Activity Solution: Introducing Spectral Densities

The *spectral density function* $f(\omega)$ of a stationary, integer–time stochastic process $X(t)$ is defined as

$$f(\omega) = \frac{1}{\pi}\left(\gamma(0) + 2\sum_{k=1}^{\infty}\gamma(k)\cos(\omega k)\right)$$

for $\omega \in (0, \pi)$, where $\gamma(k)$ is the autocovariance function of $X(t)$. We can see that $f(\omega)$ is the FT of $\gamma(k)$.

1. By applying the inverse Fourier transform and its properties in this situation, write $\gamma(k)$ in terms of $f(\omega)$.
   Since $\gamma(k)$ is an even function and defined for $k \in Z$, its inv. Fourier transform $H(\omega)$ can be written as a cosine transform

   $$\gamma(k) = \int_0^{\pi} H(\omega)\cos(\omega k)\, d\omega.$$

   Yet we know that the FT of $\gamma(k)$ is $f(\omega)$, so

   $$\gamma(k) = \int_0^{\pi} f(\omega)\cos(\omega k)\, d\omega.$$

2. Suppose a stochastic process $X(t)$ has acvf. $\gamma(k)$. The spectral density function for $\gamma(k)$ is

   $$f(\omega) = \frac{1 + \cos(\omega)}{\pi}.$$

   Write down an expression for $\gamma(k)$, leaving your answer in terms of integrals.
   We can find $\gamma(k)$, since

   $$\begin{aligned}\gamma(k) &= \int_0^{\pi}\cos(\omega k) f(\omega)\, d\omega \\ &= \frac{1}{\pi}\int_0^{\pi}\cos(\omega k)(1 + \cos(\omega))\, d\omega \\ &= \frac{1}{\pi}\int_0^{\pi}\cos(\omega k)\, d\omega + \frac{1}{\pi}\int_0^{\pi}\cos(\omega k)\cos(\omega)\, d\omega.\end{aligned}$$

3. *Clickers question:* What is Var($X(t)$)?
   Since Var($X(t)$) $= \gamma(0)$, we have

$$
\begin{aligned}
\gamma(0) &= \frac{1}{\pi}\int_0^{\pi} 1 d\omega + \frac{1}{\pi}\int_0^{\pi} \cos(\omega)\, d\omega \\
&= 1.
\end{aligned}
$$

4. *Clickers question:* What is $\gamma(1)$?
   Setting $k = 1$,

$$
\begin{aligned}
\gamma(1) &= \frac{1}{\pi}\int_0^{\pi} \cos(\omega)\, d\omega + \frac{1}{\pi}\int_0^{\pi} \cos^2(\omega)\, d\omega \\
&= \frac{1}{2}
\end{aligned}
$$

   since

$$\int_0^{\pi} \cos^2(\omega)\, d\omega = \frac{1}{2}\int_0^{\pi} (1 + \cos(2\omega))\, d\omega = \frac{1}{2}\pi.$$

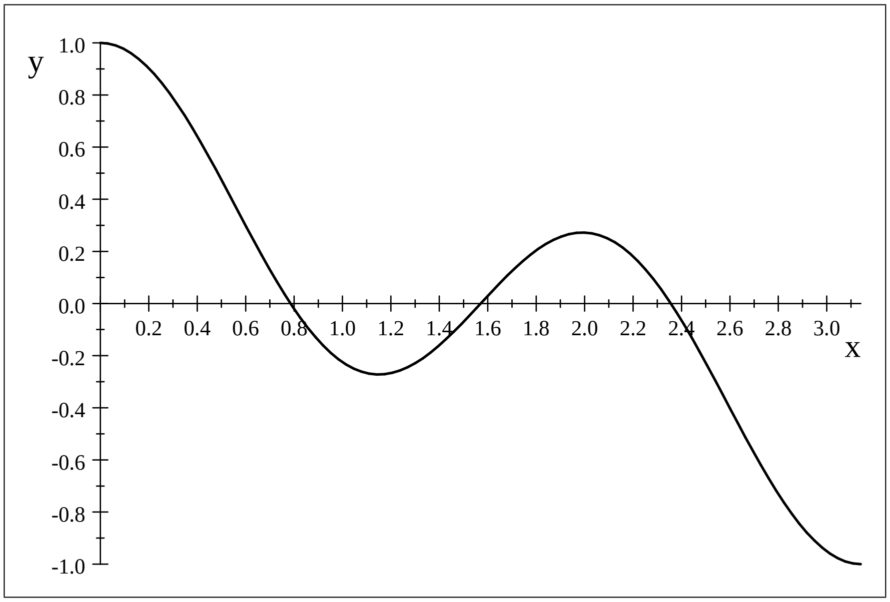
   Similarly

$$\gamma(-1) = \frac{1}{2}.$$

5. What is $\gamma(k)$ for $k \neq 0, \pm 1$?
   The acvf is zero when $k \neq 0, \pm 1$. For instance, when $k = 2$,

$$
\begin{aligned}
\gamma(2) &= \frac{1}{\pi}\int_0^{\pi} \cos(2\omega)\, d\omega + \frac{1}{\pi}\int_0^{\pi} \cos(2\omega)\cos(\omega)\, d\omega \\
&= 0
\end{aligned}
$$

To see the second integral above is zero, below is a plot of $y = \cos(2x)\cos(x)$:



When $k = 3$,

$$\gamma(3) = \frac{1}{\pi}\int_0^\pi \cos(3\omega)\, d\omega + \frac{1}{\pi}\int_0^\pi \cos(3\omega)\cos(\omega)\, d\omega.$$

The first integral above is clearly zero; the second is messier, but application of double–angle formulae can show that

$$\cos(3\omega)\cos(\omega) = 4\cos^4(\omega) - 3\cos^2(\omega),$$

and via calculus techniques

$$\int_0^\pi \cos^4(\omega)\, d\omega = \frac{3\pi}{8}.$$

Hence

$$\gamma(3) = \frac{1}{\pi}\left(4 \times \frac{3\pi}{8} - 3 \times \frac{\pi}{2}\right) = 0.$$

6. *Clickers question:* For which type of ARMA process could $\gamma(k)$ be the acvf?
As the acvf cuts off at lag 1, $X(t)$ *could be an* $MA(1)$.

7. Re-cap what you have done during this activity. Why do you think you were asked to do this activity? What have you learned?
The activity has introduced the notion of the spectral density function of a stochastic process, which is the FT of the acvf of the process. Since the acvf is an even function of integer domain, its FT is a cosine transform. A special case was explored where the spectral density function of a process was given from which acvf could be determined. It was seen that the process could be an $MA(1)$ *.*
Learning outcomes encountered in this activity include:

   (a) *Recall the properties of the Fourier transform of functions that (i) are defined only on the integers, (ii) are even and (iii) functions that satisfy both (i) and (ii).*

   (b) *Define and interpret the spectral density and spectral distribution functions for a time series.*

   (c) *Where mathematically tractable, derive the acvf of a time series model from the spectral density function.*