13.1 Histograms

Suppose $X_1 ... X_n$ are i.i.d p on $[0,1]^d$. Divide the unit cube into $B_1, ..., B_n$ cubes each with side length h, $N \leq (\frac{1}{h})^d$. Define

$$\widehat{p}_h(x) = \sum_j \frac{\widehat{\theta}_j}{h^d} I(x \in B_j) \qquad \text{where} \qquad \widehat{\theta}_j = \frac{1}{n}\sum_{i=1}^{n} I(x_i \in B_j)$$

We want to bound

$$P(\|\widehat{p}_h - p\|_\infty > \epsilon)$$

Consider :

$$\begin{aligned} &P(\max_j |\frac{\widehat{\theta}_j}{h^d} - \frac{\theta_j}{h^d}| > \epsilon) \\ &(\theta_j = \int_{B_j} p(x)dx) \\ &= P(\max_j |\widehat{\theta}_j - \theta_j| > \epsilon h^d) \end{aligned}$$

assume $|p(x) - p(y)| \leq L\|x - y\|$ then p is bounded. It follows that

$$\begin{aligned} \theta_j = \int_{B_j} p(u)du \leq c \int_{B_j} du = ch^d \\ \to \theta_j(1-\theta_j) \leq \theta_j \leq ch^d \end{aligned}$$

Now

$$\begin{aligned} P(|\widehat{\theta}_j - \theta_j| > h^d\epsilon) &\leq 2\exp(-\frac{1}{2}\frac{n\epsilon^2 h^{2d}}{\theta_j(1-\theta_j) + \epsilon h^d/3}) \\ &\leq 2exp(-\frac{1}{2}\frac{n\epsilon^2 h^{2d}}{ch^d + \epsilon h^d/3}) \\ &\leq 2exp(-cn\epsilon^2 h^d) \\ &(c = \frac{1}{2(c+\frac{1}{3})}) \end{aligned}$$

$$P(\max_j |\widehat{\theta}_j - \theta_j| > h^d\epsilon) \leq 2h^{-d}exp(-cn\epsilon^2 h^d) = \Pi_n$$

Now

$$\begin{aligned} p(x) - p_n(x) &= p(x) - \frac{\int_{B_j} p(u)du}{h^d} \\ &= \frac{1}{h^d} \int_{B_j} (p(x) - p(u))du \\ \to |\cdot| &\le \frac{1}{h^d} \int_{B_j} |p(x) - p(u)|du \\ &\le \frac{1}{h^d} Lh\sqrt{d} \int du = Lh\sqrt{d} \end{aligned}$$

wp $1 - \Pi_n$

$$\begin{aligned} \|\widehat{p}_h - p\|_\infty \le \|\widehat{p}_h - p_h\|_\infty + \|p_h - p\|_\infty \\ \le \epsilon + L\Pi h \\ \epsilon = \sqrt{\frac{1}{cnh^d} \log(\frac{2}{\delta h^d})} \\ \delta = \Pi n \end{aligned}$$

wp $1 - \delta$,

$$\begin{aligned} \|\widehat{p}_h - p\|_\infty \le \sqrt{\frac{1}{cnh^d} \log(\frac{2}{\delta h^d})} + L\sqrt{d}h \\ h = (\frac{c}{n})^{\frac{1}{2+d}} \end{aligned}$$

wp $1 - \Pi_n$.

$$\|\widehat{p}_h - p\|_\infty = O([\frac{logn}{n}]^{\frac{1}{2+d}})$$

13.2 Martingales

Let $(\Omega, \mathscr{F})$. Define sequence $\mathscr{F}_i$ of σ-algebras s.t. $\mathscr{F}_i \subset \mathscr{F}_{i+1}$ and $\mathscr{F}_0 = \sigma\{\Omega\}$. We call $((\mathscr{F}_i, (X_i))$ a martingale w.r.t $(\mathscr{F}_i)$ if X_i is $\mathscr{F}_i$ measurable and $E[X_i|\mathscr{F}_{i-1}] = X_{i-1}$ where $\mathscr{F}_i = \sigma(X_1, X_2 ... X_i)$. Note the following

- $E[X_i] = \mu$
- If $E[X_i|\mathscr{F}_{i-1}] = 0$ call it a martingale difference.

Example 13.1. $Y_0 = E[X_0], Y_1 = X_i - X_{i-1}$

Example 13.2. $\exists X$ *s.t. can create* $\mathscr{F}_i \subset \sigma[X]$, $X_i = E[X|\mathscr{F}_i$.

13.3 Azuma's inequality

Let $((\mathscr{F}_i), (y_i))$ be a martingale difference s.t. for (a_i) and (b_i) , $P(Y_i \in [a_i, b_i]) = 1$ for all i. Then for $\delta > 0$,

$$P(\frac{1}{n}|\sum Y_i| \ge \delta) \le 2 \exp^{-\frac{n\delta^2}{\sum_{i=1}^{n}(b_i - a_i)^2}}$$

Pf: sketch

$$Ee^{\lambda \sum\limits_{i=1}^{n} Y_i} = E[E[e^{\lambda \sum\limits_{i=1}^{n} Y_i} | \mathscr{F}_{n-1}]] = E[e^{\lambda \sum\limits_{i=1}^{n}} Y_i E[e^{\lambda Y_n} | \mathscr{F}_{n-1}]$$

Let $Z = f(X_1, ...X_n)$.

Theorem 13.3 (McDiamird's)**.** *If f satisfies "bounded differences" :*

$$\sup_{x_1, x_2 ... x_n, x' \in X^{n+1}} |f(x_1, ..., x_n) - f(x_2, ..., x_{i-1}, x, x_{i+1}, ..., x_n)| < C_i$$

Then $\delta > 0$, $P(Z - E[Z] > \delta) \leq \exp^{-\frac{2\delta}{\sum c_i^2}}$

Pf : sketch
$Y_0 = E[Z]$, $Y_1 = E[Z|X_1...X_i] = E[Z|X_1...X_{i-1}]$.