Test Flight Problem: Set Solutions

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A1. FALSE. If $n \geq 2$, then for any m, we have that $3m+5n \geq 13$ (since $3m+5n \geq 3m+10 \geq 13$). Thus, the only way to find such a solution for n in the natural numbers would be when $n = 1$.
Substituting, we have $3m + 5 \cdot 1 = 3m + 5 = 12$, or $3m = 7$.
But since there is no natural number m satisfying this equation, we have proved the result.

A2. TRUE. Without loss of generality, we may assume the consecutive five integers may be written in the form:

$n-2$, $n-1$, n, $n+1$, $n+2$. If we sum these integers, we have $5n$, which is divisible by 5. Hence, we have proved the result.

A3. TRUE. We may rewrite n^2+n+1, as $n\cdot(n+1)+1$. If n is even, then $n+1$ is odd. If n is odd, then $n+1$ is even. In either case, $n \cdot (n+1)$ is even because the product of an even and odd number is even. Hence, we may write $n \cdot (n+1) + 1$ as $2k+1$, which is odd. Hence, we have proved the result.

A4. Recall from the remainder theorem: if a,b are integers with $b > 0$, then there exist unique integers q, r such that $a = bq + r$ and $0 \leq r < b$. If we let $b = 4$ (and $n = q$), then we have the statement that

$a = 4n + r$ with $0 \leq r < 4$. If $r = 0$ or 2, then we have $a = 4n$ or $a = 4n + 2$, which are even natural numbers. If $r = 1$ or 3, we have that $a = 4n+1$ or $a = 4n+3$, which are odd natural numbers. Since a is any odd natural number, satisfying the antecedent, we have that it must be of one of the following forms $a = 4n + 1$ or $a = 4n + 3$. Hence, we have proved the result.

A5. Recall from the remainder theorem: if a,b are integers with $b > 0$, then there exist unique integers q,r such that $a = bq + r$ and $0 \leq r < b$. If we take $b = 3$, then we have the statement that $a = 3q + r$ with $0 \leq r < 3$. Expanding out (and letting $n = a$), we have that $n = 3q$, or $n = 3q + 1$, or $n = 3q + 2$.
Let's now write n, $n+2$, and $n+4$ in these forms:
n is either $3q$, $3q+1$, or $3q+2$. $n+2$ is either $3q+2$, $3q+3$, or $3q+4$. $n+4$ is either $3q+4$, $3q+5$, or $3q+6$.
But we see that in each of the forms, there exists an element which is divisible by 3
i.e. if n, $3|3q$ and if $n+2$, $3|(3q+3)$, and if $n+4$, $3|(3q+6)$. Hence, we have proved the result.

A6. By contradiction, assume there exists $n > 3$, such that n, $n+2$, and $n+4$ are prime. But from the proof of #5, we have just shown that one of n, $n+2$, $n+4$ must be divisible by 3. And since $n > 3$, 3 is not one of the primes. Thus, one of n, $n+2$, $n+4$ is not prime. Hence, we have proved the result.

A7. Let the sum, $2+2^2+2^3+...+2^n$, be denoted by S. Multiplying by 2, we have that $2S = 2^2 + 2^3 + ... + 2^{n+1}$. Subtracting S from $2S$, we have that $S = 2^{n+1} - 2$, which was to be proved.

A8. For any given > 0, there exists an n where for all $m \geq n$, $|a_m - L| < \epsilon$. The statement that Ma_n tends to ML as n tends to infinity is equivalent to saying that for any given
$\epsilon_1 > 0$, there exists an n where all $m \geq n$, $|Ma_m - ML| < \epsilon_1$. This simplifies to $|M|a_m - L| \iff M|a_m - L| < \epsilon_1 \iff |a_m - L| < \frac{\epsilon_1}{M}$.
This will be true if we take $\epsilon_1/M = \epsilon$, and find such an n. Since we can always do this, we have proved the result.

A9. Let $A_n = (0,{}^1\!/_n)$. We have that A_n is a subset of A_1 since $(0,{}^1\!/_n)$ is a subset of (0,1). Suppose that x is an element of (0,1). We can always find a natural number m such that ${}^1\!/_m < x$. But that means that x is not an element of A_m. Hence, x is not an element of the intersection of A_n where n is a natural number. Since we can always find this number m, we must necessarily have that intersection of A_n is empty. Hence, we have proved the result.

A10.Let $A_n = [0,{}^1\!/_n)$. We may write this set as $0 \cup B_n$, where $B_n =(0,{}^1\!/_n)$. The intersection of A_n for n in the natural numbers may thus be written as $0 \cup (\cap B_n)$. But since we have proved from #9 that intersection of all B_n is the empty set, we have that $0 \cup \emptyset = \{0\}$.Hence, we have proved the result.