24.1 Finishing upper bounds

Example 24.1. *$K(u) = \left(\frac{9}{8} - \frac{15}{8}u^2\right) I(|u| \le 1)$ is Kernel of order 3.*

$K(u) = \sum_{m=0}^{\ell} \phi_m(0)\phi_m(u)\mu(u)$ is a kernel of order ℓ if one satisfies the following conditions:

1. ϕ_m is polynomial of degree m.
2. $\{\phi_m\}$ is a basis for $L_2([-1,1])$.
3. $\forall u : \mu(u) \ge 0$ and $\mu(0) = 1$.
4. $\int \phi_j(u)\phi_k(u)\mu(u)du = I[j = k]$.

In case when integrated risk is equal to $\mathbb{E}\left[\int (\widehat{p}_h(x_0) - p(x_0))^2 dx\right]$.

Theorem 24.2. *Suppose*

1. *$\int K^2(u)du < \infty$.*
2. *K is kernel of order ℓ.*
3. *$\left(\int (p^{(\ell)}(x+t) - p^{(\ell)}(x))^2 dx\right)^{\frac{1}{2}} \le L|t|^{\beta-\ell}$.*
4. *$\int |u|^\beta |K(u)|du < \infty$.*

Then with $h^* = \alpha n^{-\frac{1}{2\beta+1}}$

$$\sup_p \mathbb{E}\left[\int (\widehat{p}_{h^*}(x) - p(x))^2 dx\right] \le C n^{\frac{-2\beta}{2\beta+1}}.$$

Theorem 24.3. *Let $y_i = f(x_i) + \epsilon_i$, under lots of assumptions... $\sup_{f\in\mathcal{F}} \mathbb{E}\left[\int (\widehat{f}_h(x) - f(x))^2 dx\right] \le C n^{\frac{-2\beta}{2\beta+1}}$ where $\widehat{f}_h(x) = \frac{\sum Y_i K\left(\frac{x-x_i}{h}\right)}{\sum K\left(\frac{x-x_i}{h}\right)}$.*

24.2 Lower bounds

Ingredients:

- Class of "parameters" Θ.
- A family of $\{P_\theta : \theta \in \Theta\}$.

- A semi distance on Θ, $d : \Theta \times \Theta \to [0, \infty)$ (satisfies triangle inequality).

Goal: try to show (rate minimax)

$$\liminf_{n\to\infty} \inf_{\widehat{\theta}_n} \sup_{\theta\in\Theta} \mathbb{E}\big[w(\psi_n^{-1} d(\theta_n, \theta))\big] \geq C > 0\,.$$

Theorem 24.4. *If $Y_i = f(x_i) + \epsilon_i$, $f \in W(\beta, \ell)$ (Sobolev functions)*

$$\liminf_{n\to\infty} \inf_{\widehat{f}} \sup_{f\in W(\beta,\ell)} \mathbb{E}\left[n^{\frac{\beta}{2\beta+1}} \int (\widehat{f}_n(x) - f(x))^2 dx\right] \geq C > 0$$

We take $w(u) = u^2$ and $\psi_n^{-1} = n^{\frac{-beta}{2\beta+1}}$.

General schemes: Fano's method.

1. From expectations to probabilities

$$\mathbb{E}\Big[w(\psi_n^{-1} d(\widehat{\theta}, \theta))\Big] \geq w(A) \Pr[\psi_n^{-1} d(\widehat{\theta}, \theta) \geq A] = w(A) \Pr[d(\widehat{\theta}, \theta) \geq \psi_n A]$$

   Thus we need only prove lower bound on $\inf_{\widehat{\theta}} \sup_\theta \Pr(d(\widehat{\theta}, \theta) > s)$.

2. Reduction to finite set of parameters $\inf_{\widehat{\theta}} \sup_\theta \Pr(d(\widehat{\theta}, \theta) \geq s) = \inf_{\widehat{\theta}} \max_{\theta\in\{\theta_1,\ldots,\theta_n\}} \Pr(d(\widehat{\theta}, \theta) \geq s)\,.$

3. Convert from estimation to testing. Suppose $\forall k \neq j : d(\theta_j, \theta_k) \geq 2s$ then for any estimator ψ and j:

$$\Pr_{\theta_j}(d(\widehat{\theta}, \theta_j) \geq s) \geq \Pr(\psi^* \neq j)$$

   Example 24.5. *$X_1, \ldots, X_n \sim \mathcal{N}(\theta, 1)$, $\theta \in \Theta = \mathbb{R}$, $P_\theta = \{\mathcal{N}(\theta, 1) : \theta \in \Theta\}$, $d(\widehat{\theta}, \theta) = |\widehat{\theta} - \theta|$, $w(u) = u$, and $P_{\theta_i} = P_i$.*

   want to prove $\inf_\psi \max_{j=0,1} P_j(\psi^* \neq j) \geq C > 0$, but need $|\theta_0 - \theta_1| \geq 2s$. Notice that $P_0(\overline{x}) = \prod_{i=1}^n \mathcal{N}(x_i \mid \theta_0, 1)$ and $\Pr(\psi \neq 0) = \Pr(\psi = 1)$.

$$\int I(\psi = 1) dP_0 = \int I(\psi = 1) \frac{p_0(x)}{p_1(x)} p_1(x) dx \geq \tau \int I\left(\psi = 1 \wedge \frac{p_0(x)}{p_1(x)} \geq \tau\right) p_1(x) dx \geq \tau(\pi - \alpha_1)$$

   where $\pi = P_1(\psi = 1)$ and $\alpha_1 = P_1\left(\frac{p_0(\overline{x})}{p_1(\overline{x})} < \tau\right)$.

$$p_* = \inf \max P_j(\psi \neq j) \geq \min_{0\leq\pi\leq 1} \max(\tau(\pi - \alpha_1), 1 - \pi) = \frac{\tau(1 - \alpha_1)}{1 + \tau}\,.$$

   It means $p_* \geq \sup_{\tau>0} \frac{\tau}{\tau+1} P_1\left(\frac{p_0(x)}{p_1(x)} \geq \tau\right)$, notice that $P_1\left(\frac{p_0(\overline{x})}{p_1(\overline{x})} \geq \tau\right) = P_1(\sqrt{n}\overline{x} + \frac{n}{2}(\theta_0^2 - \theta_1^2) > \tau)$. If $\theta_0 = \frac{1}{\sqrt{n}}, \theta_1 = 0$ then $P_1(\sqrt{n}\overline{x} + \frac{1}{2} > \tau) = 1 - \Phi(\tau - \frac{1}{2})$ and $d(\theta_0, \theta_1) \geq \psi_n = \frac{1}{\sqrt{n}}$. Thus $\liminf_{n\to\infty} \inf_\theta \sup_{\theta\in\Theta} \mathbb{E}\Big[w(\sqrt{n}|\widehat{\theta} - \theta|)\Big] \geq c > 0$