Chapter 4: An Introduction to the Frequency Domain

Stat 443: Time Series and Forecasting

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Introduction

So far, our study of time series has concentrated on the *time domain* aspects, such as the acf and time–dependent parametric models. Now we introduce a complementary approach, in the *frequency domain.*

The methods we discuss will be applicable to stationary, real–valued processes only.

A simple model

Consider the model

$$X(t) = R\cos(\omega t + \theta),$$

in which

- t represents time,
- R is the *amplitude,*
- θ is the *phase,*
- ω is the *(angular) frequency.*

Now $\omega t + \theta$ is measured in radians, and ω is the number of radians per unit time.

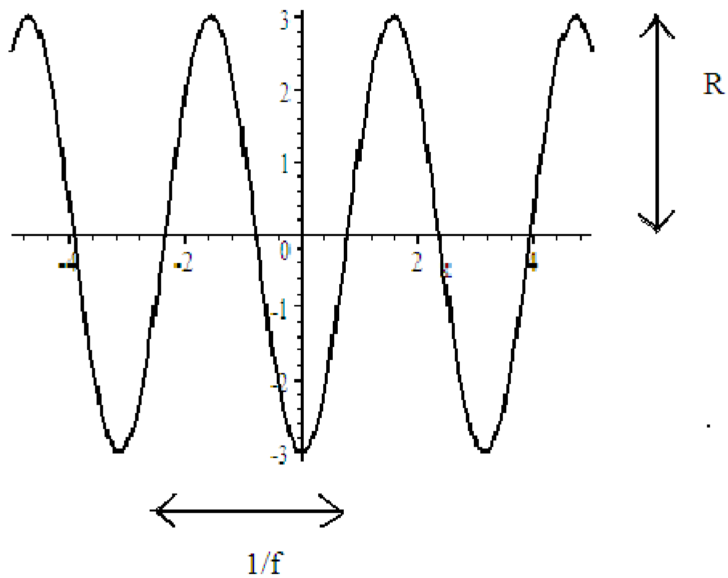
We define the *frequency* of the model to be

$$f := \frac{\omega}{2\pi},$$

being the number of cycles per unit time. This makes

$$\frac{1}{f} = \frac{2\pi}{\omega}$$

the *wavelength* (or *period)* of X, this being the time taken to complete a cycle.



Plot of $3\cos(2t+\pi)$.

Naturally the model for $X(t)$ is purely deterministic (i.e., non-random) if R, ω and θ are constants, so some noise could be added,

$$X(t) = R\cos(\omega t + \theta) + Z(t)$$

where $Z(t)$ is white noise with variance σ^2.

Rather than just a single frequency component, perhaps a process could be a collection at different frequencies:

$$X(t) = Z(t) + \sum_{j=1}^{k} R_j \cos\left(\omega_j t + \theta_j\right), \qquad (1)$$

where R_j is the amplitude at frequency ω_j, θ_j the corresponding phase, for $j = 1, \ldots, k$.

Example 0.1 *The sales figures for a product may have various seasonal components – weekly, monthly and yearly – and perhaps other cyclical variations of less obvious periods.*

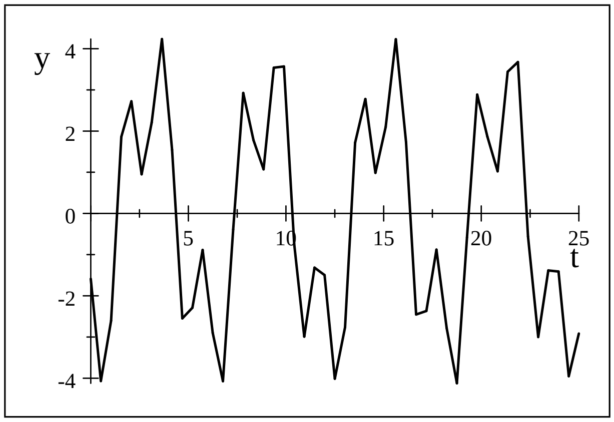
Example 0.2 *Consider the function*

$$y = 2\cos(\pi t + \pi/4) + 3\cos(\pi t/3 + \pi)$$

plotted below. Since $2\cos(\pi t + \pi/4)$ *attains is maximum displacement when*

$$t = \frac{7}{4}, \frac{15}{4}, \frac{23}{4}, \ldots,$$

and $3\cos(\pi t/3+\pi)$ *is 3 when* $t = 3 + 6k,\ k \in \mathbb{Z},\ y$ *never takes the value 5.*



Plot of $2\cos(\pi t + \pi/4) + 3\cos(\pi t/3 + \pi)$

Now $X(t)$ defined by (1) is not stationary, since $E(X(t))$ depends on t. However, if we have at least one of either

1. $\left\{R_j : j = 1, \ldots, k\right\}$ are a set of uncorrelated variables,

2. $\theta_j \sim U(0, 2\pi)$, for all $j = 1, \ldots, k$

but that these variables are then fixed for each realisation, then $X(t)$ is a stationary process.

Recalling the trig identity

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

we see

$$\cos\left(\omega_j t + \theta_j\right) = \cos\left(\omega_j t\right)\cos\left(\theta_j\right) - \sin\left(\omega_j t\right)\sin\left(\theta_j\right)$$

for $j = 1, \ldots, k$.

So

$$X(t) = Z(t) + \sum_{j=1}^{k} \left(a_j \cos\left(\omega_j t\right) + b_j \sin\left(\omega_j t\right)\right)$$

where

$$\begin{aligned} a_j &= R_j \cos\left(\theta_j\right) \\ b_j &= -R_j \sin\left(\theta_j\right). \end{aligned}$$

Spectral representations

In the model (1), why should we stop at a finite number of cyclical components? That is, why limit k to being finite?

Letting $k \rightarrow \infty$, it can be shown that a stationary integer–time process $X(t)$ can be written in the form

$$X(t) = \int_0^\pi \cos(\omega t)\, dU(\omega) + \int_0^\pi \sin(\omega t)\, dV(\omega) \quad (2)$$

where U and V are uncorrelated stochastic processes with orthogonal increments. This is the *spectral representation* of $X(t)$.

Remark 0.1 *The integrals in (2) are so–called* stochastic *integrals, and you may not be familiar with such integration. No problem, as (2) is of no direct use to us here. Interpret the result as indicating that any stationary stochastic process can be expressed approximately as a weighted sum of trigonemetric functions, the weights being random variables.*

If $X(t)$ were a continuous–time process, the upper limits in the integrals in (2) would be $+\infty$. However, if $t \in \{0, 1, 2, \ldots\}$ and $k \in \mathbb{Z}^+$ then

$$\cos((\omega + k\pi)t) = \begin{cases} \cos(\omega t) & k \text{ even} \\ \\ \cos((\pi - \omega)t) & k \text{ odd} \end{cases}$$

since any angle is of the form $\omega + k\pi$ for some $\omega \in (0, \pi)$.

