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# Laser Technology – Exercise Sheet 2

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## 1 Solid angle and radiance

a) Which solid angle does the moon cover on the sky? Its diameter is 3476 km, its average distance from earth is 384 392 km.

The solid angle is defined as

$$\Omega = \frac{A}{R^2} \tag{1}$$

where $A$ is a surface area on a sphere with radius $R$. When looking from earth to the moon we approximately see a disc in the sky. Thus, we seek for the solid angle of a cone with opening angle $\theta_0$ which is depicted in Fig 1. In order to obtain the area $A$ we simply need to integrate over the cap of the cone marked in red in Fig. 1 which corresponds to the projection of the disc onto the sphere. In spherical coordinates the surface element is $\mathrm{d}A = R^2 \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi \, (= R^2\mathrm{d}\Omega)$ and we have

$$A = \int_{\theta=0}^{\theta_0}\int_{\phi=0}^{2\pi} R^2 \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi = R^2 2\pi \left(-\cos\theta\right)\Big|_0^{\theta_0} = R^2 2\pi\left[1-\cos(\theta_0)\right] \tag{2}$$

with the elevation or inclination angle $\theta \in [0, \pi]$ and the azimut angle $\phi \in [0, 2\pi]$. With equation (1) the solid angle of the cone is

$$\Omega = 2\pi\left[1-\cos(\theta_0)\right]. \tag{3}$$

The angle $\theta_0$ can be found by trigonometry and is $\tan(\theta_0) = r_\mathrm{M}/\mathrm{r}_\mathrm{EM}$ with the radius of the moon $r_\mathrm{M}$ and the distance earth to moon $r_\mathrm{EM}$ yielding $\Omega = 6.42 \times 10^{-5}$.
Since $r_\mathrm{M} \ll \mathrm{r}_\mathrm{EM}$ the opening angle of the cone is $\theta_0 \ll 1$ and we can write $\theta_0 \approx r_\mathrm{M}/\mathrm{r}_\mathrm{EM}$. In addition, we can use a Taylor expansion to approximate $\cos(\theta_0) \approx 1 - \theta_0^2/2$. By doing so we find

$$\Omega \approx 2\pi\left[1-\left(1-\frac{r_\mathrm{M}^2}{2r_\mathrm{EM}^2}\right)\right] = \frac{\pi r_\mathrm{M}^2}{r_\mathrm{RE}^2} = 6.42\times10^{-5} \tag{4}$$

and observe that the solid angle of a cone can be well approximated as the area of a disk over the squared distance $R^2$ when the opening angle is small, i.e. the curvature of the cap can be neglected.

b) Compute the radiance of an LED with a surface area of 1 mm$^2$ emitting an optical power of 1 mW uniformly into the half-space (i.e., 0° $\leq \theta \leq$ 90°). How big is the intensity and energy density measured in 1 m distance at $\theta$ = 0°?

A light source emitting uniformly (i.e., $L = const$) into the half-space ($\Omega = 2\pi$) is called a Lambertian source. We obtain

$$L = \frac{P}{2\pi A_{LED}} = 159.15\,\mathrm{W\,m^{-2}\,sr^{-1}}. \tag{5}$$

The emitted power in $\theta$=0° direction is $\Delta P = LA_{LED}\Delta\Omega$, where the solid angle $\Delta\Omega$ is given by $\Delta\Omega = \Delta A/r^2$, with $\Delta A$ being the (infinitesimal) reference area where the intensity is measured, and $r$=1 m. Therefore, the intensity can be calculated as

$$I = \frac{\Delta P}{\Delta A} = \frac{LA_{LED}}{\Delta A}\Delta\Omega = \frac{LA_{LED}}{\Delta A}\frac{\Delta A}{r^2} = \frac{LA_{LED}}{r^2} = 1.59\times10^{-4}\frac{W}{m^2}. \tag{6}$$

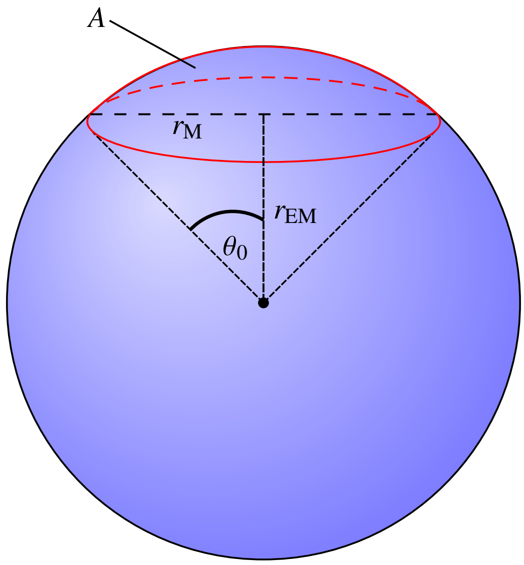


**Figure 1** Schematic of the cone with opening angle $\theta_0$.

The optical power hits the reference area $\Delta A$ perpendicularly, thus $I = wc/n$ (with $n$=1) and

$$w = I\frac{n}{c} = 5.31 \times 10^{-13}\,\mathrm{J\,m^{-3}}. \tag{7}$$

## 2 Thermal equilibrium

a) The energy levels of the hydrogen atom are given by $E_n$ = –13.6 eV$/n^2$, $n$ = 1, 2, . . . , and degeneracy $g_n = n^2$. For $T$ = 300 K, how big is the occupation probability for the levels $n$ = 1, 2 and 3?

