MTH 411: Theory of Ordinary Differential Equation Assignment

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1 Q1:

Theorem: Suppose f is continuous on an open set $U \subset \mathbb{R} \times \mathbb{R}^n$. Let $(t_0, x_0) \in U$, and ϕ be a function defined on an open set I of $\mathbb{R}$ such that $t_0 \in I$. Then ϕ is a solution of the IVP (1.3) if, and only if,

1. $\forall t \in I$, $(t, \phi(t)) \in U$.
2. ϕ is continuous on I.
3. $\forall t \in I$, $\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds$.

Proof:
($\Rightarrow$) Let us suppose that $\phi' = f(t, \phi)$ for all $t \in I$ and that $\phi(t_0) = x_0$. Then for all $t \in I$, $(t, \phi(t)) \in U$ (i). Also, ϕ is differentiable and thus continuous on I (ii). Finally,

$$\phi'(s) = f(s, \phi(s))$$

so integrating both sides from t_0 to t,

$$\phi(t) - \phi(t_0) = \int_{t_0}^{t} f(s, \phi(s))\, ds$$

and thus

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds$$

hence (iii).
($\Leftarrow$) Assume i), ii), and iii). Then ϕ is differentiable on I, and $\phi'(t) = f(t, \phi(t))$ for all $t \in I$. From (iii), $\phi(t_0) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds = x_0$.

2 Q2:

Claim: If $f \in C^k$, then all solutions ϕ of the differential equation $\frac{dx}{dt} = f(t, x)$ are of class C^{k+1}.

Proof:
Assume f is C^k, and let $\phi(t)$ be a solution to the given differential equation on some open interval I containing t_0. We want to show that ϕ is C^{k+1} on I.

1. Existence of Derivatives: Since ϕ is a solution, it is differentiable. We want to show that ϕ' (the first derivative) exists and is continuous.

2. Differentiability: Differentiate the given differential equation with respect to t:

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt} f(t, \phi(t))$$

This gives us $\frac{d^2\phi}{dt^2} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt}$.

3. Continuity of Derivatives: Since f is C^k, its partial derivatives up to order k are continuous. Also, $\frac{dx}{dt} = f(t, \phi(t))$ implies that $\frac{dx}{dt}$ is continuous.

4. Inductive Step: Repeat the process for higher-order derivatives. In general, you will have:

$$\frac{d^{k+1}\phi}{dt^{k+1}} = (\text{a sum of terms involving derivatives of } f \text{ and } \phi \text{ up to order } k)$$

Again, continuity follows from the continuity of partial derivatives of f and lower-order derivatives of ϕ.

Therefore, by induction, all derivatives of ϕ up to order $k+1$ exist and are continuous, implying that ϕ is of class C^{k+1}.

This completes the modified proof, specifically tailored for the differential equation $\frac{dx}{dt} = f(t, x)$.