Activity Solution: A Sinusoidal Model

For some constants α and β, consider the model

$$X(t) = \alpha \cos(\omega t) + \beta \sin(\omega t) + Z(t) \tag{1}$$

where $Z(t)$ is white noise and ω is a fixed frequency. We take a sample

$$\mathbf{x}' = (x(1), x(2), \ldots, x(N))$$

at fixed time periods, and assume that N is even. The vector

$$\boldsymbol{\theta} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

contains the parameters to be estimated.

1. *Clickers question:* Following ideas you may have encountered in STAT 306, for some matrix A the model (1) can be written

   $$E(\mathbf{X}) = A\boldsymbol{\theta},$$

   where $\mathbf{X} = (X(1), \ldots, X(N))'$. Find the matrix A.
   Well we have

   $$\begin{aligned} \mathbf{X} &= (X(1), \ldots, X(N))', \\ A &= \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ \cos(2\omega) & \sin(2\omega) \\ \vdots & \vdots \\ \cos(N\omega) & \sin(N\omega) \end{pmatrix}. \end{aligned}$$

2. *Clickers question:* The least squares estimate of $\boldsymbol{\theta}$ based on data $\mathbf{x}$ is given by

   $$\hat{\boldsymbol{\theta}} = (A'A)^{-1} A'\mathbf{x}.$$

   Suppose we restrict possible frequencies in the model to $\omega_p = \frac{2\pi p}{N}$ for $p = 1, 2, \ldots, N/2 - 1$, the Fourier frequencies (the case $p = N/2$ can be handled separately). Find $A'A$ and $(A'A)^{-1}$ in this case. (Hint: You

need identities given in the previous class.)
Now $A'A$ is

$$\begin{pmatrix} \sum_{t=1}^{N} \cos^2(\omega_p t) & \sum_{t=1}^{N} \cos(\omega_p t)\sin(\omega_p t) \\ \sum_{t=1}^{N} \cos(\omega_p t)\sin(\omega_p t) & \sum_{t=1}^{N} \sin^2(\omega_p t) \end{pmatrix}$$

and for $p \neq \frac{N}{2}$, by identities (b), (c) and (d), this is

$$\begin{pmatrix} \frac{N}{2} & 0 \\ 0 & \frac{N}{2} \end{pmatrix}.$$

Therefore

$$(A'A)^{-1} = \begin{pmatrix} \frac{2}{N} & 0 \\ 0 & \frac{2}{N} \end{pmatrix}.$$

3. *Clickers question:* Hence find $\hat{\boldsymbol{\theta}}$. (Hint: Treat the case $p = N/2$ separately.) Comment on the results.
Now

$$A'\mathbf{x} = \begin{pmatrix} \sum_{t=1}^{N} x(t)\cos(\omega_p t) \\ \sum_{t=1}^{N} x(t)\sin(\omega_p t) \end{pmatrix}.$$

Hence

$$\begin{aligned} \hat{\boldsymbol{\theta}} &= (A'A)^{-1} A'\mathbf{x} \\ &= \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} \hat{\alpha} &= \frac{2}{N}\sum_{t=1}^{N} x(t)\cos(\omega_p t) \\ \hat{\beta} &= \frac{2}{N}\sum_{t=1}^{N} x(t)\sin(\omega_p t). \end{aligned}$$

When $p = N/2$,

$$\begin{aligned} \hat{\alpha} &= \frac{1}{N}\sum_{t=1}^{N} (-1)^t x(t), \\ \hat{\beta} &= 0 \end{aligned}$$

since the $\sin(\pi t)$ terms disappear. Note the estimates of the coefficients in the model are the Fourier coefficients derived in the previous activity.

4. Re-cap what you have done during this activity. Why do you think you were asked to do this activity? What have you learned?
We have seen that for a simple stochastic sinusoidal model at a single fixed frequency, given data from the model we can estimate the parameters involve via least squares. Moreover, the estimates of the coefficients in the model are not difficult to derive mathematically and are so–called Fourier coefficients explored in a previous activity. This motivates the study of Fourier coefficients and their use in model fitting in the frequency domain.
The learning outcome explored in the activity is:

    (a) *Describe the role of the Fourier series coefficients in the fitting of a sinusoidal model to a time series by least squares.*