Influence of Aggregate Claims Models on VaR and TVaR Risk Measures

Actsc 966 Project

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Abstract

This paper derives explicitly the distributions of six aggregate claim models and calculates the VaR and TVaR of each of them, under five different confidence levels. The first four models assume that the claim amounts follow a discrete uniform distribution; the first three examine the case that the number of claims is a member of the (a,b,0) class (i.e. Poisson,binomial and negative binomial); the fourth model examines the case that the number of claims follows a logarithmic distribution, which is a member of the (a,b,1) class. We then give a general comparison of these four models and pose their practicality weaknesses (most notably that we assume the claim amount is discrete). Unlike the first four models, the last two ones assume that the aggregate claim amount is the sum of two smaller independent aggregate claim amounts, in which each one has claim amounts exponentially distributed and claim sizes following geometric or negative binomial distributions. An interesting property of these two models is that they can be expressed in Phase-type form, which gives an elegant representation of the models, yet may pose some stability issues. We conclude by giving a brief comparison of the different models, addressing practicality issues of these models and the main challenges of finding and deriving more powerful ones.

1. Introduction

In this day and age insurance - in its many forms - plays a heavy role in the wellbeing of several societies (especially developed ones), by providing their individuals/corporations with financial/service coverage in case of the occurrence of certain events (e.g. earthquake, fire, death, etc.). This has become increasingly important nowadays mainly due to the increasingly competitive lifestyle and the increasing interdependence between the different elements of society (e.g. the plunge of the oil prices affects the unemployment rate in the private sector; thus the risk of losing one's job has increased). Thus, it is of society's main interest to ensure that the insurance sector continues to thrive. Otherwise, a new form of safety protocols would need to be established.

Naturally, as the insurance sector is becoming more privatized, the need for developing effective marketing strategies has become a key element for the success of insurance companies. However, next to this, insurance companies must assess their financial risk exposure, in order to optimally design insurance policies, while minimizing the risk of bankruptcy. This is especially important for this type of business, since insurance companies are constantly facing the nature of odds (i.e. the filing of too many claims than accounted for), and hence face more risk in going out of business overnight. Hence, an important key factor in developing sound insurance policies is to accurately model aggregate claim models, in order to evaluate the risk exposure of their business, without falling into ruin.

This, however, is no easy task and it is usually the case that the more realistic a model is required to be, the less mathematically tractable it becomes. This paper gives an idea of this. We derive the probability distributions of six different models, starting with simplistic ones (discrete cases) and then proceeding to more mathematically demanding models (continuous cases). We calculate two types of risk measures (the VaR and TVaR) for each model and comment on the results, by giving comparisons between the models on the basis of both their mathematical tractability and practicality. We begin by giving the mathematical notations and main assumptions used in this paper.

2. Mathematical Notations and Assumptions

N: The number of claims. A discrete (counting) non-negative random variable with a pf p_k and pgf $P_N(z)$.

***X_i*:** The claim amount of the ith individual. All X_i's are non-negative, i.i.d. and independent of N. Can be discrete with pf f_i:=Pr{X_1=i}, or continuous with pdf and cdf $f_X(x)$ and $F_X(x)$ respectively.

***S*:** Total claim amount, i.e. $S = \sum_{i=1}^{N} X_i$. For the discrete case (if and only if X is discrete), S has a pf g_k=Pr{S=k}. For the continuous case (if and only if X is continuous) S has pdf, cdf and mgf $f_S(s)$, $F_S(s)$ and $M_S(t)$ respectively. Note that $g_0=P_N(f_0)$.

$VaR_\alpha(S)$: Value at risk, for some confidence level α, mathematically defined as:

$$\inf\{x \in \Re \mid \Pr\{X \le x\} \ge \alpha\}.$$

$E[(S-d)_+]$: Expected stop loss premium d. where:

$$(S-d)_+ = \begin{cases} 0 & \text{if } S \le d \\ S-d & \text{if } S > d \end{cases}$$

$TVaR_\alpha(S)$: Total value at risk, for some confidence level α, mathematically defined as:

$$TVaR_\alpha(S) = \frac{\int_\alpha^1 VaR_q(S)dq}{1-\alpha}$$

Note that it can be proved that:

$$\begin{aligned} TVaR_\alpha(S) &= VaR_\alpha(S) + \frac{E[(S - VaR_\alpha(S))_+]}{1-\alpha} \\ &= VaR_\alpha(S) + \frac{E[S] - E[S \wedge VaR_\alpha(S)]}{1-\alpha} \end{aligned}$$

which will be used in repeatedly in this paper.

Now we present the six models in which we derive their distributions and calculate their VaR and TVaR values for α=0.95, 0.96, 0.97, 0.98, and 0.99.

3. Compound Discrete Models

X_i is discrete uniform taking values 0,1,…,101, then the pf is

$$f_i = P(X_i = x) = \begin{cases} 1/101, & x = 0,1,\ldots 100 \\ 0 & otherwise \end{cases}$$

The difference between the following four models lies in the distribution of N. They are (a, b, 0) class or (a, b, 1) class. For comparative reasons, we assume that $E[N]$=100 for all the models.

For compound models with N is (a, b, 0) class or (a, b, 1) class, we use Panjer's recursive formula to calculate a pf of S, g_k=Pr{S=k}.

$$g_x = \frac{\sum_{y=1}^{x}(a+by/x)f_y \cdot g_{x-y}}{1-a\cdot f_0}, \qquad \text{for } (a,b,0) \text{ case.}$$

$$g_x = \frac{(p_1-(a+b)p_0)\cdot f_x + \sum_{y=1}^{x}(a+by/x)f_y \cdot g_{x-y}}{1-a\cdot f_0}, \quad \text{for } (a,b,1) \text{ case.}$$

Where starting point is $g_0 = \Pr(S=0) = P_N(f_0)$

Instead of using both formula, we use Panjer's recursive formula for (a, b, 1) case which is more general formula. In (a, b, 0) case, we set the first term $(p_1-(a+b)p_0)$ equal to 0.

