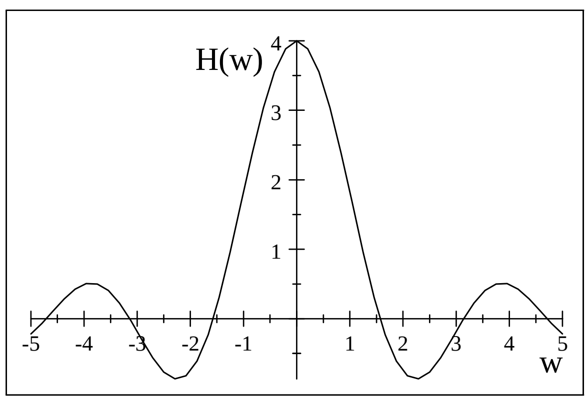
Exercises 4: Solutions

1. The FT of $f(x)$ here is

$$\begin{aligned}
H(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-\mathbf{i}\omega t} dt \\
&= \int_{-c}^{c} 1 \times e^{-\mathbf{i}\omega t} dt \\
&= \int_{-c}^{c} \cos(-\omega t)\, dt + \mathbf{i} \int_{-c}^{c} \sin(-\omega t)\, dt \\
&= \int_{-c}^{c} \cos(\omega t)\, dt \\
&= \left[\frac{\sin(\omega t)}{\omega} \right]_{-c}^{c} \\
&= \frac{\sin(\omega c)}{\omega} - \frac{\sin(-\omega c)}{\omega} \\
&= \frac{2\sin(\omega c)}{\omega},
\end{aligned}$$

for $\omega \neq 0$, with $H(0) = 2c$. The case with $c = 2$ is plotted below. Note the function takes its highest values at low frequencies.



$H(\omega)$ when $c = 2$.

2. Here we have

$$\begin{aligned}
\pi H(\omega) &= \sum_{t=-\infty}^{\infty}\left(-\frac{1}{3}\right)^{|t|} e^{-\mathbf{i}\omega t} \\
&= 1+\sum_{t=1}^{\infty}\left(-\frac{1}{3}e^{\mathbf{i}\omega}\right)^{t}+\sum_{t=1}^{\infty}\left(-\frac{1}{3}e^{-\mathbf{i}\omega}\right)^{t} \\
&= 1-\frac{e^{\mathbf{i}\omega}}{3}\left(\frac{1}{1+e^{\mathbf{i}\omega}/3}\right)-\frac{e^{-\mathbf{i}\omega}}{3}\left(\frac{1}{1+e^{-\mathbf{i}\omega}/3}\right) \\
&= 1-\frac{e^{\mathbf{i}\omega}}{3+e^{\mathbf{i}\omega}}-\frac{e^{-\mathbf{i}\omega}}{3+e^{-\mathbf{i}\omega}} \\
&= 1-\left(\frac{e^{\mathbf{i}\omega}\left(3+e^{-\mathbf{i}\omega}\right)+e^{-\mathbf{i}\omega}\left(3+e^{\mathbf{i}\omega}\right)}{\left(3+e^{\mathbf{i}\omega}\right)\left(3+e^{-\mathbf{i}\omega}\right)}\right) \\
&= 1-\left(\frac{2+3\left(e^{\mathbf{i}\omega}+e^{-\mathbf{i}\omega}\right)}{9+3e^{-\mathbf{i}\omega}+3e^{\mathbf{i}\omega}+1}\right) \\
&= 1-\left(\frac{2+6\cos(\omega)}{10+3\left(e^{-\mathbf{i}\omega}+e^{\mathbf{i}\omega}\right)}\right) \\
&= 1-\left(\frac{2+6\cos(\omega)}{10+6\cos(\omega)}\right) \\
&= \frac{10+6\cos(\omega)-2-6\cos(\omega)}{10+6\cos(\omega)}.
\end{aligned}$$

Consequently, we find

$$H(\omega)=\frac{8}{\pi\left(10+6\cos(\omega)\right)}$$

as stated in a lecture.

