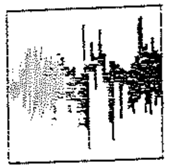
PROAKIS, D.G. MANOLAKIS, Digital Signal Processing.



5

The Discrete Fourier Transform: Its Properties and Applications

Frequency analysis of discrete-time signals is usually and most conveniently performed on a digital signal processor, which may be a general-purpose digital computer or specially designed digital hardware. To perform frequency analysis on a discrete-time signal $\{x(n)\}$, we convert the time-domain sequence to an equivalent frequency-domain representation. We know that such a representation is given by the Fourier transform $X(\omega)$ of the sequence $\{x(n)\}$. However, $X(\omega)$ is a continuous function of frequency and therefore, it is not a computationally convenient representation of the sequence $\{x(n)\}$.

In this section we consider the representation of a sequence $\{x(n)\}$ by samples of its spectrum $X(\omega)$. Such a frequency-domain representation leads to the discrete Fourier transform (DFT), which is a powerful computational tool for performing frequency analysis of discrete-time signals.

5.1 FREQUENCY DOMAIN SAMPLING: THE DISCRETE FOURIER TRANSFORM

Before we introduce the DFT, we consider the sampling of the Fourier transform of an aperiodic discrete-time sequence. Thus, we establish the relationship between the sampled Fourier transform and the DFT.

5.1.1 Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

We recall that aperiodic finite-energy signals have continuous spectra. Let us consider such an aperiodic discrete-time signal $x(n)$ with Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{5.1.1}$$

Suppose that we sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ radians between successive samples. Since $X(\omega)$ is periodic with period 2π, only samples in the fundamental frequency range are necessary. For convenience, we take N equidistant samples in the interval $0 \le \omega < 2\pi$ with spacing $\delta\omega = 2\pi/N$, as shown in Fig. 5.1. First, we consider the selection of N, the number of samples in the frequency domain.

If we evaluate (5.1.1) at $\omega = 2\pi k/N$, we obtain

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.2}$$

The summation in (5.1.2) can be subdivided into an infinite number of summations, where each sum contains N terms. Thus

$$\begin{aligned} X\left(\frac{2\pi}{N}k\right) &= \cdots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ &\quad + \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \cdots \\ &= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-j2\pi kn/N} \end{aligned}$$

If we change the index in the inner summation from n to $n - lN$ and interchange the order of the summation, we obtain the result

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n - lN)\right] e^{-j2\pi kn/N} \tag{5.1.3}$$

for $k = 0, 1, 2, \ldots, N-1$.

The signal

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \tag{5.1.4}$$

obtained by the periodic repetition of $x(n)$ every N samples, is clearly periodic with fundamental period N. Consequently, it can be expanded in a Fourier

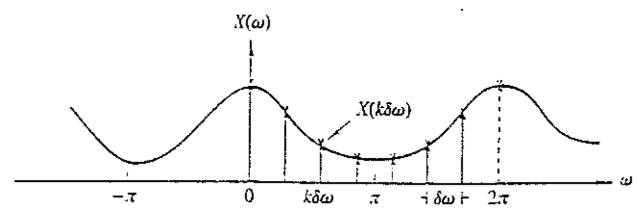


Figure 5.1 Frequency-domain sampling of the Fourier transform.

series as

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.5}$$

with Fourier coefficients

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.6}$$

Upon comparing (5.1.3) with (5.1.6), we conclude that

$$c_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right) \qquad k = 0, 1, \ldots, N-1 \tag{5.1.7}$$

Therefore,

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.8}$$

The relationship in (5.1.8) provides the reconstruction of the periodic signal $x_p(n)$ from the samples of the spectrum $X(\omega)$. However, it does not imply that we can recover $X(\omega)$ or $x(n)$ from the samples. To accomplish this, we need to consider the relationship between $x_p(n)$ and $x(n)$.

Since $x_p(n)$ is the periodic extension of $x(n)$ as given by (5.1.4), it is clear that $x(n)$ can be recovered from $x_p(n)$ if there is no aliasing in the time domain, that is, if $x(n)$ is time-limited to less than the period N of $x_p(n)$. This situation is illustrated in Fig. 5.2, where without loss of generality, we consider a finite-duration

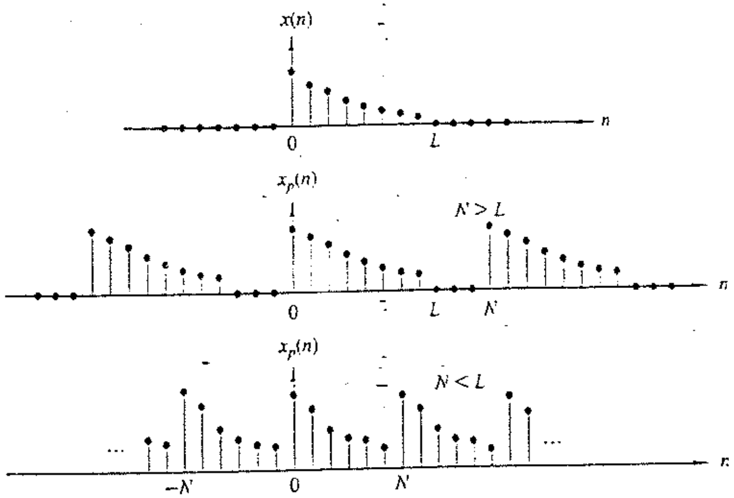


Figure 5.2 Aperiodic sequence $x(n)$ of length L and its periodic extension for $N \geq L$ (no aliasing) and $N < L$ (aliasing).

sequence $x(n)$, which is nonzero in the interval $0 \leq n \leq L-1$. We observe that when $N \geq L$,

$$x(n) = x_p(n) \qquad 0 \leq n \leq N-1$$

so that $x(n)$ can be recovered from $x_p(n)$ without ambiguity. On the other hand, if $N < L$, it is not possible to recover $x(n)$ from its periodic extension due to *time-domain aliasing*. Thus, we conclude that the spectrum of an aperiodic discrete-time signal with finite duration L, can be exactly recovered from its samples at frequencies $\omega_k = 2\pi k/N$, if $N \geq L$. The procedure is to compute $x_p(n)$, $n = 0, 1, \ldots, N-1$ from (5.1.8); then

$$x(n) = \begin{cases} x_p(n), & 0 \leq n \leq N-1 \\ 0, & \text{elsewhere} \end{cases} \tag{5.1.9}$$

and finally, $X(\omega)$ can be computed from (5.1.1).

As in the case of continuous-time signals, it is possible to express the spectrum $X(\omega)$ directly in terms of its samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$. To derive such an interpolation formula for $X(\omega)$, we assume that $N \geq L$ and begin with (5.1.8). Since $x(n) = x_p(n)$ for $0 \leq n \leq N-1$,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N} \qquad 0 \leq n \leq N-1 \tag{5.1.10}$$

If we use (5.1.1) and substitute for $x(n)$, we obtain

$$\begin{aligned} X(\omega) &= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N} \right] e^{-j\omega n} \\ &= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{-j(\omega - 2\pi k/N)n} \right] \end{aligned} \tag{5.1.11}$$

The inner summation term in the brackets of (5.1.11) represents the basic interpolation function shifted by $2\pi k/N$ in frequency. Indeed, if we define

$$\begin{aligned} P(\omega) &= \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N} \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\ &= \frac{\sin(\omega N/2)}{N \sin(\omega/2)} e^{-j\omega(N-1)/2} \end{aligned} \tag{5.1.12}$$

then (5.1.11) can be expressed as

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) P\left(\omega - \frac{2\pi}{N}k\right) \qquad N \geq L \tag{5.1.13}$$

The interpolation function $P(\omega)$ is not the familiar $(\sin\theta)/\theta$ but instead, it is a periodic counterpart of it, and it is due to the periodic nature of $X(\omega)$. The phase shift in (5.1.12) reflects the fact that the signal $x(n)$ is a causal, finite-duration sequence of length N. The function $\sin(\omega N/2)/(N\sin(\omega/2))$ is plotted in Fig. 5.3 for $N=5$. We observe that the function $P(\omega)$ has the property

$$P\left(\frac{2\pi}{N}k\right) = \begin{cases} 1, & k=0 \\ 0, & k=1,2,\ldots,N-1 \end{cases} \tag{5.1.14}$$

Consequently, the interpolation formula in (5.1.13) gives exactly the sample values $X(2\pi k/N)$ for $\omega = 2\pi k/N$. At all other frequencies, the formula provides a properly weighted linear combination of the original spectral samples.

The following example illustrates the frequency-domain sampling of a discrete-time signal and the time-domain aliasing that results.

Example 5.1.1

Consider the signal

$$x(n) = a^n u(n) \qquad 0<a<1$$

The spectrum of this signal is sampled at frequencies $\omega_k = 2\pi k/N$, $k=0,1,\ldots,N-1$. Determine the reconstructed spectra for $a=0.8$ when $N=5$ and $N=50$.

Solution The Fourier transform of the sequence $x(n)$ is

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1-ae^{-j\omega}}$$

Suppose that we sample $X(\omega)$ at N equidistant frequencies $\omega_k = 2\pi k/N$, $k=0,1,\ldots,N-1$. Thus we obtain the spectral samples

$$X(\omega_k) \equiv X\left(\frac{2\pi k}{N}\right) = \frac{1}{1-ae^{-j2\pi k/N}} \qquad k=0,1,\ldots,N-1$$

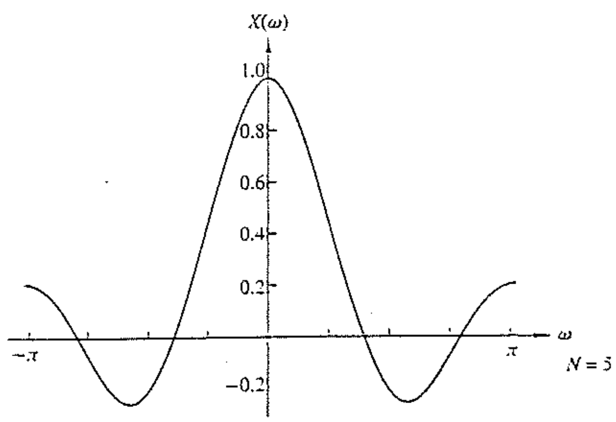


Figure 5.3 Plot of the function $[\sin(\omega N/2)]/[N\sin(\omega/2)]$.

The periodic sequence $x_p(n)$, corresponding to the frequency samples $X(2\pi k/N)$, $k=0,1,\ldots,N-1$, can be obtained from either (5.1.4) or (5.1.8). Hence

$$\begin{aligned} x_p(n) &= \sum_{l=-\infty}^{\infty} x(n-lN) = \sum_{l=-\infty}^{0} a^{n-lN} \\ &= a^n \sum_{l=0}^{\infty} a^{lN} = \frac{a^n}{1-a^N} \qquad 0\le n\le N-1 \end{aligned}$$

where the factor $1/(1-a^N)$ represents the effect of aliasing. Since $0<a<1$, the aliasing error tends toward zero as $N\to\infty$.

For $a=0.8$, the sequence $x(n)$ and its spectrum $X(\omega)$ are shown in Fig. 5.4a and b, respectively. The aliased sequences $x_p(n)$ for $N=5$ and $N=50$ and the corresponding spectral samples are shown in Fig. 5.4c and d, respectively. We note that the aliasing effects are negligible for $N=50$.

If we define the aliased finite-duration sequence $\hat{x}(n)$ as

$$\hat{x}(n) = \begin{cases} x_p(n), & 0\le n\le N-1 \\ 0, & \text{otherwise} \end{cases}$$

then its Fourier transform is

$$\begin{aligned} \hat{X}(\omega) &= \sum_{n=0}^{N-1} \hat{x}(n)e^{-j\omega n} = \sum_{n=0}^{N-1} x_p(n)e^{-j\omega N} \\ &= \frac{1}{1-a^N}\cdot\frac{1-a^N e^{-j\omega N}}{1-ae^{-j\omega}} \end{aligned}$$

Note that although $\hat{X}(\omega) \neq X(\omega)$, the sample values at $\omega_k = 2\pi k/N$ are identical. That is,

$$\hat{X}\left(\frac{2\pi}{N}k\right) = \frac{1}{1-a^N}\cdot\frac{1-a^N}{1-ae^{-j2\pi kN}} = X\left(\frac{2\pi}{N}k\right)$$

5.1.2 The Discrete Fourier Transform (DFT)

The development in the preceding section is concerned with the frequency-domain sampling of an aperiodic finite-energy sequence $x(n)$. In general, the equally spaced frequency samples $X(2\pi k/N)$, $k=0,1,\ldots,N-1$, do not uniquely represent the original sequence $x(n)$ when $x(n)$ has infinite duration. Instead, the frequency samples $X(2\pi k/N)$, $k=0,1,\ldots,N-1$, correspond to a periodic sequence $x_p(n)$ of period N, where $x_p(n)$ is an aliased version of $x(n)$, as indicated by the relation in (5.1.4), that is,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \tag{5.1.15}$$

When the sequence $x(n)$ has a finite duration of length $L\le N$, then $x_p(n)$ is simply a periodic repetition of $x(n)$, where $x_p(n)$ over a single period is

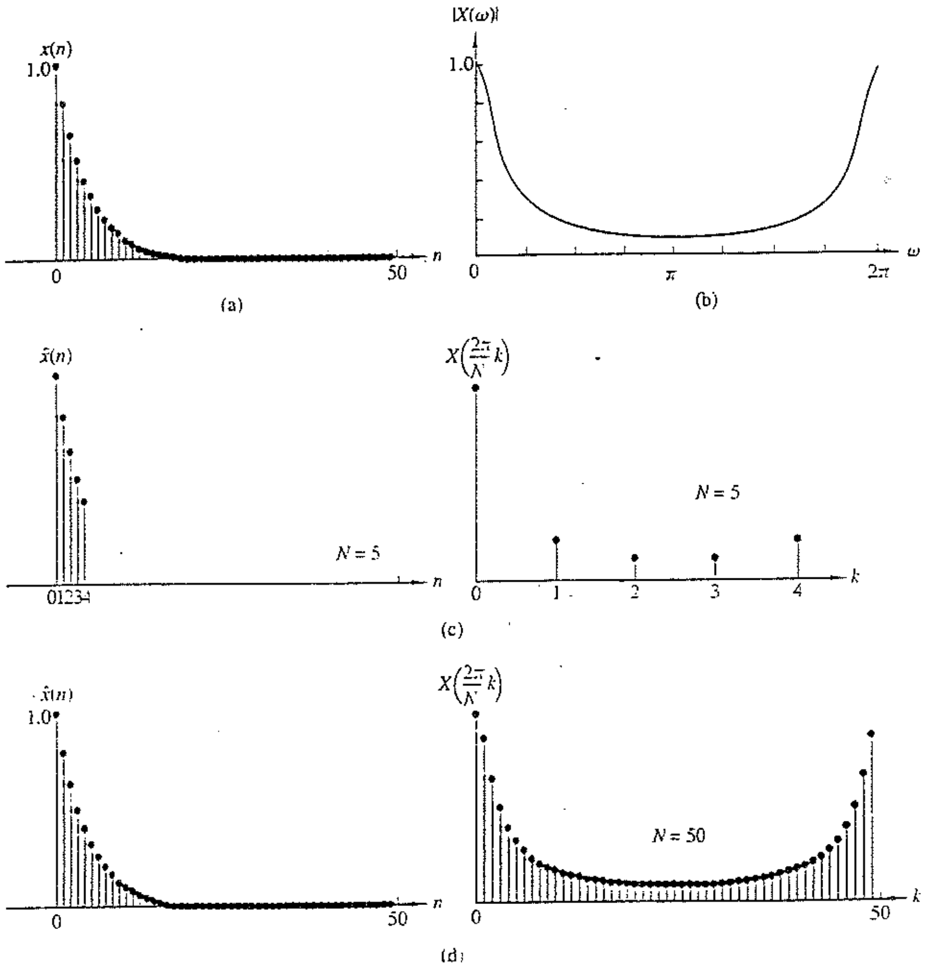


Figure 5.4 (a) Plot of sequence $x(n) = (0.8)^n u(n)$; (b) its Fourier transform (magnitude only); (c) effect of aliasing with $N = 5$; (d) reduced effect of aliasing with $N = 50$.

given as

$$x_p(n) = \begin{cases} x(n), & 0 \le n \le L-1 \\ 0, & L \le n \le N-1 \end{cases} \qquad (5.1.16)$$

Consequently, the frequency samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$, uniquely represent the finite-duration sequence $x(n)$. Since $x(n) \equiv x_p(n)$ over a single period (padded by $N - L$ zeros), the original finite-duration sequence $x(n)$ can be obtained from the frequency samples $\{X(2\pi k/N)\}$ by means of the formula (5.1.8).

