Fun With Cats — Homework 3

Epis, Monos, and Abstract Structures

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This homework mainly draws from Chapter 2 of Awodey's 2nd edition. The first few problems are for getting comfortable with definitions and may be skipped (just be sure you *could* do them). The next batch are what we will consider the formal homework set. Finally, there are some extra problems in case you get bored. The last one proves Brouwer's Fixed Point Theorem but has a good deal of background reading (so it is rather long).

1 SUGGESTED EXERCISES

Problem 2.1. Show that a function between sets is an epi if and only if it is surjective. Conclude that the isos in **Sets** are exactly the epi-monos.

Note: *We pretty much did this one in class but it is worth going through again.*

Solution

Suppose $A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C$ are arrows in **Sets**.

$($**epi** $\Rightarrow$ **onto**$)$ If f is an epi then a diagram implies $gf = hf \iff g = h$. If there is a b in B that is not in $f(A)$ then g and h may map b to different values and this information is lost in the composition, whence $gf = hf$ even though $g \neq h$, a contradiction.

$($**onto** $\Rightarrow$ **epi**$)$ Conversely, suppose that f is surjective and suppose that $gf = hf$. If $g \neq h$ then there is some $b \in B$ such that $f(b) \neq h(b)$. By surjectivity there is some element $a \in A$ so that $a \overset{f}{\mapsto} b$ and we have $(g \circ f)(a) = g(f(a)) = g(b) \neq h(b) = h(f(a)) = (h \circ f)(a)$, contradicting our hypothesis that $gh = hf$. $\square$

Problem 2.2. Show that in a poset category, all arrows are both monic and epic.

Note: *Again, this is something we have shown in class (more or less) but will be good to solidify definitions and work through forms of categorical arguments.*

Solution

For monos, suppose that we have objects a, b, c in a preorder P with an arrow configuration

$$a \underset{h}{\overset{g}{\rightrightarrows}} b \xrightarrow{f} c.$$

Since P is a poset there can be at most one arrow between a and b corresponding to the boolean query $a \leq ?b$, whence $g = h$. The same argument proves the result for epis. $\square$

Problem 2.3. Show that inverses are unique; that is, given arrow $f : A \to B$ and inverses $g, g' : B \to A$ satisfying the inverse property of f, show that $g = g'$.

More generally, show that if f has right inverse $r : B \to A$ and left inverse $l : B \to A$ then $r = l$.

Solution

The generalization of the problem shows that inverses must be unique. Given respectively right and left inverses $r, l : B \to A$ of arrow $f : A \to B$ we have

$$l = l \circ \mathbb{1}_B = l \circ (f \circ r) = (l \circ f) \circ r = \mathbb{1}_B \circ r = r,$$

as desired. $\square$

2 Exercises Proper

Problem 2.4. With regard to commutative triangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \tag{2.1}$$

in any category $\mathbf{C}$, show that

(a) if f and g are isos (resp monos, resp epis), then so is h;

(b) if h is monic, so is f;

(c) if h is epic, so is g;

(d) if h is monic, g need not be.

Solution

(i) **Isos** If f, g are isos then there exist inverses g', f' such that $f'f = 1$ and $g'g = 1$. Then we compute

$$(f'g')gf = f'(g'g)f = f'1f = f'f = 1. \tag{2.2}$$

Therefore $f'g'$ is a left inverse of of gf; a similar proof shows that it is also a right inverse.

Monos Suppose that f, g are monos and let $a, a' : X \to A$ be arbitrary arrows. Suppose $(gf)a = (gf)a'$. Then $g(fa) = g(fa')$ and $fa = fa'$ (g is monic) which in turn implies $a = a'$ (f is monic).

