Introduction

In this lecture we will study sub-Gaussian Random Variables and study their properties.

11.1 Sub-Gaussian Random Variables

Many important classes of random variables have tail probabilities decreasing at least as rapidly as normally distributed random variables. In order to facilitate the exploration of this phenomenon, we find it useful to formalize the notion of a sub-Gaussian random variable.

Definition 11.1. *(Sub Gaussian) Let X be a Random Variable, If $\exists \sigma > 0$ such that, $\mathbb{E}[e^{(\lambda x)}] \leq e^{\frac{\sigma^2 \lambda^2}{2}}$, then we call X σ-Sub Gaussian.*

We will now show a few examples and prove that, they are indeed sub-Gaussian.

Example 11.2. *Gaussian Random Variable with mean 0 and variance σ^2 is σ sub-Gaussian.*

Example 11.3. *(Radamachar Variable) $\mathbb{P}[X = 1] = \mathbb{P}[X = -1] = 1/2$ is sub-Gaussian with $\sigma = 1$*

We will prove that Radamachar Variable is indeed 1-Sub Gaussian.

Proof.

$$\begin{aligned}
\mathbb{E}[e^{\lambda X}] &= e^{-\lambda}\mathbb{P}[X = -1] + e^{\lambda}\mathbb{P}[X = 1] \\
&= -1/2(e^{-\lambda} + e^{\lambda}) \\
&= \cosh(\lambda) \\
&= \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} \\
&\leq \sum_{k=0}^{\infty} \frac{(\lambda^2)^k}{2^k k!} \\
&= e^{\lambda^2/2}
\end{aligned}$$

Thus $\sigma = 1$ ■

Example 11.4. *Uniform Distribution between $[-a, a]$ is a-sub Gaussian.*

We will prove this.

Proof.

$$\begin{aligned}
\mathbb{E}[e^{\lambda X}] &= -1/2a(e^{-\lambda x} + e^{\lambda x}) \\
&= \sinh(a\lambda) \\
&= \sum_{k=0}^{\infty} \frac{a\lambda^{2k}}{(2k+1)!} \\
&= e^{a^2\lambda^2/2}
\end{aligned}$$

Thus $\sigma = a$. ∎

11.2 Properties of Sub-Gaussian Random Variables

In this section we will provide a few properties of the Sub-Gaussian Random Variables. We will not prove all the results.

Theorem 11.5. *Following statements hold true.*

1. *If X_1 is σ_1-sub Gaussian, then aX_1 is $|a|\sigma_1$ sub-Gaussian.*
2. *If X_1 is σ_1-sub Gaussian and X_2 is σ_2-sub Gaussian, then $X_1 + X_2$ is $(\sigma_1 + \sigma_2)$ sub Gaussian.*
3. *If X_1 is σ_1-sub Gaussian and X_2 is σ_2-sub Gaussian, and X_1 and X_2 are independent then - $X_1 + X_2$ is $\sqrt{\sigma_1^2 + \sigma_2^2}$- sub Gaussian.*

Proof. .

1. $\mathbb{E}[e^{\lambda a X}] \leq e^{\lambda^2 a^2 \sigma^2/2}$.
2. Follows from Independence.
3. $$\begin{aligned}
\mathbb{E}[e^{\lambda(X_1+X_2)}] &\leq \mathbb{E}[e^{\lambda X_1 p}]^{1/p}\mathbb{E}[e^{\lambda X_2 q}]^{1/q} \\
&\leq (e^{\sigma_1^2/2\lambda^2 p^2})^{1/p}(e^{\sigma_1^2/2\lambda^2 q^2})^{1/q} \\
&= e^{\lambda^2/2(p\sigma_1^2 + q\sigma_2^2)} \\
&= e^{\lambda^2(\sigma_1+\sigma_2)^2/2}
\end{aligned}$$

∎

Theorem 11.6. *Suppose $\mathbb{E}[X = 0]$, then following are equivalent.*

1. *$\mathbb{P}[X > \delta]$ **OR** $\mathbb{P}[-X \geq \delta] \leq e^{-\delta^2/2\sigma^2}$*
2. *For every integer $q \geq 1$, $\mathbb{E}[X^{2q}] \leq 2q!(2\sigma^2)^q \leq q!(4\sigma^2)^q$*
3. *$\exists \alpha$ such that $\mathbb{E}[e^{\alpha X^2}] \leq 2$.*
4. *$\mathbb{E}[e^{\lambda X}] \leq e^{\sigma^2\lambda^2/2}$ for some σ^2*

Proof. 4 $\implies$ 1 due to Chernoff Bound. 3 $\implies$ 4 and 4 $\implies$ 3 for $\sigma^2 \in [2/\alpha, 4/\alpha]$. We omit the remaining proofs. ∎

11.3 Bounded Random Variables

Bounded variables are an important class of sub-Gaussian random variables. The sub-Gaussian property of bounded random variables is established by the following lemma:

Lemma 11.7. *(Hoeffding's Inequality) Let X be a Random Variable such that $\mathbb{E}[X] = 0$ and $\mathbb{P}[X \in [a, b]] = 1$, then:* $\mathbb{E}[e^{\lambda X}] \leq e^{\lambda^2 (b-a)^2/8}$

Proof. Write X as a convex combination a and b. Define $\alpha = \frac{X-a}{b-a}$. $X = \alpha b + (1 - \alpha)X$, $\alpha \in [0, 1]$. By Convexity we can write:

$$
\begin{aligned}
e^{\lambda x} &\leq \alpha e^{\lambda b} + (1-\alpha)e^{\lambda a} \\
&= \frac{X-a}{b-a}e^{\lambda b} + \frac{b-\lambda}{b-a}e^{\lambda a} \\
\mathbb{E}[e^{\lambda x}] &\leq \frac{-a}{b-a}e^{\lambda b} + \frac{b}{b-a}e^{\lambda a} \\
&= e^{g(u)} \quad (u = \lambda(b-a))
\end{aligned}
$$

$g(u) = -\gamma u + \log(1 - \gamma + \gamma e^2)$ where $\gamma = -a/(b-a)$

Note that,

$$g(0) = g'(0) = 0 \text{and} g''(u) \leq 1/4.$$

By Taylor's Theorem:

$$\exists x \in (0, u) \textit{ such that } g(u) = g(0) + ug'(0) + u^2/2g''(u)$$

Note that,

$$u^2/2g''(u) \leq u^2/8 = \lambda^2(b-a)^2/8$$

Thus we have

$$\mathbb{E}[e^{-\lambda x}] \leq e^{g(u)} \leq e^{\frac{\lambda^2(b-a)^2}{8}}$$

■