It is important to note that *zero padding* does not provide any additional information about the spectrum $X(\omega)$ of the sequence $\{x(n)\}$. The L equidistant samples of $X(\omega)$ are sufficient to reconstruct $X(\omega)$ using the reconstruction formula (5.1.13). However, padding the sequence $\{x(n)\}$ with $N - L$ zeros and computing an N-point DFT results in a "better display" of the Fourier transform $X(\omega)$.

In summary, a finite-duration sequence $x(n)$ of length L [i.e., $x(n) = 0$ for $n < 0$ and $n \ge L$] has a Fourier transform

$$X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n} \qquad 0 \le \omega \le 2\pi \qquad (5.1.17)$$

where the upper and lower indices in the summation reflect the fact that $x(n) = 0$ outside the range $0 \le n \le L-1$. When we sample $X(\omega)$ at equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, 2, \ldots, N-1$, where $N \ge L$, the resultant samples are

$$\begin{aligned} X(k) &\equiv X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{L-1} x(n)e^{-j2\pi kn/N} \\ X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1 \end{aligned} \qquad (5.1.18)$$

where for convenience, the upper index in the sum has been increased from $L-1$ to $N-1$ since $x(n) = 0$ for $n \ge L$.

The relation in (5.1.18) is a formula for transforming a sequence $\{x(n)\}$ of length $L \le N$ into a sequence of frequency samples $\{X(k)\}$ of length N. Since the frequency samples are obtained by evaluating the Fourier transform $X(\omega)$ at a set of N (equally spaced) discrete frequencies, the relation in (5.1.18) is called the *discrete Fourier transform* (DFT) of $x(n)$. In turn, the relation given by (5.1.10), which allows us to recover the sequence $x(n)$ from the frequency samples

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N-1 \qquad (5.1.19)$$

is called the *inverse DFT* (IDFT). Clearly, when $x(n)$ has length $L < N$, the N-point IDFT yields $x(n) = 0$ for $L \le n \le N-1$. To summarize, the formulas for the DFT and IDFT are

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1 \qquad (5.1.18)$$

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \qquad n = 0, 1, 2, \ldots, N-1 \qquad (5.1.19)$$

Example 5.1.2

A finite-duration sequence of length L is given as

$$x(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$

Determine the N-point DFT of this sequence for $N \ge L$.

Solution The Fourier transform of this sequence is

$$\begin{aligned} X(\omega) &= \sum_{n=0}^{L-1} x(n) e^{-j\omega n} \\ &= \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2} \end{aligned}$$

The magnitude and phase of $X(\omega)$ are illustrated in Fig. 5.5 for $L = 10$. The N-point DFT of $x(n)$ is simply $X(\omega)$ evaluated at the set of N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. Hence

$$\begin{aligned} X(k) &= \frac{1-e^{-j2\pi kL/N}}{1-e^{-j2\pi k/N}} \qquad k = 0, 1, \ldots, N-1 \\ &= \frac{\sin(\pi kL/N)}{\sin(\pi k/N)} e^{-j\pi k(L-1)/N} \end{aligned}$$

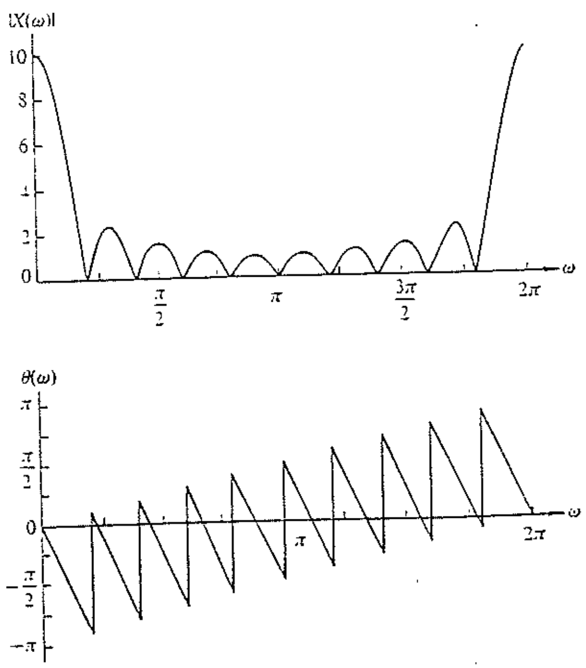


Figure 5.5 Magnitude and phase characteristics of the Fourier transform for signal in Example 5.1.2.

If N is selected such that $N = L$, then the DFT becomes

$$X(k) = \begin{cases} L, & k = 0 \\ 0, & k = 1, 2, \ldots, L-1 \end{cases}$$

Thus there is only one nonzero value in the DFT. This is apparent from observation of $X(\omega)$, since $X(\omega) = 0$ at the frequencies $\omega_k = 2\pi k/L$, $k \ne 0$. The reader should verify that $x(n)$ can be recovered from $X(k)$ by performing an L-point IDFT.

Although the L-point DFT is sufficient to uniquely represent the sequence $x(n)$ in the frequency domain, it is apparent that it does not provide sufficient detail to yield a good picture of the spectral characteristics of $x(n)$. If we wish to have better picture, we must evaluate (interpolate) $X(\omega)$ at more closely spaced frequencies, say $\omega_k = 2\pi k/N$, where $N > L$. In effect, we can view this computation as expanding the size of the sequence from L points to N points by appending $N - L$ zeros to the sequence $x(n)$, that is, zero padding. Then the N-point DFT provides finer interpolation than the L-point DFT.

Figure 5.6 provides a plot of the N-point DFT, in magnitude and phase, for $L = 10$, $N = 50$, and $N = 100$. Now the spectral characteristics of the sequence are more clearly evident, as one will conclude by comparing these spectra with the continuous spectrum $X(\omega)$.

5.1.3 The DFT as a Linear Transformation

The formulas for the DFT and IDFT given by (5.1.18) and (5.1.19) may be expressed as

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.20}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.21}$$

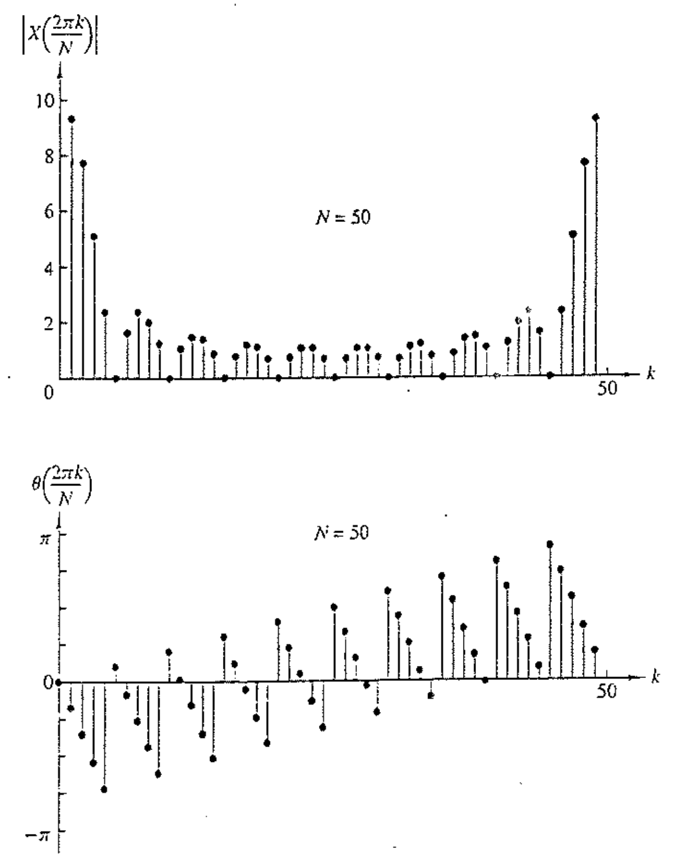
where, by definition,

$$W_N = e^{-j2\pi/N} \tag{5.1.22}$$

which is an Nth root of unity.

We note that the computation of each point of the DFT can be accomplished by N complex multiplications and $(N-1)$ complex additions. Hence the N-point DFT values can be computed in a total of N^2 complex multiplications and $N(N-1)$ complex additions.

It is instructive to view the DFT and IDFT as linear transformations on sequences $\{x(n)\}$ and $\{X(k)\}$, respectively. Let us define an N-point vector $\mathbf{x}_N$ of the signal sequence $x(n)$, $n = 0, 1, \ldots, N-1$, an N-point vector $\mathbf{X}_N$ of frequency



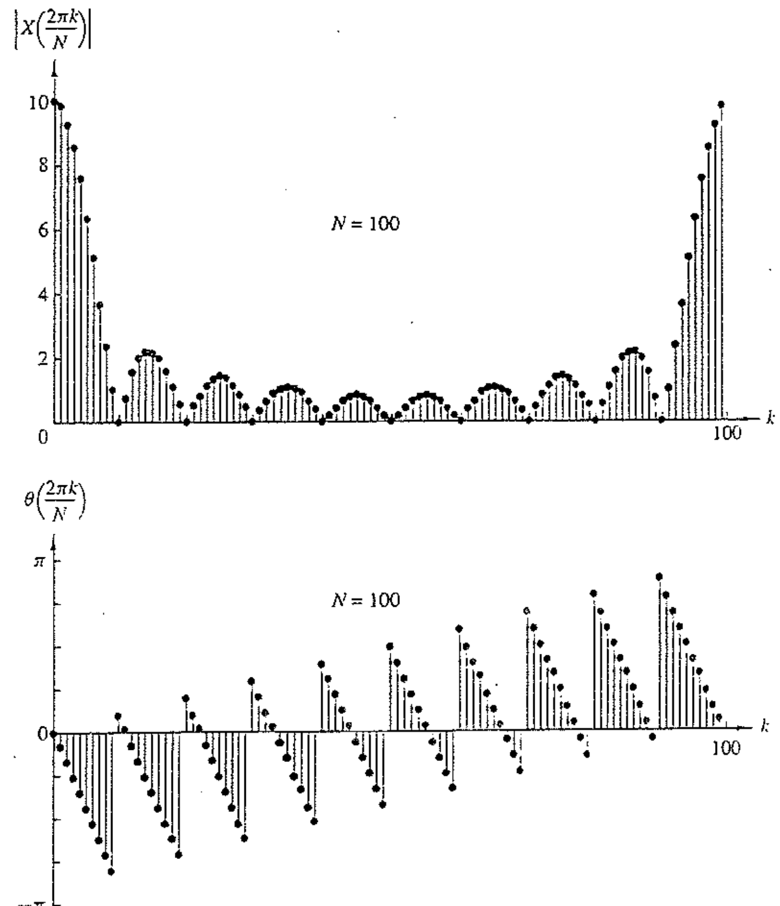
(a)

Figure 5.6 Magnitude and phase of an N-point DFT in Example 6.4.2: (a) $L = 10$, $N = 50$; (b) $L = 10$, $N = 100$.

samples, and an $N \times N$ matrix $\mathbf{W}_N$ as

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{5.1.23}$$



(b)

Figure 5.6 *continued*

With these definitions, the N-point DFT may be expressed in matrix form as

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N \tag{5.1.24}$$

where $\mathbf{W}_N$ is the matrix of the linear transformation. We observe that $\mathbf{W}_N$ is a symmetric matrix. If we assume that the inverse of $\mathbf{W}_N$ exists, then (5.1.24) can be inverted by premultiplying both sides by $\mathbf{W}_N^{-1}$. Thus we obtain

$$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N \tag{5.1.25}$$

But this is just an expression for the IDFT.

In fact, the IDFT as given by (5.1.21), can be expressed in matrix form as

$$\mathbf{x}_N = \frac{1}{N} \mathbf{W}_N^* \mathbf{X}_N \tag{5.1.26}$$

where $\mathbf{W}_N^*$ denotes the complex conjugate of the matrix $\mathbf{W}_N$. Comparison of (5.1.26) with (5.1.25) leads us to conclude that

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^* \tag{5.1.27}$$

which, in turn, implies that

$$\mathbf{W}_N\mathbf{W}_N^* = N\mathbf{I}_N \tag{5.1.28}$$

where $\mathbf{I}_N$ is an $N \times N$ identity matrix. Therefore, the matrix $\mathbf{W}_N$ in the transformation is an orthogonal (unitary) matrix. Furthermore, its inverse exists and is given as $\mathbf{W}_N^*/N$. Of course, the existence of the inverse of $\mathbf{W}_N$ was established previously from our derivation of the IDFT.

Example 5.1.3

Compute the DFT of the four-point sequence

$$x(n) = (0 \quad 1 \quad 2 \quad 3)$$

Solution The first step is to determine the matrix $\mathbf{W}_4$. By exploiting the periodicity property of $\mathbf{W}_4$ and the symmetry property

$$W_N^{k+N/2} = -W_N^k$$

the matrix $\mathbf{W}_4$ may be expressed as

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Then

$$\mathbf{X}_4 = \mathbf{W}_4\mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

The IDFT of $\mathbf{X}_4$ may be determined by conjugating the elements in $\mathbf{W}_4$ to obtain $\mathbf{W}_4^*$ and then applying the formula (5.1.26).

The DFT and IDFT are computational tools that play a very important role in many digital signal processing applications, such as frequency analysis (spectrum analysis) of signals, power spectrum estimation, and linear filtering. The importance of the DFT and IDFT in such practical applications is due to a large extent on the existence of computationally efficient algorithms, known collectively as fast

Fourier transform (FFT) algorithms, for computing the DFT and IDFT. This class of algorithms is described in Chapter 6.

5.1.4 Relationship of the DFT to Other Transforms

In this discussion we have indicated that the DFT is an important computational tool for performing frequency analysis of signals on digital signal processors. In view of the other frequency analysis tools and transforms that we have developed, it is important to establish the relationships between the DFT to these other transforms.

Relationship to the Fourier series coefficients of a periodic sequence. A periodic sequence $\{x_p(n)\}$ with fundamental period N can be represented in a Fourier series of the form

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi nk/N} \qquad -\infty < n < \infty \tag{5.1.29}$$

where the Fourier series coefficients are given by the expression

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.30}$$

If we compare (5.1.29) and (5.1.30) with (5.1.18) and (5.1.19), we observe that the formula for the Fourier series coefficients has the form of a DFT. In fact, if we define a sequence $x(n) = x_p(n)$, $0 \le n \le N-1$, the DFT of this sequence is simply

$$X(k) = Nc_k \tag{5.1.31}$$

Furthermore, (5.1.29) has the form of an IDFT. Thus the N-point DFT provides the exact line spectrum of a periodic sequence with fundamental period N.

Relationship to the Fourier transform of an aperiodic sequence. We have already shown that if $x(n)$ is an aperiodic finite energy sequence with Fourier transform $X(\omega)$, which is sampled at N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$, the spectral components

$$X(k) = X(\omega)|_{\omega=2\pi k/N} = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.32}$$

are the DFT coefficients of the periodic sequence of period N, given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \tag{5.1.33}$$

Thus $x_p(n)$ is determined by aliasing $\{x(n)\}$ over the interval $0 \le n \le N-1$. The finite-duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.1.34}$$

bears no resemblance to the original sequence $\{x(n)\}$, unless $x(n)$ is of finite duration and length $L \le N$, in which case

$$x(n) = \hat{x}(n) \qquad 0 \le n \le N-1 \tag{5.1.35}$$

Only in this case will the IDFT of $\{X(k)\}$ yield the original sequence $\{x(n)\}$.

