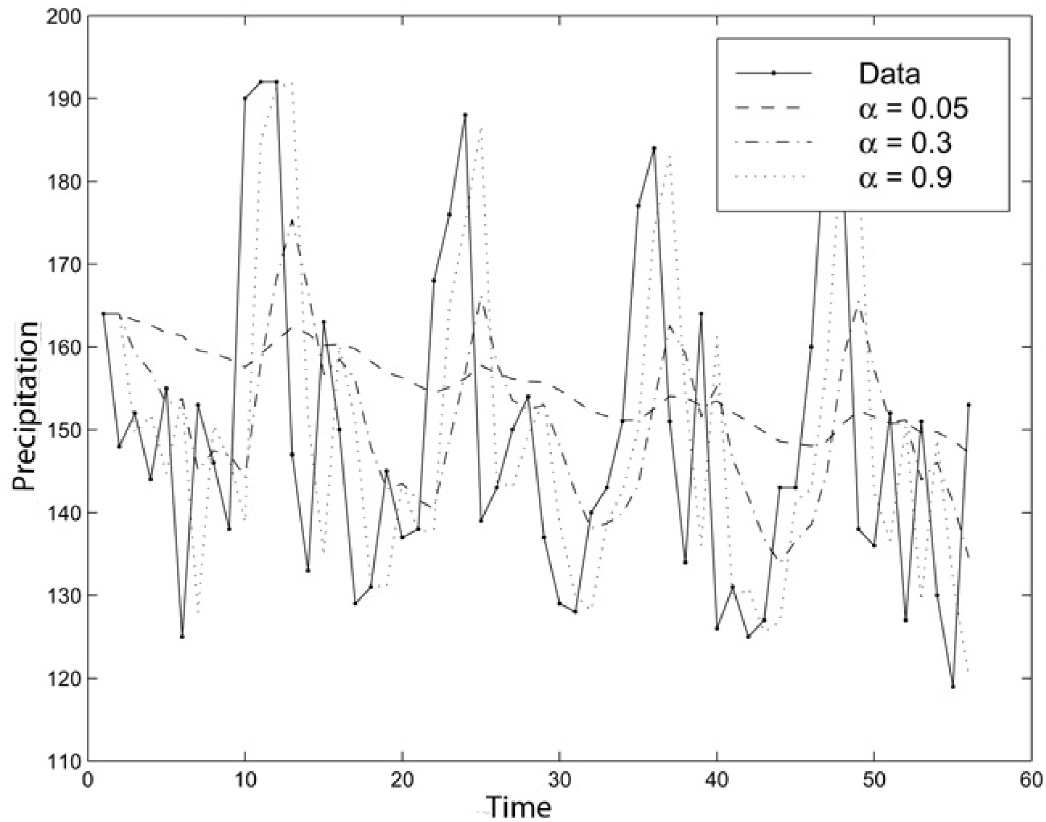
Activity Solution: Exponential Smoothing, Holt, and Holt–Winters Method

This activity extends the method known as exponential smoothing to cover non-stationary effects such as trends and seasonal variation.

1. *Clickers question:* Consider the following figure showing precipitation levels (in mm) on a city over 58 months. In addition to the data, fitted exponential smoothing models with $\alpha = 0.05$, $\alpha = 0.3$, and $\alpha = 0.9$ are shown. Compare and contrast the different fitted models. Which is least noisy (i.e, the smoothest)? Which best captures the seasonality?



The case with $\alpha = 0.9$ puts much weight on the last value, so the predicted series follows the original closely but looks as though it lags *behind one time unit. As is usual, the smallest value of α gives the smoothest model giving a good prediction of the overall "trend".*

