axiomy - základy

$\wedge$ konjunkce $\vee$ disjunkce

if $\Rightarrow$ then

$\Rightarrow$ implikace

1 1	1
1 0	0
0 1	1
0 0	1

-.. / .-- / ~~....~~ / .. / -..

a v i d

—... / . / —. / . / ...

..— / .... / ———/

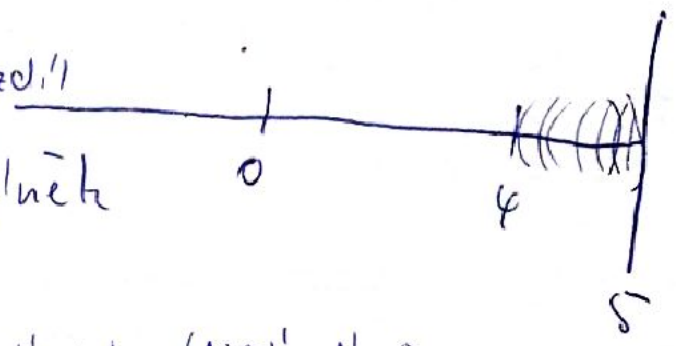
potenční množina

$B \in P(A) \Leftrightarrow B \subseteq A$ $B = \{1,2\}$ $P(B) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$A \cup B = \{x;\ x \in A \vee x \in B\}$ - komutativní, asociativní zákon $a+b = b+a$ $a+(b+c) = (a+b)+c$

$A \cap B = \{x;\ x \in A \wedge x \in B\}$

$A \setminus B = \{x;\ x \in A \wedge x \notin B\}$ → rozdíl



$A'_n \Leftrightarrow \Pi \setminus A$ doplněk

$(A \cup B)' \Leftrightarrow x \notin A \vee x \in B$

$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$ $(A \setminus B)' = A' \cup B$

$A \triangle B = (A \setminus B) \cup (B \setminus A)$

symetrický rozdíl

$[x_1, x_2] = \{\{x_1, x_2\}, \{x_2\}\}$ $(-\infty, 5)$ $(4.999, 5)$

kartézský součin

$A \times B = \{[x,y];\ x \in A \wedge y \in B\}$ $A = \{1\}$ $B = \{2\}$ $A \times B = \{[1,2]\}$

$A \times A = A^2 = \{[1,1]\}$

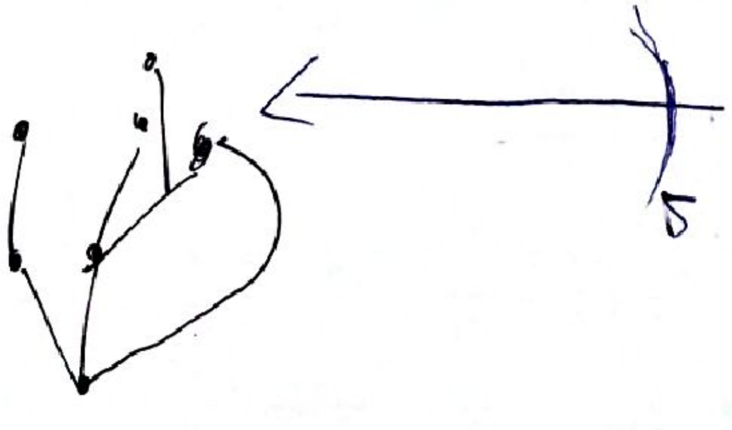
~~AxA~~

$A = |n|$

$B = |m|$

$|P(A) \times P(B)| = 2^{n+m}$

$|P(A \times B)| = 2^{n \cdot m}$



$A = \{1, 2, 3\}$ $\quad P(A) = \{\emptyset \{1\}, \{2\}\{3\} \dots\}$

$B = \{*, \Delta, O, \heartsuit\}$ $\quad P(B) = \{\emptyset, \{*\}, \{\Delta\}, \{O\}, \{\heartsuit\}\}$

$A \times B = \{[1, *], [1, \Delta], [1, O], [1, \heartsuit], [2, *] \dots, \{3, *\}, \dots [3, \heartsuit]\}$

$B \times A = \{[*, 1] \dots [\heartsuit, 3]\}$

$P(A \times B) = \{\emptyset, \{[1, *]\}, \{[1, \Delta]\}, \{[1, O]\}, \dots \{[1, *], [1, \Delta]\} \dots\}$

$P(A) \times P(B) = \{(\emptyset, \emptyset), (\emptyset, \{*\}) \dots (\emptyset, \{*, \Delta, O, \heartsuit\}) \dots\}$

$|A \times B| = 12$

$|P(A \times B)| = 2^{12}$

$|\underset{2^3}{P(A)} \times \underset{2^4}{P(B)}| = 2^{3+4} = 2^7$

Př. Klobouky všem špatně $\quad P = \dfrac{n \leftarrow \text{přijatelné}}{m! \leftarrow \text{všechny}}$

$n=1 \quad \frac{0}{1}$

$n=2 \quad \frac{1}{2} \qquad n=3 \quad \frac{1}{3}$

$n=4 \quad \frac{3}{24} = \frac{1}{8}$

alespoň 1 dostane svůj

$1 - \dfrac{x}{n!}$

$|A_1 \cup A_2 \cup \dots A_n|$

$|A_i| = n \cdot (n-1)! = n!$

$|A_{i_2}| = \binom{n}{2}(n-2)!$

$|A_{i_3}| = \binom{n}{3}(n-3)!$

$\dfrac{n!}{(n-2)! \cdot 2!}$ $\quad$ kolik? $\quad$ trojpočník

$$1 - \frac{x}{n!} = 1 - \left(\frac{n! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! \dots + (-1)^{n+1}\binom{n}{n}(n-n)!}{n!}\right) \qquad \frac{\cancel{n!}}{\cancel{(n-2)!} \cdot 2!} \cdot \frac{\cancel{n-2!}}{\cancel{n!}}$$

$$\cancel{= 1 - \frac{n!}{n!}} -$$

$$= 1 - \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \dots + (-1)^{n+1} \cdot \frac{1}{n!}\right)$$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \dots + (-1)^n \cdot \frac{1}{n!}$$

Kombinatorika

Př: třída -> 35 žáků -> kolik zp. se mohou seřadit?

$\underline{35}\cdot\underline{34}\cdot\underline{33}\cdot\underline{32} \ldots \underline{2}\cdot\underline{1} = 35!$ -> $\boxed{\text{Permutace (bez opakování) } n!}$

Př:
50 postelí pro 35 lidí (1 na 1)

$\underline{50}\ \underline{49}\ \underline{48} \ldots (50-35+1)=16 \rightarrow \frac{50!}{(50-35)!} = \binom{50}{35}\cdot 35!$

Pascalův Δ

1 $n=0$
1 1 $n=1$
1 2 1 $n=2$
1 3 3 1 $n=3$

$\binom{n}{k} = \frac{n!}{(n-k)!\cdot k!}$ $\binom{4}{2} = \frac{4!}{2!\cdot 2!} = \frac{4\cdot 3\cdot 2}{2\cdot 2} = 6$

Kombinace
~~Variace~~ (bez opakování) $\binom{n}{k} = \frac{n!}{(n-k)!\cdot k!}$

Př:
vybrat ze 50 postelí pro 35 lidí
kolik zp.

$\frac{50\cdot 49\cdot\ldots\cdot 16}{35!}$ $\frac{\frac{50!}{15!}}{35!} = \frac{50!}{35!\cdot 15!} = \binom{50}{15}$

Potenční množina

$|A| = n$

$|P(A)| = 2^n$ -> 1. důkaz binomická věta

$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}+\binom{n}{n}$

$\binom{n}{0}\cdot 1^n\cdot 1^0 + \binom{n}{1}1^{n-1}\cdot 1^1 + \binom{n}{2}1^{n-2}\cdot 1^2 + \ldots + \binom{n}{n-1}1^1\cdot 1^{n-1} + \binom{n}{n}1^0\cdot 1^n = (1+1)^n =$
$= 2^n$

Př: 1, 2, 3 -> 5 ciferné číslo $= 3^5$

Př: 5 barevných kuliček, beru 1 a vracím dohr. 3

$5\cdot 5\cdot 5 = 5^3$

$\boxed{\text{Variace s opakováním } n^k}$

Př: 5 barev, 3 tahám pak vracím, záleží na barvě ne na pořadí

B	M	Ž	Č	Z
•	•	••		•
	•••			

$\bullet|\bullet|\bullet|\bullet||$
$|\bullet\bullet\bullet||||$
$||\bullet\bullet||\bullet$

$\binom{7}{3}\ \binom{7}{4}$ $\binom{5-1+3}{3}$

kombinace s opakováním $\binom{n-1+k}{k}$

Pj.:

TRANTÁRIA změnit písmenka na jiné slova

$\frac{9!}{2!\cdot 2!\cdot 2!}$ MISSISSIPPI $\frac{11!}{4!\cdot 4!\cdot 2!}$

! Dirichletov princip n objektů do m tříd, kde $n > m$, potom je ve tříd alespoň 2

Pj.:

Ve skupině ~~jsou~~ ve alespoň 2 co znají stejně lidí?

$n=2$ 1 1 0 0 $n=3$ 0-2 0-2 0-2 $n=4$ 0-3

Pj.:

kolik je přirozených čísel < 100 které jsou dělitelné ~~od~~ 2 nebo 3.

1 ... 99

$|A_2| = 49$

$|A_3| = 33$

$2\cdot 1$ $2\cdot 49$

1, 2, 3 .. 98, 99

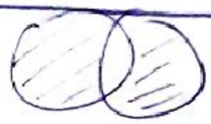
$3\cdot 1$ $3\cdot 33$

V =

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

$|A_1 \cup A_2| = 49 + 33 - 16 = 66$

Př 1)



$A = \{1, 4, 5, 6\}$

$B = \{3, 4, 6, 7\}$

$C = \{2, 5, 6\}$

$B \setminus C = \{3, 4, 7\}$

$A \cup B = \{1, 3, 4, 5, 6, 7\}$

$A \cap C = \{5, 6\}$

$C \setminus B = \{2, 5\}$

$A \setminus (C \setminus B) = \{1, 4, 6\}$

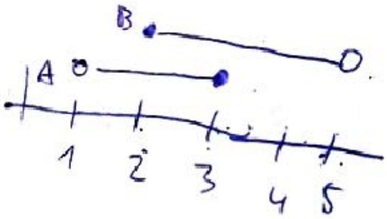
$A \cap (B \setminus C) = \{4\}$

$A \,\Delta\, B = \{1, 3, 5, 7\}$

$A \times C = \{ [1,2], [1,5), (1,6), (4,2), (4,5), (4,6), (5,2), (5,5), (5,6), (6,2), (6,5), (6,6) \}$

$(A \times C) \setminus B = A \times C$

2 Př:



$A = (1, 3\rangle$

$B = \langle 2, 5)$

$A \cup B = (1, 5)$

$A \cap B = \langle 2, 3\rangle$

$B \setminus A = \cancel{\langle 2,3) \cup (3,5)} \; (3, 5)$

$A \setminus (A \setminus B) = (1, 3\rangle \setminus (1, 2) = \langle 2, 3)$

$B \setminus (B \setminus A) = \langle 2, 5) \setminus (3, 5) = \langle 2, 3)$

$A \setminus (B \setminus A) = (1, 3\rangle \setminus (3, 5) = (1, 3\rangle$

$B \setminus (A \setminus B) = \langle 2, 5) \setminus (1, 2) = \langle 2, 5)$

$(A \setminus B) \cap (B \setminus A) = (1, 2) \cap (3, 5) = \emptyset$

$A \,\Delta\, B = (1, 2) \cup (3, 5)$

4Př:

$M = \{1, 2, 3, 4, 5, 6\}$

$X, Y \subseteq M$

$X = \{2, 6, x, 7\}$

$Y = \{2, x, z, u\}$

$X \cap Y' = \{5, 6\} \Rightarrow y = 5$

$X' \cap Y = \{1, 4\} \Leftrightarrow z, u - 1, 4$ $\Big\} \; x = 3$

3Př

$X = ?$

$Y = ?$

$X \cap Y = \emptyset$

$X \cup Y = \{1, 2, 3, 4, 5, 6, 7\}$

$\forall a \in X \; \exists b \in Y;\; b = a + 4$

① $X = \{1, 2, 3\}$ $Y = \{4, 5, 6, 7\}$ ② $X = \emptyset$ $Y = \{1 \ldots 7\}$

③

$X = \{1\}$ $Y = \{2 \ldots 7\}$

$X = \{2\}$ $Y = \{1, 3 \ldots 7\}$

5Př:

pro jaké $x \in \mathbb{R}$ mají $A \cap B \neq \emptyset$, $A = \langle \frac{x-1}{2}, 3 \rangle$

$B = (-\infty, x+2)$

$x + 2 \geq \frac{x-1}{2}$

$2x + 4 \geq x - 1$

$x \geq -5$ $x \in \langle -5, +\infty)$

$\frac{x-1}{2} < 3$

$x < 7$ $x \notin (7, +\infty)$

$x \in \langle -5, \infty) \cap (-\infty, 7)$

$x \in \langle -5, 7)$

6 Př:

$A = \langle \frac{4x}{3}; \infty)$

$B = \langle -11, \frac{6x-2}{4} \rangle$

$A \cap B = \emptyset$

$\frac{6x-2}{4} > -11$

$6x > -42$

$x > -7 \Rightarrow x \in (-7, \infty)$

$\frac{6x-2}{4} < \frac{4x}{3}$

$\cancel{12x-6<}$

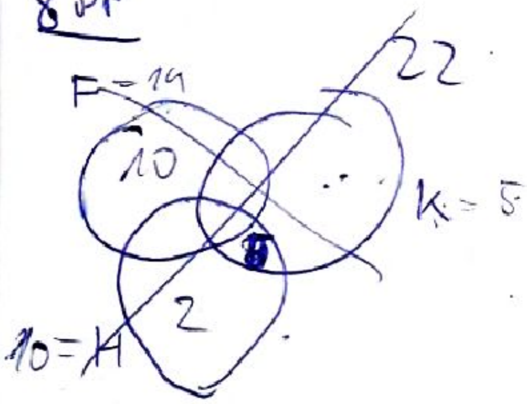
$\frac{3x-1}{2} < \frac{4x}{3}$

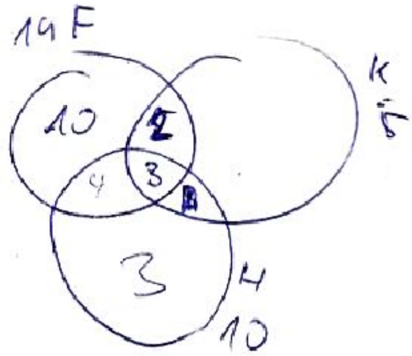
$x \in (-\infty, 3) \cap (-7, \infty) = (-7, 3)$

$9x - 3 < 8x$

$x < 3 \Rightarrow x \in (-\infty, 3)$

8 Př





15 Př

a) $T = (0, \infty)$, $A_t = \langle -t, t \rangle$ $\quad \bigcap_{t \in T} A_t \quad \bigcup_{t \in T} A_t$

$t = 1$

$0 \leftarrow \qquad \rightarrow \infty$

$A_1 = \langle -1, 1 \rangle$

$A_{0,5} = \langle -\frac{1}{2}, \frac{1}{2} \rangle$

$\vdots$

$A_{10^{-6}} = \langle -10^{-6}, 10^{-6} \rangle$

$A_1 = \langle -1, 1 \rangle$

$A_{10} = \langle -10, 10 \rangle$

$A_{100} = \langle -100, 100 \rangle$

$A_{10^6} = \langle -10^6, 10^6 \rangle$

$\bigcup_{t \in T} A_t = \mathbb{R}$

$\bigcap_{t \in T} A_t = \{0\}$

e) $T = (0,1)$, $A_t = \left(1 - \frac{1}{1-t}, 2 + \frac{1}{t}\right)$

$$\frac{1}{2}$$

$0 \leftarrow \quad \rightarrow 1$

$A_{\frac{1}{4}} = \left(-\frac{1}{3}, 6\right)$

$A_{\frac{1}{10}} = \left(-\frac{1}{9}, 12\right)$

$A_{\frac{1}{2}} = (-1, 4)$

$A_{\frac{3}{4}} = \left(-3, \frac{10}{3}\right)$

$A_{0{,}9} = \left(-9, \frac{28}{9}\right)$

$A_{0{,}99} = \left(-99, \frac{298}{99}\right)$

$A_{0{,}999} = \left(-999, \frac{2998}{999}\right)$

$$\bigcup_{t \in T} A_t = \mathbb{R}$$

$$\bigcap_{t \in T} A_t = \langle 0, 3 \rangle$$

Kolik je přirozených č. < 100, které nejsou dělitelné 2, 3, 5

← viz min. př. (1–99)

$|A_2| = 49$

$|A_3| = 33$

$|A_5| = 19$

$$|A_2 \cup A_3 \cup A_5| = |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5|$$

$$= 49 + 33 + 19 - 16 - 9 - 6 + 3 = 73$$

$\underline{99 - 73 = 26}$

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

DŮKAZY

typy

- přímý
- nepřímý
- důkaz sporem
- matematickou indukcí

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přímý důkaz

- součet dvou lichých čísel je číslo sudé

a – liché, b – liché $\Rightarrow$ a+b = sudé

pokud a je liché $\Rightarrow \exists k \in \mathbb{Z};\ a = 2 \cdot k + 1$

b – " – $\Rightarrow \exists l \in \mathbb{Z};\ b = 2l + 1$

$a + b = (2k+1) + (2l+1) = 2(k+l+1) \Rightarrow$ a+b je sudé

Dokaž:

$A \setminus (A \setminus B) = A \cap B$

① $A \setminus (A \setminus B) \subseteq A \cap B$

$x \in A \setminus (A \setminus B) \Rightarrow x \in A \wedge (x \notin (A \setminus B)) \Rightarrow x \in A \wedge (x \notin A \wedge x \in B)$

$\Rightarrow (x \in A \underset{0}{\wedge} x \notin A) \vee (x \in A \underset{1}{\wedge} x \in B) \Rightarrow x \in A \cap B$

② $A \cap B \subseteq A \setminus (A \setminus B)$

$x \in A \cap B \Rightarrow x \in A \wedge x \in B \Rightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \notin A)$

$\Rightarrow x \in A \wedge (x \notin A \vee x \in B)$

$\Rightarrow x \in A \wedge x \notin A \setminus B \rightarrow x \in A \setminus (A \setminus B)$

Dokaž: $P(A \cap B) \subseteq P(A) \cap P(B)$

$A = \{1\}$

$B = \{1, 2\}$

$A \cap B = A$

$P(A) = \{\emptyset, \{1\}\}$

$P(B) = \{\emptyset, \{1\}, \{1, 2\}\}$

$P(A \cap B) = P(A)$

$P(A) \cap P(B) = P(A)$

$\cancel{P(A) \cap P(B) \subseteq P(A \cap B)}$

$P(A \cap B) \subseteq P(A) \cap P(B)$

① $M \in P(A \cap B) \Rightarrow M \subseteq A \cap B \Rightarrow M \subseteq A \wedge M \subseteq B \Rightarrow$

$\Rightarrow M \in P(A) \wedge M \in P(B)$

$\Rightarrow M \in P(A) \cap P(B)$

$P(A) \cap P(B) \subseteq P(A \cap B)$

② $M \in P(A) \cap P(B) \Rightarrow M \in P(A) \wedge M \in P(B)$

$\Rightarrow M \subseteq A \wedge M \subseteq B \Rightarrow M \subseteq A \cap B$

$\Rightarrow M \in P(A \cap B)$

Zjisti jestli platí $P(A \cup B) = P(A) \cup P(B)$

$A = \{1\}$
$B = \{1,2\}$

$A \cup B = B$
$P(A \cup B) = P(B)$

① $P(A \cup B) \subseteq P(A) \cup P(B)$

$M \in P(A \cup B) \Rightarrow M \subseteq A \cup B \not\Rightarrow M \subseteq A \vee M \subseteq B$

↳ NEPLATÍ

② $P(A) \cup P(B) \subseteq P(A \cup B)$ -> PLATÍ

$M \in P(A) \cup P(B) \rightarrow M \in P(A) \vee M \in P(B) \Rightarrow M \subseteq A \vee M \subseteq B \Rightarrow M \subseteq A \cup B$
$\Rightarrow M \in P(A \cup B)$

$A = \{1\}$
$B = \{2\}$
$A \cup B = \{1,2\}$

$P(A) = \{\emptyset, \{1\}\}$
$P(B) = \{\emptyset, \{2\}\}$

$P(A) \cup P(B) = \{\emptyset, \{1\}, \{2\}\}$
$P(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

Matematická indukce (ukážeme pro n, potom platí pro $n+1$)

$n \in \mathbb{N}$; $1+2+\ldots+n = \frac{1}{2} n \cdot (n+1)$

Indukční předpoklad

$n=1$ L.S. $= 1$, P.S. $= 1$ ✓

předpok. že $1+2+\ldots n = \frac{1}{2} n(n+1)$ I.P

$1+2+\ldots n + (n+1) = \frac{1}{2} \cdot (n+1)((n+1)+1)$ $(n+2)$

$\underbrace{1+2+\ldots+n+(n+1)} = \underbrace{\frac{1}{2} \cdot n \cdot (n+1)} + (n+1) = \left(\frac{1}{2} n + 1\right)(n+1)$
$= (n+1) \cdot \frac{1}{2}(n+2)$
$= \frac{1}{2}(n+1)(n+2)$

$n \in \mathbb{N}$ $\quad \frac{1}{3} k \, (4k^2-1)=$

$1^2+3^2 \dots + (2n-1)^2 = \frac{1}{3} n \, (4n^2-1)$ $\qquad ((2k+1)-1)((2k+1)+1)$

$\underline{n=1}$ $\quad L.S. = 1^2 = 1$ $\qquad (2 \cdot k+1)^2-(1)^2$

$P.S. = \frac{1}{3} \cdot 3 = 1$ ✓

Predpokladám, že $1^2+3^2+ \dots (2k-1)^2 = \frac{1}{3} k \, (4k^2-1)$ IP

dokazuji, že $1^2+3^2 \dots + (2k-1)^2 + (2k+1)^2 = \frac{1}{3}(k+1)(4 \cdot (k+1)^2-1)$

$$\underbrace{1^2+3^2 \dots + (2k-1)^2}+(2k+1)^2 = \underbrace{\tfrac{1}{3} k \, (4k^2-1)} + (2k+1)^2 =$$

$$= \frac{1}{3} k \cdot (2k-1)(2k+1) + (2k+1)^2 =$$

$$= (2k+1)\left(\frac{1}{3}(2k-1) \cdot k + (2k+1)\right) =$$

$$= (2k+1) \cdot \frac{1}{3}\left((2k-1)k + 3(2k+1)\right) =$$

$$= \frac{1}{3}(2k+1)(2k^2-k+6k+3) =$$

$$= \frac{1}{3}(2k+1)(2k^2+5k+3) =$$

$$= \frac{1}{3}(2k+1)(2k+3)(k+1)$$

$2 \setminus (2k+1)$

① ✗

② Predp., že $2 \setminus (2k+1)$; doh., že $2 \setminus (2(k+1)+1) = 2 \setminus (2k+3)$

$2k+3 = 2k+1+3 \Rightarrow 2 \setminus (2k+3)$

$$P(P(P(\emptyset))) = ?$$

$P(\emptyset) = \{\emptyset\}$

$P(P(\emptyset)) = P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

$P(P(P(\emptyset))) = P(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

Dokaž, že platí $|A| = n \Rightarrow |P(A)| = 2^n$

① $A = \{1,2\}$ $P(A) = 4$

$B = \{1,2,3\}$ $P(B) = 8$

Binární relace

lib. podmnožina kartézského součinu

R je relace, píšeme aRb $[a,b] \in R$, prvek a je v relaci s prvkem b

$2 < 7$ $[2,7] \in <$

$D(R) \subseteq A$
$H(R) \subseteq B$
R je relace z množiny A do množ. B

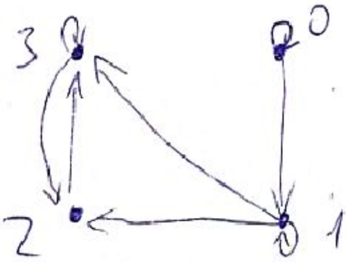
~~Pokud~~
Příklad:

R na A $R \subseteq A \times A = A^2$

$A = \{0,1,2,3\}$ a $R = \{[0,0], [0,1], [1,1], [1,2], [1,3], [2,3], [3,3], [3,2]\}$

$D(R) = ?$ $H(R) = ?$

D \ H	0	1	2	3
0	1	1	0	0
1	0	1	1	1
2	0	0	0	1
3	0	0	1	1



$D(R) = \{0,1,2,3\}$

$H(R) = \{0,1,2,3\}$

$A = \{m \in \mathbb{N}; m \leq 5\}$ $R = \{(m,n) \in A \times B; m = n+1\}$

$B = \{n \in \mathbb{N}; n \leq 6\}$ $S = \{(m,n) \in B \times A; m = 2n+1\}$

$A = \{0,1,2,3,4,5\}$ $R = \{(1,0), (2,1), (3,2), (4,3), (5,4)\}$

$B = \{0,1,2,3,4,5,6\}$ $S = \{(1,0), (3,1), (5,2)\}$

$D(R) = \{1,2,3,4,5\}$
$H(R) = \{0,1,2,3,4\}$
$D(S) = \{1,3,5\}$
$H(S) = \{0,1,2\}$

Definice

Nechť R je relace na A, tak i $\overline{R} = A^2 \setminus R$, $\overline{R}$ je doplňková množina k relaci R na A

R	0	1	2	3
0	1	1	0	0
1	0	1	1	1
2	0	0	0	1
3	0	0	1	1

$\overline{R}$	0	1	2	3
0	0	0	1	1
1	1	0	0	0
2	1	1	1	0
3	1	1	0	0

$R = \{(0,0), (0,1), (1,1), (1,2), (1,3), (2,3), (3,3), (3,$

Nechť ~~taki~~

Identická relace k A $\Delta_A = \{(a,a); a \in A\}$

Δ_A	1	2	3
1	1	0	0
2	0	1	0
3	0	0	1

Inverzní relace $R^{-1} = \{[a,b]; [b,a] \in R\}$

R^{-1}	0	1	2	3
0	1	0	0	0
1	1	1	0	0
2	0	1	0	1
3	0	1	1	1

$R^{-1} = \{(0,0), (1,0), (1,1), (2,1) \ldots (2,3)\}$

$(R^{-1})^{-1} = R$

$(R^{-1})^{-1} = R$

Důkaz:

$(R^{-1})^{-1} \subseteq R$

$[x,y] \in (R^{-1})^{-1} \Leftrightarrow [y,x] \in R^{-1} \Leftrightarrow [x,y] \in R$

$R = \{(1,2),(2,3),(2,4)\}$ $S = \{(2,2),(5,3),(1,5),(3,4),(1,7)\}$

$R \cup S = \{(1,2),(2,3),(2,4),(2,2),(5,3)\dots\}$

$R \cap S = \emptyset$ stále relace

$R \setminus S = R$

$R \circ S = \{[a,c];\ \exists b\ [a,b] \in S \wedge [b,c] \in R\}$ složená relace z rel R a S

$R = \{(1,2),(1,1)\}$

$S = \{(2,3),(1,3)\}$

$R \circ S = \emptyset$

$S \circ R = \{(1,3)\}$ $(1,2)-(2,3) \to (1,3)$ "Lepím od zadu"

$A = \{0,1,2\}$

$B = \{a,b\}$ R po S

$R = A \times B$ $R \circ S = \{(a,a),(a,b),(b,b),(b,a)\}$

$S = B \times A$ $S \circ R = \{(0,0),(0,2),(1,1),(1,2),(2,0),(2,2),(2,1)\}$

$(R \circ S) \circ T = R \circ (S \circ T)$

$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

$(,,\subseteq" \Rightarrow)$ $(,,\supseteq" \Leftarrow)$

$(x,y) \in (R \circ S)^{-1} \Leftrightarrow (y,x) \in (R \circ S) \Leftrightarrow \exists z;\ (y,z) \in S \wedge (z,x) \in R \Leftrightarrow$

$\Leftrightarrow \exists z;\ (z,y) \in S^{-1} \wedge (x,z) \in R^{-1} \Leftrightarrow (x,y) \in S^{-1} \circ R^{-1}$

Reflexivní relace R na množině A; $\forall a \in A;\ aRa$

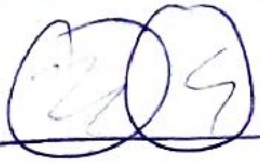
$\begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix}$ $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$ $\to$ diagonální $\Rightarrow$ reflexivní

refl. $\not\Rightarrow$ diag.

