Matrice

$A \in \mathbb{R}^{m \times n}$

$Im(A) = \{ y \in \mathbb{R}^m \;\; \exists x \in \mathbb{R}^n \text{ a. î. } y = Ax \}$

$rang(A) = dim(Im(A))$ $\rightarrow$ dim reprezintă nr. vectorilor din bază

$Im\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \{ x \in \mathbb{R}^2 \mid x = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \alpha \in \mathbb{R} \} \Rightarrow dim\left(Im\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\right) = 1$

$Ker(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$

$Ker([1 \;\; 2]) = \{ x \in \mathbb{R}^2 \mid x_1 + 2x_2 = 0 \}$

Teoremă:

$Im(A) \perp Ker(A^T)$ și orice $x \in \mathbb{R}^m$ se descompune $x = u + v$, $u \in Im(A)$, $v \in Ker(A^T)$

$$\mathbb{R}^m = Im(A) \oplus Ker(A^T)$$

Liniar dependenta

$$\begin{bmatrix} & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \cdots x \\ \cdots \\ a^n x \end{bmatrix} = \begin{bmatrix} \sum_{j=1} a_j x_j \\ \cdots \end{bmatrix} \Big\downarrow i$$

Liniar independenta: $\nexists\, \alpha \in \mathbb{R}^m$ a. î. $\exists \alpha_i \neq 0$ a. î.

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$

$\exists \alpha_i \neq 0 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (coloanele v_1, v_2) $\qquad \alpha_1 v_1 + \alpha_2 v_2 = \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_2 \end{bmatrix} = 0$

liniar independenti

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \;\; 1] \qquad \underbrace{1}_{\alpha_1} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{v_1} \underbrace{-1}_{\alpha_2} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{v_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u \cdot v^T$ rang 1

$Im(uv^T) = \{ y = u \underbrace{v^T x}_{\alpha \in \mathbb{R}} \mid x \in \mathbb{R}^n \} =$

$= span\{u\} \leftarrow dim\ 1$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{cases} \Rightarrow x_1 = -x_2$

$x = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} = span\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = Ker\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$

$$A \cdot x = \begin{bmatrix} a^1 \\ \cdots \\ a^n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a^1 x \\ \cdots \\ a^n x \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_j^1 x_j \\ \cdots \\ \sum_{j=1} a_j^n x_j \end{bmatrix} \Big\downarrow i$$

$GAXPY(A, x, y) \rightarrow Ax + y$

Fie $U \cdot B \cdot x + y$, $\underset{n \times n}{U}$ sup tr., $\underset{n \times n}{B}$ inf. bidiag., $x, y \in \mathbb{R}^n$

Calculați prod folosind eficient GAXPY. Detaliați GAXPY

$r = \underbrace{U \cdot B}_{M} \cdot x + y$ | Remarcăm $U \cdot B$ înmulțire MM $O(n^3)$, $M \cdot x$ înmulțire MV $O(n^2)$ $> O(n^3)$

$z = Bx \quad O(n^2)$
$r = Uz + y \quad O(n^2)$ $> O(n^2)$

```
GAXPYbd(B, x, 0){                O(n)
    r1 = b11 + x1
    for i = 2:n
      | ri = b_{i,i-1} x_{i-1} + b_{i,i} x_i
}
```

```
GAXPYtr(U, z, y){                O(n^2)
  r = 0
  for i = 1:n{
    for j = i:n{
       ri = ri + u_ij x_j
    }
    ri = ri + yi
}
```

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ - \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} u_{1j} x_j \\ \\ \sum_{j=n}^{n} u_{ij} x_j \end{bmatrix}_n \Big\downarrow i$$

Sisteme de ecuații liniar pătratice

A singulară (neinversabilă)
A nesingulară (inversabilă)

Nesingulară dc. $\rightarrow det(A) = 0$
$\rightarrow$ col liniar dependente
$\rightarrow$ nucleu $Ker(A)$ nenul

Rang $\rightarrow$ nr maxim de coloane liniar independente

I Matricea A superior bidiagonală

$$\begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ 0 & a_{22} & a_{23} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$$

$$\begin{cases} a_{11} x_1 + a_{12} x_2 = b_1 \\ a_{22} x_2 + a_{23} x_3 = b_2 \\ \cdots \\ a_{nn} \cdot x_n = b_n \end{cases}$$

```
BDtris(A, b){
  x := b
  xn = xn / ann
  for i = n-1 : -1 : 1 {
      xi = xi - a_{i,i+1} x_{i+1}
      xi = xi / a_ii
  }
}
```