Example 0.3 *If* $t = 1$ *and*

$$\theta = \frac{\pi}{6} + k\pi,$$

then, for example

$$\cos\left(\frac{\pi}{6} + 2\pi\right) = \cos\left(\frac{\pi}{6}\right) = \cos\left(30^\circ\right)$$

and

$$\cos\left(\frac{\pi}{6} + \pi\right) = \cos\left(\pi - \frac{\pi}{6}\right) = \cos(150^\circ).$$

Hence any contributing frequency outside the range $(0, \pi)$ will be *aliased* by a frequency in $(0, \pi)$. So taking $\omega \in (0, \pi)$ in (2) suffices for integer–time processes.

Definition 0.1 *The frequency $\omega = \pi$ is called the* Nyquist frequency.

More generally, when a discrete–time process has time δt between each observation, then the Nyquist frequency is

$$\frac{\pi}{\delta t}.$$

Equation (2) tells us that each frequency in the range $(0, \pi)$ may contribute to the variation of a stationary, discrete–time process.

Fourier transforms

We outline here some essential tools and theory for analysis in the frequency domain. In particular, we review the Fourier transform of a function. This describes the behaviour of a function in the frequency domain.

These ideas may be familiar to you, and they have broad applications, particularly in electrical engineering. The Fourier transform is closely related to the *Laplace transform*, though that will not be required.

Suppose $h(t)$ is function of the real variable t. For completeness, $h(t)$ may be complex–valued. As usual, let $\mathbf{i} = (-1)^{\frac{1}{2}}$.

Definition 0.2 *The* Fourier transform *(FT) of the function $h(t)$ is*

$$H(\omega) := \int_{-\infty}^{\infty} h(t)\, e^{-\mathbf{i}\omega t} dt.$$

Now $H(\omega)$ is finite if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty,$$

an is typically complex–valued.

The *inverse Fourier transform* (inv. FT) is given by

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, e^{\mathbf{i}\omega t} d\omega.$$

$H(\omega)$ and $h(t)$ are referred to as *Fourier transform pair.*

Remark 0.2 *Definitions may differ slightly here, with the multiple* $\frac{1}{2\pi}$ *either being on* $H(\omega)$ *or split into* $(2\pi)^{-\frac{1}{2}}$ *on both* $h(t)$ *and* $H(\omega)$ *being common alternatives.*

Some special cases

Some special cases will be particularly relevant to us.

(a) If $h(t)$ is defined only for integer valued t, then its FT is

$$H(\omega) = \sum_{t=-\infty}^{\infty} h(t)\, e^{-\mathbf{i}\omega t}$$

defined only for $\omega \in [-\pi, \pi]$ (here, though any domain of width 2π is possible, as $h(t)\, e^{-\mathbf{i}\omega t}$ is a periodic function of ω with period 2π). This is the *discrete time Fourier transform* of $h(t)$.

The (discrete time) inv. FT is

$$h(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega)\, e^{\mathbf{i}\omega t} d\omega.$$

(b) If $h(t)$ is *even*, in that

$$h(t) = h(-t)$$

for all t, and real–valued, then let its FT be defined as

$$H(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} h(t)\, e^{-\mathbf{i}\omega t} dt$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_{-\infty}^{\infty} h(t)\cos(\omega t)\,dt \\
&\qquad -\mathbf{i}\frac{1}{\pi}\int_{-\infty}^{\infty} h(t)\sin(\omega t)\,dt \\
&= \frac{2}{\pi}\int_{0}^{\infty} h(t)\cos(\omega t)\,dt,
\end{aligned}
$$

recalling that $\cos(x)$ is even and $\sin(x)$ is *odd,* in that

$$\sin(x) = -\sin(-x)$$

for all x. Now $H(\omega)$ is a real–valued, even function of ω.

The inverse FT is

$$
\begin{aligned}
h(t) &= \frac{1}{2}\int_{-\infty}^{\infty} H(\omega)\,e^{\mathbf{i}\omega t}d\omega \\
&= \int_{0}^{\infty} H(\omega)\cos(\omega t)\,d\omega.
\end{aligned}
$$

(c) When $h(t)$ is both even and defined only for integer t, combining (a) and (b),

$$H(\omega) = \frac{1}{\pi}\left(h(0) + 2\sum_{t=1}^{\infty} h(t)\cos(\omega t)\right)$$

for $\omega \in [0, \pi]$. The inv. FT is

$$h(t) = \int_0^{\pi} H(\omega)\cos(\omega t)\, d\omega.$$

Strictly, in (a) and (b), if $h(t)$ is non-zero for a infinite number of $t \in \mathbb{Z}$, in order for $H(\omega)$ to be defined we require

$$\sum_{t=-\infty}^{\infty} |h(t)| < \infty.$$

Sufficient, and weaker, is that

$$\sum_{t=-\infty}^{\infty} h(t)^2 < \infty.$$

Example 0.4 *Let*

$$h(t) = \left(\frac{1}{3}\right)^{|t|}$$

for $t \in \mathbb{Z}$. *Note that*

$$\begin{aligned}
\sum_{t=-\infty}^{\infty} |h(t)| &= \sum_{t=-\infty}^{\infty} \left(\frac{1}{3}\right)^{|t|} \\
&= 1 + 2\sum_{t=1}^{\infty} \left(\frac{1}{3}\right)^{t} \\
&= 1 + 2 \times \frac{1/3}{2/3} \\
&< \infty.
\end{aligned}$$

Hence, by (c),

$$\begin{aligned}
H(\omega) &= \frac{1}{\pi}\left(h(0) + 2\sum_{t=1}^{\infty} h(t)\cos(\omega t)\right) \\
&= \frac{1}{\pi}\left(1 + 2\sum_{t=1}^{\infty} \left(\frac{1}{3}\right)^{t} \cos(\omega t)\right) \\
&= \frac{1}{\pi}\left(1 + \frac{(1 - 3\cos(\omega))}{(3\cos(\omega) - 5)}\right) \\
&= \frac{4}{\pi(5 - 3\cos(\omega))}.
\end{aligned}$$

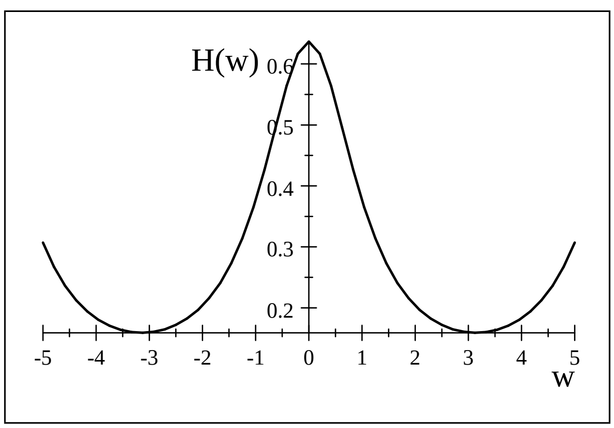
The closed form given above comes from writing

$$
\begin{aligned}
H(\omega) &= \frac{1}{\pi}\sum_{t=-\infty}^{\infty}\left(\frac{1}{3}\right)^{|t|} e^{-\mathbf{i}\omega t} \\
&= \frac{1}{\pi}\left(1+\sum_{t=-\infty}^{-1}\left(\frac{1}{3}\right)^{-t} e^{-\mathbf{i}\omega t}+\sum_{t=1}^{\infty}\left(\frac{1}{3}\right)^{t} e^{-\mathbf{i}\omega t}\right)
\end{aligned}
$$