1) <u>Model 1: N is Poisson: Poi (100)</u>

$$p_n = \frac{e^{-\lambda}\cdot\lambda^n}{n!} = \frac{e^{-100}\cdot 100^n}{n!}, \quad n = 0,1,\cdots$$

$$p_n = \frac{\lambda}{n}\cdot p_{n-1} = \frac{100}{n}\cdot p_{n-1}, \quad n = 1,2\cdots \quad \Rightarrow \quad a = 0, \quad b = 100$$

$$P_N(z) = e^{\lambda(z-1)} \quad \Rightarrow \quad g_0 = P_N(f_0) = e^{100\cdot(\frac{1}{101}-1)}$$

With the aid of the software package MATLAB (see Appendix), the VaR values for the corresponding alpha values are:

α	0.95	0.96	0.97	0.98	0.99
VaR	5,973	6,039	6,120	6,228	6,401

Also, E[S-VaR] $_+$ and TVaR are found to be:

α	0.95	0.96	0.97	0.98	0.99
E[*S*-VaR]+	13.13	10.17	7.35	4.69	2.18
TVaR	6,236	6,293	6,365	6,462	6,619

2) <u>Model 2: N is Binomial: bin(125,0.8)</u>

$$p_n = \binom{m}{n} \cdot p^n (1-p)^{m-n} = \binom{125}{n} \cdot 0.8^n (1-0.8)^{125-n}, \quad n = 0,1,\cdots$$

$$p_n = \left(-\frac{p}{1-p} + \frac{(m+1)\cdot \frac{p}{1-p}}{n}\right) \cdot p_{n-1}$$

$$= \left(-\frac{0.8}{1-0.8} + \frac{(125+1)\cdot \frac{0.8}{1-0.8}}{n}\right) \cdot p_{n-1}, \quad n = 1,2,\cdots \quad \Rightarrow \quad a = -4, \quad b = 504$$

$$P_N(z) = (1+p(z-1))^m \quad \Rightarrow \quad g_0 = P_N(f_0) = (1+0.8\cdot(\frac{1}{101}-1))^{125}$$

With the aid of the software package MATLAB (see Appendix), the VaR values for the corresponding alpha values are:

α	0. 95	0. 96	0. 97	0. 98	0. 99
VaR	5, 607	5, 646	5, 694	5, 758	5, 860

E[*S*-VaR] $_+$ and TVaR are found to be:

α	0.95	0.96	0.97	0.98	0.99
$E[S-VaR]_+$	7.73	5.99	4.32	2.73	1.25
TVaR	5,762	5,796	5,838	5,895	5,985

3) Model 3: N is Negative Binomial: NB(5.20)

$$p_n = \binom{n+r-1}{n} \cdot \left(\frac{1}{1+\beta}\right)^r \left(\frac{\beta}{1+\beta}\right)^n = \binom{n+5-1}{n} \cdot \left(\frac{1}{1+20}\right)^5 \left(\frac{20}{1+20}\right)^n, \quad n = 0,1,\cdots$$

$$p_n = \left(\frac{\beta}{1+\beta} + \frac{(r-1)\cdot\frac{\beta}{1+\beta}}{n}\right)\cdot p_{n-1}$$

$$= \left(\frac{20}{1+20} + \frac{(5-1)\cdot\frac{20}{1+20}}{n}\right)\cdot p_{n-1} \quad n = 1,2,\cdots \quad \Rightarrow \quad a = \frac{20}{21}, \ b = \frac{80}{21}$$

$$P_N(z) = (1-\beta(z-1))^{-r} \quad \Rightarrow \quad g_0 = P_N(f_0) = (1-20\cdot(\frac{1}{101}-1))^{-5}$$

With the aid of the software package MATLAB (see Appendix), the VaR values for the corresponding alpha values are:

α	0.95	0.96	0.97	0.98	0.99
VaR	9,291	9,659	10,125	10,765	11,823

E[S-VaR]$_+$ and TVaR are found to be:

α	0.95	0.96	0.97	0.98	0.99
$E[S-VaR]_+$	78.23	61.73	45.52	29.72	14.42
TVaR	10,856	11,202	11,642	12,251	13,265

4) <u>Model 4: N is Logarithmic</u>

The pf of Logarithmic distribution is $p_n = \dfrac{\left(\dfrac{\beta}{1+\beta}\right)^n}{n\cdot\log(1+\beta)}$, $n = 1,2,\cdots$. There is no mass point at n=0. First, we calculate the parameter β to get mean of N equal to 100.

$$E[N] = \sum_{n=1}^{\infty} n\cdot p_n = \sum_{n=1}^{\infty} n\frac{\left(\dfrac{\beta}{1+\beta}\right)^n}{n\cdot\log(1+\beta)} = \frac{1}{\log(1+\beta)}\sum_{n=1}^{\infty}\left(\frac{\beta}{1+\beta}\right)^n = \frac{\beta}{\log(1+\beta)} = 100$$

Using trial-and-error method, we get $\beta \approx 647.46$. Thus, we have

$$p_n = \left(\frac{\beta}{1+\beta} + \frac{-\dfrac{\beta}{1+\beta}}{n}\right)\cdot p_{n-1}$$

$$= \left(\frac{647.46}{1+647.46} + \frac{-\dfrac{647.46}{1+647.46}}{n}\right)\cdot p_{n-1},\quad n = 2,3,\cdots,\quad \Rightarrow\quad a = \frac{647.46}{648.46},\quad b = -\frac{647.46}{648.46}$$

Because the recursive relationship starts from n=2, the logarithmic distribution is a member of the (a,b,1) class. Additionally, the values $p_0 = 0,\quad p_1 = \dfrac{647.46/648.46}{\log(648.46)}$ are used to calculate Panjer's recursive formula.

With the aid of the software package MATLAB (see Appendix), the VaR values for the corresponding alpha values are:

α	0.95	0.96	0.97	0.98	0.99
VaR	25,187	29,252	34,786	43,073	58,320

Also, $E[\ S\text{-}VaR\ ]_+$ and TVaR are found to be:

α	0.95	0.96	0.97	0.98	0.99
$E[S\text{-}VaR]_+$	1,042	860	667	463	244
TVaR	46,016	50,738	57,031	66,243	82,765

5) <u>Comparing Results of Models 1-4</u>

Recall that models for claim number random variable N Poisson(100), Bin(125,0.8), NB(5,20), and Logarithmic(647.48) have same expected value 100. However, VaR and TVaR of aggregate claim S using the same distribution of claim size X differ quite significantly. We gather the results of the four models in the following tables for model comparison:

[VaR]

α	0.95	0.96	0.97	0.98	0.99
Poisson	5,973	6,039	6,120	6,228	6,401
Bin	5,607	5,646	5,694	5,758	5,860
NB	9,291	9,659	10,125	10,765	11,823
Log	25,187	29,252	34,786	43,073	58,320

[TVaR]

α	0.95	0.96	0.97	0.98	0.99
Poisson	6,236	6,293	2,365	6,462	6,619
Bin	5,782	5,796	5,838	5,895	5,985
NB	10,856	11,202	11,642	12,251	13,265
Log	46,016	50,738	57,031	66,243	82,765

As shown above, the VaR and TVaR values are smallest in binomial case and largest in Logarithmic case. The main reason behind this is tail thickness: the binomial distribution has the thinnest tail here (and hence the smallest VaR and TVaR values) and the Logarithmic distribution has the thickest tail here (and hence the largest VaR and TVaR values). Hence, in terms of practicality, the most appealing model here would be the aggregate claims model with the logarithmic distribution. However, with the X's taking discrete, finite values, one find difficulty in finding empirical cases which models such as those can fit. We now turn to more realistic cases, where X is modeled as a continuous random variable.

4. Compound Continuous Models

Assume $S=S_1+S_2$, where S_1 and S_2 are independent continuous aggregate Claims. Because S_1 and S_2 are assumed to be independent, we have $M_S(t)=M_{S_1}(t)M_{S_2}(t)$.

