Bayard Walsh Algorithms Midterm

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1 Question 1: Hopping Game

Solution:
Input: A sequence of $n+1$ positions, where some of the positions j in the range 1 to $n-1$ have an obstacle.

Desired Output: the minimum number of moves required to win the game (or "can't win" if it's impossible)

General Approach: Create an array of length n, where every index represents the minimum steps needed to reach index n from 0, which is our start point. The array will function similar to a Dijkstra path. Set every value to ∞ to start, and then set $d[0] = 0$, because the first index represents the starting position. Starting from $i = 0$ and looping forwards, we first check if $A[i]$ is an obstacle. If so, $A[i]$ is always unreachable, so we leave as ∞. If not, we have $p = d[i]$, or we record the minimum steps to reach $d[i]$ from 0. Then we consider the positions $i+1, i+2, i+3, i+4$ represented as k, or the positions that we could reach from taking $1, 2, 3$, or 4 steps from i. For a given position k, if $A[k]$ is an obstacle, we leave as ∞, because it is unreachable. If it is not an obstacle, we update it to be the min of $d[k]$ and $p+1$ (shortest path to $d[i]$ plus one step). If uninitialized, it will always be $p+1$, because values are first set to ∞, so $min = p+1$. If not, we take the min, because there may be many paths to reach a certain position k and we only want the shortest one in $d[k]$. We repeat this approach for every 4 values ahead of i for every i, and gradually build an array $d[]$ so that any given value $i \in \{1, 2 \cdots n\}$ is either ∞ if unreachable, or $d[i]$ which is the shortest path from the start at a given step. We then check the last value, which will be the min steps to the end, if it is possible. If $M_p = \infty$, we know that there is no possible path.

Algorithm($A[1, 2 \cdots n-1]$**)** *// array of positions indicating obstacle*
if $n \leq 4$ *// edge case checking*
    return 1
$d[n] = \infty$ *// create n sized array for min steps to position array, all positions $\{0, 1, 2 \cdots n-1\} \in d[]$ initialized to ∞. Note the 0 index is for starting spot*

$d[0] = 0$ // ***NOTE:*** *Set* $d[0]$ *to* 0
$i = 0$ // *Starts at* 0
while $i < n$ **do:**
    if ($A[i] \neq 1$ **do:**
        $k = i + 1$ // *check values in front of* i
        $p = d[i]$ // *minimum steps to reach* i
        while $k \leq i + 4$ **AND** $k < n$ **do:**
            // *checks each of the* 4 *spots in front of* i*, and* $k < n$ *so no outside indexing*
            if $A[k] \neq 1$ **do:** // *check for obstacle, if no obstacle, possible path*
                $d[k] = min\{d[k], p + 1\}$ // *update min steps to* $d[k]$*. Only change if shorter*

            $k = k + 1$ // *increment* k

    $i = i + 1$ // *increment* i
$M_p = min\{d[n-4], d[n-3], d[n-2], d[n-1]\}$ // *min value of paths one step from end*
$M_p = M_p + 1$ // *one more step required to reach* n
if $M_p \neq \infty$ **do:** //*check if a path is possible*
    return M_p
else: //*if no paths from* $\{d[n-4], d[n-3], d[n-2], d[n-1]\}$*, no paths are possible*
    return "can't win"

Run time: In terms of Run time analysis, the algorithm performs in $O(n)$. First it initializes an array of length n, setting all values to ∞. This is $O(n)$. Second, as it makes one pass through $A[1, 2 \cdots n-1]$ in the while loop, from the start of the loop to the end. While there is a nested while loop, this performs a maximum of 4 steps per i iteration, to check the potential steps forward from i. Inside the loop there are constant operations, such as updating the distance value and increasing the while loop counters. Therefore, we can consider the loop to have $O(4n)$ run time, simplifying to $O(n)$. After the loop, we take the minimum values from the last 4 positions and add one (each is within one step from end), to find the best path, and min is semi-constant operation. Finally, we check for infinity and either return M_p or impossible. Therefore, we have $O(n) + O(4N)$ which simplifies to $O(n)$.

