Exercises 5: Solutions

1. Here

$$\begin{aligned}\gamma_{XY}(k) &= \operatorname{Cov}(X(t), Y(t+k)) \\ &= \operatorname{Cov}(Z(t)+\beta_1 Z(t-1), Z(t+k)+\beta_2 Z(t+k-1)) \\ &= \begin{cases} \sigma^2\beta_1 & k=-1 \\ \sigma^2(1+\beta_1\beta_2) & k=0 \\ \sigma^2\beta_2 & k=+1 \\ 0 & \text{else.}\end{cases}\end{aligned}$$

Recalling for an MA(1) process $X(t) = Z(t) + \beta Z(t-1)$ we have

$$\operatorname{Var}(X(t)) = \sigma^2\left(1+\beta^2\right),$$

the cross–correlation function here is

$$\rho_{XY}(k) = \begin{cases} \frac{\beta_1}{\sqrt{\left(1+\beta_1^2\right)\left(1+\beta_2^2\right)}} & k=-1 \\ \frac{(1+\beta_1\beta_2)}{\sqrt{\left(1+\beta_1^2\right)\left(1+\beta_2^2\right)}} & k=0 \\ \frac{\beta_2}{\sqrt{\left(1+\beta_1^2\right)\left(1+\beta_2^2\right)}} & k=+1 \\ 0 & \text{else.}\end{cases}$$

Note that $\rho_{XY}(k) = \rho_{YX}(-k)$. When $\beta_1 = 0.6$ and $\beta_2 = -0.6$ we find

$$\rho_{XY}(k) = \begin{cases} 0.4412 & k=-1 \\ 0.4706 & k=0 \\ -0.4412 & k=+1 \\ 0 & \text{else.}\end{cases}$$

2. Since

$$\begin{aligned}X(t+k) &= (1-\alpha B)^{-1} Z(t+k) \\ &= Z(t+k) + \alpha Z(t+k-1) + \alpha^2 Z(t+k-2) + \cdots\end{aligned}$$

the cross–covariance function γ_{ZX} is

$$\begin{aligned}\gamma_{ZX}(k) &= E(Z(t)X(t+k)) \\ &= \begin{cases} \alpha^k\sigma^2 & k\geq 0 \\ 0 & k<0.\end{cases}\end{aligned}$$

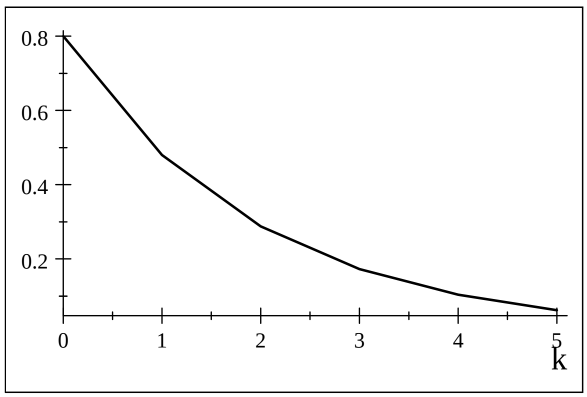
Recalling that

$$\text{Var}\left(X\left(t\right)\right) = \frac{\sigma^2}{\left(1-\alpha^2\right)}$$

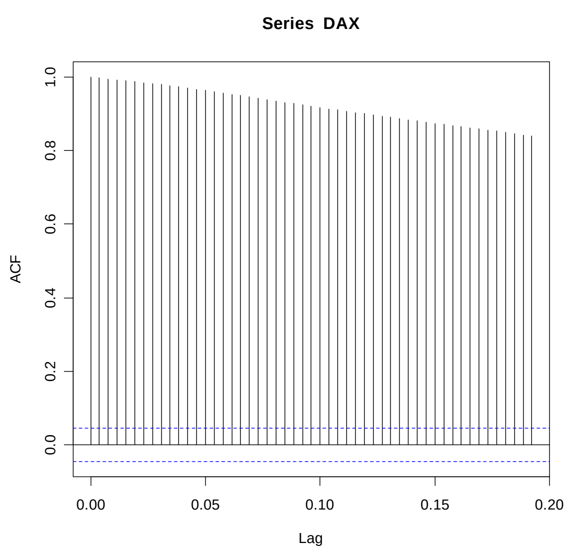
we have

$$\rho_{ZX}\left(k\right) = \begin{cases} \alpha^k\sqrt{1-\alpha^2} & k \geq 0 \\ 0 & k < 0. \end{cases}$$