Let $S_1=\sum_{i=0}^{N_1} X_i$ and $S_2=\sum_{i=0}^{N_2} Y_i$, where X_i ~ exp(1/20) and Y_i ~ exp(1/25).

Assume N_1 ~ NB(1,150), that implies S_1 satisfies the following theorem:

Theorem (compound geometric-exponential)

If $S=\sum_{i=0}^{N} X_i$, where X_1, X_2,… are i.i.d. and have exponential distribution with mean θ, and the counting random variable N has a geometric distribution with mean β. Assuming X_i and N are independent, the cdf of S is

$$F_S(s)=1-\frac{\beta}{1+\beta}e^{-\frac{s}{\theta(1+\beta)}} \;; \qquad x\geq 0$$

It is easy to get the mgf of S, $M_S(t)=\frac{1}{1+\beta}+\frac{\beta}{1+\beta}\frac{1}{1-\theta(1+\beta)t}$

We now derive the distribution of the last two models, calculating the VaR and TVaR values of the above mentioned confidence levels.

1) Model 5: N_2 ~NB(1,80)

The counting random variables N_1 and N_2 have geometric distributions with $\beta_1 = 150$ and $\beta_2 = 80$. Let θ_1=E[X_1]=20 and θ_2=E[Y_1]=25. There are two methods.

Method 1. Based on the above theorem, one gets:

$$
\begin{aligned}
M_S(t) &= M_{S_1}(t)M_{S_2}(t) \\
&= \left[\frac{1}{1+\beta_1}+\frac{\beta_1}{1+\beta_1}\frac{1}{1-\theta_1(1+\beta_1)t}\right]\left[\frac{1}{1+\beta_2}+\frac{\beta_2}{1+\beta_2}\frac{1}{1-\theta_2(1+\beta_2)t}\right] \\
&= \frac{1}{(1+\beta_1)(1+\beta_2)}+\frac{\beta_1}{(1+\beta_1)(1+\beta_2)}\left[\frac{1}{1-\theta_1(1+\beta_1)t}\right] \\
&\quad +\frac{\beta_1\beta_2}{(1+\beta_1)(1+\beta_2)}\left[\frac{1}{1-\theta_1(1+\beta_1)t}\frac{1}{1-\theta_2(1+\beta_2)t}\right] \\
&\quad +\frac{\beta_2}{(1+\beta_1)(1+\beta_2)}\left[\frac{1}{1-\theta_2(1+\beta_2)t}\right] \\
&= A_0 + A_1M_1(t) + A_2M_1(t)M_2(t) + A_3M_2(t)
\end{aligned}
$$

$$
\textit{where}\quad A_0 = \frac{1}{(1+\beta_1)(1+\beta_2)},\quad A_1 = \frac{\beta_1}{(1+\beta_1)(1+\beta_2)},
$$

$$
A_2 = \frac{\beta_1\beta_2}{(1+\beta_1)(1+\beta_2)},\quad A_3 = \frac{\beta_2}{(1+\beta_1)(1+\beta_2)}
$$

M_1 is the mgf of exponential distribution with mean $\theta_1(1+\beta_1)$

M_2 is the mgf of exponential distribution with mean $\theta_2(1+\beta_2)$

We denote $Z_1 \sim \exp(\lambda_1)$, $\lambda_1 = (\theta_1(1+\beta_1))^{-1}$

$Z_2 \sim \exp(\lambda_2)$, $\lambda_2 = (\theta_2(1+\beta_2))^{-1}$

then the survival function of S can be represented as

$$
S(x) := 1 - F_S(x) = A_1P(Z_1 > x) + A_2P(Z_1 + Z_2 > x) + A_3P(Z_2 > x)
$$

Let $Z = Z_1 + Z_2$, then Z has a hypoexponential distribution.

$$
\begin{aligned}
f_Z(z) &= \int_0^z \lambda_1 e^{-\lambda_1(z-z_2)} \lambda_2 e^{-\lambda_2 z_2} dz_2 \\
&= \lambda_1 \lambda_2 e^{-\lambda_1 z} \int_0^z e^{-(\lambda_2-\lambda_1) z_2} dz_2 \\
&= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 z} \left[1 - e^{-(\lambda_2-\lambda_1) z} \right] \\
&= \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_1 e^{-\lambda_1 z} - \frac{\lambda_1}{\lambda_2 - \lambda_1} \lambda_2 e^{-\lambda_2 z}
\end{aligned}
$$

One can simplify the survival function of S as follows:

$$
\begin{aligned}
S(x) &= A_1 P(Z_1 > x) + A_2 \left[\frac{\lambda_2}{\lambda_2 - \lambda_1} P(Z_1 > x) - \frac{\lambda_1}{\lambda_2 - \lambda_1} P(Z_2 > x) \right] + A_3 P(Z_2 > x) \\
&= \left(A_1 + A_2 \frac{\lambda_2}{\lambda_2 - \lambda_1} \right) P(Z_1 > x) + \left(A_3 - A_2 \frac{\lambda_1}{\lambda_2 - \lambda_1} \right) P(Z_2 > x) \\
&= p_1 \ P(Z_1 > x) + p_2 \ P(Z_2 > x) \\
&\textit{where } \ p_1 = A_1 + A_2 \frac{\lambda_2}{\lambda_2 - \lambda_1}, \quad p_2 = A_3 - A_2 \frac{\lambda_1}{\lambda_2 - \lambda_1}
\end{aligned}
$$

Since survival function S is continuous and strictly decreasing on $[0, +\infty)$, $\mathrm{VaR}_\alpha(x)$ is the unique solution to $F(S \leq x) = \alpha$. With the aid of the software package MATLAB (see Appendix), the VaR values for the corresponding alpha values are:

α	0. 95	0. 96	0. 97	0. 98	0. 99
VaR	12, 058	12, 766	13, 672	14, 940	17, 088

As for the TVaR, since TVaR = VaR + E[S-VaR]$_+$ / (1 $-\alpha$), and since one can show that, in this case:

$$
\begin{aligned}
E[S-d]_+ &= \int_d^\infty S(x) dx \\
&= \int_d^\infty p_1 e^{-\lambda_1 x} + p_2 e^{-\lambda_2 x} dx \\
&= \frac{p_1}{\lambda_1} e^{-\lambda_1 d} + \frac{p_2}{\lambda_2} e^{-\lambda_2 d}
\end{aligned}
$$

E[S-VaR]$_+$ and TVaR are found to be:

α	0. 95	0. 96	0. 97	0. 98	0. 99

E[S-VaR]+	156.1357	124.419	92.913	61.6356	30.6268
TVaR	15,181	15,876	16,769	18,022	20,151

Method 2. An alternative and more elegant way for calculating VaR and TVaR is by deriving the distribution of S and $(S\text{-}d)_+$ in Phase-type form, which fortunately is possible here (due to S being a weighted combination of exponential and hypoexponential random variables).

