HW 7 Discrete

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1 Conditional Probability

a)A five-card poker hand is a straight if it consists of five cards with consecutive ranks (Aces can be high or low). For example, a hand with cards ranked Ace, 2, 3, 4, 5 would be a straight, but a hand with ranks 6, 10, J, Q, K would not be, as would any hand with multiple cards of the same rank. Find the probability that a poker hand is a straight, given that it has five cards of different ranks.

Let $Pr(A)$ be the probability that a five card hand is a straight. Given aces are low as well as high, consider the lowest and highest possible minimum value a straight hand can have. The lowest minimum would be Ace, forming Ace,2,3,4,5. The highest minimum would be 10, forming 10,J,Q,K,Ace. The number of cards between Ace and 10 (including each of the cards) is 10. Therefore there are 10 possible minimum values for a straight, or 10 possible straights (ignoring ordering and suit). Now, consider suit. For each of the 5 cards, there are 4 possible suits. Therefore, the number of possible straight hands would be $10 * \binom{4}{1}^5$. Divide this by all possible five card hands to get the probability that a five card hand is a straight.
$Pr(A) = (10 * \binom{4}{1}^5)/\binom{52}{5}$
Let $Pr(B)$ be the probability that a hand has five cards of different ranks. As you progress along your card picking process, you remove 4 potential cards every time a card is picked from the numerator (all cards of that rank), and one card from the denominator (the card that was picked, so that you update the current deck) for 5 cards.
$Pr(B) = (52/52) * (48/51) * (44/50) * (40/49) * (36/48)$
Consider $Pr(A \cap B)$. For $Pr(A)$, every hand that is a straight must also have five cards of different ranks by definition, so every value in $Pr(A)$ must be in $Pr(B)$. However, not every hand that has five cards of different ranks is a straight, so not every value in $Pr(B)$ must also be in $Pr(A)$. Therefore, $Pr(A)$ would be a subset of $Pr(B)$. As $Pr(A)$ is a subset of $Pr(B)$, we can reduce $Pr(A \cap B)$ to $Pr(A)$, so $Pr(A \cap B) = Pr(A)$
We apply the conditional probability formula. $Pr(A \cap B)/Pr(B)$, or $Pr(A)/Pr(B)$. This equals $((10 * \binom{4}{1}^5)/\binom{52}{5})/((52/52) * (48/51) * (44/50) * (40/49) * (36/48))$, which is the probability that a poker hand is a straight, given that it has five

cards of different ranks.

b) If you flip a fair coin 10 times, what is the probability that the first toss is Heads, given that you get at least 2 Heads?

Let $Pr(A)$ be the probability first toss is heads. This will be 1/2, as it is a single coin flip.
Let $Pr(B)$ be the probability at least 2 flips out of 10 are heads. For this, consider all cases where there are under 2 heads out of 10 flips. There is 1 possibility where there are no heads flipped (every coin out of 10 flips tails), and there are 10 possibilities where there are exactly one heads flipped (every coin out of except one flips tails, repeated for each coin). Divide this by all possible outcomes of flipping 10 coins (2^{10}) to get the probability. Therefore, when flipping 10 coins, there is a $11/(2^{10})$ possibility that under 2 heads are shown. The compliment gives the possibility that at least 2 heads are shown. Therefore, the probability at least 2 flips out of 10 are heads is $1 - (11/(2^{10}))$, or $(2^{10} - 11)/2^{10}$
Consider $Pr(A \cap B)$, or the probability that the first toss is Heads and at least 2 out of 10 coins flipped show Heads. The first coin must show heads for all cases of $A \cap B$. Therefore, consider the following 9 coins and determine the probability that there is at least one heads (as the first coin will always be heads, we only need to determine the probability that there is one other heads in the following set of 9). Consider the cases where after 9 flips NONE show heads. There is one case where this occurs, when every coin flips tails. The probability of this occurring would be $1/2^9$. We take the compliment, to find that $1 - (1/2^9)$, or $((2^9 - 1)/2^9)$ to be the probability of there being one heads after flipping 9 coins. However, we must also account for the probability that first coin flips heads, so we multiply by 1/2. Therefore, the probability of $A \cap B$ occurring is $((2^9 - 1)/2^9) * 1/2$ or $(2^9 - 1)/2^{10}$
To determine the conditional probability of the first toss is heads given that you get at least 2 Heads, we have
$((2^9 - 1)/2^{10})/((2^{10} - 11)/2^{10})$

2 Law of Total Probability

An urn contains four coins C1, C2, C3, C4, biased to show Heads with probability 1/2, 1/3, 1/4, 1/5 respectively. You pick a random coin and toss it twice.
(a) What is the probability you get Heads twice?