The Boltzmann distribution is

$$p_n\left(T\right) = \frac{n^2}{Z\left(T\right)} \exp\left(-\frac{E_n}{k_{\mathrm{B}}T}\right),$$

with

$$Z\left(T\right) = \sum_m m^2 \exp\left(-\frac{E_m}{k_{\mathrm{B}}T}\right).$$

To avoid too extreme numbers (e.g., in the pocket calculator), we can also write

$$p_n\left(T\right) = \frac{n^2}{\sum_m m^2 \exp\left(\frac{E_n - E_m}{k_{\mathrm{B}}T}\right)}.$$

With $-13.6$ eV$= 2.179 \times 10^{-18}$ J and $k_{\mathrm{B}}T \approx 4.142 \times 10^{-21}$ J, we obtain $p_1 = 1$, $p_2 = 1.7774 \times 10^{-171} \approx 0$, $p_3 = 7.4141 \times 10^{-203} \approx 0$.

## 3 Mode density

a) Derive the 1-dimensional mode density in a resonator of length $L$ analogously to our derivation of the three-dimensional mode density (i.e., assume periodic boundary conditions, make plane wave ansatz, and count modes, assuming $L \to \infty$). Does the mode density change when assuming boundary conditions $\mathbf{E}(0) = \mathbf{E}(L) = 0$?

The wave equation in 1D is

$$\partial_z^2 \mathbf{E} = \frac{n^2}{c^2} \partial_t^2 \mathbf{E} \,. \tag{8}$$

We define periodic boundary conditions

$$\begin{aligned} \mathbf{E}(z,t) &= \mathbf{E}(z+L,t) \,, \\ \partial_z \mathbf{E}(z,t) &= \partial_z E(z+L,t) \,. \end{aligned} \tag{9}$$

We solve Eq. (8) with a plane wave ansatz:

$$\mathbf{E} = \mathbf{E}_0 \exp\left[\mathrm{i}\left(k_z z - 2\pi f t\right)\right] . \tag{10}$$

Inserting into Eq. (8) yields the condition

$$k_z^2 = \frac{n^2}{c^2} \left(2\pi f\right)^2 . \tag{11}$$

From the periodic boundary conditions Eq. (9), we obtain with $\Delta k = 2\pi/L$

$$k_z = n_z \Delta k \tag{12}$$

with $n_z = \pm 1, \pm 2, \ldots$, yielding

$$n_z = \pm \frac{Lnf}{c} \tag{13}$$

and a frequency spacing between two adjacent modes $n_z$ and $n_z + 1$

$$\Delta f = \frac{c}{Ln}. \tag{14}$$

The number of modes in a resonator with length $L$ and in a frequency interval $0..f$ is then

$$M = 2\frac{f}{\Delta f} = 2\frac{Ln}{c} f,$$

and the spectral mode density is

$$M_f = \frac{dM}{df} = 2\frac{Ln}{c}.$$

If we additionally take into account two orthogonal polarizations of the field, we get another factor of 2.

For assuming boundary conditions $\mathbf{E}(0) = \mathbf{E}(L) = 0$, we solve Eq. (8) with a standing wave ansatz

$$\mathbf{E} = \mathbf{E}_0 \sin\left(k_z z\right) \exp\left(-\mathrm{i}2\pi f t\right) , \tag{15}$$

and obtain again Eq. (11). The boundary conditions require $k_z$ values according to Eq. (12), but now with $\Delta k = \pi/L$ and $n_z = 1, 2, \ldots$, i.e., the spacing of the $k_z$ values is denser by a factor of 2. But now only positive values $n_z$ are allowed since a mode $n_z$ is identical to a mode $-n_z$ (apart from a "$-$" sign, see Fig. 2). We thus obtain again the same number of modes.

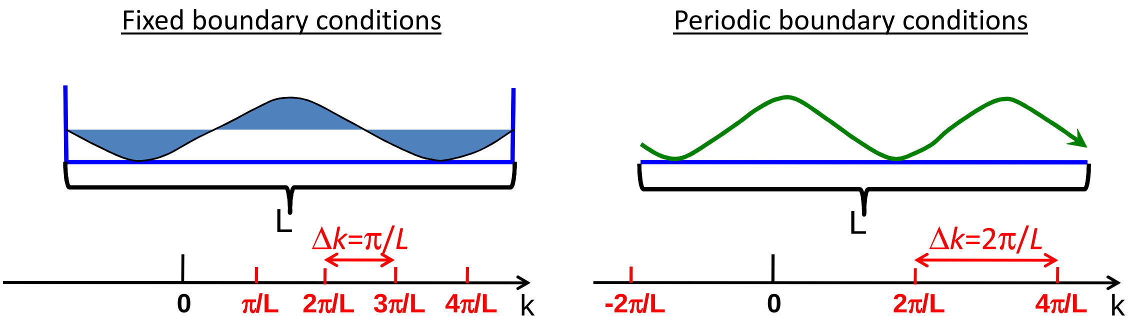


**Figure 2** Comparison of the modes for the different boundary conditions.