Relationship to the z-transform. Let us consider a sequence $x(n)$ having the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{5.1.36}$$

with a ROC that includes the unit circle. If $X(z)$ is sampled at the N equally spaced points on the unit circle $z_k = e^{j2\pi k/N}$, $0, 1, 2, \ldots, N-1$, we obtain

$$\begin{aligned} X(k) &\equiv X(z)|_{z=e^{j2\pi nk/N}} \qquad k = 0, 1, \ldots, N-1 \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi nk/N} \end{aligned} \tag{5.1.37}$$

The expression in (5.1.37) is identical to the Fourier transform $X(\omega)$ evaluated at the N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$, which is the topic treated in Section 5.1.1.

If the sequence $x(n)$ has a finite duration of length N or less, the sequence can be recovered from its N-point DFT. Hence its z-transform is uniquely determined by its N-point DFT. Consequently, $X(z)$ can be expressed as a function of the DFT {X(k)} as follows

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} x(n)z^{-n} \\ X(z) &= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}\right]z^{-n} \\ X(z) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\sum_{n=0}^{N-1}\left(e^{j2\pi k/N}z^{-1}\right)^n \\ X(z) &= \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{j2\pi k/N}z^{-1}} \end{aligned} \tag{5.1.38}$$

When evaluated on the unit circle, (5.1.38) yields the Fourier transform of the finite-duration sequence in terms of its DFT, in the form

$$X(\omega) = \frac{1-e^{-j\omega N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{-j(\omega-2\pi k/N)}} \tag{5.1.39}$$

This expression for the Fourier transform is a polynomial (Lagrange) interpolation formula for $X(\omega)$ expressed in terms of the values $\{X(k)\}$ of the polynomial at a set of equally spaced discrete frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. With some algebraic manipulations, it is possible to reduce (5.1.39) to the interpolation formula given previously in (5.1.13).

Relationship to the Fourier series coefficients of a continuous-time signal. Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period $T_p = 1/F_0$. The signal can be expressed in a Fourier series

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0} \tag{5.1.40}$$

where $\{c_k\}$ are the Fourier coefficients. If we sample $x_a(t)$ at a uniform rate $F_s = N/T_p = 1/T$, we obtain the discrete-time sequence

$$\begin{aligned} x(n) \equiv x_a(nT) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0nT} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kn/N} \\ &= \sum_{k=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} c_{k-lN}\right]e^{j2\pi kn/N} \end{aligned} \tag{5.1.41}$$

It is clear that (5.1.41) is in the form of an IDFT formula, where

$$X(k) = N\sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k \tag{5.1.42}$$

and

$$\tilde{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN} \tag{5.1.43}$$

Thus the $\{\tilde{c}_k\}$ sequence is an aliased version of the sequence $\{c_k\}$.

5.2 PROPERTIES OF THE DFT

In Section 5.1.2 we introduced the DFT as a set of N samples $\{X(k)\}$ of the Fourier transform $X(\omega)$ for a finite-duration sequence $\{x(n)\}$ of length $L \le N$. The sampling of $X(\omega)$ occurs at the N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, 2, \ldots, N-1$. We demonstrated that the N samples $\{X(k)\}$ uniquely represent the sequence $\{x(n)\}$ in the frequency domain. Recall that the DFT and inverse DFT (IDFT) for an N-point sequence $\{x(n)\}$ are given as

$$\text{DFT: } X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} \qquad k = 0, 1, \ldots, N-1 \tag{5.2.1}$$

$$\text{IDFT: } x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn} \qquad n = 0, 1, \ldots, N-1 \tag{5.2.2}$$

where W_N is defined as

$$W_N = e^{-j2\pi/N} \tag{5.2.3}$$

In this section we present the important properties of the DFT. In view of the relationships established in Section 5.1.4 between the DFT and Fourier series, and Fourier transforms and z-transforms of discrete-time signals, we expect the properties of the DFT to resemble the properties of these other transforms and series. However, some important differences exist, one of which is the circular convolution property derived in the following section. A good understanding of these properties is extremely helpful in the application of the DFT to practical problems.

The notation used below to denote the N-point DFT pair $x(n)$ and $X(k)$ is

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

5.2.1 Periodicity, Linearity, and Symmetry Properties

Periodicity. If $x(n)$ and X(k) are an N-point DFT pair, then

$$x(n+N) = x(n) \quad \text{for all } n \tag{5.2.4}$$

$$X(k+N) = X(k) \quad \text{for all } k \tag{5.2.5}$$

These periodicities in $x(n)$ and $X(k)$ follow immediately from formulas (5.2.1) and (5.2.2) for the DFT and IDFT, respectively.

We previously illustrated the periodicity property in the sequence $x(n)$ for a given DFT. However, we had not previously viewed the DFT $X(k)$ as a periodic sequence. In some applications it is advantageous to do this.

Linearity. If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then for any real-valued or complex-valued constants a_1 and a_2,

$$a_1x_1(n) + a_2x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} a_1X_1(k) + a_2X_2(k) \tag{5.2.6}$$

This property follows immediately from the definition of the DFT given by (5.2.1).

Circular Symmetries of a Sequence. As we have seen, the N-point DFT of a finite duration sequence, $x(n)$ of length $L \le N$ is equivalent to the N-point DFT of a periodic sequence $x_p(n)$, of period N, which is obtained by periodically extending $x(n)$, that is,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \tag{5.2.7}$$

Now suppose that we shift the periodic sequence $x_p(n)$ by k units to the right. Thus we obtain another periodic sequence

$$x_p'(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN) \tag{5.2.8}$$

The finite-duration sequence

$$x'(n) = \begin{cases} x_p'(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.2.9}$$

is related to the original sequence $x(n)$ by a circular shift. This relationship is illustrated in Fig. 5.7 for $N = 4$.

In general, the circular shift of the sequence can be represented as the index modulo N. Thus we can write

$$\begin{aligned} x'(n) &= x(n-k, \text{modulo } N) \\ &\equiv x((n-k))_N \end{aligned} \tag{5.2.10}$$

For example, if $k = 2$ and $N = 4$, we have

$$x'(n) = x((n-2))_4$$

which implies that

$$\begin{aligned} x'(0) &= x((-2))_4 = x(2) \\ x'(1) &= x((-1))_4 = x(3) \\ x'(2) &= x((0))_4 = x(0) \\ x'(3) &= x((1))_4 = x(1) \end{aligned}$$

Hence $x'(n)$ is simply $x(n)$ shifted circularly by two units in time, where the counterclockwise direction has been arbitrarily selected as the positive direction. Thus we conclude that a circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension, and vice versa.

The inherent periodicity resulting from the arrangement of the N-point sequence on the circumference of a circle dictates a different definition of even and odd symmetry, and time reversal of a sequence.

An N-point sequence is called circularly *even* if it is symmetric about the point zero on the circle. This implies that

$$x(N-n) = x(n) \qquad 1 \le n \le N-1 \tag{5.2.11}$$

An N-point sequence is called circularly *odd* if it is antisymmetric about the point zero on the circle. This implies that

$$x(N-n) = -x(n) \qquad 1 \le n \le N-1 \tag{5.2.12}$$

The time reversal of an N-point sequence is attained by reversing its samples about the point zero on the circle. Thus the sequence $x((-n))_N$ is simply given as

$$x((-n))_N = x(N-n) \qquad 0 \le n \le N-1 \tag{5.2.13}$$

This time reversal is equivalent to plotting $x(n)$ in a clockwise direction on a circle.

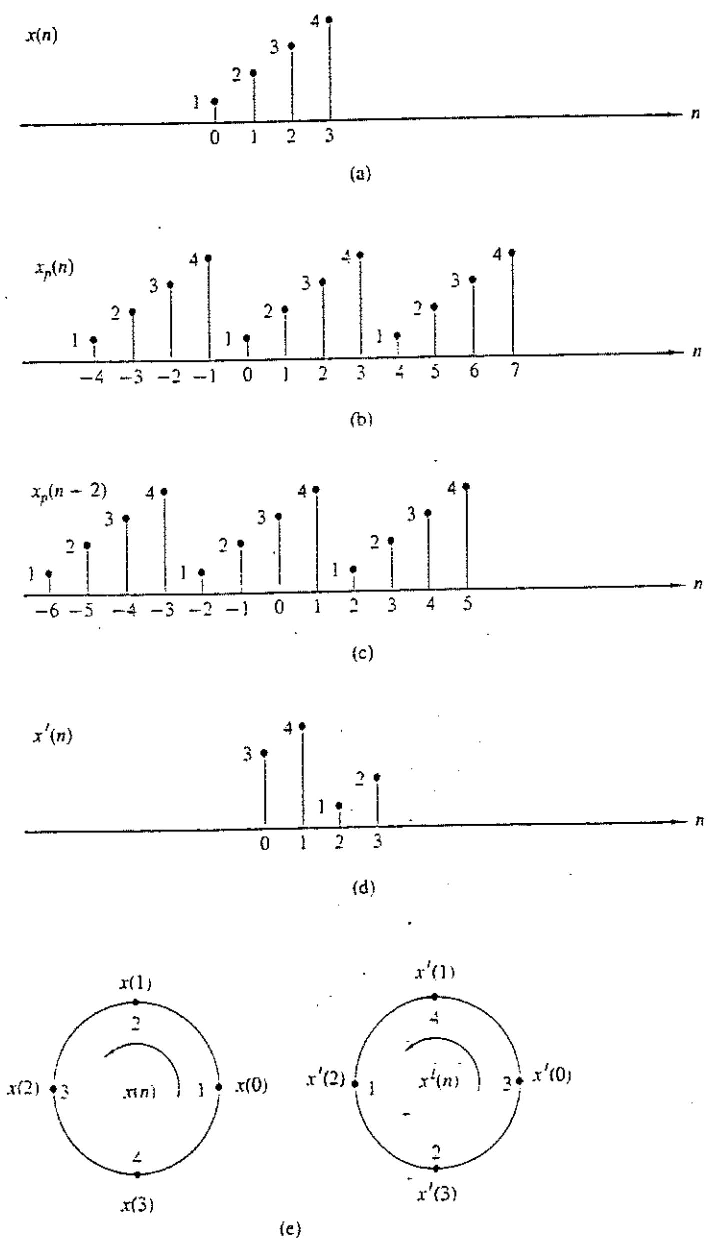


Figure 5.7 Circular shift of a sequence.

An equivalent definition of even and odd sequences for the associated periodic sequence $x_p(n)$ is given as follows

$$\begin{aligned} \text{even:}\quad x_p(n) &= x_p(-n) = x_p(N-n) \\ \text{odd:}\quad x_p(n) &= -x_p(-n) = -x_p(N-n) \end{aligned} \tag{5.2.14}$$

If the periodic sequence is complex-valued, we have

$$\begin{aligned} \text{conjugate even:}\quad x_p(n) &= x_p^*(N-n) \\ \text{conjugate odd:}\quad x_p(n) &= -x_p^*(N-n) \end{aligned} \tag{5.2.15}$$

These relationships suggest that we decompose the sequence $x_p(n)$ as

$$x_p(n) = x_{pe}(n) + x_{po}(n) \tag{5.2.16}$$

where

$$\begin{aligned} x_{pe}(n) &= \tfrac{1}{2}[x_p(n) + x_p^*(N-n)] \\ x_{po}(n) &= \tfrac{1}{2}[x_p(n) - x_p^*(N-n)] \end{aligned} \tag{5.2.17}$$

Symmetry properties of the DFT. The symmetry properties for the DFT can be obtained by applying the methodology previously used for the Fourier transform. Let us assume that the N-point sequence $x(n)$ and its DFT are both complex valued. Then the sequences can be expressed as

$$x(n) = x_R(n) + jx_I(n) \qquad 0 \le n \le N-1 \tag{5.2.18}$$

$$X(k) = X_R(k) + jX_I(k) \qquad 0 \le k \le N-1 \tag{5.2.19}$$

By substituting (5.2.18) into the expression for the DFT given by (5.2.1), we obtain

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos\frac{2\pi kn}{N} + x_I(n) \sin\frac{2\pi kn}{N} \right] \tag{5.2.20}$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n) \sin\frac{2\pi kn}{N} - x_I(n) \cos\frac{2\pi kn}{N} \right] \tag{5.2.21}$$

Similarly, by substituting (5.2.19) into the expression for the IDFT given by (5.2.2), we obtain

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1} \left[X_R(k) \cos\frac{2\pi kn}{N} - X_I(k) \sin\frac{2\pi kn}{N} \right] \tag{5.2.22}$$

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1} \left[X_R(k) \sin\frac{2\pi kn}{N} + X_I(k) \cos\frac{2\pi kn}{N} \right] \tag{5.2.23}$$

Real-valued sequences. If the sequence $x(n)$ is real, it follows directly from (5.2.1) that

$$X(N-k) = X^*(k) = X(-k) \tag{5.2.24}$$

Consequently, $|X(N-k)| = |X(k)|$ and $\angle X(N-k) = -\angle X(k)$. Furthermore, $x_I(n) = 0$ and therefore $x(n)$ can be determined from (5.2.22), which is another form for the IDFT.

Real and even sequences. If $x(n)$ is real and even, that is,

$$x(n) = x(N-n) \qquad 0 \le n \le N-1$$

then (5.2.21) yields $X_I(k) = 0$. Hence the DFT reduces to

$$X(k) = \sum_{n=0}^{N-1} x(n) \cos\frac{2\pi kn}{N} \qquad 0 \le k \le N-1 \tag{5.2.25}$$

which is itself real-valued and even. Furthermore, since $X_I(k) = 0$, the IDFT reduces to

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \cos\frac{2\pi kn}{N} \qquad 0 \le n \le N-1 \tag{5.2.26}$$

Real and odd sequences. If $x(n)$ is real and odd, that is,

$$x(n) = -x(N-n) \qquad 0 \le n \le N-1$$

then (5.2.20) yields $X_R(k) = 0$. Hence

$$X(k) = -j\sum_{n=0}^{N-1} x(n) \sin\frac{2\pi kn}{N} \qquad 0 \le k \le N-1 \tag{5.2.27}$$

which is purely imaginary and odd. Since $X_R(k) = 0$, the IDFT reduces to

$$x(n) = j\frac{1}{N}\sum_{k=0}^{N-1} X(k) \sin\frac{2\pi kn}{N} \qquad 0 \le n \le N-1 \tag{5.2.28}$$

Purely imaginary sequences. In this case, $x(n) = jx_I(n)$. Consequently, (5.2.20) and (5.2.21) reduce to

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin\frac{2\pi kn}{N} \tag{5.2.29}$$

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n) \cos\frac{2\pi kn}{N} \tag{5.2.30}$$

We observe that $X_R(k)$ is odd and $X_I(k)$ is even.

If $x_I(n)$ is odd, then $X_I(k) = 0$ and hence $X(k)$ is purely real. On the other hand, if $x_I(n)$ is even, then $X_R(k) = 0$ and hence $X(k)$ is purely imaginary.

