Activity Solution: Moving Average Processes

The i.i.d. sequence $Z(t)$ has mean zero and variance σ^2. Recall we defined a stochastic process $X(t)$ by

$$X(t) = Z(t) + 0.3Z(t-1) + 0.2Z(t-2) + 0.1Z(t-3).$$

1. The model $X(t)$ is an MA(q) for which value of q?
   3

2. *Clickers question:* Find $\mathrm{Cov}(X(t), X(t+2))$.

$$\begin{aligned} \mathrm{Cov}(X(t), X(t+2)) &= \mathrm{Cov}(Z(t) + 0.3Z(t-1) + 0.2Z(t-2) + 0.1Z(t-3), \\ &\quad Z(t+2) + 0.3Z(t+1) + 0.2Z(t) + 0.1Z(t-1)) \\ &= \sigma^2 (0.2 + 0.3 \times 0.1) \\ &= 0.23\sigma^2. \end{aligned}$$

3. *Clickers question:* Find $\mathrm{Cov}(X(t), X(t+3))$.

$$\begin{aligned} \mathrm{Cov}(X(t), X(t+3)) &= \mathrm{Cov}(Z(t) + 0.3Z(t-1) + 0.2Z(t-2) + 0.1Z(t-3), \\ &\quad Z(t+3) + 0.3Z(t+2) + 0.2Z(t+1) + 0.1Z(t)) \\ &= 0.1\sigma^2. \end{aligned}$$

4. What is $\mathrm{Cov}(X(t), X(t+k))$ when $k > 3$?
   0

5. Is $X(t)$ stationary?
   Yes!

6. *Clickers question:* What is $\rho(k)$ for $X(t)$?

$$\begin{aligned} \rho(k) &= \frac{\gamma(k)}{\gamma(0)} \\ &= \begin{cases} 1 & k = 0 \\ 0.33 & k = \pm 1 \\ 0.20 & k = \pm 2 \\ 0.088 & k = \pm 3 \\ 0 & |k| > 3. \end{cases} \end{aligned}$$

7. Suppose $E\left(Z\left(t\right)\right)=1$ in the above. What, if anything, would change?
Only $E\left(X\left(t\right)\right)$ *would change, this being then*

$$\begin{aligned} E\left(X\left(t\right)\right) &= 1+0.3+0.2+0.1 \\ &= 1.6. \end{aligned}$$

8. Re-cap what you have done in the above activity. What do the results suggest about the behaviour of the acf for an MA(q)?
We have explored the autocorrelation structure of a moving average process of order 3. We saw that the autocorrelation function (acf) "cut off" at lag 3, in that for lags greater than three in absolute value the acf is zero, while for other lags the acf is non-zero. This suggests a general property of moving average processes.
Learning outcomes encountered in this activity are:

   (a) *Identify a moving average process of order p, i.e., an MA(q).*

   (b) *Derive the mean, variance, and autocovariance function of an MA(q) process.*