Symetrická relace; $\forall a,b \in A;\ aRb \Rightarrow bRa$

Tranzitivní relace; $\forall a,b,c \in A;\ aRb \wedge bRc \Rightarrow aRc$

$A = \{1, 2, 3\}$



$\{(1,2), (1,3), (1,1)\}$

Př:

$A = \{1, 2, 3, 4, 5, 6, 7\}$

$B = \{2, 4, 6, 8\}$

$C = \{3, 6, 9\}$

$A \Delta (B \Delta C) = A \Delta \{2, 3, 4, 8, 9\} = \{1, 5, 6, 7, 8, 9\}$

$(A \Delta B) \Delta C = \{1, 3, 5, 7, 8\} \Delta C = \{1, \cancel{3}, 5, 6, 7, 9, 8\}$

$B \setminus (A \times C) = B \setminus \emptyset = \{2, 4, 6, 8\}$

Viz 15 Př:

① $\bigcap_{t \in T} A_t = \{0\}$

1a) $0 \in \bigcap_{t \in T} A_t$

0 musí patřit do $\forall$ množin A_t; $t \in (0, \infty)$

$\langle -t, t \rangle = \{x \in \mathbb{R};\ -t \le x \le t\}$

$t > 0 > -t$

$-t < 0 < t$

Zřejmě pro $\forall t \in T$ platí, že $t > 0$, potom $-t < 0$, potom pro $\forall t \in T$ platí $-t < 0 < t$

$\Rightarrow \forall t \in T;\ 0 \in (-t, t) \subseteq \langle -t, t \rangle \Rightarrow 0 \in \langle -t, t \rangle$

$\Rightarrow \forall t \in T;\ 0 \in A_t$

1b) pokud $x \neq 0 \Rightarrow x \in \bigcap_{t \in T} A_t$

Stačí najít 1 množinu do které ~~x nepatří~~ $x \notin A_t$

Nechť $x \in \mathbb{R}$ (libovolné, ale ~~nezáporné~~) $\wedge\ x \neq 0 \Rightarrow \frac{|x|}{2} > 0 \wedge \left(|x| > \frac{|x|}{2} \Rightarrow x > \frac{|x|}{2} \vee x < -\frac{|x|}{2}\right.$

$\Rightarrow t = \frac{|x|}{2},\ -t = -\frac{|x|}{2},\ t > 0$

$A_{\frac{|x|}{2}} = \langle -\frac{|x|}{2}, \frac{|x|}{2} \rangle \Rightarrow x \in A_{\frac{|x|}{2}}$

① $\bigcup_{t \in T} A_t = \mathbb{R}$

Pro $\forall x \in \mathbb{R}$ (od nepřítele) třeba dokázat že $\exists t \in T;\ x \in A_t$

Nechť $x \in \mathbb{R}; \Rightarrow |x| > 0 \Rightarrow |x| \in T \Rightarrow$ použijeme $t = |x|$ a $x \in A_t = A_{|x|}$

$x \neq 0$

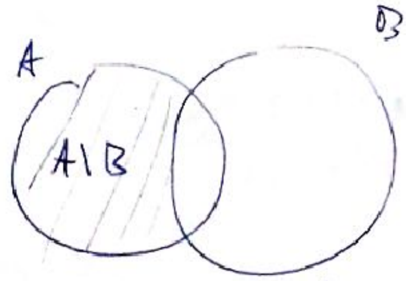
když $x = 0$, potom $t = 1 \Rightarrow A_t = \langle -1, 1 \rangle$

13 Př:

$A \setminus B = (A \cup B) \cap B$

a) rovnost platí

b) rovnost neplatí



a)

$A = \emptyset$ $A \setminus B = \emptyset$ ✓

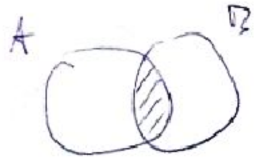
$B = \emptyset$ $(A \cup B) \cap B = \emptyset$

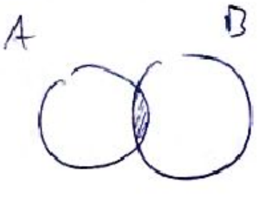
b)

$A = \{1, 2\}$ $A \setminus B = A = \{1, 2\}$

$B = \{*\}$ $(A \cup B) \cap B = \{*\}$

$A \setminus (A \setminus B) = B \setminus (B \setminus A)$





$A = \{1, 2\}$

$B = \{2, 3\}$ $A \setminus (A \setminus B) = A \setminus \{1\} = \{2\}$

$B \setminus (B \setminus A) = B \setminus \{3\} = \{2\}$

Důkaz:

$A \setminus (A \setminus B) = B \setminus (B \setminus A)$

"$\subseteq$"

$x \in A \setminus (A \setminus B) \Leftrightarrow x \in A \wedge x \notin (A \setminus B) \Leftrightarrow x \in A \wedge (x \notin A \vee x \in B) \Leftrightarrow$

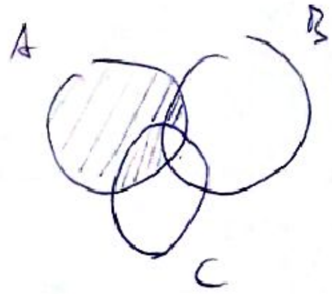
$\Leftrightarrow (x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B) \Leftrightarrow$

$\Leftrightarrow (x \in B \wedge x \notin B) \vee (x \in B \wedge x \in A) \Leftrightarrow$

$\Leftrightarrow (x \in B) \wedge (x \notin B \vee x \in A) \Leftrightarrow x \in B \wedge x \notin B \setminus A \Leftrightarrow$

$\Leftrightarrow x \in B \setminus (B \setminus A)$

$A \setminus (B \cup C) \overset{?}{=} A \setminus (B \cap C)$



$A = \{1,2,3\}$
$B = \{4,5,6\}$
$C = \{*,+\}$

$A \setminus (B \cup C) = A = \{1,2,3\}$ ✓
$A \setminus (B \cap C) = A = \{1,2,3\}$

$A = \{1,2,3,4\}$ $\quad A \setminus (B \cup C) = \{1,2,3\}$ ✗
$B = \{4,5\}$ $\quad A \setminus (B \cap C) = \{1,2,3,4\}$
$C = \{5,6\}$

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2)\}$
$S = \{(2,1), (2,2), (2,3), (3,1), (3,2), (4,1)\}$

$R^{-1} = \{(1,1), (2,1), (3,1), (1,2), (2,2)\}$
$S^{-1} = \{(1,2), (2,2), (3,2), (1,3), (2,3), (1,4)\}$
$R \circ S = \{(2,1), (2,2), (2,3), \cancel{(2,1)}, \cancel{(2,2)}, (3,1), (3,2), (3,3), (4,1), (4,2), (4,3)$
$S \circ R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$
$R^{-1} \circ S^{-1} = (S \circ R)^{-1}$

$(R \circ S)^{-1} = \{(1,2), (2,2), (3,2), (1,3), (2,3), \cancel{(1,3)}, (3,3), (1,4), (2,4), (3,4)\}$

$(S \circ R)^{-1} = \{(1,1), (2,1), (3,1), (1,2), (2,2), (3,2)\}$

$0 \notin \mathbb{N}$

$A = \{n \in \mathbb{N}, n < 10\} = \{1,2,3,4,5,6,7,8,9\}$

$B = \{m \in \mathbb{N}; m \leq 12\} = \{1, 2, \dots 12\}$ $\quad (4,5), (5,6), (6,7), (7,8), (8,9)$

$R = \{(m,n) \in A \times B, m+1 = n\} = \{(1,2), (2,3), (3,4), \dots (9,10)\}$

$S = \{(m,n) \in A \times B; m^2 = n\} = \{(1,1), (2,4), (3,9)\}$

$R \circ R = \{(1,3), (2,4), (4,6), (5,7), \overset{(6,8)}{} (7,9), (8,10)\}$
$R \circ S = \{(1,2), (2,5), (3,10)\}$
$S \circ R = \{(1,4), (2,9)\}$

Antisimetrická relace

$\forall a,b \in A;\ aRb \wedge bRa \Rightarrow a=b$

Ireflexivní relace

~~Ud Ava~~ $\forall a \in A;\ a\overline{R}a$

0	0	1
1	0	0
1	1	0

Souvislá

$\forall a,b\ a \neq b \Rightarrow aRb \vee bRa$ [illegible]

trichotomická

$a,b \in A$ musí platit právě jeden: $a=b$, aRb, bRa

$R = \{(0,0),(0,1),(1,0),(1,1),(2,2),(2,3),(3,2)\}$ $A = \{0,1,2,3\}$

R	0	1	2	3
0	1	1	0	0
1	1	1	0	0
2	0	0	1	1
3	0	0	1	0

Ref: není $3 \in A$ ale $3 \notin R$

Sym: je, $\forall a,b \in A;\ (a,b) \in R \Rightarrow (b,a) \in R$

Tranz: není $(3,2) \in R \wedge (2,3) \in R$ ale $(3,3) \notin R$

Antisym.: není $(0,1) \in R \wedge (1,0) \in R$, ale $0 \neq 1$

Iref: není $(1,1) \in R$

$M = \{1,2,3\}$

Ref

	1	2	3
1	1	0	1
2	1	1	0
3	0	1	1

Sym

	1	2	3
1	0	0	1
2	0	1	0
3	1	0	0

tran:

	1	2	3
1	1	1	1
2	1	1	1
3	1	1	1

Antisym

	1	2	3
1	1	1	0
2	0	0	0
3	0	1	1

iref:

	1	2	3
1	0	1	0
2	0	0	1
3	1	1	0

Sym a antisym

	1	2	3
1	0	0	0
2	0	1	0
3	0	0	1

$\neg$sym a $\neg$antisym

	1	2	3
1	0	1	1
2	0	1	0
3	1	0	1

ref a iref

	1	2	3
1	X		
2			
3			

$\neg$ref a $\neg$iref

	1	2	3
1	1	1	1
2	0	0	0
3	1	1	1

$R = \{(x,y) \in \mathbb{Z}^2 ; x^2 = y\}$

Ref:
není $(2,2) \in R$
$2^2 \neq 2$

Sym:
není $(2,4) \in R$ ale $(4,2) \notin R$
$4^2 \neq 2$

Tranz:
není $(2,4) \in R \wedge (4,16) \in R$
ale $(2,16) \in R \to 2^2 \neq 16$

iref:
není $(0,0) \in R$

$S = \{(x,y) \in \mathbb{Z}^2 ; |x-y| \geq 3\}$

ref:
není $(2,2) \notin S$
$(|2-2| = 0) \leq 3$

sym:
je
*

Trans:
není
např. $(1,4) \in S$ $(|1-4| \geq 3)$
$(1,0) \notin S$ $(|1-0| \not\geq 3)$

iref:

*
nechť $(x,y) \in \mathbb{Z}^2 ; |x-y| \geq 3$
$(x,y) \in S \Rightarrow |x-y| \geq 3 \Rightarrow |x-y| = |y-x| \Rightarrow$
$|y-x| \geq 3 \Rightarrow (y,x) \in S$

$M = \{1,2,3\}$ $R = \{(1,1),(1,2),(2,3)\}$
$S_1 \to$ ref a sym; $R \subseteq S_1$, S_1 - nejm. možná
$S_1 \{(1,1),(1,2),(2,3),(2,2),(3,3),(2,1),(3,2)\}$

reflexivní uzávěr
$R \cup \{(x,x) \mid x \in R\}$ $S^* = \{(1,1),(1,2),(2,3),(2,2),(3,3)\}$
symetrický uzávěr
$R \cup \{(x,y) \mid (x,y) \in R \vee (y,x) \in R$ $S_* = \{(1,1),(1,2),(2,3),(2,1),(3,2)\}$

$S_2 \to$ tranz; $R \subseteq S_2$; S_2 nejmenší možná

$S_2 = \{(1,1),(1,2),(2,3) \mid (1,3)\}$

$M = \{0,1,2,3\}$ $R = \{(0,1),(1,2),(2,0),(2,3)\}$
Tranzitivní uzávěr
$R^+ = \{(0,1),(1,2),(2,0),(2,3) \mid (0,2),(1,0),(1,3),(2,1) \mid (0,0),(0,3),(1,1),(2,2)\}$
$R^+ = R \cup \{\dots\}$

reflexivní uzávěr

$\rho(R_1 \cap R_2) = \rho(R_1) \cap \rho(R_2)$

$(x,y) \in \rho(R_1 \cap R_2) \Leftrightarrow (x,y) \in R_1 \cap R_2 \vee x = y \Leftrightarrow \big((x,y) \in R_1 \wedge (x,y) \in R_2\big) \vee x = y \Leftrightarrow$

$\Leftrightarrow \big((x,y) \in R_1 \vee x = y\big) \wedge \big((x,y) \in R_2 \vee x = y\big) \Leftrightarrow$

$\Leftrightarrow (x,y) \in \rho(R_1) \wedge (x,y) \in \rho(R_2) \Leftrightarrow (x,y) \in \rho(R_1) \cap \rho(R_2)$

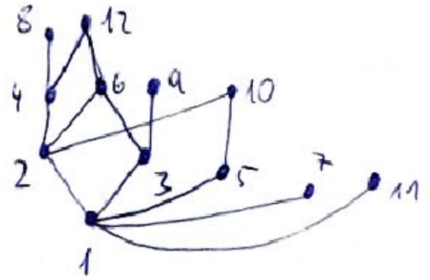
Uspořádané množiny

$R \subseteq M \times M$; R je reflexivní, antisymetrická, tranzitivní

$(M, \preceq)$

lineární uspořádání -> $\forall$ prvky jsou porovnatelné

$M = \{1, 2, 3 \dots 12\}$ $\quad (a,b) \in R \Leftrightarrow a | b$ -> has. diag. $\quad 1 | 2 \Leftrightarrow \frac{2}{1}$



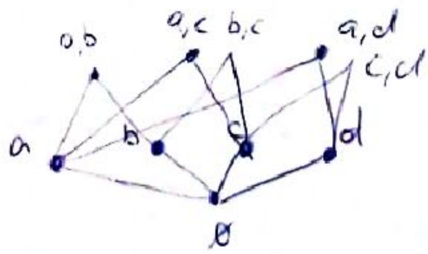
$M = \{a, b, c, d\}$

$P(M) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \dots\}$

$(P(M), \subseteq)$

$A \subseteq B \wedge B \subseteq A \Leftrightarrow A = B$

$A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$



I Pojmy uspořádaných množin

1 $(M, \preceq)$

y pod x

X je MINIMÁLNÍ pro $\forall y$ ani $y \preceq x$; $x \preceq y$

minimální -> nic pod ním

MAXIMÁLNÍ

maximální -> nic nad ním

NEJMENŠÍ

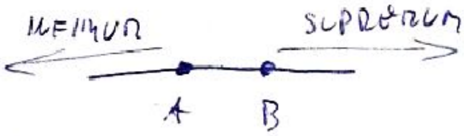
-> všechny nad

NEJVĚTŠÍ -> všechny pod

DOLNÍ OHRANIČENÍ

HORNÍ OHRANIČENÍ

INFIMUM -> největší dolní ohraničení



SUPREMUM - největší horní ohraničení

Ledničky	Ceny	E.T.
i	i	A+
ε	i	A++

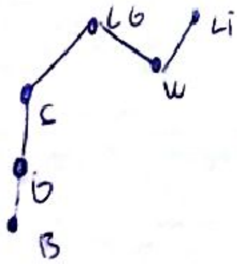
$(L, \prec)$

$x \prec y$

$L_i \prec L_j \Leftrightarrow C_i \leq C_j$

$C_i \leq C_j \wedge C_j < C_i \wedge L_i \neq L_j$

$L_i \prec L_j \Leftrightarrow C_i \leq C_j \wedge P_{ri} \geq P_{rj}$



dobré uspořádání – ∀ podmnožiny mají nejmenší prvek

kvazi-uspořádání – jen tranzitivní a reflexivní

Systém S podmnožin množiny A se nazývá rozklad

$\emptyset \notin S$

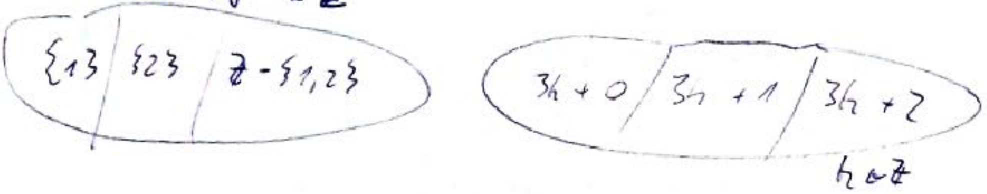
$\forall B, C \in S;\ B \neq C \Rightarrow B \cap C = \emptyset$

$\bigcup S = A$

$M = \{1, 2\}$ 

$M = \{1, 2, 3\}$

Rozklad množiny $\mathbb{Z}$



Relace ekvivalence

symetrická, reflexivní, tranzitivní

$A = \{0, 1, 2\}$

R, S, T

$R_1 = \{(0,0), (1,1), (2,2)\}$ R, S, T -> je relace ekvivalence

$R_2 = \{(0,0), (1,1), (2,2), (1,2), (2,1)\}$ R, S, T -> rel. ekv.

$R_3 = \{$ ——//—— $(0,1), (1,0)\}$ R, S, $\cancel{T}$ -> $(0,1) \in R_3 \wedge (1,2) \in R_3; (0,2) \in R_3$

$A = \{0, 1, 2\}$

$R = \{(2,2), (0,0), (0,1), (1,0), (1,1)\}$

$R = \{(0,0), (1,1), (2,2), (0,1), (0,2), (1,0), (2,0), (2,1), (1,2)\}$

$\mathbb{Z}$ a v relaci s b

$\forall a \in \mathbb{Z}$

$a \equiv b \Leftrightarrow 6 \mid (a-b)$

R - zřejmě $\forall a \in \mathbb{Z}; a-a=0 \wedge 6 \mid 0 \Rightarrow 6 \mid (a-a) \Leftrightarrow a \equiv a$

S - nechť $a \equiv b \Rightarrow 6 \mid (a-b) \Rightarrow 6 \mid -(a-b) \Rightarrow 6 \mid (b-a) \Rightarrow \underline{b \equiv a}$

T nechť $a \equiv b \wedge b \equiv c \Rightarrow 6 \mid (a-b) \wedge 6 \mid (b-c) \Rightarrow 6 \mid \left((a-b)+(b-c)\right) \Rightarrow$

$\Rightarrow 6 \mid (a-c) \Rightarrow \underline{a \equiv c}$

„$\equiv$" je rel. EKV na $\mathbb{Z}$

$\mathbb{Z}/_{\equiv} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

$\bar{0} - \{-6, 0, 6\}$ $6k+0$

$\bar{1} = \{-5, 1, 7 \ldots\}$ $6k+1$

$\bar{2} = \{-4, 2, 8 \ldots\}$ $6k+2$

$\bar{3} = \{\ldots -3, 3, 9 \ldots\}$ $6k+3$

$\bar{4} = \{\ldots -2, 4, 10 \ldots\}$ $6k+4$

$\bar{5} = \{\ldots -1, 5, 11 \ldots\}$ $6k+5$

$R_1 = \{(x,y) \in \mathbb{R}^2; \frac{x}{y} = 0\}$ -> je není refl. $(1,1) \notin R_1$ $\frac{1}{1} \neq 0$

$R_2 = \{(x,y) \in \mathbb{R}^2; \frac{x}{y} = 1\}$ -> není refl. $(0,0) \notin R_2$ $\frac{0}{0} \neq 1$

$R_3 = \{(x,y) \in \mathbb{R}^2; |x-y| \geq 0\}$ -> je EK. $|x-y| \geq 0$ pro $\forall (x,y) \in \mathbb{R}$

$R_4 = \{(x,y) \in \mathbb{R}^2; x = y\}$ -> je EK.

