1. **Bifurcation Diagram for the Logistic Map**

The *logistic map* is defined by the equation

$$x_{i+1} = \mu x_i(1 - x_i). \tag{1}$$

Its behaviour for different values of μ is totally different. To get an overview over the different domains, we will analyze it with the help of a so-called *bifurcation diagram*.

(a) Implement the logistic map and try it for different values of μ. Try to find the case of a fixed point, multiple attractors and chaotic behaviour. ($x_0 = 0.9$ is a good initial value for the iterations.)

(b) Generate the bifurcation diagram for the logistic map in the following steps:

1. For the bifurcation diagram we need to evaluate the logistic map for different values of μ and take the final x-value of each evaluation. To ensure that any transients have vanished, you should use around 200 iterations for each evaluation. Write a function that computes these final x-values for an arbitrary value of μ.
2. To ensure that we capture the general case, we should compute the above samples for a range of initial values for each value of μ, different random start values x_0 should be used in the range $x_0 \in (0, 1)$. Rewrite your function to take this into account.
3. Now collect the samples for different values of $\mu \in [1, 4]$ by splitting the interval into equal parts. 1000 parts are a reasonable choice.
4. Plot your results as a scatter plot, i.e. single points without lines. The x-coordinate is μ, the y-coordinate shows the final x-values. The resulting plot is the bifurcation diagram for the logistic map.

Once you have plotted the bifurcation diagram, feel free to explore some of its features, like self-similarity and intermittency. You can try to use more points, e.g. 100000 in the interval $\mu \in [1, 4]$ to observe more details.

2. **Chaos in dynamical systems**

We want to study a realistic pendulum, i.e. we will *not* make the approximation $\sin\theta \approx \theta$, and we will include friction and a driving force,

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin\theta - \alpha\frac{d\theta}{dt} + f\cos\omega t\,.$$

(a) Study the free pendulum, i.e. $\alpha = 0, f = 0$, use $\omega_0 = 1$. Solve the differential equation with the Runge-Kutta IV algorithm (as shown in the lecture) for suitable initial conditions and a suitable step size. Plot the time evolution of θ and make a phase space plot, compare with the analytical solution for small initial values of θ.

(b) Use different random start parameters and combine the results in one phase-space diagram. You should use at least 30 different start-parameter sets.

(c) Include friction, e.g. $\alpha = 0.1$. Again, study the evolution in coordinate space and in phase space.

(d) Include both friction and a driving force. Vary both f and ω to observe the phase space structures that were discussed in the lecture: *limit cycles, predictable attractors, strange attractors, chaotic paths, mode locking*. You could start with $\omega = 2.0$.

(e) Instead of a driving force $f \cos \omega t$ we now introduce a $\sin \theta$-dependent driving force,

$$\frac{d^2\theta}{dt^2} = -(\omega_0^2 + f \cos \omega t) \sin \theta - \alpha \frac{d\theta}{dt} \,,$$

which corresponds to a vibrating pivot or a variation of ω_0. Produce a bifurcation diagram by plotting against $0 < f < 2$ the magnitude of the velocity of the pendulum at $\theta = 0$ after a suitably long period (to wait for transients to die off). In the resulting diagram, you should see bifurcation structures, but not as cleanly as in the case of the logistic map.
Show that for e.g. $\omega = 20\omega_0$ the configuration where the pendulum points up can become stable (you could try $f = 30$).