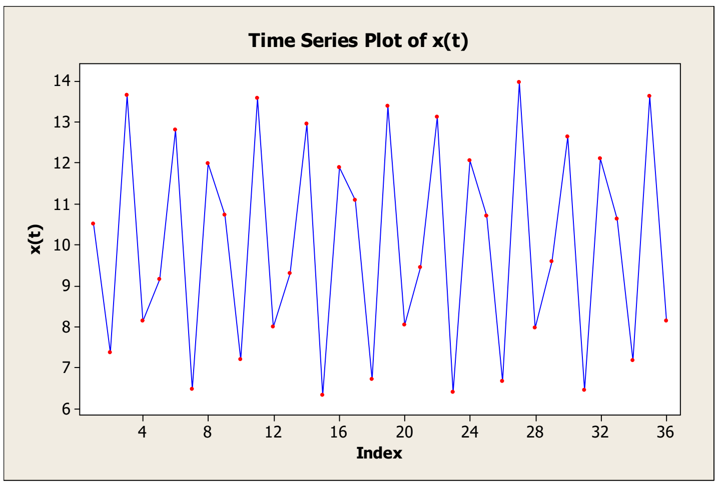
# Activity Solution: Modifying the Periodogram 1

As we have seen, the raw periodogram $I(\omega)$ is not a good estimator of the underlying spectrum of a series as it is inconsistent. Here we explore a common method for modifying the periodogram to give a consistent estimator of $f(\omega)$.

We explore the method as applied to the series $x(t)$ plotted below.




Note that in practice this series is rather too short for the methods to be robustly applied, there being only 36 observations. The sample avcf for the series above is below (with $c_0 = 1$):

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c_k$ | $-0.71$ | $0.029$ | $0.63$ | $-0.89$ | $0.64$ | $-0.035$ | $-0.55$ | $0.79$ | $-0.57$ | $0.043$ | $0.46$ | $-0.67$ |

We consider here how to *truncate and transform* to improve the properties of the periodogram. Since

$$I(\omega_p) = \frac{1}{\pi}\left(c_0 + 2\sum_{k=1}^{N-1} c_k \cos(\omega_p k)\right),$$

attach weights $\{\lambda_k\}$ to the $c_k$ which decrease with $k$, giving an estimator of the form

$$\hat{f}_T(\omega_p) = \frac{1}{\pi}\left(\lambda_0 c_0 + 2\sum_{k=1}^{M} \lambda_k c_k \cos(\omega_p k)\right).$$

The weights $\{\lambda_k\}$ are called the *lag window*, and $M < N$ is the *truncation point.*
A common choice is $M = 2\sqrt{N}$. Taking $M = 12$ as suggested, we will work out the estimator $\hat{f}_T(\omega_p)$, for $p = 0, 1, \ldots, N/2 = 18$, using the *Tukey window:*

$$\lambda_k = \frac{1}{2}\left(1 + \cos\left(\frac{\pi k}{12}\right)\right)$$

for $k = 0, 1 \ldots, 12$.

1. *Clickers question:* To start, complete the table for the values of $\lambda_k$ below. Each member of your group should compute (at least) one value.

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_k$ | 1 | 0.98 | | 0.85 | 0.75 | 0.63 | | 0.37 | 0.25 | 0.15 | 0.067 | 0.017 | |

*The completed table is*

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_k$ | 1 | 0.98 | 0.93 | 0.85 | 0.75 | 0.63 | 0.50 | 0.37 | 0.25 | 0.15 | 0.067 | 0.017 | 0 |

*For instance,*

$$\lambda_5 = \frac{1}{2}\left(1 + \cos\left(\frac{5\pi}{12}\right)\right) = 0.629\,.$$

2. *Clickers question:* To compute $\hat{f}_T(\omega_p)$ for $p = 0, 1 \ldots, 18$, we need to find the sums in the final column of the following table, in which the column labels are the values of $k$ and $\omega_p = \pi p/18$. Complete the final column, with each member of your group computing at least one final

value:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | Sum |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_k c_k \cos(\omega_1 k)$ | | | | | | | | | | | | 0 | -0.500 |
| $\lambda_k c_k \cos(\omega_2 k)$ | | | | | | | | | | | | 0 | -0.498 |
| $\lambda_k c_k \cos(\omega_3 k)$ | | | | | | | | | | | | 0 | -0.499 |
| $\lambda_k c_k \cos(\omega_4 k)$ | | | | | | | | | | | | 0 | -0.499 |
| $\lambda_k c_k \cos(\omega_5 k)$ | | | | | | | | | | | | 0 | |
| $\vdots$ | | | | | | | | | | | | 0 | -0.494 |
| | | | | | | | | | | | | 0 | -0.499 |
| | | | | | | | | | | | | 0 | -0.476 |
| | | | | | | | | | | | | 0 | -0.483 |
| | | | | | | | | | | | | 0 | -0.511 |
| | | | | | | | | | | | | 0 | -0.156 |
| | | | | | | | | | | | | 0 | 0.900 |
| | | | | | | | | | | | | 0 | |
| $\vdots$ | | | | | | | | | | | | 0 | 2.089 |
| $\lambda_k c_k \cos(\omega_{15} k)$ | | | | | | | | | | | | 0 | |
| $\lambda_k c_k \cos(\omega_{16} k)$ | | | | | | | | | | | | 0 | -0.050 |
| $\lambda_k c_k \cos(\omega_{17} k)$ | | | | | | | | | | | | 0 | -0.424 |
| $\lambda_k c_k \cos(\omega_{18} k)$ | | | | | | | | | | | | 0 | -0.419 |

