MTH 412: Functional Analysis Assignment

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1 Q1: Prove the third condition for Normed Space X $= \mathbb{R}^2$

Statement: Let $X = \mathbb{R}^2$ for arbitiary $\bar{x}, \bar{y}$,

where $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$, and with $\alpha \in \mathbb{R}$ define the operation, additon and scalar multiplication as:
additon:

$$\bar{x} + \bar{y} = (x_1 + y_1, x_2 + y_2)$$

scalar multiplication:

$$\alpha\bar{x} = (\alpha x_1, \alpha x_2)$$

with these definitions, $\mathbb{R}^2$ is a vector space. for each $\bar{x} \in X$, we define the maximum norm

$$\|\bar{x}\|_\infty = \max\{|x_1|, |x_2|\}$$

is a normed on $\mathbb{R}^2$.
Prove that the third condition for normed space is satisfied.

Proof:

Let $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ be arbitrary elements of $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$,
N3: $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$, then,

$$\|x + y\|_\infty =$$

$$\begin{aligned}
&= \|x_1 + y_1, x_2 + y_2\| \\
&= \max\{|x_1 + y_1|, |x_2 + y_2|\} \\
&= \max_{1<i<2}(\|x_i\| + \|y_i\|) \\
&\leq \max_{1<i<2}(\|x_i\|) + \max(\|y_i\|) \\
&\leq \|\bar{x}\|_\infty + \|\bar{y}\|_\infty \\
&= \|x + y\|_\infty \leq \|\bar{x}\|_\infty + \|\bar{y}\|_\infty
\end{aligned}$$

2 Q2: Prove that X = $\mathbb{R}$ is a Normed Space

Statement: Let $X = \mathbb{R}$ for arbitiary $\bar{x}$,,

where $\bar{x} = (x_1, x_2, x_3, x_4, \ldots x_n) \in \mathbb{R}^n$, and with $\alpha \in \mathbb{R}$, then $\|\bar{x}\|_p = \|(x_1, x_2, x_3, x_4, \ldots x_n)\|_p$ $= (\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}}$ verify that $\|.\|$ is a norm.

Proof:

N1: $\|x\| > 0$, iff $x = 0$

$(\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}} \geq 0 \|\bar{x}\|_p \geq 0$ clearly because absolute value of any value is greater than or equal to 1

Also,

if $\bar{x} = 0 \implies \forall 1 < i < n \implies (\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}} = 0$

N2: $\|\alpha\bar{x}\|_p = |\alpha|\|\bar{x}\|$

$$\begin{aligned}
\|\alpha\bar{x}\|_p = \|\alpha(x_1, x_2, x_3, x_4, \dots x_n)\|_p &= (\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}} \\
&= \Big(|\alpha|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{\frac{1}{p}} \\
&= |\alpha| \Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{\frac{1}{p}} \\
&= |\alpha|\|\bar{x}\|_p
\end{aligned}$$

N3: $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

We need Holder's inequality

$$\begin{aligned}
\|\bar{x} + \bar{y}\|_p^p &= \|(x_1 + y_1) + (x_2 + y_2) + \dots + (x_n + y_n)\| \\
&= \sum_{i=1}^{n} |x_i + y_i|^p = \sum_{i=1}^{n} |x_i + y_i|^{p-1}|x_i + y_i| \\
& ApplyingHolder'sTheorem \\
&\leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1} \\
\leq \Big[(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}(\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)})^{\frac{1}{q}}\Big] &+ \Big[(\sum_{i=1}^{n} |y_i|^p)^{\frac{1}{p}}(\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)})^{\frac{1}{q}}\Big] \\
&\leq \Big(\|x\|_p + \|y\|_p\Big)\Big(\sum_{i=1}^{n} |x_i + y_i|\Big)^{\frac{p}{q}} \\
&\implies \|\bar{x} + \bar{y}\|_p = \|\bar{x}\|_p + \|\bar{y}\|_q
\end{aligned}$$