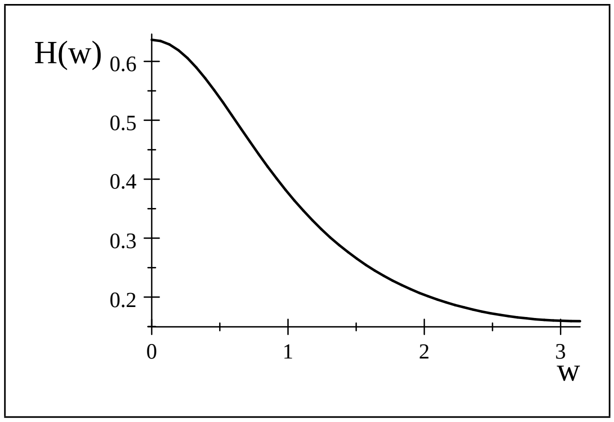
and recalling that

$$\cos(\omega)=\frac{e^{\mathbf{i}\omega}+e^{-\mathbf{i}\omega}}{2}.$$

The FT is plotted below:



Note that $H(\omega)$ is positive, even and periodic, with period 2π. Consequently it suffices to plot $H(\omega)$ on the range $[0,\pi]$, as below:



Observe that $H(\omega)$ is largest at low frequencies.

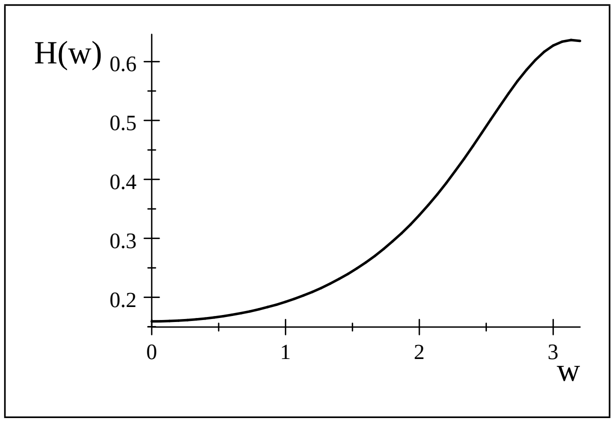
Example 0.5 *Modify the above so that*

$$h(t) = \left(-\frac{1}{3}\right)^{|t|}$$

for $t \in \mathbb{Z}$. Then $h(t)$ is even, but oscillates, *with successive values being of opposite sign. In other respects the example is similar to the previous one, with (exercise!)*

$$\begin{aligned} H(\omega) &= \frac{1}{\pi}\left(1 + 2\sum_{t=1}^{\infty}\left(-\frac{1}{3}\right)^{t}\cos(\omega t)\right) \\ &= \frac{8}{\pi\left(10 + 6\cos(\omega)\right)}. \end{aligned}$$

Now a plot of this is:



Note that this is greatest for high frequencies, as might be expected for an oscillating series.

The spectral distribution

The frequency domain behaviour of a time series is often expressed in terms of two key functions, the *spectral distribution function* and the *spectral density function*, defined here. These functions are closely related to the autocovariance function of a series.

Recall that a stationary, integer–time process $X(t)$ can be written as

$$X(t) = \int_0^\pi \cos(\omega t)\, dU(\omega) + \int_0^\pi \sin(\omega t)\, dV(\omega)$$

where U and V are uncorrelated stochastic processes with orthogonal increments. This *spectral representation* of $X(t)$ tells us that all frequencies in $(0, \pi)$ can contribute to the fluctuations of $X(t)$.

Now $U(\omega)$ and $V(\omega)$ are of no direct interest here, as we prefer to work with the *spectral distribution function,* $F(\omega)$, defined by

$$\gamma(k) = \int_0^\pi \cos(\omega k)\, dF(\omega),$$

for $k = 0, \pm 1, \ldots$. This is the spectral representation of $\gamma(k)$.

Think of $F(\omega)$ as being the **contribution to the variance of $X(t)$ by frequencies up to ω**, $0 < \omega \leq \pi$.

Note

$$\begin{aligned}\gamma(0) &= \sigma_X^2 \\ &= \int_0^\pi 1 dF(\omega) \\ &= F(\pi).\end{aligned}$$

Note that

- Between 0 and π the function $F(\omega)$ is non-decreasing and non-negative, and behaves like a c.d.f.

- If $X(t)$ has some deterministic frequency component, at ω_0 say, then F will have a jump discontinuity at ω_0.

- The *normalised spectral distribution function* is

$$F^*(\omega) := \frac{F(\omega)}{\sigma_X^2}$$

and so gives the *proportion* of variance in $X(t)$ accounted for by frequencies in $(0, \omega)$. As

$$F^*(\pi) = 1,$$

F^* has the properties of a c.d.f.

The spectral density function

We will assume that $F(\omega)$ is differentiable for $\omega \in (0, \pi)$, and so define the *spectral density function* as

$$f(\omega) := \frac{dF(\omega)}{d\omega}.$$

This function is sometimes called the *(power) spectrum*.

The functions f and F relate like a p.d.f. and a c.d.f., so we can write

$$\gamma(k) = \int_0^\pi \cos(\omega k) f(\omega) \, d\omega, \tag{3}$$

which is an “ordinary” integral.

With $k = 0$,

$$\begin{aligned} \gamma(0) &= \sigma_X^2 \\ &= \int_0^{\pi} f(\omega)\, d\omega \\ &= F(\pi). \end{aligned}$$

A peak in $f(\omega)$ around ω_0 indicates that frequencies at or around ω_0 greatly contribute to the variation in $X(t)$.

Recall that $\gamma(k)$ is an even function and defined for $k \in \mathbb{Z}$, therefore (see previous section) its inv. Fourier transform can be written

$$\gamma(k) = \int_0^{\pi} H(\omega) \cos(\omega k)\, d\omega$$

where $H(\omega)$ is the FT of $\gamma(k)$. But the above equation is (3), and so

$$\begin{aligned} H(\omega) &= f(\omega) \\ &= \frac{1}{\pi}\left(\gamma(0) + 2\sum_{k=1}^{\infty}\gamma(k)\cos(\omega k)\right) \end{aligned}$$

for $\omega \in (0, \pi)$. That is, $f(\omega)$ is the FT of $\gamma(k)$. Hence $f(\omega)$ and $\gamma(k)$ form a Fourier transform pair.

Strictly for the above to be well–defined at ω, we require

$$\sum_{k=1}^{\infty}\gamma(k)\cos(\omega k) < \infty.$$

Necessary therefore is that

$$\lim_{k\to\infty}\gamma(k)\cos(\omega k) = 0$$

but this will not occur at $\omega = \omega_0$ if $X(t)$ has a deterministic component at that frequency. Now $F(\omega)$ would have a discontinuity at ω_0 in that case, and f is not defined at ω_0.