With $\underline{\gamma} = [A_1 + A_2, A_3]$, $T = \begin{bmatrix} -\lambda_1 & A_2\lambda_1/(A_1 + A_2) \\ 0 & -\lambda_2 \end{bmatrix}$ and $\underline{e}' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have

$$F_S(s) = 1 - \underline{\gamma}\exp(-Ts)\underline{e}'$$

Also, one can show that $(S\text{-}d)_+$ is of Phase-type with:

$$F_{(S-d)_+}(s) = 1 - \underline{\gamma_d}\exp(-Ts)\underline{e}', \text{ and } E[(S-d)_+] = -\underline{\gamma_d}T^{-1}\underline{e}'$$

where $\underline{\gamma_d} = \gamma\exp(-Td)$

This method makes computation less tedious, but may have accuracy limitations if T is of a higher dimension (such as 5 dimensions).

2) ***Model 6: N_2 ~NB(2,40)***

The counting random variables N_1 and N_2 have a geometric distribution NB (1, β_1 = 150) and a negative binomial distribution NB (2, β_2 = 40), respectively.

Let θ_1=E[X_1]=20 and θ_2=E[Y_1]=25. As in **Model 5**, there are two methods to solve the problem.

One can derive the mgf of S to be:

$$
\begin{aligned}
M_S(t) &= M_{S_1}(t)M_{S_2}(t) \\
&= \left[\frac{1}{1+\beta_1}+\frac{\beta_1}{1+\beta_1}\frac{1}{1-\theta_1(1+\beta_1)t}\right]\left[\frac{1}{1+\beta_2}+\frac{\beta_2}{1+\beta_2}\frac{1}{1-\theta_2(1+\beta_2)t}\right]^2 \\
&= \frac{1}{(1+\beta_1)(1+\beta_2)^2}+\frac{\beta_1}{(1+\beta_1)(1+\beta_2)^2}\left[\frac{1}{1-\theta_1(1+\beta_1)t}\right] \\
&\quad +\frac{2\beta_2}{(1+\beta_1)(1+\beta_2)^2}\left[\frac{1}{1-\theta_2(1+\beta_2)t}\right] \\
&\quad +\frac{2\beta_1\beta_2}{(1+\beta_1)(1+\beta_2)^2}\left[\frac{1}{1-\theta_1(1+\beta_1)t}\frac{1}{1-\theta_2(1+\beta_2)t}\right] \\
&\quad +\frac{\beta_2{}^2}{(1+\beta_1)(1+\beta_2)^2}\left[\frac{1}{1-\theta_2(1+\beta_2)t}\right]^2 \\
&\quad +\frac{\beta_1\beta_2{}^2}{(1+\beta_1)(1+\beta_2)^2}\left[\frac{1}{1-\theta_1(1+\beta_1)t}\right]\left[\frac{1}{1-\theta_2(1+\beta_2)t}\right]^2 \\
&= A_0 + A_1M_1(t) + A_2M_2(t) + A_3M_1(t)M_2(t) + A_4M_2(t)^2 + A_5M_1(t)M_2(t)^2
\end{aligned}
$$

$$
\begin{aligned}
\textit{where}\quad A_0 &= \frac{1}{(1+\beta_1)(1+\beta_2)^2},\quad A_1 = \frac{\beta_1}{(1+\beta_1)(1+\beta_2)^2}, \\
A_2 &= \frac{2\beta_2}{(1+\beta_1)(1+\beta_2)^2},\quad A_3 = \frac{2\beta_1\beta_2}{(1+\beta_1)(1+\beta_2)^2}, \\
A_4 &= \frac{\beta_2{}^2}{(1+\beta_1)(1+\beta_2)^2},\quad A_5 = \frac{\beta_1\beta_2{}^2}{(1+\beta_1)(1+\beta_2)^2},
\end{aligned}
$$

M_1 is the mgf of exponential distribution with mean $\theta_1(1+\beta_1)$

M_2 is the mgf of exponential distribution with mean $\theta_2(1+\beta_2)$

Then the survival function of S can be written as

$$S(x) = A_1 P(Z_1 > x) + A_2 P(Z_2 > x) + A_3 P(Z_1 + Z_2 > x) \\ + A_4 P(2Z_2 > x) + A_5 P(Z_1 + 2Z_2 > x)$$

where $Z_1 \sim \exp(\lambda_1)$, $\lambda_1 = (\theta_1(1+\beta_1))^{-1}$

$Z_2 \sim \exp(\lambda_2)$, $\lambda_2 = (\theta_2(1+\beta_2))^{-1}$

We know the survival function of Z_1+Z_2 from **Model 5**, and that $2Z_2$ will follow gamma distribution $G(2, 1/\lambda_2)$. So the only task here is the distribution of Z_1+2Z_2.

Let $Z = Z_1 + Z_3$, where $Z_3 = 2Z_2$, so Z_3 follows gamma distribution $G(2, 1/\lambda_2)$. Then the pdf of Z could be

$$\begin{aligned}
f_Z(z) &= \int_0^z \lambda_1 e^{-\lambda_1(z-z_3)} \lambda_2^{\,2} z_3 e^{-\lambda_2 z_3} dz_3 \\
&= \lambda_1 \lambda_2^{\,2} e^{-\lambda_1 z} \int_0^z z_3 e^{-(\lambda_2-\lambda_1)z_3} dz_3 \\
&= \lambda_1 \lambda_2^{\,2} e^{-\lambda_1 z} \left[-\frac{1}{\lambda_2 - \lambda_1} \int_0^z z_3 de^{-(\lambda_2-\lambda_1)z_3} \right] \\
&= \lambda_1 \lambda_2^{\,2} e^{-\lambda_1 z} \left[-\frac{ze^{-(\lambda_2-\lambda_1)z}}{\lambda_2 - \lambda_1} + \frac{1}{\lambda_2 - \lambda_1} \int_0^z e^{-(\lambda_2-\lambda_1)z_3} dz_3 \right] \\
&= \lambda_1 \lambda_2^{\,2} e^{-\lambda_1 z} \left[-\frac{ze^{-(\lambda_2-\lambda_1)z}}{\lambda_2 - \lambda_1} + \frac{1}{(\lambda_2 - \lambda_1)^2} \left[1 - e^{-(\lambda_2-\lambda_1)z}\right] \right] \\
&= \frac{\lambda_2^{\,2}}{(\lambda_2-\lambda_1)^2} \lambda_1 e^{-\lambda_1 z} - \frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2} \lambda_2 e^{-\lambda_2 z} - \frac{\lambda_1}{\lambda_2 - \lambda_1} \lambda_2^{\,2} z e^{-\lambda_2 z}
\end{aligned}$$