Correctness:
Feasibility: In terms of feasibility, we prove two aspects:
First: the algorithm will always return a path if a path is possible
Proof: Because the algorithm begins with $d[0] = 0$, and builds paths forwards, any position that can be reached from the start will be updated with a non ∞ amount, because the loop checks each of the 4 values in front of every valid index (a valid index is reachable from index 0) and adds them if they are non-obstacles. Therefore, because any valid path must move at

within $1-4$ values ahead of another valid position, the algorithm will incrementally do this, and if there is a valid path eventually then it will be added to $min\{d[n-4], d[n-3], d[n-2], d[n-1]\}$ and returned by the algorithm. Note if any of $\{d[n-4], d[n-3], d[n-2], d[n-1]\} \neq \infty$, a path is possible.
Second: the algorithm will never return a path if a path is impossible
Proof: If there is not a valid path from the start to the end, then there must be some point where you cannot move $1, 2, 3$ or 4 steps from a valid position, say p towards another valid position on the path to n. If this is the case, then when the algorithm arrives at p it will check the $1, 2, 3$ or 4 steps ahead, and if none are valid then it will not update any steps. Therefore, all the values after p will equal ∞, which will **return "can't win"**.
Therefore the algorithm will always always return a path if possible and won't if impossible.

Optimality: First, **Lemma: For each $u \in A[1, 2 \cdots n-1]$, $d[u]$ is the length of the fewest steps from 0 (the starting point) to u**
Proof by induction. If $n \not> 4$, we return 1 by edge case checking. Therefore, our base case is $n = 5$, in which case $d[u] = 1$ for each $\{d[1], d[2], d[3], d[4]\}$ that doesn't have an obstacle, and 2 for overall min steps to n, which is optimal, meaning that $d[u]$ has offered the fewest steps from 0 for each $\{d[1], d[2], d[3], d[4]\}$.
Induction: Now suppose this holds for $n \geq 5$. Let v be an arbitrary position which the algorithm has iterated over. Let p be the move-sequence made by the algorithm to reach v, such that $|p| = d[u]$. Let p' be any other move-sequence to reach v. We want to show that p' cannot be shorter than p. Let u be the position of the last move made by both p and p' before the paths diverge. Therefore, let p go to some position x and p' go to some other position y, both which are reachable from u. However, as the algorithm records every possible move for a given index (it checks each of the 4 spaces ahead for any given i in the array), the algorithm will either record this step p' in $d[y]$ or $d[y]$ will contain a count of a shorter path to y. At the next step of p', at position y the algorithm will once again consider all valid possible moves from y. Therefore, for every step s in p', the algorithm will have a path, $d[s] \leq |p'_s|$ where p'_s is the number of steps to reach s given by p'. Finally, at some point p' will reach v, meaning that it is within 4 positions of some other valid position. As the algorithm loops through every position, then it will also consider this position, meaning it will consider path p', so $d[v] = min\{|p|, |p'|\}$,and as $|p| = d[u]$, $|p \not> p'|$. Therefore, $|p'|$ is not shorter than $d[v]$, meaning that for any arbitrary $u \in d[u]$, the distance to $d[u]$ is the length of the fewest steps from 0.
Proof: The Algorithm Provides the Fewest Steps to n if a path is possible. As proven by the previous lemma, for each $u \in A[1, 2 \cdots n-1]$, $d[u]$ is the length of the fewest steps from 0 (the starting point) to u. Therefore, consider the paths $d[n-4], d[n-3], d[n-2], d[n-1]$. Note that each of these paths is exactly one step away from n, and that these are the only paths that can reach n in one step. Secondly, as each of these values are the minimum paths to the positions $n-4, n-3, n-2, n-1$, and any valid path must go through one of these 4 positions to reach n. Therefore the minimum

path must go through at least one of these positions. Therefore, if we taken the $min\{d[n-4], d[n-3], d[n-2], d[n-1]\}$, and add 1 (each path is equally 1 step away, so it will still minimize), we will find our shortest path possible to n. However, if the $min = \infty$, we know that it wasn't possible to reach any of these positions from 0, or the start, and as we iterated through the entire array, we know that there cannot be any possible paths to n, so we **return "can't win"**.

2 Question 2: MST with one modified edge

Note: I used e_* instead of e^*. In regards to any work on the problem $e_* = e^*$. Thanks!

First, we know that T is the only MST for w, because the edges are distinct. Second, we know that T' is the only MST for w' because edges are also distinct. I will show that at its largest $k = 1$, after moving from w to w'.

