Example 6.1 (Max-Flow and Min-Cut). *For a graph $G = (V, E)$ with an edge limit function $c : E \to \mathbb{R}_{\geq 0}$ (sometimes this pair is called network) and vertices v_s, v_t; source and sink accordingly. The Max-Flow is the following problem.*

$$\begin{cases} & \max\limits_{x \in \mathbb{R}^E} \sum\limits_{j \in V} f_{sj} & \\ s.t. & 0 \leq f_{ij} \leq c_{ij} & \forall (i,j) \in E \\ & \sum\limits_{(i,k) \in E} f_{ik} = \sum\limits_{(k,i) \in E} f_{ki} & \forall k \in V \setminus \{v_s, v_t\} \end{cases}$$

Dual version of this problem:

$$\begin{cases} & \min\limits_{(b,x) \in \mathbb{R}^E \times \mathbb{R}^V} \sum\limits_{(i,j) \in E} b_{ij} c_{ij} & \\ s.t. & b_{ij} + x_j - x_i \geq 0 & \forall (i,j) \in E \\ & b \geq 0, x_s = 1, x_t = 0 & \end{cases}$$

Suppose $x \in \{0,1\}^V$ at the solution. If we consider two sets $A = \{i : x_i = 0\}$ and $B = \{i : x_i = 1\}$ then inequalities $b_{ij} \geq x_i - x_j$ is equivalent to $b_{ij} \geq I(i \in A, j \in B)$. Obviously, for the optimum b this inequalities are equalities $b_{ij} = I(i \in A, j \in B)$. We can reformulate our problem as find of partition (cut) V into two disjoint sets A and B such that $v_s \in A$, $v_t \in B$ and $\sum\limits_{i \in A, j \in B} c_{ij} \to \min$; the last term is called *capacity of the cut* (A, B). Any such cut gives an upper bound on the value of max-flow in the network.

Our Dual of Max-Flow was LP-relaxation of this problem. Such wise

$$\text{value of Max-Flow} \leq \text{optimal of } LP\text{-relaxation} \leq \text{capacity of Min-Cut}\,.$$

Theorem 6.2 (Max-Flow Min-Cut Theorem). *The optimal flow is equal to the capacity of an optimal cut.*

In this case Primal and Dual gas the same optimal values. It's truth in general case for linear programs (see Farkas' lemma in **?**); this property is called "strong duality".

6.1 Lagrangian

6.1.1 Linear programs

Return to LP (alternate derivation)

$$\begin{cases} & \min\limits_x c^T x \\ \text{s.t.} & Ax = b \\ & Gx \geq h \end{cases}$$

For any u and $v \geq 0$ if x is feasible:

$$c^T x \geq c^T x + u^T (Ax - b) + v^T (Gx - h) =: L(x, u, v)\,.$$

If C is a feasible set, f^* is optimal value:

$$f^* \geq \min_{x \in C} L(x, u, v) \geq \min_x L(x, u, v) =: g(u, v)\,.$$

$$g(u,v) = \begin{cases} -b^T u - h^T v & \text{, if } c = -A^T u - G^T v \\ -\infty & \text{, otherwise} \end{cases}$$

Maximize g over u, $v \geq 0$ provide lower bound. We call $L(x,u,v)$ the Lagrangian, $g(u,v)$ the Lagrange dual function.

6.1.2 General Case

$$\begin{cases} & \min f(x) \\ \text{s.t.} & h_i(x) \leq 0 \quad i \in [m] \\ & l_j(x) = 0 \quad i \in [r] \end{cases}$$

$$L(x,u,v) = f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{r} v_j l_j(x) \ \ (u \geq 0)$$

$$f^* \geq \min_x L(x,u,v) =: g(u,v)$$

g is concave.

Example 6.3.

$$\begin{cases} & \min\limits_x \frac{1}{2} x^T Q x + c^T x \\ s.t. & Ax = b \\ & x \geq 0 \end{cases}$$

$$L(x,u,v) = \frac{1}{2} x^T Q x + c^T x - u^T x + v^T (Ax - b)$$

$$x^* = -Q^{-1}(c - u^T A^T v)$$

$$L(x^*,u,v) = -\frac{1}{2}(c - u + A^T v) Q^{-1} (c - u + A^T v) - b^T v$$

what if $Q \succeq 0$?

- ***Case 1:*** $c - u + A^T v \in col(Q)$

  $$x = -Q^T(c - u + A^T v) \implies P(c - u + A^T v) = 0$$

  where P is projection onto $nul(Q)$.

- ***Case 2:*** $c - u + A^T v \notin col(Q) \implies c - u + A^T v = z_1 + z_2$ *where* $z_1 \in col(Q), z_2 \in nul(Q)$ *and* $z_2 \neq 0$.

$$g(u,v) = \begin{cases} -\frac{1}{2}(c - u + A^T v) Q^{-1} (c - u + A^T v) - b^T v & , \textit{Case 1} \\ -\infty & , \textit{Case 2} \end{cases}$$