Remark 0.3 *Definitions vary in different books, ostensibly in the factors involving either* $1/\pi$ *or* $1/2\pi$*. If we define* $f(\omega)$ *for* $\omega \in (-\pi, \pi)$*, the constant* $1/2\pi$ *should appear before the summation. As* $\gamma(k)$ *is an even function, there is little point in considering –ve frequencies.*

Example 0.6 *Let the spectral representation for an acvf* $\gamma(k)$ *be*

$$f(\omega) = \frac{1 + \cos(\omega)}{\pi}.$$

From this, we can find $\gamma(k)$*, since*

$$\begin{aligned}
\gamma(k) &= \int_0^\pi \cos(\omega k) f(\omega)\, d\omega \\
&= \frac{1}{\pi}\int_0^\pi \cos(\omega k)(1 + \cos(\omega))\, d\omega \\
&= \frac{1}{\pi}\int_0^\pi \cos(\omega k)\, d\omega + \frac{1}{\pi}\int_0^\pi \cos(\omega k)\cos(\omega)\, d\omega.
\end{aligned}$$

Now when $k=0$, *the above is*

$$\begin{aligned}\gamma(0) &= \frac{1}{\pi}\int_0^{\pi} 1 d\omega + \frac{1}{\pi}\int_0^{\pi}\cos(\omega)\, d\omega \\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\gamma(1) &= \frac{1}{\pi}\int_0^{\pi}\cos(\omega)\, d\omega + \frac{1}{\pi}\int_0^{\pi}\cos^2(\omega)\, d\omega \\ &= \frac{1}{2}\end{aligned}$$

since

$$\int_0^{\pi}\cos^2(\omega)\, d\omega = \frac{1}{2}\int_0^{\pi}(1+\cos(2x))\, dx = \frac{1}{2}\pi.$$

Similarly

$$\gamma(-1) = \frac{1}{2}.$$

However $\gamma(k) = 0$ *for* $k \neq 0, \pm 1$. *So non-zero correlation exists only for lags* ± 1.

The *normalised spectral density function* is

$$f^*(\omega) = \frac{dF^*(\omega)}{d\omega}$$

$$= \frac{f(\omega)}{\sigma_X^2}.$$

Therefore

$$f^*(\omega) = \frac{1}{\pi}\left(1 + 2\sum_{k=1}^{\infty} \rho(k)\cos(\omega k)\right)$$

and so is the FT of the autocorrelation function of $X(t)$. Now $f^*(\omega)$

- behaves like a p.d.f.:

  $$f^*(\omega)\, d\omega$$

  is the *proportion* of variance of $X(t)$ due to frequencies in $(\omega, \omega + d\omega)$;

- is an even function.

Examples

Having introduced the idea of a spectral distribution, we describe some key examples. The spectral density function, sometimes called the *spectrum*, of several important stochastic processes are derived.

We conclude with an additive property of the spectra of independent processes.

Recall the definition of the spectral density function $f(\omega)$ of a stationary stochastic process $X(t)$, which is that

$$f(\omega) = \frac{1}{\pi}\left(\gamma(0) + 2\sum_{k=1}^{\infty}\gamma(k)\cos(\omega k)\right)$$

for $\omega \in (0, \pi)$, where $\gamma(k)$ is the acvf. of $X(t)$. Basically, $f(\omega)$ is the FT of $\gamma(k)$. Recall also that the normalised spectrum,

$$f^*(\omega) = \frac{f(\omega)}{\sigma_X^2},$$

where $\sigma_X^2 =$Var$(X(t))$, is the FT of $\rho(k)$, the acf.

We outline some examples here, including some key special cases.

White noise

If $Z(t)$ is white noise, with variance σ^2, then

$$\gamma(k) = \begin{cases} \sigma^2 & k = 0 \\ \\ 0 & k \neq 0. \end{cases}$$

So the spectrum is

$$f(\omega) = \frac{\sigma^2}{\pi}.$$

That is, the spectral density is constant over $(0, \pi)$.

MA(1) **processes**

For the MA(1)

$$X(t) = Z(t) + \beta Z(t-1)$$

the acf is

$$\rho(k) = \begin{cases} 1 & k = 0 \\ \frac{\beta}{1+\beta^2} & k = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

So the normalised spectrum here is

$$f^*(\omega) = \frac{1}{\pi}\left(1 + \frac{2\beta\cos(\omega)}{1+\beta^2}\right),$$

and the spectral density is

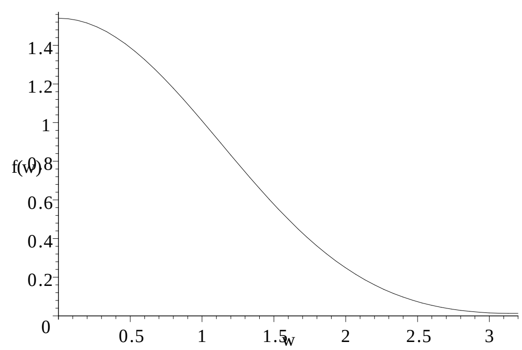
$$f(\omega) = \left(1+\beta^2\right)\sigma^2 f^*(\omega).$$

Now the shape of the spectrum is determined by the sign of β.

The case $\beta = 1$ was met earlier,

$$f^* (\omega) = \frac{1 + \cos (\omega)}{\pi},$$

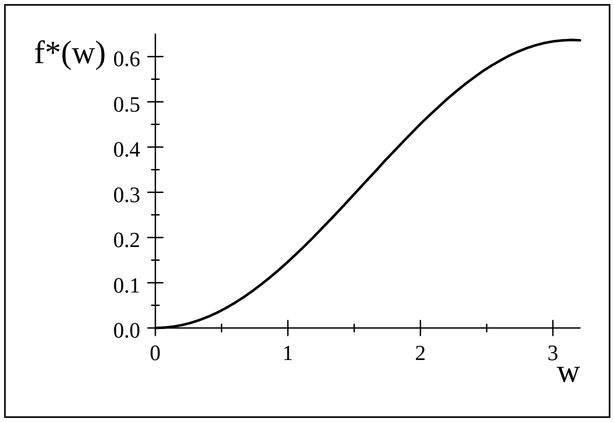
being dominated by low frequencies,



However when $\beta = -1$, the spectrum peaks at high frequencies,

$$f^* (\omega) = \frac{1 - \cos (x)}{\pi}$$

which is



AR(1) process

The process

$$X(t) = \alpha X(t-1) + Z(t)$$

has acvf.

$$\gamma(k) = \sigma_X^2 \alpha^{|k|}$$

for $k = 0, \pm 1, \ldots$. So

$$f(\omega) = \frac{\sigma_X^2}{\pi}\left(1 + \sum_{k=1}^{\infty} \alpha^k e^{-\mathbf{i}k\omega} + \sum_{k=1}^{\infty} \alpha^k e^{\mathbf{i}k\omega}\right)$$

$$= \quad \frac{\sigma_X^2}{\pi}\left(1+\frac{\alpha e^{-\mathbf{i}\omega}}{1-\alpha e^{-\mathbf{i}\omega}}+\frac{\alpha e^{\mathbf{i}\omega}}{1-\alpha e^{\mathbf{i}\omega}}\right).$$

