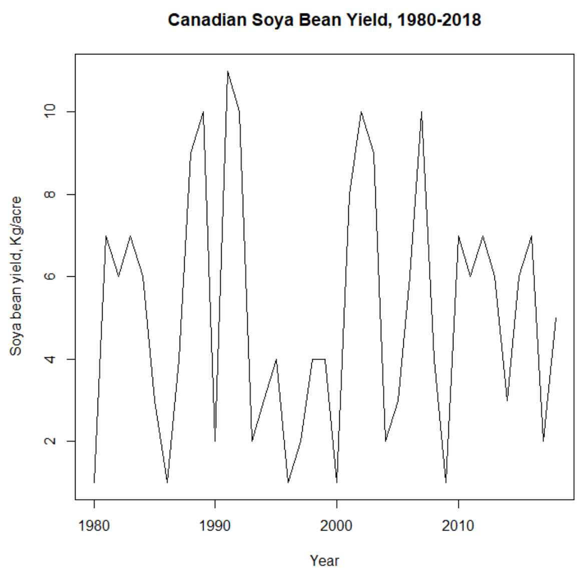
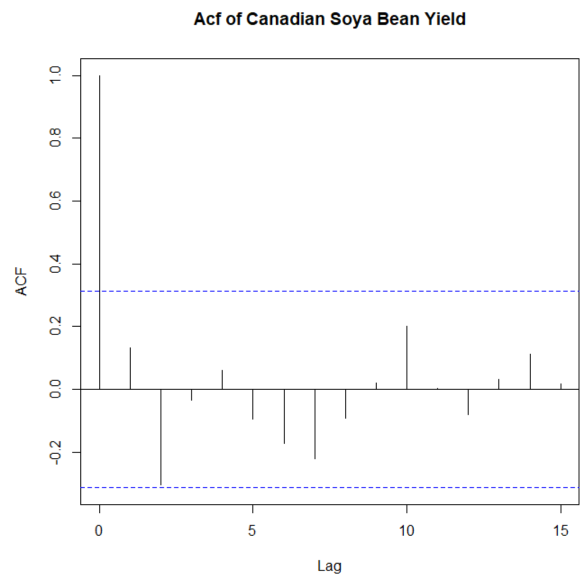
In the following, $Z(t)$ denotes a white noise process with mean zero and variance σ^2 unless otherwise stated. For questions 1, 2, 3, and 5, circle your answers clearly. If you make a mistake, indicate your answers clearly in the margin. Ambiguous responses will be considered incorrect.

1. Below are the time series plot and the correlogram of the annual Canadian soya bean yields (in kg per acre) for years 1980 to 2018 inclusive.





Based on the above information, what would you conclude about the time series? (Circle all that apply.)

(a) **The series is stationary.**

(b) The series is non-stationary, as it has a clear trend.

(c) Applying ∇ to the series would probably remove non-stationary effects.

(d) The series has a clear seasonal effect.

(e) **The series may be white noise.**

2. Suppose we define the stochastic process $X(t)$ by

$$X(t) = Z(t) + 0.4Z(t-1) + 0.2Z(t-2) + 0.1Z(t-3).$$

Which of the following are true for $X(t)$? (Circle all that apply.)

(a) $*E(X(t)) = 0.*$

(b) $\mathrm{Var}(X(t)) = 0$.

(c) $*X(t)$ is stationary.*

(d) The acf for $X(t)$ would "cut–off" at lag 2.

(e) The process would be a suitable model for a time series with a seasonal effect with period three.

3. Consider the SARIMA model

$$\left(1 - \alpha_1 B - \alpha_2 B^2\right) W(t) = \left(1 + \beta B^{12}\right) Z(t)$$