3. Recalling that

$$\gamma(k)=\sigma_X^2\alpha^{|k|}$$

we have

$$
\begin{aligned}
f(\omega) &= \frac{\sigma_X^2}{\pi}\left(1+\sum_{k=1}^{\infty}\alpha^k e^{-\mathbf{i}k\omega}+\sum_{k=1}^{\infty}\alpha^k e^{\mathbf{i}k\omega}\right) \\
&= \frac{\sigma_X^2}{\pi}\left(1+\frac{\alpha e^{-\mathbf{i}\omega}}{1-\alpha e^{-\mathbf{i}\omega}}+\frac{\alpha e^{\mathbf{i}\omega}}{1-\alpha e^{\mathbf{i}\omega}}\right) \\
&= \frac{\sigma_X^2}{\pi}\left(1+\frac{\left(1-\alpha e^{-\mathbf{i}\omega}\right)\left(1-\alpha e^{\mathbf{i}\omega}\right)+\alpha e^{-\mathbf{i}\omega}\left(1-\alpha e^{\mathbf{i}\omega}\right)+\alpha e^{\mathbf{i}\omega}\left(1-\alpha e^{-\mathbf{i}\omega}\right)}{\left(1-\alpha e^{-\mathbf{i}\omega}\right)\left(1-\alpha e^{\mathbf{i}\omega}\right)}\right) \\
&= \frac{\sigma_X^2}{\pi}\left(\frac{1-\alpha^2}{1-\alpha e^{-\mathbf{i}\omega}+\alpha^2-\alpha e^{\mathbf{i}\omega}}\right) \\
&= \frac{\sigma_X^2}{\pi}\left(\frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos(\omega)}\right) \\
&= \frac{\sigma^2}{\pi\left(1-2\alpha\cos(\omega)+\alpha^2\right)}
\end{aligned}
$$

since

$$\sigma_X^2 = \frac{\sigma^2}{\left(1-\alpha^2\right)}.$$

where σ^2 is the variance of $Z(t)$. The normalised spectrum is therefore

$$f^*(\omega) = \frac{1}{\pi}\left(\frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos(\omega)}\right).$$

4. Let σ^2 denote $\mathrm{Var}(Z(t))$ as usual.

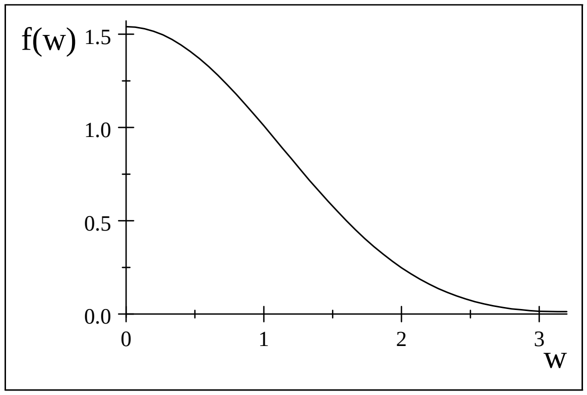
(i) Note here that

$$
\gamma(k) = \begin{cases} \sigma^2(1+1+0.04) & k=0 \\ \sigma^2(1+0.2) & k=1 \\ 0.2\sigma^2 & k=2 \\ 0 & k>2 \end{cases}
$$

with $\gamma(-k)=\gamma(k)$ as usual. So

$$f(\omega) = \frac{\sigma^2}{\pi}\left(2.04+2\left(1.2\cos(\omega)+0.2\cos(2\omega)\right)\right).$$

The case with $\sigma^2 = 1$ is plotted below:



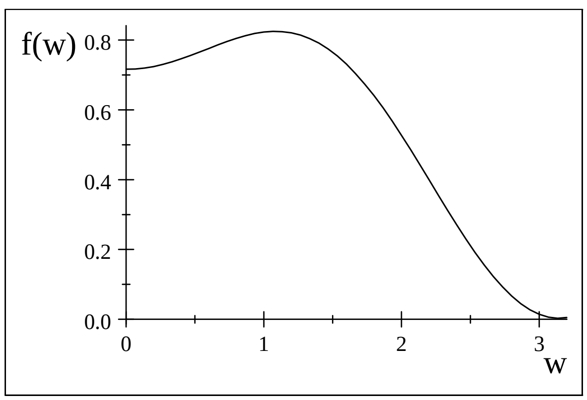
(ii) Now we have

$$\gamma(k) = \begin{cases} \sigma^2\left(1+0.8^2+0.3^2\right) & k=0 \\ \sigma^2\left(0.8-0.8\times 0.3\right) & k=\pm 1 \\ \sigma^2\left(-0.3\right) & k=\pm 2 \\ 0 & |k|>2, \end{cases}$$

so

$$f(\omega) = \frac{\sigma^2}{\pi}\left(1.73 + 2\left(0.56\cos(\omega) - 0.3\cos(2\omega)\right)\right).$$

