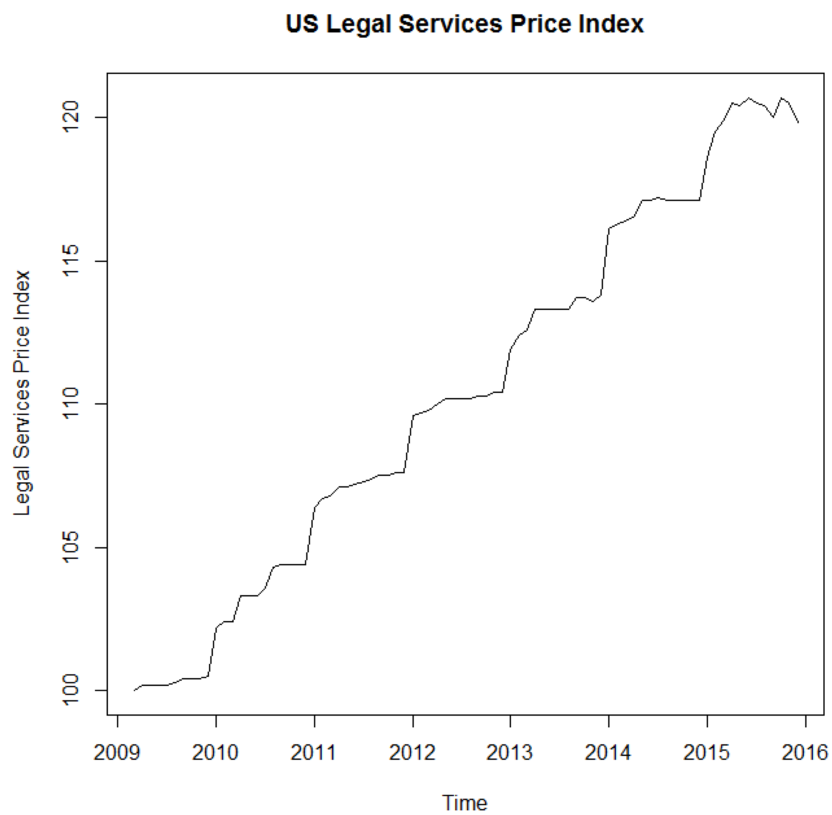
In the following, $Z(t)$ denotes a white noise process with mean zero and variance σ^2 unless otherwise stated. For questions 1–5, circle your answers clearly. If you make a mistake, indicate your answers clearly in the margin. Ambiguous responses will be considered incorrect.

1. Below is a plot of the price index for legal services in the US, monthly between March 2009 and December 2015. Each observation relates to the final day of a month, and is relative to the baseline value on 31st March 2009, which is taken as 100.



Based on the above information, what would you conclude about the time series? (Circle all that apply.)

(a) The series is stationary.

(b) **The series appears to have a trend.**

(c) Applying ∇ to the series would probably remove all non-stationary effects.

(d) Square rooting the series would probably remove all non-stationary effects.

(e) **The series appears to have a seasonal effect.**

2. Suppose we define the stochastic process $X(t)$ by

$$X(t) = X(t-1) + Z(t)$$

where $E(Z(t)) = 1$ for all t and $X(1) = Z(1)$. Which of the following is/are true for $X(t)$? (Circle all that apply.)

(a) $E(X(t)) = 0$.

(b) $\mathrm{Var}(X(t))$ does not depend on t.

(c) $X(t)$ is stationary.