for some constants α_1, α_2, and β, where

$$W(t) = \nabla \nabla_{12} X(t).$$

This is a SARIMA model of order

(a) $(0,1,0) \times (0,1,2)$

(b) $(0,0,2) \times (0,1,1)_{12}$

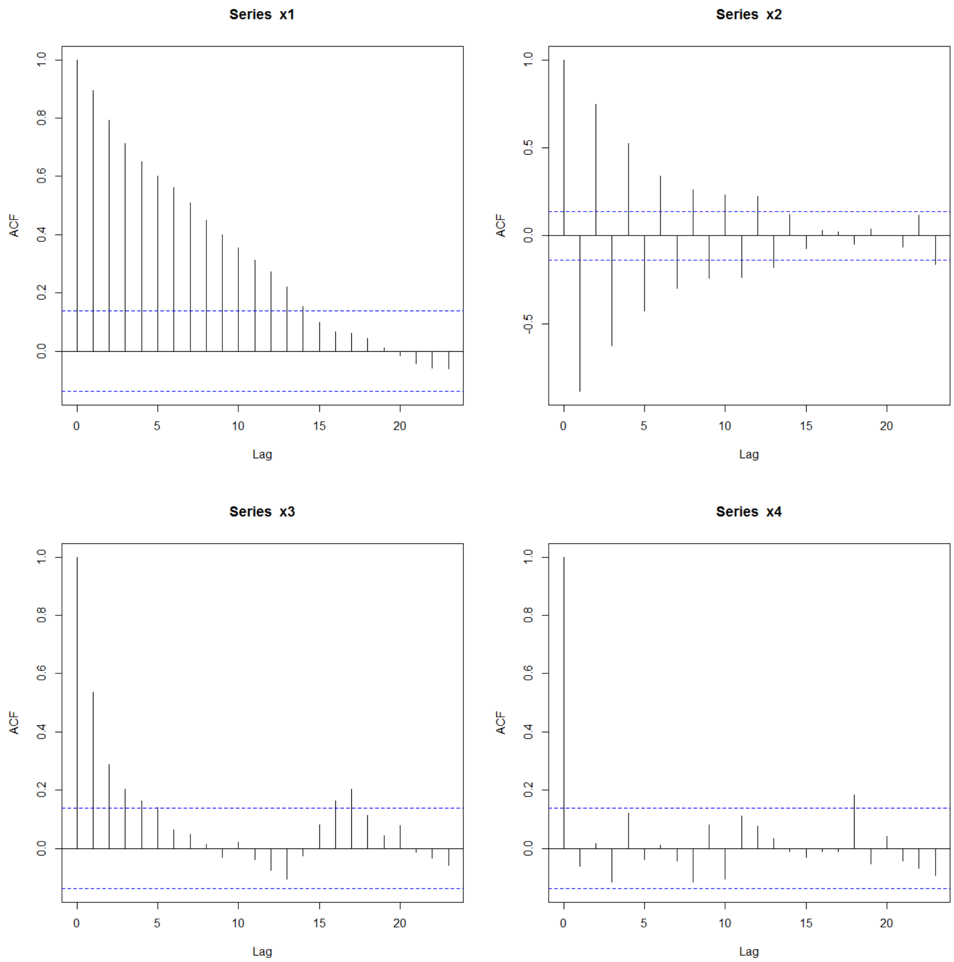
(c) $(1,1,0) \times (0,0,2)_{12}$

(d) $*(2,1,0) \times (0,1,1)_{12}*$

(e) $(0,1,1) \times (0,1,2)_{12}$

4. Four AR(1) processes were simulated with $\alpha = -0.9, 0.5, -0.1$ and 0.9. In each case, 200 observations were generated. The autocorrelation plots are given below. Decide which series corresponds to each value of

α.