Let $Pr(A)$ be defined as the event that a randomly selected coin flipped twice shows heads both times.
Let the probability that a respective coin from Cn (where Pn corresponds to Cn) is picked be defined as $Pr(P1), Pr(P2), Pr(P3), Pr(P4)$.
As you pick a **single** random coin from the set of the 4, each event is disjoint,

and $Pr(P1) \cap Pr(P2) \cap Pr(P3) \cap Pr(P4)$ covers all of Ω (any coin picked will be from that group). Therefore, picking a coin forms a partition, $P1...P4$, where each respective Pn equals 1/4.
Let the probability that a flipped coin from Cn show heads after it is flipped **Twice** given it is picked from the hat for each respective coin be as follows $Pr(A|P1) = 1/4, Pr(A|P2) = 1/9, Pr(A|P3) = 1/16, Pr(A|P4) = 1/25$. These values are determined by squaring their respective probabilities.
With partition $P1...P4$, apply Law of Total Probability
Therefore, we have $Pr(A) = (Pr(A|P1) * Pr(P1)) + (Pr(A|P2) * Pr(P2)) + (Pr(A|P3) * Pr(P3)) + (Pr(A|P4) * Pr(P4))$
This can be simplified to
$Pr(A) = (1/4 * 1/4) + (1/9 * 1/4) + (1/16 * 1/4) + (1/25 * 1/4)$.

b) Now assume that you have tossed the coin and it showed Heads twice. Based on this information, what is the probability that you C1 is the coin you picked from the hat?

Let $Pr(B)$ be the probability that the coin has shown heads twice. This is calculated in part A. Let
$Pr(B) = (1/4 * 1/4) + (1/9 * 1/4) + (1/16 * 1/4) + (1/25 * 1/4)$.
Let $Pr(A)$ be the probability that you have picked $C1$ from the urn. As there are 4 coins and you pick randomly, $Pr(A) = 1/4$.
Let $Pr(A \cap B)$ be the probability that you pick $C1$ from the hat **and** flip 2 heads in a row. There is a 1/4 chance of picking $C1$, and then to flip heads twice given you've picked $C1$ there would be a 1/4 chance of getting heads. Therefore, to pick $C1$ and then flip heads twice is the probability is 1/16.
$Pr(A \cap B) = 1/16$
Applying conditional probability formula, we have the following probability that we picked $C1$ given we flipped heads twice:
$(1/16)/((1/4 * 1/4) + (1/9 * 1/4) + (1/16 * 1/4) + (1/25 * 1/4))$

3 Distributions Warm-Up

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4 Independence of R.V.s

a)
View X and Y as functions with the following described behavior. Let $X = x$ for its r.v. over the sample space $\{1, 2, 3, 4, 5\}$. Therefore, $Pr(X = j) = 1/5$ for all $j \in \{1, 2, 3, 4, 5\}$, and will return $\{1, 2, 3, 4, 5\}$ for $\{1, 2, 3, 4, 5\}$ respectively.
Let $Y = ((x + 3) \bmod 5) + 1$, where mod 5 is defined as the remainder after dividing by 5. Therefore, $Pr(Y = j) = 1/5$ for all $j \in \{1, 2, 3, 4, 5\}$, and will

return $\{5, 1, 2, 3, 4\}$ for $\{1, 2, 3, 4, 5\}$ respectively.
Through these functions that model r.v.s, both X and Y take on every value in $\{1, 2, 3, 4, 5\}$ with a non-zero probability, have identical distributions, $(pX(x) = pY(y) = 1/5$ for every value in $\{1, 2, 3, 4, 5\})$, and $Pr(X = Y) = 0$ for every value.

b) Now suppose that we add the following requirement to the above list: X and Y are independent. Prove that it is impossible to construct X and Y satisfying this requirement.

In order for X and Y to be independent, $Pr(X = x, Y = y) = Pr(X = x)Pr(Y = y)$ for all $x, y \in \mathbb{R}$
Consider 2, which is in both of the r.v. distributions.
Therefore, we have $Pr(X = 2, Y = 2) = Pr(X = 2)Pr(Y = 2)$
If $Pr(X = 2, Y = 2)$, which implies $Pr(X = Y)$ for 2. As X and Y are never equal by propriety three, then $Pr(X = Y) = 0$, therefore $Pr(X = 2, Y = 2) = 0$. However, as X and Y take on every value in $\{1, 2, 3, 4, 5\})$ with non-zero probability, both $Pr(X = 2) > 0$ and $Pr(Y = 2) > 0$. Therefore, $Pr(X = 2)Pr(Y = 2) \neq 0$. If $Pr(X = 2, Y = 2) = 0$ and $Pr(X = 2)Pr(Y = 2) \neq 0$, then $Pr(X = 2, Y = 2) \neq Pr(X = 2)Pr(Y = 2)$, so it is impossible that X and Y are independent with the given properties above.

c) This part is unrelated to the first two. Suppose we roll a fair six-sided die three times, and let X be the sum total of the rolls and Y be the number of rolls that were even. Prove that X and Y are not independent.

