Activity Solution: Examples of Spectral Densities/Fourier Coefficients

The spectral density function $f(\omega)$ of a stationary stochastic process $X(t)$ is defined as

$$f(\omega) = \frac{1}{\pi}\left(\gamma(0) + 2\sum_{k=1}^{\infty}\gamma(k)\cos(\omega k)\right)$$

for $\omega \in (0, \pi)$, where $\gamma(k)$ is the acvf. of $X(t)$. Basically, $f(\omega)$ is the FT of $\gamma(k)$. The normalized spectrum is

$$f^*(\omega) = \frac{f(\omega)}{\sigma_X^2},$$

where $\sigma_X^2 = \text{Var}(X(t))$, is the FT of $\rho(k)$, the acf.

The aim here is to aid appreciation of the spectral density by consideration of some key special cases.

1. Consider first the MA(1) process $X(t) = Z(t) + \beta Z(t-1)$.

   (a) Remind yourself of, or derive if need be, the acvf and acf of $X(t)$.
   The acf is

   $$\rho(k) = \begin{cases} 1 & k = 0 \\ \frac{\beta}{1+\beta^2} & k = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (b) *Clickers question:* Write down the normalized spectrum and spectral density function here.
   So the normalized spectrum here is

   $$f^*(\omega) = \frac{1}{\pi}\left(1 + \frac{2\beta\cos(\omega)}{1+\beta^2}\right),$$

   and the spectral density is

   $$f(\omega) = \left(1+\beta^2\right)\sigma^2 f^*(\omega).$$

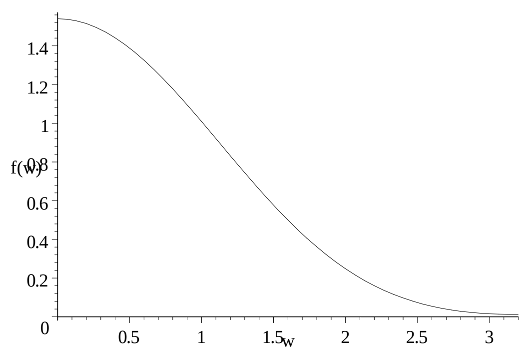
   Now the shape of the spectrum is determined by the sign of β.

(c) Plot the normalized spectral density when $\beta = 1$. Briefly comment on what you observe.

This case was seen last class, with

$$f^*(\omega) = \frac{1 + \cos(\omega)}{\pi},$$

being dominated by low frequencies,

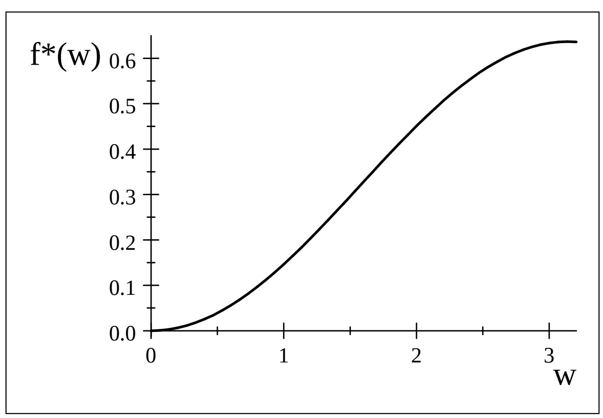


(d) Plot the normalized spectral density when $\beta = -1$.Briefly comment on what you observe.

However when $\beta = -1$, the spectrum peaks at high frequencies,

$$f^*(\omega) = \frac{1 - \cos(x)}{\pi}$$

which is



2. The AR(1) process

$$X(t) = \alpha X(t-1) + Z(t)$$

has a spectral density that can be written (see exercises 4)

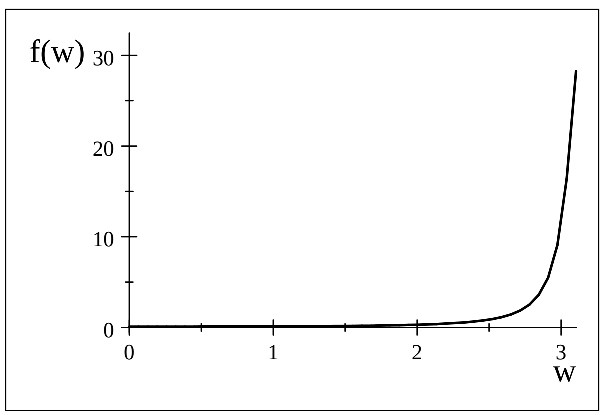
$$f(\omega) = \frac{\sigma^2}{\pi\left(1 - 2\alpha\cos(\omega) + \alpha^2\right)}.$$

Plot the cases below when $\sigma^2 = 1$, and in each case comment on the observed spectral density.