Binární relace

$[a,b] \in f \wedge [a,c] \in f \Rightarrow b = c$

~~fce~~ n

obraz množiny

$f(M) = \{b; \exists a \in M\ f(a) = b\}$

úplný vzor množiny

$f^{-1}(P) = \{a; \exists b \in P\ f(a) = b\}$

$R = \{(x,y) \in \mathbb{R}^2 ; |x| + |y| = 1$ není zobrazení $(0,1) \in R \wedge (0,-1) \in R$; $1 \neq -1$

$S = \{(x,y) \in R ; x + |y| = 1\}$ není —— // ——

$T = \{(x,y) \in R ; |x| + y = 1\}$

15e) Př:

$T = (0,1)$ $A_t = \left(1 - \frac{1}{1-t}, 2 + \frac{1}{t}\right)$

$t = \frac{1}{2}$

0 ↙ ↘ 1

$-\infty \leftrightarrow 0$ $\quad -\infty \leftrightarrow 3$

$\bigcup_{t \in T} A_t = \mathbb{R}$

$\bigcap_{t \in T} A_t = \langle 0, 3 \rangle$

① $\bigcap_{t \in T} A_t = \langle 0, 3 \rangle$

(1a) Dokážeme že $\forall x \in \langle 0, 3 \rangle, x \in \bigcap_{t \in T} A_t$

$\forall x \in \langle 0, 3 \rangle \ \forall t \in T ; x \in A_t$

stačí ukázat, že $1 - \frac{1}{1-t} < 0 \wedge 3 < 2 + \frac{1}{t}$

$\forall t \in T, \forall t \in (0,1)$

$1 - \frac{1}{1-t} < x < 2 + \frac{1}{t}$

$1 - \frac{1}{1-t} \overset{?}{<} 0$

$\frac{1-t-1}{1-t} \overset{?}{<} 0$

$\frac{-t}{1-t} \overset{?}{<} 0$ PRAVDA

$t \in (0,1) ; -t \in (-1, 0)$

$1 - t > 0$

$3 < 2 + \frac{1}{t}$

$1 - \frac{1}{t} < 0$

$\frac{t-1}{t} < 0$ Pravda

(1b) dokazujeme že $\langle 0,3 \rangle$ ... $x \notin \langle 0, 3 \rangle \Rightarrow x \notin \bigcap_{t \in T} A_t$

$x = 2 + \frac{1}{t}$

$\frac{1}{x-2} = t$

$x > 3$

$x - 2 > 3 - 2$

$x > 1$

$x = 1 - \frac{1}{1-t}$

$\frac{1}{1-t} = 1 - x$

$1 - t = \frac{1}{1-x}$

$t = 1 - \frac{1}{1-x} = \frac{1-x-1}{1-x} = \frac{-x}{1-x} = \frac{x}{x-1}$

$1 - \frac{1}{1 - \left(\frac{x}{x-1}\right)}$

$1 - \frac{x-1}{-1} = x$

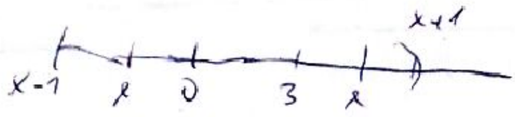
Nechť $x \in (3,\infty) \Rightarrow$ ak položíme $t = \frac{1}{x-2}$, tak $A_t = (\frac{1}{2}\frac{1}{x-3}, x)$

zřejmě $x \notin A_t$ a t jsme zjistili, že patří do $(0,1)$

Nechť $x \in (-\infty, 0) \Rightarrow$ položíme $t = \frac{x}{x-1}$ ($\in (0,1)$ ověřeno)

$$A_t = (x, \frac{3x-1}{x}) \Rightarrow x \notin A_t$$

② $\bigcup_{t \in T} A_t = \mathbb{R}$



nechť $x \in \langle 0,3 \rangle \Rightarrow$ položíme $t = \frac{1}{2}$, $A_{\frac{1}{2}} = (-1, 4)$

nechť $x > 3 \Rightarrow$ položíme $t = \frac{1}{x-1}$, $A_t = (\quad, x+1)$

$x+1 > x$ potom $x \in A_t$

$$1 - \frac{1}{1-t} = x - 1$$

$$t = 1 - \frac{1}{2-x} = \frac{2-x-1}{2-x} = \frac{1-x}{2-x} \in T$$

ak $x < 0 \Rightarrow t = \frac{1-x}{2-x} \in T \Rightarrow A_t = (x-1, \quad)$

$x - 1 < x \Rightarrow x \in A_t$

$T = (0,1)$

$A_t = (t, t+1)$ DÚ

$\bigcap_{t \in T} A_t$, $\bigcup_{t \in T} A_t$

$R = \{(x,y) \in \mathbb{R}^2; y = 2x\}$

$S = \{(x,y) \in \mathbb{R}^2; x^2 = y\}$

$T = \{(x,y) \in \mathbb{R}^2; y = \sqrt{x}\}$

$R \circ S$

$S \circ R$

$S \circ T$

$T \circ S$

$R \circ T$

$T \circ R$

$R \circ S$:

$x \xrightarrow{S} x^2 \xrightarrow{R} 2(x^2)$ $\quad D(f) = \mathbb{R}$

$D(f) = \mathbb{R}$ $\quad H(f) = \langle 0, \infty)$

$H(f) = \langle 0, \infty)$

$S \circ R$:

$x \xrightarrow{R} 2x \xrightarrow{S} 4x^2$

$D(f) = \mathbb{R}$

$H(f) = \langle 0, \infty)$

$R \circ T$

$x \xrightarrow{T} \sqrt{x} \xrightarrow{R} 2\sqrt{x}$

$D(f) = \langle 0, \infty)$

$H(f) = $ —//—

$T \circ R$

$x \xrightarrow{R} 2x \xrightarrow{T} \sqrt{2x}$

$D(f) = H(f) = \langle 0, \infty)$

$S \circ T$

$x \xrightarrow{T} \sqrt{x} \xrightarrow{S} (\sqrt{x})^2 = x \wedge x \in \langle 0, \infty) = H(f) = D(f)$

$T \circ S$

$x \xrightarrow{S} x^2 \xrightarrow{T} \sqrt{x^2} = |x|$ $\quad D(f) = \mathbb{R}$ $\quad H(f) = \langle 0, \infty)$

R, S na $A = \{1, 2, 3\}$

$R \circ S = \Delta_A$

$R \circ S = \{(1,1), (2,2), (3,3)\}$

	1	2	3
1	1	0	0
2	0	1	0
3	0	0	1

$A = \{1, 2, 3\}$

R - S, R, T

	1	2	3
1	1	1	0
2	1	1	0
3	0	0	1

$A = \{1, 2, 3\}$

R - $T, \neg R, \neg$ ref

	1	2	3
1	1	0	0
2	0	0	0
3	0	0	1

$R, \neg S, T$

	1	2	3
1	1	1	0
2	0	1	0
3	0	0	1

R, S_1, S_2

$R \circ (S_1 \cap S_2) = (R \circ S_1) \cap (R \circ S_2)$

$R = \{(1,1)\}$

$S_1 = \{(1,1)\}$ $\quad R \circ (S_1 \cap S_2) = R \circ S_1 = (1,1)$

$S_2 = \{(1,1)\}$ $\quad (R \circ S_1) \cap (R \circ S_2) = (1,1)$

$R_1 = \{(1,1)\}$ $\quad R \circ (S_1 \cap S_2) = \emptyset$

$S_1 = \{(1,2)\}$ $\quad R \circ S_1 \cap R \circ S_2 = \emptyset$

$S_2 = \{(1,3)\}$

$(1,1)$ $\quad R = \{(2,1),(4,1)\}$

$(2,1)$ $\quad S_1 = \{(1,2)\}$

$(3,1)$ $\quad S_2 = \{(1,4)\}$

$\left.\right\}$ $R \circ (S_1 \cap S_2) = \emptyset$

$R \circ S_1 \cap R \circ S_2 \neq \emptyset$

$R \circ \overset{a}{(S_1 \setminus S_2)} = \overset{b}{(R \circ S_1) \setminus (R \circ S_2)}$

$R =$

$S_1 = \{(1,1)\} \to \overset{a}{\emptyset} \mid \overset{b}{\emptyset}$

$S_2 =$

$R = \{(2,1),(4,1)\}$

$S_1 = \{(1,2)\}$ $\quad \to \{(1,1)\} \mid \emptyset$

$S_2 = \{(1,4)\}$

f

g $\quad f \circ g = h$

h $\quad f(x) = 3x - 1$

$g(x) = ?$

$h(x) = x^{10}$

$x \xrightarrow{g} \overset{\frac{x^{10}}{3}+1}{\bigcirc} \xrightarrow{f} 3\bigcirc - 1 = x^{10}$

$3 x^{10} - 1$

$3 \cdot \frac{x^{10}}{3} - 1$

$3 \cdot \left(\frac{x^{10}}{3} + 1\right) - 1 \Rightarrow x^{10}$

$$x \xrightarrow{g} 3x-1 \xrightarrow{f} \left(\frac{x}{3}+\frac{1}{3}\right)^{10} = x^{10}$$

$$\left(\frac{3x-1}{3}+\frac{1}{3}\right)^{10}$$

T

$x T y \Leftrightarrow |x|-|y| \leq 0$

R, S, T

$\forall x \in \mathbb{R};\ xTx$

$\forall x \in \mathbb{R};\ |x|-|x| \leq 0 \Rightarrow \forall x \in \mathbb{R};\ xTx \rightarrow$ je reflex.

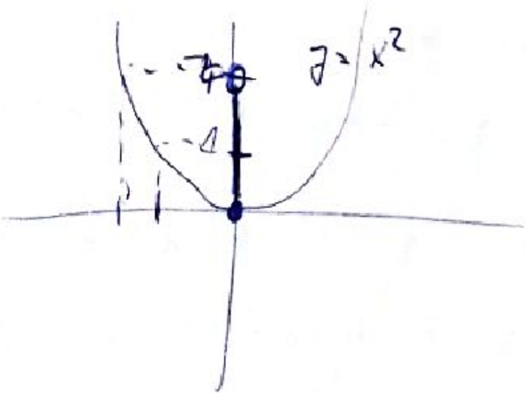
Nechť $xTy \Rightarrow |x|-|y| = |y|-|x| \leq 0 \rightarrow$ neplatí není sym.

Je Trans

---

nechť $(x,y) \in T \wedge (x,z) \in T \Rightarrow |x|-y = 1 \wedge |x|+z = 1 \Rightarrow$

$\Rightarrow |x| = 1-y \wedge |x| = 1-z \Rightarrow 1-y = 1-z \Rightarrow y = z$ Hurá!!!



obraz množiny:

$$F(M) = \{b;\ \exists a \in M\ F(a) = b\}$$

$F: \mathbb{R} \to \mathbb{R} \wedge F(x) = x^2$

$F = \{(x,y) \in \mathbb{R}^2;\ y = x^2\}$

$F(\langle -1, 2)) = \langle 0, 4)$

$F(\langle -1, 1\rangle) = \langle 0, 1\rangle$

$F(\mathbb{R}) = \langle 0, \infty)$

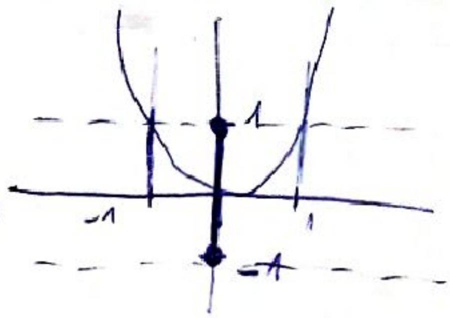
$F^{-1}(\langle -1, 1\rangle)$

$F(4) = 16$

$F(\{4\}) = \{16\}$

Úplný vzor množiny

$F^{-1}(P) = \{a,\ \exists b \in P\ F(a) = b\}$



$F(\langle -1,1\rangle) = F^{-1}(\langle 0,1\rangle) = \langle -1,1\rangle$

$F^{-1}(\langle 0,2\rangle) = \langle -\sqrt{2}, \sqrt{2}\rangle$

f, g sú zobr. $\overset{?}{\Rightarrow}$ $f \cup g$ je zobr.

Ne, např: $f = \{(1,1), (2,2)\}$ – zobr.
$g = \{(1,2), (2,1)\}$ – zobr.

$f \cup g = \{(1,1), (2,2), (1,2), (2,1)\}$ není zobr.
$(1,1) \in f \cup g \wedge (1,2) \in f \cup g$ ale $1 \neq 2$

$f \cap g$ je zobr?

Nechť $(a,b) \in f \cap g \wedge (a,c) \in f \cap g \Rightarrow$

$\Rightarrow ((a,b) \in f \wedge (a,b) \in g) \wedge ((a,c) \in f \wedge (a,c) \in g) \Rightarrow$

$\Rightarrow (a,b) \in f \wedge (a,c) \in f \wedge (a,b) \in g \wedge (a,c) \in g \xRightarrow{f, g \text{ jsou zobr.}}$

$\Rightarrow \underline{b = c}$

$A \Rightarrow B$

$B' \Rightarrow A'$

prosté zobrazení (injekce)

$\forall a,b \in D(f),\ a \neq b \Rightarrow f(a) \neq f(b)$

surjekce $A \to B$ $H(f) = B$

bijekce zobrazení A na B

$f: A \to B$ $\quad$ f je prosté zobr. $\Rightarrow f^{-1}$ je zobr.

Nechť $(a,b) \in f^{-1} \wedge (a,c) \in f^{-1} \Rightarrow (b,a) \in f \wedge (c,a) \in f \Rightarrow$

injekce

$a \neq b \Rightarrow f(a) \neq f(b)$

$f(a) = f(b) \Rightarrow a = b$

$(a,c) \in f \wedge (b,c) \in f \Rightarrow a = b$

$\Rightarrow f(b) = a \wedge f(c) = a \overset{f \text{ je prosté}}{\Rightarrow} \underline{\underline{b = c}}$

f, g jsou zobr $\overset{?}{\Rightarrow} f \circ g$ je zobr.

Nechť $(a,b) \in f \circ g \wedge (a,c) \in f \circ g \Rightarrow \exists d;\ (a,d) \in g \wedge (d,b) \in f \ \wedge$

$\exists e;\ (a,e) \in g \wedge (e,c) \in f \Rightarrow$

$\Rightarrow \exists d,e;\ \underline{(a,d) \in g \wedge (a,e) \in g} \wedge (d,b) \in f \wedge (e,c) \in f$

$\Rightarrow \underline{d = e} \wedge (d,b) \in f \wedge (e,c) \in f \Rightarrow (e,b) \in f \wedge (e,c) \in f \Rightarrow b = c$

$f: A \to B;\ f(x) = |x|$

$A = \{-2, -1, 0, 1, 2\}$

$B = \{0, 1, 2\}$

$a \sim b \Leftrightarrow f(a) = f(b)$

ref: ✓ $\quad (0,0)\ (1,1)\ (-1,-1)\ (2,2)\ (-1,1)\ (1,-1)\ (2,-2)\ (-2,2)$

sym: ✓

tran: ✓

rozklad:

$A/_\sim = \{\bar{0}, \bar{1}, \bar{2}\}$

mohutnost (kardinalita) množin

konečné

nekonečné

- spočitatelné ($\exists$ bijekce s $\mathbb{N}$)

- nespo ($\nexists$ ...)

$R = \{(1,2),(2,3),(3,1),(1,4)\}$

? R - tranz.? X

tranz. uzávěr ?

$R^+ = \{(1,2),(2,3),(3,1),(1,4); (1,3),(2,1),(3,4); (1,1),(2,4),(3,3)\}$ (3,2) (2,2)

R je tranz $\Leftrightarrow$ $R \circ R \subseteq R$

$\boxed{\mathbb{N}}$ - rel na $\mathbb{N}$

$(a,b) \mathbb{N} (c,d) \Leftrightarrow a+d = b+c$

ref. $\forall a,b \in \mathbb{N}; a+b = b+a \Rightarrow \forall a,b \in \mathbb{N}; (a,b) \sim (a,b)$ ✓

sym. ✓ Nechť $(a,b) \sim (c,d) \Rightarrow a+d = b+c \Rightarrow b+c = d+a \Rightarrow (c,d) \sim (a,b)$

tranz:

$(a,b) \mathbb{N} (c,d) \wedge (c,d) \sim (e,f) \Rightarrow a+d = b+c \wedge c+f = d+e \Rightarrow$

$\Rightarrow a+d+c+f = b+c+d+e$

$a+f = b+e$

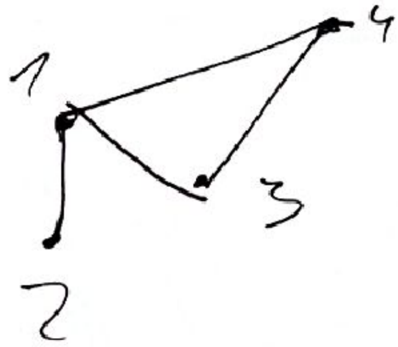
$(a,b) \sim (e,f)$ ✓

PŘEDNÁŠK

Teorie grafů

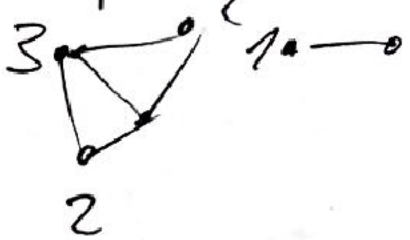
$G(V, E)$

$\{\ \}$



$V = \{1, 2, 3, 4\}$ $E = \{\{1, 2\}, \{1, 4\}, \{1, 3\}, \{3,$

stupeň vrcholu = počet paliček



{F i j 1

d-regulární graf → stejné stupně

→ 2 regulární graf

úplný graf → z každého do každého

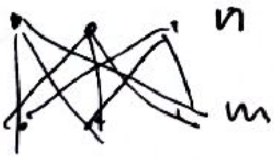
úpl

$\frac{n \cdot (n-1)}{2} = \binom{n}{2} = \binom{4}{2} =$

$=$

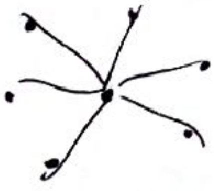
paliček

Úplný bipartitní graf

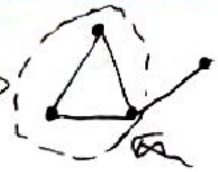


$n \cdot m$

hvězda



Podgrafy



$V(H) \subseteq V(G)$

$H \subseteq G$

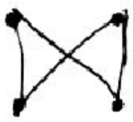
Indukovaný graf

← indukovaný

→ podgraf není indukovaný

Stejnost grafů

=



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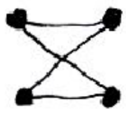
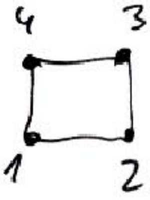
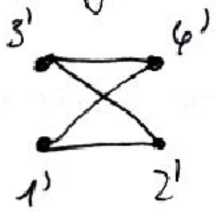


Isomorfismus ($\simeq$) grafů





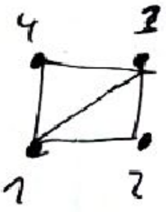
$f(1) = 1'$

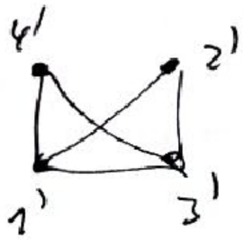
$f(2) = 2'$

$1 - 2$

$f(1) - f(2)$

$\quad 1' \qquad 2'$





CVIČENÍ

$xTy \iff |x|-|y| \le 0$

$\forall x \in \mathbb{R};\ |x|-|x| \le 0 \Rightarrow \forall x \in \mathbb{R};\ xTx$

tranz: $xTy \wedge yTz \Rightarrow |x|-|y| \le 0 \wedge |y|-|z| \le 0$

$(|x|-|y|)+(|y|-|z|) \le 0 \Rightarrow$

$\Rightarrow |x|-|z| \le 0 \Rightarrow xTz$

c.b.t.d.

$|x| \le |y| \wedge |y| \le |z| \Rightarrow |x| \le |z| \Rightarrow |x|-|z| \le 0 \Rightarrow xTz$

$R = \{(1,1), (2,1), (3,2), (3,3), (4,1)\}$

Je zobr?

$aRb \wedge aRc \Rightarrow b = c \Rightarrow$ není zobr

$R = \{(x,y) \in \mathbb{R}^2;\ x^2+y^2 = 1\}$

$S = \{(x,y) \in \mathbb{R}^2;\ x+y^2 = 1\}$ Jsou ZOBR?

$T = \{(x,y) \in \mathbb{R}^2;\ x^2+y = 1\}$

$R \to (0,1) \in R \wedge (0,-1) \in R \Rightarrow 1 \neq -1 \Rightarrow$ NENÍ ZOBR.

$S \to$ S S —||—

$T \to y = 1 - x^2$

$(x,y) \in T \wedge (x,z) \in T \Rightarrow y = z$

$x^2+y=1 \wedge x^2+z=1 \Rightarrow x^2+y = x^2+z$

$\Rightarrow y = z$ c.b.t.d.

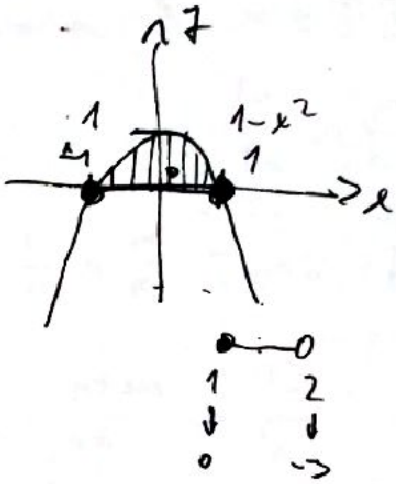
$f(x) = 1 - x^2$ ↙ hledám obraz

$f(\langle 1, 2 \rangle) = \langle -3, 0 \rangle$

$f(\langle -1, 1 \rangle) = \langle 0, 1 \rangle$

$f(\mathbb{R}) = \cancel{(-\infty, \infty) = H(f)}$

$(-\infty, 1\rangle = H(f)$

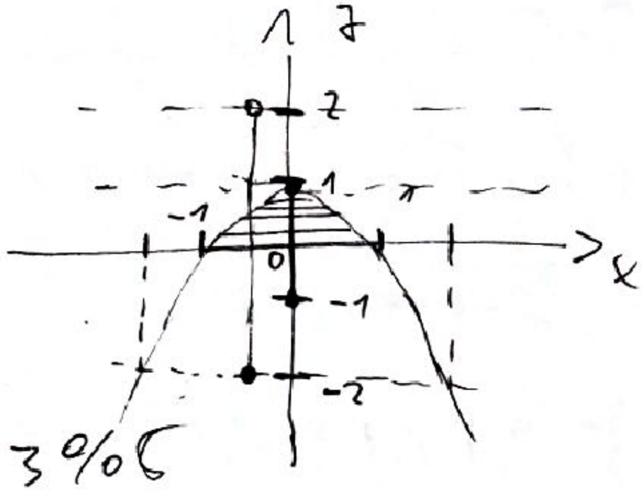


↙ hledám vzor

$f^{-1}(\langle -1, 1 \rangle) = \langle -\sqrt{2}, \sqrt{2} \rangle$

$f^{-1}(\langle -2, 2 \rangle) =$

$= f^{-1}(\langle -2, 1 \rangle) = \langle -\sqrt{3}, \sqrt{3} \rangle$



$-2 = 1 - x^2$

$x^2 = 3$

$x = \sqrt{3}$

$3 \% 2 = 3 \% 6$

$(a,b) \in \mathbb{N}^2$

$a \sim b \Leftrightarrow 3 \mid (a + 2b)$

$\overline{0}$	$\overline{2}$	$\overline{1}$
0	2	1
3	2	1
6	5	4
9	8	7
$3k$	$3k+1$	$3k+2$

+1	+2	+0	+1
$1 \sim 1$	$2 \sim 2$	$3 \sim 3$	$4 \sim 1$
$1 \sim 4$	$2 \sim 5$	$3 \sim 6$	$4 \sim 4$
$1 \sim 7$	8	9	7
10	11	12	10
13	14	15	13

$3k+1 \sim 3l+1 \rightarrow$ stejný zbytek

Zřejmě $\forall a \in \mathbb{N}$ platí, že $3 \mid 3a \Rightarrow$

$\Rightarrow \forall a \in \mathbb{N}_i\ 3 \mid (a + 2a) \Rightarrow a \in \mathbb{N}_i\ a \sim a$

Nechť $a \sim b \Rightarrow 3 \mid (a + 2b)$

$2 \sim 3 \wedge 3 \sim 1$ ale $2 \not\sim 1$

$\{0, 1, 2 \dots 9\}$

$a \sim b \Leftrightarrow 10a + b$ je prvočíslo

$(0,0) \notin \sim \Rightarrow \sim$ není ekv.

$2 \sim 3$ protože 23 je prvoč. ale $3 \not\sim 2$ protože 32 není prvoč $\rightarrow$ není sym

$1 \sim 1$

$\left.\begin{matrix} 1 \sim 3 \\ 3 \sim 1 \end{matrix}\right\}$ není antisym

$\{\frac{m}{n}; m \in \mathbb{Z}, n \in \mathbb{N}^*\}$ $\quad \frac{1}{2} \sim \frac{2}{4} \quad 1\cdot 4 = 2\cdot 2$

$\frac{m}{n} \sim \frac{m'}{n'} \iff m n' = m' n$

ref: zřejmě $\frac{m}{n} \sim \frac{m}{n}$ protože $m \cdot n = n \cdot m$

$\frac{m'}{n'} \sim \frac{m}{n} = m'n = mn'$

sym $\frac{m}{n} \sim \frac{m'}{n'} \Rightarrow mn' = m'n \Rightarrow m'n = m \cdot n' \Rightarrow \frac{m'}{n'} = \frac{m}{n}$

$\frac{m}{n} \sim \frac{p}{q} \wedge \frac{p}{q} \sim \frac{r}{s} \Rightarrow mq = np \wedge ps = qr \Rightarrow$

$\Rightarrow \frac{mq}{n} = p \wedge ps = qr \Rightarrow \frac{mq}{n} \cdot s = qr \Rightarrow$

můžu dělit $(n, q, s \neq 0)$ $\Rightarrow \frac{ms}{n} = r \Rightarrow$

$n, q, s \in \mathbb{N}^*$ $\quad \Rightarrow ms = rn \Rightarrow \frac{m}{n} \sim \frac{r}{s}$

$\frac{m}{n} \sim \frac{p}{q} \iff mq = np$

$T = (0,1)$

$A_t = (t, t+1)$

0 $\swarrow$ $\frac{1}{2}$ $\Rightarrow$ 1

$A_{\frac{1}{2}}$ $\quad A_{\frac{1}{2}} = (\frac{1}{2}, \frac{3}{2})$

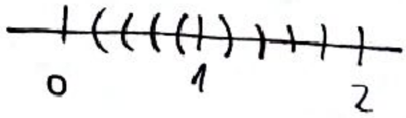
$A_{\frac{1}{2}} = (\frac{1}{2}, \frac{3}{2})$ $\quad A_{0,9} = (0,9; 1,9)$

$A_{0,1} = (0,1; 1,1)$ $\quad A_{0,9999} = (0,9999; 1,9999)$

$\downarrow \; 1 \quad \downarrow \; 1 \quad \downarrow \; 2$

$A_{0,001} = (0,001; 1,001)$

$\downarrow \; 0 \quad \downarrow \; 0 \quad \downarrow \; 1$



$\bigcap_{t \in T} A_t = \{1\}$

$\bigcup_{t \in T} A_t = (0,2)$

$1 \in \bigcap_{t\in T} A_t$ $\forall t \in T;\ 1 > t$, $1 < t+1$ $A_t = (t, t+1) = \{x;\ t < x < t+1\}$

① zřejmě pro $\forall t\in T;\ t<1<t+1 \Rightarrow \forall t\in T;\ 1\in A_t \Rightarrow 1\in\bigcap_{t\in T}A_t$

② $x\neq 1,\ x\notin \bigcap_{t\in T} A_t \Rightarrow$ potř. dokázat:

když $x\neq 1 \Rightarrow \exists t\in T;\ x\notin A_t$

Nechť $x\in(-\infty,0\rangle \Rightarrow t=\frac12 \Rightarrow A_{\frac12}=(\frac12,\frac32) \Rightarrow x\notin \bigcap_{t\in T}A_t$

$\forall x\in(-\infty,0) \Rightarrow x\notin(\frac12,\frac32)$

Nechť $x\in(0,1) \Rightarrow t=x \Rightarrow A_t=(t,t+1)=(x,x+1)$

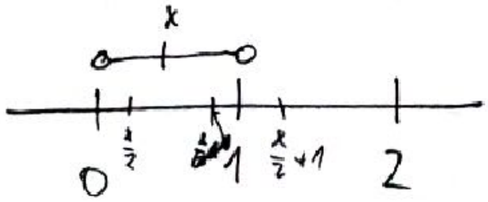
zřejmě $x\notin A_t \Rightarrow x\notin\bigcap_{t\in T}A_t$

Nechť $x\in (1,2) \Rightarrow t = x-1;\ A_t(t;t+1)=(x-1,x)$

Nechť $x\in(2,\infty) \Rightarrow t=\frac12 \Rightarrow x\notin A_{\frac12}\ (\frac12,\frac32) \Rightarrow \forall x\notin\bigcap_{t\in T}A_t$

① $\forall x\in(0,2);\ x\in\bigcup_{t\in T}A_t$

② $\forall x\notin(0,2);\ x\notin\bigcup_{t\in T}A_t$



(platí pro $x<2$)

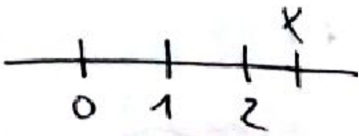
① T

Nechť $x\in(0,2) \Rightarrow \frac x2\in(0,1)$ a $\frac x2 < x$ a $\frac x2+1 > x$

