Activity Solution: AR(1) Model Inference

Recall from the previous activity that

$$\mathrm{Var}\left(\bar{x}\right) = \frac{\sigma_X^2}{N}\left(1 + 2\sum_{k=1}^{N-1}\left(1 - \frac{k}{N}\right)\rho\left(k\right)\right)$$

where the data are N observations from a stationary process with variance σ_X^2 and acf $\rho\left(\cdot\right)$.

1. Consider the stochastic process

$$X\left(t\right) = \frac{1}{2}X\left(t-1\right) + Z\left(t\right)$$

where $Z\left(t\right)$ is white noise as usual. Using the result from above, show that for this process the variance of the mean of a sample $x\left(1\right), \ldots, x\left(N\right)$ from $X\left(t\right)$ would be

$$\mathrm{Var}\left(\bar{x}\right) = \frac{\sigma_X^2}{N}\left(1 + 2\left(\sum_{k=1}^{N-1}\left(\frac{1}{2}\right)^k - \frac{1}{N}\sum_{k=1}^{N-1}k\left(\frac{1}{2}\right)^k\right)\right).$$

Recall that for an AR(1), provided $|\alpha| < 1$ we have $\rho\left(k\right) = \alpha^{|k|}$. Put $\rho(k) = \left(\frac{1}{2}\right)^k$ into the equation for Var$(\bar{x})$ from above.

2. *Clickers question:* It can be shown that

$$\frac{1}{N}\sum_{k=1}^{N-1}k\left(\frac{1}{2}\right)^k$$

tends to zero as N grows large. What about $\sum_{k=1}^{N-1}\left(\frac{1}{2}\right)^k$ for large N?
It can be seen that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N-1}k\left(\frac{1}{2}\right)^k = 0,$$

and by summing the G.P. (remember

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

if $|r| < 1$*) we have*

$$\lim_{N\to\infty} \sum_{k=1}^{N-1} \left(\frac{1}{2}\right)^k = 1.$$

3. *Clickers question:* For large N approximate the variance of the mean of a sample from $X(t)$. Comment on the result.
   Use the results from above to simplify the expression for $\mathrm{Var}(\bar{x})$. *Obviously the variance of the sample mean here is asymptotically three times larger than for the variance of the sample mean of i.i.d. data.*

4. We have seen that for $X(t)$ above the autocorrelation in the process makes the sample mean less accurate as an estimator of the mean μ. This is not true for all values of α. For

   $$X(t) = \alpha X(t-1) + Z(t)$$

   with parameter $|\alpha| < 1$, derive an approximate formula for $\mathrm{Var}(\bar{x})$ when the sample size N is large. What does this tell us when α is positive? What about when α is negative? Compare the above with the result for $\mathrm{Var}(\bar{x})$ when the data are i.i.d., and try to provide some intuition to the results.
   It can be seen that for large N

   $$\begin{aligned} \mathrm{Var}(\bar{x}) &\approx \frac{\sigma_X^2}{N}\left(1 + \frac{2\alpha}{1-\alpha}\right) \\ &= \frac{\sigma_X^2}{N}\left(\frac{1+\alpha}{1-\alpha}\right). \end{aligned}$$

   So the "scaling factor" on the variance (from the i.i.d. case) is

   $$\frac{1+\alpha}{1-\alpha}.$$

   Positive α *leads to increased variance, negative* α *to reduced variance. In other words, negatively correlated series provide MORE information about the mean* μ *than i.i.d. data. This is intuitive – negatively correlated series will tend to avoid sequences of observations on the same side of* μ.

5. *Clickers question:* The first 200 terms of a time series gave the following results:

k	1	2	3	4	5
acf r_k	–0.80	0.67	–0.52	0.39	–0.31
pacf $\hat{\alpha}_{kk}$	–0.80	0.085	0.112	–0.046	–0.061

The mean of the observed series was $\bar{x} = 0.03$, and $c_0 = 3.34$. What type of model would you suggest here? Explain your answer.
Note that the sample acf r_k appears to decay here, whereas the pacf cuts–off after the first lag. This strongly suggests an $AR(1)$ model.

6. *Clickers question:* Suggest estimates of the parameter(s) in the model you have proposed.
We would estimate the parameter α_1 by $r_1 = -0.8$, and using a method of moments estimator of σ^2 we would take

$$\begin{aligned} \hat{\sigma}^2 &= c_0 \left(1 - \hat{\alpha}_1^2\right) \\ &= 3.34 \left(1 - 0.64\right) \\ &= 1.2024. \end{aligned}$$

This ad hoc estimate is unlikely to have good properties, but is available to us with the information we have.

7. *Clickers question:* In your proposed model, you probably chose $\mu = 0$. Can you suggest a *test* for whether the mean is zero here? You can assume that the Central Limit Theorem holds, and then perform a test at the 5% significance level.
Well $\bar{x} = 0.03$, and in recalling the previous activity, the estimate of the standard deviation (i.e., the standard error) for this is

$$\sqrt{\frac{c_0 \left(1 + r_1\right)}{N \left(1 - r_1\right)}} = 0.04.$$

Now since $|\bar{x}/0.04| < 1.96$, we would accept that μ is zero at the 5% significance level.

8. Finally, write down your suggested model here.
So we would try as an initial model the $AR(1)$

$$X\left(t\right) = -0.8X\left(t - 1\right) + Z\left(t\right)$$

where $Z\left(t\right) \sim N\left(0, 1.2024\right)$.

9. Re-cap what you have done during this activity. What were you asked to do? Why were you asked to do this? What did you learn?

   We explored how the sample mean from an autoregressive model or order 1 behaves. In particular, we derived how the variance of the sample mean of data from an AR(1) process behaves and saw that its value is a constant multiple of the variance of a sample mean from an independent sample. The constant multiplier depends on the autoregressive parameter, and may be greater than or less than unity.
   We went on to consider model fitting and estimation from summary statistics taken, in fact, from simulated $AR(1)$ data. We deduced how to estimate the parameters in the model and derived a test for whether the mean of the process is zero.
   Learning outcomes encountered in this activity include:

   (a) *Recall the main properties of the sample acvf and acf for a time series.*

   (b) *Describe the issues related to fitting AR models, in particular the choice of the order and approaches to parameter estimation.*