UMASS AMHERST

Fun With Cats — Homework 2

June 9, 2018

Hi all, here are problems for this week. I've typed them up since I know that not everyone has the current text, and I also wanted to add a couple extra problems from Mac Lane as 'extra credit'.

The first few problems are on the easier side, mainly ensuring that you understand definitions. Problems 11 and 12 make you work with UMPs which is much more useful than watching someone prattle on about them. I've also included three problems at the end which are a bit harder but should be doable. Problem 2 in particular is a good one as this has a very nice 'algebraic' solution that comes up in a number of contexts, including a proof of Brouwer's Fixed Point Theorem. The last of the Mac Lane problems is fairly simple—there's really only one thing you can do.

CHAPTER 1: CATEGORIES

Problem 5. For any category $\mathbf{C}$, define a functor $U : \mathbf{C}/C \to \mathbf{C}$ from the slice category over an object C that "forgets about C." Find a functor $F : \mathbf{C}/C \to \mathbf{C}^{\to}$ to the arrow category such that $\mathbf{dom} \circ F = U$.[1] $\square$

Problem 6. Construct the "coslice category" $C/\mathbf{C}$ of a category $\mathbf{C}$ under an object C from the slice category $\mathbf{C}/C$ and the "dual category" operation $-^{\mathrm{op}}$.

Problem 7. Let $\{a, b\}$ be any set with exactly 2 elements a and b. Define a functor $F : \mathbf{Sets}/2 \to \mathbf{Sets} \times \mathbf{Sets}$ with $F(f : X \to 2) = (f^{-1}(a), f^{-1}(b))$. Is this an isomorphism of categories? What about the analogous situation with a one-lement set $1 = \{a\}$ instead of 2? $\square$

[1] Recall that $\mathbf{C}^{\to}$ is equipped with two functors

$$\mathbf{C} \xleftarrow{\mathbf{dom}} \mathbf{C}^{\to} \xrightarrow{\mathbf{cod}} \mathbf{C}\ .$$

Problem 9. Describe the free categories on the following graphs by determining their objects, arrows, and composition operations.

(a)

$$\begin{tikzcd} a \arrow[r, "e"] & b \end{tikzcd}$$

(b)

$$\begin{tikzcd} a \arrow[r, shift left, "e"] & b \arrow[l, shift left, "f"] \end{tikzcd}$$

(c)

$$\begin{tikzcd} a \arrow[r, "e"] \arrow[dr, "g"'] & b \arrow[d, "f"] \\ & c \end{tikzcd}$$

(d)

$$\begin{tikzcd} a \arrow[r, shift left, "e"] & b \arrow[l, shift left, "h"] \arrow[d, "f"] & d \\ & c \arrow[ul, "g"] & \end{tikzcd}$$

□

Problem 10. How may free categories on graphs are there which have exactly six arrows? Draw the graphs that generate these categories. □

Problem 11. Show that the free monoid functor

$$M : \mathbf{Sets} \to \mathbf{Mon}$$

exists in two different ways:

(a) Assume the particular choice $M(X) = X^*$ and define its effect

$$M(f) : M(A) \to M(B)$$

on a function $f : A \to B$ to be

$$M(f)(a_1 \cdots a_k) = f(a_1) \cdots f(a_k), a_i \in A.$$

(b) Assume only the UMP of the free monoid and use it to determine M on functions, showing the result to be a functor.

Reflect on how these two approaches are related. $\square$

Problem 12. Verify the UMP for free categories on graphs. Specifically, let $\mathbf{C}(G)$ be the free category on the graph G and $i : G \to U(\mathbf{C}(G))$ be the 'inclusion' graph homomorphism taking vertices and edges to themselves. Show that for any category $\mathbf{D}$ and graph homomorphism $f : G \to U(\mathbf{D})$, there is a unique functor

$$\bar{h} : \mathbf{C}(G) \to \mathbf{D}$$

with

$$U(\bar{h}) \circ i = h.$$

Note: refer to the UMP for all the movers and shakers in the above problem. $\square$

Some (Slightly Adapted) Problems from Mac Lane

Note that these are more difficult (and thus more fun).

Problem 1. Recall the following categories:

- $\mathbf{1}$:

$$*$$

- $\mathbf{2}$:

$$* \longrightarrow \star$$

- $\mathbf{3}$:

$$\begin{tikzcd} * \arrow[r] \arrow[dr] & \star \arrow[d] \\ & \circ \end{tikzcd}$$

Given a category $\mathbf{C}$, describe the functors of each of the following forms:

- $\mathbf{1} \to \mathbf{C}$
- $\mathbf{2} \to \mathbf{C}$
- $\mathbf{3} \to \mathbf{C}$

What is a nice way to characterize these functors? That is, how would you characterize the class of functors $\mathbf{1} \to \mathbf{C}$? $\mathbf{2} \to \mathbf{C}$? $\mathbf{3} \to \mathbf{C}$? $\square$

Problem 2. Recall that a group G is *abelian* if $ab = ba$ for all $a, b \in G$. Let $\mathbf{Ab}$ be the category of abelian groups and group homomorphisms and $\mathbf{Grp}$ be the category of groups and group homomorphisms. For a group G define the *center* of G, denoted $Z(G)$, to be the set of all $x \in G$ that commutes with all of G; that is,

$$Z(G) = \{x \in G : \forall y \in G xy = yx\}.$$

It is a fact that $Z(G)$ is a subgroup of G *(proof!)*. Show that there is no functor sending each group G to its center. (**hint:** recall that S_n is permutation group on n elements, called the symmetric group. Consider $S_2 \to S_3 \to S_2$).

Note: in the statement of the problem in Mac Lane (which is much shorter and has none of the background I give) it is a bit ambiguous what is meant by "send each group to its center". However there is only one reasonable interpretation of this functor: this F would map each group G to its center, taking each element $z \in Z(G)$ to itself. In mathematics we call this a retract of G onto $Z(G)$: that is, $F \circ F = F$, this term being rooted in topology (as far as I know). We will see more of retractions in section 2.1 next Thursday. If I recall I'll bring in my copy of Hatcher's Algebraic Topology as this gives motivation for the name (much of the motivation for the subject comes from topology so I will continue to reference it).

Speaking of topology, this problem above has a very similar flavor to (one of) the (many) proof(s) of the following fixed point theorem:

Theorem 1 (Brouwer)**.** *Every continuous function from a disk to itself has a fixed point.*

We won't prove this since it involves some background in fundamental groups but the punchline of the proof is the same as the proof here. □

Problem 3. Find two different functors $F : \mathbf{Grp} \to \mathbf{Grp}$ such that the object map $F(G) = G$ is the identify. □