The case when $\sigma = 1$ is shown below:



5. We saw in Exercises 2 that for the model

$$X(t) = \alpha X(t-1) + Z(t) + \beta Z(t-1)$$

we have

$$\begin{aligned} \gamma(0) &= \frac{\sigma^2\left(1+\beta^2+2\alpha\beta\right)}{\left(1-\alpha^2\right)}, \\ \gamma(1) &= \frac{\sigma^2\left(\alpha+\beta+\alpha\beta^2+\alpha^2\beta\right)}{\left(1-\alpha^2\right)} \end{aligned}$$

and for $k \geq 2$,

$$\gamma(k) = \alpha\gamma(k-1).$$

Hence

$$\begin{aligned} f(\omega) &= \frac{1}{\pi}\left(\gamma(0) + 2\sum_{k=1}^{\infty}\alpha^{k-1}\gamma(1)\cos(k\omega)\right) \\ &= \frac{\sigma^2}{\pi(1-\alpha^2)}\left(1+\beta^2+2\alpha\beta+2\left(\alpha+\beta+\alpha\beta^2+\alpha^2\beta\right)\sum_{k=1}^{\infty}\alpha^{k-1}\cos(k\omega)\right). \end{aligned}$$

Now using $\cos(k\omega) = \left(e^{\mathbf{i}k\omega} + e^{-\mathbf{i}k\omega}\right)/2$, we see

$$\begin{aligned} \sum_{k=1}^{\infty}\alpha^{k-1}\cos(k\omega) &= \frac{1}{2\alpha}\sum_{k=1}^{\infty}\alpha^{k}\left(e^{\mathbf{i}k\omega} + e^{-\mathbf{i}k\omega}\right) \\ &= \frac{\cos(\omega)-\alpha}{1+\alpha^2-2\alpha\cos(\omega)} \end{aligned}$$

on summing the g.p.; hence

$$\begin{aligned} f(\omega) &= \frac{\sigma^2}{\pi}\left(\frac{\left(1+\beta^2+2\alpha\beta\right)\left(1+\alpha^2-2\alpha\cos(\omega)\right)+2\left(\alpha+\alpha\beta(\alpha+\beta)(\cos(\omega)-\alpha)\right)}{(1-\alpha^2)(1+\alpha^2-2\alpha\cos(\omega))}\right) \\ &= \frac{\sigma^2\left(1+2\beta\cos(\omega)+\beta^2\right)}{\pi(1+\alpha^2-2\alpha\cos(\omega))} \end{aligned}$$

after some manipulation.

6. Here

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{8\pi^2}{3}$$

and for $n \neq 0$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} x^2\cos(nx)\,dx \\ &= \frac{1}{\pi}\left[x^2\left(\frac{\sin(nx)}{n}\right) - 2x\left(-\frac{\cos(nx)}{n^2}\right) + 2\left(-\frac{\sin(nx)}{n^3}\right)\right]_0^{2\pi} \\ &= \frac{4}{n^2}. \end{aligned}$$

Furthermore,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} x^2\sin(nx)\,dx \\ &= \frac{1}{\pi}\left[x^2\left(\frac{-\cos(nx)}{n}\right) - 2x\left(-\frac{\sin(nx)}{n^2}\right) + 2\left(\frac{\cos(nx)}{n^3}\right)\right]_0^{2\pi} \\ &= -\frac{4\pi}{n}. \end{aligned}$$

So that

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos(nx) - \frac{4\pi}{n} \sin(nx) \right).$$