(d) **The mean of $X(t)$ varies with time.**

(e) **The process $X(t) - X(t-1)$ is stationary.**

3. Consider the model

$$(1 - 0.3B)\left(1 - 0.2B^4\right) W(t) = \left(1 - 0.1B^4\right) Z(t)$$

where

$$W(t) = \nabla\nabla_4 X(t).$$

This is a SARIMA model of order

(a) $(0,1,0) \times (0,1,2)_1$

(b) $(0,0,2) \times (0,1,1)_4$

(c) *$(1,1,0) \times (1,1,1)_4$ *

(d) $(2,0,1) \times (1,1,1)_5$

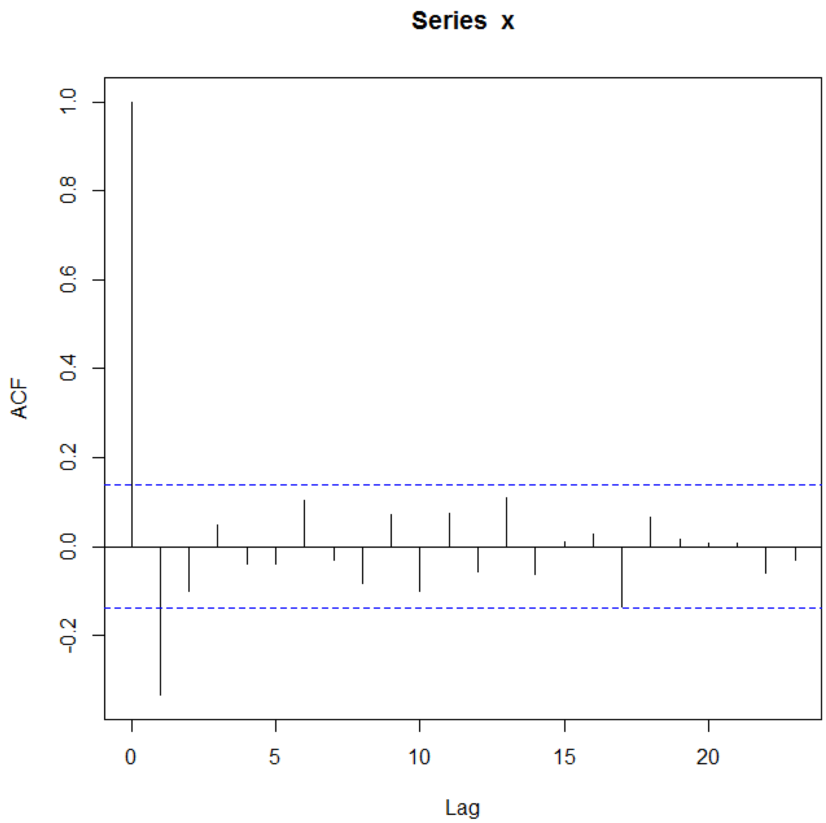
(e) $(1,0,2) \times (0,1,1)_4$

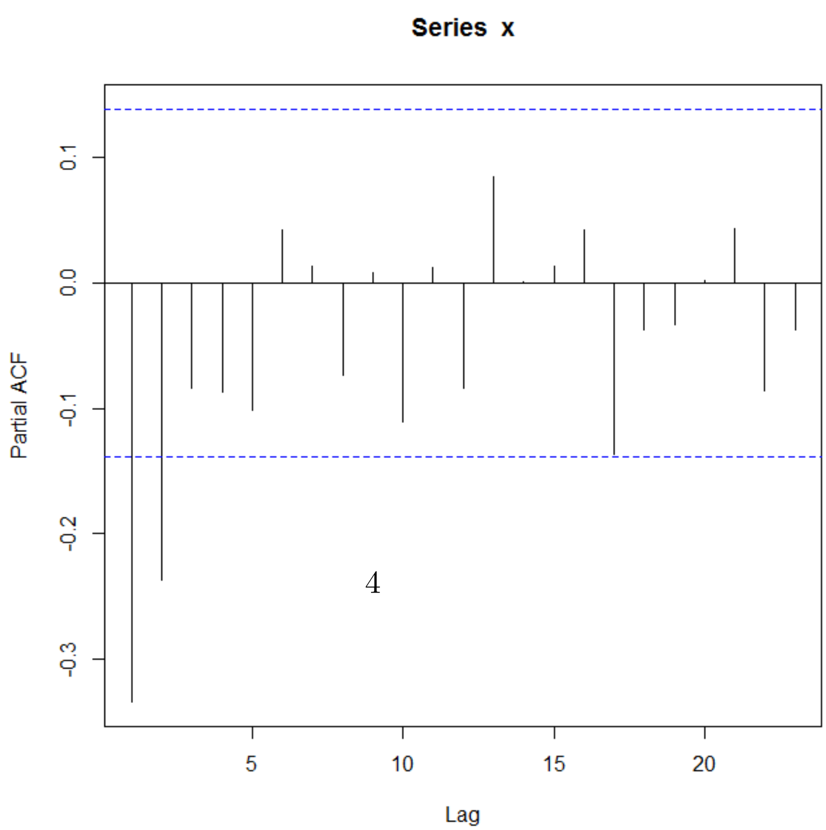
4. The model in the previous question can be written as an ARMA(p, q) model where the order (p, q) is

   (a) $(9, 1)$
   (b) $(9, 4)$
   (c) $(4, 9)$
   (d) $*(10, 4)*$
   (e) $(4, 10)$

5. The following are the sample acf and pacf from a time series labelled x

that contains 200 values.





Based on the above information, which of the following models would you consider the most appropriate for the data?