Now we simplify the survival function of S as

$$
\begin{aligned}
S(x) &= A_1 S(Z_1) + A_2 S(Z_2) + A_3\left[\frac{\lambda_2}{\lambda_2-\lambda_1}S(Z_1) - \frac{\lambda_1}{\lambda_2-\lambda_1}S(Z_2)\right] \\
&\quad + A_4 S(Z_3) + A_5\left[\frac{\lambda_2{}^2}{(\lambda_2-\lambda_1)^2}S(Z_1) - \frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2}S(Z_2) - \frac{\lambda_1}{\lambda_2-\lambda_1}S(Z_3)\right] \\
&= \left[A_1 + A_3\frac{\lambda_2}{\lambda_2-\lambda_1} + A_5\frac{\lambda_2{}^2}{(\lambda_2-\lambda_1)^2}\right]S(Z_1) \\
&\quad + \left[A_2 - A_3\frac{\lambda_1}{\lambda_2-\lambda_1} - A_5\frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2}\right]S(Z_2) \\
&\quad + \left[A_4 - A_5\frac{\lambda_1}{\lambda_2-\lambda_1}\right]S(Z_3) \\
&= p_1 S(Z_1) + p_2 S(Z_2) + p_3 S(Z_3)
\end{aligned}
$$

$$
\begin{aligned}
\textit{where}\quad p_1 &= A_1 + A_3\frac{\lambda_2}{\lambda_2-\lambda_1} + A_5\frac{\lambda_2{}^2}{(\lambda_2-\lambda_1)^2} \\
p_2 &= A_2 - A_3\frac{\lambda_1}{\lambda_2-\lambda_1} - A_5\frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2} \\
p_3 &= A_4 - A_5\frac{\lambda_1}{\lambda_2-\lambda_1} \\
\textit{and}\quad & Z_1 \sim \exp(\lambda_1),\ Z_2 \sim \exp(\lambda_2),\ Z_3 \sim G(2, 1/\lambda_2)
\end{aligned}
$$

One can use the survival function of S to calculate VaR. We have:

$$
\begin{aligned}
S(x) &= p_1 e^{-\lambda_1 x} + p_2 e^{-\lambda_2 x} + p_3\int_x^\infty \lambda_2{}^2 y e^{-\lambda_2 y}dy \\
&= p_1 e^{-\lambda_1 x} + p_2 e^{-\lambda_2 x} + p_3\int_x^\infty -\lambda_2 y d e^{-\lambda_2 y} \\
&= p_1 e^{-\lambda_1 x} + p_2 e^{-\lambda_2 x} + p_3\left[\lambda_2 x e^{-\lambda_2 x} + \int_x^\infty \lambda_2 e^{-\lambda_2 y}dy\right] \\
&= p_1 e^{-\lambda_1 x} + p_2 e^{-\lambda_2 x} + p_3\left[\lambda_2 x e^{-\lambda_2 x} + e^{-\lambda_2 x}\right] \\
&= p_1 e^{-\lambda_1 x} + (p_2 + p_3) e^{-\lambda_2 x} + p_3 \lambda_2 x e^{-\lambda_2 x}
\end{aligned}
$$

Using MATLAB, the values of VaR are found to be:

α	0.95	0.96	0.97	0.98	0.99
VaR	11,476	12,151	13,022	14,247	16,341

Furthermore, one can show that:

$$
\begin{aligned}
E[S-d]_{+} &= \int_d^\infty S(x)dx \\
&= \int_d^\infty p_1 e^{-\lambda_1 x} + (p_2 + p_3)e^{-\lambda_2 x} + p_3\lambda_2 x e^{-\lambda_2 x} dx \\
&= \frac{p_1}{\lambda_1} e^{-\lambda_1 d} + \frac{p_2 + p_3}{\lambda_2} e^{-\lambda_2 d} + p_3 \int_d^\infty - x d e^{-\lambda_2 x} \\
&= \frac{p_1}{\lambda_1} e^{-\lambda_1 d} + \frac{p_2 + p_3}{\lambda_2} e^{-\lambda_2 d} + p_3 \left[d e^{-\lambda_2 d} + \int_d^\infty e^{-\lambda_2 x} dx \right] \\
&= \frac{p_1}{\lambda_1} e^{-\lambda_1 d} + \frac{p_2 + p_3}{\lambda_2} e^{-\lambda_2 d} + p_3 \left[d e^{-\lambda_2 d} + \frac{1}{\lambda_2} e^{-\lambda_2 d} \right] \\
&= \frac{p_1}{\lambda_1} e^{-\lambda_1 d} + \frac{p_2 + 2p_3}{\lambda_2} e^{-\lambda_2 d} + p_3 d e^{-\lambda_2 d}
\end{aligned}
$$

And so, E[S-VaR]$_+$ and TVaR can be shown to be:

α	0.95	0.96	0.97	0.98	0.99
E[S-VaR]$_+$	151.1734	120.8919	90.6411	60.4122	30.2004
TVaR	14,499	15,174	16,043	17,268	19,361

Similarly in this model, one can express S and $(S\text{-}d)_+$ in Phase-type form.

With $\underline{\gamma} = [A_2 + A_3 + A_4 + A_5, 0, A_1]$, $p_{12} = \dfrac{A_4 + A_5}{A_2 + A_3 + A_4 + A_5}$,

$p_{12} = \dfrac{A_3}{A_2 + A_3 + A_4 + A_5}$ and $p_{23} = \dfrac{A_5}{A_4 + A_5}$

$T = \begin{bmatrix} -\lambda_2 & p_{12}\lambda_2 & p_{13}\lambda_2 \\ 0 & -\lambda_2 & p_{23}\lambda_2 \\ 0 & 0 & -\lambda_1 \end{bmatrix}$ and $\underline{e}' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, we have

$$F_S(s) = 1 - \underline{\gamma} \exp(-Ts)\underline{e}'$$

Also, as was shown in **Model 5**, one can show that (S-d)$_+$ is of Phase-type with:

$F_{(S-d)_+}(s) = 1 - \underline{\gamma_d} \exp(-Ts)\underline{e}'$, and $E[(S-d)_+] = -\underline{\gamma_d} T^{-1} \underline{e}'$

where $\underline{\gamma_d} = \gamma \exp(-Td)$

Similar to what has been mentioned in the previous model, T being 3 by 3 does not pose numerical inaccuracy issues; however, having T in higher dimensions would (S was originally represented by a 5 by 5 T matrix, but the calculations were less accurate).

3) __Comparing Model 5 and Model 6__

By looking at the tables above, one finds that **Model 5** has a higher risk measure than **Model 6** (in VaR and TVaR). This is an appealing feature - next to it being mathematically tractable - as insurance companies are risk averse and tend to less optimistic models. Hence, one would be more prone to choose **Model 5** over **Model 6** in this case, given that N has the same expected value.

4. Conclusion

We have looked at six models, where the claim amounts follow simple distributions. For the case where the amounts followed a discrete uniform distribution, the number of claims were members of the $(a,b,0)$ or $(a,b,1)$ class. Because of Panjer's formula, one can calculate the VaR and TVaR values for all members of an (a,b,m) class. The main challenge would be the computational complexity of some of these distributions, especially the heavy tailed ones (for instance, it takes about 3 minutes to run the program for the logarithmic case). Regarding the practicality issue, one would be reluctant to use discrete aggregate claim models, as amounts in reality are rarely in such a form. Thus, one would be more prone to use continuous distributions for the claim amounts. And so, one would be more prone to use Models 5 and 6 over the first four.

