MTH 412: Functional Analysis Assignment

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Contents

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1 Normed Linear Space

Definition 1:

Let X be a vector space over the scalar field $K = \mathcal{R}$, then a function $\|.\| : X \to \mathcal{R}$, defined by

$$f : X \times X \to X$$

and

$$f : K \times X \to X$$

, called respectively addition and scalar multiplication, and are defined for arbitrary $x, y \in X, \lambda \in K$ then $x + y \in X$ and $\lambda.x \in X$ such that the following conditions are satisfied:

1. $x + y = y + x$ for all $x, y \in X$ (Commutative)
2. $(x + y) + z = x + (y + z)$ for all $x, y, z \in X$ (Associative)
3. There exists an element $0 \in X$ such that $x + 0 = x$ for all $x \in X$ (Identity)
4. For each $x \in X$, there exists an element $-x \in X$ such that $x + (-x) = 0$ (Inverse)
5. $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in K$ and $x, y \in X$ (Distributive)
6. $(\lambda + \mu)x = \lambda x + \mu x$ for all $\lambda, \mu \in K$ and $x \in X$ (Distributive)
7. $(\lambda\mu)x = \lambda(\mu x)$ for all $\lambda, \mu \in K$ and $x \in X$ (Associative)
8. $1.x = x$ for all $x \in X$ (Identity)

Then X is called a linear space over K or a vector space over K.If K is a set of real numbers X is called a real vector space. If K is a set of complex numbers X is called a complex vector space.

Definition 2:

Let x be a non-empty set, k be a scalar field ($K = \mathcal{R}$). Suppose that we have a function $\|.\| : X \to \mathcal{R}$, then $\|.\|$ is called a norm on X if it satisfies the following conditions:

1. $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in K$.
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. (Triangle Inequality)

Proof

Statement: Let $X = \mathbb{R}^2$ for arbitiary $\bar{x}, \bar{y}$,

where $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$, and with $\alpha \in \mathbb{R}$ define the operation, addition and scalar multiplication as:

$$\bar{x} + \bar{y} = (x_1 + y_1, x_2 + y_2)$$

scalar multiplication:

$$\alpha\bar{x} = (\alpha x_1, \alpha x_2)$$

with these definitions, $\mathbb{R}^2$ is a vector space. for each $\bar{x} \in X$, we define the maximum norm

$$\|\bar{x}\|_\infty = \max\{|x_1|, |x_2|\}$$

is a normed on $\mathbb{R}^2$.

N1:

$$\begin{aligned} \|x\|_\infty \geq 0 \quad &\text{for all} \quad x \in \mathbb{R}^2 \\ \|x\|_\infty = \max\{|x_1|, |x_2|\} \geq 0 \quad &\text{for all} \quad x \in \mathbb{R}^2 \\ \|x\|_\infty = \max\{|x_1|, |x_2|\} = 0 \quad &\text{if and only if} \quad x = 0 \end{aligned}$$

N2:

$$\|\alpha x\|_\infty = \max\{|\alpha x_1|, |\alpha x_2|\} = |\alpha| \max\{|x_1|, |x_2|\} = |\alpha|\|x\|_\infty$$

N3:

Proof:

Let $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ be arbitrary elements of $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$,
$\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$, then,

$$\begin{aligned} \|x + y\|_\infty = & \\ &= \|x_1 + y_1, x_2 + y_2\| \\ &= \max\{|x_1 + y_1|, |x_2 + y_2|\} \\ &= \max_{1<i<2}(\|x_i\| + \|y_i\|) \\ &\leq \max_{1<i<2}(\|x_i\|) + \max(\|y_i\|) \\ &\leq \|\bar{x}\|_\infty + \|\bar{y}\|_\infty \\ &= \|x + y\|_\infty \leq \|\bar{x}\|_\infty + \|\bar{y}\|_\infty \end{aligned}$$

Q2: Prove that X = $\mathbb{R}$ is a Normed Space

Statement: Let $X = \mathbb{R}$ for arbitrary $\bar{x}$,,

where $\bar{x} = (x_1, x_2, x_3, x_4, \dots x_n) \in \mathbb{R}^n$, and with $\alpha \in \mathbb{R}$, then $\|\bar{x}\|_p = \|(x_1, x_2, x_3, x_4, \dots x_n)\|_p$
$= (\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}}$ verify that $\|.\|$ is a norm.