TABLE 5.1 SYMMETRY PROPERTIES OF THE DFT

N-Point Sequence $x(n)$, $0 \le n \le N-1$	N-Point DFT
$x(n)$	$X(k)$
$x^*(n)$	$X^*(N-k)$
$x^*(N-n)$	$X^*(k)$
$x_R(n)$	$X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$
$jX_I(n)$	$X_{co}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$
$x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x^*(N-n)]$	$jX_I(k)$
Real Signals	
Any real signal $x(n)$	$X(k) = X^*(N-k)$ $X_R(k) = X_R(N-k)$ $X_I(k) = -X_I(N-k)$ $\lvert X(k)\rvert = \lvert X(N-k)\rvert$ $\angle X(k) = -\angle X(N-k)$
$x_{ce}(n) = \frac{1}{2}[x(n) + x(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x(N-n)]$	$jX_I(k)$

The symmetry properties given above may be summarized as follows:

$$\begin{aligned} x(n) &= x_R^e(n) + x_R^o(n) + jx_I^e(n) + jx_I^o(n) \\ &\ \ \updownarrow \\ X(k) &= X_R^e(k) + X_R^o(k) + jX_I^e(k) + jX_I^o(k) \end{aligned} \tag{5.2.31}$$

All the symmetry properties of the DFT can easily be deduced from (5.2.31). For example, the DFT of the sequence

$$x_{pe}(n) = \tfrac{1}{2}[x_p(n) + x_p^*(N-n)]$$

is

$$X_R(k) = X_R^e(k) + X_R^o(k)$$

The symmetry properties of the DFT are summarized in Table 5.1. Exploitation of these properties for the efficient computation of the DFT of special sequences is considered in some of the problems at the end of the chapter.

5.2.2 Multiplication of Two DFTs and Circular Convolution

Suppose that we have two finite-duration sequences of length N, $x_1(n)$ and $x_2(n)$. Their respective N-point DFTs are

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.2.32}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.2.33}$$

If we multiply the two DFTs together, the result is a DFT, say $X_3(k)$, of a sequence $x_3(n)$ of length N. Let us determine the relationship between $x_3(n)$ and the sequences $x_1(n)$ and $x_2(n)$.

We have

$$X_3(k) = X_1(k)X_2(k) \qquad k = 0, 1, \dots, N-1 \tag{5.2.34}$$

The IDFT of $\{X_3(k)\}$ is

$$\begin{aligned} x_3(m) &= \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)e^{j2\pi km/N} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2(k)e^{j2\pi km/N} \end{aligned} \tag{5.2.35}$$

Suppose that we substitute for $X_1(k)$ and $X_2(k)$ in (5.2.35) using the DFTs given in (5.2.32) and (5.2.33). Thus we obtain

$$\begin{aligned} x_3(m) &= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2(l)e^{-j2\pi kl/N}\right]e^{j2\pi km/N} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] \end{aligned} \tag{5.2.36}$$

The inner sum in the brackets in (5.2.36) has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a = 1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases} \tag{5.2.37}$$

where a is defined as

$$a = e^{j2\pi(m-n-l)/N}$$

We observe that $a = 1$ when $m - n - l$ is a multiple of N. On the other hand, $a^N = 1$ for any value of $a \neq 0$. Consequently, (5.2.37) reduces to

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = m - n + pN = ((m-n))_N, \quad p \text{ an integer} \\ 0, & \text{otherwise} \end{cases} \tag{5.2.38}$$

If we substitute the result in (5.2.38) into (5.2.36), we obtain the desired expression for $x_3(m)$ in the form

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N \qquad m = 0, 1, \dots, N-1 \tag{5.2.39}$$

The expression in (5.2.39) has the form of a convolution sum. However, it is not the ordinary linear convolution that was introduced in Chapter 2, which relates the output sequence $y(n)$ of a linear system to the input sequence $x(n)$ and the impulse response $h(n)$. Instead, the convolution sum in (5.2.39) involves the index

$((m-n))_N$ and is called *circular convolution*. Thus we conclude that multiplication of the DFTs of two sequences is equivalent to the circular convolution of the two sequences in the time domain.

The following example illustrates the operations involved in circular convolution.

Example 5.2.1

Perform the circular convolution of the following two sequences:

$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

Solution Each sequence consists of four nonzero points. For the purposes of illustrating the operations involved in circular convolution, it is desirable to graph each sequence as points on a circle. Thus the sequences $x_1(n)$ and $x_2(n)$ are graphed as illustrated in Fig. 5.8(a). We note that the sequences are graphed in a counterclockwise direction on a circle. This establishes the reference direction in rotating one of the sequences relative to the other.

Now, $x_3(m)$ is obtained by circularly convolving $x_1(n)$ with $x_2(n)$ as specified by (5.2.39). Beginning with $m = 0$ we have

$$x_3(0) = \sum_{n=0}^{3} x_1(n)x_2((-n))_N$$

$x_2((-n))_4$ is simply the sequence $x_2(n)$ folded and graphed on a circle as illustrated in Fig. 5.8(b). In other words, the folded sequence is simply $x_2(n)$ graphed in a clockwise direction.

The product sequence is obtained by multiplying $x_1(n)$ with $x_2((-n))_4$, point by point. This sequence is also illustrated in Fig. 5.8(b). Finally, we sum the values in the product sequence to obtain

$$x_3(0) = 14$$

For $m = 1$ we have

$$x_3(1) = \sum_{n=0}^{3} x_1(n)x_2((1-n))_4$$

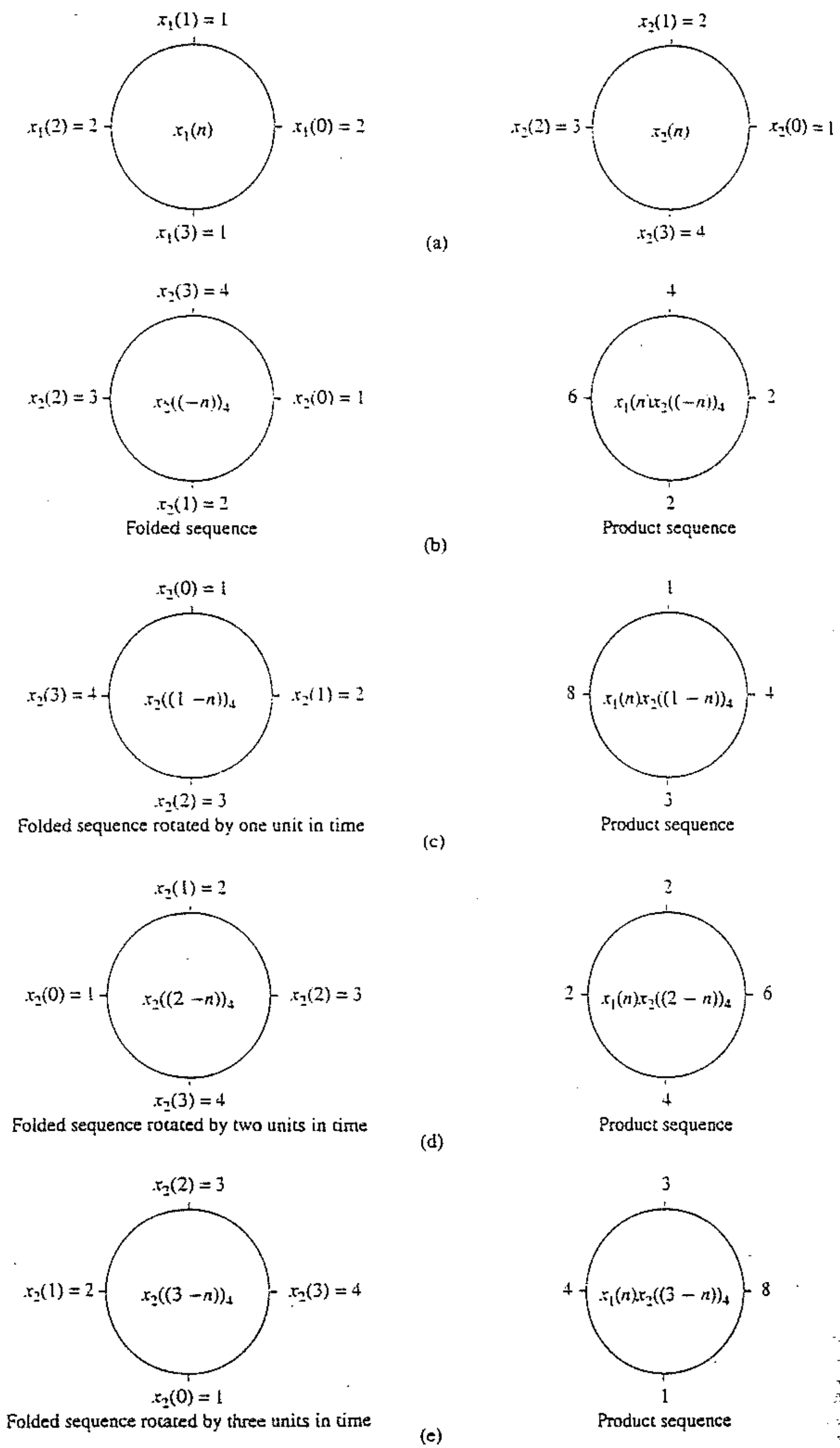
It is easily verified that $x_2((1-n))_4$ is simply the sequence $x_2((-n))_4$ rotated counterclockwise by one unit in time as illustrated in Fig. 5.8(c). This rotated sequence multiplies $x_1(n)$ to yield the product sequence, also illustrated in Fig. 5.8(c). Finally, we sum the values in the product sequence to obtain $x_3(1)$. Thus

$$x_3(1) = 16$$

For $m = 2$ we have

$$x_3(2) = \sum_{n=0}^{3} x_1(n)x_2((2-n))_4$$

Now $x_2((2-n))_4$ is the folded sequence in Fig. 5.8(b) rotated two units of time in the counterclockwise direction. The resultant sequence is illustrated in Fig. 5.8(d)



along with the product sequence $x_1(n)x_2((2-n))_4$. By summing the four terms in the product sequence, we obtain

$$x_3(2) = 14$$

For $m = 3$ we have

$$x_3(3) = \sum_{n=0}^{3} x_1(n)x_2((3-n))_4$$

The folded sequence $x_2((-n))_4$ is now rotated by three units in time to yield $x_2((3-n))_4$ and the resultant sequence is multiplied by $x_1(n)$ to yield the product sequence as illustrated in Fig. 5.8(e). The sum of the values in the product sequence is

$$x_3(3) = 16$$

We observe that if the computation above is continued beyond $m = 3$, we simply repeat the sequence of four values obtained above. Therefore, the circular convolution of the two sequences $x_1(n)$ and $x_2(n)$ yields the sequence

$$x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$$

From this example, we observe that circular convolution involves basically the same four steps as the ordinary *linear convolution* introduced in Chapter 2: *folding* (time reversing) one sequence, *shifting* the folded sequence, *multiplying* the two sequences to obtain a product sequence, and finally, *summing* the values of the product sequence. The basic difference between these two types of convolution is that, in circular convolution, the folding and shifting (rotating) operations are performed in a circular fashion by computing the index of one of the sequences modulo N. In linear convolution, there is no modulo N operation.

The reader can easily show from our previous development that either one of the two sequences may be folded and rotated without changing the result of the circular convolution. Thus

$$x_3(m) = \sum_{n=0}^{N-1} x_2(n)x_1((m-n))_N \qquad m = 0, 1, \ldots, N-1 \tag{5.2.40}$$

The following example serves to illustrate the computation of $x_3(n)$ by means of the DFT and IDFT.

Example 5.2.2

By means of the DFT and IDFT, determine the sequence $x_3(n)$ corresponding to the circular convolution of the sequences $x_1(n)$ and $x_2(n)$ given in Example 5.2.1.

Solution First we compute the DFTs of $x_1(n)$ and $x_2(n)$. The four-point DFT of $x_1(n)$ is

$$X_1(k) = \sum_{n=0}^{3} x_1(n)e^{-j2\pi nk/4} \qquad k = 0, 1, 2, 3$$

$$= 2 + e^{-j\pi k/2} + 2e^{-j\pi k} + e^{-j3\pi k/2}$$

Thus

$$X_1(0) = 6 \qquad X_1(1) = 0 \qquad X_1(2) = 2 \qquad X_1(3) = 0$$

The DFT of $x_2(n)$ is

$$\begin{aligned} X_2(k) &= \sum_{n=0}^{3} x_2(n) e^{-j2\pi nk/4} \qquad k = 0, 1, 2, 3 \\ &= 1 + 2e^{-j\pi k/2} + 3e^{-j\pi k} + 4e^{-j3\pi k/2} \end{aligned}$$

Thus

$$X_2(0) = 10 \qquad X_2(1) = -2 + j2 \qquad X_2(2) = -2 \qquad X_2(3) = -2 - j2$$

When we multiply the two DFTs, we obtain the product

$$X_3(k) = X_1(k) X_2(k)$$

or, equivalently,

$$X_3(0) = 60 \qquad X_3(1) = 0 \qquad X_3(2) = -4 \qquad X_3(3) = 0$$

Now, the IDFT of $X_3(k)$ is

$$\begin{aligned} x_3(n) &= \sum_{k=0}^{3} X_3(k) e^{j2\pi nk/4} \qquad n = 0, 1, 2, 3 \\ &= \tfrac{1}{4}(60 - 4e^{j\pi n}) \end{aligned}$$

Thus

$$x_3(0) = 14 \qquad x_3(1) = 16 \qquad x_3(2) = 14 \qquad x_3(3) = 16$$

which is the result obtained in Example 5.2.1 from circular convolution.

We conclude this section by formally stating this important property of the DFT.

Circular convolution. If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then

$$x_1(n) \, \textcircled{N} \, x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k) X_2(k) \tag{5.2.41}$$

where $x_1(n) \, \textcircled{N} \, x_2(n)$ denotes the circular convolution of the sequence $x_1(n)$ and $x_2(n)$.

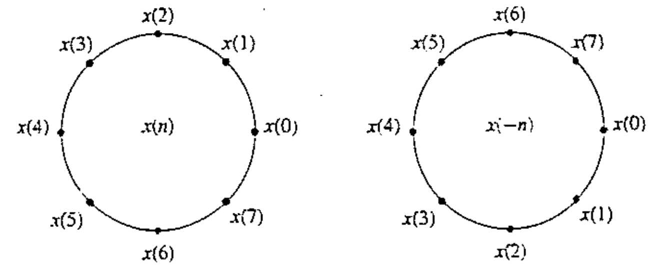


Figure 5.9 Time reversal of a sequence.

5.2.3 Additional DFT Properties

Time reversal of a sequence. If

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

then

$$x((-n))_N = x(N-n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X((-k))_N = X(N-k) \tag{5.2.42}$$

Hence reversing the N-point sequence in time is equivalent to reversing the DFT values. Time reversal of a sequence $x(n)$ is illustrated in Fig. 5.9.

Proof. From the definition of the DFT in (5.2.1) we have

$$\text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n) e^{-j2\pi kn/N}$$

If we change the index from n to $m = N - n$, then

$$\begin{aligned} \text{DFT}\{x(N-n)\} &= \sum_{m=0}^{N-1} x(m) e^{-j2\pi k(N-m)/N} \\ &= \sum_{m=0}^{N-1} x(m) e^{j2\pi km/N} \\ &= \sum_{m=0}^{N-1} x(m) e^{-j2\pi m(N-k)/N} = X(N-k) \end{aligned}$$

We note that $X(N-k) = X((-k))_N$, $0 \le k \le N-1$.

Circular time shift of a sequence. If

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

then

$$x((n-l))_N \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)e^{-j2\pi kl/N} \tag{5.2.43}$$

Proof. From the definition of the DFT we have

$$\begin{aligned}\text{DFT}\{x((n-l))_N\} &= \sum_{n=0}^{N-1} x((n-l))_N e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N} \\ &\quad + \sum_{n=l}^{N-1} x(n-l)e^{-j\pi kn/N}\end{aligned}$$

But $x((n-l))_N = x(N-l+n)$. Consequently,

$$\begin{aligned}\sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N} &= \sum_{n=0}^{l-1} x(N-l+n)e^{-j2\pi kn/N} \\ &= \sum_{m=N-l}^{N-1} x(m)e^{-j2\pi k(m+l)/N}\end{aligned}$$

Furthermore,

$$\sum_{n=l}^{N-1} x(n-l)e^{-j2\pi kn/N} = \sum_{m=0}^{N-1-l} x(m)e^{-j2\pi k(m+l)/N}$$

Therefore,

$$\begin{aligned}\text{DFT}\{x((n-l))\} &= \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(m+l)/N} \\ &= X(k)e^{-j2\pi kl/N}\end{aligned}$$

Circular frequency shift. If

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

then

$$x(n)e^{j2\pi ln/N} \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X((k-l))_N \tag{5.2.44}$$

Hence, the multiplication of the sequence $x(n)$ with the complex exponential sequence $e^{j2\pi kn/N}$ is equivalent to the circular shift of the DFT by l units in frequency. This is the dual to the circular time-shifting property and its proof is similar to the latter.