For these models, we were able to derive S in the form of n-point mixture distributions, and due to having them in the form of hypoexponential mixtures, we were able to derive them in Phase-type form. This on the one hand is very elegant and compact, but on the other hand may be less accurately computed (i.e. stability issues) if the convolutions of the distributions are high. These models are more realistic than the discrete cases, but may fall in several criteria.

This is mainly due to the fact that insurance companies seek models that have VaR and TVaR values neither overoptimistic nor unreflective of reality. For instance, it would be more realistic to consider $S=S_1+S_2+\ldots+S_n$, where the S_i's are not necessary independent. The main issue here, of course, would be being able to derive/compute probability distributions of this model (e.g. pdf, cdf, mgf,…) and some of its risk measures (e.g VaR, TVAR,…). Therefore, although the aim is not to find the most pessimistic risk-measure values, it is nonetheless imperative to use models that are both more dynamic in nature and have high risk measure values orienting the company to hedging against worst-case-scenarios.

Appendix: MATLAB Code

Model 1-4 Function- PanjerFormula:

```
function [s,VaR,ESd,TVaR] = PanjerFormula(a, b, g0, i1, upper, alpha)
f0=1/101;
for j=1:100
    f(j)=1/101;
end
for j=101:upper
    f(j)=0;
end
g(1) =(1/(1-a*f0))*(i1*f(1)+(a+b)*f(1)*g0);
for k = 2:upper
    g(k) = 0;
   for j = 1:k-1
        g(k) = g(k)+(a+b*j/k)*f(j)*g(k-j);
    end
    g(k) = (1/(1-a*f0))*(i1*f(k)+g(k)+(a+b)*f(k)*g0);
    s=g0+sum(g);
    if s>=alpha
        break
    end
end
VaR = k;
for i = 1:VaR
    d(i) = i;
end
Ed=d*g'+VaR*(1-s);
ES = 5000;
ESd = ES-Ed; %E[(S-d)+]
TVaR = VaR+ESd/(1-alpha);
```

Model 1:

```
%poisson(100)
a = 0;
b = 100;
g0 = exp(100*(1/101-1));
i1 = 0; %Poisson is (a,b,0) class
upper = 10000;
for t = 1:5
    alpha = 0.95+(t-1)/100;
    [s,VaR,ESd,TVaR] = PanjerFormula(a, b, g0, i1, upper, alpha); %defined earlier
    s1(t) = s;
```

```
    VaR1(t) = VaR;
    ESd1(t) = ESd;
    TVaR1(t) = TVaR;
end
```

Model 2:

```
%binomial(125,0.8)
a = -4;
b = 504;
g0 = (1+0.8*(1/101-1))^125;
i1 = 0; %Binomial is (a,b,0) class
upper = 10000;
for t = 1:5
    alpha = 0.95+(t-1)/100;
    [s,VaR,ESd,TVaR] = PanjerFormula(a, b, g0, i1, upper, alpha);
    s2(t) = s;
    VaR2(t) = VaR;
    ESd2(t) = ESd;
    TVaR2(t) = TVaR;
end
```

Model 3:

```
%negative binomial(5,20)
a = 20/21;
b = 80/21;
g0 = (1-20*(1/101-1))^(-5);
i1 = 0; %NB is (a,b,0) class
upper = 20000;
for t = 1:5
    alpha = 0.95+(t-1)/100;
    [s,VaR,ESd,TVaR] = PanjerFormula(a, b, g0, i1, upper, alpha);
    s3(t) = s;
    VaR3(t) = VaR;
    ESd3(t) = ESd;
    TVaR3(t) = TVaR;
end
```

Model 4:

```
%logarithmic(647.46) %calculate using trial-and-error method to get mean 100
beta = 647.46;
a = beta/(1+beta);
b = -beta/(1+beta);
g0 = 1-log(1-beta*(1/101-1))/log(1+beta);
i1 = beta/(1+beta)/log(1+beta); %Pr(N=1)
```

```
upper = 80000;
for t = 1:5
    alpha = 0.95+(t-1)/100;
    [s,VaR,ESd,TVaR] = PanjerFormula(a, b, g0, i1, upper, alpha);
    s4(t) = s;
    VaR4(t) = VaR;
    ESd4(t) = ESd;
    TVaR4(t) = TVaR;
end
```

Model 5: Traditional Method

```
theta1=20;%EX
theta2=25;%EY
beta1=150;
beta2=80;
EZ1=theta1*(1+beta1);
EZ2=theta2*(1+beta2);

lambda1=1/EZ1;
lambda2=1/EZ2;

%MGF_S(t)=A0+A1*MGF_{Z1}+A2*MGF_{Z1+Z2}+A3*MGF_{Z2}
A0=1/((1+beta1)*(1+beta2));%Mass point Pr{S=0}
A1=beta1/((1+beta1)*(1+beta2));%Coefficient of Z1
A2=(beta1*beta2)/((1+beta1)*(1+beta2));%Coefficient of Z1+Z2
A3=beta2/((1+beta1)*(1+beta2));%Coefficient of Z2

%Deriving the Distribution of S
p1=((A1+A2)*lambda2-A1*lambda1)/(lambda2-lambda1);
p2=(A3*lambda2-(A2+A3)*lambda1)/(lambda2-lambda1);
p=[p1 p2];%Entrance probabilities to hyper-exponential
lambda=[1/lambda1; 1/lambda2];%parameter of expomential

EX_S=p*lambda;
Var_S=2*p*lambda.^2-EX_S^2;

%Finding VaR, E[(S-VaR)+] and TVaR
alpha_95=0.95;
VaR_95=12058.25;%By trial and error
F_S=1-p*exp(-lambda.^(-1)*VaR_95);%CDF of S at VaR value (used for checking if equals to alpha
EX_S_ded95=p.*lambda'*exp(-lambda.^(-1)*VaR_95);%E[(S-VaR)+]
TVaR_95=VaR_95+EX_S_ded95/(1-alpha_95);%TVaR
```

```
alpha_96=0.96;
VaR_96=12765.91;%By trial and error
F_S=1-p*exp(-lambda.^(-1)*VaR_96);%CDF of S at VaR value (used for checking if equals to alpha
EX_S_ded96=p.*lambda'*exp(-lambda.^(-1)*VaR_96);%E[(S-VaR)+]
TVaR_96=VaR_96+EX_S_ded96/(1-alpha_96);%TVaR

alpha_97=0.97;
VaR_97=13672.15;%By trial and error
F_S=1-p*exp(-lambda.^(-1)*VaR_97);%CDF of S at VaR value (used for checking if equals to alpha
EX_S_ded97=p.*lambda'*exp(-lambda.^(-1)*VaR_97);%E[(S-VaR)+]
TVaR_97=VaR_97+EX_S_ded97/(1-alpha_97);%TVaR

alpha_98=0.98;
VaR_98=14940.04;%By trial and error
F_S=1-p*exp(-lambda.^(-1)*VaR_98);%CDF of S at VaR value (used for checking if equals to alpha
EX_S_ded98=p.*lambda'*exp(-lambda.^(-1)*VaR_98);%E[(S-VaR)+]
TVaR_98=VaR_98+EX_S_ded98/(1-alpha_98);%TVaR

alpha_99=0.99;
VaR_99=17088.30;%By trial and error
F_S=1-p*exp(-lambda.^(-1)*VaR_99);%CDF of S at VaR value (used for checking if equals to alpha
EX_S_ded99=p.*lambda'*exp(-lambda.^(-1)*VaR_99);%E[(S-VaR)+]
TVaR_99=VaR_99+EX_S_ded99/(1-alpha_99);%TVaR
```