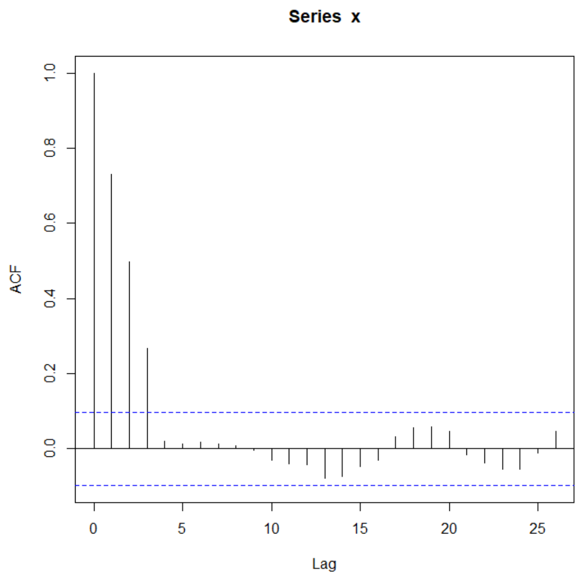


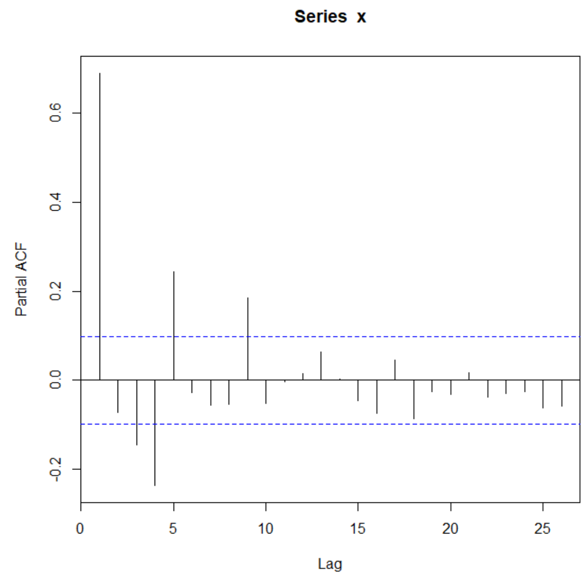
(a) Top left, series x1, $\alpha =$

(b) Top right, series x2, $\alpha =$

(c) Bottom right, series x4, $\alpha =$
(Group version: Bottom **left**, **series x3,** $\alpha =$)
x1 has $\alpha = 0.9$, *x2 has* $\alpha = -0.9$, *x3 has* $\alpha = 0.5$, *and x4 took* $\alpha = -0.1$. *(3 marks)*

5. The following are the sample acf and pacf from a stationary time series labelled x that contains 400 values.





Based on the above information, which of the following models would you consider the most appropriate for the data?

(a) AR(2)

(b) AR(3)

(c) MA(1)

(d) *MA(3) $*$

(e) ARMA$(3,3)$

6. The time series x1 from Q4 has sample variance 2.78. Using the model you suggested in Q4, find the method of moments estimate of σ^2, the variance of the white noise process in the model. Show your working clearly.
For an AR(1), $X(t) = \alpha X(t-1) + Z(t)$, *and since the process is stationary we know*

$$\text{Var}(X(t)) = \frac{\sigma^2}{1-\alpha^2}.$$

Equating the variance on the left to 2.78 suggests the estimate of σ^2 *to be*

$$2.78\left(1-0.9^2\right)=0.528.$$

(2 marks)

Group version: The time series **x3** from Q4 has sample variance 2.78. Using the model you suggested in Q4, find the method of moments estimate of σ^2, the variance of the white noise process in the model. Show your working clearly.

For an $AR(1)$, $X(t)=\alpha X(t-1)+Z(t)$, *and since the process is stationary we know*

$$\operatorname{Var}(X(t))=\frac{\sigma^2}{1-\alpha^2}.$$

Equating the variance on the left to 2.78 suggests the estimate of σ^2 *to be*

$$2.78\left(1-0.5^2\right)=2.085.$$

7. The quarterly sales (in 10^6 dollars, $x(t)$) of a company over four years are given below, along with the centred moving average series Cm(t) and the corresponding differences:

Year	Quarter (t)	$x(t)$	Cm(t)	$x(t)-$Cm(t)
1	1	4.8		
1	2	4.1		
1	3	6.0	5.4750	0.5250
1	4	6.5	5.7375	0.7625
2	5	5.8	5.9750	–0.1750
2	6	5.2	6.1875	–0.9875
2	7	6.8	6.3250	0.4750
2	8	7.4	6.4000	1.0000
3	9	6.0	6.5375	–0.5375
3	10	5.6	6.6750	–1.0750
3	11	7.5	6.7625	0.7375
3	12	7.8	6.8375	0.9625
4	13	6.3	6.9375	–0.6375
4	14	5.9	7.0750	–1.1750
4	15	8.0		
4	16	8.4		

(a) Assuming an additive seasonal model, estimate the unadjusted seasonal indices for the first quarter.

The following table illustrates the calculation:

	Q1	Q2	Q3	Q4
	*	*	0.525	$\cdots$
	–0.175	–0.987		
	–0.537	–1.075		
	–0.638	–1.175	*	*
Mean:	–0.450	–1.079		

(2 marks)

(b) Given that the unadjusted seasonal indices for quarters 2, 3, and 4 are –1.079, 0.579, and 0.908 respectively, find the four adjusted seasonal indices.