$\frac x2 < x < \frac x2+1$

$\downarrow$

$x\in(\frac x2,\frac x2+1)=A_{\frac x2} \Rightarrow x\in\bigcup_{t\in T}A_t$



② Třeba dokázat, že když $x\notin(0,2) \Rightarrow x\notin$ do žádné množ. A_t

Nechť $x\in\langle 2,\infty) \Rightarrow t<1 \Rightarrow t+1<2$ a proto ani jedno číslo ≥ 2 nepatří do A_t

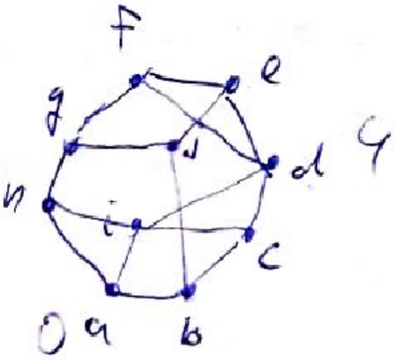
pravý konec

Nechť $x\in(-\infty,0\rangle \Rightarrow t>0$, pak když $x\geq 0$ nemůže patřit do A_t

2/13 a
2/14

~~2. sada DU prem. bod~~

FIFO / LIFO



Fronta
do šířky

a h i b g x d c j x f x c f e x e e

zásobník
do hloubky

a h i b   a 3 b 3 c 2 d
2 i 2 j 1 i
1 h 1 a

Vzdálenost v grafu (nejmenší počet hran)

vážený graf (hrana má číslo)

$A_t = \left(1, 1+\frac{1}{2-t}\right)$; $t \in (a,b)$; $\bigcap_{t \in (a,b)} A_t = ?$ $\bigcup_{t \in (a,b)} A_t = ?$

$\underset{?}{t \in (a,b)}$ $t \in (-\infty, 2)$

1

$1, \cap (x \in 2)$

sled grafu
- počet hran od 1 - n

tah grafu
- nelze opakovat hrany

Eulerova charakteristika

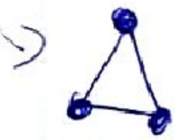
Graf lze nakreslit jedním tahem, když vrcholy jsou sudé

Rovinný graf $\qquad$ vrcholy + stěny - hrany = 2

Jednoduchý rovinný graf (jedna hrana mezi vrcholy)

$v \geq 3 \Rightarrow 3v-6$ hran

vrcholy $\quad (v \geq 3 \Rightarrow 2v-4$ hran bez $\Delta)$

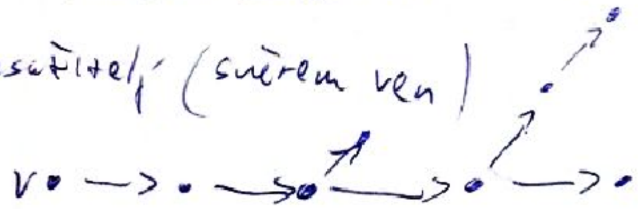


$2 \cdot h \geq 3 \cdot f$ (stěna)
3 $\quad$ 2

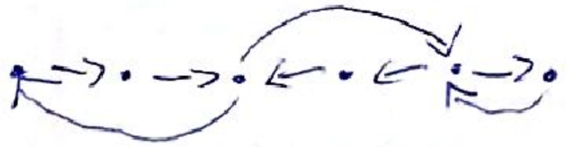
Není rovinný, když obsahuje $K_{3,3}$ v K_5

slabá souvislost (řešíme hrany ne šipky)

dosažitelný (směrem ven)



silná souvislost



$U \simeq V$ je relace ekvivalence $\Leftrightarrow U \to V$ a $V \to U$

Definice sítě a toku

zdroj ... stok

rezerva $Z \xrightarrow[+1]{3/4} \bullet \xrightarrow[+1]{1/2} \bullet \xleftarrow[+2]{2/3} \bullet \xrightarrow[+1]{1/2} S$ → nenasycená

rezerva $\boxed{+1}$ $Z \xrightarrow[+0]{4/4} \bullet \xrightarrow[+0]{2/2} \bullet \xleftarrow[+1]{1/3} \bullet \xrightarrow[+0]{2/2} S$ → nasycená

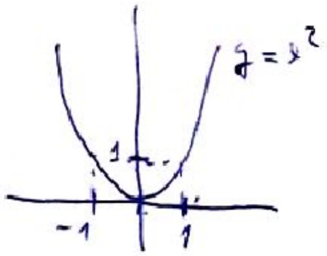
---

izomorfní grafy (rovinně nakreslit)

alespoň 1 na hledání nejkratší cesty

—— // —— kostry

$A \overset{?}{=} f^{-1}(f(A))$



$f(x) = x^2$

$A = \langle 0, 1 \rangle$

$L = \langle 0, 1 \rangle$

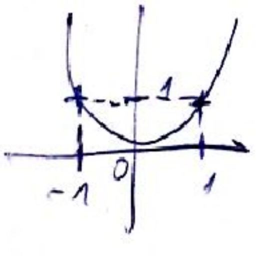
$P = f^{-1}(f(A)) = f^{-1}(\langle 0, 1 \rangle) \neq \langle -1, 1 \rangle$

$f(A) = f(\langle 0, 1 \rangle) = \langle 0, 1 \rangle$

$A = \langle -1, 1 \rangle$

$f^{-1}(f(\langle -1, 1 \rangle)) = f^{-1}(\langle 0, 1 \rangle) = \langle -1, 1 \rangle$

$f(A_1 \cap A_2) \overset{?}{=} f(A_1) \cap f(A_2)$



$A_1 = \langle 0, 1 \rangle$

$A_2 = \langle -1, 0 \rangle$

$f(A_1 \cap A_2) = f(\{0\}) = \{0\}$

$f(A_1) \cap f(A_2) = \langle 0, 1 \rangle \cap \langle 0, 1 \rangle = \langle 0, 1 \rangle$

~~$f(A_1)\cap$~~ $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ $\quad f(M) = \{y ; \exists x \in M ; f(x) = y\}$

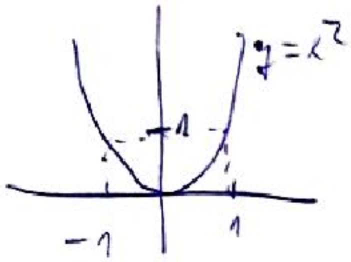
nech: $y \in f(A_1 \cap A_2) \Rightarrow \exists x \in A_1 \cap A_2 ; f(x) = y \Rightarrow \exists x ; x \in A_1 \wedge x \in A_2 \wedge f(x) = y \Rightarrow$

$\Rightarrow y \in f(A_1) \wedge y \in f(A_2) \Rightarrow y \in f(A_1) \cap f(A_2)$

„$\supseteq$"

$y \in f(A_1) \cap f(A_2) \Rightarrow y \in f(A_1) \wedge y \in f(A_2) \Rightarrow \exists x_1 \in A_1 ; f(x_1) = y ; \exists x_2 \in A_2 ; f(x_2) =$

$f(A_1 \setminus A_2) \overset{?}{=} f(A_1) \setminus f(A_2)$



$f(x) = x^2$

$A_1 = \langle 0, 1\rangle$

$A_2 = \langle -1, 0\rangle$

$f(A_1 \setminus A_2) = f((0, 1\rangle) = (0; 1\rangle$

$f(A_1) = \langle 0, 1\rangle$

$f(A_2) = \langle 0, 1\rangle$

$f(A_1) \setminus f(A_2) = \emptyset$

$A_1 = \langle 0, 1\rangle$

$A_2 = \langle 0, 1\rangle$

$f(A_1 \setminus A_2) = f(\emptyset) = \emptyset$

$f(A_1) \setminus f(A_2) = \langle 0, 1\rangle \setminus \langle 0, 1\rangle = \emptyset$

$A_1 = \langle 1, 2\rangle$

$A_2 = \langle 3, 4\rangle$

$f(A_1 \setminus A_2) = f(A_1) = \langle 2, 4\rangle$

$f(A_1) \setminus f(A_1) = \langle 2, 4\rangle \setminus \langle 9, 16\rangle = \langle 2, 4\rangle$

Binární operace (A×A do A)

$\mathbb{N}$ +, ~~-~~, *, ~~:~~

komutativní
asociativní

Neutrální prvek $a+\boxed{0}=0+a=a$ $a\cdot\boxed{1}=1\cdot a$

Inverzní prvek X

$\mathbb{Z}$ +, ·, -

$\mathbb{Q}$ +, ·, -, ÷ $\approx \mathbb{R}$

komut

	a	b	c
a	a	b	c
b	b	a	a
c	c	a	a

asoc

n.p.

	a	b	c
a	a	b	c
b	b	a	a
c	c	a	a

a = n.p.

inv

	a	b	c
a	a	b	c
b	b		a
c	c	a	

kolik může být operace neutrálních prvků ?

$e_1 \neq e_2$ $\quad a \circ e_1 = a = e_1 \circ a \quad \forall a \in A$

$a \circ e_2 = e_2 \circ a = a$

$e_1 = e_1 \circ e_2 = e_2 \Rightarrow$ ~~sl~~ $e_1 = e_2$ V rozporu s předpokladem

$\circ$ - asoc. s neut. pr. $\wedge$ $\exists a$, které má 2 inv. prvky

$\cancel{a \circ a' = a' \circ a = e}$

$a \circ b = b \circ a = e$ $\quad$ b, c jsou inv. pr.

$a \circ c = c \circ a = e$

$b \neq c$

$b = e \circ b = (c \circ a) \circ b = c \circ (b \circ a) = c \circ e = c$

op. n.p. / inv. pr. je max 1 } je asoc?

Algebry s jednou operací

Grupoid -> ~~jen~~ platí uzavřenost

G - nosič grupoidu

H - nosič podgrupoidu

$H \subseteq G$

$a * b = a \circ b ; \forall a, b \in H$

$M = \{1,2,3\}$

$(M,\circ)$ – grupoid

a) právě 1 podgrupoid

	1	2	3
1	2	1	1
2	1	3	1
3	1	1	1

$\{1,2,3\}$

~~Hele~~

b) právě 2 podgrupoidy

	1	2	3
1	1	3	2
2	3	3	2
3	3	3	1

$\{1,2,3\}$ $\{1\}$

~~$\{1,2\}$~~

$\circ$	1	2	3
1	1	3	2
2	1	2	1
3	1	1	1

$(\{1,2,3\},\circ)$ ✓

$(\{1\},\circ)$ ✓

$(\{2\},\circ)$ ✓

$\circ$	1	2	3
1	2	1	
2	1	1	
3			

$(\{1,2\},\circ)$ →

$(\{2,3\},\circ)$ → NEJDE

$(\{1,2,3\},\circ)$ ✓

pologrupa → asociativní operace

pologrupa s neutrálním prvkem → **monoid**

monoid v kterém mají všichni inverzní → **grupa**

grupa s komutativní operací je **komutativní** nebo **Abelova grupa**

$(\mathbb{Z}, \circ)$ je to GRUPA?

$a \circ b = a + b - 2$

1) uzavřenost

nechť $a, b \in \mathbb{Z} \Rightarrow a+b-2 \in \mathbb{Z} \rightarrow a \circ b \in \mathbb{Z}$.

2) asociativnost

$(a \circ b) \circ c = (a+b-2) \circ c = (a+b-2) + c - 2 = a+b+c-4$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$a \circ (b \circ c) = a \circ (b+c-2) = a + (b+c-2) - 2 = a+b+c-4$

3) neutrální prvek

$a \circ e = e \circ a = a$

$a \circ e = \; a + e - 2 = a$
$\qquad\qquad e = 2$

$e \circ a = a$
$e + a - 2 = a$
$e = 2$

4) inverzní prvek

$a \circ a' = e$
$a \circ a' = 2$
$a + a' - 2 = 2$
$a' = 4 - a \in \mathbb{Z}$

(uzavřenost)
$\rightarrow$ grupoid

$\downarrow$ + asoc

$\rightarrow$ pologrupa

$\downarrow$ + NP

$\rightarrow$ monoid

$\downarrow$ + IP

$\rightarrow$ grupa

$\downarrow$ + kom.

$a \circ b = a + b - 2 = b + a - 2 = b \circ a \rightarrow$ abelova grupa

---

$\mathbb{Z}$

$a \sim b \Leftrightarrow 2 \mid (a - b)$

zbytek

$\bar{0}$	$\bar{1}$
sudé	liché

$\mathbb{Z}/_2 = \{\bar{0}, \bar{1}\}$

$(\mathbb{Z}_2, +)$

+	0	1
0	0	1
1	1	0

Grupa

$(\mathbb{Z}_3, +)$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Grupa

$(\mathbb{Z}_3 \setminus \{0\}, \cdot)$

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$\cup / \{0\}$

	1	2
1	1	2
2	2	1

Grupa

$(Z_4, +)$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

A. GRUPA

$(Z_4, \cdot)$

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$(Z_n, +) \to$ GRUPA

$(Z_p, +) \to$ grupa

$(Z_p \setminus \{0\}, \cdot) \to$ GRUPA

pravidla

	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

$F_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $F_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $F_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ F

$F_4 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $F_5 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $F_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

	F_1	F_2	F_3	F_4	F_5	F_6
F_1	F_1	F_2	F_3	F_4	F_5	F_6
F_2	F_2	F_1	F_5	F_6	F_3	F_4
F_3	F_3	F_4	F_1	F_2	F_6	F_5
F_4	F_4	F_3	F_6	F_5	F_1	F_2
F_5	F_5	F_6	F_2	F_1	F_4	F_3
F_6	F_6	F_5	F_4	F_3	F_2	F_1

$F_2 \circ F_3$ $F \neq F$ f f

$1 \to \underline{2} \to 3$

lepidlo

$F_5 \circ F_3 = F_2$ $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$1 \to 2 \to 1$

$2 \to 1 \to 3$

$3 \to 3 \to 2$

$F_4 \circ F_5 = F_1$

1 2 3

1 2 3

$F_5 \circ F_4 = F_1$ $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$4 \to 2 \to 1$

$2 \to 3 \to 2$

$3 \to 1 \to 3$

F_1
F_2
F_3
F_4
F_5
F_6

$\left.\begin{matrix}F_1\\F_2\\F_3\\F_4\\F_5\end{matrix}\right\} \circ \left.\right\} \{F_1, F_3\} \begin{cases} = (F_1, F_3) \\ = (F_2, F_5) \\ = (F_3, F_1) \\ = (F_4, F_6) \\ = (F_5, F_2) \\ = (F_6, F_4) \end{cases}$

F_1	F_3
F_2	F_5
F_4	F_6

$(\{F_1, F_3\}, \circ)$ levý rozklad

$a \circ b \overset{?}{=} a \circ c$

$a' \circ (a \circ b) = a' \circ (a \circ c)$

$(a' \circ a) \circ b = (a' \circ a) \circ c$

$e \circ b = e \circ c$

$b = c$

musí vyjít 2 různé

$1 \circ 2$
$1 \circ 3$ jsou jiný

Largangeova věta. Počet prvků podgrupy se ~~který~~ ~~jako~~ počet dělitelem počtu prvků grupy ~~dělitelem~~ ~~počtu~~

$6 \to 2, 2, 2$

$6 \to 3, 3$

$6 \to 1, 1, 1, 1, 1, 1$

$6 \to 6$

pravý rozklad

$\{F_1, F_3\} \circ F_1 = \{F_1, F_3\}$

$\circ F_2 = \{F_2, F_4\}$

$\circ F_3 = \{F_3, F_1\}$

$\circ F_4 = \{F_4, F_2\}$

$\circ F_5 = \{F_5, F_6\}$

$\circ F_6 = \{F_6, F_5\}$

$\boxed{L = P \quad \text{normální}}$

$(\mathbb{Z}, +)$ – grupa

$H = \{3 \cdot x;\ x \in \mathbb{Z}\} \rightarrow \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$

? $(H, +)$ je podgrupa $(\mathbb{Z}, +)$?

Nechť H je neprázdná $\subseteq G$
$(H, \circ)$ je podgrupa $(G, \circ)$

$\forall a, b \in H;\ a \circ b^{-1} \in H$

$a \circ b' = 3k + (-3l) = 3(k - l) \in H$

$a + 3x + (-a) = 3x \in H \rightarrow$ normální podgrupa

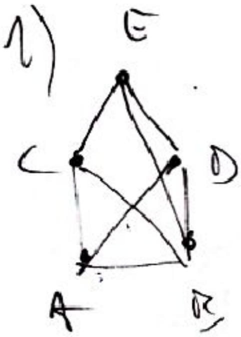
$a = 3k$

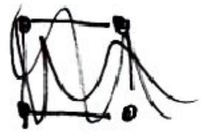
$b = 3l \rightarrow b' = -3l$

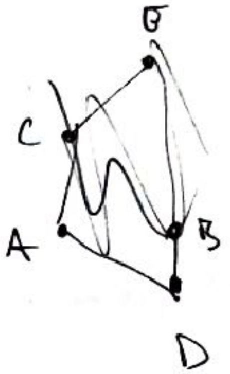
$3v - 6$ △

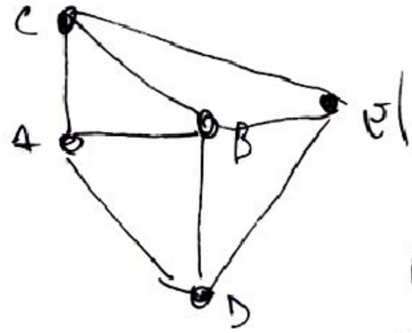
$2v - 4$ ~~△~~

1)

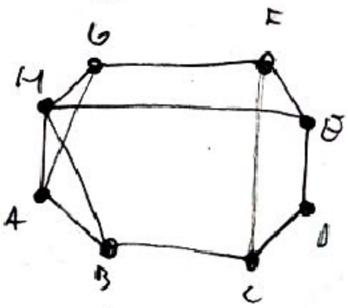


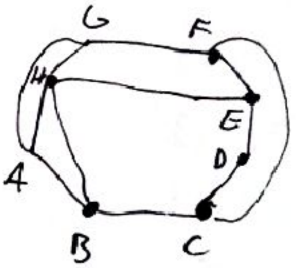




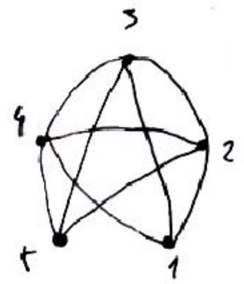


2)





3)

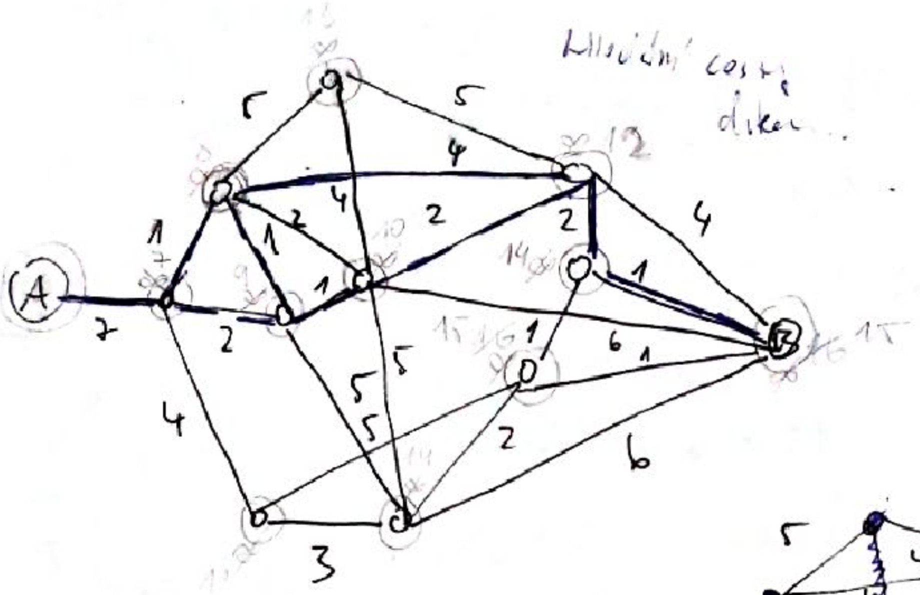


$\rightarrow 3 \cdot 5 - 6 = 9$

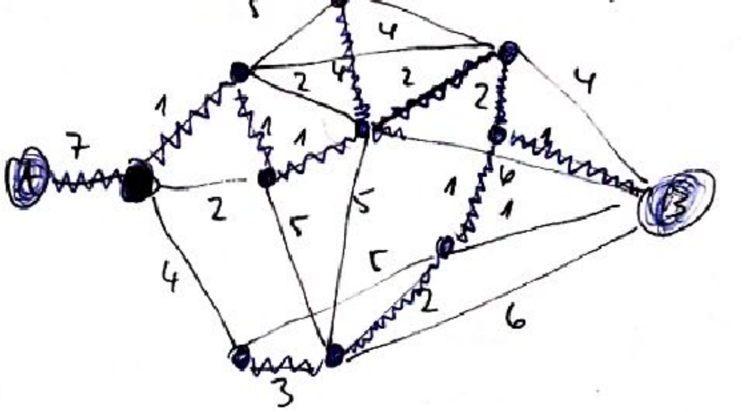
4)

→ podrozdělení K_5
není rovinný

Hledání cesty díky...



Hladový alg.



$A = \{1, 2, \emptyset, (1,2)\}$

$B = \{(2,1), \emptyset, 2\}$

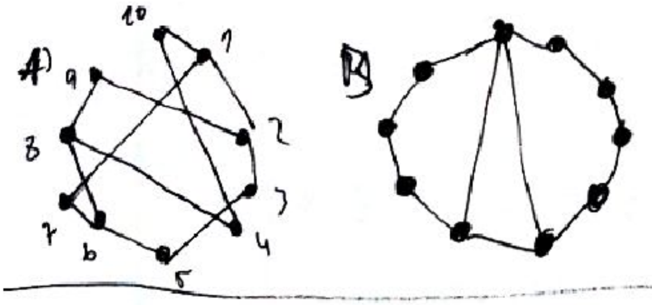
$C = \{\{1\}, \{2\}, \emptyset\}$

$(A \cap C) \times B$

$B \setminus A$

$A \cap C) \times B = \emptyset \times B = \{(\emptyset, (2,1)), (\emptyset, \emptyset), (\emptyset, 2)\}$

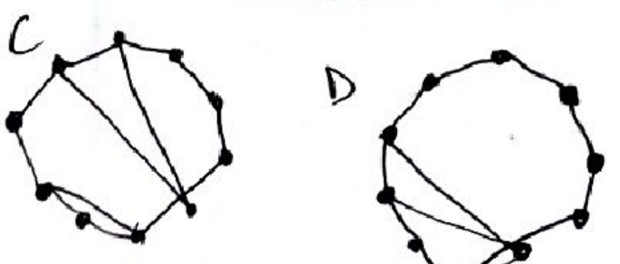
$B \setminus A = \{(2,1)\}$



A 2 2 2 2 2 3 3 3 3

B 2 2 2 2 2 2 2 3 3 4

$A \not\cong B$ $B \not\cong C$ $B \not\cong D$



C 2 2 2 2 2 2 3 3 3 3

D 2 2 2 2 2 2 3 3 3 3

$A \not\cong C$ → A nemá Δ, C má 2 Δ

$D \not\cong C$ $D \not\cong A$

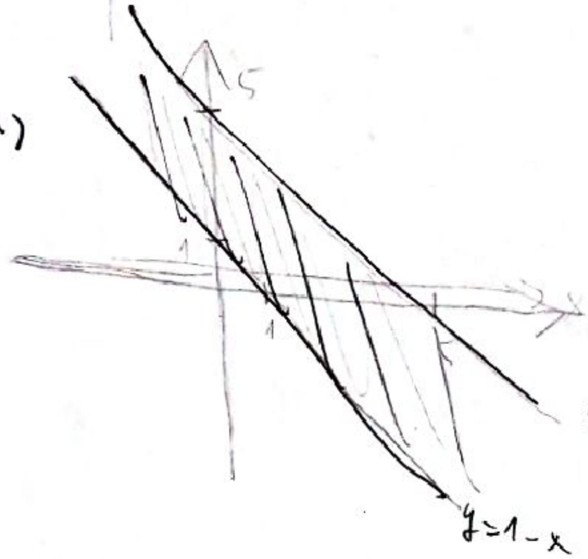
4)

a) $R = \{(x,y) \in \mathbb{R}^2;\ 1 \le x+y \le 5\}$ $\quad x+y \in \langle 1, 5 \rangle$

b) $S = \{(x,y) \in \mathbb{R}^2;\ 4 \le x^2+y^2 \le 16\}$

c) $R = \{(x,y) \in \mathbb{R}^2;\ |x+y| \le 3\}$

a)

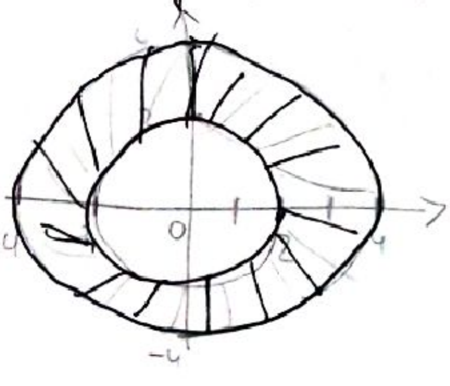


$1 \le x+y \wedge x+y \le 5$

$y = 1 - x \qquad y = 5 - x$

b) $4 \le x^2+y^2 \wedge x^2+y^2 \le 16$

$x^2+y^2 = 4 \wedge x^2+y^2 = 16$

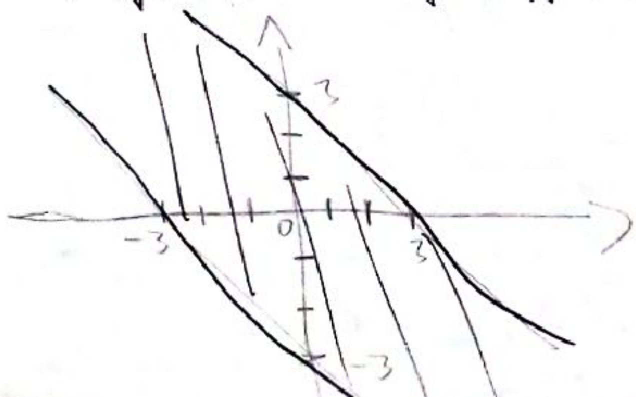


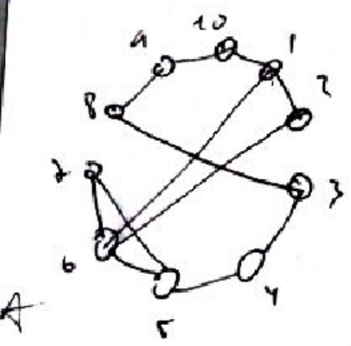
c) $|x+y| \le 3$

$|x+y| = 3 \to x+y=3 \vee x+y=-3 \to -3 \le x+y \le +3$

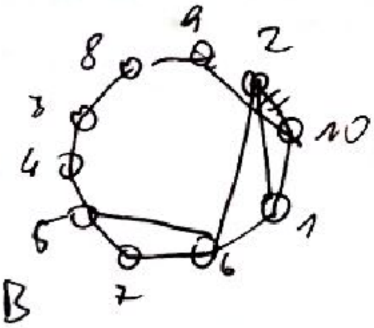
$x+y=3$

$x+y=-3$





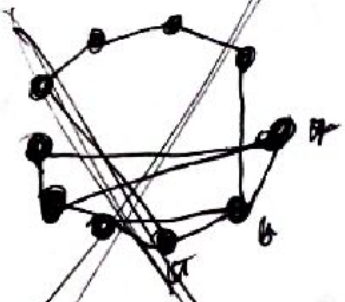
2 222222 334



2222222334

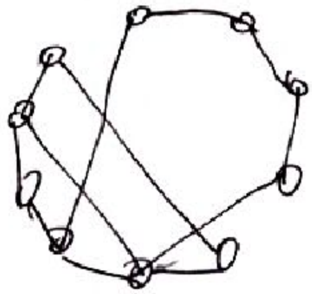
$\boxed{A \simeq B}$

$A \sim C$
$\Downarrow$
$B \sim C$



2222222334

D



2222222334

$\to$ nema Δ $\Rightarrow$ $A \not\simeq D$

$|x-2|$

$R = \{(x,y) \in \mathbb{Z}^2,\ |x-2| + |y+4| = 3\}$

0	3
1	2

$|x-2| = 0 \to x = 2$

$|y+1| = 3 \to y = 2 \vee y = -4$

$[2,2],\ [2,-4]$

$|x-2| = 1 \to x = 1, 3$

$|y+1| = 2$

Co platí v grupě   A.P.