*The completed table is*

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | Sum |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_k c_k \cos(\omega_1 k)$ | | | | | | | | | | | | 0 | -0.500 |
| $\lambda_k c_k \cos(\omega_2 k)$ | | | | | | | | | | | | 0 | -0.498 |
| $\lambda_k c_k \cos(\omega_3 k)$ | | | | | | | | | | | | 0 | -0.499 |
| $\vdots$ | | | | | | | | | | | | 0 | -0.499 |
| | | | | | | | | | | | | 0 | -0.492 |
| | | | | | | | | | | | | 0 | -0.494 |
| | | | | | | | | | | | | 0 | -0.499 |
| | | | | | | | | | | | | 0 | -0.476 |
| | | | | | | | | | | | | 0 | -0.483 |
| | | | | | | | | | | | | 0 | -0.511 |
| | | | | | | | | | | | | 0 | -0.156 |
| | | | | | | | | | | | | 0 | 0.900 |
| | | | | | | | | | | | | 0 | 2.018 |
| | | | | | | | | | | | | 0 | 2.089 |
| | | | | | | | | | | | | 0 | 1.035 |
| $\vdots$ | | | | | | | | | | | | 0 | -0.050 |
| $\lambda_k c_k \cos(\omega_{17} k)$ | | | | | | | | | | | | 0 | -0.424 |
| $\lambda_k c_k \cos(\omega_{18} k)$ | | | | | | | | | | | | 0 | -0.419 |

3. *Clickers question:* Now we must compute

$$\hat{f}_T(\omega_p) = \frac{1}{\pi}\left(1 + 2\sum_{k=1}^{12} \lambda_k c_k \cos(\omega_p k)\right)$$

for each frequency $\omega_p$. Complete the table below with these values:

| $p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\hat{f}_T(\omega_p)$ | $\approx 0$ | 0.001 | $\approx 0$ | $\approx 0$ | 0.005 | 0.002 | | 0.002 | 0.001 |

| $p$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| $\hat{f}_T(\omega_p)$ | -0.0070 | 0.219 | 0.891 | | | | 0.286 | 0.0484 | 0.0515 |

*The completed table is*

| $p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\hat{f}_T(\omega_p)$ | $\approx 0$ | 0.001 | $\approx 0$ | $\approx 0$ | 0.005 | 0.002 | $\approx 0$ | 0.002 | 0.001 |

| $p$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| $\hat{f}_T(\omega_p)$ | -0.0070 | 0.219 | 0.891 | 1.603 | 1.648 | 0.977 | 0.286 | 0.0484 | 0.0515 |

4. *Clickers question:* Comment on the shape of the estimate $\hat{f}_T$ here.
*The spectrum appears to be dominated by frequencies in the range* $(0.6\pi, 0.8\pi)$. *Observe too that the estimate is negative for one frequency, something that is possible with the Tukey window. In fact the model used to simulate the data was*

$$X(t) = 2\cos\left(\frac{3\pi t}{4}\right) + 3\sin\left(\frac{2\pi t}{3}\right) + Z(t)$$

*where* $Z(t) \sim N(0, 0.4)$.

5. *Clickers question:* If in fact asymptotically

$$\frac{n\hat{f}_T(\omega)}{f(\omega)} \sim \chi_n^2,$$

approximately, where

$$n = \frac{2N}{\sum_{k=-M}^{M} \lambda_k^2},$$

show how you could use this result here to construct 95% confidence intervals for $f(\omega)$.
*Well here*

$$n = \frac{72}{9} = 8.$$

*Hence with* $\chi_8^2(\alpha)$ *the* $100\alpha\%$ *point of* $\chi_8^2$, *we have*

$$P\left(\chi_8^2(0.025) < \frac{8\hat{f}_T(\omega)}{f(\omega)} < \chi_8^2(0.975)\right) = 0.95,$$

*consequently a 95% confidence interval for* $f(\omega)$ *is*

$$\left(\frac{8\hat{f}_T(\omega)}{\chi_8^2(0.975)}, \frac{8\hat{f}_T(\omega)}{\chi_8^2(0.025)}\right).$$

6. Re-cap what you have done during this activity. Why do you think you were asked to do this activity? What have you learned?
*A method for modifying the periodogram has been introduced that involves truncating (i.e., cutting off) the sample acvf and weighting the*

*values taken. A special case was worked through to de-mystify the formulae. The method not only permits pointwise estimation of the spectrum, but also confidence intervals via an asymptotic result.*
*Learning outcomes encountered include:*

(a) *Describe and explain methods for modifying the periodogram, in particular approaches that transform and truncate the periodogram.*

(b) *Construct confidence intervals for a spectrum based on a consistent estimator following a modification of the periodogram.*