2. An improvement in cases where the series appears to show a trend splits the model into two components: a *level* $L(t)$ and a *trend* $T(t)$. Holt (1957) suggested taking

$$L(t) = \alpha x(t) + (1-\alpha)(L(t-1) + T(t-1)),$$

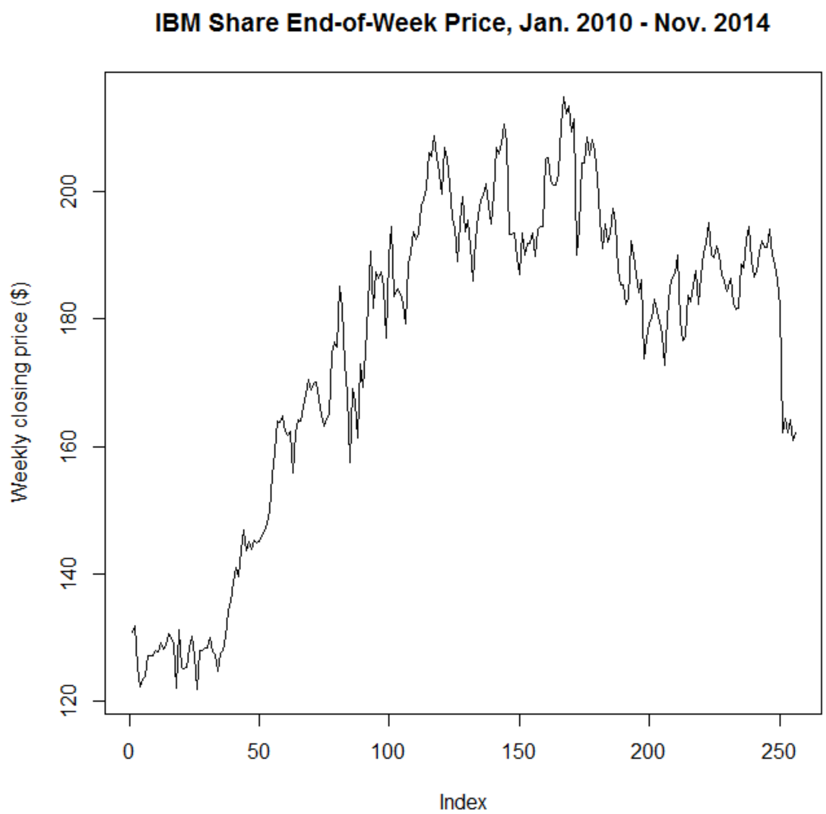
and

$$T(t) = \beta (L(t) - L(t-1)) + (1-\beta) T(t-1),$$

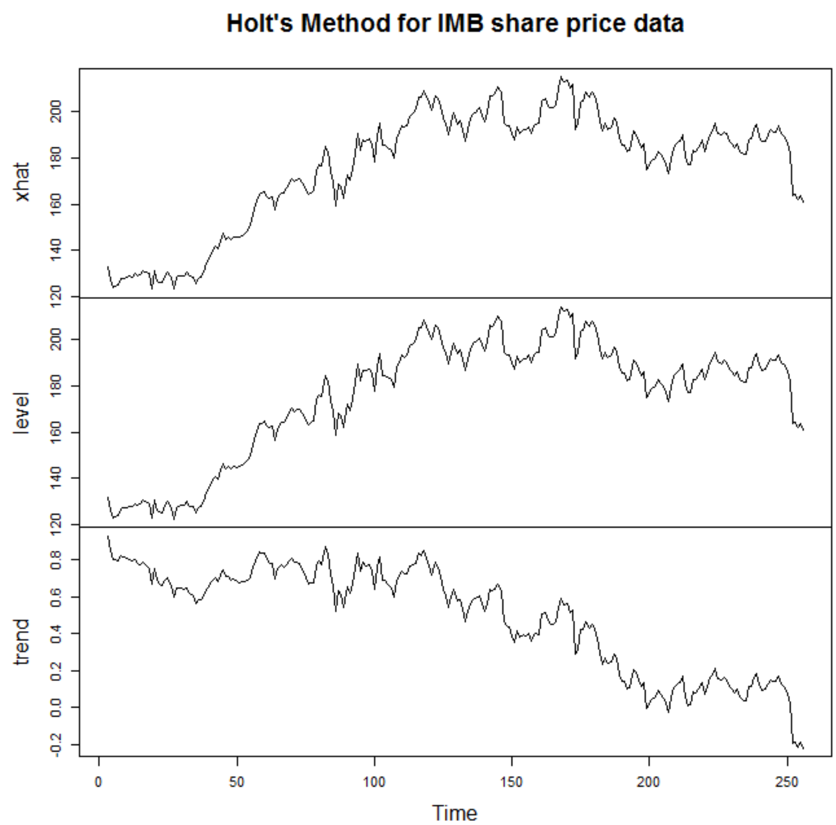
for some constants α and β. Explain why the above approach is plausible. (Hint: Consider first the special case with no trend.)
Think of the level $L(t)$ as being the expected value of $X(t)$; the value is taken as a weighted sum of the most recent observation and the previous value of the level (plus the trend at the last time point, should there be a trend). The trend at time t is taken as a weighted average of the change in the level over the last unit time interval and the estimate at time $t-1$. The weights in the two defining equations are not the same in general, that is, $\alpha \neq \beta$ in general. The level follows the mean of the series, and the estimates of $L(t)$ give the smoothed version of the series.