(a) $\alpha = -0.9$
Here we plot,

$$f(\omega) = \frac{1}{\pi\left(1.81 + 1.8\cos(\omega)\right)}$$

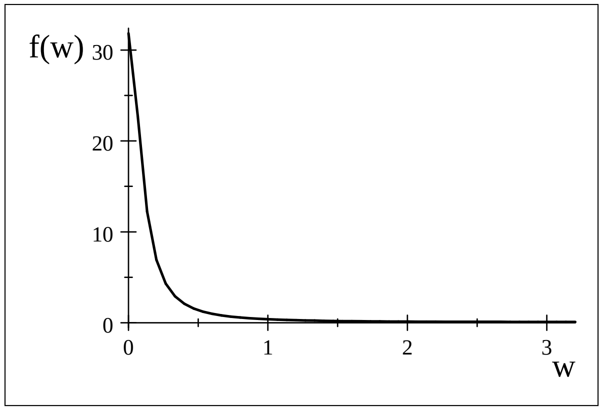
shown below:



The spectrum is dominated by high frequencies, not surprising since the series would tend to alternate signs.

(b) $\alpha = 0.9$,
Conversely here we plot

$$f(\omega) = \frac{1}{\pi\left(1.81 - 1.8\cos(\omega)\right)},$$

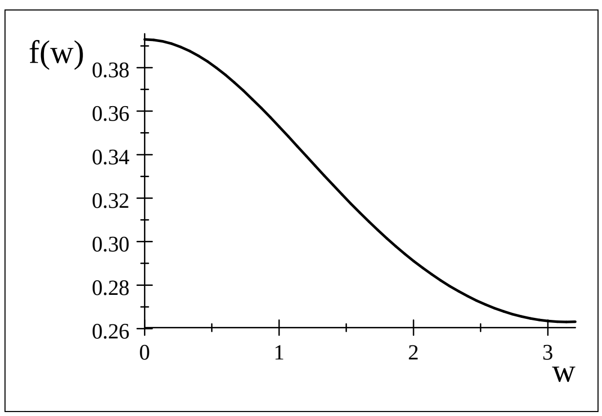


Low frequencies dominate here; again this is to be expected, since the series would be smooth and tend to exhibit low frequency variation as consecutive values tend to be highly correlated.

(c) $\alpha = 0.1$.
Finally we plot,

$$f(\omega) = \frac{1}{\pi\,(1.01 - 0.2\cos(x))}$$

and observe a spectrum where low frequencies contribute most to the variation in the process, but less pronounced than in (b).



3. What happens in the limit for an AR(1) as $\alpha \to 1$? What can you say about the spectrum in this limiting case? Is the spectrum defined for all $\omega \in [0, \pi]$?
As $\alpha \to 1$, *the* $AR(1)$ *approaches a random walk process, which suggests*

that

$$f(\omega) = \frac{\sigma^2}{2\pi\left(1 - \cos(\omega)\right)}$$

can be taken as the spectrum for a random walk. However the acvf. for this process is not absolutely summable, so strictly the spectrum does not exist. That said, apart from at $\omega = 0$, the above is a well–defined function.

Fourier Coefficients: Let N be an even, positive integer. Denote the Fourier frequencies $\omega_p = 2\pi p/N$, for $p = 0, 1, 2, \ldots, N/2$.

