HW 8 Discrete

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1 Linearity of Expectation

a) Find the expected number of stops elevator makes on its way up.

We want N= X1 + X2 + X3 + X4 + X5 + X6 + X7 + X8 + X9 + X10, where Xi represents the probability that someone gets off on that floor, and N represents the overall number of floors stopped at. By linearity of expectation we have E[N] = E[X1] + E[X2] ... + E[X10].
With 4 people, 10 possible floors, and assuming everyone gets off at a floor, the probability that no one stops at a single floor is $((10-1)/10)^4$. Therefore, the probability that someone does stop at that floor would be $1-((10-1)/10)^4$, which would be the expectation of Xi.
Therefore, for E[N], we multiply by 10 to sum for all overall floors. Therefore, we have $10*(1-((10-1)/10)^4)$, or 3.439 expected stops.

b) Suppose we toss a coin n times. The sequence of outcomes can be divided into runs (blocks of H's or blocks of T's). Find the expected number of runs.

Max number of runs will also be the max number of times that the run breaks +1 (the first coin flip cannot be a break). As every coin has the same probability of being a break, we can estimate that for every coin flipped after the first coin, there is a 1/2 chance of forming a break. Considering linearity of expectation, we have E[n] = E[X1]+E[X2] ... + E[Xi] (where X1 is the SECOND coin flipped), where each E[Xi]=1/2.
When summing these values together, we get $(N-1)/2$ expected breaks when flipping N coins. We start from N-1 as we cannot form a break on the first flip. In order to determine the amount of RUNS, we take the expected amount of breaks and add 1, to account for the first flip. Therefore, we have $((N-1)/2)+1$

2 Chebyshev and Chernoff

Consider the following game: We flip a fair coin 2000 times, and we win 1 for each time it shows Heads. We don't lose any money for the other rolls. Apply both Chebyshev's and Chernoff's to bound the probability that we are t or further from the expected value (above or below) for t = 10, 100, 150. (You should give six different numbers for your answer.)

Let x be the amount of money won. With 2000 coins flips, E[x]= 2000 * 1/2= 1000 dollars
$V(x) = 2000 * 1/2 * (1 - 1/2) = 500$ dollars
Case where t=10:
Chebyshev: $Pr(|X - 1000| \geq 10) \leq 500/10^2$, which equals **5**
Chernoff: $Pr(|X - 1000| \geq 10) \leq 2/e^{2(10^2)/2000}$, which equals approximately 1.809674836
Case where t=100:
Chebyshev: $Pr(|X - 1000| \geq 100) \leq 500/100^2$, which equals 0.05
Chernoff: $Pr(|X - 1000| \geq 100) \leq 2/e^{2(100^2)/2000}$, which equals approximately $.9079985 * 10^{-4}$
Case where t=150:
Chebyshev: $Pr(|X - 1000| \geq 150) \leq 500/150^2$, which equals 0.0222222...
Chernoff: $Pr(|X - 1000| \geq 150) \leq 2/e^{2(150^2)/2000}$, which equals $3.383795834 * 10^{-10}$

3 Markov's Inequality

a) Suppose that it is in fact possible for someone to have negative income, but that no one will have an income below -25,000 (but the average income is still 50,000). What is the largest fraction of the population that can make 75,000 or more?

Let A be the percentage of the population with 75,000 or more. Let B be the percentage of the population not earning 75,000. In the most extreme case (which would determine the largest fraction of the population that can make 75,000), everyone in population B would earn -25,000. With an average earning of 50,000, we have the following equation:
50,000=75,000*A + (-25,000 * B).
We have that A+B=1, as they are percentages that add up to 1.
By substitution we have $50,000 = 75,000 * (1 - B) + (-25,000 * B)$. Dividing by 50,000 simplifies to $1 = 3/2(1 - B) - 1/2(B)$. This simplifies to $-1/2 = -2B$, or $B = 1/4$. Therefore, the largest fraction of the population that can make 75,000 (or A) would be 1-(1/4) or 3/4.

B) Now suppose that not only is everyone's income positive, but that an especially generous country guarantees that everyone makes at least 35,000 per year. Assuming the average income is 50,000 as before, now find the largest fraction of the population that can make 75,000 or more.

Let A be the percentage of the population with 75,000 or more. Let B be the percentage of the population not earning 75,000. In the most extreme case (which would determine the largest fraction of the population that can make 75,000), everyone in population B would earn 35,000. With an average earning of 50,000, we have the following equation:
50,000=75,000*A + (35,000 * B).
We have that A+B=1, as they are percentages that add up to 1.
By substitution we have $50,000 = 75,000 * (1 - B) + (35,000 * B)$. Dividing by 50,000 simplifies to $1 = 3/2(1 - B) + 7/10(B)$. This simplifies to $-1/2 = -8/10B$, or $B = 5/8$. Therefore, the largest fraction of the population that can make 75,000 would be 1-(5/8) or 3/8.

C) Based on the reasoning for part (b) state and prove a tighter version of Markov's inequality of the following form:
As we know $X \geq c > 0$, by given. Let $Y = X - c$. We know Y is positive, so we apply Markov's. We have Markov's inequality: $Pr(Y \geq t) \leq E(Y)/t$.
By substitution, we have $Pr(X - c \geq t) \leq E(X - c)/t$.
By linearity of expectation, we have $E(X - c) = E(X) - E(c)$, so we have $Pr(X - c \geq t) \leq (E(X) - E(c))/t$
We have $Pr(X \geq t + c) \leq (E(X) - E(c))/t$
Let $a = t + c$. By substitution, we have $Pr(X \geq a) \leq (E(X) - E(c))/(a - c)$
As C is a constant, we have E(c)=c
We have $Pr(X \geq a) \leq (E(X) - c)/(a - c)$ as our bound
To prove that this is tighter, we must have $(E(X) - c)/(a - c) \leq E(X)/a$ (current Markov bond)
Assume $(E(X) - c)/(a - c) \leq E(X)/a$ to be true.
We have $(E(X) - c) * a \leq E(X) * (a - c)$, which becomes $-ca \leq -c * E(X)$, and then $a \geq E(X)$. As $E(X)/a$ defines the probability in Markov's, and probability can be as much as 1, $1 \geq E(X)/a$. Therefore, $a \geq E(X)$. Therefore, $(E(X) - c)/(a - c) \leq E(X)/a$, so $Pr(X \geq a) \leq (E(X) - c)/(a - c)$ is a tighter bound than Markov's.

4 Graph Theory

For each of the following, either exhibit a graph with the given degree sequence, or explain why no graph can have this degree sequence. The number of vertices is listed for convenience.
a) 5,5,5,5,5,5,5, $|V| = 7$
Consider the sum of all vertices, which would be 35 in this case. By the handshaking lemma,

$$\sum_{v \in V} deg(v) = 2|E|$$

As $35 \neq 2|E|$ for any integer E (because 35 is odd), therefore it is impossible

for a graph to have this degree sequence.
b) 2,2,3,3,4,$|V| = 5$

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c) 1,3,4,4,5,5,$|V| = 6$.
It is impossible to make a graph with the following vertices sequence. This is because one of the nodes has one edge, and two of the nodes have five edges. In a graph with six nodes, if a node has five edges then it must be connected to every other node in the graph. However, if there is a node with one edge then it can only be connected to one other node. As two nodes must be connected to every other node in the graph and one node can only be connected to a single node then it is impossible to compose the graph.

5 Contributor List

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