Math Course Report Sheet

Your Name

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Course Information

- Course Name: Math 411
- Instructor: Prof. J.O Fatokun
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Synopsis

- Existence and Unique of Solutions of Differential Equations
- Lipshitiz Condition
  Continuity
  Stability
- Reduction of n_{th} order ODE to a system
- Linear System

1 Lipshitiz Condition

Definition 1: A function $f(x, y)$ is said to satisfy a Lipshitiz condition at x $\in (a, b)$, if there exist a constant $M > 0$ and $\epsilon > 0$ such that $|x - y| \leq \epsilon$ and $y \in (a, b)$, then $|f(x, y) - f(x, y)| \leq M|x - y|$

where M is called the Lipshitiz constant and is generally represented as L.

Definition 2: A function $f(x, y)$ issaid to satisfy a Lipshitiz condition in a vaiable Y on a rectangle $R : a \leq x \leq b, c \leq y \leq d$ if there exist a constant $L > 0$ such that $|f(x, y) - f(x, y)| \leq L|y - y|$ for all (x, y) in R.

Example 1:

Show that the function $f(x, y) = \frac{2y}{x} + x^2e^x$ satisfy a Lipshitiz condition in the variable y on the rectangle $D : 1 \leq x \leq 2$.

Solutions

Let there exist two arbitiary point $\in$ D, i.e $(x, y_1) and (x, y_2)$, then

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| &= \left| \frac{2y_1}{x} + x^2e^x - \frac{2y_2}{x} - x^2e^x \right| \\ &= \left| \frac{2y_1 - 2y_2}{x} - x^2e^x \right| \\ &= \frac{2|y_1 - y_2|}{x} + |x^2e^x| \\ &\leq \frac{2|y_1 - y_2|}{1} + x^2e^x \\ &= 2|y_1 - y_2| + x^2e^x \\ &= L|y_1 - y_2| \quad (\text{where } L = 2 + x^2e^x) \end{aligned} \tag{1}$$

Hence $L = 2$

Definition 3: A set D, $D \subseteq R^2$, is said to be convex if whenever two point (x_1, y_1) and (x_2, y_2) belongs to D, $\exists \lambda \in [0, 1]$, then the point $(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2)$ also belongs to D.

Theorem 1:

Suppose that $f(x, y)$ is defined on a convex set $D \subset R^2$, if a constant $L > 0$, exist with jacobian $|f_y(x, y)| \leq L$ for all $(x, y) \in D$, then $f(x, y)$ satisfies a Lipshitiz condition in the variable y on D with Lipshitiz constant L.

1.0.1 Existence and Uniqueness of Solutions of Differential Equations

Theorem 2:

Suppose that domain $D = (x, y) : a \leq x \leq b, -\infty < y < \infty$ and f(x,y) is continous in D and satisfies a Lipshitiz condition in the variable y on D with Lipshitiz constant L. Then the initial value problem:

$$y^{'} = f(x, y), y(\alpha) = \beta, a \leq x \leq b, \tag{2}$$

has a unique solution $y(x) \forall x \in [a, b]$

Example 2:

Show that their is a unique solution to the following I.V.P, using the lipshitz condition approach.
$y^{'} = 1 + xsin(xy); 0 \leq x \leq 2; y(0) = 0$

Solution:

Let $y_1, y_2 \in D$, Using mean value Theorem

$$\frac{f(x, y_2) - f(x, y_1)}{y_2 - y_1} = \frac{\partial f(x, \epsilon)}{\partial y}, \epsilon \in (y_1, y_2) \tag{3}$$

$$\begin{aligned} y\prime = 1 + xsin(xy); 0 \leq x \leq 2; y(0) = 0 \\ \frac{\partial f(x, \epsilon)}{\partial y} = x^2 cos(\epsilon x) \\ f(x, y_2) - f(x, y_1) = x^2 cos(\epsilon x)(y_2 - y_1) \\ x = 2 : \\ |f(x, y_2) - f(x, y_1)| = 2^2 |y_2 - y_1| \end{aligned}$$

