13.1 Histograms

Suppose $X_1 ... X_n$ are i.i.d p on $[0,1]^d$. Divide the unit cube into $B_1, ..., B_n$ cubes each with side length h, $N \leq (\frac{1}{h})^d$. Define

$$\widehat{p}_h(x) = \sum_j \frac{\widehat{\theta}_j}{h^d} I(x \in B_j) \qquad \text{where} \qquad \widehat{\theta}_j = \frac{1}{n}\sum_{i=1}^n I(x_i \in B_j)$$

We want to bound

$$\mathbb{P}\left(\|\widehat{p}_h - p\|_\infty > \epsilon\right)$$

Consider :

$$\begin{aligned} &\mathbb{P}\left(\max_j \left|\frac{\widehat{\theta}_j}{h^d} - \frac{\theta_j}{h^d}\right| > \epsilon\right) && \left(\theta_j = \int_{B_j} p(x)dx\right) \\ &= \mathbb{P}\left(\max_j \left|\widehat{\theta}_j - \theta_j\right| > \epsilon h^d\right) \end{aligned}$$

assume $|p(x) - p(y)| \leq L\|x - y\|$ then p is bounded. It follows that

$$\begin{aligned} \theta_j = \int_{B_j} p(u)du \leq c\int_{B_j} du = ch^d \\ \to \theta_j(1-\theta_j) \leq \theta_j \leq ch^d \end{aligned}$$

Now

$$\begin{aligned} \mathbb{P}\left(\left|\widehat{\theta}_j - \theta_j\right| > h^d\epsilon\right) &\leq 2\exp\left(-\frac{1}{2}\frac{n\epsilon^2 h^{2d}}{\theta_j(1-\theta_j) + \epsilon h^d/3}\right) \\ &\leq 2\exp\left(-\frac{1}{2}\frac{n\epsilon^2 h^{2d}}{ch^d + \epsilon h^d/3}\right) \\ &\leq 2\exp\left(-cn\epsilon^2 h^d\right) && \left(c \to \frac{1}{2(c+1/3)}\right) \end{aligned}$$

$$\mathbb{P}\left(\max_j |\widehat{\theta}_j - \theta_j| > h^d\epsilon\right) \leq 2h^{-d}\exp\left(-cn\epsilon^2 h^d\right) =: \Pi_n$$

Now

$$\begin{aligned} p(x) - p_n(x) &= p(x) - \frac{\int_{B_j} p(u)du}{h^d} \\ &= \frac{1}{h^d}\int_{B_j}(p(x) - p(u))du \\ \to |\cdot| &\leq \frac{1}{h^d}\int_{B_j}|p(x) - p(u)|du \\ &\leq \frac{1}{h^d}Lh\sqrt{d}\int du = Lh\sqrt{d} \end{aligned}$$

wp $1-\Pi_n$

$$\begin{aligned}\|\widehat{p}_h - p\|_\infty &\le \|\widehat{p}_h - p_h\|_\infty + \|p_h - p\|_\infty \\ &\le \epsilon + L\sqrt{d}h.\end{aligned}$$

Therefore, wp at least $1-\delta$,

$$\|\widehat{p}_h - p\|_\infty \le \sqrt{\frac{1}{cnh^d}\log\left(\frac{2}{\delta h^d}\right)} + L\sqrt{d}h.$$

Choosing $h = (\frac{c}{n})^{\frac{1}{2+d}}$ gives that, wp at least $1-\delta$,

$$\|\widehat{p}_h - p\|_\infty = O\left(\left[\frac{\log n}{n}\right]^{\frac{1}{2+d}}\right)$$

13.2 Martingales

Let $(\Omega, \mathscr{F})$. Define sequence $\mathscr{F}_i$ of σ-algebras s.t. $\mathscr{F}_i \subset \mathscr{F}_{i+1}$ and $\mathscr{F}_0 = \sigma\{\Omega\}$. We call $((\mathscr{F}_i, (X_i))$ a martingale w.r.t $(\mathscr{F}_i)$ if X_i is $\mathscr{F}_i$ measurable and $E[X_i|\mathscr{F}_{i-1}] = X_{i-1}$ where $\mathscr{F}_i = \sigma(X_1, X_2, ..., X_i)$. Note the following

- $E[X_i] = \mu$
- If $E[X_i|\mathscr{F}_{i-1}] = 0$ call it a martingale difference.

Example 13.1. $Y_0 = E[X_0], Y_1 = X_i - X_{i-1}$

Example 13.2. *$\exists X$ s.t. can create $\mathscr{F}_i \subset \sigma[X]$, $X_i = E[X|\mathscr{F}_i$.*

13.3 Azuma's inequality

Let $((\mathscr{F}_i), (y_i))$ be a martingale difference s.t. for (a_i) and (b_i) , $P(Y_i \in [a_i, b_i]) = 1$ for all i. Then for $\delta > 0$,

$$\mathbb{P}\left(\frac{1}{n}\left|\sum Y_i\right| \ge \delta\right) \le 2\exp\left\{-\frac{n\delta^2}{\sum_{i=1}^n (b_i - a_i)^2}\right\}$$

Proof. (sketch)

$$E\left[e^{\lambda\sum_{i=1}^n Y_i}\right] = E\left[E\left[e^{\lambda\sum_{i=1}^n Y_i} \mid \mathscr{F}_{n-1}\right]\right] = E\left[e^{\lambda\sum_{i=1}^{n-1} Y_i} E\left[e^{\lambda Y_n} \mid \mathscr{F}_{n-1}\right]\right]$$ ■

Theorem 13.3 (McDiamird's)**.** *Let $Z = f(X_1, ...X_n)$. If f satisfies "bounded differences" :*

$$\sup_{x_1, x_2 \dots x_n, x' \in X^{n+1}} |f(x_1, ..., x_n) - f(x_2, ..., x_{i-1}, x', x_{i+1}, ..., x_n)| < c_i$$

Then $\delta > 0$, $P(Z - E[Z] > \delta) \le \exp\left(-\frac{2\delta}{\sum c_i^2}\right)$

Proof. (sketch)
Take $Y_0 = E[Z]$, $Y_1 = E[Z|X_1...X_i] = E[Z|X_1...X_{i-1}]$ and use Azuma. ■