15.1 Martingales

Let $(\Omega, \mathscr{F})$. Define sequence $\mathscr{F}_i$ of σ-algebras s.t. $\mathscr{F}_i \subset \mathscr{F}_{i+1}$ and $\mathscr{F}_0 = \sigma\{\Omega\}$. We call $((\mathscr{F}_i, (X_i))$ a martingale w.r.t $(\mathscr{F}_i)$ if X_i is $\mathscr{F}_i$ measurable and $E[X_i|\mathscr{F}_{i-1}] = X_{i-1}$ where $\mathscr{F}_i = \sigma(X_1, X_2...X_i)$. Note the following

- $E[X_i] = \mu$
- If $E[X_i|\mathscr{F}_{i-1}] = 0$ call it a martingale difference.

Example 15.1. $Y_0 = E[X_0], Y_1 = X_i - X_{i-1}$

Example 15.2. *$\exists X$ s.t. can create $\mathscr{F}_i \subset \sigma[X]$, $X_i = E[X|\mathscr{F}_i$.*

15.2 Azuma's inequality

Let $((\mathscr{F}_i), (y_i))$ be a martingale difference s.t. for (a_i) and (b_i) , $P(Y_i \in [a_i, b_i) = 1$ for all i. Then for $\delta > 0$,

$$P(\frac{1}{n}|\sum Y_i| \leq \delta) \leq 2\exp^{-\frac{n\delta^2}{\frac{1}{n}\sum_{i=1}^{n}(b_i - a_i)^2}}$$

Pf: sketch

$$Ee^{\lambda \sum_{i=1}^{n} Y_i} = E[E[e^{\lambda \sum_{i=1}^{n} Y_i}|\mathscr{F}_{n-1}]] = E[e^{\lambda \sum_{i=1}^{n}} Y_i E[e^{\lambda Y_n}|\mathscr{F}_{n-1}]$$

Let $Z = f(X_1, ...X_n)$.

Theorem 15.3 (McDiamird's)**.** *If f satisfies "bounded differences" :*

$$\sup_{x_1, x_2...x_n, x' \in X^{n+1}} |f(x_1, ..., x_n) - f(x_2, ..., x_{i-1}, x, x_{i+1}, ..., x_n)| < C_i$$

Then $\delta > 0$, $P(X - E[Z] > \delta) \leq \exp^{-\frac{2\delta}{\sum c_i^2}}$

Pf : sketch
$Y_0 = E[Z]$, $Y_1 = E[Z|X_1...X_i] = E[Z|X_1...X_{i-1}]$.