Now $f(x)$ has a jump discontinuity at $x = 0$, and the series above takes the value

$$\begin{aligned} \frac{f(0+) + f(0-)}{2} &= \frac{1}{2}\left(0 + 4\pi^2\right) \\ &= 2\pi^2 \end{aligned}$$

at $x = 0$. At this point the series reduces to

$$\frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}.$$

Equating these gives

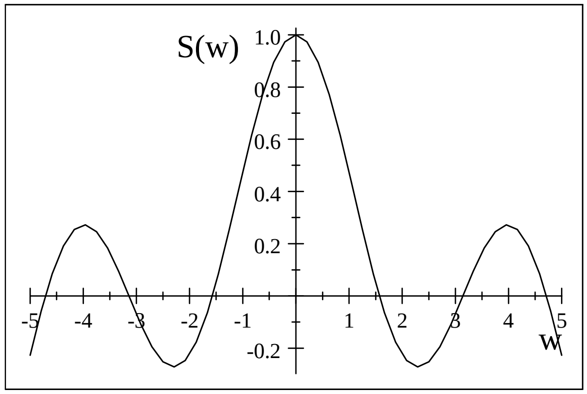
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

7. Putting $z = e^{-\mathbf{i}\omega}$ gives

$$\begin{aligned} G\left(e^{-\mathbf{i}\omega}\right) &= \sum_{k=-\infty}^{\infty} \gamma(k)\, e^{-\mathbf{i}\omega k} \\ &= \gamma(0) + 2\sum_{k=1}^{\infty} \gamma(k) \cos(\omega k) \\ &= \pi f(\omega). \end{aligned}$$

8. Clearly, since $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$,

$$\begin{aligned} S_1(\omega) &= 1, \\ S_2(\omega) &= \frac{\sin\left(\frac{\omega}{2} + \frac{\omega}{2}\right)}{2\sin\left(\frac{\omega}{2}\right)} = \cos\left(\frac{\omega}{2}\right), \\ S_3(\omega) &= \frac{\sin\left(\frac{3\omega}{2}\right)}{3\sin\left(\frac{\omega}{2}\right)} = \frac{1 + 2\cos(\omega)}{3}, \\ S_4(\omega) &= \frac{\sin(2\omega)}{4\sin\left(\frac{\omega}{2}\right)} = \cos\left(\frac{\omega}{2}\right)\cos(\omega). \end{aligned}$$



Plot of $S_4(\omega)$.

9. Note that

$$\begin{aligned}
\sum_{t=1}^{N} e^{\mathbf{i}\omega t} &= \frac{e^{\mathbf{i}\omega}\left(1-e^{\mathbf{i}\omega N}\right)}{\left(1-e^{\mathbf{i}\omega}\right)} \\
&= \frac{e^{\mathbf{i}\omega(N+1)/2}\left(e^{\mathbf{i}\omega N/2}-e^{-\mathbf{i}\omega N/2}\right)}{\left(e^{\mathbf{i}\omega/2}-e^{-\mathbf{i}\omega/2}\right)} \\
&= Ne^{\mathbf{i}\omega(N+1)/2} S_N(\omega)
\end{aligned}$$

and the above is zero if $\omega = 2\pi j/N$ for some integer $j \in \{-(N-1), \ldots, 0, \ldots, N-1\}$. Hence, equating real parts,

$$\sum_{t=1}^{N} \cos(\omega t) = N \cos\left(\frac{(N+1)\omega}{2}\right) S_N(\omega)$$

and equating the imaginary parts,

$$\sum_{t=1}^{N} \sin(\omega t) = N \sin\left(\frac{(N+1)\omega}{2}\right) S_N(\omega).$$

Both vanish when ω is a Fourier frequency. Similarly,

$$\begin{aligned}
\sum_{t=1}^{N} \cos(\omega t)\sin(\omega t) &= \frac{1}{2}\sum_{t=1}^{N} \sin(2\omega t) \\
&= \frac{N}{2} \sin((N+1)\omega) S_N(2\omega)
\end{aligned}$$

and this vanishes at a Fourier frequency. Lastly

$$\begin{aligned}\sum_{t=1}^{N}\cos(\omega_1 t)\cos(\omega_2 t) &= \frac{1}{2}\sum_{t=1}^{N}\left(\cos(\omega_1+\omega_2 t)\right)+\cos\left((\omega_1-\omega_2)t\right) \\ &= \frac{N}{2}\left(\cos\left(\frac{(N+1)(\omega_1+\omega_2)}{2}\right)S_N(\omega_1+\omega_2)\right. \\ &\qquad \left.+\cos\left(\frac{(N+1)(\omega_1-\omega_2)}{2}\right)S_N(\omega_1-\omega_2)\right)\end{aligned}$$

and the bracketed expression vanishes when ω_1 and ω_2 are different Fourier frequencies. Establishing (d) is very similar.