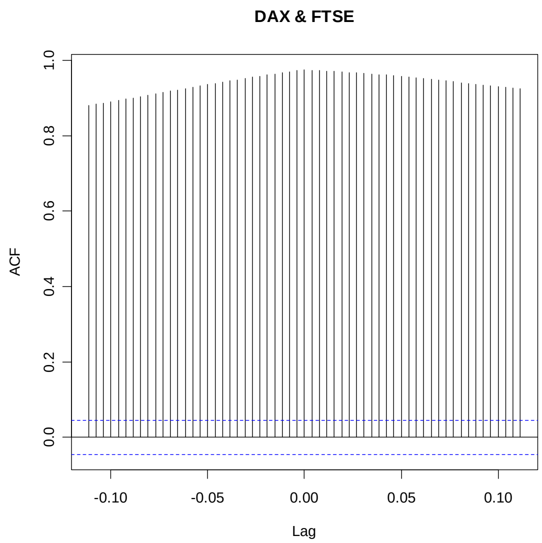
When $\alpha = 0.6$, this decays as shown below:



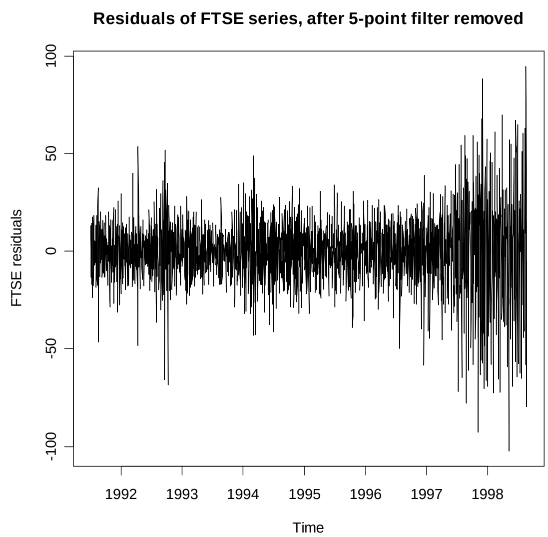
3. A plot of each series individually indicates both series have risen sharply over the time period in question, the trend in each case being at least linear, if not of a higher rate of increase. Not surprisingly, the acf in each case is relatively uninformative, that for the DAX being shown below:



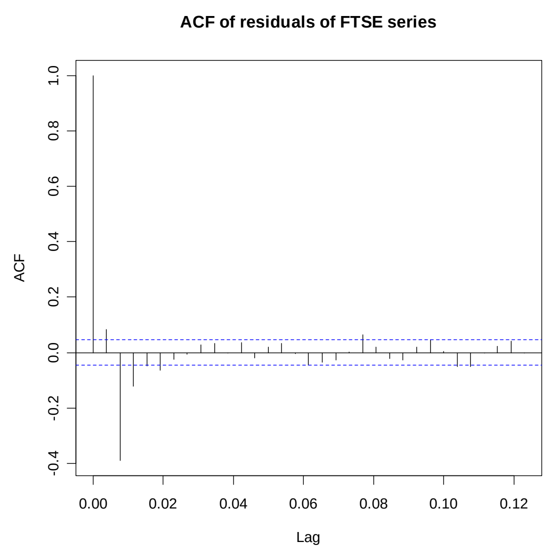
Note here, as usual, R takes its lag values in terms of a standard time unit, in this case seemingly "working days per year", of which there are about 252. The cross–correlation is equally unenlightening:



Obviously at high lags significant negative correlations would be seen, though this sheds no light other than what is given by the respective time plots. Applying a five–point moving average filter to each series, then subtracting this to leave the residual process, removes most of the pattern in each data set and is partially successful in pre–whitening. For instance, the residual FTSE series is plotted below:

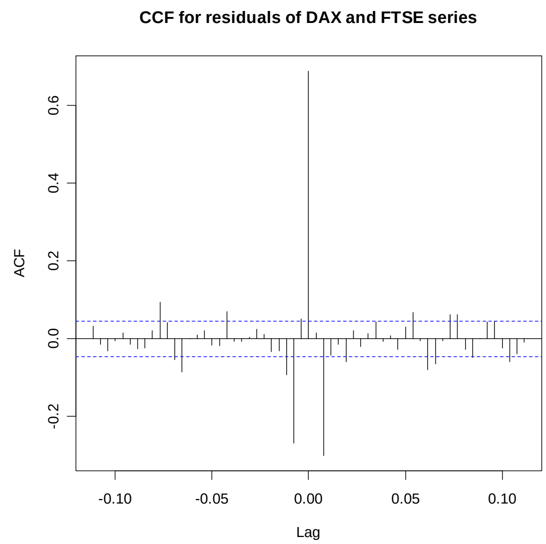


There is apparent increased volatility (also seen in the DAX residuals) towards the end of the series, though no obvious trend remains.