<u>Model 5: Phase-Type Method</u>

```
theta1=20; theta2=25;beta1=150;beta2=80;
EZ1=theta1*(1+beta1);EZ2=theta2*(1+beta2);
lambda1=1/EZ1;lambda2=1/EZ2;
%MGF_S(t)=A0+A1*MGF_{Z1}+A2*MGF_{Z1+Z2}+A3*MGF_{Z2}
A0=1/((1+beta1)*(1+beta2));%Mass point Pr{S=0}
A1=beta1/((1+beta1)*(1+beta2));%Coefficient of Z1
A2=(beta1*beta2)/((1+beta1)*(1+beta2));%Coefficient of Z1+Z2
A3=beta2/((1+beta1)*(1+beta2));%Coefficient of Z2
%Deriving the Distribution using phase-type of dimension 2
gamma=[A1+A2 A3];%Entrance probabilities to transient states, they add up to 1-A0
t0=[A1/(A1+A2)*lambda1 lambda2];%Absorbing states
T=[-lambda1 A2/(A1+A2)*lambda1; 0 -lambda2];%Transient states
e1=[1;1];
EX_S=-gamma*inv(T)*e1;
```

```
Var_S=2*gamma*T^(-2)*e1-EX_S^2;
%Finding VaR, E[(S-VaR)+] and TVaR
alpha_95=0.95;
VaR_95=12058;%By trial and error
F_S=1-gamma*expm(T*VaR_95)*e1;%CDF of S at VaR value (used for checking if equals to alpha
gamma_ded95=gamma*expm(T*VaR_95);%Gamma vector with deductible of VaR_xx
EX_S_ded95=-gamma_ded95*inv(T)*e1;%E[(S-VaR)+]
TVaR_95=VaR_95+EX_S_ded95/(1-alpha_95);%TVaR
alpha_96=0.96;
VaR_96=12766;%By trial and error
F_S=1-gamma*expm(T*VaR_96)*e1;%CDF of S at VaR value (used for checking if equals to alpha
gamma_ded96=gamma*expm(T*VaR_96);%Gamma vector with deductible of VaR_xx
EX_S_ded96=-gamma_ded96*inv(T)*e1;%E[(S-VaR)+]
TVaR_96=VaR_96+EX_S_ded96/(1-alpha_96);%TVaR
alpha_97=0.97;
VaR_97=13672;%By trial and error
F_S=1-gamma*expm(T*VaR_97)*e1;%CDF of S at VaR value (used for checking if equals to alpha
gamma_ded97=gamma*expm(T*VaR_97);%Gamma vector with deductible of VaR_xx
EX_S_ded97=-gamma_ded97*inv(T)*e1;%E[(S-VaR)+]
TVaR_97=VaR_97+EX_S_ded97/(1-alpha_97);%TVaR
alpha_98=0.98;
VaR_98=14940;%By trial and error
F_S=1-gamma*expm(T*VaR_98)*e1;%CDF of S at VaR value (used for checking if equals to alpha)
gamma_ded98=gamma*expm(T*VaR_98);%Gamma vector with deductible of VaR_xx
EX_S_ded98=-gamma_ded98*inv(T)*e1;%E[(S-VaR)+]
TVaR_98=VaR_98+EX_S_ded98/(1-alpha_98);%TVaR

alpha_99=0.99;
VaR_99=17088;%By trial and error
F_S=1-gamma*expm(T*VaR_99)*e1;%CDF of S at VaR value (used for checking if equals to alpha)
gamma_ded99=gamma*expm(T*VaR_99);%Gamma vector with deductible of VaR_xx
EX_S_ded99=-gamma_ded99*inv(T)*e1;%E[(S-VaR)+]
TVaR_99=VaR_99+EX_S_ded99/(1-alpha_99);%TVaR
```

Model 6: Traditional Method

```
theta1=20;%Mean of X1
theta2=25;%Mean of X2
beta1=150;
beta2=40;
```

```
EX_Z1=theta1*(1+beta1);
EX_Z2=theta2*(1+beta2);
lambda1=1/EX_Z1;
lambda2=1/EX_Z2;
A0=1/((1+beta2)^2*(1+beta1));%Coefficient of Mass point at zero
A1=beta1/((1+beta2)^2*(1+beta1));%Coefficient of MGF_{Z1}
A2=2*beta2/((1+beta2)^2*(1+beta1));%Coefficient of MGF_Z2
A3=2*beta1*beta2/((1+beta2)^2*(1+beta1));%Coefficient of MGF_{Z2+Z1}
A4=beta2^2/((1+beta2)^2*(1+beta1));%Coefficient of MGF_{Z2+Z2}
A5=beta1*beta2^2/((1+beta2)^2*(1+beta1));%Coefficient of MGF_{Z2+Z2+Z1}


%We Have MGF_S=A0 + A1*MGF_{Z1} + A2*MGF_Z2 + A3*MGF_{Z2+Z1} +
A4*MGF_{Z2+Z2} + A5*MGF_{Z2+Z2+Z1}

%Finding Distribution of S
p1=A1+A3*lambda2/(lambda2-lambda1)+A5*lambda2^2/(lambda2-lambda1)^2;
p2=A2-A3*lambda1/(lambda2-lambda1)-A5*lambda1*lambda2/(lambda2-lambda1)^2;
p3=A4-A5*lambda1/(lambda2-lambda1);
p=[p1 p2 p3];
lambda=[1/lambda1; 1/lambda2; 1/lambda2];
n=[1; 1; 2];
Var_n=n+n.^2;

EX_S=p*(lambda.*n);%Expected Value of S
Var_S=p*(Var_n.*lambda.^2)-EX_S^2;%Variance of S

%Calculating VaR, E[(S-VaR)+]and TVaR
alpha_95=0.95;
VaR_95=11475.64;%VaR by trial and error
F_S=1-p*exp(-lambda.^(-1)*VaR_95)-p3*lambda2*VaR_95*exp(-lambda2*VaR_95);%CDF of S
E_S_VaR_95=(p.*lambda')*(exp(-lambda.^(-1)*VaR_95).*n)+p3*VaR_95*exp(-lambda2*VaR_95);
%E[(S-VaR)+]
TVaR_95=VaR_95+E_S_VaR_95/(1-alpha_95);%TVaR

alpha_96=0.96;
VaR_96=12151.35;%VaR by trial and error
F_S=1-p*exp(-lambda.^(-1)*VaR_96)-p3*lambda2*VaR_96*exp(-lambda2*VaR_96);%CDF of S
E_S_VaR_96=(p.*lambda')*(exp(-lambda.^(-1)*VaR_96).*n)+p3*VaR_96*exp(-lambda2*VaR_96);
%E[(S-VaR)+]
TVaR_96=VaR_96+E_S_VaR_96/(1-alpha_96);%TVaR

alpha_97=0.97;
VaR_97=13021.61;%VaR by trial and error
```

```
F_S=1-p*exp(-lambda.^(-1)*VaR_97)-p3*lambda2*VaR_97*exp(-lambda2*VaR_97);%CDF of S
E_S_VaR_97=(p.*lambda')*(exp(-lambda.^(-1)*VaR_97).*n)+p3*VaR_97*exp(-lambda2*VaR_97);
%E[(S-VaR)+]
TVaR_97=VaR_97+E_S_VaR_97/(1-alpha_97);%TVaR

alpha_98=0.98;
VaR_98=14247.29;%VaR by trial and error
F_S=1-p*exp(-lambda.^(-1)*VaR_98)-p3*lambda2*VaR_98*exp(-lambda2*VaR_98);%CDF of S
E_S_VaR_98=(p.*lambda')*(exp(-lambda.^(-1)*VaR_98).*n)+p3*VaR_98*exp(-lambda2*VaR_98);
%E[(S-VaR)+]
TVaR_98=VaR_98+E_S_VaR_98/(1-alpha_98);%TVaR

alpha_99=0.99;
VaR_99=16341.45;%VaR by trial and error
F_S=1-p*exp(-lambda.^(-1)*VaR_99)-p3*lambda2*VaR_99*exp(-lambda2*VaR_99);%CDF of S
E_S_VaR_99=(p.*lambda')*(exp(-lambda.^(-1)*VaR_99).*n)+p3*VaR_99*exp(-lambda2*VaR_99);
%E[(S-VaR)+]
TVaR_99=VaR_99+E_S_VaR_99/(1-alpha_99);%TVaR
```

