HW 6 Discreet

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1 Combinatorial Proofs

A)$\binom{2n}{2}$ equals $2 * \binom{n}{2} + n^2$
Consider $\binom{n}{2}$ as the amount of ways of choosing distinct pairs in a set of length n. Therefore $\binom{2n}{2}$ is the amount of ways of choosing distinct pairs in a set of double length n.
Consider the pairs generated from $\{1, 2, 3\}$, or $\binom{3}{2}$ in other words. There would be 3 potential pairs (1,2), (2,3), (1,3).
Consider the pairs generated from $\{4, 5, 6\}$, or $\binom{3}{2}$ in other words. There would be 3 potential pairs (4,5), (5,6), (4,6).
Adding the counts of these pairs together, we get $2 * \binom{3}{2}$ (or $2 * \binom{n}{2}$ more generally), which represents the amount of pairs made between a set of length $2 * 3$ (or more general $2 * n$) where pairs are only selected within each half.
In order to count the ways of selecting distinct pairs in the overall set, for every value in the first half of the set, it has another value to form a pair with in the second half.
Therefore, for every value in $\{1, 2, 3\}$, it can be paired with every value in $\{4, 5, 6\}$ to form another distinct pair. Therefore there would be $3 * 3$ (or n^2 more generally) pairs made from combining each half.
Using this line of logic of counting the amount of pairs of size n, and then counting the amount of pairs when merging the two halves, we arrive at $2 * \binom{n}{2} + n^2$ ways of ways of choosing distinct pairs in a set of length 2n, which equals $\binom{2n}{2}$.

B) Solution:
Consider j to be an index moving along a set of n elements (represented by the following notation)

$$\sum_{j=k}^{n-m+k}$$

As $j = k$, then k is it's starting element. Consider j to form a partition between the set of size n to form two other sets. The size of the first set will be $j - 1$ (all numbers to the left of j) and the size of the second set will be $n - j$ (all numbers to the right of j). Let $k - 1$ be the amount of values we want to pick on the left

side of j. Let $m-k$ be the amount of values we want to pick on the right side of j. Therefore $\binom{j-1}{k-1}$ represents all possible ways of counting $k-1$ elements to left of j, and $\binom{n-j}{m-k}$ represents all possible ways of counting $m-k$ elements from the right of j. Let us include j in our set of picked elements. Therefore, with each iteration of j, we pick 1 (j itself) $+(k-1)$ (left elements) $+(m-k)$ (right elements), or m total elements.
If we want to pick $k-1$ elements to the left of j (not including j itself), then j must start iterating at k. Furthermore, if we want to pick $m-k$ elements from the right of j (not including j itself), then j must end iterating at while there are at least $m-k$ elements on the right, which is calculated by $n-m+k$. As the sigma notation uses these starting and ending parameters, then it will sum up all the possible ways of forming m sized subsets from a set of size n. Therefore, it is equal to $\binom{n}{m}$

C) $\binom{\binom{n}{2}}{2}$ equals $3*(\binom{n+1}{4})$

Consider $\binom{\binom{n}{2}}{2}$ to be the number of ways of picking two distinct sets of two values (or pairs) from set N.
When considering two distinct sets of pairs from the set N, there are two cases
1) The sets overlap on 1 value
2) The sets do not overlap on any values
Let's consider sets that do not overlap first.
$\binom{n}{4}$ is all the ways of choosing 4 numbers out of a set of n
Consider values a,b,c,d as the 4 numbers that are picked. Pick a to be the first number. Then there are 3 different potential pairs that can be formed with a (either b,c,d). As order does not matter with sets, this approach will describe all ways of picking distinct sets. Therefore, there are 3 ways of forming sets from 4 values.
Therefore $3*\binom{n}{4}$ defines all possible ways of forming 2 pairs from 4 numbers out of a set of n where no values overlap
Let's consider sets that do overlap on one number
$\binom{n}{3}$ is all the ways of choosing 4 numbers out of a set of n, where one value overlaps (as one is actually only picking 3 numbers from N). We'll get to which number is the duplicate in the next step, but for now choosing 4 with one duplicate is essentially choosing 3.
For each set of 3 numbers, there are 3 ways to make all combinations of duplicate pair sets. This is calculated by picking each of the respective 3 numbers to be the duplicate. Consider a,b,c. Pick A. Then the sets will be (a,b) and (a,c). Repeat this for each of the three values. Therefore, there are 3 ways to define each combination of pair sets where one number is a duplicate.
Therefore $3*\binom{n}{3}$ defines all possible ways of forming two distinct pairs from 4 numbers out of a set of n where ONE value overlaps.
As picking two distinct pairs must have either 1 value of overlap or no values of overlap (as defined previously), then $3*\binom{n}{4}+3*\binom{n}{3}$ defines all the ways of picking two distinct sets of two values (or pairs) from set N.