Complex-conjugate properties. If

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

then

$$x^*(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X^*((-k))_N = X^*(N-k) \tag{5.2.45}$$

The proof of this property is left as an exercise for the reader. The IDFT of $X^*(k)$ is

$$\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)e^{j2\pi kn/N} = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi k(N-n)/N}\right]^*$$

Therefore,

$$x^*((-n))_N = x^*(N-n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X^*(k) \tag{5.2.46}$$

Circular correlation. In general, for complex-valued sequences $x(n)$ and $y(n)$, if

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

and

$$y(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} Y(k)$$

then

$$\tilde{r}_{xy}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \tilde{R}_{xy}(k) = X(k)Y^*(k) \tag{5.2.47}$$

where $\tilde{r}_{xy}(l)$ is the (unnormalized) circular crosscorrelation sequence, defined as

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)y^*((n-l))_N$$

Proof. We can write $\tilde{r}_{xy}(l)$ as the circular convolution of $x(n)$ with $y^*(-n)$, that is,

$$\tilde{r}_{xy}(l) = x(l) \, (N) \, y^*(-l)$$

Then, with the aid of the properties in (5.2.41) and (5.2.46), the N-point DFT of $\tilde{r}_{xy}(l)$ is

$$\tilde{R}_{xy}(k) = X(k)Y^*(k)$$

In the special case where $y(n) = x(n)$, we have the corresponding expression for the circular autocorrelation of $x(n)$,

$$\tilde{r}_{xx}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \tilde{R}_{xx}(k) = |X(k)|^2 \tag{5.2.48}$$

Multiplication of two sequences. If

$$x_1(n) \xleftrightarrow[N]{\text{DFT}} X_1(k)$$

and

$$x_2(n) \xleftrightarrow[N]{\text{DFT}} X_2(k)$$

then

$$x_1(n)x_2(n) \xleftrightarrow[N]{\text{DFT}} \frac{1}{N} X_1(k) \; Ⓝ \; X_2(k) \tag{5.2.49}$$

This property is the dual of (5.2.41). Its proof follows simply by interchanging the roles of time and frequency in the expression for the circular convolution of two sequences.

Parseval's theorem. For complex-valued sequences $x(n)$ and $y(n)$, in general, if

$$x(n) \xleftrightarrow[N]{\text{DFT}} X(k)$$

and

$$y(n) \xleftrightarrow[N]{\text{DFT}} Y(k)$$

then

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)Y^*(k) \tag{5.2.50}$$

Proof. The property follows immediately from the circular correlation property in (5.2.47). We have

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \tilde{r}_{xy}(0)$$

and

$$\tilde{r}_{xy}(l) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{R}_{xy}(k) e^{j2\pi kl/N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k)Y^*(k) e^{j2\pi kl/N}$$

Hence (5.2.50) follows by evaluating the IDFT at $l = 0$.

The expression in (5.2.50) is the general form of Parseval's theorem. In the special case where $y(n) = x(n)$, (5.2.50) reduces to

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 \tag{5.2.51}$$

TABLE 5.2 PROPERTIES OF THE DFT

Property	Time Domain	Frequency Domain
Notation	$x(n), y(n)$	$X(k), Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular convolution	$x_1(n) \; Ⓝ \; x_2(n)$	$X_1(k)X_2(k)$
Circular correlation	$x(n) \; Ⓝ \; y^*(-n)$	$X(k)Y^*(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \; Ⓝ \; X_2(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

which expresses the energy in the finite-duration sequence $x(n)$ in terms of the frequency components $\{X(k)\}$.

The properties of the DFT given above are summarized in Table 5.2.

5.3 LINEAR FILTERING METHODS BASED ON THE DFT

Since the DFT provides a discrete frequency representation of a finite-duration sequence in the frequency domain, it is interesting to explore its use as a computational tool for linear system analysis and, especially, for linear filtering. We have already established that a system with frequency response $H(\omega)$, when excited with an input signal that has a spectrum $X(\omega)$, possesses an output spectrum $Y(\omega) = X(\omega)H(\omega)$. The output sequence $y(n)$ is determined from its spectrum via the inverse Fourier transform. Computationally, the problem with this frequency-domain approach is that $X(\omega)$, $H(\omega)$, and $Y(\omega)$ are functions of the continuous variable ω. As a consequence, the computations cannot be done on a digital computer, since the computer can only store and perform computations on quantities at discrete frequencies.

On the other hand, the DFT does lend itself to computation on a digital computer. In the discussion that follows, we describe how the DFT can be used to perform linear filtering in the frequency domain. In particular, we present a computational procedure that serves as an alternative to time-domain convolution. In fact, the frequency-domain approach based on the DFT, is computationally more efficient than time-domain convolution due to the existence of efficient algorithms for computing the DFT. These algorithms, which are described in Chapter 6, are collectively called fast Fourier transform (FFT) algorithms.

5.3.1 Use of the DFT in Linear Filtering

In the preceding section it was demonstrated that the product of two DFTs is equivalent to the circular convolution of the corresponding time-domain sequences. Unfortunately, circular convolution is of no use to us if our objective is to determine the output of a linear filter to a given input sequence. In this case we seek a frequency-domain methodology equivalent to linear convolution.

Suppose that we have a finite-duration sequence $x(n)$ of length L which excites an FIR filter of length M. Without loss of generality, let

$$x(n) = 0, \qquad n < 0 \text{ and } n \geq L$$

$$h(n) = 0, \qquad n < 0 \text{ and } n \geq M$$

where $h(n)$ is the impulse response of the FIR filter.

The output sequence $y(n)$ of the FIR filter can be expressed in the time domain as the convolution of $x(n)$ and $h(n)$, that is

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \tag{5.3.1}$$

Since $h(n)$ and $x(n)$ are finite-duration sequences, their convolution is also finite in duration. In fact, the duration of $y(n)$ is $L + M - 1$.

The frequency-domain equivalent to (5.3.1) is

$$Y(\omega) = X(\omega)H(\omega) \tag{5.3.2}$$

If the sequence $y(n)$ is to be represented uniquely in the frequency domain by samples of its spectrum $Y(\omega)$ at a set of discrete frequencies, the number of distinct samples must equal or exceed $L + M - 1$. Therefore, a DFT of size $N \geq L + M - 1$, is required to represent $\{y(n)\}$ in the frequency domain.

Now if

$$\begin{aligned} Y(k) &\equiv Y(\omega)|_{\omega=2\pi k/N} & k = 0, 1, \ldots, N-1 \\ &= X(\omega)H(\omega)|_{\omega=2\pi k/N} & k = 0, 1, \ldots, N-1 \end{aligned}$$

then

$$Y(k) = X(k)H(k) \qquad k = 0, 1, \ldots, N-1 \tag{5.3.3}$$

where $\{X(k)\}$ and $\{H(k)\}$ are the N-point DFTs of the corresponding sequences $x(n)$ and $h(n)$, respectively. Since the sequences $x(n)$ and $h(n)$ have a duration less than N, we simply pad these sequences with zeros to increase their length to N. This increase in the size of the sequences does not alter their spectra $X(\omega)$ and $H(\omega)$, which are continuous spectra, since the sequences are aperiodic. However, by sampling their spectra at N equally spaced points in frequency (computing the N-point DFTs), we have increased the number of samples that represent these sequences in the frequency domain beyond the minimum number (L or M, respectively).

Since the $N = L + M - 1$-point DFT of the output sequence $y(n)$ is sufficient to represent $y(n)$ in the frequency domain, it follows that the multiplication of the N-point DFTs $X(k)$ and $H(k)$, according to (5.3.3), followed by the computation of the N-point IDFT, must yield the sequence $\{y(n)\}$. In turn, this implies that the N-point circular convolution of $x(n)$ with $h(n)$ must be equivalent to the linear convolution of $x(n)$ with $h(n)$. In other words, by increasing the length of the sequences $x(n)$ and $h(n)$ to N points (by appending zeros), and then circularly convolving the resulting sequences, we obtain the same result as would have been obtained with linear convolution. Thus with zero padding, the DFT can be used to perform linear filtering.

The following example illustrates the methodology in the use of the DFT in linear filtering.

Example 5.3.1

By means of the DFT and IDFT, determine the response of the FIR filter with impulse response

$$h(n) = \{\underset{\uparrow}{1}, 2, 3\}$$

to the input sequence

$$x(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$$

Solution The input sequence has length $L = 4$ and the impulse response has length $M = 3$. Linear convolution of these two sequences produces a sequence of length $N = 6$. Consequently, the size of the DFTs must be at least six.

For simplicity we compute eight-point DFTs. We should also mention that the efficient computation of the DFT via the fast Fourier transform (FFT) algorithm is usually performed for a length N that is a power of 2. Hence the eight-point DFT of $x(n)$ is

$$\begin{aligned} X(k) &= \sum_{n=0}^{7} x(n)e^{-j2\pi kn/8} \\ &= 1 + 2e^{-j\pi k/4} + 2e^{-j\pi k/2} + e^{-j3\pi k/4} \qquad k = 0, 1, \ldots, 7 \end{aligned}$$

This computation yields

$$X(0) = 6 \qquad X(1) = \frac{2+\sqrt{2}}{2} - j\left(\frac{4+3\sqrt{2}}{2}\right)$$

$$X(2) = -1 - j \qquad X(3) = \frac{2-\sqrt{2}}{2} + j\left(\frac{4-3\sqrt{2}}{2}\right)$$

$$X(4) = 0 \qquad X(5) = \frac{2-\sqrt{2}}{2} - j\frac{4-3\sqrt{2}}{2}$$

$$X(6) = -1 + j \qquad X(7) = \frac{2+\sqrt{2}}{2} + j\left(\frac{4+3\sqrt{2}}{2}\right)$$

The eight-point DFT of $h(n)$ is

$$H(k) = \sum_{n=0}^{7} h(n)e^{-j2\pi kn/8}$$
$$= 1 + 2e^{-j\pi k/4} + 3e^{-j\pi k/2}$$

Hence

$$H(0) = 6, \quad H(1) = 1 + \sqrt{2} - j\left(3 + \sqrt{2}\right), \quad H(2) = -2 - j2$$
$$H(3) = 1 - \sqrt{2} + j\left(3 - \sqrt{2}\right), \quad H(4) = 2$$
$$H(5) = 1 - \sqrt{2} - j\left(3 - \sqrt{2}\right), \quad H(6) = -2 + j2$$
$$H(7) = 1 + \sqrt{2} + j\left(3 + \sqrt{2}\right)$$

The product of these two DFTs yields $Y(k)$, which is

$$Y(0) = 36, \quad Y(1) = -14.07 - j17.48 \quad Y(2) = j4 \quad Y(3) = 0.07 + j0.515$$
$$Y(4) = 0, \quad Y(5) = 0.07 - j0.515 \quad Y(6) = -j4 \quad Y(7) = -14.07 + j17.48$$

Finally, the eight-point IDFT is

$$y(n) = \sum_{k=0}^{7} Y(k)e^{j2\pi kn/8} \quad n = 0, 1, \ldots, 7$$

This computation yields the result

$$y(n) = \{\underset{\uparrow}{1}, 4, 9, 11, 8, 3, 0, 0\}$$

We observe that the first six values of $y(n)$ constitute the set of desired output values. The last two values are zero because we used an eight-point DFT and IDFT, when, in fact, the minimum number of points required is six.

Although the multiplication of two DFTs corresponds to circular convolution in the time domain, we have observed that padding the sequences $x(n)$ and $h(n)$ with a sufficient number of zeros forces the circular convolution to yield the same output sequence as linear convolution. In the case of the FIR filtering problem in Example 5.3.1, it is a simple matter to demonstrate that the six-point circular convolution of the sequences

$$h(n) = \{\underset{\uparrow}{1}, 2, 3, 0, 0, 0\} \tag{5.3.4}$$

$$x(n) = \{\underset{\uparrow}{1}, 2, 2, 1, 0, 0\} \tag{5.3.5}$$

results in the output sequence

$$y(n) = \{\underset{\uparrow}{1}, 4, 9, 11, 8, 3\} \tag{5.3.6}$$

which is the same sequence obtained from linear convolution.

It is important for us to understand the aliasing that results in the time domain when the size of the DFTs is smaller than $L + M - 1$. The following example focuses on the aliasing problem.

Example 5.3.2

Determine the sequence $y(n)$ that results from the use of four point DFTs in Example 5.3.1.

Solution The four-point DFT of $h(n)$ is

$$H(k) = \sum_{n=0}^{3} h(n)e^{-j2\pi kn/4}$$

$$H(k) = 1 + 2e^{-j\pi k/2} + 3e^{-jk\pi} \quad k = 0, 1, 2, 3$$

Hence

$$H(0) = 6, \quad H(1) = -2 - j2, \quad H(2) = 2, \quad H(3) = -2 + j2$$

The four-point DFT of $x(n)$ is

$$X(k) = 1 + 2e^{-j\pi k/2} + 2e^{-j\pi k} + 3e^{-j3\pi k/2} \quad k = 0, 1, 2, 3$$

Hence

$$X(0) = 6, \quad X(1) = -1 - j, \quad X(2) = 0, \quad X(3) = -1 + j$$

The product of these two four-point DFTs is

$$\hat{Y}(0) = 36, \quad \hat{Y}(1) = j4, \quad \hat{Y}(2) = 0, \quad \hat{Y}(3) = -j4$$

The four-point IDFT yields

$$\hat{y}(n) = \frac{1}{4}\sum_{k=0}^{3} \hat{Y}(k)e^{j2\pi kn/4} \quad n = 0, 1, 2, 3$$
$$= \tfrac{1}{4}(36 + j4e^{j\pi n/2} - j4e^{j3\pi n/2})$$

Therefore,

$$\hat{y}(n) = \{\underset{\uparrow}{9}, 7, 9, 11\}$$

The reader can verify that the four-point circular convolution of $h(n)$ with $x(n)$ yields the same sequence $\hat{y}(n)$.

If we compare the result $\hat{y}(n)$, obtained from four-point DFTs with the sequence $y(n)$ obtained from the use of eight-point (or six-point) DFTs, the time-domain aliasing effects derived in Section 5.2.2 are clearly evident. In particular, $y(4)$ is aliased into $y(0)$ to yield

$$\hat{y}(0) = y(0) + y(4) = 9$$

Similarly, $y(5)$ is aliased into $y(1)$ to yield

$$\hat{y}(1) = y(1) + y(5) = 7$$

All other aliasing has no effect since $y(n) = 0$ for $n \geq 6$. Consequently, we have

$$\hat{y}(2) = y(2) = 9$$
$$\hat{y}(3) = y(3) = 11$$

Therefore, only the first two points of $\hat{y}(n)$ are corrupted by the effect of aliasing [i.e., $\hat{y}(0) \neq y(0)$ and $\hat{y}(1) \neq y(1)$]. This observation has important ramifications in the discussion of the following section, in which we treat the filtering of long sequences.

5.3.2 Filtering of Long Data Sequences

In practical applications involving linear filtering of signals, the input sequence $x(n)$ is often a very long sequence. This is especially true in some real-time signal processing applications concerned with signal monitoring and analysis.

Since linear filtering performed via the DFT involves operations on a block of data, which by necessity must be limited in size due to limited memory of a digital computer, a long input signal sequence must be segmented to fixed-size blocks prior to processing. Since the filtering is linear, successive blocks can be processed one at a time via the DFT and the output blocks are fitted together to form the overall output signal sequence.

