4.1 Terminology

Definition 4.1. *A convex optimization problem (program)*

$$\begin{cases} \min\limits_{x \in D} f(x) \\ \textit{subject to} \quad g_i(x) \le 0 \quad \forall i \in [1:m] \\ \qquad\qquad\quad\; Ax = b \end{cases},$$

where f, g_i *are convex and* $S = \operatorname{dom} f \cap \operatorname{dom} g_i$.

In this definition:

- f – criteria or objective function.
- g_i – inequality constraints.
- x is a *feasible point* if it satisfies the conditions, namely $x \in D$, $g_i(x) \le 0$, and $Ax = b$.
- $\min f$ over feasible points points – *optimal value* f^*.
- If x is feasible and $f(x) = f^*$ then x is an *optimum* (solution, minimizer).
- Feasible x is a *local optimum* if $\exists R > 0$ such that $\forall y \in V_R(x) \;\; f(x) \le f(y)$.
- If x is feasible and $f(x) \le f^* + \varepsilon$ then x is ε*-suboptimal*.
- If x is feasible and $g_k(x) = 0$ then g_k is *active* at x (otherwise inactive).

Properties:

- Solution set X_{opt} is convex.
- If f is strictly convex then the solution is unique.
- For convex optimization problems all local optima are global.
- The set of feasible points is convex.

Example 4.2. *Lasso.* $\min\limits_{\beta} \|y - X\beta\|_2^2$ *subject to* $\|\beta\|_1 \le s$.

- $g_1(\beta) = \|\beta\|_1 - s$ *– convex, no equality constraints.*
- X *is* $p \times n$ *matrix*
  - *If* $n \ge p$ *and* X *is full rank then* $\nabla^2 f(\cdot) = 2X^\mathsf{T} X$ *is positive definite matrix. The function is strictly convex, therefore the solution is unique.*
  - *If* $p > n$ *then* $\exists \beta \ne 0$ *such that* $X\beta = 0 \implies$ *multiple solutions.*