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Ensuring Optimal Premium Pricing

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1 Introduction

In an era of overpopulation, light-speed communication and an unprecedented pace of technological advancement and global awareness, the stakes of ‘living up to the standard’ and avoiding life’s setbacks have become significant issues to address. In a world where money, competition and opportunity are the main drives of life, the need safety and security (from issues such as unemployment, car accident, property damage, illness, etc.) has become ever more essential. And so, an entire discipline - turning these issues and needs into economic goods - emerged, currently taking its high place in society, giving people the opportunity(commodity) to e-nsure that some of life’s setbacks do not pull them down to oblivion.

Insurance, especially in developed nations, has become an essential and obligatory component(living expense). For instance, in Canada, one cannot buy a car without insurance. In most student residential houses, there is a requirement to pay a ‘safety’ deposit, and so on. Thus, the bottom-line is that the insurance sector plays a vital role in society, and without it, the pace of human progress may be significantly hindered; if there is no insurance, taking risks will be done with far greater hesitation on a mass scale, reducing the incentive to ‘chance for change’. Therefore, it is essential for experts in the field (of insurance) to continually adjust to public and individual needs.

The terminologies ‘insurance policy’, ‘purchasing insurance’ or ‘insuring your...’ are all basically about making a contractual agreement between two entities (e.g. a person and an insurance company). The insured/policyholder (the one to be protected against a possible event known as a risk: car accident, death, etc.) pays to the insurer (the one responsible for protecting the insured when the event occurs) an amount known as a premium (could be as a lump-sum or in instalments). If the underlying event occurs, the insured files for what is known as a claim ($x for repairing a damaged car, flight cancellation, etc.). Thus, as Norberg states in his paper, on which this project will focus, the ‘very aim of insurance is to “spread risk” ’ by making the insured hold another party liable (partially or entirely) for a certain loss, if an even occurs.

The main issue of this is knowing **how to price insurance policies**. For instance, assuming that the insurance policies are of the same type, the main question lies in how much should a person with a high risk factor (highly liable to have the underlying event occurring) pay as premium, as opposed to someone with a low risk factor (unlikely to have the underlying event occurring). It is of commonsense, from the viewpoint of the insurer, that a person of high-risk should be charged with a higher premium that a person of low-risk. Hence, one of the main goals of an insurer is to develop accurate mathematical models in deriving suitable premiums for each policyholder, taking into account the risk he/she/it/they bear(s).

This paper discusses the historical and mathematical emergence and evolution of premium pricing, which is known as credibility theory. This research document articulates the main topics addressed in article written in the the late 70's: *The Credibility Approach to Experience Rating*, by the renowned academic Ragnar Norberg (who currently teaches in the London School of Economics). It is also important to note that although that article was written nearly forty years ago, the main significant contribution to the literature in credibility theory had already occurred during this time. One awaits until now for another breakthrough.

2 Mathematical Challenge

The two important issues in developing a premium model are:

1. Having a model that is mathematically tractable.
2. Finding and incorporating relevant historical data of the policyholder into the model.

The first issue is a general challenge that is present in all fields that depend on the usage of mathematical modelling (e.g. financial hedging strategies, weather forecasting, gambling, etc.). One may consider it an axiom that there is a tradeoff between having a realistic model and a mathematically simple one. Because of this, especially during the early years of credibility theory, many statisticians and actuaries (insurance mathematicians) turned to Bayesian statistics - which was considered as unconventional, and received a lot of criticism in the early stages of the literature (1920's to 1970's) - as an attempt to apply credibility models in the real world. And so, in most cases, one must find the appropriate balance between having a realistic model and a mathematically feasible one.

Regarding the second issue (which can be seen as a subset of the first), it is important that one can tailor premium models to individual risks (i.e. take into account information of the policyholder and adjust the model's parameters accordingly). As Norberg mentions: credibility theory 'explores certain principles and methods for adjusting insurance premiums as claims experience is obtained'. He demonstrates this in his paper by giving a numerical example of a list of policyholders who are assumed to have risk homogeneity (they have roughly the same liability of filing a valid claim), and hence are given the same premium amount, end up not being homogeneous as we look at their records throughout time (some of them filed six claims in ten years, while other have not filed a single one). Using a statistical test (the chi-squared test statistic), one rejects the notion that these policyholders have homogeneous risks, and thus, it would be inappropriate to assume that the risks amongst the individuals are similar. In fact, not incorporating this could lead to a bias in premium pricing. Also, next to this, by charging 'low-risk' people relatively high premiums, they would probably be more prone to cover the risk themselves or seek another insurance agency. Hence, taking heterogeneity of insurers -through incorporating historical experience- into account would help optimise policy's marketing strategy, and thus, maximise revenue.

