

Example 5.1 (Max-Flow and Min-Cut). For a graph $G = (V, E)$ with an edge limit function $c : E \rightarrow \mathbb{R}_{\geq 0}$ (sometimes this pair is called network) and vertices v_s, v_t ; source and sink accordingly. The Max-Flow is the following problem.

$$\left\{ \begin{array}{ll} \max_{x \in \mathbb{R}^E} & \sum_{j \in V} f_{sj} \\ \text{s.t.} & 0 \leq f_{ij} \leq c_{ij} \quad \forall (i, j) \in E \\ & \sum_{(i, k) \in E} f_{ik} = \sum_{(k, i) \in E} f_{ki} \quad \forall k \in V \setminus \{v_s, v_t\} \end{array} \right\}$$

Dual version of this problem:

$$\left\{ \begin{array}{ll} \min_{(b, x) \in \mathbb{R}^E \times \mathbb{R}^V} & \sum_{(i, j) \in E} b_{ij} c_{ij} \\ \text{s.t.} & b_{ij} + x_j - x_i \geq 0 \quad \forall (i, j) \in E \\ & b \geq 0, x_s = 1, x_t = 0 \end{array} \right\}$$

Suppose $x \in \{0, 1\}^V$ at the solution. If we consider two sets $A = \{i : x_i = 0\}$ and $B = \{i : x_i = 1\}$ then inequalities $b_{ij} \geq x_i - x_j$ is equivalent to $b_{ij} \geq I(i \in A, j \in B)$. Obviously, for the optimum b this inequalities are equalities $b_{ij} = I(i \in A, j \in B)$. We can reformulate our problem as find of partition (cut) V into two disjoint sets A and B such that $v_s \in A, v_t \in B$ and $\sum_{i \in A, j \in B} c_{ij} \rightarrow \min$; the last term is called *capacity of the cut* (A, B) . Any such cut gives an upper bound on the value of max-flow in the network.

Our Dual of Max-Flow was LP -relaxation of this problem. Such wise

$$\text{value of Max-Flow} \leq \text{optimal of } LP\text{-relaxation} \leq \text{capacity of Min-Cut}.$$

Theorem 5.2 (Max-Flow Min-Cut Theorem). The optimal flow is equal to the capacity of an optimal cut.

In this case Primal and Dual has the same optimal values. It's truth in general case for linear programs (see Farkas' lemma in [Boyd and Vandenberghe \(2004\)](#)); this property is called "strong duality".

5.1 Lagrangian

5.1.1 Linear programs

Return to LP (alternate derivation)

$$\left\{ \begin{array}{ll} \min_x & c^T x \\ \text{s.t.} & Ax = b \\ & Gx \geq h \end{array} \right\}$$

For any u and $v \geq 0$ if x is feasible:

$$c^T x \geq c^T x + u^T (Ax - b) + v^T (Gx - h) =: L(x, u, v).$$

If C is a feasible set, f^* is optimal value:

$$f^* \geq \min_{x \in C} L(x, u, v) \geq \min_x L(x, u, v) =: g(u, v).$$

$$g(u, v) = \begin{cases} -b^T u - h^T v & , \text{ if } c = -A^T u - G^T v \\ -\infty & , \text{ otherwise} \end{cases}$$

Maximize g over $u, v \geq 0$ provide lower bound. We call $L(x, u, v)$ the Lagrangian, $g(u, v)$ the Lagrange dual function.

5.1.2 General Case

$$\begin{cases} \min f(x) \\ \text{s.t.} & h_i(x) \leq 0 \quad i \in [m] \\ & l_j(x) = 0 \quad i \in [r] \end{cases}$$

$$L(x, u, v) = f(x) + \sum_{i=1}^m u_i h_i(x) + \sum_{j=1}^r v_j l_j(x) \quad (u \geq 0)$$

$$f^* \geq \min_x L(x, u, v) =: g(u, v)$$

g is concave.

Example 5.3.

$$\begin{cases} \min_x \frac{1}{2} x^T Q x + c^T x \\ \text{s.t.} & A x = b \\ & x \geq 0 \end{cases}$$

$$L(x, u, v) = \frac{1}{2} x^T Q x + c^T x - u^T x + v^T (A x - b)$$

$$x^* = -Q^{-1}(c - u^T A^T v)$$

$$L(x^*, u, v) = -\frac{1}{2}(c - u + A^T v)Q^{-1}(c - u + A^T v) - b^T v$$

what if $Q \succeq 0$?

- **Case 1:** $c - u + A^T v \in \text{col}(Q)$

$$x = -Q^T(c - u + A^T v) \implies P(c - u + A^T v) = 0$$

where P is projection onto $\text{nul}(Q)$.

- **Case 2:** $c - u + A^T v \notin \text{col}(Q) \implies c - u + A^T v = z_1 + z_2$ where $z_1 \in \text{col}(Q)$, $z_2 \in \text{nul}(Q)$ and $z_2 \neq 0$.

$$g(u, v) = \begin{cases} -\frac{1}{2}(c - u + A^T v)Q^{-1}(c - u + A^T v) - b^T v & , \text{ Case 1} \\ -\infty & , \text{ Case 2} \end{cases}$$

References

BOYD, S., AND VANDENBERGHE, L. (2004), *Convex optimization*, Cambridge university press.