3. *Clickers question:* The end–of–week closing prices for an IBM stock (in \$) between January 2010 and November 2014 inclusive are plotted

below.



Using Holt's method in R, the fitted values, along with estimates of the level and trend, are plotted below:



Given that the estimates of the last values of the level and trend are respectively 162.0582 and –0.2081 (in \$), what are the forecasts for the next two values of the time series?

The one–step forecast is

$$\begin{aligned}\hat{x}(256,1) &= 162.0582 - 0.2081 \\ &= 161.85\end{aligned}$$

dollars. The two–step forecast is

$$\begin{aligned}\hat{x}(256,2) &= 162.0582 - 2 \times 0.2081 \\ &= 161.64.\end{aligned}$$

4. An extension of Holt's method adds a seasonal effect of period p, denoted $I(t)$ here, a periodic function with period p. When the seasonal effect is additive, the model is

$$\begin{aligned} L(t) &= \alpha(x(t) - I(t-p)) \\ &\qquad + (1-\alpha)(L(t-1) + T(t-1)), \\ T(t) &= \beta(L(t) - L(t-1)) + (1-\beta)T(t-1), \\ I(t) &= \gamma(x(t) - L(t)) + (1-\gamma)I(t-p), \end{aligned}$$

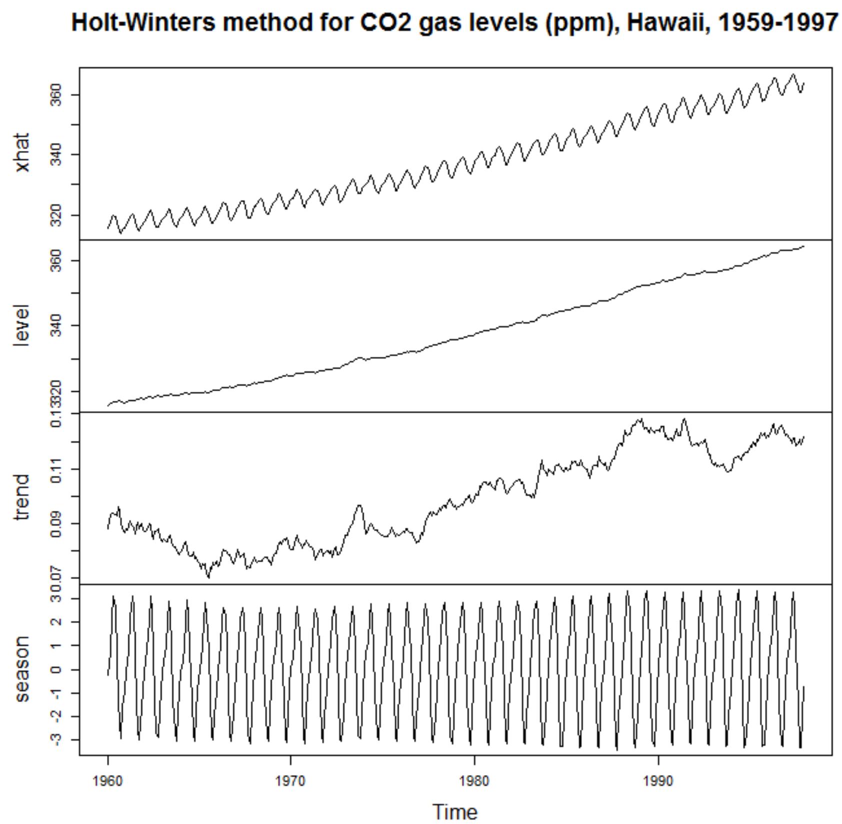
for some smoothing parameters α, β, and γ. Discuss why the above is a natural extension to Holt's method when the data exhibit an additive seasonal effect of period p. How would you modify the above when the seasonal effect is multiplicative?