Therefore, for the wellbeing of insurance companies, the main priority of actuaries in the field of credibility theory is to develop and apply effective premium models that incorporate claims experience of the underlying policyholder. The main mathematical models that are used (until today) for this are of the following form, which are also known as 'credibility formulas' (originally proposed by Whitney in 1918):
$\widetilde{\theta}_i = z\overline{\theta}_i + (1 - z)\mu$

where $\widetilde{\theta}_i$ is the premium -known as credibility premium- charged to the ith policyholder, $\widetilde{\theta}_i$ is the ith policyholder's experience (known as 'risk experience'); μ is a general experience (known as 'class experience') chosen/derived by the model makers; and z is the 'credibility factor', lying between (and including) zero and one, giving the appropriate weight to the policyholder's experience.

A good choice of z is on that is able to take into account factors such as how much historical data the insurer has on him/her/it/them, and aberrational fluctuations (in some years, there could be a high claim amount, due to an external event, which may not reflect the policyholder's true risk liability). Thus, the main objective of these 'credibility formulas', are to strike ' "a balance between class-experience on the one hand and risk-experience on the other hand" '. The following two sections give the two main paradigms of 'credibility formulas' in the theory.

3 Limited Fluctuation Theory (American Credibility Theory)

Initiated by Americans (and ergo the name American credibility), this theory developed in two steps. The difference between these steps is the choice of z, and hence the weighting of both $\widetilde{\theta}_i$ and μ. μ here was taken to be a 'manual premium' which was a fixed number obtained from an index manual, depending on certain characteristics of the ith policyholder (e.g. age, gender, income, etc.).

The first step, founded by Mowbray in 1914, z was taken to be either zero or one. This is known as full credibility, where z is set equal to one - and hence the insurer charges a premium based entirely on the policyholder's experience - if the number of years (age of the policy) is greater than a specific number. This is done mathematically as follows:

We set $z = 1$, if $P(|\overline{\theta}_i - \theta_i \leq k\theta_i)| \geq 1 - \epsilon$ is true
(i.e. 'if the probability is at least $1-\epsilon$ that its value is within $100k\% from \theta_i$').
By assuming that $\overline{\theta}_i$ follows a binomial distribution with mean θ_i and variance $\theta_i(1-\theta_i)/n$, one can employ the normal approximation rule - for a large n - and derive that

$z = 1$ if $n \geq n_0 = \left(\frac{\Phi^{-1}(\epsilon)^2(1-\theta_i)}{k^2\theta_i}\right)$ is true.

However, this model has its limitations, as z is severely restricted to two values, and policymakers would realistically be more inclined to using a value in between the two extremes. And so, experts in the field modified Mowbray's equation, using $z's$ that can take any value between, and including, zero and one. This is known as partial credibility. A popular partial credibility formula for z, used at the time, was as follows:

$z = \min\{\left(\frac{n}{n_0}\right)^{1/2}, 1\}$

However, in spite of this, the main weaknesses present in partial (and full) credibility is that there is no mathematical basis for choosing ϵ and k, as well as the vague notion of choosing μ. Therefore, there was a need to have credibility formulas whose parameters would be derived from sound mathematical bases. Thus, the a second and superior paradigm of 'credibility formulas' was established. The next section gives the main gist of it.

4 The Greatest Accuracy Theory (European Credibility Theory)

Strangely, in spite of its name, this type of credibility theory was also founded by Americans. This method is far more mathematically rigourous (and sophisticated) than American credibility. The main idea of it is as follows:

Assume that the risk characteristic parameter Θ_i - that influences the likelihood of a claim amount, X_i, to occur - is an outcome of random variables determined by a distribution, known as a 'structural distribution' $U(\theta)$. This was developed by Bülmann in 1969, and as Norberg states '$U(\theta)$ may be interpreted as the probability that a [policyholder] picked at random from the population has a claim probability per year not greater than θ'. In the paper, the policyholder is a driver, and so, '$U(\theta)$ =proportion of drivers i in the population with $\theta_i \leq \theta$. Therefore, the main art here would be choosing an appropriate $U(\theta)$, which would be tractable and as realistic as possible.

Thanks mainly to Bülmann, actuaries and policymakers are able estimate credibility premiums in 'credibility formula' form, using the European credibility approach. This was done under two subsequent frameworks: the Bülmann framework and the Bülmann-Straub (B-S for short) framework. The following two subsections explain each one in detail.

4.1 The Bülmann Model (1967, 1969)

Let x_{ji} correspond to the total claim amount of the ith policyholder in period j; x_i correspond to the claim information of policyholder i; and θ_i be 'an unknown parameter representing latent risk characteristics' for policyholder i. These are the main assumptions of the model:

1. $X_{1i}|\Theta_i, X_{2i}|\Theta_, \ldots$ are i.i.d (independent and identically distributed).
2. $(\theta_1, x_1), (\theta_2, x_2), \ldots$ are i.i.d.