Now since $-0.450-1.079+0.579+0.908 = -0.042$, *adding 0.0105 to each term we find the adjusted seasonal indices as*

$$
\begin{aligned}
S(1) &= -0.439 \\
S(2) &= -1.068 \\
S(3) &= 0.590 \\
S(4) &= 0.919.
\end{aligned}
$$

(2 marks)

Group version: (a) Assuming an additive seasonal model, given that the unadjusted estimates of the four seasonal indices are respectively -0.450, –1.079, 0.579, and 0.908, find the adjusted seasonal indices.

Now since $-0.450-1.079+0.579+0.908 = -0.042$, *adding 0.0105 to each term we find the adjusted seasonal indices as*

$$
\begin{aligned}
S(1) &= -0.439 \\
S(2) &= -1.068 \\
S(3) &= 0.590 \\
S(4) &= 0.919.
\end{aligned}
$$

(2 marks)

(c) Regressing the de-seasonalised data on time, the trend is

$$T(t) = 5.150 + 0.146t.$$

Forecast the revenue figures for the last quarter of year 5.

The forecast for year 5 quarter 4 is (in \$$10^6$'s):

$$\hat{x}(16,4) = 5.150 + 0.146 \times 20 + 0.919 = 8.989$$

(2 marks)

Group version: (b) Regressing the de-seasonalised data on time, the trend is

$$T(t) = 5.150 + 0.146t.$$

Forecast the revenue figures for the **first two** quarters of year 5.
The forecasts for year 5 quarters 1 and 2 are (in \$$10^6$'s):

$$\begin{aligned} \hat{x}(16,1) &= 5.150 + 0.146 \times 17 - 0.439 = 7.193 \\ \hat{x}(16,2) &= 5.150 + 0.146 \times 18 - 1.068 = 6.710 \end{aligned}$$

(2 marks)

8. Let

$$X(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + Z(t)$$

for some constants β_0, β_1, and β_2.

(a) Is $X(t)$ stationary? Explain your answer clearly.

No, since

$$E(X(t)) = \beta_0 + \beta_1 t + \beta_2 t^2,$$

which depends on t. (2 marks)

(b) Is $\nabla X(t)$ stationary? Explain your answer clearly.

No, since

$$\begin{aligned} \nabla X(t) &= X(t) - X(t-1) \\ &= \beta_1 - \beta_2 + 2\beta_2 t + Z(t) - Z(t-1) \end{aligned}$$

has expectation

$$E\left(\nabla X\left(t\right)\right)=\beta_1-\beta_2+2\beta_2 t,$$

which depends on t. (2 marks)

8. Group version: Let

$$X\left(t\right)=\beta_0+\beta_1 t+\beta_2 t^2+Z\left(t\right)$$

for some constants β_0, β_1, and β_2. Let $Y\left(t\right)=\nabla^2 X\left(t\right).$

(a) Find the expectation of $Y\left(t\right).$
Now

$$\nabla^2 X\left(t\right)=2\beta_2+Z\left(t\right)-2Z\left(t-1\right)+Z\left(t-2\right),$$

which has expectation $2\beta_2$. (2 marks)

(b) Find the autocovariance function of $Y\left(t\right).$
By noting that $\nabla^2 X\left(t\right)$ is an MA(3) or otherwise, we find

$$\begin{aligned}
\gamma\left(0\right) &= 6\sigma^2,\\
\gamma\left(1\right) &= \mathrm{Cov}\left(Z\left(t\right)-2Z\left(t-1\right)+Z\left(t-2\right),Z\left(t-1\right)-2Z\left(t-2\right)+Z\left(t-3\right)\right)\\
&= -4\sigma^2,\\
\gamma\left(2\right) &= \mathrm{Cov}\left(Z\left(t\right)-2Z\left(t-1\right)+Z\left(t-2\right),Z\left(t-2\right)-2Z\left(t-3\right)+Z\left(t-4\right)\right)\\
&= \sigma^2
\end{aligned}$$

and $\gamma\left(k\right)=0$ for $|k|\geq 3$. (3 marks)

(c) Is $Y\left(t\right)$ a stationary process?
Yes. (1 mark)
The original model is nonstationary, but taking differences twice leaves a process that can be modelled by a stationary ARMA (in particular, an MA(2)).