(a) AR(1)
(b) AR(3)
(c) *MA(1)$*$
(d) MA(2)
(e) ARMA(2, 2)

6. The time series x from Q5 has the following summary statistics:

k	1	2	3	4	5
acf r_k	–0.334	–0.099	0.050	–0.039	–0.040
pacf $\hat{\alpha}_{kk}$	–0.334	–0.237	–0.084	–0.087	–0.101

The mean of the observed series was $\bar{x} = -0.0258$, and $c_0 = 3.186$. Assuming the underlying model has mean zero, obtain method of moments estimates of *two* parameters for the model you selected in Q5. Explain your reasoning clearly.
For an MA(1), $X(t) = \beta Z(t-1) + Z(t)$, we know

$$\begin{aligned}\operatorname{Var}(X(t)) &= \sigma^2\left(1+\beta^2\right), \\ \rho(1) &= \frac{\beta}{1+\beta^2}.\end{aligned}$$

Equating the acf value on the left to –0.334 suggests the estimate of β to be solutions of

$$0 = 0.334\beta^2 + \beta + 0.334.$$

Solution are $\hat{\beta} = -0.383$ or -2.611. (2 marks) *As only $\hat{\beta} = -0.383$ gives an invertible model, we use this estimate.* (1 mark, for explaining choice.) *Then the method of moments estimate of σ^2 is*

$$\begin{aligned}\hat{\sigma}^2 &= \frac{c_0}{1+\hat{\beta}^2} \\ &= \frac{3.186}{1+(-0.383)^2} \\ &= 2.778\end{aligned}$$

(2 marks. In fact data are simulated from MA(1) with $\beta = -0.40$ and $\sigma^2 = 2.5$.)

7. Define the process $X(t)$ by the following:

$$X(t) = 1.6X(t-1) - 0.8X(t-2) - 0.1X(t-3) + Z(t)$$

(a) Given that $X(t)$ is stationary, *derive* the Yule–Walker equations for the process.
Take the defining equation and multiply both sides by $X(t-k)$ for $k = 1, 2, 3,$ giving:

$$\begin{aligned}
X(t)X(t-1) &= 1.6X(t-1)^2 - 0.8X(t-2)X(t-1) - 0.1X(t-3)X(t-1) \\
&\quad + Z(t)X(t-1) \\
X(t)X(t-2) &= 1.6X(t-1)X(t-2) - 0.8X(t-2)^2 - 0.1X(t-3)X(t-2) \\
&\quad + Z(t)X(t-2) \\
X(t)X(t-3) &= 1.6X(t-1)X(t-3) - 0.8X(t-2)X(t-3) - 0.1X(t-3)^2 \\
&\quad + Z(t)X(t-3).
\end{aligned}$$

Now taking expectations on both sides and dividing by σ_X^2 we have

$$\begin{aligned}
\rho(1) &= 1.6\rho(0) - 0.8\rho(1) - 0.1\rho(2) \\
\rho(2) &= 1.6\rho(1) - 0.8\rho(0) - 0.1\rho(1) \\
\rho(3) &= 1.6\rho(2) - 0.8\rho(1) - 0.1\rho(0)
\end{aligned}$$

since $E(X(t)X(t-k))/\sigma_X^2 = \rho(k)$ here. Since $\rho(0) = 1$ we have

$$\begin{aligned}
\rho(1) &= 1.6 - 0.8\rho(1) - 0.1\rho(2) \\
\rho(2) &= 1.6\rho(1) - 0.8 - 0.1\rho(1) = 1.5\rho(1) - 0.8 \\
\rho(3) &= 1.6\rho(2) - 0.8\rho(1) - 0.10.
\end{aligned}$$

(6 marks: 3 for Y-W equations, 3 for derivation).

(b) Hence find the autocorrelation function for $X(t)$ at lags 1 and 2 (to two decimal places).
So

$$\begin{aligned}
\rho(1) &= 1.6 - 0.8\rho(1) - 0.1(1.5\rho(1) - 0.8) \\
&= 1.6 - 0.8\rho(1) - 0.15\rho(1) + 0.08
\end{aligned}$$

and hence $\rho(1) = 0.86$ *and then* $\rho(2) = 1.5\rho(1) - 0.8 = 0.49$. (2 marks)

Group version: Given that the autocorrelation at lag 1 for $X(t)$ is 0.86, find the autocorrelation function at lags 2 and 3.
As $\rho(1) = 0.86$ *and then* $\rho(2) = 1.5\rho(1) - 0.8 = 0.49$.*Further,*

$$\rho(3) = 1.6 \times 0.49 - 0.8 \times 0.86 - 0.1 = -0.004.$$

(2 marks)

(c) A business analyst is interested in monitoring and modelling the price per tonne for a particular commodity. Having de-trended and de-seasonalised the monthly price series, she conjectures that the model in this question might be a suitable model for the resulting series. You calculate the first three values of the sample autocorrelation function to be $(r_0 = 1)$ $r_1 = 0.847$, $r_2 = 0.454$, $r_3 = -0.006$. Based on the information you have, does the analyst's proposed model appear reasonable? Explain your answer.
As the sample acf values nearly match the values expected for the model, the proposed model seems reasonable. (2 marks, 1 for affirming the model choice, 1 for justification. Note for individual version $\rho(3)$ could be found to strengthen the argument.)