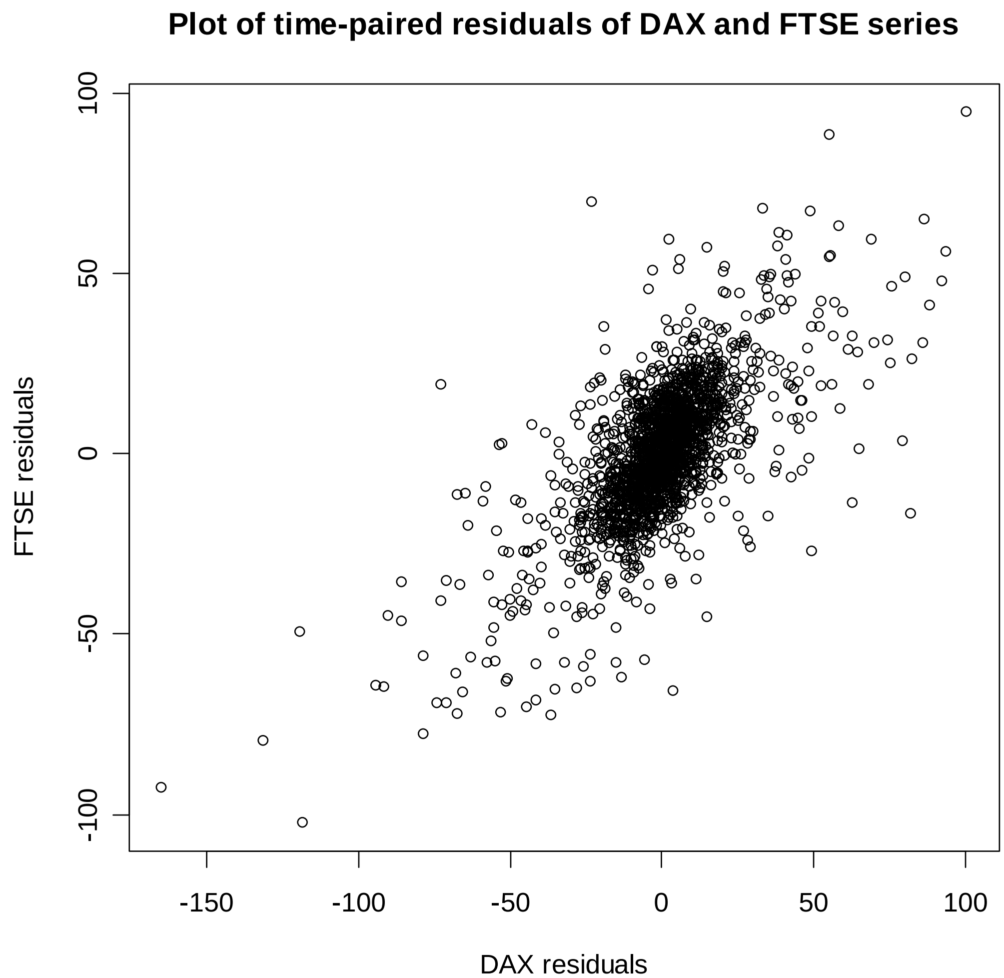


Notable on the acf for both sets of residuals is a significant negative correlation at lag 2. This tentatively indicates some delayed response to local changes in the values, manifested by a shift in the opposite direction. One might postulate that an increase in the value of an index encourages traders to sell their interests in the index the following day, leading to a drop in the value as the market becomes flooded with shares. The effect would be in the opposite direction following a drop in the index. This is, I emphasise, just conjecture, though appears a plausible explanation.

The cross–correlation of residuals is below (note R seemingly takes its definition of $r_{XY}(k)$ to be our definition with k replaced by $-k$) :



There is substantial correlation at lag 0, of 0.6871 in fact, as indicated by the scatter plot of residuals below:



The negative correlations at lags $+2$ and -2 are likely to be artefacts of the individual residuals having negative correlations at lags ± 2, and with the two series being highly correlated the negative correlations are inherited in the cross–correlation. Both series are being directly influenced by similar factors, so the correlation structure observed is not surprising.