Proof:

N1: $\|x\| > 0$, iff $x = 0$

$(\sum_{i=1}^{n} \|x_i\|^{p})^{\frac{1}{p}} \geq 0 \|\bar{x}\|_p \geq 0$ clearly because absolute value of any value is greater than or equal to 1

Also,

if $\bar{x} = 0 \implies \forall 1 < i < n \implies (\sum_{i=1}^{n} \|x_i\|^{p})^{\frac{1}{p}} = 0$

N2: $\|\alpha\bar{x}\|_p = |\alpha|\|\bar{x}\|$

$$\begin{aligned}
\|\alpha\bar{x}\|_p = \|\alpha(x_1, x_2, x_3, x_4, \ldots x_n)\|_p = (\sum_{i=1}^{n} \|x_i\|^{p})^{\frac{1}{p}} \\
= \left(|\alpha|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \|x_i\|^{p}\right)^{\frac{1}{p}} \\
= |\alpha| \left(\sum_{i=1}^{n} \|x_i\|^{p}\right)^{\frac{1}{p}} \\
= |\alpha|\|\bar{x}\|_p
\end{aligned}$$

N3: $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

We need Holder's inequality

$$\begin{aligned}
\|\bar{x} + \bar{y}\|_p^p = \|(x_1 + y_1) + (x_2 + y_2) + \cdots + (x_n + y_n)\| \\
= \sum_{i=1}^{n} |x_i + y_i|^p = \sum_{i=1}^{n} |x_i + y_i|^{p-1}|x_i + y_i| \\
Applying Holder's Theorem \\
\leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1} \\
\leq \left[(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)})^{\frac{1}{q}} \right] + \left[(\sum_{i=1}^{n} |y_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)})^{\frac{1}{q}} \right] \\
\leq \left(\|x\|_p + \|y\|_p \right) \left(\sum_{i=1}^{n} |x_i + y_i| \right)^{\frac{p}{q}} \\
\implies \|\bar{x} + \bar{y}\|_p = \|\bar{x}\|_p + \|\bar{y}\|_q
\end{aligned}$$

Recall in Holder's inequality: $\frac{1}{p} + \frac{1}{q} = 1$,1 - $\frac{1}{p} = \frac{p-1}{p} = 1$

$p = q(p-1)$

2 Completeness of Normed Linear Space

Recall that if $(E, ||.||)$ is a normed linear space, the norm $||.||$ always induces a matrix ρ on E given by:

$\rho(x, y) = ||x - y||, \forall x, y \in E$, with this it is clear that (E, ρ) becomes a matrix space.

Recall also that a sequence x_n in matrix space (E, ρ) is said to be Cauchy if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\rho(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

3 Linear maps and Functionals

Definition 1:

Let X and Y be two vector spaces over the same scalar field K. A function $T : X \to Y$ is called a linear map or linear transformation if it satisfies the following conditions:

1. $T(x + y) = T(x) + T(y)$ for all $x, y \in X$ (Additivity)

2. $T(\lambda x) = \lambda T(x)$ for all $\lambda \in K$ and $x \in X$ (Homogeneity)

   which is also interpreted as
   $T(\lambda x + \mu y) = \alpha T(x) + \beta T(y)$
   for all $\lambda, \mu \in K$ and $x, y \in X$.

Definition 2:

If in Definition 1, the linear space Y is replaced by the scalar field K, then T is called a linear functional on X. $i.e$ $T : X \to K$

Example 1:

Let $X = C[a, b]$ as a space of all real-valued continuous functions on $[a, b]$. Define $T : X \to \mathcal{R}$ by $Tf(t) = \int_1^0 f(x)dx$ for all $f \in X$. Then T is a linear functional on X.

Solution:

Let $f, g \in X$ and $\lambda \in \mathcal{R}$, then

$T(\mu f + \lambda g) = \int_1^0 \mu(f(x) + \lambda g(x))dx = \int_1^0 \mu f(x)dx + \lambda \int_1^0 g(x)dx = \mu Tf + \lambda Tg$

$\implies T$ is a linear functional on X.