I.P.

zákon o krácení (sudoku)

$\Downarrow$

Lag. ~~v~~ (Lagrangeova věta

$\Downarrow$

podgrupy

normální   ~~nenormální~~ není

Podgrupa $(H, \circ)$ grupy $(G, \circ)$ je NORMÁLNÍ PODGRUPA pokud libovolné $a \in G$, $h \in H$; $a \circ h \circ a^{-1} \in H$

Morfizmy

$*$		
	0	0
	0	0

$\circ$		
	1	1
	1	1

$h(a * b) = h(a) \circ h(b)$ -> morfizmus grupy A do grupy B

$h(x * y) = h(x) \circ h(y)$

① $h(0) = 0$
$h(1) = 0$

② $h(1) = 0$
$h(0) = 1$

surjekce -> pokrýje H
bijekcie -> každý obraz má vzor

$h(0 * 0) = h(0) \circ h(0)$

$0 = h(0) \neq 0 \circ 0 = 1$

$h(0 * 0) \overset{?}{=} h(0) \circ h(0)$

$1 = h(0) = h(1 * 0) = h(1) \circ h(0) = 0 \circ 1 = 1$

$1 = h(0) = h(1 * 1) = h(1) \circ h(1) = 0 \circ 0 = 1$

$1 = h(0) = h(\_ * \_) = h(\_) \circ h(\_) = 1$

$*$	0	1
0	0	0
1	1	0

$\circ$	0	1
0	1	0
1	0	0

$h(x*y) = h(x)\circ h(y)$ h je bijekce

① $\cancel{h(0)=0,\ h(1)=1}$

② $h(0)=1$, $h(1)=0$

$0 = h(0) = h(0*0) \neq h(0)\circ h(0) = 1$

$(\mathbb{Z}_4, +)$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\rightarrow$

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$h(0)=1$
$h(2)=4$
$h(1)=2$
$h(3)=3$

pojmenuji

jsou izomorfní

	1	2	4	3
1	1	2	4	3
2	2	4	3	1
4	4	3	1	2
3	3	1	2	4

$\rightarrow$

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

morfizmy

$h(a*b) = h(a)\circ h(b)$ $\rightarrow$ morfizmus

mono – " – injektivní
epi – " – surjektivní
izo – " – bijektivní
endo – " – morfizmus A do A
auto – " – izomorfizmus A na A

$(\mathbb{Z}_4, +)$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	3

$\dotplus$	0	1
0	0	1
1	1	0

$h(a+b) = h(a) \dotplus h(b)$ a h je surjektivní

$h(2) = h(1+1) = h(1) \dotplus h(1) = 0 \dotplus 0 = 0$

$h(3) = h(1+2) = h(1) \dotplus h(2) = 0 \dotplus 0 = 0$

$h(0) = 0$
$h(1) = 0$
$h(2) = 0$
$h(3) = 0$

$h(0) = 0 \Rightarrow$ sud
$h(1) = 1 \Rightarrow$ lich
$h(2) = 0$
$h(3) = 1$

$h(2) = h(1+1) = h(1) \dotplus h(1) = 0$

$h(3) = h(1+2) = h(1) \dotplus h(2) = 1$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$h(0) = 0$
$h(1) = 0$
$h(2) = 0$
$h(3) = 0$

$h(0) = 0$
$h(1) = 1$ surj ✓
$h(2) = 2$
$h(3) = 0$

$h(0) = h(2+2) \neq h(2) + h(2) = 2 + 2 = 1$

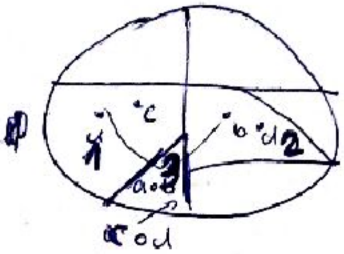
$h(0) = 0$ $h(2) = 1$
$h(1) = 2$ $h(3) = 0$

$h(2) = h(1+1) = h(1) + h(1) = 1$

$h(3) = h(1+2) = h(1) + h(2) = 2 + 1 = 0$

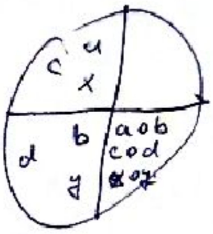
(X)

R je ekv na X =>



X

$1 \circ 2 \Rightarrow 3$



∘ je operace na X -> když je slušná operace, R je kongruence

$[a,b] \in R \wedge [c,d] \in R \Rightarrow [a \circ c, b \circ d] \in R$

$\mathbb{Z}$

$(\mathbb{Z}_3, +)$

$(\mathbb{Z}_3, \circ)$

∘	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$r: a\,r\,b \Leftrightarrow 3 \mid (a-b)$

R ✓
S ✓ } ekv
T ✓

$\bar{0}$	$\bar{1}$	$\bar{2}$
0	1	2
3	4	5
6	7	8
⋮	⋮	11

$a\,r\,b \wedge c\,r\,d \Rightarrow (a+c)\,r\,(b+d)$

$3 \mid (a-b) \wedge 3 \mid (c-d) \Rightarrow 3 \mid (a+c)-(b+d) = 3 \mid ((a-b)+(c-d))$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

kongruence $R = \{[x,y];\ x,y \in G \wedge y^{-1} \in H\}$

∘	f_1	f_2	…
f_1	.	.	.
f_2	.	.	.
⋮			

Svaz $\wedge$ průsek

$\vee$ spojení

modulární:

$\forall x,y,z \in X;\ x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$

distributivní

$\forall x,y,z \in X;$

komplement

komplementární

ak distributivní $\overset{?}{\Rightarrow}$ je modulární

DŮKAZ:

nechť $x \leq z \Rightarrow x \vee (y \wedge z) \overset{distr.}{=} (x \vee y) \wedge (x \vee z) =$

$= (x \vee y) \wedge (z)$

Ak modulární $\overset{?}{\Rightarrow}$ v.h.

Ak dus nie je modulární $\overset{?}{\Rightarrow}$ není dist.

pětiúhelník

$X = \{1,2,3,4,12\}$, rel. děl. ($\wedge$)

? svaz

? alg?

$\overline{1} = 12$ $\vee$ → největší (12)

$\overline{12} = 1$ $\wedge$ → nejmenší (1)

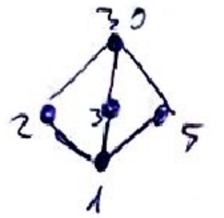
$\overline{2} = 3$

$\left.\begin{matrix} 4 \vee 3 = 12 \\ 4 \wedge 3 = 1 \end{matrix}\right\}$ $\overline{4} = 3$, $\overline{3} = 4$

$\left.\begin{matrix} 2 \leq 4 \Rightarrow 2 \vee (3 \wedge 4) \overset{?}{\neq} (2 \vee 3) \wedge 4 \\ 2 \vee 1 \neq 12 \wedge 4 \\ 2 \neq 4 \end{matrix}\right\}$ není modulární ~~komplementární~~

⇒ není distributivní

$X = \{1, 2, 3, 5, 30\}$ (D_n)



$\bar{1} = 30$ $\quad \bar{5} = 2,\ 3$

$\overline{30} = 1$

$\bar{2} = 3,\ 5$

$\bar{3} = 2,\ 5$

svaz je komplementární (každý prvek má komplement)

$* \quad 7$

$1 \leq 2$

$1 \vee (3 \wedge 2) = (1 \vee 3) \wedge 2$

$1 \vee 1 = 3 \wedge 2$

$1 = 1$

svaz je modulární

$2 \vee (3 \wedge 5) \overset{?}{=} (2 \vee 3) \wedge (2 \vee 5)$

$2 \vee 1 \overset{?}{\neq} 30 \wedge 30$

$2 \overset{?}{\neq} 30$

svaz není distributivní

když je dist => každý prvek max 1 komplement

Důkaz:

$a \wedge \bar{a}_1 = 0 \quad a \wedge \bar{a}_2 = 0 \qquad \bar{a}_1 \neq \bar{a}_2$

$a \vee \bar{a}_1 = 1 \quad a \vee \bar{a}_2 = 1$

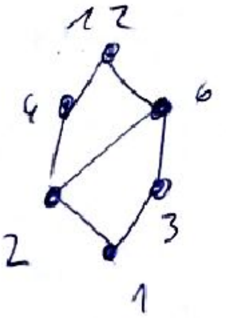
$\bar{a}_1 = \bar{a}_1 \vee 0 = \bar{a}_1 \vee (a \wedge \bar{a}_2) = (\bar{a}_1 \vee a) \wedge (\bar{a}_1 \vee \bar{a}_2)$

$= (a \vee \bar{a}_2) \wedge (\bar{a}_1 \vee \bar{a}_2) = (a \wedge \bar{a}_1) \vee \bar{a}_2 =$

$= 0 \vee \bar{a}_2 = \bar{a}_2$

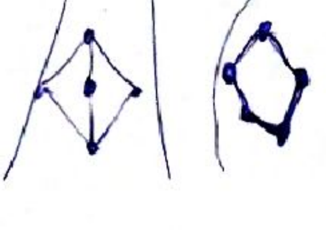
$X = \{1, 2, 3, 4, 6, 12\}$ kompl.? | modul. | distr.

X | Dí | Dí



modulární <=> neobsahuje podsvaz izomorfní s N_5

distributivní <=> neobsahuje podsvaz izom. s M_5 ani s N_5



$(a\circ b)\circ c \underset{asoc}{=} a\circ(b\circ c)$ $\qquad$ $\underset{komu}{a\circ b = b\circ a}$

1) k A

$\circ$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

2) K, ¬A

$\circ$	a	b	c
a	a	b	c
b	b	a	a
c	c	a	a

2)
¬k A

$\circ$	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

4) ¬k ¬A

$\circ$	a	b	c
a	b	a	a
b	a	a	a
c	b	a	c

1)

$\circ$	a	b	c
a	a	b	c
b	b	b	b
c	c	b	b

$\circ$	a	b	c
a	a	b	c
b	b	c	c
c	c	c	c

2)

$b\circ(b\circ c) \neq (b\circ b)\circ c$

$b\circ a \neq a\circ c$ ¬A

$b \neq c$

3)

$\circ$	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

$a\circ a = a$

$a\circ b = b$

$a\circ c = c$

4)

$R = \{(0,1), (0,3), (1,2), (1,3), (2,1), (3,0)\}$

$R^+ =$

$= \{(0,1), (0,3), (1,2), (1,3), (2,1), (3,0) \vdots (0,2), (1,1), (1,0), (2,2), (2,3), (3,1)$
$(3,3) \vdots (0,0), (2,0), (3,2)\}$

o	a	b	c	d
a	b	b	b	b
b	a	c	a	b
c	c	d	a	b
d	a	a	c	a

o	a	b	c	d
a	a	c	a	c
b	c	b	a	a
c	b	a	d	a
d	a	a	a	b

o	a	b	c	d
a	a	c	d	b
b	c	b	a	a
c	b	a	c	a
d	b	a	a	d

1 podgrupoid

$(\{a,b,c,d\}, \circ)$

$x^2 \to (x^2)^2 = x^2$

3 podgrupoidy

$(\{a,b,c,d\}, \circ)$
$(\{a\}, \circ)$
$(\{b\}, \circ)$

5 podgrupoidů

$(\{a,b,c,d\}, \circ)$
$(\{a\}, \circ)$
$(\{b\}, \circ)$
$(\{c\}, \circ)$
$(\{d\}, \circ)$

skládání funkcí (je to operace?)

F_1 množ. lin. fcí

$f(x) = ax+b \quad a \neq 0$
$g(x) = cx+d \quad c \neq 0$

$x \xrightarrow{g} cx+d \xrightarrow{f} a(cx+d)+b$
$= acx + ad + b$
$\neq 0$

$x \xrightarrow{f} ax+b \xrightarrow{g} c(ax+b)+d$
$= acx + cb + d$
$\neq 0$ ✓

F_2 množ. kons. fcí.

$f, g \in F_2 \overset{?}{\Rightarrow} f \circ g \in F_2$

$f(x) = c_1 \quad g(x) = c_2 \quad \forall x \in \mathbb{R}$

$x \xrightarrow{g} c_2 \xrightarrow{f} c_1$

$x \xrightarrow{f} c_1 \xrightarrow{g} c_2$

$f \circ g = c_1 \quad \forall x \in \mathbb{R}$

$g \circ f = c_2 \quad \forall x \in \mathbb{R}$ ✓

F_3 množ. lin. fcí

$f(x) = ax^2 + bx + c$
$g(x) = px^2 + qx + r$

$x \xrightarrow{g} px^2 + qx + r \xrightarrow{f} a(px^2+bx+c)^2 + b(px^2 \ldots$
$+ c$

$f(x) = x^2$
$g(x) = x^2$

$x \xrightarrow{g} x^2 \xrightarrow{f} (x^2)^2 = x^4$ ✗

II $(\mathbb{Z}, *)$ je operace ✓ když $a, b \in \mathbb{Z} \Rightarrow 2a+b \in \mathbb{Z}$
$\Rightarrow$ je uzavřená na $\mathbb{Z}$

$a * b = 2a + b$ e 01

$a * b = b * a$

$2a + b = 2b + a$

$a = 2 \quad 2 \cdot 2 + 3 = 7$
$\neq$ není komu
$b = 3 \quad 2 \cdot 3 + 2 = 8$

$(a * b) * c = a * (b * c)$

$(2a+b) * c = a * (2b + c)$

$2(2a+b) + c = 2a + (2b + c)$

$4a + 2b + c \neq 2a + 2b + c$

$a = 1 \quad 4 \neq 2 \rightarrow$ není asoc.

$b = 0$

$c = 0$

NP:

$a * e = a = e * a$

$a * e = a \qquad e * a = a$

$2a + e = a \qquad 2a + e = a$ nemá N.P = není

$e = -a \quad \neq \quad e = 0$

musí být jen 1 N.P.

$(\mathbb{Z}, \circ)$

$a \circ b = a^3 + b^3$ → když $a, b \in \mathbb{Z}$ → $a^3 + b^3 \in \mathbb{Z}$ → $\circ$ je uzavřená na $\mathbb{Z}$ ✓

kom:

$a \circ b = a^3 + b^3 = b^3 + a^3 = b \circ a$ ✓

Asoc:

$a \circ (b \circ c) = a^3 + (b^3 + c^3)^3$ ✗

$(a \circ b) \circ c = (a^3 + b^3)^3 + c^3$

$a = 2$, $b = 3$, $c = 5$

$2^3 + (3^3 + 5^3)^3 = 8 + (27 + 125)^3 = 8 + 152^3$

$(2^3 + 3^3)^3 + 5^3 = (8 + 27)^3 + 125 = 35^3 + 125$ ✗

NP:

$a \circ e = a = e \circ a$

$a \circ e = a$

$a^3 + e^3 = a \Rightarrow e^3 = a - a^3 = \sqrt[3]{a(1-a^2)} = \sqrt[3]{a(1+a)(1-a)} \Rightarrow$ více NP.

$\Rightarrow$ nemá NP. ✗

nemusí $\in \mathbb{Z}$

DÚ

$$a * b = \frac{a + b + ab - 1}{2}$$

$$(a * b) * c = \left(\frac{a + b + ab - 1}{2}\right) * c = \frac{\left(\frac{a+b+ab-1}{2} + c\right) + \left(\frac{a+b+ab-1}{2} \cdot c\right) - 1}{2}$$

$$= \frac{\frac{a + b + ab - 1 + 2c}{2} + \frac{ac + bc + abc - c}{2} - \frac{2}{2}}{2} =$$

$$= \frac{a + b + c + ab + bc + ac + abc - 3}{4}$$

$(X, \vee, \wedge)$ - distributivní, komplementární, 0, 1 $\Big\rangle$ svaz $\Big\}$ Boolův svaz

$\Updownarrow$

$(X, \wedge, \vee, ^{-}, 0, 1) \to$ ~~Boolův svaz~~

Boolova algebra

$x \otimes (x \oplus y) \overset{?}{=} x$

$x \otimes (x \oplus y) = (x \oplus 0) \otimes (x \oplus y) = x \oplus (\underbrace{0 \otimes y}_{0}) = x \overset{\oplus}{\bullet} 0 - 0$

$[0 \otimes y \overset{?}{=} 0$

$y \otimes 0 = \bullet = y \otimes (y \otimes y') = (y \otimes y) \otimes y' = y \otimes y' = 0]$

$x \oplus (x \otimes y) = (x \otimes 1) \oplus (x \otimes y)$

$x' \otimes y' = (x \oplus y)'$

$x' \oplus y' = (x \otimes y)'$ $\qquad (x \otimes y) \otimes (x' + y') \overset{?}{=} 0$

$(x \otimes y) \otimes (x \oplus y) = ((x \otimes y) \otimes x) \oplus ((x \otimes y) \otimes y) = (\overbrace{(x \otimes x') \otimes y}^{0}) \oplus (\overbrace{(x \cdot y) \cdot y'}^{0}) = 0$

$[(x \otimes y) \otimes x' = x \otimes (y \otimes x') = x \otimes (x' \otimes y) = (x \otimes x') \otimes y = 0 \otimes y]$

(logický $\oplus$ i $\otimes$)

$y = \bar{x}_1 x_2 x_3 + x_1 \bar{x}_2 \bar{x}_3 + x_1 x_2 \bar{x}_3 + x_1 x_2 x_3 + x_1 \bar{x}_2 x_3$

$y = \bar{x}_1 x_2 x_3 + x_1 \bar{x}_2 \underbrace{(\bar{x}_3 + x_3)}_{1} + x_1 x_2 \underbrace{(x_3 + \bar{x}_3)}_{1}$ $\qquad (x + \bar{x} y = x + y)$

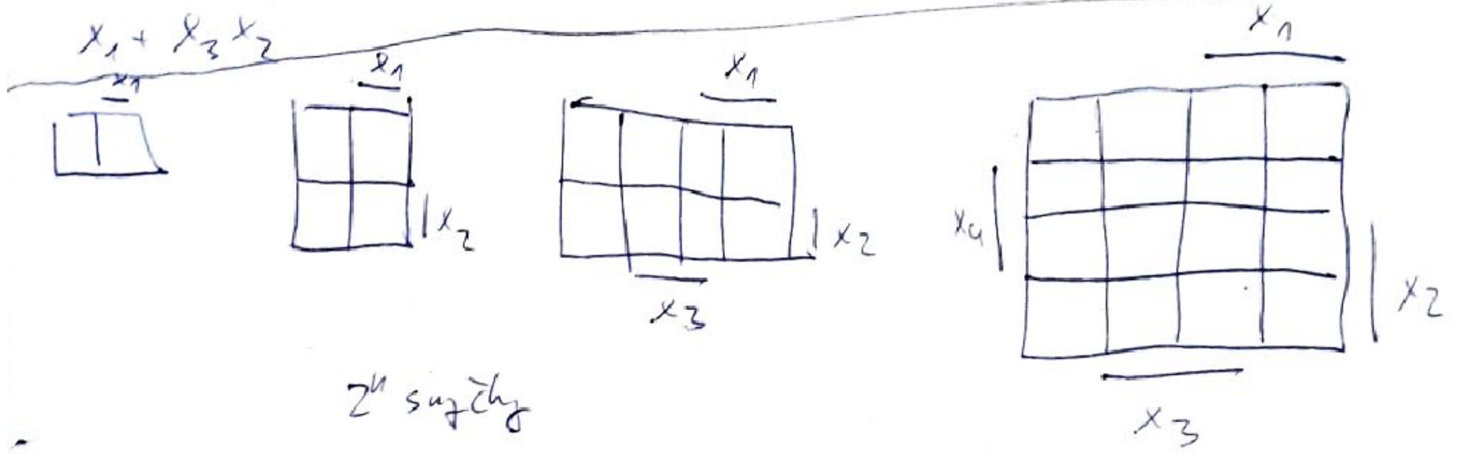
$y = \bar{x}_1 x_2 x_3 + x_1 (\bar{x}_2 + x_2) = \bar{x}_1 x_2 x_3 + x_1 = x_1 + x_2 x_3$

0	0	1	1
0	1	1	1

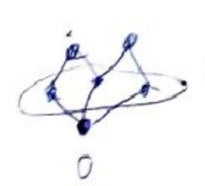
x_1, x_2, x_3

$\to \bar{x}_1 x_2 x_3 + x_1 \bar{x}_2 \bar{x}_3 + x_1 x_2 \bar{x}_3 + x_1 x_2 x_3 + x_1 \bar{x}_2 x_3$

$x_1 + x_3 x_2$



2^n sazčky



izomorfismus boolův svaz a potenční množina

$A \models\!| B$

Výrok

výroková proměnná = výroková formula $\Rightarrow$ $q \wedge p$ = výroková formula

ph(q) pm(p)

ph(p) ~~ph(q)~~

p	$\neg p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
~~q~~					

(pokud platí A tak ...)

$$\frac{\overset{1}{A}, \quad \overset{1}{A} \Rightarrow \overset{1}{B}}{\overset{1}{B}}$$

$\neg(\neg a) = a \to$ tautologie

$F = ((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow$
$\Rightarrow (p \Rightarrow r)$

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	F
0	0	0	1	1	1	1	1
1	0	0	0	1	0	0	1
0	1	0	1	0	0	1	1
1	1	0	1	0	0	0	1
0	0	1	1	1	1	1	1
1	0	1	0	1	0	1	1
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1

Tautologie

na $\langle 0,1\rangle$ $*$

$a * b = \min\{a,b\}$

? $*$ je operace na $\langle 0,1\rangle$? ✓ $\min(a;b) = a \in \langle 0,1\rangle \wedge b \in \langle 0,1\rangle \Rightarrow \begin{cases} a: a<b \\ b: b<a \end{cases}$

? $*$ je asoc?

$a \leq b \leq c$ $\qquad\qquad$ $a \leq c \leq b$

$a*(b*c) = (a*b)*c$ $\qquad$ $\vdots$

$a*b = a*c$

$a = a$

$21 \to \boxed{3} \cdot 7 = 3^1 7^1 2^0$

$12 \to \boxed{3} \cdot 2 \cdot 2 = 3^1 7^0 2^2$

$\downarrow$ $\qquad$ $\Downarrow$

3 $\qquad$ beru menší exponent

3

$*$ na $\mathbb{N}$

$a * b = NSD(a,b)$

$NSD(NSD(a,b),c) = NSD(a, NSD(b,c))$

$$\left.\begin{aligned} 42 &= 2^1 3^1 7^1 \\ 24 &= 2^3 3^1 7^0 \\ 112 &= 2^4 3^0 7^1 \end{aligned}\right\} \begin{aligned} \min\{1,2,3\} = 1 &\to 2^1 \\ - \; - \quad & 3^0 \\ - \; - \; - \quad & 7^0 \end{aligned}$$

$a = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_n^{\alpha_n}$

$b = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \dots p_n^{\beta_n}$

$c = p_1^{\gamma_1} p_2^{\gamma_2} p_3^{\gamma_3} \dots p_n^{\gamma_n}$

$NSD(NSD(a,b),c) = NSD(p_1^{\dots} p_2^{\dots} \dots p_n^{\min(\alpha_n,\beta_n)}, c)$

$NSD = p_1^{\dots} \; p_2^{\dots} \dots p_n^{\min(\min(\alpha_n,\beta_n),\gamma_n)}$

$\vdots$

$NSD(a, NSD(b,c))$

na $\langle 0,1\rangle$ $*$

$a * b = \min\{a,b\}$

? $*$ je operace na $\langle 0,1\rangle$? ✓ $\min(a;b) = a \in \langle 0,1\rangle \wedge b \in \langle 0,1\rangle = \begin{cases} a: a<b \\ b: b<a \end{cases}$

? $*$ je asoc?

