Time Series Exercises 2: Solutions

1. .

   (i) This is obvious from the definition, which is symmetric between X and Y.

   (ii)

$$\begin{aligned}\mathrm{Cov}\,(aX,bY) &= E\,(abXY)-E\,(aX)\,E\,(bY)\\ &= abE\,(XY)-abE\,(X)\,E\,(Y)\\ &= ab\mathrm{Cov}\,(X,Y)\,.\end{aligned}$$

   (iii)

$$\begin{aligned}\mathrm{Cov}\,(X+Z,Y) &= E\,((X+Z)\,Y)-E\,(X+Z)\,E\,(Y)\\ &= E\,(XY)+E\,(ZY)-E\,(X)\,E\,(Y)-E\,(Z)\,E\,(Y)\\ &= E\,(XY)-E\,(X)\,E\,(Y)+E\,(ZY)-E\,(Z)\,E\,(Y)\\ &= \mathrm{Cov}\,(X,Y)+\mathrm{Cov}\,(Z,Y)\,.\end{aligned}$$

2. Now for all t and τ,

$$\begin{aligned}\mathrm{Var}\,(\alpha X\,(t)+\beta X\,(t+\tau)) &= \alpha^2\mathrm{Var}\,(X\,(t))+\beta^2\mathrm{Var}\,(X\,(t+\tau))\\ &\qquad +2\alpha\beta\mathrm{Cov}\,(X\,(t)\,,X\,(t+\tau))\\ &= \alpha^2\sigma^2+\beta^2\sigma^2+2\alpha\beta\mathrm{Cov}\,(X\,(t)\,,X\,(t+\tau))\\ &= \left(\alpha^2+\beta^2\right)\sigma^2+2\alpha\beta\gamma\,(\tau)\\ &\geq 0\end{aligned}$$

   where $\gamma\,(\cdot)$ is the acvf. as usual. If $\alpha=\beta=1$, the above implies

$$\gamma\,(\tau)\geq-\sigma^2,$$

   and so

$$\rho\,(\tau)=\frac{\gamma\,(\tau)}{\sigma^2}\geq-1.$$

   Similarly, taking $\alpha=1$, $\beta=-1$ implies that

$$\gamma\,(\tau)\leq\sigma^2$$

   and so $\rho\,(\tau)\leq1$.

3. Suppose $X(t)$ has mean μ, say. Then

$$\begin{aligned} E(Y(t)) &= E(X(t) - X(t-1)) \\ &= \mu - \mu \\ &= 0, \end{aligned}$$

and this is independent of t. Is the acvf also independent of t? Well,

$$\begin{aligned} \operatorname{Cov}(Y(t), Y(t-k)) &= \operatorname{Cov}(X(t) - X(t-1), X(t-k) - X(t-k-1)) \\ &= \operatorname{Cov}(X(t), X(t-k)) - \operatorname{Cov}(X(t-1), X(t-k)) \\ &\quad - \operatorname{Cov}(X(t), X(t-k-1)) + \operatorname{Cov}(X(t-1), X(t-k-1)) \\ &= \gamma(k) - \gamma(k-1) - \gamma(k+1) + \gamma(k) \\ &= 2\gamma(k) - \gamma(k-1) - \gamma(k+1) \end{aligned}$$

and this does not depend on time, just the lag k. Hence the process $Y(t)$ is stationary.

4. Here $X(t) = \theta(B) Z(t)$ where

$$\theta(B) = 1 - 0.7B + 0.2B^2.$$

The roots of $\theta(B)$ are

$$\frac{0.7 \pm \sqrt{-0.31}}{0.4} = \frac{0.7 \pm \sqrt{0.31}i}{0.4},$$

that is, $1.75 \pm 1.39i$. The moduli of these roots are equal, being

$$\sqrt{1.75^2 + 1.39^2} = 2.23 > 1.$$

Hence the process is invertible.

5. Here $X(t) = \theta(B) Z(t)$ where

$$\theta(B) = 1 - 0.1B - 1.16B^2 + 0.48B^3.$$

Now $\theta(2) = 1 = 0.2 - 4.64 + 3.84 = 0$, so 2 is a root of $\theta(B)$. Dividing $\theta(B)$ by $(B-2)$ leaves $0.48B^2 - 0.2B - 0.5$, which has roots 1.25 and –0.833. As the latter is inside the unit circle, the process is not invertible.

6. Now the process here can be written $Z(t) = \phi(B) X(t)$ where

$$\phi(B) = 1 - 0.9B - 0.4B^2,$$

the roots of which being

$$\frac{0.9 \pm \sqrt{0.81 + 1.6}}{-0.8}.$$

These are –3.065 and 0.816. Since the second root is inside the unit circle, the process is not stationary.