First case: We will assume that $e_* \in T$.
Most of the work on this proof will involving picking another edge e' in the same cutset as e_*. This process will be detailed below and I will refer to the same e' and e_* throughout the first case analysis of the proof. Note that picking the **min** e' in the cut set and $\notin T$ is the best option for exchange for e_*, because if e' can't be exchanged into the MST for e_*, a greater value $\notin G$ couldn't be either.
Lemma: *For $G = (V, E)$ let $S \subseteq V$ be a cut and let e_* and e' be two edges in the cutset S, such that $e_* \in T$ and $e' \notin T$. Pick e' such that $w(e')$ is minimized over all edges that are both in the cutset S and $\notin T$. Let C be the cycle created by adding e'. Then $(\forall e \in C \setminus e')w(e) < w(e')$*
First, consider adding $e'(u, v)$ to T while we have the original weights w. As adding any edge to T will create a cycle, by its MST property, and we know all weights in w and w' are distinct positive integers, so therefore adding e' will create a cycle C between vertexes (u, v). Therefore, we apply the Cycle Property, implying that the highest-weighted edge in C does not appear in any MST for G. As $e'(u, v)$ does not appear in T, we know that it must be the highest-weighted edge in C, or else the Cycle Property would not hold, implying T is not an MST, a contradiction. Therefore, where P is the path from (u, v) in T, $(\forall w(e) \in P) < w(e')$, and subsequently $(\forall w(e) \in C \setminus e') < w(e')$

Lemma: *For $G = (V, E)$ let $S \subseteq V$ be a cut and let e_* and e' be two edges in the cutset S, such that $e_* \in T$ and $e' \notin T$. Pick e' such that $w(e')$ is minimized over all edges that are both in the cutset S and $\notin T$. If $w'(e_*) < w'(e')$ then $|T - T'| = 0$*
As we use the same constraints to pick an e' from the previous lemma, we will regard the same cycle C created above, but with the new weight function

w'. We will also consider the same path P. As $(\forall e \in P) w(e) < w(e')$, and $w(e) = w'(e)$ for all $e \neq e_*$, we know that $(\forall e \in P \setminus e_*) w'(e) < w'(e')$. If we have $w'(e_*) < w'(e')$, we have $(\forall e \in C \setminus e') w'(e) < w'(e')$, meaning that e' would be the maximum value in a cycle C, so e' could not be in any MST, meaning that we have $|T - T'| = 0$ because we can't make any exchanges.

Lemma: *For $G = (V, E)$ let $S \subseteq V$ be a cut and let e_* and e' be two edges in the cutset S, such that $e_* \in T$ and $e' \notin T$. Pick e' such that $w(e')$ is minimized over all edges that are both in the cutset S and $\notin T$. If $w'(e_*) > w'(e')$ then $|T - T'| \geq 1$*

As we use the same constraints to pick an e' from the previous lemma, we will regard the same cycle C created above, but with the new weight function w'. We will also consider the same path P. If we have $w'(e_*) > w'(e')$, and we as know that $(\forall e \in P \setminus e_*) w'(e) < w'(e')$, then when we create cycle C which contains path P by adding e', we have $(\forall e \in C) w'(e) < w'(e_*)$. Therefore, we apply the cycle property, and as $w'(e_*)$ is the max value in the cycle C, it cannot be in any MST. Therefore, we have $|T - T'| = 1$ in this exchange, as e_* is no longer $\in T$, but it is $\in T'$

Second case: We will assume that $e_* \notin T$, to cover all cases.

Most of the work on this proof will involving picking another edge e' in the same cutset as e_*. This process will be detailed below and I will refer to the same e' and e_* throughout the second case analysis of the proof. Note that picking the **max** e' in the cut set and $\in T$ is the best option for exchange for e_*, because if e' can't be taken out of the MST, a lower edge $\in T$ and in the cutset couldn't be either.

Lemma: *For $G = (V, E)$ let $S \subseteq V$ be a cut and let e_* and e' be two edges in the cutset S, such that $e_* \notin T$ and $e' \in T$. Pick e' such that $w(e')$ is maximized over all edges that are both in the cutset S and $\in T$. Let C be the cycle created by adding e_*. Then $(\forall e \in C \setminus e_*) w(e) < w(e_*)$*