10. The Fourier representation of $x(t)$ is

$$x(t)=\sum_{p=0}^{N/2}\left(a_p\cos(\omega_p t)+b_p\sin(\omega_p t)\right),$$

and squaring this gives

$$\sum_{p=0}^{N/2}a_p^2\cos^2(\omega_p t)+\sum_{p=1}^{N/2-1}b_p^2\sin^2(\omega_p t)+2\sum_{p=1}^{N/2-1}a_p b_p\cos(\omega_p t)\sin(\omega_p t).$$

Summing from $t=1$ to N, the above is

$$\sum_{p=0}^{N/2}\left(a_p^2\sum_{t=1}^{N}\cos^2(\omega_p t)\right)+\sum_{p=1}^{N/2-1}\left(b_p^2\sum_{t=1}^{N}\sin^2(\omega_p t)\right)+2\sum_{p=1}^{N/2-1}\left(a_p b_p\sum_{t=1}^{N}\cos(\omega_p t)\sin(\omega_p t)\right).$$

Hence, using the orthogonality relations and the fact that $a_0=\bar{x}$, we have

$$\begin{aligned}\sum_{t=1}^{N}x(t)^2 &= N\bar{x}^2+Na_{N/2}^2+\sum_{p=1}^{N/2-1}a_p^2\frac{N}{2}+\sum_{p=1}^{N/2}b_p^2\frac{N}{2} \\ &= N\bar{x}^2+Na_{N/2}^2+\frac{N}{2}\sum_{p=1}^{N/2-1}R_p^2.\end{aligned}$$

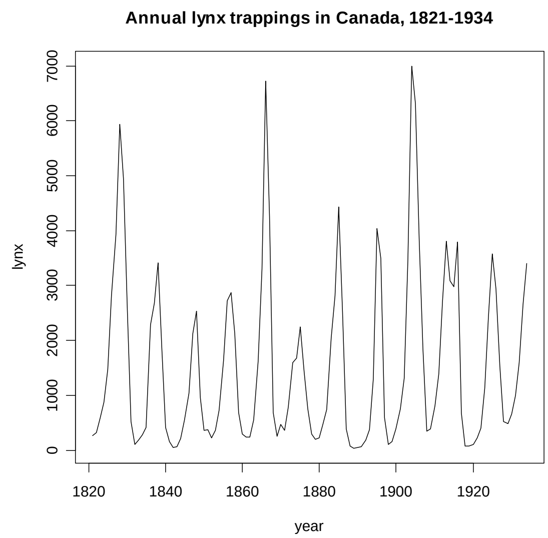
Consequently,

$$\begin{aligned}\sum_{t=1}^{N}x(t)^2-N\bar{x}^2 &= \sum_{t=1}^{N}(x(t)-\bar{x})^2 \\ &= Na_{N/2}^2+\frac{N}{2}\sum_{p=1}^{N/2-1}R_p^2\end{aligned}$$

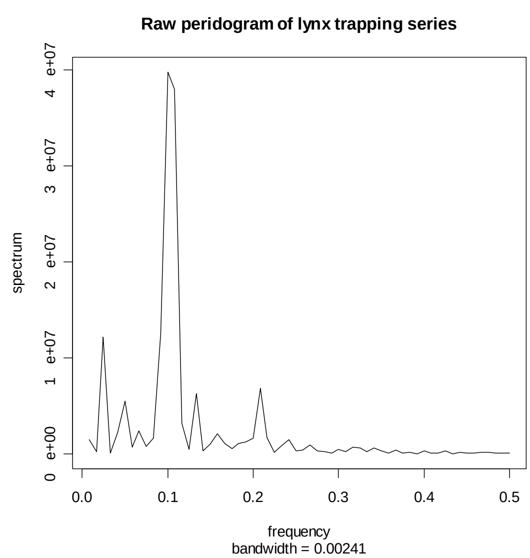
and the variance of the series (with divisor N rather than $N-1$) is

$$\frac{1}{N}\sum_{t=1}^{N}(x(t)-\bar{x})^2=a_{N/2}^2+\frac{1}{2}\sum_{p=1}^{N/2-1}R_p^2.$$