Epis As for monos. $\square$

(ii) Suppose that h is monic, and let a, a' be arbitrary arrows $a, a' : X \to A$. If $a \neq a'$ then $(gf)a = ha \neq ha' = (gf)a'$, whence $g(fa) \neq g(fa')$, and $fa \neq fa'$ and f is monic. $\square$

(iii) Suppose that h is epic, and let a, a' be arbitrary arrows $a, a' : C \to X$. If $a \neq a'$ then $a(gf) = ah \neq a'h = a'(gf)$, whence $(ag)f \neq (ag)f$, and $ag \neq a'g$ and g is epic. $\square$

Note: *If this feels redundant, it is. The only alterations I made to the above are syntactic—the proofs are identical up to duality, as we will see in the following chapter*

(iv) Consider inclusion and retraction $A \xrightarrow{i} X \xrightarrow{r} A$ in **Sets**, where $A \subsetneq X$.

Problem 2.5. Show that the following are equivalent for an arrow $f : A \to B$ in arbitrary category **C**.

(a) f is an isomorphism

(b) f is both a mono and a split epi

(c) f is both a split mono and an epi

(d) f is both a split mono and a split epi.

Solution

(a) ⇒ (b), (c) First we know that every iso f is both epic and monic, and since it has inverse f^{-1} it is evident that f is both a split mono and a split epi, whence (a) ⇒ (b) and (a) ⇒ (c).

(d) ⇒ (a) Further, if f is both a split mono and a split epi, say $fg = 1$ and $g'f = 1$ then we have $g' = g'fg = g$ and f has inverse $g' = g$; that is, f is an iso and (d) ⇒ (a).

(b) ⇒ (d), (c) ⇒ (d) Now we only need to show that (b) and (c) both imply (d), and these amount to the same proofs (up to duality). Say that f is monic and a split epi via diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd}, \qquad fg = 1$$

where g splits f. Then $f(gf) = (fg)f = f = f\mathbb{1}$ and $\mathbb{1} = gf$, so that f is a split epi and (b) ⇒ (d); the proof for (c) ⇒ (d) is similar. □

Problem 2.7. Show that a retract of a projective object is also projective.

Solution

Suppose that R is a retract of projective object P, and let $A \twoheadrightarrow B$. If $r : P \to R$ is the rectraction and $s : R \to P$ is the section then we have the following diagram

$$\begin{tikzcd} & & & A \arrow[dd, two heads, "e"] \\ R \arrow[r, "s"] & P \arrow[r, "r"] & R \arrow[ur, dashed, "\bar{f}"] \arrow[dr, "f"'] & \\ & & & B \end{tikzcd}$$

The rest follows from a diagram chase and the fact that $rs = 1$. □

Problem 2.10. Show that sets, regarded as discrete posets, are projective in **Pos** (use exercise 9). Give an example of a poset that is not projective. Show that every projective poset is discrete. Conclude that **Sets** is (isomorphic to) the "full subcategory" of projectives in **Pos**, consisting of all projective posets and all monotone maps between them.

Solution

Let P be a discrete poset[a], let $A \overset{e}{\twoheadrightarrow} B$ be an epi between posets A and B, and let $P \xrightarrow{f} B$ be an arbitrary arrow. Since e is epic there is some $a_p \in A$ such that $e(a_p) = f(p)$.[b]. We invoke the axiom of choice and define $\bar{f} : p \mapsto a_p$ which is trivially monic. By construction $e\bar{f} = f$ and we drink a beer to celebrate. $\square$

[a] that is, suppose that $\forall a, b \in P$, $(a \leq b \implies a = b)$
[b] epis are surjective in **Pos**

Problem 2.11. Let A be a set. Define an A-*monoid* to be a monoid M equipped with a function $m : A \to U(M)$ (recall that $U(M)$ is the forgetful functor mapping M to its underlying set, also written $|M|$). A morphism $h : (M, m) \to (N, n)$ of A-monoids is to be a monoid homomorphism $h : M \to N$ such that $U(h) \circ m = n$ (a commutative triangle). Together with the evident identities and composites, this defines a category A-**Mon** of A-monoids.

Show that an initial object in A-**Mon** is the same thing as a free monoid $M(A)$ on A. (Hint: compare their respective UMPs).