First, consider adding $e_*(u, v)$ to T while we have the original weights w. As adding any edge to T will create a cycle, by its MST property, and we know all weights in w and w' are distinct positive integers, so therefore adding e_* will create a cycle C between vertexes (u, v). Therefore, we apply the Cycle Property, implying that the highest-weighted edge in C does not appear in any MST for G. As $e_*(u, v)$ does not appear in T, we know that it must be the highest-weighted edge in C, or else the Cycle Property would not hold, implying T is not an MST, a contradiction. Therefore, where P is the path from (u, v) in T, $(\forall w(e) \in P) < w(e_*)$, and subsequently $(\forall w(e) \in C \setminus e_*) < w(e_*)$

Lemma: *For $G = (V, E)$ let $S \subseteq V$ be a cut and let e_* and e' be two edges in the cutset S, such that $e_* \notin T$ and $e' \in T$. If $w'(e_*) > w'(e')$ then $|T - T'| = 0$*

As we use the same constraints to pick an e' from the previous lemma, we will regard the same cycle C created above, but with the new weight function w'. We will also consider the same path P. As e' is the maximum value in the cutset $\in T$, when we add e_* to create a cycle, if $w'(e_*) > w'(e')$, $w'(e')$ is a smaller edge, meaning adding e_* would be a violation of the MST property. Therefore we add no edges and have $|T - T'| = 0$.

Lemma: *For $G = (V, E)$ let $S \subseteq V$ be a cut and let e_* and e' be two edges in the cutset S, such that $e_* \notin T$ and $e' \in T$. Pick e' such that $w(e')$ is maximized over all edges that are both in the cutset S and $\in T$. If $w'(e_*) > w'(e')$ then $|T - T'| \geq 1$*

As we use the same constraints to pick an e' from the previous lemma, we will regard the same cycle C created above, but with the new weight function w'. We will also consider the same path P. If we have $w'(e_*) > w'(e')$, and we as know that when we add e_* to create a cycle $\in T$, if $w'(e') > w'(e_*)$, then $w'(e_*)$ is a smaller edge, meaning exchanging e' for e_* will maintain MST property. Therefore we add e_* and we have $|T - T'| = 1$ in this exchange, as e' is no longer $\in T$, but it is $\in T'$

Now we consider if we can reach a greater $|T - T'|$ outside exhanging e_*

Lemma: *For $G = (V, E)$ let $S \subseteq V$ be a cut and let e_* and e' be two edges in the cutset S, such that either $e_* \in T$ and $e' \notin T$ **OR** $e_* \notin T$ and $e' \in T$. Therefore, both e' and e_* bridge the same two connected sub-graphs.*

By definition of a cut, we know that it is a partition of vertices into two nonempty subsets, $(S, V - S)$, where a cut set of S is the set of edges with has exactly one endpoint in S. Therefore, when considering that e' and e_* are in the same cut set (which is the only way to exchange one edge from outside and MST into a MST and maintain the MST properties), both e' and e_* have one edge in S and one edge in $V - S$. As we know that T is a minimum spanning tree, we know that

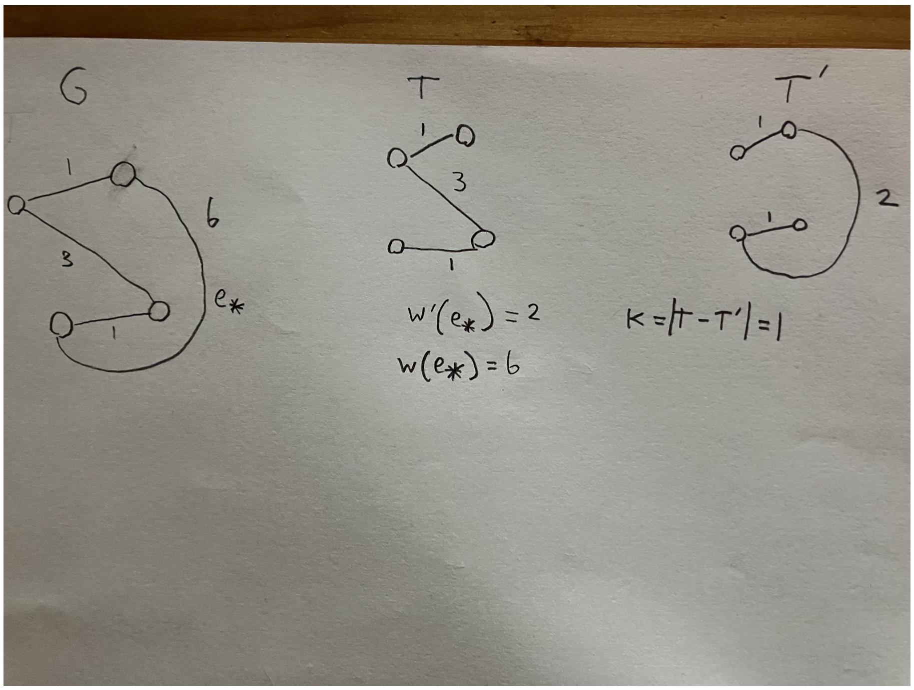
1: the only paths between the vertices in S to any in $V - S$ is through whichever edge is $\in T$ for T, or else we would have a loop, violating the spanning tree property