Exercise

1. Let $X = l_2$, and for each $x = (x_1, x_2, \dots) \in X$, define $T(x) = (0, x, x_2, x_3, \dots)$. Show that T is a linear map on X.

2. Let X, Y be two vector spaces over the same scalar field K. Show that $T : R \to R$ is a linear function, then $T : R \to R$ be given by $T(x) = 3x + 2$.

Theorem 1:

Let X and Y be two normed linear spaces over the same scalar field K. If $T : X \to Y$ is a linear map, then:

1. $T(0) = 0$
2. The range of T is given as $R(T) = \{y \in Y : y = T(x) \text{ for some } x \in X\}$ is a subspace of Y.
3. T is onto one iff $T(0) = 0 \implies x = 0$.
4. If T is one to one, then $T^{-1} : R(T) \to X$ is a linear map, and T^{-1} exists in the range $R(T)$

Proof:

1. Since T is a liinear map, then $T(\alpha x) = \alpha T(x), where \alpha = 0$, then $T(\alpha.0) = 0$, hence $T(0) = 0$.
2. We need to show that

Bound ...

Definition 3:

Let X, Y, be two normed linear spaces over the same scalar field K. A linear map $T : X \to Y$ is said to be bounded if there exists a constant $K \geq 0$ such that $x \in X$ is given by:
$\|T(x)\| \leq K\|x\|, \forall x \in X$

Theorem 2:

Let X, Y be two normed linear spaces over the same scalar field K. If $T : X \to Y$ is a linear map, then the following statements are equivalent:

1. T is continuous
2. T is continuous at 0(origin) $i.e$ if $x_n \in X$, such that $Tx_n \to 0, n \to 0$.
3. T is lipshitz $i.e$ $\exists$ a constant $k \geq 0$, such that $\forall x \in X$ on $\|T(x)\| \leq K\|x\|, \forall x \in X$.
4. if $D = \{x \in X : \|x\| \leq 1\}$ is a closed unit disc in X, then $T(D)$ is bounded. That is $\exists M \geq 0$ such that $\|T(x)\| \leq M$ for all $x \in D$.

Proof:

1. $1 \implies 2$:
   Let T be continuous at 0, then for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|x\| < \delta \implies \|T(x)\| < \epsilon$.
   Let $x_n \in X$ such that $x_n \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$ as $n \to \infty$.
   Hence, $\|T(x_n)\| \to 0$ as $n \to \infty$.
   $\implies T$ is continuous at 0.

2. $2 \implies 3$:
   Let T be continuous at 0, then for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|x\| < \delta \implies \|T(x)\| < \epsilon$.
   Let $x \in X$ be arbitrary, then $\|x\| < \delta \implies \frac{\|x\|}{\delta} < 1$.
   $\implies \|T(x)\| = \|T(\frac{\|x\|}{\delta}.\delta)\| = \frac{\|x\|}{\delta}.\|T(\delta)\| < \epsilon$.
   $\implies \|T(x)\| \leq \frac{\|x\|}{\delta}.\|T(\delta)\| < \epsilon$.
   $\implies \|T(x)\| \leq K\|x\|$, where $K = \|T(\delta)\|$.
   $\implies T$ is lipshitz.

3. $3 \implies 4$:
   Let T be lipshitz, then $\|T(x)\| \leq K\|x\|$, for all $x \in X$.
   Let $D = \{x \in X : \|x\| \leq 1\}$ be a closed unit disc in X, then $T(D)$ is bounded.
   $\implies \|T(x)\| \leq K\|x\| \leq K$, for all $x \in D$.
   $\implies T$ is bounded.

4. $4 \implies 1$:

5. $1 \implies 2$:

4 Hilbert Spaces

Definition 1:

Let E be a linear space, an inner product on E is a function $< ., . >: E \times E \to \mathcal{C}$ with values in $\mathcal{C}$ such that the following three conditions are satisfied:

1. $< x, x > \geq 0$ for all $x \in E$ and $< x, x >= 0$ if and only if $x = 0$ (Positive Definiteness)

2. $< x, y >= \overline{< y, x >}$ for all $x, y \in E$ (Conjugate Symmetry)

3. $< \lambda x + \mu y, z >= \lambda < x, z > + \mu < y, z >$ for all $x, y, z \in E$, $\lambda \in \mathcal{C}$

x,y,z $\in E; \lambda, \mu \in \mathcal{C}$

The pair $(E, < ., . >)$ is called an inner product space.