7. In this case $\phi(B) = 1 - 0.5B - 0.25B^2$, the roots of which are –3.236 and 1.236, and as both lie outside the unit circle the process is stationary. We can find the acf using the Yule–Walker equations, the solutions of which are of the form

$$\rho(k) = A_1 d_1^{|k|} + A_2 d_2^{|k|},$$

where d_1 and d_2 are the roots of

$$D^2 - \frac{1}{2}D - \frac{1}{4}.$$

Specifically, these are $\left(1+\sqrt{5}\right)/4$ and $\left(1-\sqrt{5}\right)/4$. Now as $\rho(0) = 1$, we must have $A_1 + A_2 = 1$. Furthermore, via the second Y–W equation, we see

$$\begin{aligned} \rho(1) &= \alpha_1 \rho(0) + \alpha_2 \rho(1) \\ &= A_1 \left(\frac{1+\sqrt{5}}{4}\right) + A_2 \left(\frac{1-\sqrt{5}}{4}\right) \\ &= \frac{\alpha_1}{1-\alpha_2} \\ &= \frac{2}{3}. \end{aligned}$$

Hence a second linear equation in A_1 and A_2 is

$$A_1 \left(1+\sqrt{5}\right) + A_2 \left(1-\sqrt{5}\right) = \frac{8}{3}.$$

From this, we deduce that $A_1 = \left(\sqrt{5}+3\right)/6$ and $A_2 = \left(3-\sqrt{5}\right)/6$, so the acf at lag k is

$$\rho(k) = \frac{\left(\sqrt{5}+3\right)}{6}\left(\frac{1+\sqrt{5}}{4}\right)^{|k|} + \frac{\left(3-\sqrt{5}\right)}{6}\left(\frac{1-\sqrt{5}}{4}\right)^{|k|}.$$

8. Note that here

$$\begin{aligned}\phi(B) &= 1+\alpha B - 6\alpha^2 B^2 \\ &= (1+3\alpha B)(1-2\alpha B),\end{aligned}$$

and so the roots are $-1/3\alpha$ and $1/2\alpha$. For stationarity we require $|3\alpha| < 1$ and $|2\alpha| < 1$ – therefore we need that $|\alpha| < 1/3$.

9. Let the white noise process $Z(t)$ have variance σ^2 as usual. For the ARMA process in question, observe that

$$\begin{aligned}E(X(t)Z(t-1)) &= E\left(\alpha X(t-1)Z(t-1) + Z(t)Z(t-1) + \beta Z(t-1)^2\right) \\ &= \alpha\sigma^2 + \beta\sigma^2\end{aligned}$$

and also

$$E(X(t)Z(t)) = \sigma^2.$$

The acvf at lag 0 is

$$\begin{aligned}\gamma(0) &= E\left(X(t)^2\right) - E(X(t))^2 \\ &= E((\alpha X(t-1)+Z(t)+\beta Z(t-1))(\alpha X(t-1)+Z(t)+\beta Z(t-1))) \\ &= \alpha^2 E\left(X(t-1)^2\right) + E\left(Z(t)^2\right) + 2\alpha\beta E(X(t-1)Z(t-1)) + \beta^2 E\left(Z(t-1)^2\right) \\ &= \alpha^2\gamma(0) + \sigma^2 + 2\alpha\beta\sigma^2 + \beta^2\sigma^2,\end{aligned}$$

and so

$$\left(1-\alpha^2\right)\gamma(0) = \sigma^2\left(1+\beta^2+2\alpha\beta\right),$$

from which we see

$$\gamma(0) = \frac{\sigma^2\left(1+\beta^2+2\alpha\beta\right)}{\left(1-\alpha^2\right)}.$$

The acvf at lag 1 is

$$\begin{aligned}
\gamma(1) &= \operatorname{Cov}(X(t), X(t-1)) \\
&= \operatorname{Cov}(\alpha X(t-1) + Z(t) + \beta Z(t-1), \\
&\qquad \alpha X(t-2) + Z(t-1) + \beta Z(t-2)) \\
&= \alpha^2 \gamma(1) + \alpha\sigma^2 + \alpha\beta(\alpha+\beta)\sigma^2 + \beta\sigma^2
\end{aligned}$$

where use has been made of the results given at the start and the properties of the covariance operator deduced in Q.1. Clearly then

$$\gamma(1) = \frac{\sigma^2(\alpha + \alpha\beta(\alpha+\beta) + \beta)}{(1-\alpha^2)}.$$

For $k = 2, 3, \ldots,$

$$\begin{aligned}
\gamma(k) &= \operatorname{Cov}(X(t), X(t-k)) \\
&= \operatorname{Cov}(\alpha X(t-1) + Z(t) + \beta Z(t-1), X(t-k)) \\
&= \alpha\gamma(k-1)
\end{aligned}$$

as for an AR(1) process. So the acf $\rho(k)$ is determined, dividing $\gamma(k)$ by $\gamma(0)$ for each k.