11. The plot is repeated below:



There is a clear cyclical component, established from the correlogram, with the peaks occurring roughly at ten–year intervals. This is emphasised by the raw (unlogged, unsmoothed) periodogram:



Remembering that R uses cycles per unit time (year, here), the frequency axis is

$$f = \frac{p}{114},$$

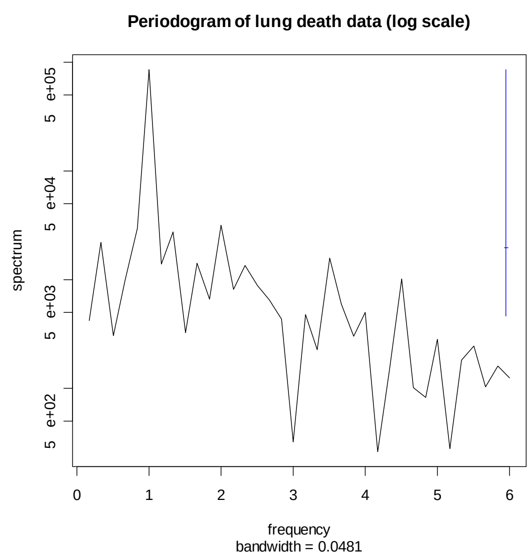
for $p = 1, 2, \ldots, 57$ (or rather would be, except R tapers the data and adds six zeroes). The peak apparent at $f = 0.1$ does of course correspond to a wavelength of 10 years, as the time plot suggests. To see the first 20 values of the spectrum and the corresponding frequencies, enter the following:

```
> lynxspec <- spec.pgram(lynx, log=''no'')

> lynxspec$freq[1:20]

> lynxspec$spec[1:20]
```

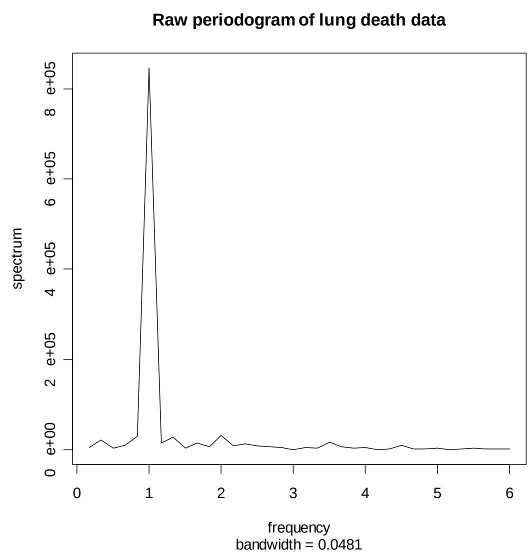
12. The plot of the time series shows clear cyclical variation, with peaks associated around each Winter, perhaps not surprisingly. R gives the periodogram on the log scale by default, as follows:



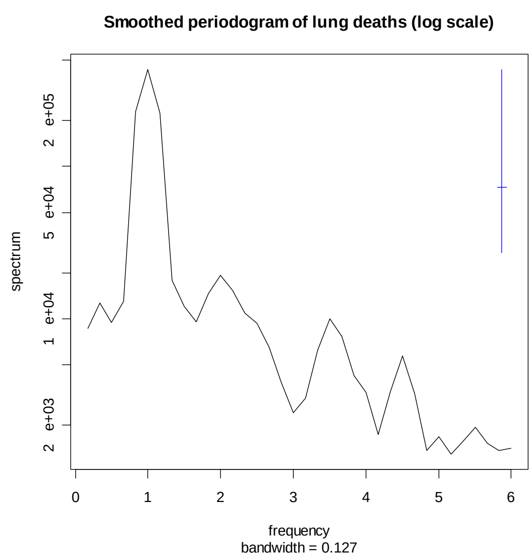
Note the sizeable peak at 1, which corresponds to a frequency of one cycle per year, as observed in the data. As usual, R defines frequency in units of cycles per unit time, so takes