Hilbert Space

A complete inner product space is called a Hilbert Space.

Example 1:

The Linear space $\mathcal{R}^n$ with $<, >$ defined for arbitrary vector $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ with $x, y \in \mathcal{R}^n$,

$< x, y >= \sum_{i=1}^{n} x_i.y_i$

Proof:

1. $< x, x >= \sum_{i=1}^{n} x_i.x_i = \sum_{i=1}^{n} x_i^2 \geq 0$ for all $x \in \mathcal{R}^n$ and $< x, x >= 0$ if and only if $x = 0$.

2. $< x, y >= \sum_{i=1}^{n} x_i.y_i = \sum_{i=1}^{n} y_i.x_i =< y, x >$ for all $x, y \in \mathcal{R}^n$.

3. $< \lambda x + \mu y, z >= \sum_{i=1}^{n} (\lambda x_i + \mu y_i).z_i = \lambda \sum_{i=1}^{n} x_i.z_i + \mu \sum_{i=1}^{n} y_i.z_i = \lambda < x, z > + \mu < y, z >$ for all $x, y, z \in \mathcal{R}^n$ and $\lambda, \mu \in \mathcal{R}$.

Basic Properties of Linear Product Space

from (Def 1), the immediate consequence of I_2 and I_3 is that for arbitrary $x, y, z \in E; \lambda, \mu \in C$.

$$\Big\langle z, \lambda x + \mu y \Big\rangle = \overline{\Big\langle z, \lambda x + \mu y \Big\rangle}$$
$$= \overline{\Big\langle (x, z)\lambda + \mu(y, z) \Big\rangle}$$
$$= \overline{\lambda}\Big\langle x, z \Big\rangle + \overline{\mu}\Big\langle y, z \Big\rangle$$

Proposition 1:

Cauchy-Schwarz Inequality: Let $(E, <.,.>)$ be an inner product space, then for all $x, y \in E$,we have:

$| < x, v > |^2 \leq \ < x, x > . < y, y >$ and $| < x, v > |^2 = \ < x, x > . < y, y >$, if and only if x and y are linearly dependent.

Proof:

Let $x, y \in E$, then for arbitrary $z \in \mathcal{C}$, we have: $|z| = 1$, such that $z < x, y >= |z| < x, y > = 1. < x, y >$,

set $a =< x, x >, b =< x, y >$ and $c =< y, y >, then for arbitiary scalar t \in \mathcal{R}, we obtain :$
$< tzx + y, tzx + y >\leq 0$
$\implies t^2 z\bar{z} < x, x > +tz < x, y > +t\bar{y,} z < y, y >\geq 0$
$\implies t^2 < a > +2t < zx, y > + < y, y >\leq 0$
$\implies t^2 a + 2tb + c \leq 0$
from theory if quadratic equation, we have that $b^2 \leq ac \implies < x, y >^2 \leq \ < x, x > . < y, y >$
$\implies | < x, y > |^2 \leq < x, x > . < y, y >$

Parallelogram Law

Let $(E, <.,.>)$ be an inner product space, then for all $x, y \in E$, we have:

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$$

Consequence: Polarization Identity

Let $(E, <.,.>)$ be an inner product space, then for all $x, y \in E$, we have:

$$< x, y >= \frac{1}{4}\Bigg(||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2 \Bigg)$$

Theorem:

Let E be a Hilbert space,and K be a closed convex subset of E, then K contains a unique vector of minimum norm.