We now describe two methods for linear FIR filtering a long sequence on a block-by-block basis using the DFT. The input sequence is segmented into blocks and each block is processed via the DFT and IDFT to produce a block of output data. The output blocks are fitted together to form an overall output sequence which is identical to the sequence obtained if the long block had been processed via time-domain convolution.

The two methods are called the *overlap-save method* and the *overlap-add method*. For both methods we assume that the FIR filter has duration M. The input data sequence is segmented into blocks of L points, where, by assumption, $L >> M$ without loss of generality.

Overlap-save method. In this method the size of the input data blocks is $N = L + M - 1$ and the size of the DFTs and IDFT are of length N. Each data block consists of the last $M - 1$ data points of the previous data block followed by L new data points to form a data sequence of length $N = L + M - 1$. An N-point DFT is computed for each data block. The impulse response of the FIR filter is increased in length by appending $L - 1$ zeros and an N-point DFT of the sequence is computed once and stored. The multiplication of the two N-point DFTs $\{H(k)\}$ and $\{X_m(k)\}$ for the mth block of data yields

$$\hat{Y}_m(k) = H(k)X_m(k) \qquad k = 0, 1, \ldots, N-1 \tag{5.3.7}$$

Then the N-point IDFT yields the result

$$\hat{Y}_m(n) = \{\hat{y}_m(0)\hat{y}_m(1)\cdots\hat{y}_m(M-1)\hat{y}_m(M)\cdots\hat{y}_m(N-1)\} \tag{5.3.8}$$

Since the data record is of length N, the first $M - 1$ points of $y_m(n)$ are corrupted by aliasing and must be discarded. The last L points of $y_m(n)$ are exactly the same as the result from linear convolution and, as a consequence,

$$\hat{y}_m(n) = y_m(n), n = M, M+1, \ldots, N-1 \tag{5.3.9}$$

To avoid loss of data due to aliasing, the last $M - 1$ points of each data record are saved and these points become the first $M - 1$ data points of the subsequent record, as indicated above. To begin the processing, the first $M - 1$ points of the first record are set to zero. Thus the blocks of data sequences are

$$x_1(n) = \{\underbrace{0, 0, \ldots, 0}_{M-1 \text{ points}}, x(0), x(1), \ldots, x(L-1)\} \tag{5.3.10}$$

$$x_2(n) = \{\underbrace{x(L-M+1), \ldots, x(L-1)}_{\substack{M-1 \text{ data points} \\ \text{from } x_1(n)}}, \underbrace{x(L), \ldots, x(2L-1)}_{L \text{ new data points}}\} \tag{5.3.11}$$

$$x_3(n) = \{\underbrace{x(2L-M+1), \ldots, x(2L-1)}_{\substack{M-1 \text{ data points} \\ \text{from } x_2(n)}}, \underbrace{x(2L), \ldots, x(3L-1)}_{L \text{ new data points}}\} \tag{5.3.12}$$

and so forth. The resulting data sequences from the IDFT are given by (5.3.8), where the first $M - 1$ points are discarded due to aliasing and the remaining L points constitute the desired result from linear convolution. This segmentation of the input data and the fitting of the output data blocks together to form the output sequence are graphically illustrated in Fig. 5.10.

Overlap-add method. In this method the size of the input data block is L points and the size of the DFTs and IDFT is $N = L + M - 1$. To each data block we append $M - 1$ zeros and compute the N-point DFT. Thus the data blocks may be represented as

$$x_1(n) = \{x(0), x(1), \ldots, x(L-1), \underbrace{0, 0, \ldots, 0}_{M-1 \text{ zeros}}\} \tag{5.3.13}$$

$$x_2(n) = \{x(L), x(L+1), \ldots, x(2L-1), \underbrace{0, 0, \ldots, 0}_{M-1 \text{ zeros}}\} \tag{5.3.14}$$

$$x_3(n) = \{x(2L), \ldots, x(3L-1), \underbrace{0, 0, \ldots, 0}_{M-1 \text{ zeros}}\} \tag{5.3.15}$$

and so on. The two N-point DFTs are multiplied together to form

$$Y_m(k) = H(k)X_m(k) \qquad k = 0, 1, \ldots, N-1 \tag{5.3.16}$$

The IDFT yields data blocks of length N that are free of aliasing since the size of the DFTs and IDFT is $N = L + M - 1$ and the sequences are increased to N-points by appending zeros to each block.

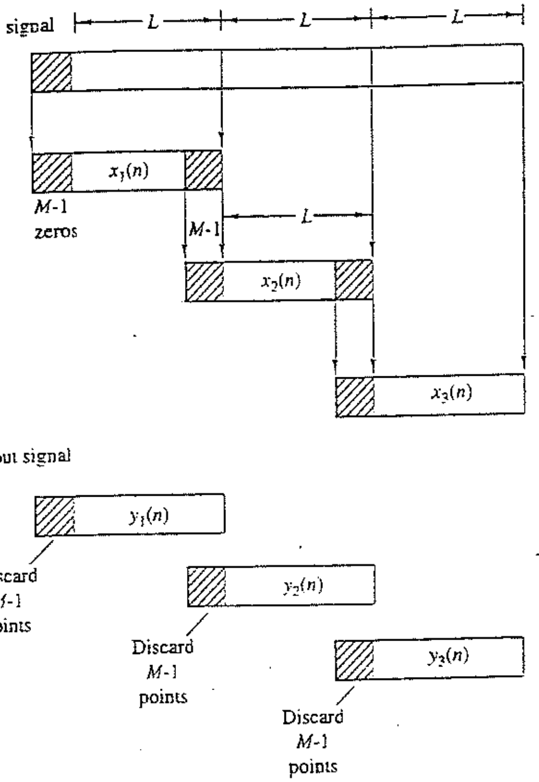


Figure 5.10 Linear FIR filtering by the overlap-save method.

Since each data block is terminated with $M-1$ zeros, the last $M-1$ points from each output block must be overlapped and added to the first $M-1$ points of the succeeding block. Hence this method is called the overlap-add method. This overlapping and adding yields the output sequence

$$y(n) = \{y_1(0), y_1(1), \ldots, y_1(L-1), y_1(L)+y_2(0), y_1(L+1)+ \\ y_2(1), \ldots, y_1(N-1)+y_2(M-1), y_2(M), \ldots\} \tag{5.3.17}$$

The segmentation of the input data into blocks and the fitting of the output data blocks to form the output sequence are graphically illustrated in Fig. 5.11.

At this point, it may appear to the reader that the use of the DFT in linear FIR filtering is not only an indirect method of computing the output of an FIR filter, but it may also be more expensive computationally since the input data must first be converted to the frequency domain via the DFT, multiplied by the DFT of the FIR filter, and finally, converted back to the time domain via the IDFT. On the contrary, however, by using the fast Fourier transform algorithm, as will be shown in Chapter 6, the DFTs and IDFT require fewer computations to compute the output sequence than the direct realization of the FIR filter in the time

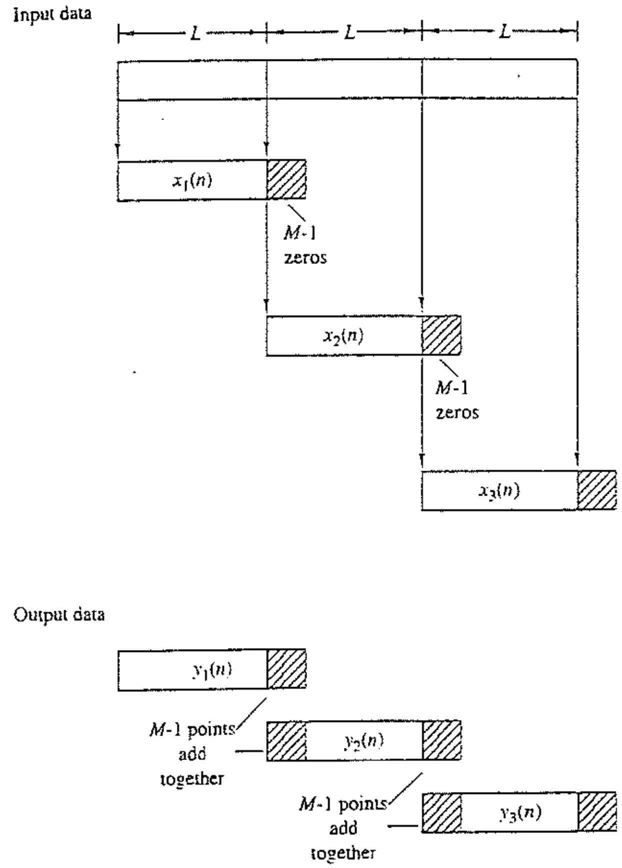


Figure 5.11 Linear FIR filtering by the overlap-add method.

domain. This computational efficiency is the basic advantage of using the DFT to compute the output of an FIR filter.

5.4 FREQUENCY ANALYSIS OF SIGNALS USING THE DFT

To compute the spectrum of either a continuous-time or discrete-time signal, the values of the signal for all time are required. However, in practice, we observe signals for only a finite duration. Consequently, the spectrum of a signal can only be approximated from a finite data record. In this section we examine the implications of a finite data record in frequency analysis using the DFT.

If the signal to be analyzed is an analog signal, we would first pass it through an antialiasing filter and then sample it at a rate $F_s \geq 2B$, where B is the bandwidth of the filtered signal. Thus the highest frequency that is contained in the sampled signal is $F_s/2$. Finally, for practical purposes, we limit the duration of the signal to the time interval $T_0 = LT$, where L is the number of samples and T

is the sample interval. As we shall observe in the following discussion, the finite observation interval for the signal places a limit on the frequency resolution; that is, it limits our ability to distinguish two frequency components that are separated by less than $1/T_0 = 1/LT$ in frequency.

Let $\{x(n)\}$ denote the sequence to be analyzed. Limiting the duration of the sequence to L samples, in the interval $0 \le n \le L-1$, is equivalent to multiplying $\{x(n)\}$ by a rectangular window $w(n)$ of length L. That is,

$$\hat{x}(n) = x(n)w(n) \tag{5.4.1}$$

where

$$w(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.4.2}$$

Now suppose that the sequence $x(n)$ consists of a single sinusoid, that is,

$$x(n) = \cos \omega_0 n \tag{5.4.3}$$

Then the Fourier transform of the finite-duration sequence $x(n)$ can be expressed as

$$\hat{X}(\omega) = \tfrac{1}{2}[W(\omega - \omega_0) + W(\omega + \omega_0)] \tag{5.4.4}$$

where $W(\omega)$ is the Fourier transform of the window sequence, which is (for the rectangular window)

$$W(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2} \tag{5.4.5}$$

To compute $\hat{X}(\omega)$ we use the DFT. By padding the sequence $\hat{x}(n)$ with $N-L$ zeros, we can compute the N-point DFT of the truncated (L points) sequence $\{\hat{x}(n)\}$. The magnitude spectrum $|\hat{X}(k)| = |\hat{X}(\omega_k)|$ for $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N$, is illustrated in Fig. 5.12 for $L = 25$ and $N = 2048$. We note that the windowed spectrum $\hat{X}(\omega)$ is not localized to a single frequency, but instead it is spread out over the whole frequency range. Thus the power of the original signal sequence $\{x(n)\}$ that was concentrated at a single frequency has been spread by the window into the entire frequency range. We say that the power has "leaked out" into the entire frequency range. Consequently, this phenomenon, which is a characteristic of windowing the signal, is called *leakage*.

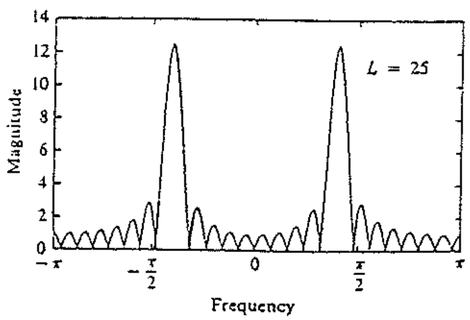


Figure 5.12 Magnitude spectrum for $L = 25$ and $n = 2048$, illustrating the occurrence of leakage.

Windowing not only distorts the spectral estimate due to the leakage effects, it also reduces spectral resolution. To illustrate this problem, let us consider a signal sequence consisting of two frequency components,

$$x(n) = \cos \omega_1 n + \cos \omega_2 n \tag{5.4.6}$$

When this sequence is truncated to L samples in the range $0 \le n \le L-1$, the windowed spectrum is

$$\hat{X}(\omega) = \tfrac{1}{2}[W(\omega - \omega_1) + W(\omega - \omega_2) + W(\omega + \omega_1) + W(\omega + \omega_2)] \tag{5.4.7}$$

The spectrum $W(\omega)$ of the rectangular window sequence has its first zero crossing at $\omega = 2\pi/L$. Now if $|\omega_1 - \omega_2| < 2\pi/L$, the two window functions $W(\omega - \omega_1)$ and $W(\omega - \omega_2)$ overlap and, as a consequence, the two spectral lines in $x(n)$ are not distinguishable. Only if $(\omega_1 - \omega_2) \ge 2\pi/L$ will we see two separate lobes in the spectrum $\hat{X}(\omega)$. Thus our ability to resolve spectral lines of different frequencies is limited by the window main lobe width. Figure 5.13 illustrates the magnitude spectrum $|\hat{X}(\omega)|$, computed via the DFT, for the sequence

$$x(n) = \cos \omega_0 n + \cos \omega_1 n + \cos \omega_2 n \tag{5.4.8}$$

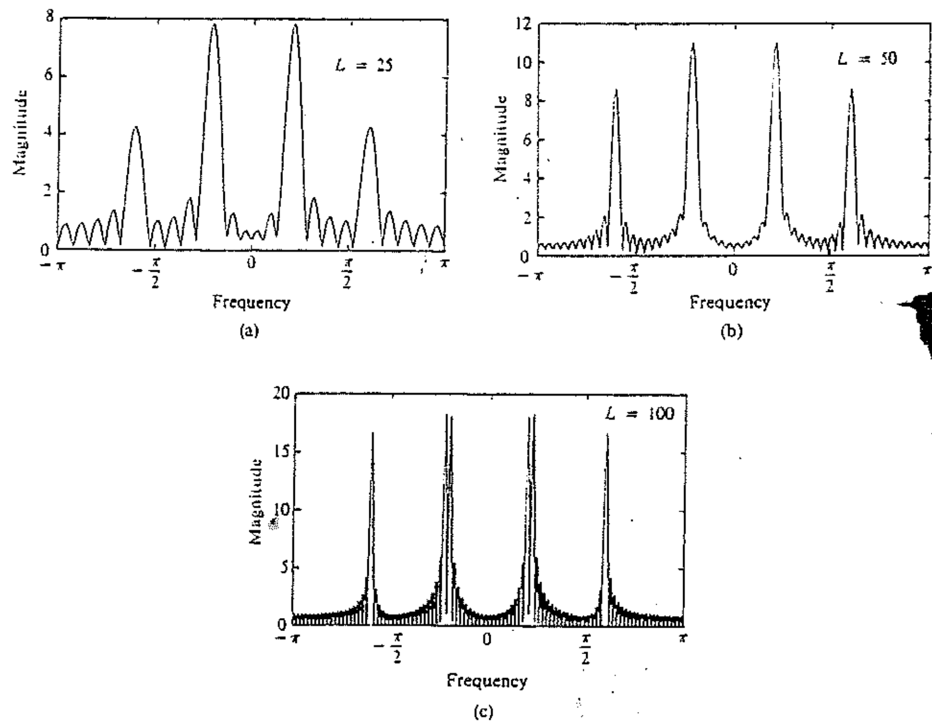


Figure 5.13 Magnitude spectrum for the signal given by (5.4.8), as observed through a rectangular window.

where $\omega_0 = 0.2\pi$, $\omega_1 = 0.22\pi$, and $\omega_2 = 0.6\pi$. The window lengths selected are $L = 25$, 50, and 100. Note that ω_0 and ω_1 are not resolvable for $L = 25$ and 50, but they are resolvable for $L = 100$.