$a \le b \le c$ $\quad\quad\quad$ $a \le c \le b$

$a*(b*c) = (a*b)*c$ $\quad\quad$ $\vdots$

$a*b = a*c$

$a = a$

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$\downarrow$ $\quad\quad$ $\Downarrow$

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$a = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_n^{\alpha_n}$

$b = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \cdots p_n^{\beta_n}$ $\quad$ $NSD(NSD(a,b),c) = NSD(p_1^{\cdots} p_2 \cdots p_n^{\min(\alpha_n,\beta_n)}, c)$

$c = p_1^{\gamma_1} p_2^{\gamma_2} p_3^{\gamma_3} \cdots p_n^{\gamma_n}$ $\quad$ $NSD = p_1^{\cdots} \; p_2^{\cdots} \cdots p_n^{\min(\min(\alpha_n,\beta_n),\gamma_n)}$

$\vdots$

$NSD(a, NSD(b,c))$

$A = \{a, b, c, d, e, f\}$

$S = \{\{a, e\}, \{b, d\}, \{c\}, \{f\}\}$

kongruence

$R = \{(a,a), (a,e), (e,e), (e,a),$
$(b,b), (b,d), (d,d), (d,b),$
$(c,c), (f,f)\}$

o	a	b	c	d	e	f
a	a	b	c	b	a	f
b	b	b	c	b	b	f
c	c	c	c	c	c	f
d	b	d	c	b	b	f
e	a	b	c	b	a	f
f	f	f	f	f	f	f

---

	a	b	c
a	a	b	c
b	b	b	c
c	c	c	c

je asoc?

	0	a	b	c	d	1
0	0	0	0	0	d	1
a	0	0	0	a	d	1
b	0	0	0	b	d	1
c	0	a	b	c	d	1
~~d~~	~~d~~	~~d~~	~~d~~	~~d~~	~~1~~	~~1~~
~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~

když 1 'd' vyjde 'd'
když více 'd' vyjde '1'
13. př.

	$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{f}$
$\bar{a}$	a	b	c	f
$\bar{b}$	b	b	c	f
$\bar{c}$	c	c	c	f
$\bar{f}$	f	f	f	f

$(\mathbb{Z}_6, +)$ $(\mathbb{Z}_3, +)$

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$f(0) = 0$
$f(1) = 1$
$f(2) = 0$
$f(3) = 0$
$f(4) =$

~~$f(1 \oplus 1) = f(1) + f(1) = 0 + 0 = 0$~~

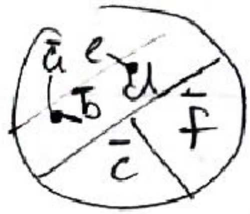
~~$f(1 \oplus 2) = f(1) + f(2) = 0 + 0 = 0$~~
~~$f(2 \oplus 2) = f(2) + f(2) = 0$~~

není epimorfismus
(musíme použít všechny 'znaky')

$0 \to 0$
$1 \to 1$
$2 \to 2$
$3 \to 0$
$4 \to 1$
$5 \to 2$

$A = \{a, b, c, d, e, f\}$

$S = \{ \{a,e\}, \{b,d\}, \{c\}, \{f\} \}$



kongruence

$R = \{ (a,a), (a,e), (e,e), (e,a), (b,b), (b,d), (d,d), (d,b), (c,c), (f,f) \}$

o	a	b	c	d	e	f
a	a	b	c	b	a	f
b	b	b	c	b	b	f
c	c	c	c	c	c	f
d	b	d	c	b	b	f
e	a	b	c	b	a	f
f	f	f	f	f	f	f

---

	a	b	c
a	a	b	c
b	b	b	c
c	c	c	c

je asoc?

	0	a	b	c	d	1
0	0	0	0	0	d	1
a	0	0	0	a	d	1
b	0	0	0	b	d	1
c	0	a	b	c	d	1
d	d	d	d	d	1	1
1	1	1	1	1	1	1

když 1 'd' vyjde 'd'
když více 'd' vyjde '1'
13. př.

	$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{f}$
$\bar{a}$	a	b	c	f
$\bar{b}$	b	b	c	f
$\bar{c}$	c	c	c	f
$\bar{f}$	f	f	f	f

$(Z_6, +)$ $(Z_3, +)$

⊕	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$f(0) = 0$
$f(1) = 1$
$f(2) = 0$
$f(3) = 0$
$f(4) =$

$f(1 \oplus 1) = f(1) + f(1) = 0 + 0 = 0$

$f(1 \oplus 2) = f(1) + f(2) = 0 + 0 = 0$

$f(2 \oplus 2) = f(2) + f(2) = 0$

není epimorfismus
(musím pokrýt všechny 'znaky')

$0 \to 0$
$1 \to 1$
$2 \to 2$
$3 \to 0$
$4 \to 1$
$5 \to 2$

p	q	r	F	DNF	KNF
0	0	0	1	• $\neg p \wedge \neg q \wedge \neg r$	
0	0	1	0	~~¬pcn~~	• $p \vee q \vee \neg r$
0	1	0	1	• $\neg p \wedge q \wedge \neg r$	
0	1	1	0		• $p \vee \neg q \vee \neg r$
1	0	0	1	• $p \wedge \neg p \wedge \neg r$	
1	0	1	0		• $\neg p \vee q \vee \neg r$
1	1	0	1	• $p \wedge q \wedge \neg r$	
1	1	1	1	• $p \wedge q \wedge r$	

$\neg(x + ? = 2)$

Disjunktivní Normální ~~Úplná~~ Forma

Konjunktivní N.. F..

DNF

$(\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)$

KNF

$(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee \neg r)$

$\{\neg, \vee\}$

$\neg A$ ✓

$A \vee B$ ✓

$A \wedge B = \neg(\neg(A \wedge B)) = \neg(\neg A \vee \neg B)$

$A \Rightarrow B = \neg(\neg(A \Rightarrow B)) = \neg(A \wedge \neg B) = \neg A \vee B$

$A \Leftrightarrow B$? $(A \Rightarrow B) \wedge (B \Rightarrow A)$

term → konstanty, proměnná

Atomická formula

term binár. rel. term

Formula predikátového počtu

$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

① $(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$

$x \in (A \cup B) \setminus C \Rightarrow x \in A \cup B \wedge x \notin C \Rightarrow (x \in A \vee x \in B) \wedge x \notin C$

$\Rightarrow (x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C)$

$\Rightarrow x \in (A \setminus C) \vee x \in (B \setminus C) \Rightarrow \underline{x \in (A \setminus C) \cup (B \setminus C)}$

② $(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C$

$x \in (A \setminus C) \cup (B \setminus C) \Rightarrow (x \in A \setminus C) \vee (x \in B \setminus C) \Rightarrow$

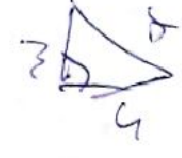
$\Rightarrow (x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C) \Rightarrow$

$\Rightarrow (x \in A \vee x \in B) \wedge x \notin C \Rightarrow$

$\Rightarrow x \in A \cup B \wedge x \notin C \Rightarrow \underline{x \in (A \cup B) \setminus C}$

čb+d

$R\circ(S_1\cap S_2) \overset{?}{=} (R\circ S_1)\cap(R\circ S_2)$



$\subseteq$

$(x,y)\in R\circ(S_1\cap S_2) \Rightarrow \exists z;\ (x,z)\in(S_1\cap S_2) \wedge (z,y)\in R \Rightarrow$

$\Rightarrow \exists z;\ (x,z)\in S_1 \wedge (x,z)\in S_2 \wedge (z,y)\in R \Rightarrow$

$\Rightarrow \exists z,\ (x,z)\in S_1 \wedge (z,y)\in R \wedge (x,z)\in S_2 \wedge (z,y)\in R \Rightarrow$

$\cancel{\Leftrightarrow} \Rightarrow \cancel{\exists z};\ (x,y)\in R\circ S_1 \wedge (x,y)\in R\circ S_2 \Rightarrow$

$\Leftrightarrow (R\circ S_1)\cap(R\circ S_2)$ ✓

$(x,y)\in (R\circ S_1)\cap(R\circ S_2) \Rightarrow (x,y)\in R\circ S_1 \wedge (x,y)\in R\circ S_2 \Rightarrow$

$\Rightarrow \exists z;\ (x,z)\in S_1 \wedge (z,y)\in R \wedge \exists \bar{z};\ (x,\bar{z})\in S_1 \wedge (x,\bar{z})\in R$

Nechť p, q jsou prvočísla, $p>3$, $q>3$. Potom číslo $\underbrace{p^2+7q^2-23}_{>3}$ není prvočíslo

DŮKAZ:

$p^2+7q^2-23 = (3k\pm1)^2+7\cdot(3l\pm1)^2-23 = 9k^2\pm6k+1+7\cdot(9l^2\pm6l+1)-23=$

$= 9k^2\pm6k+63l^2\pm42l+8-23$

$= 9k^2\pm6k+63l^2\pm42l-15 = 3(3k^2\pm2k+21l^2\pm14l-5)$

↳ není prvočíslo

V: $\sqrt{2} \notin \mathbb{Q}$ dokaz sporem

platí ... důkaz

$\neg V: \sqrt{2} = \mathbb{Q} \Rightarrow \sqrt{2} = \frac{p}{q}$; $q \neq 0$, $p, q \in \mathbb{Z}$; $NSD(p,q) = 1$

$\sqrt{2}q = p$

$2q^2 = p^2 \Rightarrow 2 | p^2 \Rightarrow 2 | p \Rightarrow p = 2k$; $k \in \mathbb{Z}$

$2q^2 = (2k)^2$

$2q^2 = 4k^2$

$q^2 = 2k^2 \Rightarrow 2 | q^2 \Rightarrow 2 | q$

spor

IMPLIKACE & DŮKAZ SPOREM

$\neg(A \Rightarrow B) = A \wedge \neg B$

V: $\forall n \in \mathbb{N}$; $2 | n \Rightarrow 2 | n^2$

$\neg V: \exists n \in \mathbb{N}$; $2 | n \wedge 2 \nmid n^2$

$\hookrightarrow n = 2k$; $k \in \mathbb{N}$

$\Downarrow$

$n^2 = (2k)^2 = 4k^2 = 2 \cdot (2k^2) \,|\, 2$

spor

přímý důkaz:

Nechť $n \in \mathbb{N}$ a $2 | n \Rightarrow n = 2k \in \mathbb{N} \Rightarrow n^2 = (2k)^2 = 2 \cdot 2k^2 \,|\, 2$

$2k^2$

č.b.d.

INDUKCE

$n = n_0$

$V(k) \Rightarrow V(k+1)$

$n \geq 1;\ n \in N$

$(n+1)(n+2) \cdot \ldots \cdot (\underbrace{n+n}_{2n}) = 2^n \cdot 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

① $n = 1$ LS: $(1+1) = 2$; PS $= 2^1 \cdot 1 = 2$; LS = PS

②

- na předp: $\underline{(k+1)(k+2)\ldots(2k)} = 2^k \cdot 1 \cdot 3 \ldots (2k-1)$

dokáži že: $((k+1)+1)\ ((k+1)+2)\ldots((k+1)+(k+1)) = 2^{(k+1)} \cdot 1 \cdot 3 \ldots (2(k+1)-1)$

$(k+2)(k+3)\ldots(2k+2) = 2^{k+1} \cdot 1 \cdot 3 \ldots (2k+1)$

$(k+2)(k+3)\ldots(k+k)(k+k+1)(k+k+2) = 2^{k+1} \cdot 1 \cdot 3 \ldots (2k+1)$

$(k+2)(k+3)\ldots(2k)(2k+1) \cdot 2 \cdot (k+1) =$

$= \underline{(k+1)(k+2)\ldots(k+k)}(2k+1) \cdot 2 =$

$= 2^{k+1} \cdot 1 \cdot 3 \ldots (2k-1) \cdot (2k+1) \cdot 2 = \underline{2^{k+1} \cdot 1 \cdot 3 \ldots (2k-1)(2k+1)}$ ✓

HURÁ!!!

$n \geq 1;\ n \in N$

$$\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$$

① $n = 2$ L: $\frac{4^2}{2+1} = \frac{16}{3}$ P: $\frac{(2\cdot 2)!}{(2!)^2} = \frac{4!}{4} = \frac{4\cdot 3\cdot 2\cdot 1}{4} = 6 = \frac{18}{3}$ L < P

② předp.: $\frac{4^k}{k+1} < \frac{(2k)!}{(k!)^2}$, dokáže $\boxed{\frac{4^{k+1}}{k+2}}$ B $< \frac{(2(k+1))!}{((k+1)!)^2}$

$$\frac{(2(k+1))!}{((k+1)!)^2} = \frac{(2k+2)(2k+1)(2k)!}{(k+1)^2 \cdot (k!)^2} = \frac{2 \cdot (k+1)(2k+1)(2k)!}{(k+1)^2 \cdot (k!)^2} = \frac{2(2k+1)(2k)!}{(k+1)(k!)^2}$$

$$\frac{2(2k+1)(2k)!}{(k+1)(k!)^2} > \frac{4^k}{k+1} \cdot \frac{2(2k+1)}{k+1}$$

A : C > B

$\frac{4^n}{n+1} \cdot \frac{2(2n+1)}{n+1} \overset{?}{>} \frac{4^{n+1}}{n+2}$ ✓

$$\frac{2\cdot(2n+1)}{(n+1)^2} \overset{?}{>} \frac{4}{n+2}$$

$$\frac{2n+1}{(n+1)^2} \overset{?}{>} \frac{2}{n+2}$$

$$\frac{2n+1}{(n+1)^2} - \frac{2}{n+2} \overset{?}{>} 0$$

$$\frac{(2n+1)(n+2) - 2(n+1)^2}{(n+1)^2(n+2)} \overset{?}{>} 0 \checkmark$$

$$(2n+1)(n+2) - 2(n+1)^2 \overset{?}{>} 0 \checkmark$$

$$2n^2+4n+n+2 - 2n^2-4n-2 \overset{?}{>} 0 \checkmark$$

$$n \overset{?}{>} 0 \checkmark$$

předchozí 7.pdf

$$\frac{2\cdot(2n+1)}{(n+1)(n!)^2} > \frac{4^n \cdot 2(2n+1)}{(n+1)^2} > \frac{4^{n+1}}{n+2} \checkmark$$

$\parallel$

$$\frac{(2(n+1))!}{((n+1)!)^2}$$

$n \geq 1,\ n \in \mathbb{N}$

$$\frac{1}{n} + \frac{1}{n+1} \cdots \frac{1}{3n-2} \geq 1$$

$n=1$: $\frac{1}{1} \geq 1$ ✓

$n=2$: LS: $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6+4+3}{12} = \frac{13}{12} \geq \frac{12}{12} = 1$ ✓ PS

$V(n) \Rightarrow V(n+1)$

Předp. $\frac{1}{n} + \frac{1}{n+1} \cdots \frac{1}{3n-2} \geq 1$, dokažte: $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n-2} + \frac{1}{3n-1} + \frac{1}{3n} + \frac{1}{3n+1} \geq 1$ ✓

$$\underbrace{\frac{1}{n+1}+\frac{1}{n+2}\dots\frac{1}{3n-2}}_{\geq 1}+\frac{1}{3n}+\frac{1}{3n+1}\geq 1+\underbrace{\frac{1}{3n-1}+\frac{1}{3n}+\frac{1}{3n+1}-\frac{1}{n}}_{\geq 0}$$

$$+\frac{1}{n} \qquad \frac{1}{3n-1} \qquad -\frac{1}{n}$$

$$\frac{1}{3n-1}+\frac{1}{3n+1}+\frac{1}{3n}-\frac{1}{n}\overset{?}{\geq} 0 \checkmark$$

$$\frac{3n(3n+1)+3n(3n-1)+(3n-1)(3n+1)-3(3n-1)(3n+1)}{\underbrace{3n(3n-1)(3n+1)}_{>0}}\overset{?}{\geq} 0 \checkmark$$

$$9n^2+3n+9n^2-1+9n^2-3n-27n^2+3\overset{?}{\geq} 0 \checkmark$$

$2 \geq 0$ SVATÁ PRAVDA

$n>1;\ n\in N$

$$1+\frac{1}{2}+\dots+\frac{1}{2^n-1}<n$$

① $n=2$

$L:\ 1+\frac{1}{2}+\frac{1}{3} = \frac{6+3+2}{6}=\frac{11}{6}<\frac{12}{6}<2 \qquad P=2=\frac{12}{6}$

② $L<P$

predp.: $1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2^k-1}<k$

dok., že: $\frac{1}{1}+\frac{1}{2}+\dots\frac{1}{2^k-1}+\frac{1}{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\dots+\frac{1}{2^{k+1}-1}<k+1$

$$\underbrace{1+\frac{1}{2}+\dots+\frac{1}{2^k-1}}_{\text{IP} \checkmark\ <k}+\underbrace{\frac{1}{2^k}+\frac{1}{2^k+1}+\dots+\frac{1}{2^{k+1}-1}}_{\frac{1}{2k}\overset{?}{\leq}1\checkmark}\leq k+1$$

$$\underbrace{\frac{1}{2^k}+\overset{\leq\frac{1}{2^k}}{\frac{1}{2^k+1}}+\dots+\frac{1}{2^{k+1}-1}}_{2^k\text{ čísel}}\leq\underbrace{\frac{1}{2^k}+\frac{1}{2^k}+\dots+\frac{1}{2^k}}_{2^k}$$

$\frac{1}{2^k}\cdot 2^k$

≤ 1

$n \in N;\ n \geq 1$

$$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} \leq 2$$

① $n = 1$

$L: 1 \quad P: 2 \quad L \leq P$

$n = 2$

$LS: \frac{1}{1^2} + \frac{1}{2^2} = 1 + \frac{1}{4} = \frac{5}{4} \leq \frac{8}{4} \leq 2$

$PS = 2 = \frac{8}{4} \qquad LS \leq PS$

Předp.:

$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} \leq 2$, dok. že: $\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2$

$$\underbrace{\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2}}_{\leq 2} + \frac{1}{(k+1)^2} \leq 2 + \frac{1}{(k+1)^2}$$

$n \geq 1,\ n \in N$

$1 + \frac{1}{2^2} + \dots + \frac{1}{n^2} \leq 2 - \frac{1}{n} \leq 2$

předp.: $1 + \frac{1}{2^2} + \dots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$

dok. že: $1 + \frac{1}{2^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{(k+1)}$

$$\underbrace{1 + \frac{1}{2^2} + \dots + \frac{1}{k^2}}_{\leq 2 - \frac{1}{k}} + \frac{1}{(k+1)^2} \underset{IP}{\leq} 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \overset{?}{\leq} 2 - \frac{1}{k+1}$$

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} \overset{?}{\leq} 2 - \frac{1}{k+1}$$

$$-\frac{1}{k} + \frac{1}{(k+1)^2} \overset{?}{\leq} -\frac{1}{k+1}$$

$$-\frac{1}{k} + \frac{1}{(k+1)^2} + \frac{1}{k+1} \overset{?}{\leq} 0$$

2SADA
2/13a
4 SADA
6, 21, 22, 23, 24

MATICE

$$x_1 + x_2 \quad - 3x_4 - x_5 = 0$$
$$x_1 - x_2 + 2x_3 - x_4 - x_5 = 0$$
$$4x_1 - 2x_2 + 6x_3 + 3x_4 - 4x_5 = 0$$
$$x_1 + 2x_2 - x_3 - 4x_4 - x_5 = 0$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & -3 & -1 & 0 \\ 1 & -1 & 2 & -1 & -1 & 0 \\ 4 & -2 & 6 & +3 & -1 & 0 \\ 1 & 2 & -4 & & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & -3 & -1 & 0 \\ 1 & -1 & 2 & -1 & -1 & 0 \\ 4 & -2 & 6 & +3 & -4 & 0 \\ 1 & 2 & -1 & -4 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 0 & -3 & -1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 0 \\ 0 & -6 & 6 & 15 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 0 & -3 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right)$$

$$\sim \quad x_1 + x_2 - 3x_4 - x_5 = 0$$
$$x_2 - x_3 - x_4 = 0 \qquad x_2 - x_3 - 0 = 0 \iff x_2 = x_3 = t$$
$$9x_4 = 0 \implies x_4 = 0$$

$$x_1 + t - 3 \cdot 0 - x_5 = 0$$
$$x_1 + t - x_5 = 0$$
$$x_1 - x_5 = t$$
$$\overset{\shortparallel}{s}$$

$$\begin{matrix} (x_1, & x_2, & x_3, & x_4, & x_5) \\ \{(s-t, & t, & t, & 0, & s) \end{matrix} \in \mathbb{R}^5,\ t, s \in \mathbb{R}\}$$

$x_1 + 3x_2 + 5x_3 - 2x_4 = 3$

$2x_1 + 7x_2 + 3x_3 + x_4 = 5$ → na konci 2 rovnice => 2 param

$x_1 + 5x_2 - 9x_3 + 8x_4 = 1$ ⇓

$5x_1 + 18x_2 + 4x_3 + 5x_4 = 12$ zvolím jak chci

součet matic => mají stejný typ

$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 7 \end{pmatrix} + B = \begin{pmatrix} 1 & 0 & 7 \\ 2 & 1 & 3 \end{pmatrix} \quad A + B = \begin{pmatrix} 3 & 3 & 11 \\ 3 & 3 & 10 \end{pmatrix}$

součin matice a prvku

$7 \cdot A = \begin{pmatrix} 14 & 21 & 28 \\ 7 & 14 & 49 \end{pmatrix}$

Transponovaná matice

$A^T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 7 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

součin matic

$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix} = B \cdot A$

$A \cdot B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

$B \cdot A = \begin{pmatrix} 0\cdot1 + 1\cdot0 & 0\cdot2 + 1\cdot1 & 0\cdot0 + 1\cdot0 \\ 1\cdot1 + 0\cdot0 & 1\cdot2 + 0\cdot1 & 1\cdot0 + 1\cdot0 \\ 1\cdot1 + 1\cdot0 & 1\cdot2 + 1\cdot1 & 1\cdot0 + 1\cdot0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix}$

Inverzní matice

$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -2 \\ 2 & 1 & 0 \end{pmatrix} \quad A^{-1} = ?$ $\boxed{A \cdot A^{-1} = \text{JEDNOTKOVÁ}} \Rightarrow A^{-1}$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \end{array}\right) \sim \dots \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -4 & -2 & 3 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$

$a_{11}x_1 + a_{12}x_2 = c_1 \rightarrow x_1 = \dfrac{c_1 - a_{12}x_2}{a_{11}}$

$a_{21}x_1 + a_{22}x_2 = c_2$

Ana Barbora Cyril Dušan

A, B, C, D
Alespoň 1ch
max 1 D

ze souv. AC půjde právě 1
B nep bez D, A nep. s D

bylo se určitě účas

A	B	C	D	
0	0	0	0	X
0	0	0	1	X
0	0	1	0	
0	0	1	1	
0	1	0	0	X
0	1	0	1	X
0	1	1	0	X
0	1	1	1	
1	0	0	0	X
1	0	0	1	X
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

určitě půjde Cyril

$F = (p \Rightarrow \neg q) \vee r$

$T = \{ \underbrace{p \Rightarrow (\neg q \vee r)}_{(1)}, \underbrace{\neg p \vee \neg q \vee r}_{(2)}, \underbrace{\underbrace{(p \Rightarrow q)}_{(a)} \wedge \underbrace{(\neg r \Rightarrow q)}_{(b)}}_{(3)} \}$

T je splnitelná? ✓

T tautologie? ×

T kontradikce? ×

F taut. důsledek T? ✓

		$\Rightarrow$
0	0	1
0	1	1
1	0	0
1	1	1

p	q	r	F	(1)	(2)	(3)	(a)	(b)
0	0	0	1	1	1	0	1	0
0	0	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	1	1	1	0	0	0
1	0	1	1	1	1	0	0	1
1	1	0	0	0	0	1	1	1
1	1	1	1	1	1	1	1	1

$\boxed{T_1 \wedge T_2 \wedge T_3 \Rightarrow F}$ když je tautologie => je taut. důsledek

p	q	r	F	DNF	KNF
0	0	0	0	~~$p \wedge q \wedge r$~~	~~$p \wedge q \vee r$~~ $p \vee q \vee r$
0	0	1	1	$\neg p \wedge \neg q \wedge r$	
0	1	0	1	$\neg p \wedge q \wedge \neg r$	
0	1	1	1	$\neg p \wedge q \wedge r$	
1	0	0	0		$\neg p \vee q \vee r$
1	0	1	0		$\neg p \vee q \vee \neg r$
1	1	0	1	$p \wedge q \wedge \neg r$	
1	1	1	1	$p \wedge q \wedge r$	~~$p \wedge q \vee r$~~

DNF

$\neg p \wedge \neg q \wedge \neg r$

DNF $= (p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r)$

KNF $= (p \vee q \vee r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r)$

$\forall\, n \geq 1;\, n \in \mathbb{N}$

$$\frac{1}{2!} + \frac{2}{3!} + \dots \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

① $n = 1$

$$L = \frac{1}{2!} = \frac{1}{2} \qquad L = P$$

$$P = \frac{1}{1+1} = 1 - \frac{1}{2} = \frac{1}{2}$$

②

Předp. že: $\frac{1}{2!} + \frac{2}{3!} + \dots \frac{k}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

doh., že: $\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!} + \frac{n+1}{(n+1+1)!} = 1 - \frac{1}{(n+2)!}$

$$\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} =$$

$$= 1 + \frac{-(n+2) + (n+1)}{(n+2)!} = 1 - \frac{1}{(n+2)!}$$

HURÁ!!!