Solution

The UMP of $M(A)$ is below, which asserts a uniqueness claim on $\bar{n}$, whence $M(A)$ is initial for A-monoid N.

$$\begin{tikzcd}
M(A) \arrow[r, "\bar{n}"] \arrow[d, "U"'] & N \arrow[d, "U"] \\
\lvert M(A) \rvert \arrow[r, "\lvert \bar{n} \rvert"] & \lvert N \rvert \\
A \arrow[u, "i"] \arrow[ur, "n"']
\end{tikzcd}$$

$\square$

3 Extra Problems

The third problem is a long one but a lot of this is background reading on fundamental groups.

Problem 1 (Mac Lane, 5.6)**.** An **idempotent** is a function satisfying $f \circ f = f$. An idempotent is said to **split** if there exist arrows g, h such that $f = hg$ and $gh = 1$.

Show that all idempotents split in **Sets**. $\square$

Solution

Let $f : A \to A$ be idempotent, so that $f^2 = f$. Define $\widetilde{A} = f(A) \subset A$ and let $\widetilde{A} \overset{s}{\hookrightarrow} A$ be the inclusion map and let $r : A \to \widetilde{A}$ be some map that fixes $\widetilde{A}$ and sends elements of $a \in A \setminus \widetilde{A}$ to $f(a)$. Then we have

$$\widetilde{A} \overset{s}{\rightarrowtail} A \overset{r}{\twoheadrightarrow} \widetilde{A}, \quad \widetilde{A} \xrightarrow{rs} \widetilde{A} = \mathbb{1}_{\widetilde{A}}$$

and $sr = f$, as desired.

Problem 2 (Mac Lane, 5.8)**.** Consider the category with objects (X, e, t), where X is a set, $e \in X$, and $t : X \to X$, and with arrows $f : (X, e, t) \to (X', e', t')$ the functions f on X to X' with $fe = e'$ and $ft = t'f$. Prove that this category has an initial object in which X is the set of natural numbers, $e = 0$, and t is the successor function. $\square$

Solution

Let $B = (X, e, t)$ be an object of $\mathbf{C}$ and let $I = (\mathbb{N}, 0, s)$ be the alleged initial object defined in the problem. There is an arrow $f : I \to B$ defined by

$$f(n) = \begin{cases} e & \text{if } n = 0 \\ (t \circ f)(n') & \text{if } n = s(n') \end{cases}.$$

Suppose $g : I \to A$ is any arrow in $\mathbf{C}$. Then by definition $g(0) = e$ and by induction $g(n) = (t \circ g)(n')$ for all $n = s(n')$, and $f = g$ is unique and I is initial. $\square$

Problem 3. (Brouwer's Fixed Point Theorem) We start off with some background and formalization of the fundamental group. In the following problem we parametrize the circle S^1 via

$$\phi : I \to S^1, \quad \phi(\alpha) = e^{2i\pi\alpha} = (\cos 2\pi\alpha, \sin 2\pi\alpha). \tag{3.1}$$

Viewed in the complex plane we write

$$1 = e^0 = \phi(0) = \phi(1) = e^{2\pi i} \in \mathbb{C},$$

and we write $(S^1, 1)$ for the pointed unit circle with point $\phi(0)$. We will abuse notation and write a function $f : (S^1, 1) \to (X, x)$ as a function of the unit interval I. We can compose maps $f, g : (S^1, 1) \to (X, x)$ by

$$f \centerdot g(\alpha) = \begin{cases} g(2\alpha) & \text{if } \alpha \in [0, 1/2) \\ f(2\alpha - 1) & \text{otherwise} \end{cases}$$

This has the affect of going around both loops in X double-time.