1. *Clickers question:* Find $\sin\left(\omega_{N/2}\right)$, $\sin(\omega_0)$, $\cos\left(\omega_{N/2}\right)$ and $\cos(\omega_0)$.
   Clearly

   $$\omega_{N/2} = \pi,$$

   so that

   $$\begin{aligned} \sin\left(\omega_{N/2}\right) &= \sin(\omega_0) = 0, \\ \cos\left(\omega_{N/2}\right) &= -\cos(\omega_0) = -1. \end{aligned}$$

2. *Clickers question:* For arbitrary ω, write down an expression in closed form for

   $$\sum_{t=1}^{N} e^{\mathbf{i}\omega t}.$$

   Recalling that $e^{2\pi\mathbf{i}} = 1$, what is $\sum_{t=1}^{N} e^{\mathbf{i}\omega_p t}$?
   Summing the G.P.,

   $$\sum_{t=1}^{N} e^{\mathbf{i}\omega t} = \frac{e^{\mathbf{i}\omega}\left(1 - e^{\mathbf{i}\omega N}\right)}{\left(1 - e^{\mathbf{i}\omega}\right)}$$

   and this vanishes at $\omega = \omega_p$, i.e.,

   $$\sum_{t=1}^{N} e^{\mathbf{i}\omega_p t} = 0$$

   as

   $$e^{2\pi\mathbf{i}} = 1.$$

   (Note that if $z \in \mathbb{C}$, the nth root of z takes n distinct values in $\mathbb{C}$, so arguments involving substituting $\omega = \omega_p$ directly are non-trivial.)

3. *Clickers question:* Recalling that

$$e^{\mathbf{i}\omega} = \cos(\omega) + \mathbf{i}\sin(\omega)$$

use your result above to find $\sum_{t=1}^{N} \cos(\omega_p t)$ and $\sum_{t=1}^{N} \sin(\omega_p t)$ (for $p \neq 0$).

From the above,

$$\begin{aligned} \sum_{t=1}^{N} e^{\mathbf{i}\omega_p t} &= \sum_{t=1}^{N} \cos(\omega_p t) + \mathbf{i} \sum_{t=1}^{N} \sin(\omega_p t) \\ &= 0 + 0\mathbf{i}, \end{aligned}$$

and both sums are zero by equating real and imaginary parts.

4. *Clickers question:* Replacing ω_p by $2\omega_p$ in the above result and using the double–angle formula for $\sin(2A)$, find

$$\sum_{t=1}^{N} \cos(\omega_p t) \sin(\omega_p t).$$

Replacing ω_p *by* $2\omega_p$ *in the above, since* $\sin(2A) = 2\cos(A)\sin(A)$,

$$\sum_{t=1}^{N} \cos(\omega_p t) \sin(\omega_p t) = \frac{1}{2} \sum_{t=1}^{N} \sin(2\omega_p t) = 0.$$

Additional simplifications arise when working with the Fourier frequencies:

(b) For each $p, q = 0, 1, 2, \ldots, N/2$,

$$\sum_{t=1}^{N} \sin(\omega_p t) \cos(\omega_q t) = 0.$$

(c)

$$\sum_{t=1}^{N} \cos(\omega_p t) \cos(\omega_q t) = \begin{cases} 0 & p \neq q \\ \frac{N}{2} & p = q \neq 0, \frac{N}{2} \\ N & p = q = 0, \frac{N}{2} \end{cases}$$

(d)

$$\sum_{t=1}^{N} \sin(\omega_p t)\sin(\omega_q t) = \begin{cases} 0 & p \neq q \\ \frac{N}{2} & p = q \neq 0, \frac{N}{2} \\ 0 & p = q = 0, \frac{N}{2} \end{cases}$$

5. Re-cap what you have done during this activity. Why do you think you were asked to do this activity? What have you learned?
The spectral density function of a stationary stochastic process has been defined as the Fourier transform of the acvf of the process. Some special cases of the spectral density were examined, including particular cases of $MA(1)$ and $AR(1)$ models. Some intuition was gained from interpreting the shapes of the spectral densities in the light of known properties of the stochastic processes.
The activity moved on to consider a set of $N/2+1$ frequencies, known as the Fourier frequencies, that partition the interval $[0, \pi]$. These frequencies have nice mathematical properties, and in particular summations of trigonometric functions at integer multiples of Fourier frequencies, and sums of products of such functions, can have simple closed form representations. We will see later that this leads to representations of time series observed at integer times.
Learning outcomes encountered in this activity include:

   (a) *Define and interpret the spectral density and spectral distribution functions for a time series.*

   (b) *Recall key properties of the spectral density and spectral distribution functions.*

   (c) *Where mathematically tractable, compute the spectral density and spectral distribution functions of a time series model.*