Proof:

Let $\gamma = \ \inf |x| : x \in K$, choose a sequence $x_{n_{n=1}}^{\infty}$ in K such that $||x_n|| \to \gamma$ as $n \to \infty$.using Parallelogram law and the convexity of K, we have the following estimate:

$$||x_n - x_m||_2 = \frac{1}{2}\Bigg(||x_n||^2 + ||x_m||^2 \Bigg) - 4||\frac{x_n + x_m}{2}||^2 \to 0, m, n \to \infty$$

since $\frac{x_n + x_m}{2} \in K$, so $||\frac{x_n + x_m}{2}||^2|| \geq \gamma$

Hence the sequence $x_{n_{n=1}}^{\infty}$ is a Cauchy sequence in K and since K is closed, thus the sequence has a limit say $x^* \in K$. Observe that $||x^*|| = ||lim x_n|| = \lim ||x_n|| = \gamma; u \in k, u \neq x^*$ and $||u|| = \gamma$, then $||x^* - u||^2 = 4\gamma^2 - 4||\frac{x^* + u}{2}||^2 \geq 0$

$4\gamma^2 \geq 4||\frac{x^*+u}{2}||^2$ $||\frac{x^*+u}{2}|| \leq \gamma$, and since the convexity of K, we have $\frac{x^*+u}{2} \in K$.Then this is a contradiction, hence $u = x^*$

Norms from Inner Product

let $\phi \neq V$, be a linear space, but necesarily a normed space. Let $< .,. >$ be an inner product on V, then for all $x \in V$, we define a function $\mu :$ V X V $\to \mathcal{R}$.
$\rho(x,y) = \begin{cases} 1 & x \neq 0 \\ 0 & x = y \end{cases}$ as a metric on V. if V is a normed linear space, the norm $||.||$ can also induce a metric on V given by $\rho(x,y) = ||x-y||$ for all $x, y \in V$.It can be shown that a a metric ρ on a metric space M, is induced if the following conditions are satisfied:

1. $\rho(x,y) \geq 0$ for all $x, y \in M$ and $\rho(x,y) = 0$ if and only if $x = y$.
2. $\rho(x,y) = \rho(y,x)$ for all $x, y \in M$.
3. $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ for all $x, y, z \in M$.

Theorem: Jordan-Von Neumann

The norm of a normed linear space is given by an inner product if and only if it satisfies the parallelogram law. that is for all $x, y \in V$, we have: $||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$

Proof:

$(\implies)$: Assume that the norm is given by an inner product, $||x||^2 =< x, x >$, then for all $x, y \in V$, then the parallelogram law is satisfied.
$||x+x||^2 + ||x-x||^2 = 2(||x||^2 + ||x||^2)$
$||2x||^2 = 4||x||^2$
$4||x||^2 = 4||x||^2$

$(\impliedby)$: Assume that the norm satisfies the parallelogram law, then for all $x, y \in V$, we have: $||x||^2 =< x, x >$ Define for arbitrary $x, y \in E$, $< x, y >= \frac{1}{4}\Bigg(||x+y||^2 - ||x-y||^2 + i||x+iy||^2 - i||x-iy||^2\Bigg)$ Observe immediately that $< x, x >= ||x||^2$ for all $x \in V$. is satisfied by definition of the above, and we that is indeed an inner product on V.

1. I_1: $< x, x >= ||x||^2 \geq 0, and ||x||^2 = 0,$ iff $, x = 0$
2. I_2: if $< x, y >= < \bar{y, x} >$ $< x, y >= \frac{1}{4}\Bigg(||x+y||^2 - ||x-y||^2 + i||x+iy||^2 - i||x-iy||^2\Bigg)$
   $= \frac{1}{4}\Bigg(||y+x||^2 - ||y-x||^2 + i||y+ix||^2 - i||y-ix||^2\Bigg)$
   $= \frac{1}{4}\Bigg(||y+x||^2 - ||y-x||^2 - i||y+ix||^2 + i||y-ix||^2\Bigg)$
   $= < \bar{y, x} >$