Now using the facts that

$$e^{\mathbf{i}\omega} = \cos(\omega)+\mathbf{i}\sin(\omega),$$

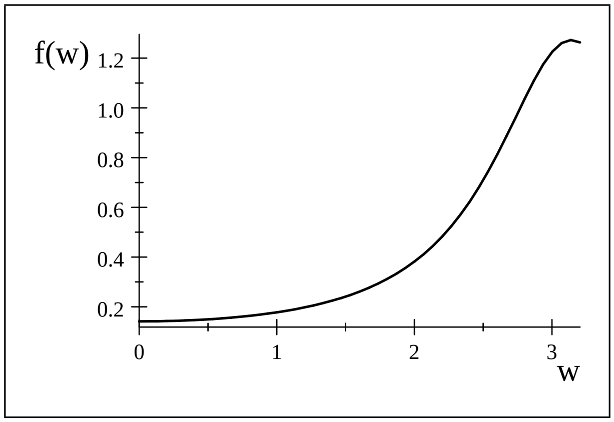
$$\cos(\omega) = \frac{e^{\mathbf{i}\omega}+e^{-\mathbf{i}\omega}}{2},$$

$$\sigma_X^2 = \frac{\sigma^2}{\left(1-\alpha^2\right)}$$

and some algebra, we find (exercises 4)

$$f(\omega)=\frac{\sigma^2}{\pi\left(1-2\alpha\cos(\omega)+\alpha^2\right)}.$$

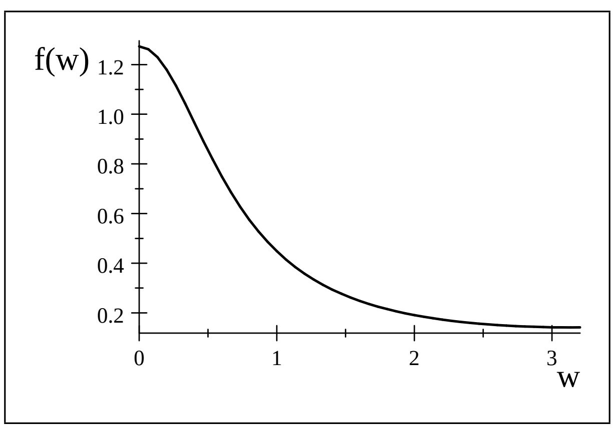
If $\alpha < 0$, the terms in $X(t)$ tend to oscillate, and we have a high–frequency spectrum:



Spectrum for AR(1) when $\sigma = 1$, $\alpha = -\frac{1}{2}$.

This would suggest a rather “ragged” series.

However if $\alpha > 0$, the spectrum is of low frequency:



AR(1) spectrum when $\sigma = 1$, $\alpha = \frac{1}{2}$.

This would be consistent with a relatively "smooth" series.

Note that as $\alpha \to 1$, the AR(1) approaches a random walk process, which suggests that

$$f(\omega) = \frac{\sigma^2}{2\pi(1 - \cos(\omega))}$$

can be taken as the spectrum for a random walk. However the acvf series for this process is not absolutely summable, so strictly the spectrum does not exist. That said, apart from at $\omega = 0$, the above is a well–defined function.

Remark 0.4 *The above suggests that where a spectrum of a series has a high peak near* $\omega = 0$*, differencing may be required.*

Fixed periodic component

A process with a deterministic sinusoidal component, such as

$$X(t) = \cos(\omega_0 t + \theta),$$

where $\omega_0 \in (0, \pi)$ is a constant and $\theta \sim U(0, 2\pi)$, but fixed for a given realisation, has acvf

$$\begin{aligned}\gamma(k) &= \frac{\cos(\omega_0 k)}{2} \\ &\nrightarrow 0\end{aligned}$$

as $k \to \infty$. Hence the spectrum is not defined (at ω_0, but is zero for other ω).

The "power" in the spectrum here is all on $\omega = \omega_0$, with the spectral distribution function having a jump at ω_0. Indeed since $E(X(t)) = 0$,

$$\begin{aligned}E\left(\cos^2(\theta)\right) &= \frac{1}{2\pi}\int_0^{2\pi} \cos^2(\theta)\, d\theta = \frac{1}{2}, \\ E(\sin(\theta)\cos(\theta)) &= \frac{1}{2\pi}\int_0^{2\pi} \sin(\theta)\cos(\theta)\, d\theta = 0,\end{aligned}$$

and $E\left(\sin^2(\theta)\right) = 1/2$ we have

$$\begin{aligned} \text{Var}(X(t)) &= E\left(X(t)^2\right) \\ &= \frac{1}{2} \end{aligned}$$

we have

$$F(\omega) = \begin{cases} 0 & \omega < \omega_0 \\ \frac{1}{2} & \omega \geq \omega_0 \end{cases}$$

which jumps (and so is non-differentiable) at ω_0.

Fixed periodic component with noise

Augmenting the process $X(t)$ above with an additive noise term $Z(t)$, so that

$$X(t) = \cos(\omega_0 t + \theta) + Z(t)$$

with ω_0 and θ are as above, we find the process has acvf

$$\gamma(k) = \begin{cases} \frac{1}{2} + \sigma^2 & k = 0 \\ \frac{\cos(\omega_0 k)}{2} & k \neq 0. \end{cases}$$

Again,

$$\gamma(k) \nrightarrow 0$$

as $k \to \infty$. As above, the spectral distribution function must jump at ω_0.

Sum of two independent processes

We demonstrate an additive property of spectra. Where required, subscripts indicate the relevant process here.

Suppose $X(t)$ and $Y(t)$ are two independent stationary process, and let

$$W(t) = X(t) + Y(t)$$

for all t. Then

$$f_W(\omega) = f_X(\omega) + f_Y(\omega).$$

To see this, note that

$$\sum_{k=-\infty}^{\infty} \gamma_W(k) e^{-\mathbf{i}\omega k} = \sum_{k=-\infty}^{\infty} \mathsf{Cov}(W(t), W(t+k)) e^{-\mathbf{i}\omega k}$$

for all t, which is

$$\sum_{k=-\infty}^{\infty} \mathsf{Cov}(X(t) + Y(t), X(t+k) + Y(t+k)) e^{-\mathbf{i}\omega k}$$

and this equals

$$\sum_{k=-\infty}^{\infty} (\mathsf{Cov}(X(t), X(t+k)) + \mathsf{Cov}(Y(t), Y(t+k))) e^{-\mathbf{i}\omega k}$$

i.e.,

$$\sum_{k=-\infty}^{\infty} \gamma_X(k) e^{-\mathbf{i}\omega k} + \sum_{k=-\infty}^{\infty} \gamma_Y(k) e^{-\mathbf{i}\omega k}.$$

Example 0.7 *Return to the process*

$$X(t) = \cos(\omega_0 t + \theta) + Z(t).$$

Using the additive property and earlier results, it follows that

$$f(\omega) = \frac{\sigma^2}{\pi} \qquad \omega \neq \omega_0$$

but is undefined elsewhere.