$\forall n > 1;\ n \in N$

$$\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n} > \frac{13}{24}$$

①

$$L = \frac{1}{2+1}+\frac{1}{2+2} = \frac{1}{3}+\frac{1}{4} = \frac{7}{12} = \frac{14}{24}$$

$$P = \frac{13}{24} \qquad L > P$$

②

Předp.: $\frac{1}{h+1}+\frac{1}{h+2}+\dots+\frac{1}{2h} > \frac{13}{24}$

dokažte: $\frac{1}{h+2}+\frac{1}{h+3}+\dots+\frac{1}{2(h)}+\frac{1}{h+(h+1)}+\frac{1}{2h+2} > \frac{13}{24}$

$$\underbrace{\frac{1}{h+1}+\frac{1}{h+2}+\dots+\frac{1}{2h}}_{\overset{?}{>}\frac{13}{24}\ \checkmark}+\frac{1}{2h+1}+\frac{1}{2h+2}-\frac{1}{h+1} > \frac{13}{24}+\frac{1}{2h+1}+\frac{1}{2h+2}-\frac{1}{h+1}$$

$$\frac{13}{24}+\frac{1}{2h+1}+\frac{1}{2h+2}-\frac{1}{h+1} \overset{?}{>} \frac{13}{24}\ \checkmark$$ Hurá!!!

$$\frac{1}{2h+1}+\frac{1}{2h+2}-\frac{1}{h+1} \overset{?}{>} 0\ \checkmark$$

$$\frac{2(h+1)+(2h+1)-(2(2h+1))}{(2h+1)\cdot 2\cdot(h+1)} \overset{?}{>} 0\ \checkmark$$

$$2h+2+2h+1-4h-2 \overset{?}{>} 0\ \checkmark$$

$$1 > 0\ \checkmark$$

A je množina 19 různých $n \in \mathbb{N}$ vybraných z aritmet. posl.
$1, 4, 7, \ldots 100$, dokažte, že v A jsou 2 různé čísla
ze součtem 104

1 a 52 nemůžu použít $\rightarrow$ dalm do A (zbyde 16č.)
tak
$\Leftarrow$ aby nebyly $x, y, x+$
aby součet nebyl 104
$\Leftarrow$ chybí 1 číslo $\Leftarrow$
musím dodat 1 číslo $\Rightarrow$ tam bude $x+y=104$
(musím mít 19č.)

Rovnice $x^2+1=-1$ NEMÁ řešení na množ. reál. č.
$\forall x \in \mathbb{R};\ x^2+1 \neq -1$

Rovnice $x^2-1=0$ má právě 1 řešení na množ. r. č.
$\exists! x \in \mathbb{R};\ x^2-1=0$

Existuje nejmenší přirozené číslo
$\exists x_{\in \mathbb{N}};\ \forall y_{\in \mathbb{N}};\ x \leq y$

Neexistuje nejm. celé číslo
$\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z};\ x > y$

$F(A_1 \cap A_2) \subseteq F(A_1) \cap F(A_2)$

$$-9x_1 + 5x_2 + 5x_3 = 1$$
$$-2x_1 + x_2 + x_3 = 0$$
$$3x_1 + 2x_2 + 3x_3 = 2$$

$$-27 + (-20) + 15 - (15 + (-18)) + (30)$$

$$A \quad \begin{array}{ccc|c} -9 & 5 & 5 & 1 \\ -2 & 1 & 1 & 0 \\ 3 & 2 & 3 & 2 \\ -9 & 5 & 5 & \\ -2 & 1 & 1 & \end{array}$$

$$|A| = -27 - 20 + 15 - 15 + 18 + 30 = 1$$

$$A_1 \quad \begin{array}{ccc} 1 & 5 & 5 \\ 0 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 5 & 5 \\ 0 & 1 & 1 \end{array} \qquad |A_1| = 3 + 0 + 10 - 0 - 10 - 2 = 1$$

$$A_2 \quad \begin{array}{ccc} -9 & 1 & 5 \\ -2 & 0 & 1 \\ 3 & 2 & 3 \\ -9 & 1 & 5 \\ -2 & 0 & 1 \end{array} \qquad |A_2| = 0 - 20 + 3 - (0 - 18 - 6) = 7$$

$$|A_3| \quad \begin{array}{ccc} -9 & 5 & 1 \\ -2 & 1 & 0 \\ 3 & 2 & 2 \\ -9 & 5 & 1 \\ -2 & 1 & 0 \end{array} \qquad |A_3| = -5$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{1}{1} = 1$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{7}{1} = 7$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-5}{1} = -5$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$\{1,2,3\}$ $\qquad f(1)$

$J(1,2,3) = 0$
$J(1,3,2) = 1$
$J(2,1,3) = 1$
$J(2,3,1) = 2$
$J(3,1,2) = 2$
$J(3,2,1) = 3$

z definice (nedoporučeno)

$$|A| = (-1)^0 a_{11}a_{22}a_{33} + (-1)^1 a_{11}a_{23}a_{32} + (-1)^1 \cdot a_{12}a_{21}a_{33} + (-1)^2 a_{12}a_{23}a_{31} + (-1)^2 a_{13}a_{21}a_{32} + (-1)^3 a_{13}a_{22}a_{31}$$

Lablaceov rozvoj

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \qquad |A^T| = |A|$$

$$\begin{array}{ccc} 1 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 2 & 0 \\ -1 & 2 & 0 \end{array} \qquad 6-(-6) = 12$$

a) rozvoj pro 2. řádek
b) rozvoj 3. ~~[illegible]~~ sloupce

a)
$$|A| = -1 \cdot (-1)^{2+1} \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} + 2 \cdot (-1)^{2+2} \cdot \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + 0 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$
$$= -1 \cdot (-1) \cdot 6 + 2 \cdot 1 \cdot (1 \cdot 3 - 0 \cdot 0) + 0$$
$$= 6 + 3 \cdot 2 = 12$$

b)
$$|A| = 0 \cdot (-1)^{1+3} \cdot \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} + 0 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} + 3 \cdot (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} =$$
$$= 0 + 0 + 3 \cdot (1 \cdot 2 - (-1) \cdot 2) = 12$$

$$A = \begin{pmatrix} 4 & 0 & -2 & 3 \\ -2 & 0 & 1 & 2 \\ 1 & 3 & -4 & -2 \\ 2 & 1 & -1 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 7 \\ -2 & 0 & 1 & 2 \\ 1 & 3 & -4 & -2 \\ 2 & 1 & -1 & 4 \end{pmatrix}$$

$$|A| = 0 + 0 + 0 + 7 \cdot \overset{(-7)}{\begin{vmatrix} -2 & 0 & 1 \\ 1 & 3 & -4 \\ 2 & 1 & -1 \end{vmatrix}} \cdot (-1)^{1+4} = \underline{\underline{49}}$$:)

HURÁ!!!

$$A = \begin{vmatrix} b+c & c+a & a+b \\ b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \end{vmatrix} = 2 \cdot \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$\begin{vmatrix} b & c+a & a+b \\ b_1 & c_1+a_1 & a_1+b_1 \\ b_2 & c_2+a_2 & a_2+b_2 \end{vmatrix} + \begin{vmatrix} c & c+a & a+b \\ c_1 & c_1+a_1 & a_1+b_1 \\ c_2 & c_2+a_2 & a_2+b_2 \end{vmatrix}$$

$$\begin{vmatrix} b & c & a+b \\ b_1 & c_1 & a_1+b_1 \\ b_2 & c_2 & a_2+b_2 \end{vmatrix} + \begin{vmatrix} b & a & a+b \\ b_1 & a_1 & a_1+b_1 \\ b_2 & a_2 & a_2+b_2 \end{vmatrix} + \overset{= 0}{\begin{vmatrix} c & c & a+b \\ c_1 & c_1 & a_1+b_1 \\ c_2 & c_2 & a_2+b_2 \end{vmatrix}} + \begin{vmatrix} c & a & a+b \\ c_1 & a_1 & a_1+b_1 \\ c_2 & a_2 & a_2+b_2 \end{vmatrix}$$

$$\vdots$$

$$\begin{vmatrix} b & c & a \\ b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \end{vmatrix} + \begin{vmatrix} c & a & b \\ c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} + \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 2 \cdot \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -2 \\ 2 & 1 & 0 \end{pmatrix}$$

$|A| = 1 \neq 0$

$A^* =$

$A^{-1} =$

$$\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & -2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{matrix}$$

$0 + 1 + 0 - 0(-2) - 2 = 1$

$a_{11} = 1$

$(-1)^2 \cdot \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0+2) = 2$

$a_{12} = 0$

$(-1)^3 \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = -1 \cdot (0+4) = -4$

$a_{13} = 1$

$(-1)^4 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 \cdot (1-2) = -1$

$a_{21} = 1$

$\vdots$

$a_{33} = 0$

$$A^* = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -2 & 3 \\ -1 & -1 & 1 \end{pmatrix}$$

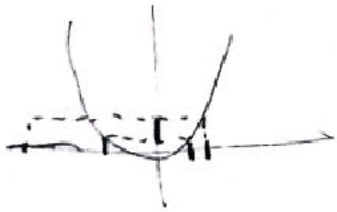
$$A^{-1} = \frac{A^*}{|A|} = A^*$$

$a_{44} = 1$

$\vdots$

$= 1$

$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$



$f(M) - \{y;\ f(x) = y\}$

$\{y;\ \exists x \in M \wedge \quad \}$

Nechť $y \in f(A_1 \cap A_2) \Rightarrow \exists_{x;\, x \in} (A_1 \cap A_2) \wedge f(x) = y \Rightarrow \exists x;\ x \in A_1 \wedge x \in$

$\wedge f(x) =$

$\Rightarrow y \in f(A_1) \wedge y \in f(A_2) \Leftrightarrow y \in f(A_1) \cap f(A_2)$

zpět nejde, mohlo by být jiné x v A_1 a v A_2

$2x + y + 3z = 2$

$3x + 2y + 4z = 2$

$x + y + z = 1$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 2 & 4 & 2 \end{pmatrix} \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right) \rightarrow \text{nemá řeš.}$$

$x+2y+z=2$
$2x+3y+2z=3$
$x+y+z=1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right) \quad 1x+2y+1z=2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

nekonečně řeš.

$x+2y+z=2$
$y=1$

$2+x+z=2$

$x+z=0$

$x=-z$

$$P=\left\{\begin{bmatrix} \overset{x}{x}, & \overset{y}{1}, & \overset{z}{-x} \end{bmatrix}\right\}$$

$x+2y+3z=1$
$4x+4y+5z=3$
$3x+y+2z=2$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 4 & 5 & 3 \\ 3 & 1 & 2 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & +4 & +7 & +1 \\ 0 & +5 & +7 & +1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 4 & 7 & 1 \\ 0 & 0 & \frac{7}{4} & \frac{1}{4} \end{array}\right)$$

$x+2y+3z=1 \Rightarrow x+3\cdot\frac{1}{7}=1 \Rightarrow x=\frac{4}{7}$

$4y+7z=1 \Rightarrow 4y+1=1 \Rightarrow y=0$

$\frac{7}{4}z=\frac{1}{4} \Rightarrow z=\frac{1}{7}$

$$P=\left\{\left[\frac{4}{7}, 0, \frac{1}{7}\right]\right\}$$

$c \in \mathbb{R}$

$$\begin{vmatrix} c & 2 & 1 \\ 2 & c & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0 \quad \begin{matrix} c & 2 & 1 \\ 2 & c & 1 \end{matrix}$$

a) najdi $c \in \mathbb{R}$, aby platilo.

b) -//- neplatilo

$$c \cdot c \cdot 1 + 2 \cdot 2 \cdot 1 + 1 \cdot 2 \cdot 1 - 2 \cdot 2 \cdot 1 - c \cdot 2 \cdot 1 - 1 \cdot c \cdot 1 =$$
$$= c^2 + 4 + 2 - 4 - 2c - c =$$

a) $= c^2 - 3c + 2 = 0$ $\qquad \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

$$(c-1)(c-2) = 0$$
$$c \in \{1, 2\} \Rightarrow \text{platí}$$

b) $c \in \mathbb{R} \setminus \{1, 2\} \Rightarrow$ neplatí

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|A|$, $|B|$, $|A^{-1}|$, $|B^{-1}|$, $|A \times A^{-1}|$, $|B \times B^{-1}|$, $|A^T \cdot A|$, $|B^T \cdot B|$,

$|A^{-1} \cdot A^T|$, $|B^{-1} \cdot B^T|$,

A^{-1}, B^{-1}

$|A| = 2 \quad |A^{-1}| = \frac{1}{2} \quad |A \times A^{-1}| = 1 \quad |A^T \cdot A| = 4 \quad |A^{-1} \cdot A^T| = 1$

$|B| = 0 \quad |B^{-1}| = X \quad |B \times B^{-1}| = X \quad |B^T \cdot B| = 0 \quad |B^{-1} \cdot B^T| = X$

$$A^{-1} = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \approx \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & -2 \\ 0 & -4 & -2 & 1 & 0 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & -2 \\ 0 & 0 & 2 & 1 & -4 & -11 \end{array}\right] \sim$$

~~B^{-1} NEEXISTUJE~~

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & \frac{1}{2} & -2 & -\frac{11}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 2 & -\frac{9}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & \frac{7}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -2 & -\frac{11}{2} \end{array}\right]$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad |A| = 2$$

$$a_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1 \to \frac{1}{2} \qquad \frac{|a_{11}|}{|A|} = \frac{1}{2}$$

$$a_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1 \to -\frac{1}{2}$$

$$a_{13} = (-1)^4 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1 \to \frac{1}{2}$$

$$a_{21} = (-1)^3 \cdot \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0 \to 0$$

$$a_{22} = (-1)^4 \cdot \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2 \to 1$$

$$a_{23} = (-1)^5 \cdot \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = -4 \to -2$$

$$a_{31} = (-1)^4 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1 \to -\frac{1}{2}$$

$$a_{32} = (-1)^5 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = -1 \to -\frac{1}{2}$$

$$a_{33} = (-1)^6 \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5 \to \frac{5}{2}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ +\frac{1}{2} & -\frac{4}{2} & \frac{5}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

není komutativní

$$A \cdot X = B \qquad A^{-1}(A \cdot X) = A^{-1} \cdot B$$

$$X = ? \qquad (A^{-1} \cdot A) \cdot X = A^{-1} B$$

$$X = A^{-1} \cdot B$$

$$X = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & -2 & \frac{5}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \\ \frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{9}{2} & \frac{9}{2} \end{bmatrix}$$

$$x + cy + 2z = 2$$
$$x + y + cz = 9$$
$$2x + y + cz = 4$$

$$\left(\begin{array}{ccc|c} 1 & c & 2 & 2 \\ 1 & 1 & c & 9 \\ 2 & 1 & c & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & c & 2 & 2 \\ 0 & 1-c & c-2 & 7 \\ 9 & 0 & 0 & -5 \end{array}\right) \sim$$

$$x = -5$$

$$-5 + cy + 2z = 2$$
$$cy + 2z = 7$$
$$\Downarrow$$
$$2z = 7 - cy$$
$$z = \frac{7 - cy}{2}$$

$$(1-c)y + (1-c) \cdot \frac{7 - cy}{2} = 7$$

$$y - cy + \frac{cz}{2} - \frac{c^2 y}{2} - 7 + cy = 7$$

$$y\left(1 - c - \frac{c^2}{2} + c\right) = 7 + 7 - \frac{c \cdot 7}{2}$$

$$y\left(1 - \frac{c^2}{2}\right) = 14 - \frac{7}{2}c$$

if $(c = \pm\sqrt{2})$: | if $(c \neq \pm\sqrt{2})$:

$$y = \frac{14 - \frac{7}{2}c}{1 - \frac{c^2}{2}}$$

$f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ $\quad V_2(\mathbb{R})$ $\qquad \bar a \cdot \bar a > 0$

$f((a_1,a_2),(b_1,b_2)) = a_1 b_1 + a_1 b_2 + a_2 b_1$

$f((a_1,a_2),(a_1,a_2)) = a_1^2 + a_1 a_2 + a_2 a_1 = a_1^2 + 2(a_1 a_2) = a(a_1 + 2a_2)$

$(a_1,a_2) = (-1,1)$

$(-1)(-1+2) = -1 \quad 1 = -1$

$\bar v = (1,2,0,-1,3)$

$\|\bar v\| = \sqrt{1\cdot 1 + 2\cdot 2 + 0\cdot 0 + (-1)(-1) + 3\cdot 3} = \sqrt{15}$

$\bar o \cdot \bar a \overset{?}{=} 0$ OBECNĚ $\quad \bar o = (\bar a + (-1)\bar a)$

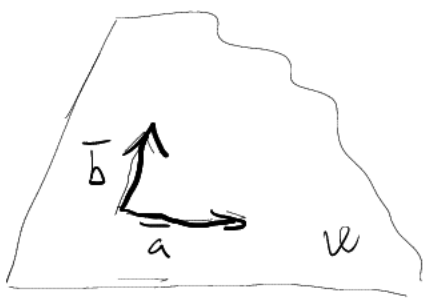
$(\bar a + (-1)\bar a)\cdot \bar a = \bar a \cdot \bar a + (-1)\cdot \bar a \cdot \bar a = \bar a \cdot \bar a - \bar a \cdot \bar a = 0$

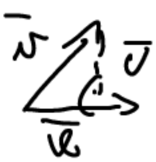
$W = \langle \bar a, \bar b \rangle$ $\qquad \bar v = (1,2,3)$

$\bar a = (-1,1,1)$, $\bar b = (1,1,1)$

určete jeho ortog. průmět do W

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$





$\bar v = \bar w + \bar u$ $\qquad \bar w = r\cdot \bar a + s \cdot \bar b$

$\bar v = r\cdot \bar a + s\cdot \bar b + \bar u$ $\quad /\cdot \bar b \quad /\cdot \bar a$

$\bar v \cdot \bar b = r\bar a \bar b + s\bar b \bar b + \bar u \bar b^{=0}$

$\bar v \cdot \bar b = r\bar a \bar b + s\bar b \bar b$

$\bar v \cdot \bar a = r\cdot \bar a \cdot \bar a + s\cdot \bar b \cdot \bar a + \bar u \cdot \bar a^{=0}$

$\bar v \cdot \bar a = r\bar a \cdot \bar a + s\cdot \bar a \bar b$

$(1,2,3)(1,1,1) = r(-1,1,1)(1,1,1) + s\cdot(1,1,1)(1,1,1) \to 6 = r\cdot 1 + s\cdot 3 \quad /\cdot(-3)$

$(1,2,3)(-1,1,1) = r(-1,1,1)(-1,1,1) + s(1,1,1)(-1,1,1) \to 4 = 3r + s$

$-14 = -8s$ $\qquad r = 6 - 3s = 6 - \frac{21}{4} = \frac{3}{4}$

$s = \frac{7}{4}$ $\qquad w = \frac{3}{4}(1,1,1) + \frac{7}{4}(1,1,1) = \left(1, \frac{5}{2}, \frac{5}{2}\right)$

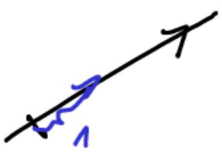
$\vec{a}_i \cdot \bar{a}_j = 0 \qquad i \neq j \qquad \bar{a}_1, \bar{a}_2, \dots, \bar{a}_n \qquad a_i \neq 0$

$\bar{a}_k = r_1 \bar{a}_1, r_2 \bar{a}_2, \dots, r_n \bar{a}_n$

$$0 \neq \bar{a}_k \cdot \bar{a}_k = \underbrace{r_1 \bar{a}_1 \bar{a}_k}_{0} + \underbrace{r_2 \bar{a}_2 \bar{a}_k}_{0} + \dots + \underbrace{r_n \bar{a}_n \bar{a}_k}_{0}$$

$\bar{v} = (1,2,3)$

$\|\bar{v}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \qquad \|\bar{w}\| = 1$



$\bar{w}$ – stejný vektor jako $\bar{v}$

$$\|\bar{w}\| = \left\| \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) \right\| =$$

$$= \sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2} =$$

$$= \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = 1$$

$(1,0)$ | $(1,0,0)$

$(0,1)$ | $(0,1,0)$

$(0,0,1)$

ortonormální báze

Ortonormální báze podprostoru

$S = \langle (1,0,1,0), (1,1,3,0), (1,0,2,2), \cancel{(3,1,6,2)} \rangle$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 0 & 2 & 2 \\ 3 & 1 & 6 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$\bar{a}_1 = (1,0,1,0) \qquad \bar{b}_1 = \bar{a}_1 = (1,0,1,0)$

$\bar{a}_2 = (0,1,2,0) \qquad \bar{b}_2 = \bar{a}_2 + x b_1$

$\bar{a}_3 = (0,0,1,2) \qquad \bar{b}_3 = \bar{a}_3 + y \bar{b}_1 + z \bar{b}_2$

$\bar{b}_2 = \bar{a}_2 + x\bar{b}_1 \quad | \cdot \bar{b}_1$

$\bar{b}_1 \bar{b}_2 = \bar{a}_2 \bar{b}_1 + x \bar{b}_1 \bar{b}_1$

$0 = (0,1,2,0)(1,0,1,0) + x(1,0,1,0)(1,0,1,0)$

$0 = 2 + x \cdot 2 \Rightarrow x = -1$

$\bar{b}_2 = (0,1,2,0) + (-1)(1,0,1,0) = (-1,1,1,0)$

$\bar{b}_1 \bar{b}_3 = \bar{a}_3 \bar{b}_1 + y \bar{b}_1 \bar{b}_1 + z b_2 b_1$

$0 = \bar{a}_3 \bar{b}_1 + y \bar{b}_1 \bar{b}_1$

$0 = (0,0,1,2)(1,0,1,0) + y(1,0,1,0)(1,0,1,0)$

$0 = 1 + y \cdot 2 \rightarrow y = -\frac{1}{2}$

$\bar{b}_3 \bar{b}_2 = \bar{a}_3 \bar{b}_2 + y \bar{b}_1 \bar{b}_2 + z \vec{b}_2 \vec{b}_2$

$0 = \bar{a}_3 \bar{b}_2 + z \bar{b}_2 \bar{b}_2$

$0 = (0,0,1,2)(-1,1,1,0) + z(-1,1,1,0)(-1,1,1,0)$

$0 = 1 + 3z \Rightarrow z = -\frac{1}{3}$

$\bar{b}_3 = \bar{a}_3 \; (-\frac{1}{2}) \bar{b}_1 + (-\frac{1}{3}) \bar{b}_2$

$\bar{b}_3 = (0,0,1,2) \; (-\frac{1}{2})(1,0,1,0) + (-\frac{1}{3})(-1,1,1,0) = (-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, 2)$

$(1,0,1,0), \; (-1,1,1,0), \; (-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, 2)$

$(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0), \; (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0), \; (-\frac{\sqrt{6}}{30}, -\frac{\sqrt{6}}{15}, -\frac{\sqrt{6}}{30}, \frac{2\sqrt{6}}{5})$

$(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0), \; (-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, 0)$

Abelova grupa $(u, +)$

$$f(r\bar{a} + s\bar{b}) = f(r\bar{a}) + f(s\bar{b}) = rf(\bar{a}) + sf(\bar{b})$$

$$f(\bar{a}+\bar{b}+\bar{c}) = f((\bar{a}+\bar{b})+\bar{c}) = f(\bar{a}+\bar{b}) + f(\bar{c}) = (f(\bar{a})+f(\bar{b}))+f(\bar{c})$$

$$f(\bar{a})+f(\bar{b})+f(\bar{c})$$

$$f(r_1\bar{a} + r_2\bar{a}_2 + \dots + r_n\bar{a}_n) = r_1 f_1(\bar{a}_1) + r_2 f_2(\bar{a}_2) + \dots + r_n f(\bar{a}_n)$$

$f_1 : \mathbb{R}^2 \to \mathbb{R}^3 \qquad V_2(\mathbb{R}) \xrightarrow{f_1} V_3(\mathbb{R})$

$f_1((x,y)) = (x, y, 1) \qquad \bar{u} = (a,b)$

$\bar{v} = (c,d)$

$$L = f_1(\bar{u}+\bar{v}) \overset{?}{=} f_1(\bar{u}) + f_1(\bar{v}) = P$$

$L: \; f_1(\bar{u}+\bar{v}) = f_1((a+c, b+d)) = (a+c, b+d, 1)$ $\qquad L \neq P$

$P: \; f_1(\bar{u}) + f_1(\bar{v}) = f_1((a,b)) + f_1((c,d)) = (a,b,1) + (c,d,1) = (a+c, b+d, 2)$

$f_2 : \mathbb{R}^2 \to \mathbb{R}^3$

$f_2((x,y)) = (x, y, x+y)$

$$f_2(\bar{u}+\bar{v}) = f_2(a+c, b+d) = (a+c, b+d, a+c+b+d)$$

$$\|$$

$$f_2(\bar{u}) + f_2(\bar{v}) = (a,b,a+b) + (c,d,c+d) = (a+c, b+d, a+b+c+d)$$

$$f_2(r\bar{a}) = f(ra, rb) = (ra, rb, r(a+b)) = r(a, b, a+b)$$

$$r f_2(\bar{a}) = r(a, b, a+b)$$

$$V(f) \underset{f^{-1}}{\overset{f}{\rightleftarrows}} W(f)$$

keď je izomorfizmus v jednom smere => izo. i v druhom

izo -> lineár., bijekcia

$$\begin{tikzcd} V(f) \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & W(f) \arrow[r, "g"] & U(f) \end{tikzcd}$$

$(2,1,1,1) = r\cdot(1,1,0,1) + s(1,0,0,0) + t(0,1,1,0) + q(0,1,1,1)$

Daná je báza $\{[1,1,0],[1,0,1],[0,-1,0]\}$ priestoru $V_3(\mathbb{R})$ a báza $\{[1,1,0,1],[1,0,0,0],[0,1,1,0],[0,1,1,1]\}$ priestoru $V_4(\mathbb{R})$. Nájdite maticu lineárneho zobrazenia $f: V_3 \to V_4$;
$[x,y,z] \to [x+y, y+z, x+z, x]$.

Riešenie

- Treba overiť, či uvedené vektory sú naozaj bázami $V_3(\mathbb{R})$ a $V_4(\mathbb{R})$. Teda matice A, B

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

upravíme na trojuholníkový tvar a ak budú všetky riadky nenulové (čo je aj náš prípad), tak sa jedná o bázy. Ak by vznikol nejaký (aspoň jeden) nulový riadok, tak sa o bázy nejedná.