$3 * \binom{n}{4} + 3 * \binom{n}{3}$ equals $3 * (\binom{n}{4} + \binom{n}{3})$
By Pascal's Identity $(\binom{n}{4} + \binom{n}{3})$ equals $\binom{n+1}{4}$
By substitution we have $3 * (\binom{n+1}{4})$
Therefore $\binom{\binom{n}{2}}{2}$ equals $3 * (\binom{n+1}{4})$

2 Axioms of Probability

a) If $B \subseteq A$,then $Pr(A \setminus B) = Pr(A) - Pr(B)$.

Let (A, Pr) be a discrete probability space in the sample space A.
As $B \subseteq A$, $B \subseteq A$ is an event in the universe A
By compliment rule, as B, A are finite sets and $B \subseteq A$, $(A \setminus B) = (B^c)$
By corollary 18.4 $Pr(B^c) = 1 - Pr(B)$
By the second Axiom of probability, and because A is the universe under consideration $Pr(A) = 1$
By substitution, we have $Pr(B^c) = Pr(A) - Pr(B)$.
Therefore, because $B \subseteq A$, $Pr(A \setminus B) = Pr(A) - Pr(B)$

b) $Pr(A \oplus B) = Pr(A) + Pr(B) - 2Pr(A \cap B)$, where $A \oplus B$ is the event that either A or B happened, but not both.

As $Pr(A \oplus B)$ is defined as the event that either A **or** B happened, **but not both**, it can be defined as $Pr(A \cup B)$ (all events in A **or** B) - $Pr(A \cap B)$ (all events in A **and** B)
Therefore, $Pr(A \oplus B) = Pr(A \cup B) - Pr(A \cap B)$
As A, B are events, by theorem 18.5, $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$
By substitution, $Pr(A \cup B) - Pr(A \cap B)$ equals $(Pr(A) + Pr(B) - Pr(A \cap B)) - Pr(A \cap B)$
Therefore $Pr(A \oplus B) = Pr(A) + Pr(B) - 2Pr(A \cap B)$

3 Inclusion-Exclusion

(a) How many numbers in $\{1 \ldots 1000\}$ are not divisible by 5 or 11?

In order to find how many numbers are in not divisible by 5 or 11 in $\{1 \ldots 1000\}$, take the compliment of how many numbers ARE divisible by 5 or 11.
Let $Pr(Ei)$ mean divisible by 5. Therefore, $Pr(Ei) = 1/5$. In $\{1 \ldots 1000\}$, this would be 200
Let $Pr(Ej)$ mean divisible by 11. Therefore, $Pr(Ej) = 1/11$. In $\{1 \ldots 1000\}$, this would be 90.
However we want $Pr(Ej \cup Ei)$. This can be calculated by $Pr(Ej \cup Ei) = Pr(Ej) + Pr(Ei) - Pr(Ej \cap Ei)$

$Pr(Ej \cap Ei) = Pr(Ei) * Pr(Ej)$. This would be $1/5 * 1/11$, or $1/55$. Over $\{1 \dots 1000\}$, this would be 18.
Therefore, we have $Pr(Ej \cup Ei) = 200 + 90 - 18$
Our compliment is 1000 (the universe of the set) - 272. Therefore there are 728 numbers in $\{1 \dots 1000\}$ are not divisible by 5 or 11.

(b) A fair 6-sided die is rolled n times. What is the probability that at least one face is never shown?

Define events E1, E2, E3, E4, E5, E6 to correspond to a face not being rolled for each of the respective six faces. We want $PR(E1 \cup E2 \cup E3 \cup E4 \cup E5 \cup E6)$ to define the probability that at least one face is never shown.
Applying Theorem 18.6 with the following calculations.
$Pr(Ei) = \binom{5n}{n}/\binom{6}{n}$, because not rolling a side of the die means picking from the other 5 faces for each roll. Use n choices as there are n rolls.
$Pr(Ei \cap Ej) = \binom{4n}{n}/\binom{6}{n}$, because not rolling two sides of the die means picking from the other 4 faces.
This pattern continues with the described pattern
3 faces=$Pr(Ei \cap Ej \cap Ek) = \binom{3n}{n}/\binom{6}{n}$
4 faces=$Pr(Ei \cap Ej \cap Ek \cap El) = \binom{2n}{n}/\binom{6}{n}$
5 faces=$Pr(Ei \cap Ej \cap Ek \cap El \cap Em) = \binom{n}{n}/\binom{6}{n}$
6 faces=$Pr(Ei \cap Ej \cap Ek \cap El \cap Em \cap Ef) = \binom{0}{n}/\binom{6}{n}$ In all cases where n!=0, this case is impossible. For the n=0 case: as it is certain that a face is never shown when the die is never rolled and this does not impact the calculation, it will be omitted
Then we apply Theorem 18.6, to get $PR(E1 \cup E2 \cup E3 \cup E4 \cup E5 \cup E6)$, we have the following calculation
$((((\binom{6}{1} * \binom{5n}{n}) - (\binom{6}{2} * \binom{4n}{n}) + (\binom{6}{3} * \binom{3n}{n}) - (\binom{6}{4} * \binom{2n}{n}) + (\binom{6}{5} * \binom{1n}{n}))/\binom{6}{n})$, which is the probability that when a fair 6-sided die is rolled n times at least one face is never shown.

4 Contributor List

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