Well-Posedness

Equation (2) is said to be well posed problem if

1. A unique solution y(x) to the problem exist
2. if there exist a constant $\epsilon_0 > 0$ and $k > 0$ such that for any ϵ satisfy the condition $\epsilon_0 > \epsilon > 0$, whenever a $\gamma(x)$ is continous with $|\gamma(x)| < \epsilon, \forall x \in [a, b]$, and when $|\gamma_0| < \epsilon$, the ivp $\frac{dz}{dx} = f(x, z) + \delta(x), a \leq x \leq b; z(a) = a + \delta_0$ has a unique solution $z(x)$ that satisfies $|z(x) - y(x)| < k\epsilon; \forall x \in [a, b]$

Theorem 3:

Suppose $D = (x, y) : a \leq x \leq b$ and $-\infty < y < \infty$,and that $f(x, y)$ is continous in D and satisfies Lipshitiz condition on D in a variable y, then the I.V.P,

2 General Theory of ODEs

Definition

A n_{th} of order ODE is a functional relationship taking the form

$$F\left(x, y(x), \frac{dy(x)}{dx}, \frac{dy^2(x)}{dx^2}, \cdots, \frac{dy^n(x)}{dx^n}\right) = 0 \tag{4}$$

that involves an independent variable $x \in I \subset R$ and unknown function $y(x) \in D \subset R^n$ of the independent variable, it derivates and derivatives up to n. For simplicity the time-dependence of y is often omitted and (4) becomes

$$F\left(x, y, y^{'}, y^{''}, \cdots, y^n\right) = 0 \tag{5}$$

Normal/Explicit ODE Form

An equation of type (4) is said to be normal or in explicit form when it is written in the form

$$y^n = F\left(x, y, y^{'}, y^{''}, y^{'''}, \cdots, y^{n-1}\right) \tag{6}$$

otherwise they are called implicit form.

Consider (6) in nominal form

$$y^{'} = F(x, y) \tag{7}$$

is referred to as First Order ODE.

Initial Value Problem

An initial value problem for (7) is given by

$$y^{'} = F\left(x, y\right), y(x_0) = y_0 \tag{8}$$

where F is continuous and real valued on a set $U \subset R \times R^n$ with $(x_0, y_0) \in U$.

An IVP for a nth order ODE takes the form

$$\begin{gathered} y^n = F\left(x, y, y^{'}, y^{''}, y^{'''}, \cdots, y^{n-1}\right) \\ y(x_0) = y_0, y^{'}(x_0) = y_0^{'}, y^{''}(x_0) = y_0^{''} \ldots y^{n-1}(x_0) = y_0^{n-1} \end{gathered} \tag{9}$$

Solutions to an ODE

A function $\phi(x)$ is said to be a solution to (7) if it satisfies this equation

$$\phi^{'}(x) = f(x, \phi(x)), \forall x \in I \subset R \tag{10}$$

an open interval $(x, \phi(x)) \in U, \forall x \in I$

Integral Form of Solution

The function

$$\phi(x) = y_0 + \int_{x_0}^{x} f(s, \phi(s))ds \tag{11}$$

is called an Integral Form of Solution to (8)

Reduction of Higher ODE to a System of First order ODE

To Reduce an nth order ODE to an equivalent first order system, we shall make some informed representation of the system and eventually make ...

Example 1:

Reduce $\frac{d^3y}{dx^3} + y^2 = 1$

Solution:

let:

$$\begin{aligned} y_1 &= y \\ y_2 &= y^{'} \\ y_3 &= y^{''} \end{aligned}$$

then:

$$\}$$

Thus the system

3 Existence of Solutions of Ordinary Differential Equations

Consider equation (8), The existence of solution if (8) will be obtained in relation the domain R by considering a subset of the time interval

$$|x - x_0| \le a$$

defined by

$$|x - x_0| \le \alpha$$