2: all the nodes in S are connected to each other in some subgraph and that all the nodes in $V - S$ are also a connected to each other in some subgraph, by the spanning tree property. In other words, whichever edge is $\in T$ for T is a bridge between two connected components.

Therefore, let A be a connected sub graph of nodes in S and let B be a connected sub graph of nodes in $V - S$. If neither e' nor e_* is $\in T'$, we would have two connected components A and B. As we know that all weights are the same except e_* from w to w', and we are weighing either e_* or e' in the MST, then A and B must also be components of T' bridged by either e_* or e'. If not, this would imply that there must be some other edge $e_i \neq e_*$ and $e_i \neq e' \in T'$ and $e_i \notin T$. However, as none of the other weights are changed, then e_i could only be greater than some other edge it would replace in T or else it would violate the MST rules for T. If e_i is greater, than it will not be in T', as that would violate the MST rules for T', because the weights haven't changed. Therefore the same two connected components must be in T and T'.

Proof: *At it's largest, $|T - T'| = k = 1$*

As proved above, e' and e_* both bridge the same two connected components for T and T'. Therefore, the only differences between the two MSTs is if e_* is removed from or added to T' from T, as shown in the case analysis above, causing a maximum difference of 1. Adding any other edge to T' would be a violation of MST attributes.
Example Below



3 Question 3: Gale-Shapley n^2

Preference Lists given below *Assume that j is some arbitrary value such that $3 < j < n-3$, j is used for the pattern*

Group A's preference lists (from most preferred to least preferred):

a_1: b_1, b_2, b_3, $\cdots$, b_{n-1}, b_n

a_2: b_2, b_3, $\cdots$, b_{n-1}, b_1, b_n

a_3: b_3, $\cdots$, b_{n-1}, b_1, b_2, b_n

$\cdots$ $\cdots$

a_j: b_j, b_{j+1}, b_{j+2}, $\cdots$, b_{n-2},b_{n-1}, b_1,b_2, $\cdots$,b_{j-1}, b_n

$\cdots \cdots$

a_{n-1}: b_{n-1},b_1, b_2, b_3, $\cdots$, b_n

a_n: b_1, b_2, b_3, $\cdots$, b_{n-1}, b_n

Group B's preference lists (from most preferred to least preferred):

b_1: a_2, a_3, a_4, $\cdots$, a_{n-1}, a_n, a_1

b_2: a_3,a_4, $\cdots$, a_n, a_1, a_2

b_3: a_4, $\cdots$, a_n, a_1, a_2, a_3

$\cdots \cdots$

b_j: a_{j+1}, a_{j+2}, $\cdots$,a_{n-1}, a_n,a_1, a_2, $\cdots$,a_j

$\cdots \cdots$

b_{n-1}: a_n,a_1, a_2, a_3, a_4, $\cdots$,a_{n-2}, a_{n-1}

b_n: a_1, a_2, a_3, a_4, $\cdots$, a_n

Order of execution: As A is proposing, let every $a_i \in A$ propose from lowest to greatest, such that the initial order of execution is $a_1, a_2, a_3 \cdots a_n$. As the first choice for every $a_i \in A \setminus a_n$ is the corresponding b_i, the moment before a_n first proposes, we will have exactly $n-1$ executions of the while loop and $n-1$ pairs, because each a_i proposes to a different b_i, and as every b_i is un-proposed, they will all accept every incoming offer. Therefore, let the first proposal of a_{n-1} be the first iteration ending with just one single (non-engaged) individual in A (which is a_n), and I will show that the Gale Shapley algorithm still takes at least $c \cdot n^2$ additional while-loop iterations to terminate.

