25.1 Ingredients

A class of "parameters" Θ

A family of distributions $\{P_\theta : \theta \in \Theta\}$

A semi-distance on Θ $d : \Theta$ x $\Theta \longrightarrow [0, \infty]$ (satisfy triangle inequality) (2 parameters are closer)

Goal (rate minimax)

$$\lim_{n\to\infty} \inf_{\theta} \inf_{\theta\in\Theta} \sup \mathbb{E}\Big[w(\psi_n^{(-1)})d(\widehat{\theta}, \theta)\Big]$$

w can be any monotonic increasing function

25.2 Scheme

1. Convert expectations to probabilities

$$\mathbb{E}\Big[w(\phi_n^{-1})d(\widehat{\theta}, \theta)\Big] \geq w(A)P(d(\widehat{\theta}, \theta) \geq \psi_n A)$$

   (using Markov's inequality)

2. Reduction to finite set, $\theta_0, ..., \theta_m \in \Theta$
   $\longrightarrow$ Won't work if the parameters are chosen randomly
   $\longrightarrow$ We need them to be "separated" in our distance d(.,.)

3. Convert from estimation to testing (show hard to test)
   (distinguish θ_j from θ_k) ¡ small
   (hard to tell difference between 2 θ's)

Example 25.1. *Regression at point x_0 Assumptions: $y_i = f(x_i)+, i = 1.., .., n$ f:[0,1]*

are iid with density P that satisfies the following

holds if are sub-gaussians

X_i are deterministic [0,1] and st A [0,1],n ¿=1 (equal amount of information all across)

25.3 Choices

1. $f \in \Theta = \sum(\beta, L)$
   Holder class, f is any function such that

$$\Rightarrow \int (f^{\beta}(x))^2 dx \leq L^2$$

(smoother derivatives)

2. P_f given by Assumptions (class of probability distributions)

3. $d(f,g) = |f(x_o) - g(x_o)|$ we want ψ to be $n^{\frac{-\beta}{2\beta+1}}$
   (this case we knew about the estimator, usually you may need to guess this and try it)
   Step 1 + 2 look at
   $$\inf_{f} \max_{f \in \{f_0, f_1\}} P_f(|\widehat{f}(x_0) - f(x_0)| \geq A\psi_n) \geq c' > 0$$
   need A, c', f_0, f_1
   $$f_0 = 0$$
   $$f_1^{(x)} = Lh_n^{\beta} K(\frac{x - x_0}{n}), x \in [0,1]$$
   with
   $$h_n = c_0 n^{\frac{-1}{2\beta+1}}$$
   choice of f and h is non trivial
   Require
   $$K \in \sum\!\!\!\!\!\!\int(\beta, \frac{1}{2}) c^{\infty}(\Re)$$
   $$K(u) > 0 \iff u \in (\frac{-1}{2}, \frac{1}{2})$$
   eg:
   $$K(u) = c_i K_0(2u)$$
   $$K_0(u) = exp(\frac{-1}{1-u^2}) I(|u| \leq 1)$$
   as n increases bump becomes shorter and narrower

1. Claimed that $f_i \in \sum(\beta, L)$, show this

2. Need $|f_0(x_0) - f_1(x_0)| \geq 2s = 2A\psi_n$ (parameters are far enough apart)

3. Need $P_f 0$, $P_f 1$ are "close" (hard to differentiate)
   Different metrics for probabilities
   last time : LRT
   this time :KL Divergence
   $$K(P_0, P_1) = \int \log \frac{dP_0}{dP_1} dP_0, if P_0 << P_1$$

1. Clear that $f_0 \in \sum(\beta, L)$ show $f_1 \in \sum(\beta, L)$ Let

$$L = \lfloor \beta \rfloor, f_1^{(l)} x = L h_n^{\beta - l} k(l) \frac{x - x_0}{h_n}$$

$$\begin{aligned} |f_1^l(x) - f_1^l(x')| &= L h_n^{\beta - l} |K^{(l)}(u) - k^{(l)}(u')| \\ &\leq h_n^{\beta - l} |u - u'|^{\frac{\beta - l}{2}} \\ &= \frac{L}{2} |x - x'|^{\beta - l} \end{aligned}$$

2.

$$\begin{aligned} |f_0(x_0) - f_1(x_0)| &= |f_1(x_0)| \\ &= L h^{\beta} k(0) \\ &= L c_0^{\beta} k(0) n^{\frac{-\beta}{2\beta + 1}} \\ &= \frac{1}{2} L c_0^{\beta} k(0) n^{\frac{-\beta}{2\beta + 1}} \end{aligned}$$

$$A = \frac{1}{2} L c_0^{\beta} k(0), \psi_n = n^{\frac{-\beta}{2\beta + 1}}$$

3.

$$K(P_{f_0}, P_{f_1}) \leq \alpha < \infty$$

$$\left[c' \geq \max\{\frac{1}{4} e^{-\alpha}, \frac{1 - \sqrt{\frac{\alpha}{2}}}{2}\} \right]$$

Alternatively you can use total variational distance.

For n large enough, $n h_n \geq 1$ and $L h_n^{\beta} k(u) \leq v_o \forall u$

$$\begin{aligned} K(P_0, P_1) &= \int .. \int \log \pi \frac{\mathbb{P}(y_i)}{P(y_i - f_1(x_i))} \pi [P(y_i) dy_i] \\ &= \sum \int \log \frac{P(y_i)}{P(y_i - f_1(x_i))} P(y_i) dy_i \\ &\leq P_* \sum_{i=1}^{n} f_1^2(x_i) \\ &= P_* L^2 h_n^{2\beta} \sum K^2 (\frac{x_i - x_0}{h_n}) \\ &\leq P_* L^2 h_n^{2\beta} K_{max}^2 \sum I(|\frac{x_i - x_0}{h_n}| \leq \frac{1}{2}) \\ &\leq P_* a_0 L^2 h_n^{2\beta} K_{max}^2 max(n h_n, 1) \\ &= P_* a_0 L^2 h_n^{2\beta + 1} K_{max}^2 n \end{aligned}$$

take,

$$C_0 = (\frac{\alpha}{P_* a_0 L^2 K_{max}^2})^{\frac{1}{2\beta + 1}}$$

Poor choice of f_1

take $x_i = \frac{1}{n}$

$$f_1(x) = (2\pi n)^{-1} sin(2\pi n x), \in \not\sum(1, 1)$$

$$\Rightarrow f_0(\frac{i}{n}) = f_1(\frac{i}{n})$$
$$\Rightarrow P_0 = P_1$$
$$\Rightarrow K(P_0, P_1) = 0$$

take

$$d(f_0, f_1) = ||f_0 - f_1||_\infty = (2\pi n)^{-1}$$

then

$$s = (4\pi n)^{-1} so \psi_n = \frac{c}{n}$$

known upper bound is

$$(\frac{logn}{n})^{\frac{1}{3}},$$

for

$$\not\sum(1,1)$$