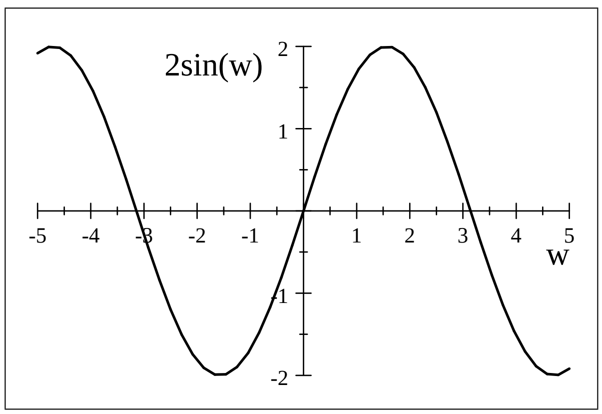
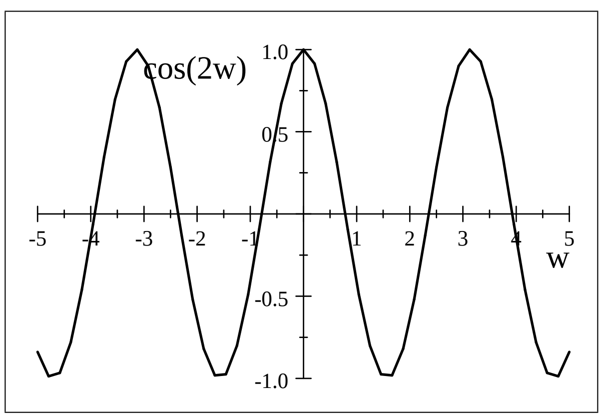
Activity Solution: Trigonometry Revision, Fourier Transfroms

This activity helps refresh you on properties of trigonometric functions, notably sin and cos. Recall that for any angles A and B we have the following:

$$\begin{aligned}
\sin^2(A) + \cos^2(A) &= 1, \\
\sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B), \\
\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B).
\end{aligned}$$

1. Sketch the functions $2\sin(\omega)$ and $\cos(2\omega)$ for $\omega \in (-2\pi, 2\pi)$.





2. *Clickers question:* Find expressions for $\sin(A-B)$ and $\cos(A-B)$. *From the above, since* $\sin(x) = -\sin(x)$ *and* $\cos(-x) = \cos(x)$ *for all*

x we have

$$\begin{aligned}
\sin(A-B) &= \sin(A)\cos(-B)+\cos(A)\sin(-B)\\
&= \sin(A)\cos(B)-\cos(A)\sin(B),\\
\cos(A-B) &= \cos(A)\cos(-B)-\sin(A)\sin(-B)\\
&= \cos(A)\cos(B)+\sin(A)\sin(B)
\end{aligned}$$

3. *Clickers question:* Using the results from above, find expressions for

$$\sin(A+B)+\sin(A-B)$$

and $\cos(A+B)+\cos(A-B)$.
By adding the formulae for sin$(A+B)$ and sin$(A-B)$ we find

$$\sin(A+B)+\sin(A-B)=2\sin(A)\cos(B)$$

and similarly

$$\cos(A+B)+\cos(A-B)=2\cos(A)\cos(B).$$

4. *Clickers question:* For $t\in\mathbb{Q}^+$, find when

$$2\cos\left(\pi t+\frac{\pi}{4}\right)$$

attains its maximum value, giving your answer in terms of $k\in\mathbb{Z}^+$.
Now $\cos(x)$ *attains its maximum value of +1 when* $x=2k\pi$ *for integer* k*. Here we require*

$$\pi t+\frac{\pi}{4}=2k\pi$$

which occurs when

$$4t+1=8k,$$

and rearranging this gives $t=(8k-1)/4$.

5. Explain for when $t\in\mathbb{Z}$ why

$$\cos((\omega+k\pi)t)=\cos(\omega t)$$

when k is even.
For $k=2,4,6,\ldots,$

$$\cos(x+k\pi)=\cos(x)$$

for all x *since cos is periodic with period* 2π.

