5.1 First Order Condition

- convex problem with differentialbe 'f'
- a feasible x is optimal iff $\nabla f(x)^T(x-y) \geq 0$, $\forall$ feasible y
- if unconstrained, the condition reduces to $\nabla f(x) = 0$

Example 5.1. $\min\limits_x \frac{1}{2}x^TQx + b^Tx + c$, $Q \succeq 0$
FOC: $\nabla f(x) = Q^Tx + b = 0$
if $Q \succ 0 \rightarrow x^* = -Q^{-1}b$
if Q *singular,* $b \in Col[Q] \rightarrow$ *no solution*
if Q *singular,* $b \notin Col[Q] \rightarrow x^* = -Q^*b + z$ *with* $z \in null[Q]$

Example 5.2. *Projection ont convex* C : $\min\limits_x \|a-x\|_2^2$ *s.t.* $x \in C$
FOC : $\nabla f(x)^T(y-x) = (x-a)^T(y-x) \geq 0, \forall y \in C \Leftrightarrow a - x \in N_2(x)$

5.2 Useful Operations

5.2.1 Partial Optimizations

$h(x) = \min\limits_y f(x,y)$ is convex if f is convex, and C is convex.

Example 5.3. $\min\limits_{x_1,x_2} f(x_1,x_2)$ $s.t. g_1(x_1) \leq 0, g_2(x_2) \leq 0 \Leftrightarrow \min\limits_{x_1} h(x_1)$ $s.t.$ $g_1(x_1) \leq 0$

5.2.2 Transformations

We can use a monotone increasing transformation $h : \mathbb{R} \rightarrow \mathbb{R}$ to change our problem:
$\min\limits_{x\in C} f(x) \Rightarrow \min\limits_{x \in C} h(f(x))$
We can use a change of variable transformation $\phi : \mathbb{R}^n \Rightarrow \mathbb{R}^m$:
$\min\limits_{x \in C} f(x) \Leftrightarrow \min\limits_{\phi(y)\in C} f(\phi(y))$

Example 5.4. *Geometric Program*
$\min\limits_x f(x) = \sum\limits_{k=1}^p \gamma_k x_1^{a_{k_1}} x_2^{a_{k_2}} .. x_n^{a_{k_n}}$ *(posynomial)*
C *: involved inequalities are in some form and equalities are affine.*
We can change above non-convex problem to the following convex problem by letting $y_i = log x_i$ *:*
$\min\limits_y \log(\sum\limits_{k=1}^p \exp^{a_{ik}^T y + b_{ik}})$ $s.t.$ $\log(\sum_{k=1}^m \exp^{a_{ik}^T y + b_{ik}}) \leq 0$ *and* $c_j^T y + d_j = 0, j = 1, ..., r$

5.2.3 Eliminate equality constraints

$\min_x f(x)$ s.t. $g_i(x) \leq 0, Ax = b$
x feasible $= My + x_0$ s.t. $Ax_0 = b$ $col[M] = null[A]$
So we can get an optimization problem with only inequality constraints :
$\min_y f[My + x_0]$ s.t. $g_i(My + x_i) = 0$

5.2.4 Slack Variables

Consider:
$\min_{x,s} f(x)$ s.t. $s \geq 0, g_i(x) + s_i = 0, Ax = b$ This problem is not convex unless g_i is affine. We can relx nonaffine constraints to get :
$\min_{x \in C} f(x) \Rightarrow \min_{x \in \tilde{C}} f(x)$, $C \subset \tilde{C}$
In this case optimum of new problem is smaller or equal to the optimum of the original problem.

5.3 Standard Problems

5.3.1 LP (Linear Programs)

$\min_x c^T x$ with affine inequlaity and affine eqaulity

Example 5.5. *Basis Pursuit*
$\min_\beta \|\beta_0\|$ *s.t.* $X\beta = y$
Above problem can be relaxed to :
$\min_\beta \|\beta\|_1$ *s.t.* $X\beta = y$.
This relaxation can be reformulated to a LP problem:
$\min_{\beta,z} 1^T z$ *s.t.* $z \geq \beta, z \geq -\beta, X\beta = y$

Example 5.6. *Dantzig selector*
$\min_\beta \|\beta\|_1$ *s.t.* $\|x^T(y - X\beta)\|_\infty \leq \lambda$

5.3.2 QP (Quadratic Programming)

Example 5.7. *Lasso, ridge regression, OLS, Portfolio Optimization*

5.3.3 SDP (Semi-Definite Programming)

$\min_{X \in S_n} tr(C^T X)$ s.t. $tr(A_i^T X) = b_i$, $X \succeq 0$
We define $X \bullet A = tr(X^T A)$ from now on.

5.3.4 Conic Program

$$\min_x c^T x$$
s.t. $Ax = b$, $D(x) + d \in K$, K a closed convex cone.

The following relationship holds between above programs:
$CP(ConicProgramming) \subset QP \subset SOCP \subset SDP \subset CP(ConvexProgramming)$

5.4 Lower bound in LP

Want to find $B \leq \min_x f(x)$

Example 5.8. *easy example*
$\min_{x,y} x + y$ *s.t.* $x + y \geq 2, x \geq 0, y \geq 0$

Example 5.9. *medium example*
$\min_{x,y} x + 3y$ *s.t.* $x + y \geq 2, x \geq 0, y \geq 0$
We transform constraints :
know $x + y \geq 2, y \geq 0 \Rightarrow x + y \geq 2, 2y \geq 0 \Rightarrow x + 3y \geq 2$

Example 5.10. *general LP*
$\min_{x,y} px + qy$ *s.t.* $x + y \geq 2, x, y \geq 0$
We transform constraints:
$ax + ay \geq 2a, bx \geq 0, cy \geq 0, a \geq 0, b \geq 0c \geq 0$
Add $(a + b)x + (a + c)y \geq 2a$
Let $(a + b) = p, (a + c) = q \Rightarrow b = 2a$
To answer the question "What is best lower bound?", Solve:
$$\max_{a,b,c} 2a$$
s.t. $a + b = p, a + c = q, a, b, c \geq 0$
The above LP is "dual" of the original LP :
$$\min_{x,y} px + qy \text{ s.t. } x + y \geq 2, x, y, \geq 0$$
We see that number of Dual variables = number of Primal constraints

Example 5.11. *Another general LP example*
$\min_{x,y}(px + qy)$ *s.t.* $x \geq 0, y \leq 1, 3x + y = 2$
We transform constraints:
$a \geq 0, b \geq 0, -by \geq -b, 3cx + cy = 2c$
Now combine:
$(a + 3c)x + (-b + c)y \geq 2c - b$
By letting p = (a+3c) and q = (-b + c) We get the following Dual:
$\max_{a,b,c} 2c - b$
s.t. $a + 3c = p, c - b = a, a, b \geq 0$

For General LP:
$$\min_x c^T x$$
s.t. $Ax = b, Gx \leq h$
$\Rightarrow$
$\max_{u,v} -b^T w - b^T v$
s.t. $-A^T u - h^T v = c, v \geq 0$