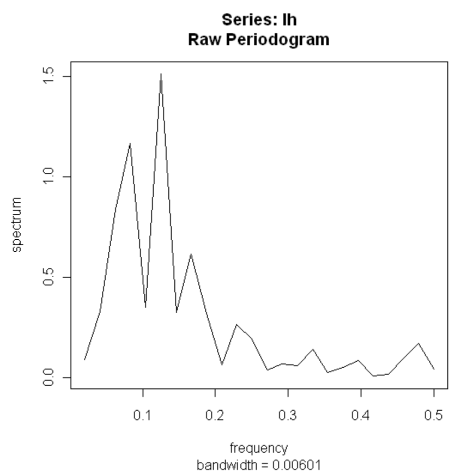
# Activity Solution: Modifying the Periodogram 2

As we have seen, the raw periodogram $I(\omega)$ is not a good estimator of the underlying spectrum of a series as it is inconsistent. Here we explore another method for modifying the periodogram to give a consistent estimator of $f(\omega)$.

We explore the method as applied to the series `lh` met earlier in the course. Recall this series contains 48 observations at equal time intervals, recording levels of a lutenizing hormone. The raw periodogram from R (with `log='no'`), $I_R(\omega)$, is below:




Some values of the periodogram $I(\omega_p) = I_R(\omega_p)/\pi$ are tabulated below, to 2 significant figures only:

| $p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $I(\omega_p)$ | 0.029 | 0.10 | 0.27 | 0.37 | 0.11 | 0.48 | 0.10 | 0.20 | 0.10 | 0.021 | ... |

*Smoothing:* Let

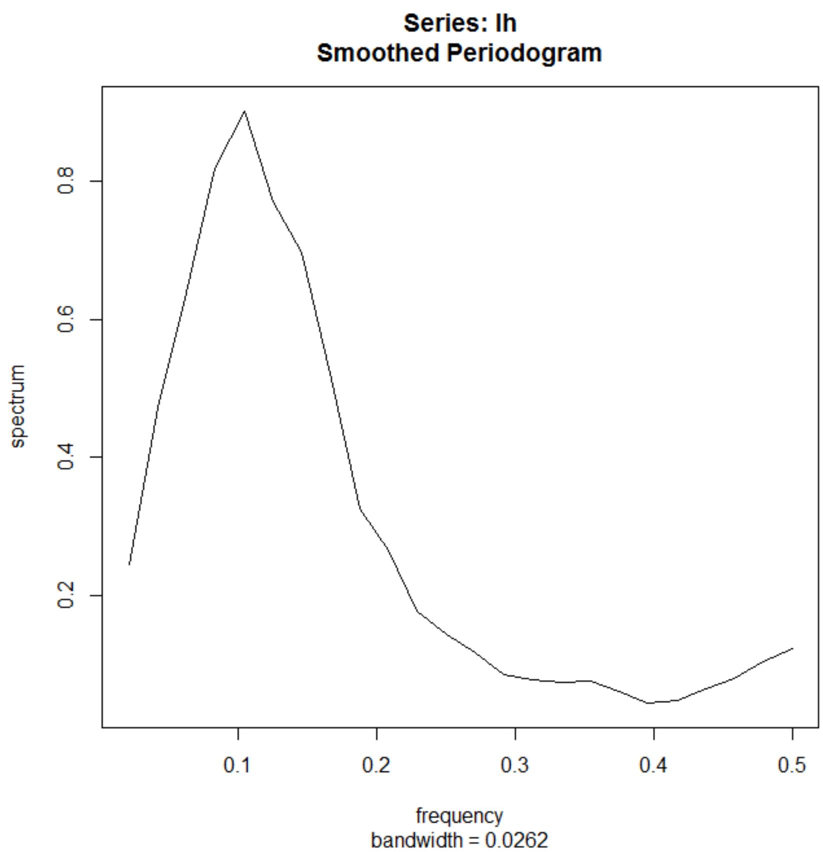
$$\hat{f}_S(\omega) = \frac{1}{m}\sum_j I(\omega_j)$$

where $\omega_j = 2\pi j/N$ and $j$ ranges over $m$ consecutive integers so that $\{\omega_j\}$ are symmetric about $\omega$. A common choice is $m = \left[2\sqrt{N}\right]$, which is 14 in this example. The formula for $\hat{f}_S$ needs to be adjusted near the end–points, though this has little impact if $N$ is large.

