Activity Solution: MA Representations

This activity should help you appreciate how to obtain the MA representation of an ARMA model, and also why that representation can be useful in forecasting. We will consider the model

$$X(t) = 0.7X(t-1) + Z(t) - 0.3Z(t-1)$$

where $Z(t)$ is white noise with known variance 0.4.

1. *Clickers question:* Express $X(t)$ as a pure MA process. That is, write $X(t)$ as
$$\left(1 + \psi_1 B + \psi_2 B^2 + \cdots\right) Z(t)$$
for some constants $\psi_1, \psi_2 \ldots$. (Hint: Recall that $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$, for $|x| < 1$.)

$$\begin{aligned} X(t) &= (1-0.3B)(1-0.7B)^{-1} Z(t) \\ &= (1-0.3B)\left(1 + 0.7B + 0.7^2B^2 + 0.7^3B^3 + \cdots\right) Z(t) \\ &= \left(1 + 0.4B + \left(0.7^2 - 0.21\right)B^2 + \left(0.7^3 - 0.3 \times 0.7^2\right)B^3 + \cdots\right) Z(t) \\ &= \left(1 + 0.4B + 0.28B^2 + 0.196B^3 + \cdots\right) Z(t) \end{aligned}$$

2. *Clickers question:* Write down a general formula for ψ_j in this case.

$$\begin{aligned} \psi_j &= 0.7^j - 0.3 \times 0.7^{j-1} \\ &= 0.7^{j-1}(0.7 - 0.3) \\ &= 0.4 \times 0.7^{j-1}. \end{aligned}$$

3. We observe data up to time N. Suppose we are interested in forecasting the value $X(N+3)$ based on values up to time N. What would be natural unbiased estimates of $Z(N+1)$, $Z(N+2)$, and $Z(N+3)$? *The most natural unbiased estimator for $Z(t)$ when $t > N$ is zero, for $Z(t)$ has mean zero.*

4. Write down an expression for the *forecast error* at lead time 3, $X(N+3) - \hat{x}(N,3)$.

$$X(N+3) - \hat{x}(N,3) = Z(N+3) + 0.4Z(N+2) + 0.4 \times 0.7Z(N+1).$$

5. *Clickers question:* Denoting the forecast error in the previous part by $e(N, 3)$, find $E(e(N, 3))$ and $\mathrm{Var}(e(N, 3))$.
Since $E(Z(t)) = 0$ then clearly $E(e(N, 3)) = 0$. Moreover

$$\begin{aligned} \mathrm{Var}(e(N, 3)) &= 0.4\left(1 + 0.4^2 + 0.28^2\right) \\ &= 0.4954. \end{aligned}$$

6. Assume now that $Z(t) \sim N(0, 0.4)$ for all t. Hence write down an approximate 95% confidence interval for your forecast of $X(N + 3)$. (More usually this would be referred to as a *prediction interval.)*
Writing the forecast as $\hat{x}(N, 3)$ we will have a 95% prediction interval for $X(N + 3)$ to be

$$\hat{x}(N, 3) \pm 1.96\sqrt{0.495}.$$

7. *Clickers question:* As the lead time l increases, what happens to the prediction interval for $X(N + l)$?
As l increases the prediction interval grows wider.

8. Re-cap what you have done during this activity. Why do you think you were asked to do this activity? What have you learned?
We have reviewed how to convert a stationary ARMA model to an MA(∞) process, finding the general form for the coefficients on the MA process. This MA process can then be used for forecasting and provides a basis for the construction of prediction intervals under the assumption that the white noise process underlying the MA process is Normal. We have seen that as the lead time increases the prediction interval will increase in width.
Learning outcomes encountered in this activity include:

   (a) *Express an ARMA(p, q) model as a pure MA process.*
   (b) *Describe and implement the Box–Jenkins approach to forecasting.*
   (c) *Compute prediction intervals for a Box–Jenkins forecast.*
   (d) *Describe the behaviour of Box–Jenkins forecasts as the lead time increases.*