Lemma: *Starting at the first proposal from a_n, with the preferences and execution given above, we will have $(n-1)^2 + 1$ iterations in the while loop.*
First, note that every $a \in A$ has b_n as its last choice preference. This is important, because once b_n is proposed to, every person in A and in B will be paired, meaning that the Gale-Shapley algorithm will terminate with a stable matching, so no more iterations of the while loop. I believe this is proved in class, however (very briefly) the proof is that if there is an unstable pair (u, v), with respect to matching M, then there are two pairs (u, v') and (u', v) such that u prefers v to v' and v prefers u to u', however this cannot occur because either u didn't propose to v (meaning v' is higher on the preference list, so not unstable) or u did propose to v and got rejected or dumped, (meaning u' is higher than u for the v preference list). This implies that there cannot be an unstable match once every person in A is matched, so we want to prevent that. Secondly, notice that every b_i has the corresponding a_i ranked last, meaning they will switch with any other partner over their current one.
Continuing, when a_n first proposes, it will propose to b_1. Because b_1 prefers a_n,

a_1 will be dumped. Now a_1 proposes to b_2, which *slightly* prefers a_1 to a_2, so a_2 is now dumped. This pattern will continue for $n-1$ steps, where each respective a_i proposes to b_{i+1}, causing a_{i+1} to get dumped and follow the same pattern. Now we get to a_{n-1} being dumped by b_{n-1} after a_{n-2} proposes to b_{n-1}. Now a_{n-1} proposes to its next choice, which is b_1, and as b_1 *slightly* prefers a_{n-1}, a_n is now dumped. At this point we have ended a "downshift" a term for a cascade of treading partners, where one of the later values in $a_1, a_2, a_3 \cdots a_n$ proposes to b_1. Because every partner in A proposes to B which is only slightly more interested in their current partner, the dumping continues, meaning that until someone purposes to b_n, the "downshift" will continue, and we loop through every value in $a_1, a_2, a_3 \cdots a_n$. A downshift will have $n-1$ iterations, because each person in $B \setminus b_n$ is proposed to each round, rejecting another person in A, forcing them to re-propose, which is $n-1$ iterations. In terms of measuring how many of these downshifts occur (considering a downshift ending when a_1 proposes again), it will be $n-1$ iterations, because a_1 eventually proposes to b_n (the first to do it), ending the algorithm, however b_n is in the n spot on the preference list for every person in A, so there will be $n-1$ overall "downshifts", where each person in A proposes to their first $n-1$ preferences before proposing to b_n. When b_n is proposed to the algorithm will terminate, because every person in A will be matched to some person in B, resulting in a stable matching. Therefore, we have $n-1$ as the iterations per downshift, and $n-1$ overall downshifts, and the final proposal to from a_1 to b_n, which is +1 iteration. Therefore, from this step, for these preferences and specific execution, we have achieved $(n-1)^2+1$ iterations of the while loop.

Proof: *The Gale Shapley algorithm still takes at least $c \cdot n^2$ additional while-loop iterations to terminate given the preferences and execution above*
We have $(n-1)^2+1$ from above. Therefore, we want a constant $c > 0$ such that for all $n \geq 3$, $(n-1)^2+1 > c \cdot n^2$. Therefore we want $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq 2$.
Therefore, we want $0 \leq c \cdot (n^2) < (n-1)^2+1$ for all $n \geq 3$. For 0, we have $((3-1)^2+1) = 5$, and as $n \geq 3$ and the function is increasing when $n \geq 3$, we have $0 \leq ((n-1)^2+1)$ for any n. As Ω gives the lower bound of a function, we want a positive constant c such that $c \cdot (n^2) < (n-1)^2+1$, or

$$c \cdot (n^2) < (n-1)^2 + 1$$

$$c \cdot (n^2) < n^2 - 2n + 2$$

$$c < (n^2 - 2n + 2)/(n^2)$$

$$c < 1 - (2/n) + (2/n^2)$$

In order to determine behavior for sufficiently large n we take derivative.

$$(2/n^2) + (4/n^3)$$

$$2/n^2 * (1 - 2/n)$$

Considering that we have the domain $(3, \infty)$, the function is always increasing on our domain. To make certain that this meets the lower bound,Choose $c = 0.1$. Therefore, when $n \geq 3$, we have

$$.1 < 1 - (2/3) + (2/3^2)$$

This supports the notion that

$$c \cdot (n^2) < (n-1)^2 + 1$$

for all $n \geq 3$ and $c = 0.1$, proving that there will be an additional $\Omega(n^2)$ loops for the problem.