UMASS AMHERST

Fun With Cats — Homework 4

Epis, Monos, and Abstract Structures

July 9, 2018

1 EXERCISES

Problem 2.13. In any category with binary products, show directly that

$$A \times (B \times C) \cong (A \times B) \times C$$

Problem 2.14. (a) For any index set I, define the product $\prod_{i \in I} X_i$ of an I-indexed family of objects $(X_i)_{i \in I}$ in a category, by giving a UMP generalizing that for binary products (the case $I = 2$).

(b) Show that in **Sets**, for any set X the set X^I of all functions $f : I \to X$ has the UMP, with respect to the "constant family" where $X_i = X$ for all $i \in I$, and thus

$$X^I \cong \prod_{i \in I} X.$$

Problem 2.17. In any category **C** with products, define the **graph** of an arrow $f : A \to B$ to be the monomorphism

$$\Gamma(f) = \langle 1_A, f \rangle : A \rightarrowtail A \times B$$

(Why is this monic?). Show that for $\mathbf{C} = \mathbf{Sets}$ this determines a functor $\Gamma : \mathbf{Sets} \to \mathbf{Rel}$ to the category **Rel** of relations, as defined in the exercies to Chapter 1. (To get an actual relation $R(f) \subset A \times B$, take the image of $\Gamma(f) : A \rightarrowtail A \times B$.)

Problem 2.18. Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Sets}$ form monoids to sets is representable. Infer that U preserves all (small) products.

Problem 3.1. In any category **C**, show that

$$A \xrightarrow[c_1]{} C \xleftarrow[c_2]{} B$$

is a coproduct diagram just if for ewvery object Z, the map

$$\begin{aligned} \hom(C,Z) &\to \hom(A,Z)\times\hom(B,Z) \\ f &\mapsto \langle f\circ c_1, f\circ c_2\rangle \end{aligned}$$

is an iso. Do this by using duality, taking the corresponding fact about products as given.

Problem 3.2. Show in detail that the ree monoid functor M preserves coproducts: for any set A, B,

$$M(A)+M(B)\cong M(A+B) \quad \text{(canonically)}.$$

Do this as indicated in the text by using the UMPs of the coproducts $A+B$ and $M(A)+M(B)$ and of free monoids.

Note: *We did this in class but it is worth working through the proof—this is a diagram chase and might be one of the more useful problems to really 'get'. Note that the diagram listed in the text is incomplete: you will have to use some type-data (i.e., that we are working with monoids and sets) to be able to leverage one of the UMPs.*

Problem 3.3. Verify that the construction given in the text of the coproduct of monoids $A+B$ as a quotient of the free monoid $M(|A|+|B|)$ really is a coproduct in the category of monoids.

Problem 3.4. Show that the product of two powerset Boolean algebras $\mathcal{P}(A)$ and $\mathcal{P}(B)$ is also a powerset, namely of the coproduct of the sets A and B,

$$\mathcal{P}(A)\times\mathcal{P}(B)\cong\mathcal{P}(A+B).$$

(Hint: determine the projections $\pi_1 : \mathcal{P}(A+B)\to\mathcal{P}(A)$ and $\pi_2 : \mathcal{P}(A+B)\to\mathcal{P}(B)$, and check that they have the UMP of the product.)

2 Additional Exercises

Problem 1 (Awodey, 2.16)**.** In the category of types $\mathbf{C}(\lambda)$ of the λ-calculus, determine the product functor $A,B\mapsto A\times B$ explicitly. Also show that for any fixed type A there is a functor $A\to(-) : \mathbf{C}(\lambda)\to\mathbf{C}(\lambda)$, taking any type X to $A\to X$.

Problem 2 (Awodey, 3.5)**.** Consider the category of proofs of a natural deduction system with disjunction introduction and elimination rules. Identify proofs under the equations

$$[p,q]\circ i_1 = p, \qquad [p,q]\circ i_2 = q$$
$$[r\circ i_1, r\circ i_2] = r$$

for any $p : A\to C, q : B\to C$, and $r : A+B\to C$. By passing to equivalence classes of proofs awith respect to the equivalence relation generated by these equations (i.e., two proofs are equivalent if you can get one from the other by removing all such "detours"). Show that the resulting category does indeed have coproducts.