Introduction to Fourier series

We review the key ideas we require from *Fourier analysis,* theory which arose in the 19th century and showed how "suitably nice" functions can be written as a sum of suitably weighted sinusoidal functions. This is sometimes termed *harmonic analysis.*

The most important application here is to functions in discrete time, which is the case we concentrate on.

Orthogonal bases

You should be familiar with the idea that, within $\mathbb{R}^3$, the vectors

$$\begin{aligned}\mathbf{v}_1 &= (1, 0, 0)', \\ \mathbf{v}_2 &= (0, 1, 0)', \\ \mathbf{v}_3 &= (0, 0, 1)'\end{aligned}$$

form an *orthogonal basis*, in that

$$\mathbf{v}_i'\mathbf{v}_j \begin{cases} = 0 & i \neq j \\ \neq 0 & i = j \end{cases}$$

and moreover each $\mathbf{x} \in \mathbb{R}^3$ can be written

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

for some constants c_1, c_2 and c_3.

Of course the triple $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not unique in forming such a basis, though is one natural choice.

We will demonstrate an analogous notion holds when considering a basis for functions with finite integer domain: sine and cosine functions of a suitable set of frequencies define an orthogonal basis.

Definition 0.3 *Real–valued functions* f_i *and* f_j *are said to be* orthogonal *on the discrete (i.e., countable) set* S *if*

$$\sum_{t \in S} f_i(t)\, f_j(t) \begin{cases} = 0 & i \neq j \\ \neq 0 & i = j. \end{cases}$$

Of key importance:

- A similar definition exists for complex functions.
- If S is continuous, summation above is replaced by integration.
- Within an N-dimensional space, any set of N orthogonal vectors forms a basis.

Fourier frequencies

Let N be a positive integer. For simplicity, we will start by assuming N is *even.*

Definition 0.4 *Let the* p*th* Fourier frequency *be*

$$\omega_p := \frac{2\pi p}{N}$$

for $p = 0, 1, 2, \ldots, N/2$.

Clearly then

$$\omega_{N/2} = \pi,$$

so that

$$\begin{aligned}
\sin\left(\omega_{N/2}\right) &= \sin\left(\omega_0\right) = 0, \\
\cos\left(\omega_{N/2}\right) &= -\cos\left(\omega_0\right) = -1.
\end{aligned}$$

Some simplifications arise when working with the Fourier frequencies:

(a) For each $p = 1, 2, \ldots, N/2$,

$$\sum_{t=1}^{N} \cos(\omega_p t) = \sum_{t=1}^{N} \sin(\omega_p t) = 0.$$

(b) For each $p, q = 0, 1, 2, \ldots, N/2$,

$$\sum_{t=1}^{N} \sin(\omega_p t) \cos(\omega_q t) = 0.$$

(c)

$$\sum_{t=1}^{N} \cos(\omega_p t) \cos(\omega_q t) = \begin{cases} 0 & p \neq q \\ \frac{N}{2} & p = q \neq 0, \frac{N}{2} \\ N & p = q = 0, \frac{N}{2} \end{cases}$$

(d)

$$\sum_{t=1}^{N} \sin(\omega_p t)\sin(\omega_q t) = \begin{cases} 0 & p \neq q \\ \frac{N}{2} & p = q \neq 0, \frac{N}{2} \\ 0 & p = q = 0, \frac{N}{2} \end{cases}$$

Note the results (a) are special cases of (b).

The proofs of the above are basically routine (see exercises): recalling that

$$e^{\mathbf{i}\omega} = \cos(\omega) + \mathbf{i}\sin(\omega)$$

note that

$$\sum_{t=1}^{N} e^{\mathbf{i}\omega t} = \frac{e^{\mathbf{i}\omega}\left(1 - e^{\mathbf{i}\omega N}\right)}{\left(1 - e^{\mathbf{i}\omega}\right)}$$

and this vanishes at $\omega = \omega_p$, i.e.,

$$\sum_{t=1}^{N} e^{\mathbf{i}\omega_p t} = 0$$

as

$$e^{2\pi \mathbf{i}} = 1.$$

Consequently

$$\begin{aligned} \sum_{t=1}^{N} e^{\mathbf{i}\omega_p t} &= \sum_{t=1}^{N} \cos(\omega_p t) + \mathbf{i} \sum_{t=1}^{N} \sin(\omega_p t) \\ &= 0 + 0\mathbf{i}, \end{aligned}$$

and (a) follows by equating real and imaginary parts. Replacing ω_p by $2\omega_p$ in the above, since $\sin(2A) = 2\cos(A)\sin(A)$,

$$\sum_{t=1}^{N} \cos(\omega_p t)\sin(\omega_p t) = \frac{1}{2}\sum_{t=1}^{N} \sin(2\omega_p t) = 0.$$

Now for $t \in \{1, \ldots, N\}$ the set of functions

$$\mathcal{F}_N := \{\sin(\omega_p t), \cos(\omega_p t) : p = 0, 1, \ldots, N/2\}$$

contains exactly N non-zero functions. Specifically, for N even, the set contains

$$\cos(\omega_0 t) = \cos(0) = 1,$$

each of

$$\sin(\omega_p t) = \sin\left(\frac{2\pi pt}{N}\right),$$

$$\cos(\omega_p t) = \cos\left(\frac{2\pi pt}{N}\right)$$

for $p = 1, \ldots, N/2 - 1$ and

$$\cos\left(\omega_{N/2} t\right) = \cos(\pi t) = (-1)^t$$

Moreover, by results (a) – (d), these functions form an orthogonal set.

Remark 0.5 *Minor modifications are required if N is odd, though main results hold. Typically $N/2$ needs to be replaced by $[N/2]$, with $[x]$ indicating the integer part of x.*

Fourier representation of a finite series

Let $x(1), \ldots, x(N)$ be a sequence of N numbers – again, assume N is even. Now by the above, for each $t = 1, \ldots, N$, it must be possible to write

$$x(t) = \sum_{q=0}^{N/2} \left(a_q \cos(\omega_q t) + b_q \sin(\omega_q t) \right) \qquad (4)$$

for some constants $\{a_q, b_q : q = 0, \ldots, N/2\}$. To find these constants,

1. multiply (4) on both sides by $\cos(\omega_p t)$

2. and then sum over $t = 1, \ldots, N$;

3. multiply (4) on both sides by $\sin(\omega_p t)$,

4. and then sum over $t = 1, \ldots, N$.

Using (b), (c) and (d) we find

$$
\begin{aligned}
a_0 &= \frac{1}{N} \sum_{t=0}^{N} x(t) = \bar{x}, \\
a_{N/2} &= \frac{1}{N} \sum_{t=1}^{N} (-1)^t x(t), \\
a_p &= \frac{2}{N} \sum_{t=1}^{N} x(t) \cos(\omega_p t), \\
b_p &= \frac{2}{N} \sum_{t=1}^{N} x(t) \sin(\omega_p t),
\end{aligned}
$$

for $p = 1, \ldots, N/2-1$. These numbers are the *Fourier coefficients.*

Therefore the *Fourier series representation* of $\{x(t) : 1 \le t \le N\}$ is

$$\begin{aligned} x(t) &= a_0 + \sum_{p=1}^{N/2-1} \left(a_p \cos(\omega_p t) + b_p \sin(\omega_p t)\right) \\ &\quad + a_{N/2} \cos(\pi t) \end{aligned} \tag{5}$$

for $t = 1, \ldots, N$. This provides the *harmonic analysis* of $x(1), \ldots, x(N)$.