3. I_3: if $<\lambda x+\mu y, z>=\lambda<x, z>+\mu<y, z>$ Consider for arbitrary $x, y, z \in E$, and a complex scalar $\lambda \in \mathcal{C}$, we have the following: $<x+z, y>=<x, y>+<z, y>$
$<\lambda x, y>=\lambda<x, y>$
observe that using equation 4, and considering the real and imaginery part seperately, we have: $Re<x+y, z>+Re<x-y, z>=\frac{1}{4}\Big\{||x+y+z||^2-||x+y-z||^2+||x-y+z||^2-||x-y-z||^2\Big\}$
expand:
$\frac{1}{4}\Big\{2||x+z||^2-2||x-z||^2\Big\}$
$=2Re<x, z>$
Hence,
$Re<x+y, z>+Re<x+y, z>=2Re<x, z>$
replace y by x: $Re<2x, z>+Re<0, z>=2Re<x, z>$
from the two result, we have:
$Re<x+y, z>+Re<x-y, z>=Re<2x, z>$
Similarly,replacing x by $\frac{1}{2}(x+y)$and y by $\frac{1}{2}(x-y)$
$Re<x, y+z>+Re<x, y-z>=Re<x, 2y>$
becomes
$Re<x, z>+Re<y, z>=Re<x+y, z>$
Similarly, for the imaginery part, we have: $Im<x+y, z>+Im<x-y, z>=Im<2x, z>$
Similarly,replacing x by $\frac{1}{2}(x+y)$and y by $\frac{1}{2}(x-y)$
$Im<x, z>+Im<y, z>=Im<x+y, z>$
Hence, we have: $<x+y, z>=<x, z>+<y, z>$ By induction for any positive integer n, we have: $<nx, Z>=n<x, z>$
$<-x, y>=-<x, y>$ and from definition, $<nx, y>=n<x, y>$

Orthonormal Set

Definition 1:

Two vectors $x, y \in E$ are said to be orthogonal if $<x, y>=0$. A set $S \subset E$ is said to be orthogonal if $<x, y>=0$ for all $x, y \in S$ with $x \neq y$, and is written as $x \perp y$., if the inner product is equal to zero. Since $<x, y>=<\bar{y, x}>$, it follows that $x \perp y$ if and only if $y \perp x$.Further more $x \perp x$ if and only if $x=0$.

Definition 2:

A set S in an inner product space E is called an orthogonal set if $x \perp y$ for all $x, y \in S$ with $x \neq y$. A set S is called an orthonormal set if S is an orthogonal set and $||x||=1$ for all $x \in S$. example: space l_2

Bessel Inequality

if $\{U_i\}_{i=1}^{\infty}$ is an orthonormal set in an inner product space E, then for all $x \in E$, we have: $\sum_{i=1}^{\infty} | < x, u_i > |^2 \leq ||x||^2$ furthermore, if $\sum_{i=1}^{\infty} | < x, u_i > |^2 = ||x||^2$, then $x = \sum_{i=1}^{\infty} < x, u_i > u_i$

The Projection Theory

Using the concept of orthogonality, we shall extend the well known elementary fact, that the shortest distance from a point to a line is the perpendicular distance from the point to the line. Let E be an inner product space, and M be a closed subspace of E. For each $x \in E$, we define the distance from x to M by: $||x - m^*|| = inf\{||x - m|| : m \in M\}$ Then m^* is unique, in fact m^* is the unique vector in M such that if and only if $x - m^* \perp M$. The vector m^* is called the projection of x on M.

Proof:

$(\Longrightarrow)$: Let $m^* \in M$ be the unique, assume for contradiction that this is not the case, then $\exists 0 \neq m_0 \in M$, that is not orthogonal to $(x - m^*)$. without loss of generality, we may assume that $||m_0|| = 1$, then $< x - m^*, m_0 > \neq 0$.since m_0 is not orthogonal to $(x - m^*)$, we let the inner product $< x - m^*, m_0 >= \alpha \neq 0$. Now, consider the vector $m^* + \lambda m_0$, where $\lambda \in \mathcal{R}$, then:

$$||x - (m^* + \lambda m_0)||^2 = ||x - m^*||^2 + \lambda^2 ||m_0||^2 - 2\lambda < x - m^*, m_0 > \ = ||x - m^*||^2 + \lambda^2 - 2\lambda\alpha$$

This contradicts the hypothesis that m^* is the unique vector in M such that $x - m^* \perp M$. Hence, m^* is unique.
$(\Longleftarrow)$: Let m^* be the unique vector in M such that $x - m^* \perp M$, then for all $m \in M, m \neq m^*$, we compute $||x - m||$

$$||x-m||^2 = ||x-m^*+m^*-m||^2 = ||x-m^*||^2+||m^*-m||^2+2 < x-m^*, m^*-m > \ = ||x-m^*||^2+||m^*-m||^2+2$$

Hence, $||x - m^*||$ is the minimum distance from x to M.

