3.1 Convex functions

Definition 3.1. *Set C is* convex *iff* $\forall x, c \in C, \forall t \in [0; 1] \quad tx + (1-t)y \in C$.

So C is convex iff for any two points in C their segment is also entirely in C.

Definition 3.2. Convex combination *of set of points $x_1, \dots, x_k \in \mathbb{R}^n$ is*

$$\left\{ \sum_{i=1}^{k} \Theta_i x_i \;\middle|\; \sum_{i=1}^{k} \Theta_i = 1,\ \forall i\ \Theta_i \in [0; 1] \right\}.$$

Definition 3.3. Convex hull *of any $C \in \mathbb{R}^n$, denoted $conv(C)$ is a union of all convex combinations of different elements of C.*

Examples:

- Empty set, point, line, segment.
- Norm ball: $\{ x \mid \|x\| < r \}$.
- Hyperplane $\left\{ x \mid a^\mathsf{T} x = b \right\}$, Affine space $\{ x \mid Ax = b \}$.
- Hyperspace: $\left\{ x \mid a^\mathsf{T} x \le b \right\}$, Polyhedron $\{ x \mid Ax \le b \}$.
- Cone such that if $x_1, x_2 \in C$ then $t_1 x_1 + t_2 x_2 \in C\ \forall t_1, t_2 \ge 0$.

Definition 3.4. *Set C is a* cone *iff* $\forall t \ge 0,\ x \in C \implies tx \in C$.

Type of cones:

- Norm cone: $\{ (x, t) \mid \|x\| \le t \}$.
- Normal cone for some C and $x \in C$: $N_C(x) = \left\{ g \mid g^\mathsf{T} x \ge g^\mathsf{T} y\ \forall y \in C \right\}$.
- Positive semidefinite cone $S^n_+ = \{ x \in S^n \mid x \succeq 0 \}$, S^n is Hilbert space.

3.2 Key properties

- Separation hyperplane. A, B are convex, nonempty, disjoint. Then $\exists a, b :\ A \subseteq \left\{ x \mid a^\mathsf{T} \le b \right\}, B \subseteq \left\{ x \mid a^\mathsf{T} \ge b \right\}$.
- Supporting hyperplane. C nonempty, convex, $x_0 \in \mathit{boundary}(C)$. Then $\exists a :\ C \subseteq \left\{ x \mid a^\mathsf{T} x \le a^\mathsf{T} x_0 \right\}$.

3.3 Operations preserving convexity

- Intersection.
- Scaling, translation. C is convex $\implies$ $aC + b$ is convex.
- Affine image and preimage. $f(x) = Ax + b$, C is convex $\implies$ $f(C), f^{-1}(C)$ are convex.
- Lots more (See Boyd and Vandenberghe (2004), chapter 2).

Example 3.5. *$A_1, \dots, A_k, B \in \mathbb{S}^n$ – symmetrical matrices. Then* $C = \left\{ x \in \mathbb{R}^k \;\middle|\; \sum\limits_{i=1}^{k} x_i A_i \preceq B \right\}$.

Proof. $f : \mathbb{R}^k \to \mathbb{S}^n$, $f(x) = B - \sum\limits_{i=1}^{k} x_i A_i$. $C = f^{-1}(S_+^n)$ – affine preimage of convex cone. ■

3.4 Convex function

Definition 3.6. *Function $f : \mathbb{R}^n \to \mathbb{R}$ is* convex *iff* $\operatorname{dom} f \subseteq \mathbb{R}^n$ *is convex and*

$$\forall x, y \in \operatorname{dom} f, t \in [0; 1] \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

Other definitions:

- f is *concave* iff $-f$ is convex.
- f is *strictly convex* iff $\forall t \in (0; 1)$ the inequality in definition is strict.
- f is *strongly convex* with parameter τ iff $f(x) - \frac{\tau}{2} \|x\|_2^2$ is convex.

Examples:

- $f(x) = \frac{1}{x}$ is strictly convex, but not strongly.
- Univariate functions:
  - e^{ax} is convex $\forall a \in \mathbb{R}$ over $\mathbb{R}$.
  - x^a convex given $a \geq 1$ or $a \leq 1$ over $\mathbb{R}_+$.
  - $\log x$ is concave over $\mathbb{R}_+$.
- Affine $a^\top x b$ is both convex and concave.
- Quadratic $\frac{1}{2} x^\top Q x + b^\top x + c$ is convex if $Q \succeq 0$.
- $\|u - Ax\|_2^2$ convex since $A^\top A \succeq 0$.
- Norms: all vector norms and most matrix norms are convex.
- Indicator function is convex. C is a convex set, then $I_C(x) = \begin{cases} 0, & x \in C \\ \infty, & otherwise \end{cases}$.
- Support function is convex $\forall C$. $I_C^*(x) = \max\limits_{y \in C} x^\top y$.

3.5 Key properties

- f is convex iff its epigraph is convex, where $epi(f) = \{ (x,t) \in \operatorname{dom} f \times R \mid f(x) \leq t \}$.
- f is convex $\implies$ all its sublevel sets are convex. $C_t = \{ x \in \operatorname{dom} f \mid f(x) \leq t \}$. The converse is false.
- Assume f is differentiable. Then f is convex iff $\operatorname{dom} f$ is convex and $\forall x, y \in \operatorname{dom} f \quad f(y) \geq f(x) + \nabla f(x)^{\mathsf{T}}(y - x)$. Essentially, it means that f's graph is above any tangent plain.
- Assume f is twice differentiable. f is convex iff $\operatorname{dom} f$ is convex and $\forall x \in \operatorname{dom} f \quad \nabla^2 f(x) \succeq 0$.

3.6 Operations preserving function convexity

- Nonnegative linear combination.
- Pointwise maximum. $\forall s \in S \quad f_s$ is convex $\implies f(x) = \max_S f_s(x)$ is also convex.
- Partial minimum. $g(x, y)$ convex over variables x, y; C convex. Then $f(x) = \min_{y \in C} g(x, y)$ is also convex. E.g., $f(x) = \max_{y \in C} \|x - y\|$ or $f(x) = \min_{y \in C} \|x - y\|$.

References

Boyd, S., and Vandenberghe, L. (2004), *Convex optimization*, Cambridge university press.