Note that

1. Representation (5) is not unique, in that other sets of frequencies could be found over which sines and cosines form an orthogonal basis ...

2. ... but the Fourier frequencies give simple forms for the coefficients.

3. There is no error term in (5) – it fits the data exactly.

4. Extending the range of t to

$$kN + s,$$

for integers k and s gives (check)

$$x(kN + s) = x(s),$$

defining a periodic series.

5. Frequency ω_p has period (wavelength)

$$\frac{2\pi}{\omega_p} = \frac{N}{p}.$$

This is not generally an integer, but a sinusoid with wavelength N/p completes p cycles over the duration of the data. So (5) decomposes the series into components each of which are repeated an integer number of times during the data.

Harmonic analysis of continuous functions

You are more likely to have met Fourier analysis for a continuous function: if $f(t)$ is a "nice" function with domain $(-\pi, \pi]$, it may be approximated by

$$\frac{a_0}{2} + \sum_{r=1}^{k} \left(a_r \cos(rt) + b_r \sin(rt) \right)$$

which $\to f(t)$ as $k \to \infty$, where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt, \\ a_r &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(rt)\, dt \\ b_r &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(rt)\, dt \end{aligned}$$

for $r = 1, 2, \ldots$.

A sinusoidal model

We illustrate the harmonic analysis discussed so far with consideration of a sinusoidal model. The model has historical relevance, being routinely fitted to data around the turn of the 20th century.

Also, motivation is given for the choice of frequencies fitted and form of the models considered.

Suppose for some constants α and β,

$$X(t) = \alpha \cos(\omega t) + \beta \sin(\omega t) + Z(t) \qquad (6)$$

where $Z(t)$ is white noise and ω is a fixed frequency.

We take a sample

$$\mathbf{x}' = (x(1), x(2), \ldots, x(N))$$

at fixed time periods, and assume that N is even. The vector

$$\boldsymbol{\theta} = \binom{\alpha}{\beta}$$

contains the parameters to be estimated.

Now the model (6) can be written

$$E(\mathbf{X}) = A\boldsymbol{\theta}$$

where

$$\mathbf{X} = (X(1), \ldots, X(N))',$$

$$A = \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ \cos(2\omega) & \sin(2\omega) \\ \vdots & \vdots \\ \\ \cos(N\omega) & \sin(N\omega) \end{pmatrix}.$$

This is clearly a linear model, and so the least squares estimate of $\boldsymbol{\theta}$ based on data $\mathbf{x}$ is given by

$$\hat{\boldsymbol{\theta}} = \left(A'A\right)^{-1} A'\mathbf{x}.$$

The largest frequency which could be fitted to the data is $\omega = \pi$, whilst the lowest completes one cycle over the whole series, i.e.,

$$\frac{2\pi}{\omega} = N,$$

so has frequency $2\pi/N$. We further restrict possible frequencies to

$$\omega_p = \frac{2\pi p}{N}$$

for $p = 1, 2, \ldots, N/2$, the Fourier frequencies met earlier.

For then $A'A$ is

$$\begin{pmatrix} \sum_{t=1}^{N} \cos^2(\omega_p t) & \sum_{t=1}^{N} \cos(\omega_p t) \sin(\omega_p t) \\ \sum_{t=1}^{N} \cos(\omega_p t) \sin(\omega_p t) & \sum_{t=1}^{N} \sin^2(\omega_p t) \end{pmatrix}$$

and for $p \neq \frac{1}{2}$, by identities (b), (c) and (d), this is

$$\begin{pmatrix} \frac{N}{2} & 0 \\ 0 & \frac{N}{2} \end{pmatrix}.$$

Therefore

$$\left(A'A\right)^{-1} = \begin{pmatrix} \frac{2}{N} & 0 \\ 0 & \frac{2}{N} \end{pmatrix}.$$

Also

$$A'\mathbf{x} = \begin{pmatrix} \sum_{t=1}^{N} x(t) \cos(\omega_p t) \\ \sum_{t=1}^{N} x(t) \sin(\omega_p t) \end{pmatrix}.$$

Hence

$$
\begin{aligned}
\hat{\boldsymbol{\theta}} &= \left(A'A\right)^{-1} A'\mathbf{x} \\
&= \binom{\hat{\alpha}}{\hat{\beta}}
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\alpha} &= \frac{2}{N}\sum_{t=1}^{N} x\left(t\right)\cos\left(\omega_p t\right) \\
\hat{\beta} &= \frac{2}{N}\sum_{t=1}^{N} x\left(t\right)\sin\left(\omega_p t\right).
\end{aligned}
$$

When $p = N/2$,

$$
\begin{aligned}
\hat{\alpha} &= \frac{1}{N}\sum_{t=1}^{N} \left(-1\right)^{t} x\left(t\right), \\
\hat{\beta} &= 0
\end{aligned}
$$

since the $\sin(\pi t)$ terms disappear.

Recall that the so–called regression SS for a linear model is the amount of "variation" in the data explained by the model – in this case periodic component at frequency ω_p. Now this can be written

$$\mathbf{x}'A\left(A'A\right)^{-1}A'\mathbf{x}=\mathbf{x}'A\hat{\boldsymbol{\theta}}$$

which is

$$\left(x\left(1\right),x\left(2\right),\ldots,x\left(N\right)\right)\begin{pmatrix}\cos\left(\omega_p\right) & \sin\left(\omega_p\right)\\ \cos\left(2\omega_p\right) & \sin\left(2\omega_p\right)\\ \vdots & \vdots\\ \\ \cos\left(N\omega_p\right) & \sin\left(N\omega_p\right)\end{pmatrix}\binom{\hat{\alpha}}{\hat{\beta}}$$

and this is

$$\frac{N\left(\hat{\alpha}^2+\hat{\beta}^2\right)}{2}$$

when $p \neq N/2$ (and equals $N\hat{\alpha}^2$ when $p = N/2$). Note that this is proportional to the square of the amplitude, $\hat{\alpha}^2+\hat{\beta}^2$, of the model fitted.

In practice, model (6) could be fitted at each Fourier frequency,

$$0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \pi,$$

the coefficients and regression SS calculated in each case.

This was the approach taken to time series analysis a century ago.