6. Explain clearly with a sketch why

$$\cos\left(\frac{\pi}{3}+\pi\right)=\cos\left(\pi-\frac{\pi}{3}\right).$$

Since cos is symmetric about π,

$$\cos\left(\frac{4\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right).$$

7. Show that for $\omega \in (0,\pi)$, when $k \in \mathbb{Z}$ is odd

$$\cos(\omega+k\pi)=\cos(\pi-\omega).$$

Now

$$\begin{aligned}\cos(\omega+k\pi) &= \cos(\omega)\cos(k\pi)-\sin(\omega)\sin(k\pi)\\ &= -\cos(\omega)\end{aligned}$$

since $\sin(k\pi)=0$. *Further*

$$\begin{aligned}\cos(\pi-\omega) &= \cos(\pi)\cos(\omega)+\sin(\pi)\sin(\omega)\\ &= -\cos(\omega).\end{aligned}$$

8. More generally, show that for when $k,t \in \mathbb{Z}$, k odd,

$$\cos((\omega+k\pi)t)=\cos((\pi-\omega)t).$$

The proof is essentially as above, since $\sin(\pi t)=0$ *for all* t *and*

$$\begin{aligned}\cos(\omega t+k\pi t) &= \cos(\omega t)\cos(kt\pi)-\sin(\omega t)\sin(kt\pi)\\ &= -\cos(\omega t)\cos(kt\pi)\end{aligned}$$

and

$$\begin{aligned}\cos(\pi t-\omega t) &= \cos(\pi t)\cos(\omega t)+\sin(\pi t)\sin(\omega t)\\ &= -\cos(t\pi)\cos(\omega t).\end{aligned}$$

When t *is even,* kt *is even and* $\cos(kt\pi)=\cos(t\pi)=1$. *Otherwise* $\cos(kt\pi)=\cos(t\pi)=-1$.

We define the *Fourier transform* (FT) of the function $h(t)$ of the real variable t to be

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-\mathbf{i}\omega t} dt.$$

This transform is finite if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty.$$

The *inverse Fourier transform* (inv. FT) is given by

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, e^{\mathbf{i}\omega t} d\omega.$$

The functions $H(\omega)$ and $h(t)$ are referred to as a *Fourier transform pair.*

In this activity, you may find the following identities useful:

$$\begin{aligned} e^{\mathbf{i}\omega} &= \cos(\omega) + \mathbf{i}\sin(\omega), \\ \cos(\omega) &= \frac{e^{\mathbf{i}\omega} + e^{-\mathbf{i}\omega}}{2}. \end{aligned}$$

1. *Clickers question:* If $h(t)$ is *even*, in that

$$h(t) = h(-t)$$

for all t, and real–valued, finds its FT.

For even functions the FT is often defined as

$$\begin{aligned} H(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} h(t)\, e^{-\mathbf{i}\omega t} dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} h(t) \cos(\omega t)\, dt \\ &\qquad -\mathbf{i}\frac{1}{\pi} \int_{-\infty}^{\infty} h(t) \sin(\omega t)\, dt \\ &= \frac{2}{\pi} \int_{0}^{\infty} h(t) \cos(\omega t)\, dt, \end{aligned}$$

recalling that $\cos(x)$ *is even and* $\sin(x)$ *is odd, in that*

$$\sin(x) = -\sin(-x)$$

for all x*. Now* $H(\omega)$ *is a real–valued, even function of* ω*, a so–called* **cosine transform**.

2. What is the form of the inverse FT in such cases?
The inverse FT is

$$h(t) = \frac{1}{2}\int_{-\infty}^{\infty} H(\omega)\, e^{\mathbf{i}\omega t} d\omega$$
$$= \int_{0}^{\infty} H(\omega)\cos(\omega t)\, d\omega.$$

3. *Clickers question:* When $h(t)$ is both even and defined only for integer t, what is the form of $H(\omega)$ for $\omega \in [0, \pi]$?
We combine the results for $h(t)$ being even and defined on the integers to give the result that $H(\omega)$ is proportional to

$$h(0) + 2\sum_{t=1}^{\infty} h(t)\cos(\omega t)$$

for $\omega \in [0, \pi]$.

4. What is the inv. FT in such a situation?
The inv. FT is
$$h(t) = \int_{0}^{\pi} H(\omega)\cos(\omega t)\, d\omega.$$