Theorem:The Projection Theorem

Let H be a Hilbert space, and M be a closed subspace of H. Then for each $x \in H$, there exists a unique vector $m^* \in M$ such that $x - m^* \perp M$.There exists a unique vector $m^* \in M$ such that $||x - m^*|| \leq ||x - m||$. furthermore m^* is the unique vector in M if and only if $(x - m^*) \perp M$.

Proof of Projection Theorem

We shall consider only the existence of a minimum vector m^*, for the uniqueness we say if $x \in M$, then choose $m^* = x$, then there's nothing to prove.

Assume that $x \neq m$, define $\delta = inf\{||x - m|| : m \in M\}$, we need to generate $m^* \in M$ with $||x - m^*|| = \delta$. Let $\{m_n\}_{n=1}^{\infty}$ be a sequence in M such that $||x - m_n|| \to \delta$ as $n \to \infty$.

Since M is closed, then $m^* = limm_n \in M$, and $||x - m^*|| = \delta$. We need to show that $x - m^* \perp M$. Let $m \in M$, then: By Parallelogram Law

$$||(m_i - x) + (x - m_j)||^2 + ||(m_i - x) - (x - m_j)||^2 = 2(||m_i - x||^2 + ||x - m_j||^2)$$

Rearranging, we have:

$$||m_i - m_j||^2 = 2(||m_i - x||^2 + ||x - m_j||^2) - 4||x - \frac{m_i + m_j}{2}||^2$$

$$\text{Since } m_i, m_j \in M, \text{ then } \frac{m_i + m_j}{2} \in M$$

Since m is a linear subspace, hence by the definition of δ $||m_i - m_j||^2 = 2(||m_i - x||^2 + ||x - m_j||^2) - 4\delta^2$

Taking the limit as $i, j \to \infty$ Hence the sequence $m_{n_{n=1}}^{\infty}$ is a Cauchy sequence in M, and since M is closed, then $m^* = limm_n \in M$, and $||x - m^*|| = \delta$. We need to show that $x - m^* \perp M$. Let $m \in M$. It follows that $x - m_j \to x - m_i$ as $j \to \infty$, so that by the uniqueness of the limit, we obtain $||x - m^*|| = \delta = \inf\{||x - m|| : m \in M\}$.
The Consequence of projection Theorem is the "Direct Sum"

Direct Sum

Let E be a vector space, E is said to be a direct of two subspaces E_1 and E_2 if $E_1 \cap E_2 = \{0\}$ and $E_1 + E_2 = E$. We write $E = E_1 \oplus E_2$. If E is a direct sum of two subspaces E_1 and E_2, then for each $x \in E$, there exists a unique pair (x_1, x_2) such that $x = x_1 + x_2$, where $x_1 \in E_1$ and $x_2 \in E_2$. The pair (x_1, x_2) is called the decomposition of x with respect to the direct sum $E = E_1 \oplus E_2$.

Definition:

For a Hilbert space H, let M be a closed subspace of H, then H is said to be the orthogonal direct sum of M and $M^{\perp}$, and is written as $H = M \oplus M^{\perp}$.

Lemma:

Let X_1 and X_2 be two complete Orthonormal sets with inner product space E, then X_1 and X_2 have the same cardinality.

Cardinality

The dimension of innner product space is the cardinality of any complete orthonormal set in B.

Definition

Let X, Y be normed linear spaces, a linear map $T : X \to Y$ is said to be isometric if it is norm preserving, that is $||Tx|| = ||x||$ for all $x \in X$.
The Linear Map T is called an isometric Isomorphism of X, Y if:

1. T is injective

2. $||Tx|| = ||x||$

If T is a bijective isometric isomorphism of X, Y, then X and Y are said to be isometrically isomorphic, and we write $X \cong Y$.

Adjoint Operators on Hilbert Spaces

Definition

Let H be a Hilbert space, and $T : H \to H$ be a linear map, then there exists a unique linear map $T^* : H \to H$ such that $< Tx, y >=< x, T^*y >$ for all $x, y \in H$. The map T^* is called the adjoint of T.

Definition: Self, Adjoint,and Unitary Operators

Let H be a Hilbert space, and $T : H \to H$ be a linear map, then T is said to be:

1. Self Adjoint if $T = T^*$

2. Unitary if $T^*T = I$

3. Normal if $TT^* = T^*T$