Recall in Holder's inequality: $\frac{1}{p} + \frac{1}{q} = 1$,1 - $\frac{1}{p} = \frac{p-1}{p} = 1$

$p = q(p-1)$

3 Q3: Proof that l_∞ is a Complete Normed Space

Step 1: Cauchy Sequences in l_∞

Consider a Cauchy sequence $(x^{(k)})$ in l_∞, where $x^{(k)} = (x_n^{(k)})$ for each k. This means that for any $\varepsilon > 0$, there exists an N such that for all $k, m \geq N$:

$$\|x^{(k)} - x^{(m)}\|_\infty = \sup_{n\in\mathbb{N}} |x_n^{(k)} - x_n^{(m)}| < \varepsilon$$

Step 2: Define the Limit Sequence

Define the sequence $x = (x_n)$ as follows:

$$x_n = \lim_{k\to\infty} x_n^{(k)}$$

Step 3: Prove $x \in l_\infty$

Now, we need to show that $x \in l_\infty$, i.e., the sequence x is bounded. Consider the Cauchy property and the fact that limits preserve inequalities:

$$|x_n| = \lim_{k\to\infty} |x_n^{(k)}| \leq \sup_{n\in\mathbb{N}} |x_n^{(k)}| = \|x^{(k)}\|_\infty$$

Since $(x^{(k)})$ is a Cauchy sequence, the right side is bounded. Therefore, $|x_n|$ is bounded for each n, implying that $x \in l_\infty$.

Step 4: Prove Convergence in l_∞

We now need to show that $x^{(k)}$ converges to x in the l_∞ norm:

$$\|x^{(k)} - x\|_\infty = \sup_{n\in\mathbb{N}} |x_n^{(k)} - x_n| \to 0 \text{ as } k \to \infty$$

This follows from the definition of x_n and the convergence of $x_n^{(k)}$ to x_n for each n.

Conclusion:

Since every Cauchy sequence in l_∞ converges to a limit in l_∞, we can conclude that l_∞ is a complete normed space.

4 Q4: Proof: Completeness of l_p for $1 < p < \infty$

Step 1: Cauchy Sequences in l_p:

Consider a Cauchy sequence $(x^{(k)})$ in l_p, where $x^{(k)} = (x_n^{(k)})$ for each k. This means that for any $\varepsilon > 0$, there exists an N such that for all $k, m \geq N$:

$$\|x^{(k)} - x^{(m)}\|_p = \left(\sum_{n=1}^{\infty} |x_n^{(k)} - x_n^{(m)}|^p\right)^{1/p} < \varepsilon$$

Step 2: Define the Limit Sequence:

Define the sequence $x = (x_n)$ as follows:

$$x_n = \lim_{k\to\infty} x_n^{(k)}$$

Step 3: Prove $x \in l_p$:

Now, we need to show that $x \in l_p$, i.e., the sequence x is in l_p. We'll use the fact that the p-th root of the sum of the p-th powers is a continuous function.

$$|x_n|^p = \lim_{k\to\infty} |x_n^{(k)}|^p \leq \left(\lim_{k\to\infty} |x_n^{(k)}|^p\right)^{1/p} = \lim_{k\to\infty} |x_n^{(k)}|$$

Since $(x^{(k)})$ is a Cauchy sequence, the right side is finite. Therefore, $|x_n|^p$ is bounded for each n, implying that $x \in l_p$.

Step 4: Prove Convergence in l_p:

We now need to show that $x^{(k)}$ converges to x in the l_p norm:

$$\|x^{(k)} - x\|_p = \left(\sum_{n=1}^{\infty} |x_n^{(k)} - x_n|^p\right)^{1/p} \to 0 \text{ as } k \to \infty$$

This follows from the definition of x_n and the convergence of $x_n^{(k)}$ to x_n for each n.

Conclusion:

Since every Cauchy sequence in l_p converges to a limit in l_p, we can conclude that l_p is a complete normed space for $1 < p < \infty$.