$$f = \frac{\omega_p}{2\pi} \times 12$$

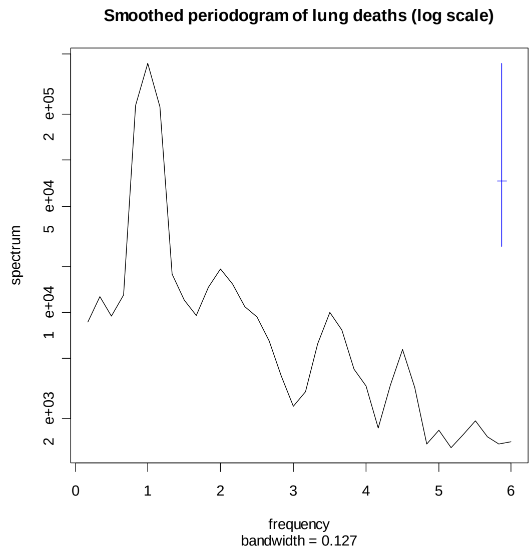
here, where $\omega_p = 2\pi p/N$ as usual, and the factor of 12 accounts for there being that number of observations per unit time (year). The peak corresponds to the sixth Fourier frequency. The cyclical effect is even more pronounced when the raw (un-logged) periodogram is plotted



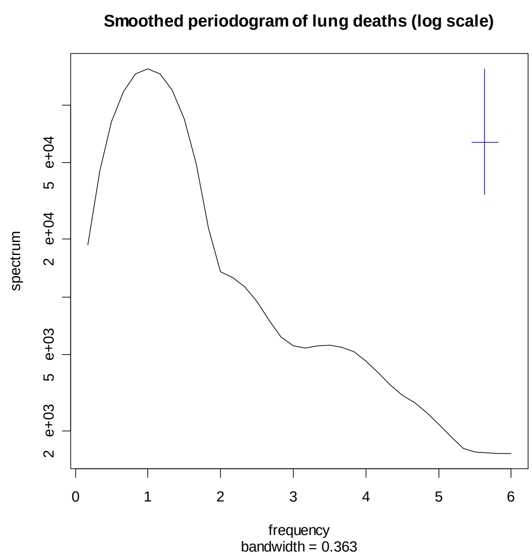
Smoothing with a 3–point filter as suggested, the (log) periodogram is shown below:



Smoothing this again, using a 5–point filter for the second run, we find:



Finally, as suggested, applying a 5-point filter followed by a 7–point filter, the periodogram is arguably over–smoothed:



The periodogram, in whatever format, confirms what could be said about the data directly from the time plot: there is a strong cyclic component in the data with cycles repeating approximately ever year. The second highest value of the periodogram occurs at the twelfth Fourier frequency, hinting at a component with a six–month cycle. This is rather unlikely – more feasible is that this second "peak" arises since the major effect is not purely sinusoidal in nature, and so constitutes a "harmonic" of a significant frequency.

13. Let

$$I(\omega) = \frac{1}{\pi}\left(c_0 + 2\sum_{k=1}^{N-1} c_k \cos(\omega k)\right)$$

define the periodogram as usual, and smooth this at the (Fourier) frequency ω with weights $\{w_j : j \in \{-p, \ldots, -1, 0, 1, \ldots, p\}\}$ such that $w_{-j} = w_j$ and $\sum_{j=-p}^{p} w_j = 1$. This gives

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\pi}\sum_{j=-p}^{p} w_j \left(c_0 + 2\sum_{k=1}^{N-1} c_k \cos\left(\left(\omega + \frac{2\pi j}{N}\right) k\right)\right) \\
&= \frac{1}{\pi}\left(c_0 + 2\sum_{k=1}^{N-1} c_k \left[w_0 \cos(\omega k)\right.\right. \\
&\qquad \left.\left. + \sum_{j=1}^{p} w_j \left(\cos\left(\omega k + \frac{2\pi jk}{N}\right) + \cos\left(\omega k - \frac{2\pi jk}{N}\right)\right)\right]\right).
\end{aligned}$$

Now using

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

we find

$$\hat{f}(\omega) = \frac{1}{\pi}\left(c_0 + 2\sum_{k=1}^{N-1} \lambda_k c_k \cos(\omega k)\right)$$

where

$$\lambda_k = w_0 + \sum_{j=1}^{p} w_j \cos\left(\frac{2\pi kj}{N}\right).$$

BD