Now, although the desired premium an insurer would like to derive for policyholder i for the following year $n+1$ is
$m(\theta_i) = E[X_{n+1,i}]|\theta_i]$, θ_i is unknown. Thus, the aim becomes calculating
$\widetilde{\widetilde{m}}_i := E[X_{n+1,i}|x_1, \ldots, x_n] = E[E[X_{n+1,i}|x_1, \ldots, x_n, \theta_i]]$
$= E[X_{n+1,i}|x_1]$ (by Assumption 2)
$= E[E[X_{n+1,i}|x_1, \theta_i]] = E[E[X_{n+1,i}|x_1, \theta_i]|x_i]$
$= E[E[X_{n+1,i}|\theta_i]|x_i] = E[m(\Theta_i)|x_1]$ (by Assumption 1)

Also, instead of having a manual premium μ, one can use an arbitrary premium $\breve{m}(x_i)$. In order to find a suitable $\breve{m}$, one can use the following criterion (proposed by Bülmann in 1967):

find the $\breve{m}$ that minimises $E[(m(\Theta_i) - \breve{m}(X_i))^2]$

The main problem with this equation is its mathematical tractability. Consequently, a further simplification was used: by assuming that $\breve{m}$ is of the linear form $b\overline{x}_{.i} + a$, where

$b\overline{x}_{.i} := \frac{\sum_{j=1}^{n} xji}{n}$.

The objective, therefore, became estimating a and b. A nice result of using this simplification is that an estimate of the credibility premium $\widetilde{\widetilde{m}}_i$, denoted as $\widetilde{m}_i$ can be expressed in 'credibility formula' form. More specifically:

$\widetilde{m}_i = \zeta\overline{x}_{.i} + (1-\zeta)\mu$, with

$\zeta = \frac{nVar(E[X_{ji}|\theta_i])}{nVar(E[X_{ji}|\theta_i]) + E[Var(X_{ji}|\theta_i)]}$ and $\mu = E[m(\theta_i)] = E[X_{ji}]$.

The result of having $\widetilde{m}_i$ in this form is that:

1. As n increases, and hence, as claim experience of the policyholder is gained, the credibility factor ζ increases, and thus, the credibility premium estimate $\widetilde{m}_i$, for policyholder i becomes more inclined to his/her/its/their historical claim average.

2. ζ is inversely proportional to $E[Var(X_{ji}|\theta_i)]$, which means that the credibility premium $\widetilde{m}_i$ leans more towards the 'manual premium' μ when the variation is high ($\overline{x}_{.i}$ becomes less reliable in this case, and the model favours μ).

The final piece to the puzzle in this model is selecting/deriving the distribution of θ_i, $U(\theta)$ for without it, one will not be able to estimate μ nor ζ. In practice, 'U can only be acquired from statistical analysis of risks in the collective (i.e. of all policyholders for all previous periods), the collateral (i.e. θ's of all policyholders for all previous periods) as well as the current one'. This is where Bayesian and Frequentist statisticians mainly differ.

Especially when there is not enough relevant historical data, and/or numerical simulation is infeasible (financially, technologically, mathematically, etc.), many *Bayesisist* actuaries pick their U based on educated judgment. Possible practical reasons behind this is that insurers want to be able to charge premiums for as many customers as possible without the need to wait several years to have enough data to calculate them. In this case, U is labelled as a prior distribution, and the first target would be to calculate the posterior distribution $f_{\theta|X_1,\ldots,X_n}$ and then eventually calculate the marginal distribution of the X_i's. Hence, the Bayesian approach is an appealing method to use to calculate credibility premiums when little historical or mathematical tractable information is available for the collateral risk parameter θ_i.

And so, as one could see, the Bülmann model was quite a breakthrough in credibility theory. However, notwithstanding this fact, there are several weakness, most notably due to the assumption that the conditional distributions of the policyholder across time are i.i.d, where in reality, it is rarely the case. This is mainly due to the fact that there usually are some periods where the policyholder experiences higher volatility than other periods, increasing the liability of issuing a valid claim to the insurer (e.g. economic recession, spread of an epidemic, bad weather,etc.). And so, for this reason, a superior model was developed: the Bülmann-Straub (B-S) model, which relaxes the i.i.d assumption. The next section explains the structure and components of this model.