To reduce leakage, we can select a data window $w(n)$ that has lower sidelobes in the frequency domain compared with the rectangular window. However, as we describe in more detail in Chapter 8, a reduction of the sidelobes in a window $W(\omega)$ is obtained at the expense of an increase in the width of the main lobe of $W(\omega)$ and hence a loss in resolution. To illustrate this point, let us consider the Hanning window, which is specified as

$$w(n) = \begin{cases} \frac{1}{2}(1 - \cos\frac{2\pi}{L-1}n), & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.4.9}$$

Figure 5.14 shows $|\hat{X}(\omega)|$ for the window of (5.4.9). Its sidelobes are significantly smaller than those of the rectangular window, but its main lobe is approximately twice as wide. Figure 5.15 shows the spectrum of the signal in (5.4.8), after it is windowed by the Hanning window, for $L = 50$, 75, and 100. The reduction of the sidelobes and the decrease in the resolution, compared with the rectangular window, is clearly evident.

For a general signal sequence $\{x(n)\}$, the frequency-domain relationship between the windowed sequence $\hat{x}(n)$ and the original sequence $x(n)$ is given by the convolution formula

$$\hat{X}(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)W(\omega-\theta)d\theta \tag{5.4.10}$$

The DFT of the windowed sequence $\hat{x}(n)$ is the sampled version of the spectrum $X(\omega)$. Thus we have

$$\begin{aligned} \hat{X}(k) &\equiv \hat{X}(\omega)|_{\omega=2\pi k/N} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)W\left(\frac{2\pi k}{N}-\theta\right)d\theta \qquad k = 0, 1, \ldots, N-1 \end{aligned} \tag{5.4.11}$$

Just as in the case of the sinusoidal sequence, if the spectrum of the window is relatively narrow in width compared to the spectrum $X(\omega)$ of the signal, the window function has only a small (smoothing) effect on the spectrum $X(\omega)$. On the other hand, if the window function has a wide spectrum compared to the width of

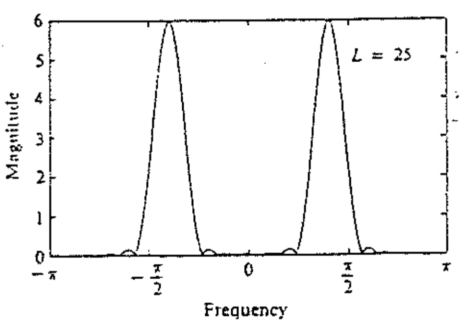


Figure 5.14 Magnitude spectrum of the Hanning window.

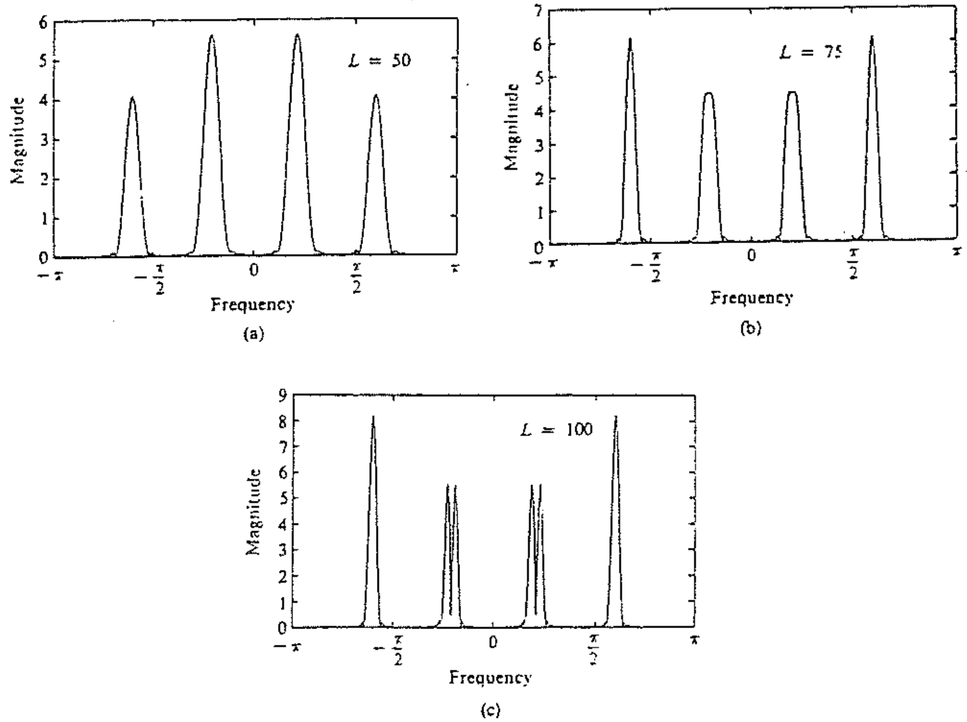


Figure 5.15 Magnitude spectrum of the signal in (5.4.8) as observed through a Hanning window.

$X(\omega)$, as would be the case when the number of samples L is small, the window spectrum masks the signal spectrum and, consequently, the DFT of the data reflects the spectral characteristics of the window function. Of course, this situation should be avoided.

Example 5.4.1

The exponential signal

$$x_a(t) = \begin{cases} e^{-t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

is sampled at the rate $F_s = 20$ samples per second, and a block of 100 samples is used to estimate its spectrum. Determine the spectral characteristics of the signal $x_a(t)$ by computing the DFT of the finite-duration sequence. Compare the spectrum of the truncated discrete-time signal to the spectrum of the analog signal.

Solution The spectrum of the analog signal is

$$X_a(F) = \frac{1}{1 + j2\pi F}$$

The exponential analog signal sampled at the rate of 20 samples per second yields the sequence

$$x(n) = e^{-nT} = e^{-n/20}, \qquad n \geq 0$$
$$= (e^{-1/20})^n = (0.95)^n, \qquad n \geq 0$$

Now, let

$$x(n) = \begin{cases} (0.95)^n, & 0 \leq n \leq 99 \\ 0, & \text{otherwise} \end{cases}$$

The N-point DFT of the $L = 100$ point sequence is

$$\hat{X}(k) = \sum_{k=0}^{99} \hat{x}(n) e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1$$

To obtain sufficient detail in the spectrum we choose $N = 200$. This is equivalent to padding the sequence $x(n)$ with 100 zeros.

The graph of the analog signal $x_a(t)$ and its magnitude spectrum $|X_a(F)|$ are illustrated in Fig. 5.16(a) and (b), respectively. The truncated sequence $x(n)$ and its $N = 200$ point DFT (magnitude) are illustrated in Fig. 5.16(c) and (d), respectively.

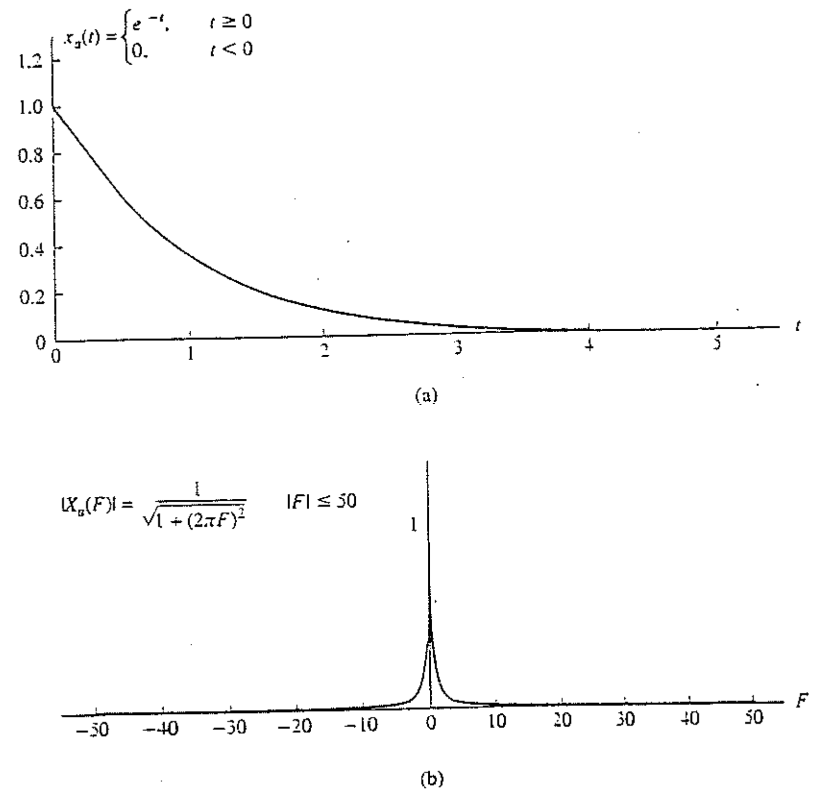


Figure 5.16 Effect of windowing (truncating) the sampled version of the analog signal in Example 5.4.1.

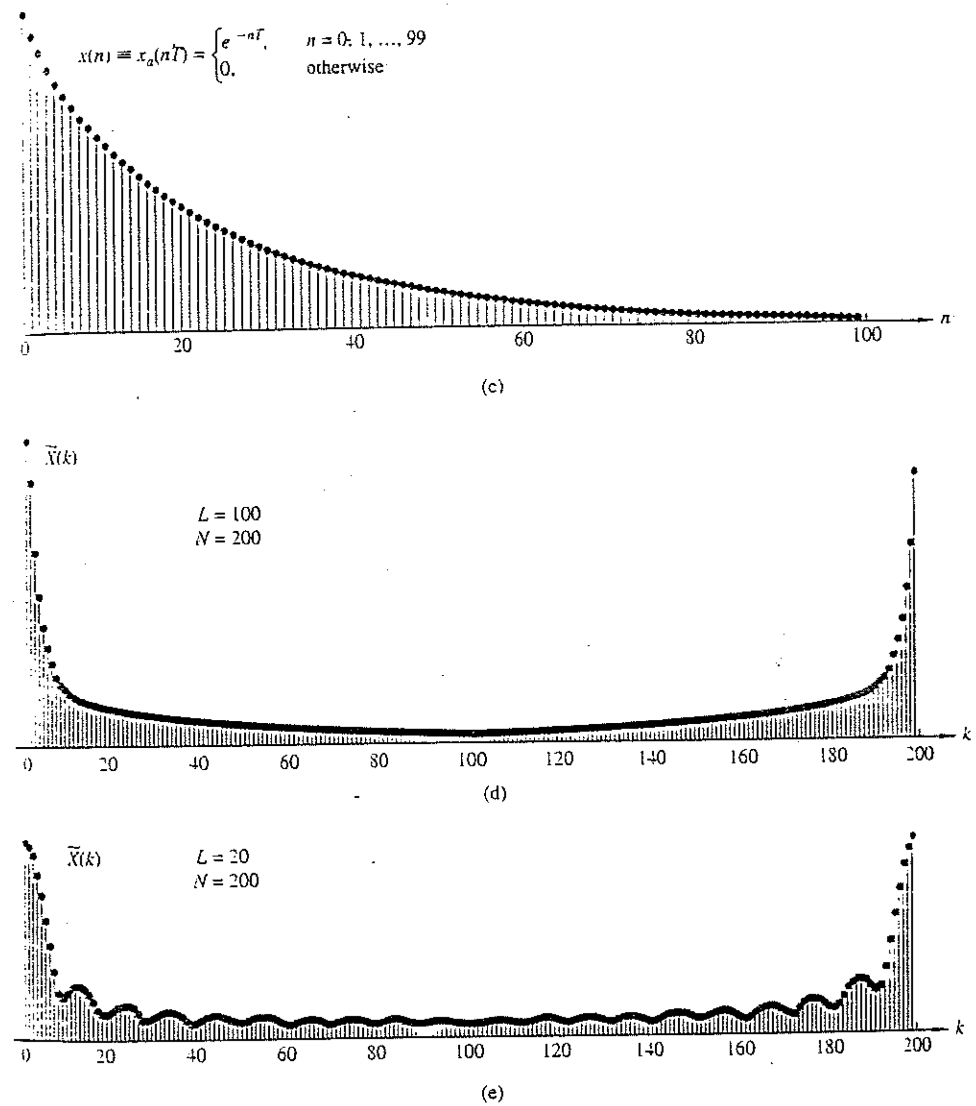


Figure 5.16 *Continued*

In this case the DFT $\{X(k)\}$ bears a close resemblance to the spectrum of the analog signal. The effect of the window function is relatively small.

On the other hand, suppose that a window function of length $L = 20$ is selected. Then the truncated sequence $x(n)$ is now given as

$$\hat{x}(n) = \begin{cases} (0.95)^n, & 0 \leq n \leq 19 \\ 0, & \text{otherwise} \end{cases}$$

Its $N = 200$ point DFT is illustrated in Fig. 5.16(e). Now the effect of the wider spectral window function is clearly evident. First, the main peak is very wide as a result of the wide spectral window. Second, the sinusoidal envelope variations in the spectrum away from the main peak are due to the large sidelobes of the rectangular window spectrum. Consequently, the DFT is no longer a good approximation of the analog signal spectrum.

SUMMARY AND REFERENCES

The major focus of this chapter was on the discrete Fourier transform, its properties and its applications. We developed the DFT by sampling the spectrum $X(\omega)$ of the sequence $x(n)$.

Frequency-domain sampling of the spectrum of a discrete-time signal is particularly important in the processing of digital signals. Of particular significance is the DFT, which was shown to uniquely represent a finite-duration sequence in the frequency domain. The existence of computationally efficient algorithms for the DFT, which are described in Chapter 6, make it possible to digitally process signals in the frequency domain much faster than in the time domain. The processing methods in which the DFT is especially suitable include linear filtering as described in this chapter and correlation, and spectrum analysis, which are treated in Chapters 6 and 12. A particularly lucid and concise treatment of the DFT and its application to frequency analysis is given in the book by Brigham (1988).

PROBLEMS

5.1 The first five points of the eight-point DFT of a real-valued sequence are $\{0.25, 0.125 - j0.3018, 0, 0.125 - j0.0518, 0\}$. Determine the remaining three points.

5.2 Compute the eight-point circular convolution for the following sequences.

(a) $x_1(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$

$$x_2(n) = \sin\frac{3\pi}{8}n \qquad 0 \le n \le 7$$

(b) $x_1(n) = (\frac{1}{4})^n \qquad 0 \le n \le 7$

$$x_2(n) = \cos\frac{3\pi}{8}n \qquad 0 \le n \le 7$$

(c) Compute the DFT of the two circular convolution sequences using the DFTs of $x_1(n)$ and $x_2(n)$.

5.3 Let $X(k)$, $0 \le k \le N-1$, be the N-point DFT of the sequence $x(n)$, $0 \le n \le N-1$. We define

$$\hat{X}(k) = \begin{cases} X(k), & 0 \le k \le k_c, \; N-k_c \le k \le N-1 \\ 0, & k_c < k < N-k_c \end{cases}$$

and we compute the inverse N-point DFT of $\hat{X}(k)$, $0 \le k \le N-1$. What is the effect of this process on the sequence $x(n)$? Explain.

5.4 For the sequences

$$x_1(n) = \cos\frac{2\pi}{N}n \qquad x_2(n) = \sin\frac{2\pi}{N}n \qquad 0 \le n \le N-1$$

determine the N-point:

(a) Circular convolution $x_1(n) \textcircled{N} x_2(n)$

(b) Circular correlation of $x_1(n)$ and $x_2(n)$

(c) Circular autocorrelation of $x_1(n)$

(d) Circular autocorrelation of $x_2(n)$

5.5 Compute the quantity

$$\sum_{n=0}^{N-1} x_1(n)x_2(n)$$

for the following pairs of sequences.