The estimate for the level at time t is as for Holt's method, but the value of $x(t)$ is adjusted by deducting the estimate of the seasonal effect at time $t-p$; since the value of $I(t)$ requires an estimate of $L(t)$, the most recent estimate of $I(\cdot)$ that can be used at time t is $I(t-p)$. In this way the level is seasonally adjusted, and the trend depends only on the level. New values of the seasonal index are found by a weighted average of the current (de-meaned) observation and the estimate at p time units previous. The model for data with a multiplicative seasonal effect of period p has

$$\begin{aligned} L(t) &= \alpha\left(\frac{x(t)}{I(t-p)}\right) + (1-\alpha)(L(t-1) + T(t-1)), \\ T(t) &= \beta(L(t) - L(t-1)) + (1-\beta)T(t-1), \\ I(t) &= \gamma\left(\frac{x(t)}{L(t)}\right) + (1-\gamma)I(t-p). \end{aligned}$$

5. *Clickers question:* The plot below shows the above method applied to the monthly levels of CO_2 gas, in parts per million (ppm), taken between 1959 and 1997 in Hawaii. Was the seasonal effect taken as

additive or multiplicative?



Given that the estimates of the seasonal effect for January and February 1997 are 0.2215 and 0.9553 respectively, and the December 1997 estimates of the level and trend components were 364.762 and 0.1247 respectively, what are forecasted values for January and February 1998? *The regular nature of the seasonal variation over each cycle suggests an additive model is most natural. Using such a model, we have*

$$\begin{aligned}\hat{x}(N,1) &= 364.762+0.1247+0.2215\\ &= 365.11\end{aligned}$$

and

$$\begin{aligned}\hat{x}(N,2) &= 364.762+2\times 0.1247+0.9553\\ &= 365.97\end{aligned}$$

(both in ppm).

6. Re-cap what you have done during this activity. Why do you think you were asked to do this activity? What have you learned?
   We have met three related approaches for forecasting series. Exponential smoothing is a simple approach for stationary series and takes the next forecast values to be a weighted sum of previous observations, the weights exponentially decreasing like geometric distribution probabilities. A little algebra shows that for this method the next forecast value is just a weighted sum of the last value and its forecast based on the previous observations. Holt's method is similar but incorporates an additive trend. A further extension is Holt–Winters method, which allows inclusion of either an additive or multiplicative seasonal effect of known period.
   Learning outcomes encountered in this activity include:

   (a) *Explain the principles underlying exponential smoothing as a forecasting method.*

   (b) *Describe the role of the parameter α in exponential smoothing, and the criteria for how this parameter can be chosen.*

   (c) *Explain the principles underlying Holt's method (double exponential smoothing) as a forecasting method.*

   (d) *Describe the role of the parameters α and β in Holt's method, and the criteria for how these parameter can be chosen.*

   (e) *Apply Holt's method to forecast future values of a time series, given necessary parameter estimates.*

   (f) *Explain the principles underlying Holt–Winters forecasting method for series with additive or multiplicative seasonal components.*

   (g) *Describe the roles of the parameters α, β, and γ in Holt–Winters forecasting method.*

   (h) *Apply Holt–Winters method to forecast future values of a time series, given necessary parameter estimates.*
   The original reference for the methods described here is:

Holt, C. C. (1957): Forecasting trends and seasonals by exponentially weighted moving averages. *ONR Research Memorandum*, Carnegie Institute of Technology **52**.