1. **Quantum Wave Functions via Path Integrals: The Metropolis Algorithm (Extension)**

   Consider the task from the last exercise sheet and extend your program.

   (a) **Improved Action**
   Modify your program to use an improved action: Replace the derivative $\frac{\partial f(x)}{\partial x} \approx \frac{f_{i+1}-f_i}{a}$ in the lattice action by a derivative expanded to higher order:

$$\frac{\partial^2 f(x)}{\partial x^2} = \Delta_x^{(2)} f(x) - \frac{a^2}{12}\left(\Delta_x^{(2)}\right)^2 f(x) + \mathcal{O}(a^4)\,, \tag{1}$$

$$\Delta_x^{(2)} f(x) = \frac{f(x+a) + f(x-a) - 2f(x)}{a^2}\,. \tag{2}$$

   This is a formula for the *second derivative*. To use this formula you need to rewrite the action in terms of $x\ddot{x}$ instead of $\dot{x}^2$. This can be done by a trivial integration by parts since we assume that the wave function is the same at both ends of the lattice.
   Note: In the case of an interacting gauge field theory, additional corrections are introduced to take the UV cutoff into account.

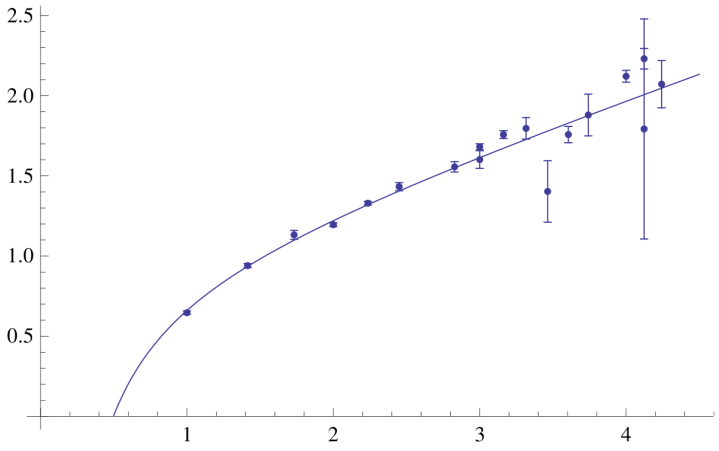
   (b) **The Improved Numerical Derivative**
   Use Mathematica to show that the factor $-1/12$ for the a^2 term in the above approximation is correct in the sense that it cancels the $\mathcal{O}(a^2)$ error produced by $\Delta_x^{(2)} f(x)$. Show also that to cancel the a^4 term, you would need to add $\frac{a^4}{90}\left(\Delta_x^{(2)}\right)^3 f(x)$.

   Hint: Remember how we derived improved numerical derivatives in CPI.

(c) **Lattice Gauge Theory***

You can find a detailed introduction into Lattice QCD in the publication of G. Lepage "Lattice QCD for Novices"[1]. An implementation of this approach to calculate gluons on a lattice by C. Lehner can be found in the moodle. Download and compile the programs. The `e7` program calculates the gluon-field configurations. As this takes some time, precalculated field configurations can be found in `/mount/share/lattice` on the CIP pool computers. The `e9` program calculates the $q\bar{q}$-potential from the gluon configurations as shown in the lecture. It reads the gluon configurations from folders called `wilson` or `improved` in the current working directory. The `e9` program calculates the potential for all four configurations (Wilson/Improved action, non-smeared/smeared links) discussed in the note. Produce a plot of the QCD potential and verify that it follows

$$V_{QCD}(r) = \frac{a}{r} + b \cdot r + c\,. \tag{3}$$



You can use the `plotPotential.py` skeleton to load the potentials calculated with `e9` from the corresponding output files.

[1] http://arxiv.org/pdf/hep-lat/0506036