)\cos(\psi_1) - \sin(\psi_2)\cos(\phi)\sin(\psi_1) & \cos(\psi_2)\sin(\psi_1) + \sin(\psi_2)\cos(\phi)\cos(\psi_1) & \sin(\psi_2)\sin(\phi) \\ -\sin(\psi_2)\cos(\psi_1) - \cos(\psi_2)\cos(\phi)\sin(\psi_1) & -\sin(\psi_2)\sin(\psi_1) + \cos(\psi_2)\cos(\phi)\cos(\psi_1) & \cos(\psi_2)\sin(\phi) \\ \sin(\phi)\sin(\psi_1) & -\sin(\phi)\cos(\psi_1) & \cos(\phi) \end{pmatrix} \tag{30}$$

The Euler angles $\psi_1, \phi, \psi_2$:

$$\begin{array}{lllll} \phi \neq 0, \pi & |M_{2,2}| \neq 1 & \psi_1 = \text{atan2}(M_{2,0}, -M_{2,1}) & \phi = \cos^{-1}(M_{2,2}) & \psi_2 = \text{atan2}(M_{0,2}, M_{1,2}) \\ \phi = 0 & M_{2,2} = 1 & \psi_1 = \text{atan2}(M_{0,1}, M_{0,0}) & \phi = 0 & \psi_2 = 0 \\ \phi = \pi & M_{2,2} = -1 & \psi_1 = \text{atan2}(M_{0,1}, M_{0,0}) & \phi = \pi & \psi_2 = 0 \end{array} \tag{31}$$

## A.9 Pitch - Yaw - Pitch

The direction cosine matrix $M(\theta_1, \psi, \theta_2)$:

$M(\theta_1, \psi, \theta_2) =$

$$\begin{pmatrix} \cos(\theta_2)\cos(\psi)\cos(\theta_1) - \sin(\theta_2)\sin(\theta_1) & \cos(\theta_2)\sin(\psi) & -\cos(\theta_2)\cos(\psi)\sin(\theta_1) - \sin(\theta_2)\cos(\theta_1) \\ -\sin(\psi)\cos(\theta_1) & \cos(\psi) & \sin(\psi)\sin(\theta_1) \\ \sin(\theta_2)\cos(\psi)\cos(\theta_1) + \cos(\theta_2)\sin(\theta_1) & \sin(\theta_2)\sin(\psi) & -\sin(\theta_2)\cos(\psi)\sin(\theta_1) + \cos(\theta_2)\cos(\theta_1) \end{pmatrix} \tag{32}$$

The Euler angles $\theta_1, \psi, \theta_2$:

$$\begin{array}{lllll} \psi \neq 0, \pi & |M_{1,1}| \neq 1 & \theta_1 = \text{atan2}(M_{1,2}, -M_{1,0}) & \psi = \cos^{-1}(M_{1,1}) & \theta_2 = \text{atan2}(M_{2,1}, M_{0,1}) \\ \psi = 0 & M_{1,1} = 1 & \theta_1 = \text{atan2}(M_{2,0}, M_{0,0}) & \psi = 0 & \theta_2 = 0 \\ \psi = \pi & M_{1,1} = -1 & \theta_1 = \text{atan2}(M_{2,0}, M_{2,2}) & \psi = \pi & \theta_2 = 0 \end{array} \tag{33}$$

## A.10 Pitch - Roll - Pitch

The direction cosine matrix $M(\theta_1, \phi, \theta_2)$:

$M(\theta_1, \phi, \theta_2) =$

$$\begin{pmatrix} \cos(\theta_2)\cos(\theta_1) - \sin(\theta_2)\cos(\phi)\sin(\theta_1) & \sin(\theta_2)\sin(\phi) & -\cos(\theta_2)\sin(\theta_1) - \sin(\theta_2)\cos(\phi)\cos(\theta_1) \\ \sin(\phi)\sin(\theta_1) & \cos(\phi) & \sin(\phi)\cos(\theta_1) \\ \sin(\theta_2)\cos(\theta_1) + \cos(\theta_2)\cos(\phi)\sin(\theta_1) & -\cos(\theta_2)\sin(\phi) & -\sin(\theta_2)\sin(\theta_1) + \cos(\theta_2)\cos(\phi)\cos(\theta_1) \end{pmatrix} \tag{34}$$