(a) $x_1(n) = x_2(n) = \cos\frac{2\pi}{N}n \qquad 0 \le n \le N-1$

(b) $x_1(n) = \cos\frac{2\pi}{N}n \qquad x_2(n) = \sin\frac{2\pi}{N}n \qquad 0 \le n \le N-1$

(c) $x_1(n) = \delta(n) + \delta(n-8) \qquad x_2(n) = u(n) - u(n-N)$

5.6 Determine the N-point DFT of the Blackman window

$$w(n) = 0.42 - 0.5\cos\frac{2\pi n}{N-1} + 0.08\cos\frac{4\pi n}{N-1} \qquad 0 \le n \le N-1$$

5.7 If $X(k)$ is the DFT of the sequence $x(n)$, determine the N-point DFTs of the sequences

$$x_c(n) = x(n)\cos\frac{2\pi kn}{N} \qquad 0 \le n \le N-1$$

and

$$x_s(n) = x(n)\sin\frac{2\pi kn}{N} \qquad 0 \le n \le N-1$$

in terms of $X(k)$.

5.8 Determine the circular convolution of the sequences

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

$$x_2(n) = \{\underset{\uparrow}{4}, 3, 2, 2\}$$

using the time-domain formula in (5.2.39).

5.9 Use the four-point DFT and IDFT to determine the sequence

$$x_3(n) = x_1(n) \textcircled{N} x_2(n)$$

where $x_1(n)$ and $x_2(n)$ are the sequence given in Problem 5.8.

5.10 Compute the energy of the N-point sequence

$$x(n) = \cos\frac{2\pi kn}{N} \qquad 0 \le n \le N-1$$

5.11 Given the eight-point DFT of the sequence

$$x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & 4 \le n \le 7 \end{cases}$$

compute the DFT of the sequences:

(a) $x_1(n) = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le 4 \\ 1, & 5 \le n \le 7 \end{cases}$

(b) $x_2(n) = \begin{cases} 0, & 0 \le n \le 1 \\ 1, & 2 \le n \le 5 \\ 0, & 6 \le n \le 7 \end{cases}$

5.12 Consider a finite-duration sequence

$$x(n) = \{\underset{\uparrow}{0}, 1, 2, 3, 4\}$$

(a) Sketch the sequence $s(n)$ with six-point DFT

$$S(k) = W_2^k X(k) \qquad k = 0, 1, \ldots, 6$$

(b) Determine the sequence $y(n)$ with six-point DFT $Y(k) = \text{Re}\{X(k)\}$.
(c) Determine the sequence $v(n)$ with six-point DFT $V(k) = \text{Im}\{X(k)\}$.

5.13 Let $x_p(n)$ be a periodic sequence with fundamental period N. Consider the following DFTs:

$$x_p(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

$$x_p(n) \overset{\text{DFT}}{\underset{3N}{\longleftrightarrow}} X_3(k)$$

(a) What is the relationship between $X_1(k)$ and $X_3(k)$?
(b) Verify the result in part (a) using the sequence

$$x_p(n) = \{\cdots 1, 2, 1, 2, \underset{\uparrow}{1}, 2, 1, 2 \cdots\}$$

5.14 Consider the sequences

$$x_1(n) = \{\underset{\uparrow}{0}, 1, 2, 3, 4\} \qquad x_2(n) = \{\underset{\uparrow}{0}, 1, 0, 0, 0\} \qquad s(n) = \{\underset{\uparrow}{1}, 0, 0, 0, 0\}$$

and their 5-point DFTs.

(a) Determine a sequence $y(n)$ so that $Y(k) = X_1(k)X_2(k)$.
(b) Is there a sequence $x_3(n)$ such that $S(k) = X_1(k)X_3(k)$?

5.15 Consider a causal LTI system with system function

$$H(z) = \frac{1}{1 - 0.5z^{-1}}$$

The output $y(n)$ of the system is known for $0 \le n \le 63$. Assuming that $H(z)$ is available, can you develop a 64-point DFT method to recover the sequence $x(n)$, $0 \le n \le 63$? Can you recover all values of $x(n)$ in this interval?

5.16* The impulse response of an LTI system is given by $h(n) = \delta(n) - \frac{1}{4}\delta(n - k_0)$. To determine the impulse response $g(n)$ of the inverse system, an engineer computes the N-point DFT $H(k)$, $N = 4k_0$, of $h(n)$ and then defines $g(n)$ as the inverse DFT of $G(k) = 1/H(k)$, $k = 0, 1, 2, \ldots, N-1$. Determine $g(n)$ and the convolution $h(n) * g(n)$, and comment on whether the system with impulse response $g(n)$ is the inverse of the system with impulse response $h(n)$.

5.17* Determine the eight-point DFT of the signal

$$x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$$

and sketch its magnitude and phase.

5.18 A linear time-invariant system with frequency response $H(\omega)$ is excited with the periodic input

$$x(n) = \sum_{k=-\infty}^{\infty} \delta(n - kN)$$

Suppose that we compute the N-point DFT $Y(k)$ of the samples $y(n)$, $0 \le n \le N - 1$ of the output sequence. How is $Y(k)$ related to $H(\omega)$?

5.19 *DFT of real sequences with special symmetries*

(a) Using the symmetry properties of Section 5.2 (especially the decomposition properties), explain how we can compute the DFT of two real symmetric (even) and two real antisymmetric (odd) sequences simultaneously using an N-point DFT only.
(b) Suppose now that we are given four real sequences $x_i(n)$, $i = 1, 2, 3, 4$, that are all symmetric [i.e., $x_i(n) = x_i(N - n)$, $0 \le n \le N - 1$]. Show that the sequences

$$s_i(n) = x_i(n + 1) - x_i(n - 1)$$

are antisymmetric [i.e., $s_i(n) = -s_i(N - n)$ and $s_i(0) = 0$].
(c) Form a sequence $x(n)$ using $x_1(n)$, $x_2(n)$, $s_3(n)$, and $s_4(n)$ and show how to compute the DFT $X_i(k)$ of $x_i(n)$, $i = 1, 2, 3, 4$ from the N-point DFT $X(k)$ of $x(n)$.
(d) Are there any frequency samples of $X_i(k)$ that cannot be recovered from $X(k)$? Explain.

5.20 *DFT of real sequences with odd harmonics only* Let $x(n)$ be an N-point real sequence with N-point DFT $X(k)$ (N even). In addition, $x(n)$ satisfies the following symmetry property:

$$x\left(n + \frac{N}{2}\right) = -x(n) \qquad n = 0, 1, \ldots, \frac{N}{2} - 1$$

that is, the upper half of the sequence is the negative of the lower half.

(a) Show that

$$X(k) = 0 \qquad k \text{ even}$$

that is, the sequence has a spectrum with odd harmonics.
(b) Show that the values of this odd-harmonic spectrum can be computed by evaluating the $N/2$-point DFT of a complex modulated version of the original sequence $x(n)$.

5.21 Let $x_a(t)$ be an analog signal with bandwidth $B = 3$ kHz. We wish to use a $N = 2^m$-point DFT to compute the spectrum of the signal with a resolution less than or equal to 50 Hz. Determine (a) the minimum sampling rate, (b) the minimum number of required samples, and (c) the minimum length of the analog signal record.

5.22 Consider the periodic sequence

$$x_p(n) = \cos\frac{2\pi}{10}n \qquad -\infty < n < \infty$$

with frequency $f_0 = \frac{1}{10}$ and fundamental period $N = 10$. Determine the 10-point DFT of the sequence $x(n) = x_p(n)$, $0 \le n \le N-1$.

5.23 Compute the N-point DFTs of the signals

(a) $x(n) = \delta(n)$

(b) $x(n) = \delta(n - n_0)$ $\quad 0 < n_0 < N$

(c) $x(n) = a^n$ $\quad 0 \le n \le N-1$

(d) $x(n) = \begin{cases} 1, & 0 \le n \le N/2 - 1 \, (N \text{ even}) \\ 0, & N/2 \le n \le N-1 \end{cases}$

(e) $x(n) = e^{j(2\pi/N)k_0 n}$ $\quad 0 \le n \le N-1$

(f) $x(n) = \cos\frac{2\pi}{N}k_0 n$ $\quad 0 \le n \le N-1$

(g) $x(n) = \sin\frac{2\pi}{N}k_0 n$ $\quad 0 \le n \le N-1$

(h) $x(n) = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \quad 0 \le n \le N-1 \end{cases}$

5.24 Consider the finite-duration signal

$$x(n) = \{1, 2, 3, 1\}$$

(a) Compute its four-point DFT by solving explicitly the 4-by-4 system of linear equations defined by the inverse DFT formula.

(b) Check the answer in part (a) by computing the four-point DFT, using its definition.

5.25 (a) Determine the Fourier transform $X(\omega)$ of the signal

$$x(n) = \{1, 2, 3, \underset{\uparrow}{2}, 1, 0\}$$

(b) Compute the 6-point DFT $V(k)$ of the signal

$$v(n) = \{3, 2, 1, 0, 1, 2\}$$

(c) Is there any relation between $X(\omega)$ and $V(k)$? Explain.

5.26 Prove the identity

$$\sum_{l=-\infty}^{\infty} \delta(n + lN) = \frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi/N)kn}$$

(*Hint:* Find the DFT of the periodic signal in the left-hand side.)

5.27 *Computation of the even and odd harmonics using the DFT* Let $x(n)$ be an N-point sequence with an N-point DFT $X(k)$ (N even)

(a) Consider the time-aliased sequence

$$y(n) = \begin{cases} \displaystyle\sum_{l=-\infty}^{\infty} x(n + lM), & 0 \le n \le M-1 \\ 0, & \text{elsewhere} \end{cases}$$

What is the relationship between the M-point DFT $Y(k)$ of $y(n)$ and the Fourier transform $X(\omega)$ of $x(n)$?

(b) Let

$$y(n) = \begin{cases} x(n) + x\left(n + \dfrac{N}{2}\right), & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

and

$$y(n) \xrightarrow[N/2]{\text{DFT}} Y(k)$$

Show that $X(k) = Y(k/2)$, $k = 2, 4, \ldots, N-2$.

(c) Use the results in parts (a) and (b) to develop a procedure that computes the odd harmonics of $X(k)$ using an $N/2$-point DFT.

5.28* *Frequency-domain sampling* Consider the following discrete-time signal

$$x(n) = \begin{cases} a^{|n|}, & |n| \le L \\ 0, & |n| > L \end{cases}$$

where $a = 0.95$ and $L = 10$

(a) Compute and plot the signal $x(n)$.

(b) Show that

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = x(0) + 2\sum_{n=1}^{L} x(n)\cos\omega n$$

Plot $X(\omega)$ by computing it at $\omega = \pi k/100$, $k = 0, 1, \ldots, 100$.

(c) Compute

$$c_k = \frac{1}{N}X\left(\frac{2\pi}{N}K\right) \qquad k = 0, 1, \ldots, N-1$$

for $N = 30$.

(d) Determine and plot the signal

$$\tilde{x}(n) = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$$

What is the relation between the signals $x(n)$ and $\tilde{x}(n)$? Explain.

(e) Compute and plot the signal $\tilde{x}_1(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$, $-L \le n \le L$ for $N = 30$. Compare the signals $\tilde{x}(n)$ and $\tilde{x}_1(n)$.

(f) Repeat parts (c) to (e) for $N = 15$.

5.29* *Frequency-domain sampling* The signal $x(n) = a^{|n|}$, $-1 < a < 1$ has a Fourier transform

$$X(\omega) = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

(a) Plot $X(\omega)$ for $0 \le \omega \le 2\pi$, $a = 0.8$.

Reconstruct and plot $X(\omega)$ from its samples $X(2\pi k/N)$, $0 \le k \le N-1$ for:

(b) $N = 20$

(c) $N = 100$

(d) Compare the spectra obtained in parts (b) and (c) with the original spectrum $X(\omega)$ and explain the differences.

(e) Illustrate the time-domain aliasing when $N = 20$.

5.30* *Frequency analysis of amplitude-modulated discrete-time signal* The discrete-time signal

$$x(n) = \cos 2\pi f_1 n + \cos 2\pi f_2 n$$

where $f_1 = \frac{1}{18}$ and $f_2 = \frac{5}{128}$, modulates the amplitude of the carrier

$$x_c(n) = \cos 2\pi f_c n$$

where $f_c = \frac{50}{128}$. The resulting amplitude-modulated signal is

$$x_{am}(n) = x(n) \cos 2\pi f_c n$$

(a) Sketch the signals $x(n)$, $x_c(n)$, and $x_{am}(n)$, $0 \le n \le 255$.
(b) Compute and sketch the 128-point DFT of the signal $x_{am}(n)$, $0 \le n \le 127$.
(c) Compute and sketch the 128-point DFT of the signal $x_{am}(n)$, $0 \le n \le 99$.
(d) Compute and sketch the 256-point DFT of the signal $x_{am}(n)$, $0 \le n \le 179$.
(e) Explain the results obtained in parts (b) through (d), by deriving the spectrum of the amplitude-modulated signal and comparing it with the experimental results.

5.31* The sawtooth waveform in Fig. P5.31 can be expressed in the form of a Fourier series as

$$x(t) = \frac{2}{\pi}\left(\sin \pi t - \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t - \frac{1}{4}\sin 4\pi t \cdots\right)$$

(a) Determine the Fourier series coefficients c_k.
(b) Use an N-point subroutine to generate samples of this signal in the time domain using the first six terms of the expansion for $N = 64$ and $N = 128$. Plot the signal $x(t)$ and the samples generated, and comment on the results.

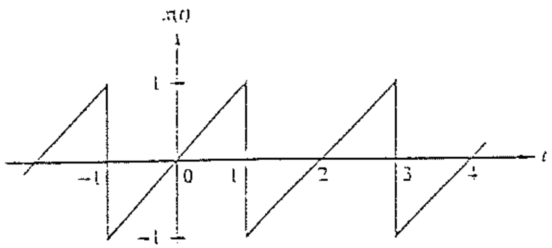


Figure P5.31

5.32 Recall that the Fourier transform of $x(t) = e^{j\Omega_0 t}$ is $X(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$ and the Fourier transform of

$$p(t) = \begin{cases} 1, & 0 \le t \le T_0 \\ 0, & \text{otherwise} \end{cases}$$

is

$$P(j\Omega) = T_0 \frac{\sin \Omega T_0/2}{\Omega T_0/2} e^{-j\Omega T_0/2}$$

(a) Determine the Fourier transform $Y(j\Omega)$ of

$$y(t) = p(t)e^{j\Omega_0 t}$$

and roughly sketch $|Y(j\Omega)|$ versus Ω.

(b) Now consider the exponential sequence

$$x(n) = e^{j\omega_0 n}$$

where ω_0 is some arbitrary frequency in the range $0 < \omega_0 < \pi$ radians. Give the most general condition that ω_0 must satisfy in order for $x(n)$ to be periodic with period P (P is a positive integer).

(c) Let $y(n)$ be the finite-duration sequence

$$y(n) = x(n)w_N(n) = e^{j\omega_0 n} w_N(n)$$

where $w_N(n)$ is a finite-duration rectangular sequence of length N and where $x(n)$ is not necessarily periodic. Determine $Y(\omega)$ and roughly sketch $|Y(\omega)|$ for $0 \le \omega \le 2\pi$. What effect does N have in $|Y(\omega)|$? Briefly comment on the similarities and differences between $|Y(\omega)|$ and $|Y(j\Omega)|$.

(d) Suppose that

$$x(n) = e^{j(2\pi/P)n} \qquad P \text{ a positive integer}$$

and

$$y(n) = w_N(n)x(n)$$

where $N = lP$, l a positive integer. Determine and sketch the N-point DFT of $y(n)$. Relate your answer to the characteristics of $|Y(\omega)|$.

(e) Is the frequency sampling for the DFT in part (d) adequate for obtaining a rough approximation of $|Y(\omega)|$ directly from the magnitude of the DFT sequence $|Y(k)|$? If not, explain briefly how the sampling can be increased so that it will be possible to obtain a rough sketch of $|Y(\omega)|$ from an appropriate sequence $Y(k)$.