1. *Clickers question:* Taking $m = 5$, find $\hat{f}_S(\omega)$ at frequency $\omega = \pi/4$ for the lh data.
   *We find $\pi/4 = 12\pi/48$ is the 6th Fourier frequency, so we have*

$$\begin{aligned}\hat{f}_S\left(\frac{\pi}{4}\right) &= \frac{1}{5}\sum_{j=-2}^{2} I\left(\omega_6 + \frac{2\pi j}{48}\right) \\ &= \frac{1}{5}(0.37 + 0.11 + 0.48 + 0.10 + 0.20) \\ &= 0.252.\end{aligned}$$

   *Below is the smoothed periodogram from $R$ :*




2. *Clickers question:* Assuming the asymptotic distribution results for the

raw periodogram, what is the distribution of

$$\frac{2m\hat{f}_S(\omega_p)}{f(\omega_p)}?$$

*Recall that approximately*

$$\frac{2I(\omega_p)}{f(\omega_p)} \sim \chi_2^2,$$

*and so*

$$\frac{2m\hat{f}_S(\omega_p)}{f(\omega_p)} = \frac{2\sum_j I(\omega_j)}{f(\omega_p)} \sim \chi_{2m}^2,$$

*since the summation* $\sum_j I(\omega_j)$ *contains* $m$ *(approximately) independent terms.*

3. *Clickers question:* Describe for our example how to use $\hat{f}_S(\omega_p)$ to produce 95% confidence intervals for $f(\omega_p)$.
   *Similar as for* $\hat{f}_T(\omega_p)$, *we have 95% confidence intervals given by the form*

$$\left(\frac{10\hat{f}_S(\omega_p)}{\chi_{10}^2(0.975)}, \frac{10\hat{f}_S(\omega_p)}{\chi_{10}^2(0.025)}\right).$$

4. *Clickers question:* Let

$$I(\omega) = \frac{1}{\pi}\left(c_0 + 2\sum_{k=1}^{47} c_k \cos(\omega k)\right)$$

   define the periodogram of the lh series. We have smoothed this at the (Fourier) frequency $\omega$. Writing the frequencies over which the summation above is taken as

$$\left(\omega \pm \frac{2\pi j}{N}\right)k$$

   for $j \in \{-2, -1, 0, 1, 2\}$ and recalling the form of $\cos(A \pm B)$, show that the smoothing gives

$$\hat{f}_S(\omega) = \frac{1}{\pi}\left(c_0 + 2\sum_{k=1}^{47} \lambda_k c_k \cos(\omega k)\right)$$

for some $\{\lambda_k : k = 1, \ldots, 47\}$. Find the values of $\lambda_k$ in terms of cosines of frequencies that are proportional to $k$.

*As directed, let*

$$\begin{aligned}
\hat{f}_S(\omega) &= \frac{1}{\pi} \sum_{j=-2}^{2} \frac{1}{5} \left( c_0 + 2 \sum_{k=1}^{47} c_k \cos\left( \left( \omega + \frac{2\pi j}{48} \right) k \right) \right) \\
&= \frac{1}{\pi} \left( c_0 + 2 \sum_{k=1}^{47} c_k \left[ \frac{1}{5} \cos(\omega k) \right. \right. \\
&\qquad \left. \left. + \sum_{j=1}^{2} \frac{1}{5} \left( \cos\left( \omega k + \frac{2\pi j k}{48} \right) + \cos\left( \omega k - \frac{2\pi j k}{48} \right) \right) \right] \right).
\end{aligned}$$

*Now using*

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

*here we see*

$$\begin{aligned}
\cos\left( \omega k + \frac{2\pi j k}{48} \right) &= \cos(\omega k)\cos\left( \frac{2\pi j k}{48} \right) - \sin(\omega k)\sin\left( \frac{2\pi j k}{48} \right), \\
\cos\left( \omega k - \frac{2\pi j k}{48} \right) &= \cos(\omega k)\cos\left( \frac{2\pi j k}{48} \right) + \sin(\omega k)\sin\left( \frac{2\pi j k}{48} \right)
\end{aligned}$$

*and so we find*

$$\hat{f}_S(\omega) = \frac{1}{\pi} \left( c_0 + 2 \sum_{k=1}^{47} \lambda_k c_k \cos(\omega k) \right)$$

*where*

$$\lambda_k = \frac{1}{5} \left( 1 + 2\cos\left( \frac{2\pi k}{48} \right) + 2\cos\left( \frac{4\pi k}{48} \right) \right).$$

5. What does the above tell us about the smoothing of a periodogram?
*This shows that the two methods of modifying the periodogram are equivalent in the sense that smoothing is a special case of truncating and transforming.*

6. Re-cap what you have done during this activity. Why do you think you were asked to do this activity? What have you learned?

*We have explored how to modify the (raw) periodogram by smoothing, i.e., by taking a weighted average of consecutive terms. In a special case we saw that the smoothing was equivalent to an apparently alternative method for modifying the periodogram, namely truncating and transforming. It was far from apparent previously that the two approaches can be equivalent.*
*Learning outcomes encountered in this activity are:*

(a) *Describe and explain methods for modifying the periodogram, in particular approaches that (i) transform and truncate and (ii) smooth the periodogram.*

(b) *Construct confidence intervals for a spectrum based on a consistent estimator following a modification of the periodogram.*