- Určíme si obrazy bázy $V_3(\mathbb{R})$ podľa daného predpisu:
$f([1,1,0]) = \boxed{[2,1,1,1],}$
$f([1,0,1]) = [,1,1,2,1],$
$f([0,-1,0]) = [-1,-1,0,0].$
- Teraz už stačí tieto obrazy vyjadriť ako lin. kombinácie bázy $V_4(\mathbb{R})$. Na prednáške bolo podrobne urobené. Teda
$[2,1,1,1] = 0.[1,1,0,1] + 2.[1,0,0,0] + 0.[0,1,1,0] + 1.[0,1,1,1];$
$[1,1,2,1] = (-1).[1,1,0,1] + 2.[1,0,0,0] + 0.[0,1,1,0] + 2.[0,1,1,1];$
$[-1,-1,0,0] = (-1).[1,1,0,1] + 0.[1,0,0,0] + (-1).[0,1,1,0] + 1.[0,1,1,1].$
- Pomocou skalárov v lin. kombináciách obrazov dostaneme maticu lin. zobrazenia:

$$M_f = \begin{pmatrix} 0 & 2 & 0 & 1 \\ -1 & 2 & 0 & 2 \\ -1 & 0 & -1 & 1 \end{pmatrix}$$

$2 = r + s$
$1 = r + t + q$
$1 = t + q$
$1 = r + q$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right)$$

$r = 0$
$s = 2$
$t = 0$
$q = 1$

Nech $\overline{a_1} = [1,1], \overline{a_2} = [1,2]$ sú vektory $V_2(\mathbb{R})$ a $\overline{b_1} = [1,-1,2], \overline{b_2} = [2,-2,4]$ sú vektory $V_3(\mathbb{R})$. Zrejme $\overline{a_1}, \overline{a_2}$ sú nezávislé a teda tvoria bázu $V_2(\mathbb{R})$, preto existuje jediné lineárne zobrazenie z $V_2(\mathbb{R})$ do $V_3(\mathbb{R})$ tak, aby $f(\overline{a_1}) = \overline{b_1}, f(\overline{a_2}) = \overline{b_2}$. Nájdite maticu tohoto zobrazenia vzhľadom na bázu $\{\overline{a_1}, \overline{a_2}\}$ a jednotkovú bázu $V_3(\mathbb{R})$. Nájdite maticu tohoto zobrazenia vzhľadom na jednotkové bázy.

Riešenie

- Prvá časť úlohy je jednoduchá. Riadky hľadanej matice sú vlastne vektory $\overline{b_1}, \overline{b_2}$, teda $M_f = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \end{pmatrix}$
- V druhej časti si musíme vyjadriť prvky jednotkovej bázy $V_2(\mathbb{R})$ ako lin. kombinácie vektorov $\overline{a_1}, \overline{a_2}$. Teda $[1,0] = r.[1,1] + s[1,2]$ a pre r, s dostaneme: $r = 2, s = -1$. Pre $[0,1]$ máme nasledovné vyjadrenie pomocou $\overline{a_1}, \overline{a_2}$: $[0,1] = (-1).\overline{a_1} + 1.\overline{a_2}$.

①
$$f((1,1)) = (1,-1,2) = 1\cdot(1,0,0) - 1\cdot(0,1,0) + 2(0,0,1)$$
$$f((1,2)) = (2,-2,4) = $$
$$= 2(1,0,0) + (-2)(0,1,0) + 4(0,0,1)$$
$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \end{pmatrix}$$

② $f((1,0)) = \ldots$

$f((1,2)) = \ldots$

$$f((r,s)) = f((r\bar{a}_1 + s\bar{a}_2)) = \ldots$$
$$= r f(\bar{a}_1) + s f(\bar{a}_2)$$

$$(1,0) = r\bar{a}_1 + s\bar{a}_2$$
$$(1,0) = r(1,1) + s(1,2)$$
$$1 = r+s$$
$$0 = r+2s$$
$$-1 = s,\ r = 2$$

$$f((1,0)) = 2f((1,1)) + (-1)f((1,2))$$
$$= 2(1,-1,2) - 1(2,-2,4) = (0,0,0)$$

$$(0,1) = s(1,1) + r(1,2)$$
$$\left.\begin{aligned} 0 &= s+r \\ 1 &= s+2r \end{aligned}\right\} \begin{aligned} r &= -1 \\ s &= 1 \end{aligned}$$

$$f((0,1)) = (-1)f((1,1)) + 1\cdot f((1,2)) =$$
$$= -1(1,-1,2) + 1\cdot(2,-2,4) = (1,-1,2)$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

$\bar{a}_1 \mid \bar{b}_1 \ldots\ 1\ 0 \mid \ldots$

$\bar{a}_2 \mid \bar{b}_2 \ldots\ 0\ 1 \mid \ldots$

Lepší výpočet

$$\left(\begin{array}{cc|ccc} 1 & 1 & 1 & -1 & 2 \\ 1 & 2 & 2 & -2 & 4 \end{array}\right) \sim \left(\begin{array}{cc|ccc} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right)$$

$f((7,8)) = f(7\cdot(1,0) + 8\cdot(0,1)) = 7\cdot f((1,0)) + 8 f((0,1))$

$7\cdot(0,0,0) + 8(1,-1,2) = (8,-8,16)$

Nech $f: V_2 \to V_3$ je lineárne zobrazenie $V_2(\mathbb{R})$ do $V_3(\mathbb{R})$, ktoré má maticu

$$M_f = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$ násobení matic

nech $g: V_3 \to V_2$ je lineárne zobrazenie $V_3(\mathbb{R})$ do $V_2(\mathbb{R})$, ktoré má maticu

$$M_g = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$ *Určme obrazy jednotkovej bázy priestoru $V_2(\mathbb{R})$ v zobrazení $g \circ f$.*

$f: (1,0) \to (a_{11}, a_{12}, a_{13})$

$f: (0,1) \to (a_{21}, a_{22}, a_{23})$

$g: (1,0,0) \to (b_{11}, b_{12})$

$g: (0,1,0) \to (b_{21}, b_{22})$

$g\,(0,0,1) \to (b_{31}, b_{32})$

$M\; g\circ f = ?$

$g\circ f((1,0)) = g(f(1,0)) = g(a_{11}, a_{12}, a_{13}) = g(a_{11}(1,0,0) + a_{12}(0,1,0) + a_{13}(0,0,1)) =$

$= a_{11} g(1,0,0) + a_{12}\, g(0,1,0) + a_{13}\, g(0,0,1) =$

$= a_{11}(b_{11}, b_{12}) + a_{12}(b_{21}, b_{22}) + a_{13}(b_{31}, b_{32}) =$

$= (a_{11}b_{11} + a_{12}b_{21} + a_{31}b_{31},\ a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32})$

Riešenie. Na určenie matice potrebujeme obrazy jednotkovej bázy v zobrazení $g \circ f$, teda:
$g \circ f([1,0]) = g(f([1,0])) = g([a_{11}, a_{12}, a_{13}]) =$
$= g(a_{11}[1,0,0] + a_{12}[0,1,0] + a_{13}[0,0,1]) =$
$= a_{11}g([1,0,0]) + a_{12}g([0,1,0]) + a_{13}g([0,0,1]) =$
$= a_{11}[b_{11}, b_{12}] + a_{12}[b_{21}, b_{22}] + a_{13}[b_{31}, b_{32}] =$
$= [a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}, a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}]$ Analogicky dostaneme
$g \circ f([0,1]) = [a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}, a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}]$
Teda vidíme, že maticu zobrazenia $g \circ f$ dostaneme vynásobením matíc $M_f \cdot M_g$.

V $\mathbb{R}^2$ je dané lineárne zobrazenie obrazmi bázy $\langle[1,0],[0,1]\rangle$ takto: $f([1,0]) = [4,2]$, $f([0,1]) = [-3,-1]$. Nájdite všetky vektory, ktoré pri tomto zobrazení nezmenia smer (len veľkosť, prípadne orientáciu).

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(a,b) = h\cdot(a,b)$

$f(a,b) = f(a(1,0) + b(0,1)) = a f(1,0) + b f(0,1) = a(4,2) + b(-3,-1) \Rightarrow$

$\Rightarrow 4a - 3b = h\cdot a$ $\quad h \in \mathbb{R}$

$2a - 1b = h\cdot b$

$$2a - 2b = ka - kb$$
$$2(a-b) = k(a-b)$$
$$2 = k \qquad a = b$$
$$\left(a, \tfrac{2}{3}b\right) \qquad (a, a)$$

$$4a - 3b = 2a$$
$$2a - b = 2b$$
$$2a = 3b \Rightarrow a = \tfrac{3}{2}b$$

Definícia. Nech A je štvorcová matica stupňa n. Hovoríme, že vektor $\overline{x}$ je **vlastný vektor** ak existuje $\lambda \in \mathbb{R}$ tak, aby $A.\overline{x} = \lambda.\overline{x}$. Číslo λ nazývame **vlastná hodnota** matice A.

Zrejme

$$A\overline{x} - \lambda.\overline{x} = \mathbf{0}$$
$$A\overline{x} - \lambda.I_n.\overline{x} = \mathbf{0}$$
$$(\lambda.I_n - A).\overline{x} = \mathbf{0}.$$

$$I \cdot \lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \lambda$$

Posledná rovnica sa nazýva **charakteristcká rovnica** prislúchajúca matici A.
Takáto rovnica má riešenie práve vtedy, keď $det(\lambda.I_n - A) = 0$. $(*)$
Výraz na ľavej strane rovnice (*) sa nazýva **charakteristický polynóm**, jeho korene $\lambda_1, \ldots, \lambda_n$ sa nazývajú **vlastné hodnoty** matice A a vektor $\overline{x}$ je **vlastný vektor.** Množina všetkých vlastných vektorov matice A tvorí **vlastný podpriestor** matice A.

Určte vlastné hodnoty a vektory k matici

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{pmatrix}$$

$$\begin{vmatrix} \lambda-1 & 0 & 0 \\ 1 & \lambda-3 & 0 \\ -3 & -2 & \lambda+2 \end{vmatrix} \begin{matrix} \lambda-1 & 0 & 0 \\ 1 & \lambda-3 & 0 \end{matrix} = (\lambda-1)(\lambda-3)(\lambda+2) - (0)$$

$$(\lambda-1)(\lambda-3)(\lambda+2) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 3, \quad \lambda_3 = -2$$

$$\lambda_1 = 1 \quad \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -3 & -2 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ -3 & -2 & 3 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & -8 & 3 & 0 \end{array}\right)$$

$$-8y + 3r = 0$$

$$\left\{\left(\tfrac{3}{4}r, \tfrac{3}{8}r, r\right)\right\} \quad r \in \mathbb{R} \quad \longleftarrow \quad x = \tfrac{3}{4}r \quad \Longleftarrow \quad y = \tfrac{3}{8}r$$

$\lambda_2 = 3$ $\left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & -2 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & -2 & 5 & 0 \end{array}\right) \to x = 0$

$z = r \to -2y + 5r = 0$

$y = \frac{5}{2} r$

$\{(0, \frac{5}{2} r, r), r \in \mathbb{R}\}$

$\lambda_3 = -2$ $\left(\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 1 & -5 & 0 & 0 \\ -3 & -2 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -5 & 0 & 0 \\ 0 & -15 & 0 & 0 \\ 0 & -17 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \end{array}\right) \sim$

$\{(0, 0, r) ; r \in \mathbb{R}\} \leftarrow$ $y = 0$

$x = 0$

Určte vlastné hodnoty a vektory k matici

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

Riešenie.

- Zostavíme charakteristickú rovnicu:
$$\begin{pmatrix} \lambda & 0 & -2 \\ 0 & \lambda - 2 & 0 \\ -2 & 0 & \lambda - 3 \end{pmatrix} = \mathbf{0}$$
- Vyjadríme si determinant matice na ľavej strane rovnice a zistíme, kedy je rovný 0. Po úprave dostaneme:
$$(\lambda - 2).(\lambda - 4).(\lambda + 1) = 0.$$
Potom $\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = -1$.

$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$ $\begin{pmatrix} \lambda & 0 & -2 \\ 0 & \lambda-2 & 0 \\ -2 & 0 & \lambda-3 \end{pmatrix}$

$\begin{matrix} \lambda & 0 & -2 \\ \lambda & \lambda-2 & 0 \end{matrix}$

$\lambda(\lambda-2).(\lambda-3) - (4(\lambda-2)) =$

$= (\lambda-2)(\lambda(\lambda-3)-4) =$

$= (\lambda-2)(\lambda^2-3\lambda-4) =$

$= (\lambda-2)(\lambda-4)(\lambda+1)$

$\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = -1$

- Určíme vlastné vektory pre λ_1. Teda vyriešime sústavu $(\lambda_1.I_3 - A).\overline{x} = \mathbf{0}$.
$$\left(\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
Riešením je teda množina vektorov $\{[0, r, 0]; r \in \mathbb{R}\}$.
- Podobne určíme vlastné vektory pre λ_2.
$$\left(\begin{array}{ccc|c} 4 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ -2 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
Riešením je teda množina vektorov $\{[\frac{1}{2} r, 0, r]; r \in \mathbb{R}\}$.
- A na záver určíme aj vlastné vektory pre λ_3.
$$\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & -3 & 0 & 0 \\ -2 & 0 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
Riešením je teda množina vektorov $\{[-2r, 0, r]; r \in \mathbb{R}\}$.

- Vyberme teraz z každej množiny vektorov jeden tak, aby sme dostali trojicu normovaných ortogonálnych vektorov: $\overline{x_1} = [0,1,0], \overline{x_2} = [\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}], \overline{x_3} = [\frac{-2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}]$. Zostavme maticu H tak, že jej stĺpcami budú postupne vektory $\overline{x_1}, \overline{x_2}, \overline{x_3}$. Zrejme:

$$H = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

Overte, že

$$H^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}}_{H^{-1}} \cdot \underbrace{\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}}_{H} = B =$$

$\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = -1$

$$= \begin{pmatrix} 0 & 2 & 0 \\ \frac{4}{\sqrt{5}} & 0 & \frac{8}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Podobnost matic $\rightarrow$ ekvivalence

Definícia. Nech A, B sú štvorcové matice stupňa n. Hovoríme, že sú **podobné** ak existuje štvorcová regulárna matica P stupňa n taká, že platí: $B = P^{-1}.A.P$.
Veta. Podobné matice majú rovnaké vlastné hodnoty.
Pozor, opačná implikácia vo všeobecnosti neplatí!
Poznámka. Relácia podobnosti je zrejme reflexívna, symetrická a tranzitívna, teda je to ekvivalencia. Množina štvorcových matíc sa teda rozpadá na disjunktné triedy navzájom podobných matíc.

Nájdite vlastné čísla matíc. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
Sú tieto matice podobné?

Riešenie. Po úprave zistíme, že obe matice majú rovnaké vlastné číslo $\lambda = 1$. Všimnime si prvú z nich, je to jednotková matica. Preto platí:

$$P^{-1}.E.P = (P^{-1}.E).P = P^{-1}.P = E.$$

Teda jednotková matica je podobná iba sama so sebou. Našli sme príklad matíc, ktoré majú rovnaké vlastné čísla a nie sú podobné.

$x + cy + 2z = 2$

$x + y + cz = 9$

$2x + y + cz = 4$

$$\left(\begin{array}{ccc|c} 1 & c & 2 & 2 \\ 1 & 1 & c & 9 \\ 2 & 1 & c & 4 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & c & 2 & 2 \\ 1 & 1 & c & 9 \\ 1 & 0 & 0 & -5 \end{array}\right) \Rightarrow x = -5$$

$-5 \cdot 1 + y + cz = 9$

$y + cz = 14$

$-5 + cy + 2z = 2$

$cy + 2z = 7$

$y + cz = 14 \Rightarrow y = 14 - cz$

$$\left(\begin{array}{ccc|c} 1 & c & 2 & 2 \\ 0 & 1-c & c-2 & 7 \\ 0 & 1-2c & c-4 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & c & 2 & 2 \\ 0 & 1-c & c-2 & 7 \\ 0 & 1 & c & 14 \end{array}\right) / \cdot (-2) \sim \left(\begin{array}{ccc|c} 1 & c & 2 & 2 \\ 0 & 1 & c & 14 \\ 0 & 0 & c^2-2 & 14c-7 \end{array}\right) / \cdot (c-1)$$

$cy + 2z = 7$

$c(14 - cz) + 2z = 7$

$14c - c^2z + 2z = 7$

$c^2z - 14c - 2z + 7 = 0$

$z(c^2 - 2) - 14c + 7 = 0$

$c \neq \pm\sqrt{2}$ $\qquad$ $c = \sqrt{2}$ $\quad \emptyset$

$z = \dfrac{14c-7}{c^2-2}$ $\qquad$ $c = -\sqrt{2}$

$$\begin{vmatrix} 1 & c & 2 \\ 0 & 1 & c \\ 2 & 1 & c \end{vmatrix} = (c + 2 + 2c^2) - (4 + c + c^2) =$$

$= c^2 - 2 \rightarrow c = \pm\sqrt{2}$

$\rightarrow \mathbb{R} \setminus \{\pm\sqrt{2}\}$

1 c 2

0 1 c

$$\begin{vmatrix} 2 & c & 2 \\ 9 & 1 & c \\ 14 & 1 & c \end{vmatrix} \qquad x = \frac{|A_1|}{|A|} = -5$$

2 c 2

9 1 c

$(1,3,6)$, $(1,-4,1)$

$(3,10,b)$

a) lin. komb.

b) nebyl lin. komb.

---

a) ①

$r+s=3 \quad \rightarrow \quad r=3-s$

$3r-4s=10 \qquad 3\cdot(3-s)-4s=10$

$6r+s=b \qquad 9-3s-4s=10$

$9-7s=10$

$s=-\frac{1}{7}$

$r-\frac{1}{7}=3$

$r=3+\frac{1}{7}=\frac{22}{7}$

$6\cdot\frac{22}{7}+\left(-\frac{1}{7}\right)=b$

$b=\frac{131}{7}$

②
$$\begin{vmatrix} 1 & 3 & 6 \\ 1 & -4 & 1 \\ 3 & 10 & b \end{vmatrix} = -7b+131 \overset{?}{=} 0$$

$b=\frac{131}{7}$

③
$$\begin{pmatrix} 1 & 3 & 6 \\ 1 & -4 & 1 \\ 3 & 10 & b \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 6 \\ 0 & -7 & -5 \\ 0 & 1 & b-18 \end{pmatrix}$$

Př: doplňte na bázi $V_3(\mathbb{R})$

$\{(1,2,3),(4,4,1),(3,1,2)\}$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 4 & 1 \\ 3 & 1 & 2 \end{vmatrix} = (8+12+6)-(16+1+36) = 26-53 \neq 0 \Rightarrow \text{vektory jsou nezávislé}$$

$1 \; 2 \; 3$

$4 \; 4 \; 1$

$\Downarrow$

tvoří bázi

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 4 & 1 \\ 3 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -11 \\ 0 & -5 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{11}{4} \\ 0 & 0 & \frac{27}{4} \end{pmatrix}$$ nezávislé ✓

další př:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 7 & 5 \\ 0 & 0 & 0 \end{vmatrix} \quad \{(1,2,3), (x,y,z), \boxed{(0,0,1)}\} \to \text{přidám a mám nezávislé}$$

pokaždé přirozeně

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (0,1,0) \qquad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (0, 1, 0) + (0, 0, 1)$$

libov. číslo (u prostřední 1 v $(0,1,0)$)

$V = \langle (1,b,1), (4,4,1), (1,1,b) \rangle$

a) dimenze = 2

b) dimenze = 3

---

a) $$\begin{vmatrix} 1 & b & 1 \\ 4 & 4 & 1 \\ 1 & 1 & b \end{vmatrix} = (4b + 4 + b) - (4b^2 + 1 + 4) = 5b + 4 - 4b^2 - 5 =$$

$$\begin{matrix} 1 & b & 1 \\ 4 & 4 & 1 \end{matrix}$$

$$= -4b^2 + 5b - 1$$

$$b = \frac{-5 \pm 3}{-8} = \begin{cases} \frac{1}{4} \\ 1 \end{cases}$$

$\langle (2,5,3), (4,4,2), (3,2,5) \rangle$

$$\begin{vmatrix} 2 & 5 & 3 \\ 4 & 4 & 2 \\ 3 & 2 & 5 \end{vmatrix} = (40 + 24 + 30) - (36 + 8 + 100) = 94 - 144 < 0$$

$$\begin{matrix} 2 & 5 & 3 \\ 4 & 4 & 2 \end{matrix}$$

$\neq 0 \Rightarrow$ nezávislé

jsou 3

$\Downarrow$

gen. 3 prost.

$$S = \langle \overset{\bar a_1}{(1,0,2,0)}, \overset{\bar a_2}{(0,1,0,-7)}, \overset{\bar a_3}{(3,-2,3,14)} \rangle$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -7 \\ 3 & -2 & 3 & 14 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -7 \\ 0 & -2 & -3 & 14 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & -3 & 0 \end{pmatrix} \Rightarrow \text{je baze}$$

$$\underline{\bar b_1 = \bar a_1 = (1,0,2,0)}$$

$$\bar b_2 = \bar a_2 + x b_1 \quad / \cdot \bar b_1$$

$$\underset{\text{ortog}}{\bar b_2 \bar b_1} = \underset{\text{součin}}{\bar a_2 \bar b_1} + x \bar b_1 \bar b_1$$

$$0 = 0 + x \cdot 5 \Rightarrow x = 0 \qquad \underline{\bar b_2 = (0,1,0,-7)}$$

$$b_2 = \bar a_2$$

$$\bar b_3 = \bar a_3 + r \cdot \bar b_1 + s \cdot \bar b_2$$

$$\bar b_3 \bar b_1 = \bar a_3 \bar b_1 + r \bar b_1 \bar b_1 + s \bar b_2 \bar b_1$$

$$0 = \bar a_3 \bar b_1 + r \bar b_1 \bar b_1 + 0$$

$$0 = 9 + r5 \Rightarrow r = -\frac{9}{5}$$

$$\bar b_3 \bar b_2 = \bar a_3 \bar b_2 + r \bar b_1 \bar b_2 + s \bar b_2 \bar b_2$$

$$0 = \bar a_3 \bar b_2 + 0 + s \bar b_2 \bar b_2$$

$$0 = -100 + s \cdot 50 \Rightarrow s = 2$$

$$\bar b_3 = (3,-2,3,14) + \left(-\frac{9}{5}\right)(1,0,2,0) + 2(0,1,0,-7) =$$

$$= \underline{\left(\frac{6}{5}, 0, -\frac{3}{5}, 0\right) = \bar b_3}$$

$$\bar c_1 = \left(\frac{1}{\sqrt5}, 0, \frac{2}{\sqrt5}, 0\right) = \left(\frac{b_{11}}{|\bar b_1|}, \frac{b_{12}}{|\bar b_1|}, \frac{b_{13}}{|\bar b_1|}, \frac{b_{14}}{|\bar b_1|}\right)$$

$$\bar c_3 = \left(\frac{2}{\sqrt5}, 0, -\frac{1}{\sqrt5}, 0\right)$$

$$\bar c_2 = \left(0, \frac{1}{\sqrt{50}}, 0, -\frac{7}{\sqrt{50}}\right)$$

$f: \mathbb{R}^3 \to \mathbb{R}^2$

$f((1,2,1)) = (1,2)$

$f((1,1,3)) = (0,3)$

$f((2,3,-1)) = (1,1)$

$f(\vec{a}+\vec{b}) = f(\vec{a}) + f(\vec{b})$

$f(r\cdot\vec{a}) = r f(\vec{a})$

$f((0,2,3))$
$f((0,4,6))$
$f((2,4,4))$ $\Big\} = ?$

$f(0,2,3) = r\,(1,2,1) + s\,(1,1,3) + t\,(2,3,-1)$, with f applied to each vector

moc složité pro více vektorů

$$\left(\begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 3 & 0 & 3 \\ 2 & 3 & -1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 2 \\ 0 & -1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & -2 & 1 & -1 \\ 0 & 0 & -5 & 0 & -4 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & \frac{6}{5} \\ 0 & 1 & 0 & 1 & \frac{3}{5} \\ 0 & 0 & 1 & 0 & \frac{4}{5} \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & \frac{3}{5} \\ 0 & 0 & 1 & 0 & \frac{4}{5} \end{array}\right)$$

$f((1,0,0)) = (-1,0)$

$f((0,1,0)) = (1,\frac{3}{5})$

$f((0,0,1)) = (0,\frac{4}{5})$

$f((0,2,3)) = 0\cdot f((1,0,0)) + 2\cdot f((0,1,0)) + 3 f((0,0,1)) =$

$= 0 + 2(1,\frac{3}{5}) + 3(0,\frac{4}{5}) = (2,\frac{18}{5})$

$f((0,4,6)) = 0 + 4\cdot(1,\frac{3}{5}) + 6\cdot(0,\frac{4}{5}) = (4,\frac{36}{5})$

$f((2,4,4)) = 2\cdot(-1,0) + 4\cdot(1,\frac{3}{5}) + 4\cdot(0,\frac{4}{5}) = (2,\frac{28}{5})$

E_1, E_2 jsou ekv.

? $E_1 \cap E_2$ je ekv ?

E_1 - R, S, T $\quad E_1 \cap E_2$ - R, S, T

E_2 - R, S, T

$\forall a \in A;\ (a,a) \in E_1$

$\forall a \in A;\ (a,a) \in E_2$

$\forall a \in A;\ (a,a) \in E_1 \cap E_2$

žřejmě $\forall a \in A;\ (a,a) \in E_1 \wedge (a,a) \in E_2 \Rightarrow \underline{(a,a) \in E_1 \cap E_2}$ ✓ R

$(a,b) \in E_1 \cap E_2 \Rightarrow (b,a) \in E_1 \cap E_2$

$(a,b) \in E_1 \cap E_2 \Rightarrow (a,b) \in E_1 \wedge (a,b) \in E_2 \Rightarrow (b,a) \in E_1 \wedge (b,a) \in E_2 \Rightarrow$

$= \underline{(b,a) \in E_1 \cap E_2}$ ✓ S

$(a,b) \in E_1 \cap E_2 \wedge (b,c) \in E_1 \cap E_2 \Rightarrow (a,c) \in E_1 \cap E_2$

$(a,b) \in E_1 \cap E_2 \wedge (b,c) \in E_1 \cap E_2 \Rightarrow$

$= \underline{(a,b) \in E_1} \wedge \underline{(a,b) \in E_2} \wedge \underline{(b,c) \in E_1} \wedge \underline{(b,c) \in E_2} \Rightarrow$

$\Rightarrow (a,c) \in E_1 \wedge (a,c) \in E_2 \Rightarrow \underline{(a,c) \in E_1 \cap E_2}$ ✓ T