II Fie A = L inferior triunghiulară

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$$

$$x_1 = \frac{b_1}{l_{11}} \quad ; \quad x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} x_j}{l_{ii}}$$

```
LTRIS(L, b){
  x := b
  for i = 1:n{
     for j = 1 : i-1{
        xi = xi - l_ij xj
     }
     xi = xi / l_ii
  }
}
```

Triangularizare

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

(coloanele a_1, a_2, a_3)

$$M_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} [1 \;\; 0 \;\; 0] = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\boxed{\text{Produs exterior: } \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} [1 \;\; 0 \;\; 0] = \begin{bmatrix} 0 \cdot 1 & 0 \cdot 0 & 0 \cdot 0 \\ 4 \cdot 1 & 4 \cdot 0 & 4 \cdot 0 \\ 2 \cdot 1 & 2 \cdot 0 & 2 \cdot 0 \end{bmatrix}}$$

$$M_{11} a_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$M_{21} = I_3 - \frac{1}{4} \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} [0 \;\; 1 \;\; 0] = I_3 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

$$M_{21} \cdot a_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

Transformări elementare

Inferior triunghiulară elementară (ITE)

$M_k = I_n - m_k e_k^T$

$m_k = [0 \ldots 0 \;\; \mu_{k+1,k} \cdots \mu_{n,k}]$

$$M_k = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ & & \cdots & & \cdots & \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & \cdots & -\mu_{k+1,k} & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & & -\mu_{nk} & \cdots & 1 \end{bmatrix}$$

M_k inversabilă, cu $M_k^{-1} = I_n + m_k e_k^T$

Eliminarea Gausiană

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$$

$$M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$$M_2 = I_3 - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [0 \;\; 1 \;\; 0] = I_3 - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$M_2 M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Acum rezolvăm $\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = M_1 M_2 b$

În general $M = M_{n-1} \cdots M_2 M_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \mu_{21} & 1 & \cdots & 0 \\ \mu_{n1} & \mu_{n2} & \cdots & 1 \end{bmatrix}$

Deci multiplicatorii μ_{ij} îi memorăm în triunghiul inf. a matricii A.

```
EG(A){
  for k = 1 : n-1{
     for i = k+1 : n{
        a_ik <- μ_ik = a_ik / a_kk
     }
     for j = k+1 : n{
        for i = k+1 : n{
           a_ij <- a_ij - μ_ik a_kj
        }
     }
}
```

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1k} & u_{1,k+1} & \cdots & u_{1n} \\ \mu_{21} & u_{22} & \cdots & u_{2k} & u_{2,k+1} & \cdots & u_{2n} \\ & & \cdots & & & \cdots & \\ \mu_{k1} & \mu_{k2} & \cdots & u_{kk} & u_{k,k+1} & \cdots & u_{kn} \\ \mu_{k+1,1} & \mu_{k+1,2} & \cdots & \mu_{k+1,k} & a_{k+1,k+1}^{(k+1)} & \cdots & a_{k+1,n}^{(k+1)} \\ & & \cdots & & & \cdots & \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nk} & a_{n,k+1}^{(k+1)} & \cdots & a_{nn}^{(k+1)} \end{bmatrix}$$

După pasul k

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1k} & \cdots & u_{1n} \\ \mu_{21} & u_{22} & \cdots & u_{2k} & \cdots & u_{2n} \\ & & \cdots & & \cdots & \\ \mu_{k1} & \mu_{k2} & \cdots & u_{kk} & \cdots & u_{kn} \\ & & \cdots & & \cdots & \\ & & \cdots & & \cdots & \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nk} & \cdots & u_{nn} \end{bmatrix}$$

În final

```
GPP(A){
  for k = 1 : n-1{
     Se det cel mai mic i_k : |a_{i_k k}| = max_{i=k:n} |a_ik|
     p(k) = i_k
     for i = k+1 : n{
        a_kj <-> a_{i_k j}
     }
     for i = k+1 : n{
        a_ik <- μ_ik = a_ik / a_kk
     }
     for j = k+1 : n{
        for i = k+1 : n{
           a_ij <- a_ij - μ_ik a_kj
        }
     }
  }
}
```