The Euler angles $\theta_1, \phi, \theta_2$:

$$\begin{array}{lllll} \phi \neq 0, \pi & |M_{1,1}| \neq 1 & \theta_1 = \text{atan2}(M_{1,0}, M_{1,2}) & \phi = \cos^{-1}(M_{1,1}) & \theta_2 = \text{atan2}(M_{0,1}, -M_{2,1}) \\ \phi = 0 & M_{1,1} = 1 & \theta_1 = \text{atan2}(M_{2,0}, M_{0,0}) & \phi = 0 & \theta_2 = 0 \\ \phi = \pi & M_{1,1} = -1 & \theta_1 = \text{atan2}(M_{2,0}, M_{2,2}) & \phi = \pi & \theta_2 = 0 \end{array} \tag{35}$$

## A.11 Roll - Yaw - Roll

The direction cosine matrix $M(\phi_1, \psi, \phi_2)$:

$M(\phi_1, \psi, \phi_2) =$

$$\begin{pmatrix} \cos(\psi) & \sin(\psi)\cos(\phi_1) & \sin(\psi)\sin(\phi_1) \\ -\cos(\phi_2)\sin(\psi) & \cos(\phi_2)\cos(\psi)\cos(\phi_1) - \sin(\phi_2)\sin(\phi_1) & \cos(\phi_2)\cos(\psi)\sin(\phi_1) + \sin(\phi_2)\cos(\phi_1) \\ \sin(\phi_2)\sin(\psi) & -\sin(\phi_2)\cos(\psi)\cos(\phi_1) - \cos(\phi_2)\sin(\phi_1) & -\sin(\phi_2)\cos(\psi)\sin(\phi_1) + \cos(\phi_2)\cos(\phi_1) \end{pmatrix} \tag{36}$$

The Euler angles $\phi_1, \psi, \phi_2$:

$$\begin{array}{lllll} \psi \neq 0, \pi & |M_{0,0}| \neq 1 & \phi_1 = \mathrm{atan2}(M_{0,2}, M_{0,1}) & \psi = \cos^{-1}(M_{0,0}) & \phi_2 = \mathrm{atan2}(M_{2,0}, -M_{1,0}) \\ \psi = 0 & M_{0,0} = 1 & \phi_1 = \mathrm{atan2}(M_{1,2}, M_{1,1}) & \psi = 0 & \phi_2 = 0 \\ \psi = \pi & M_{0,0} = -1 & \phi_1 = \mathrm{atan2}(M_{1,2}, M_{1,1}) & \psi = \pi & \phi_2 = 0 \end{array} \tag{37}$$



## A.12 Roll - Pitch - Roll

The direction cosine matrix $M(\phi_1, \theta, \phi_2)$:

$M(\phi_1, \theta, \phi_2) =$

$$\begin{pmatrix} \cos(\theta) & \sin(\theta)\sin(\phi_1) & -\sin(\theta)\cos(\phi_1) \\ \sin(\phi_2)\sin(\theta) & \cos(\phi_2)\cos(\phi_1) - \sin(\phi_2)\cos(\theta)\sin(\phi_1) & \cos(\phi_2)\sin(\phi_1) + \sin(\phi_2)\cos(\theta)\cos(\phi_1) \\ \cos(\phi_2)\sin(\theta) & -\sin(\phi_2)\cos(\phi_1) - \cos(\phi_2)\cos(\theta)\sin(\phi_1) & -\sin(\phi_2)\sin(\phi_1) + \cos(\phi_2)\cos(\theta)\cos(\phi_1) \end{pmatrix} \tag{38}$$

The Euler angles $\phi_1, \theta, \phi_2$:

$$\begin{array}{lllll} \theta \neq 0, \pi & |M_{0,0}| \neq 1 & \phi_1 = \mathrm{atan2}(M_{0,1}, -M_{0,2}) & \theta = \cos^{-1}(M_{0,0}) & \phi_2 = \mathrm{atan2}(M_{1,0}, M_{2,0}) \\ \theta = 0 & M_{0,0} = 1 & \phi_1 = \mathrm{atan2}(M_{1,2}, M_{1,1,}) & \theta = 0 & \phi_2 = 0 \\ \theta = \pi & M_{0,0} = -1 & \phi_1 = \mathrm{atan2}(M_{1,2}, M_{1,1}) & \theta = \pi & \phi_2 = 0 \end{array} \tag{39}$$