4.2 The Bülmann-Straub Model (1970)

These are the main assumption of the B-S model (note that the second assumption is identical to that of the Bülmann model):

1. $X_{1i}|\Theta_i, X_{2i}|\Theta_i, \ldots$ are independent (not necessarily identically distributed, and hence removing the time-homogeneity assumption). In this case:

   $E[X_{ji}|\theta_i] = m(\theta_i)$ and $Var(X_{ji}|\theta_i) = s^2(\theta_i)/p_{ji}$
   where $i = 1, 2, \ldots, N$ policyholders and $j = 1, \ldots, n$ periods, and p_{ji}'s are known constants (measures of 'the volume of data or risk exposure').

2. $(\theta_1, x_1), (\theta_2, x_2), \ldots$ are i.i.d., where $x_i := (x_{1i}, \ldots, x_{ni})$

Generalising $\breve{m}$ to be $\breve{m}(x_i) = g_0 + \sum_{j=1}^{n} g_j x_{ji}$, one obtains the credibility premium estimate $\widetilde{m}_i$ also in 'credibility formula' form (extracted from Norberg's paper):

$\widetilde{m}_i = \frac{p_{.i}}{p_{.i}+\chi}\overline{x}_{.i} + (1 - \frac{p_{.i}}{p_{.i}+\chi})\mu$, where

$p_{.i} = \sum_{j=1}^{n} p_{ji}$, $\overline{x}_{.i} = \frac{1}{p_{.i}} \sum_{j=1}^{n} p_{ji}x_{ji}$,

$\mu = E[m(\Theta_i)] = E[X_{ji}]$ and $\chi = E[s^2(\Theta_i)]/Var(m(\Theta_i))$.

Similarly, but more complicatedly, it is imperative to find an appropriate U to estimate μ and χ (which in turn will estimate $\frac{p_{.i}}{p_{.i}+\chi}$. By doing so, one has reached a closer step to deriving a superiorly effective credibility model. Of course, there more we can relax the underlying assumptions in the model, the closer we are to achieving this goal. The next section highlights this issue.

5 Challenges of Generalisations

So far the main breakthrough in credibility theory has been the B-S model. A possible modification to it is the generalisation of the second assumption into an s by 1 vector. This is known as the Hachemeister regression model. however, it generally no longer follows the 'credibility formula' form, and the main challenge in this model is in finding suitable estimators of the the parameters, after having chosen a U. Therefore, one hesitates in using such a model, as the cost in terms of tractability as opposed to having model flexibility (and hence realistic formulae) is significantly high.

Also, next to generalising model parameters, it may be desirable to minimise the difference between $m(\theta_i)$ and $\check{m}(x_i)$ on a higher dimension plane, preserving the assumption of a linear relationship of the 'estimator and expected square error'. This employs the use of Hilbert spaces, which on the one hand is a significantly powerful tool in deriving general expression in higher planes, but on the other hand, seldom exists beyond the theoretical realm. Hence, one it is essential to prioritise the importance of having an applicable model over having intellectually stimulating derviations that cannot surpass the abstract world.

A possible simplification to the previous two issues is that one could be to assume that $X_i|\Theta_i$ is a member of the linear exponential family, (i.e. $f(x|\theta) = a(x)c(\theta)exp(-\theta x)$) and the pdf of θ_i, $u(\theta)$ is of the following form:

$u(\theta) = \frac{c(\theta)^{n_0} e^{-\theta x_0}}{d(n_0, x_0)}$, which is the corresponding 'natural conjugate' of f,

This results that $\widetilde{\widetilde{m}}_i = \widetilde{m}$. This can be generalised to vector form (shown by Jewell in 1974), for both x_i and θ_i. However, the trade-off here is that although the linear exponential family is rich with distributions, it is nevertheless restrictive.

Apart from the importance of finding a balance between the theoretical-applicability spectrum, another important and more specific issue to note is that the focus of parameter estimation has been completely on point estimation, and not interval estimation (up until now, to my knowledge as a student in the field). It is obvious that dealing with interval estimation would be significantly harder. Furthermore, even if one is able to deduce the estimates, a question such as this arises: 'which particular point in this region should be inserted into the credibility formula [?]'. The solutions to this issue have so far only been theoretical (to my knowledge).

Therefore, it is imperative to be aware of the underlying assumptions of the credibility models used in practice and the limitations they have. One should also assess the feasibility and improvement level of generalising these models, as the sayings go ‘if it ain’t broke, don’t fix it’ and ‘deal with what you can, let go of the rest’. Thus, one (the insurer) needs to find the optimally effective premium model corresponding to his/her/its/their surrounding environment.

6 Conclusion

This paper underlined the ethos of credibility theory and its evolution up until this date. As time goes by, one should expect improvement premium modelling, especially in an age of high pace technological and academic advancement. The insurance sector will continue to play its role in providing safety and security adjusting its products to society’s needs.