Recall that the fundamental group $\pi_1((X, x))$ of a pointed topological space $(X, x) \in$ $\mathbf{Top}_*$ is the *group* of homotopy classes of maps from the pointed unit circle $(S^1, 1)$ into (X, x). We want to work with pointed topological spaces so that we can compose classes $\widetilde{g}, \widetilde{h}$ of pointed loops $g, h : (S^1, 1) \to (X, x)$ via

$$\widetilde{g} \centerdot \widetilde{h} = \widetilde{g \centerdot h},$$

the homotopy class of the composition of g, h as defined above. Explicitly, our group $\pi_1((X, x))$ is the tuple $(H, \centerdot)$ where

$$H = \hom_{\mathbf{Top}_*}((S^1, 1), (X, x)) / \simeq \tag{3.2}$$

is the set of pointed maps from S^1 to X modded out by the 'is-homotopic-to' relation $\simeq$. Note that $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ is a functor.

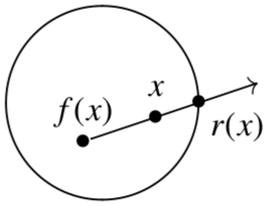
We have the following pieces of data that will be useful in the following problem:

$\mathbf{X}$	$\pi_1(\mathbf{X})$
D^2	0
S^1	$\mathbb{Z}$

Recall that Brouwer's Fixed Point Theorem states that any continuous mapping $f :$ $D^2 \to D^2$, where D^2 is the closed unit disk in $\{(x, y) : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$, has some fixed point.

There are many proofs of this; here is one by contradiction. For clarity we drop the 'pointed set' syntax though the points are implicit.

(i) For the contradiction, suppose that we have a continuous map $f : D^2 \to D^2$ that does not fix any points; that is, for all $x \in D^2$ we have $fx \neq x$. We can then draw a raw starting at $f(x)$ and passing through x); each of these rays then continues on to pass through the boundary of D^2, as in the following figure:



Define $r(x)$ to be the point of intersection of this ray. Prove that $r(x)$ is a retraction of D^2 to $S^1 \subset D^2$. You may take for granted that r is continuous (this follows from continuity of f).

(ii) We have retraction $D^2 \xrightarrow{r} S^1$ and we may extend this from the left via the inclusion map

$$S^1 \xrightarrow{i} D^2 \xrightarrow{r} S^1.$$

What is the composition ri?

(iii) Use the functoriality of π_1 to derive a contradiction in the following diagram

$$\begin{tikzcd}
S^1 \arrow[r, "i"] \arrow[d, "\pi_1"'] & D^2 \arrow[r, "r"] \arrow[d, "\pi_1"'] & S^1 \arrow[d, "\pi_1"'] \\
\pi_1(S^1) \arrow[r, "\pi_1(i)"] & \pi_1(D^2) \arrow[r, "\pi_1(r)"] & \pi_1(S^1)
\end{tikzcd} \qquad \square \tag{3.3}$$

Solution

(i) Let $x \in S_1 = \partial D^2$ be point on the boundary circle of disk D^2. Then the ray from $f(x)$ to x intersects S^1 at x and r restricts to the identity on S_1; that is,

$$r|_{S_1} = \mathbb{1}_{S_1}.$$

Choosing function $i : S^1 \to D^2$ to be the inclusion map of the boundary it is easy to see that $ri : S^1 \to S^1, x \mapsto x$. Now, using both the fact that inclusions are continuous and the assumption that r is continuous, we have shown that r splits arrow i and thus r is a retraction of D^2 onto S^1.

(ii) As noted in (i), $ri = \mathbb{1}_{S1}$

(iii) By the diagram in 3.3 we have $\pi_1(r) \circ \pi_1(i) = \pi_1(ri) = \pi_1(\mathbb{1}_{S^1}) = \mathbb{1}_{\pi_1(S^1)}$. But $\pi_1(S^1) = \mathbb{Z}$ while $\pi_1(D^2) = 0$ and thus $\pi_1(i)$ is the trivial homomorphism. In particular, $\pi_1(r) \circ \pi_1(i)$ is also trivial and cannot be the identity on $\mathbb{Z}$. $\square$