Model 6: Phase-Type Method

```
theta1=20; theta2=25;beta1=150;beta2=40;
EX_Z1=theta1*(1+beta1);EX_Z2=theta2*(1+beta2);
lambda1=1/EX_Z1;lambda2=1/EX_Z2;
A0=1/((1+beta2)^2*(1+beta1));%Coefficient of Mass point at zero
A1=2*beta2/((1+beta2)^2*(1+beta1));%Coefficient of MGF_Z2
A2=beta2^2/((1+beta2)^2*(1+beta1));%Coefficient of MGF_{Z2+Z2}
A3=2*beta1*beta2/((1+beta2)^2*(1+beta1));%Coefficient of MGF_{Z2+Z1}
A4=beta1*beta2^2/((1+beta2)^2*(1+beta1));%Coefficient of MGF_{Z2+Z2+Z1}
A5=beta1/((1+beta2)^2*(1+beta1));%Coefficient of MGF_{Z1}
%We Have MGF_S=A0 + A1*MGF_Z2 + A2*MGF_{Z2+Z2} + A3*MGF_{Z2+Z1} +
%                               A4*MGF_{Z2+Z2+Z1}+A5*MGF_{Z1}
%NOTE: A1,A2,A4 and A5 in PAPER = A5,A1,A2 and A4 HERE respectively
%Finding Distribution of S: Using Phase Type with T of Dimension 3
abs1=A1/(A1+A2+A3+A4);abs2=A2/(A2+A4);%Absorption probabilities
t_0=[abs1*lambda2 abs2*lambda2 lambda1];%Absorption States
gamma=[A1+A2+A3+A4 0 A5];%Entrance Probabilities: Either enter at First Z2 or Z1.
nonabs12=(A2+A4)/(A1+A2+A3+A4);%Probability of going to Z2
nonabs13=A3/(A1+A2+A3+A4);%Probability of going to Z1 directly
nonabs23=A4/(A2+A4);%Probability of going to Z1 after being in the first Z2 and now in the second
Z2
T=[-lambda2 nonabs12*lambda2 nonabs13*lambda2; 0 -lambda2 nonabs23*lambda2; 0 0 -lambda1];
e1=[1;1;1];
EX_S=-gamma*inv(T)*e1;%Expected Value of S
Var_S=2*gamma*T^(-2)*e1-EX_S^2;%Variance of S
```

```
%Calculating VaR, E[(S-VaR)+]and TVaR
alpha_95=0.95;
VaR_95=11476;%VaR by trial and error
F_S=1-gamma*expm(T*VaR_95)*e1;%CDF of S
gamma_95=gamma*expm(T*VaR_95);%Gamma vector of Phase Type with deductible VaR_alpha
E_S_VaR_95=-gamma_95*inv(T)*e1;%E[(S-VaR)+]
TVaR_95=VaR_95+E_S_VaR_95/(1-alpha_95);%TVaR
alpha_96=0.96;
VaR_96=12151;%VaR by trial and error
F_S=1-gamma*expm(T*VaR_96)*e1;%CDF of S
gamma_96=gamma*expm(T*VaR_96);%Gamma vector of Phase Type with deductible VaR_alpha
E_S_VaR_96=-gamma_96*inv(T)*e1;%E[(S-VaR)+]
TVaR_96=VaR_96+E_S_VaR_96/(1-alpha_96);%TVaR
alpha_97=0.97;
VaR_97=13022;%VaR by trial and error
F_S=1-gamma*expm(T*VaR_97)*e1;%CDF of S
gamma_97=gamma*expm(T*VaR_97);%Gamma vector of Phase Type with deductible VaR_alpha
E_S_VaR_97=-gamma_97*inv(T)*e1;%E[(S-VaR)+]
TVaR_97=VaR_97+E_S_VaR_97/(1-alpha_97);%TVaR
alpha_98=0.98;
VaR_98=14247;%VaR by trial and error
F_S=1-gamma*expm(T*VaR_98)*e1;%CDF of S
gamma_98=gamma*expm(T*VaR_98);%Gamma vector of Phase Type with deductible VaR_alpha
E_S_VaR_98=-gamma_98*inv(T)*e1;%E[(S-VaR)+]
TVaR_98=VaR_98+E_S_VaR_98/(1-alpha_98);%TVaR
alpha_99=0.99;
VaR_99=16341;%VaR by trial and error
F_S=1-gamma*expm(T*VaR_99)*e1;%CDF of S
gamma_99=gamma*expm(T*VaR_99);%Gamma vector of Phase Type with deductible VaR_alpha
E_S_VaR_99=-gamma_99*inv(T)*e1;%E[(S-VaR)+]
TVaR_99=VaR_99+E_S_VaR_99/(1-alpha_99);%TVaR
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