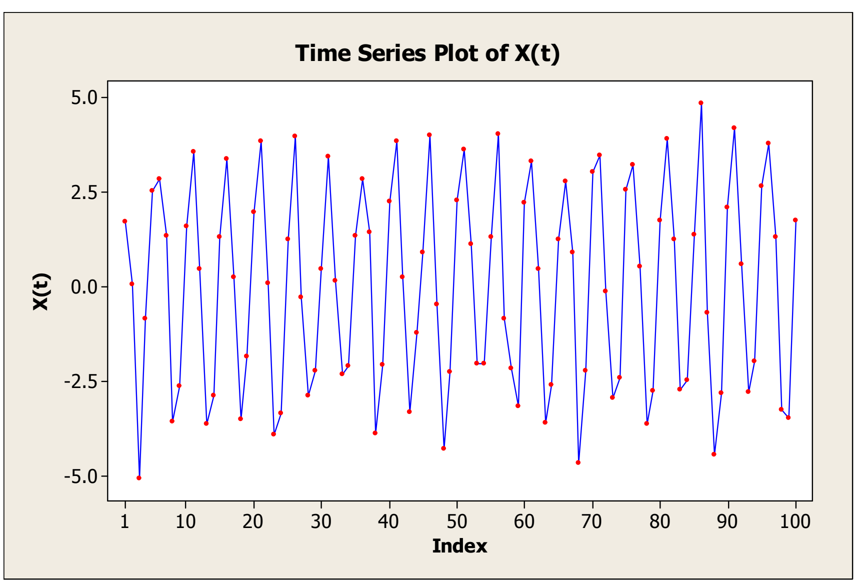
Example 0.8 *We generate* $N = 100$ *data from the model*

$$X(t) = 2\cos(\omega t) + 3\sin(\omega t) + Z(t),$$

where

$$\omega = \frac{40\pi}{100} = \frac{2\pi}{5}$$

and $Z(t) \sim N\left(0, 0.6^2\right)$.



Fitting the regression line with predictors $\cos(\omega t)$ *and* $\sin(\omega t)$ *we find*

$$X(t) = 1.849 \cos(\omega t) + 3.144 \sin(\omega t),$$

and not surprisingly both the fit and the estimation of the parameters is good. Note that the regression SS is 665.09, which is

$$\frac{100 \times \left(1.849^2 + 3.144^2\right)}{2}.$$

Models involving other frequencies will naturally perform less well; for instance, taking

$$\omega' = \frac{3\pi}{5},$$

the model is

$$X(t) = 0.134 \cos\left(\omega' t\right) - 0.0037 \sin\left(\omega' t\right)$$

with a poor fit (not surprisingly). The regression SS here is only 0.895, tiny compared to the total SS of 711.621.

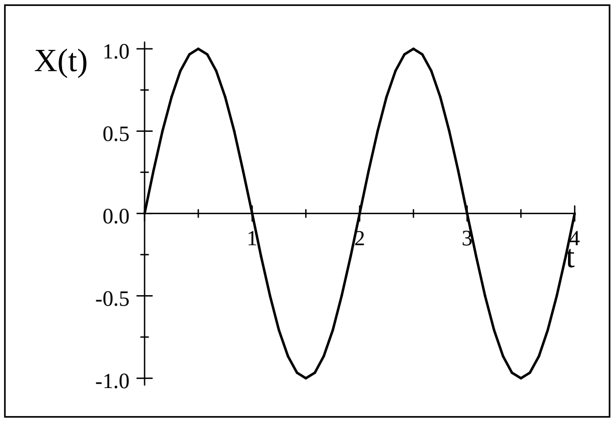
Why the Fourier frequencies?

We reiterate why the Fourier frequencies comprise a sensible set of frequencies to fit.

1. *Formulae are simpler:* recall results (a) – (d) from earlier. They were used above.

2. *The highest frequency which can be fitted to discrete data is π, the Nyquist frequency:* E.g. For daily data, frequency $\omega = \pi$ radians per day is $1/2$ cycle per day – one complete cycle every two days. Frequencies corresponding to variations *within days* will be greater than this, but unless data are collected more frequently such periodicities cannot be studied. E.g., $w = 2\pi$ has one cycle per day, but within day variations are hidden with one observation a day.
   Below is a plot of $X(t) = \sin(\pi t)$. This function

competes a cycle every two integer time points, and is exactly zero when sampled at integers.



The function $Y(t) = \sin(2\pi t)$ would complete a cycle between integer times, and so we would be unable to distinguish $X(t)$ from $Y(t)$ if observing only at integer time points.

3. *The lowest frequency which can be fitted to discrete data is $2\pi/N$, called the fundamental Fourier frequency:* This is one complete cycle per N observations – obviously we need N observations to detect this! E.g., for a year of weekly data, $N = 52$, and frequency $2\pi/N$ corresponds to $1/N$ cycles per week, which is one complete cycle per year.

Why sinusoidal models?

Why are we focusing only on sinusoidal models? Apart from some nice mathematical properties, there are two key reasons:

1. *Behaviour under change of time scale:* Return to the model
$$X(t) = R\cos(\omega t + \theta),$$
with R the amplitude, θ the phase and ω the frequency, in radians per unit time. Change the time variable linearly to
$$s = \frac{t-a}{b}$$
for some constants a and b. This alters both origin and scale. Now
$$\begin{aligned} X(t) &= X(bs+a) \\ &= R\cos(\omega bs + \omega a + \theta) \\ &= R'\cos\left(\omega' s + \theta'\right) \end{aligned}$$

where

$$\begin{aligned} R' &= R, \\ \omega' &= b\omega, \\ \theta' &= \theta + a\omega. \end{aligned}$$

So there is *no change in amplitude*, the frequency is multiplied by b, and the phase is shifted by a simple function of the frequency and the change in the time origin.

2. *Behaviour when sampled at discrete time intervals:* If sampling from the models

$$R \cos(\omega_1 t + \theta)$$

and

$$R \cos(\omega_2 t + \theta)$$

at equally spaced intervals of the length δt say, the two models are indistinguishable if

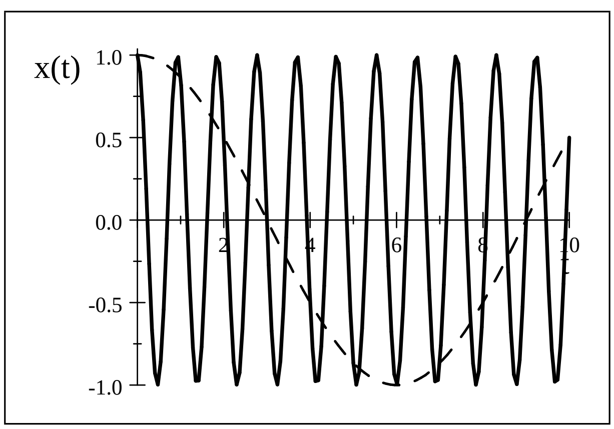
$$\omega_1 - \omega_2 = \frac{2\pi k}{\delta t}$$

for some integer k. This is known as *aliasing,* and will be discussed again later.

For example, below are plots of

$$\begin{aligned} x_1(t) &= \cos\left(\frac{\pi t}{6}\right), \\ x_2(t) &= \cos\left(\frac{13\pi t}{6}\right), \end{aligned}$$

the latter being the higher-frequency (solid bold) curve.



Note that at integer time points the two curves coincide, to be expected since

$$\frac{13\pi}{6} - \frac{\pi}{6} = 2\pi.$$