10. For our MA(1) here we know that

$$\begin{aligned}
\operatorname{Cov}(X(t), X(t-1)) &= \sigma^2\theta, \\
\operatorname{Var}(X(t)) &= \sigma^2\left(1+\theta^2\right)
\end{aligned}$$

for all t. Hence

$$\operatorname{Cov}((X(t) - \rho(1)X(t-1), X(t-2) - \rho(1)X(t-1)))$$

is

$$\begin{aligned}
&\operatorname{Cov}(X(t), X(t-2)) - \rho(1)\operatorname{Cov}(X(t), X(t-1)) \\
&\quad -\rho(1)\operatorname{Cov}(X(t-1), X(t-2)) + \rho(1)^2 \operatorname{Var}(X(t-1))
\end{aligned}$$

which by the results above simplifies to

$$-\rho(1)\sigma^2\left(2\theta - \rho(1)\left(1+\theta^2\right)\right) = -\frac{\theta^2\sigma^2}{\left(1+\theta^2\right)}$$

recalling that

$$\rho(1) = \frac{\theta}{1+\theta^2}.$$

Now the denominator of the required expression, $\mathrm{Var}(X(t) - \rho(1) X(t-1))$, is

$$\mathrm{Var}(X(t)) + \rho(1)^2 \mathrm{Var}(X(t-1)) - 2\rho(1) \mathrm{Cov}(X(t), X(t-1))$$

and this is

$$\begin{aligned} \sigma^2\left(1+\theta^2\right) + \rho(1)^2 \sigma^2\left(1+\theta^2\right) - 2\rho(1)\sigma^2\theta &= \sigma^2\left(1+\theta^2+\frac{\theta^2}{1+\theta^2}-\frac{2\theta^2}{1+\theta^2}\right) \\ &= \sigma^2\left(\frac{1+\theta^2+\theta^4}{1+\theta^2}\right) \end{aligned}$$

after a little algebra. Dividing the two expressions gives the required result.

11. Now the process can be written

$$\phi(B) X(t) = \theta(B) Z(t)$$

where

$$\phi(B) = 1 - 0.7B$$

and

$$\theta(B) = 1 - 0.1B.$$

Both these polynomials have a single root, in each case outside the unit circle, so the process is both stationary and invertible. To write $X(t)$ as a pure MA we find

$$\begin{aligned} \psi(B) &= \theta(B)\phi(B)^{-1} \\ &= (1-0.1B)(1-0.7B)^{-1} \\ &= (1-0.1B)\left(1+0.7B+0.7^2B^2+0.7^3B^3+\cdots\right) \\ &= 1+0.6B+\left(-0.1\times0.7+0.7^2\right)B^2+\left(-0.1\times0.7^2+0.7^3\right)B^3+\cdots \end{aligned}$$

and so $\psi_0 = 1$, and for $j = 1, 2, \ldots$

$$\begin{aligned} \psi_j &= -0.1\times0.7^{j-1}+0.7^j \\ &= 0.6\times0.7^{j-1}. \end{aligned}$$

Hence as an MA we can write

$$X(t) = Z(t) + 0.6 \sum_{j=1}^{\infty} 0.7^{j-1} Z(t-j).$$

Now to find $X(t)$ as a pure AR we require

$$\begin{aligned}
\pi(B) &= \phi(B)\,\theta(B)^{-1} \\
&= (1-0.7B)\left(1+0.1B+0.1^2B^2+0.1^3B^3+\cdots\right) \\
&= 1-0.6B+\left(-0.7\times 0.1+0.1^2\right)B^2+\left(-0.7\times 0.1^2+0.1^3\right)B^3+\cdots \\
&\qquad +\left(-0.7\times 0.1^{j-1}+0.1^j\right)B^j+\cdots
\end{aligned}$$

and so $\pi_0 = 1$ and for $j = 1, 2, \ldots,$

$$\pi_j = -0.6 \times 0.1^{j-1}.$$

Hence

$$\begin{aligned}
X(t) &= \sum_{j=1}^{\infty} \pi_j B^j X(t) + Z(t) \\
&= -0.6 \sum_{j=1}^{\infty} 0.1^{j-1} X(t-j) + Z(t).
\end{aligned}$$

12. Here we have $p = D = Q = 0$ and $d = q = P = 1$, with $s = 12$, and so

$$W(t) = \nabla X(t) = X(t) - X(t-1).$$

The left hand side of the model is

$$\begin{aligned}
\Phi\left(B^{12}\right)W(t) &= \left(1-\alpha B^{12}\right)\left(X(t)-X(t-1)\right) \\
&= X(t)-X(t-1)-\alpha B^{12}X(t)+\alpha B^{12}X(t-1) \\
&= X(t)-X(t-1)-\alpha X(t-12)+\alpha X(t-13)
\end{aligned}$$

for some α. The right hand side of the model is

$$(1+\beta B)\,Z(t) = Z(t) + \beta Z(t-1)$$

for some β. Therefore the model can be written

$$X(t) = X(t-1) + \alpha X(t-12) - \alpha X(t-13) + Z(t) + \beta Z(t-1).$$

BD