If X and Y are independent, then $Pr(X = x, Y = y) = Pr(X = x)Pr(Y = y)$ for all x, y $\in$ **R**
Consider $Pr(X = 18)$ and $Pr(Y = 3)$, or we roll 3 sixes in a roll.
$Pr(X = 18, Y = 3) = 1/36$, $Pr(X = 18) = 1/36$, $Pr(Y = 3) = 1/8$. As $1/36 \neq 1/36 * 1/8$, $Pr(X = x, Y = y) \neq Pr(X = x)Pr(Y = y)$ for some x, y $\in$ **R**
Therefore X and Y are not independent.

5 Independence and Cryptography

a) Determine the distribution of C.

As M and K are independent and $M \sim Bern(P)$, $K \sim Bern(1/2)$, for the Probability Mass Function, $Pr(M = 1) = 1 - p, Pr(M = 0) = p$ and $Pr(K = 1) = 1/2, Pr(K = 0) = 1 - 1/2$, or $Pr(K = 0) = 1/2$.
We have the following cases:
$Pr(C = 0, M = 0, K = 0) = (1 - p) * (1/2)$
$Pr(C = 0, M = 1, K = 1) = p * 1/2$

$Pr(C = 1, M = 0, K = 1) = (1 - p) * (1/2)$
$Pr(C = 1, M = 1, K = 0) = p * 1/2$
Therefore the distribution of C will be as follows: $P_c(0) = (1 - p)/2 + p/2$ and $P_c(1) = (1 - p)/2 + p/2$.
Therefore, $P_c(0) = 1/2$ and $P_c(1) = 1/2$
Therefore, the distribution of C is $C \sim Bern(1/2)$

b) Prove that C and M are independent.
For C and M to be independent, $Pr(C = x, M = y) = Pr(C = x)Pr(M = y)$ for all x, y $\in \mathbf{R}$
All possible C and M are listed as follows:
C=0,M=0. $Pr(C = 0, M = 0) = (1 - p) * (1/2)$ and $Pr(C = 0) = 1/2$ and $Pr(M = 0) = (1 - p)$. Therefore $Pr(C = 0, M = 0) = Pr(C = 0) * Pr(M = 0)$

C=1,M=0. $Pr(C = 1, M = 0) = (1 - p) * (1/2)$ and $Pr(C = 1) = 1/2$ and$Pr(M = 0) = (1-p)$. Therefore $Pr(C = 1, M = 0) = Pr(C = 1) * Pr(M = 0)$

C=1,M=1. $Pr(C = 1, M = 1) = p * (1/2)$ and $Pr(C = 1) = 1/2$ and $Pr(M = 1) = p$. Therefore $Pr(C = 1, M = 1) = Pr(C = 1) * Pr(M = 1)$

C=0,M=1. $Pr(C = 0, M = 1) = p * (1/2)$ and $Pr(C = 0) = 1/2$ and $Pr(M = 1) = P$. Therefore $Pr(C = 0, M = 1) = Pr(C = 0) * Pr(M = 1)$

As $Pr(C = x, M = y) = Pr(C = x)Pr(M = y)$ for all x, y $\in \mathbf{R}$, C and M are independent.

c)Prove that C and K are not necessarily independent. (More explicitly, this means: Prove there exist M, K satisfying the requirements above, but such that C and K are not independent.)

For C and K to be independent, $Pr(C = x, K = y) = Pr(C = x)Pr(K = y)$ for all x,y $\in \mathbf{R}$
Proof by counterexample: Let C=0, K=0.
Therefore, we have $Pr(C = 0, K = 0) = (1 - p) * (1/2)$, which occurs when C=0, P=0 and K=0
$Pr(C = 0) = ((1 - p) * 1/2) + (1/2 * p)$, which occurs when C=0,P=0 and K=0 **AND** when C=0,P=1 and K=1
Simplify $Pr(C = 0) = ((1 - p) * 1/2) + (1/2 * p)$ to $Pr(C = 0) = 1/2$
$Pr(K = 0) = 1/2$
Consider $Pr(C = 0, K = 0) = Pr(C = 0)Pr(K = 0)$
$(1 - p) * (1/2) \neq 1/2 * 1/2$, for all P, therefore C and K are not necessarily independent.

6 Contributors List

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