Activity Solution: Cross-Correlation

This activity aims to help you understand *cross-correlation*, as applied to two stochastic processes.

For a bivariate stochastic process $\{X(t), Y(t)\}$ we define the *cross–covariance function* at lag k to be

$$\begin{aligned}\gamma_{XY}(k) &= \operatorname{Cov}(X(t), Y(t+k)) \\ &= E((X(t)-\mu_X)(Y(t+k)-\mu_Y)).\end{aligned}$$

The *cross–correlation* at lag k is defined as

$$\rho_{XY}(k) = \frac{\gamma_{XY}(k)}{\sqrt{\gamma_X(0)\gamma_Y(0)}} = \frac{\gamma_{XY}(k)}{\sigma_X \sigma_Y}.$$

We look at a particular example here. Let

$$X(t) = \alpha X(t-1) + Z_1(t)$$

where $|\alpha| < 1$ and $Z_1(t)$ is white noise with mean 0 and variance σ_1^2, and for some integer j let

$$Y(t) = \beta X(t+j) + Z_2(t)$$

where $|\beta| < 1$ and $Z_2(t)$ is white noise with mean 0 and variance σ_2^2, independent of $X(t)$ and $Z_1(t)$. Recall that since $X(t)$ is an AR(1) we have

$$\gamma_X(k) = \frac{\sigma_1^2 \alpha^{|k|}}{(1-\alpha^2)}.$$

1. *Clickers question:* Find $\gamma_Y(0)$.
   The variance of $Y(t)$ is

$$\begin{aligned}\gamma_Y(0) &= E\left(Y(t)^2\right) \\ &= \beta^2 E\left(X(t+j)^2\right) + E\left(Z_2(t)^2\right) \\ &= \frac{\beta^2\sigma_1^2}{(1-\alpha^2)} + \sigma_2^2 \\ &= \beta^2\gamma_X(0) + \sigma_2^2.\end{aligned}$$

2. *Clickers question:* Find $\gamma_Y(k)$ in terms of $\gamma_X(k)$ when $k \neq 0$.
In general

$$\begin{aligned}
\gamma_Y(k) &= E(Y(t)Y(t+k)) \\
&= E((\beta X(t+j) + Z_2(t))(\beta X(t+j+k) + Z_2(t+k))) \\
&= \beta^2 E(X(t+j)X(t+j+k)) \\
&= \beta^2 \gamma_X(k).
\end{aligned}$$

3. *Clickers question:* Find $\gamma_{XY}(k)$ in terms of $\gamma_X(k)$. Hence find $\rho_{XY}(k)$. For what value of k is $\rho_{XY}(k)$ largest?
The cross–covariance is

$$\begin{aligned}
\gamma_{XY}(k) &= E(X(t)Y(t+k)) \\
&= E(X(t)(\beta X(t+j+k) + Z_2(t+k))) \\
&= \beta\gamma_X(k+j)
\end{aligned}$$

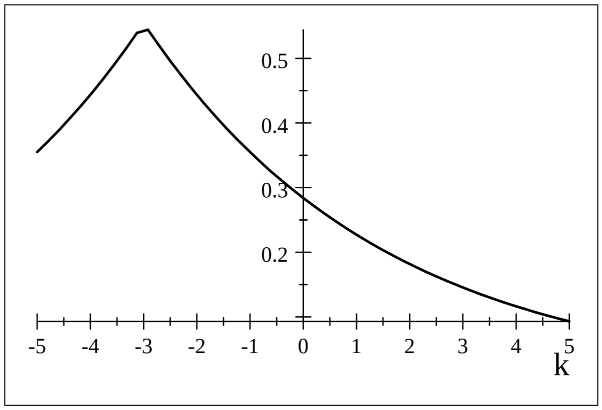
and the cross–correlation is

$$\begin{aligned}
\rho_{XY}(k) &= \frac{\beta\gamma_X(k+j)}{\left(\gamma_X(0)\left(\beta^2\gamma_X(0) + \sigma_2^2\right)\right)^{\frac{1}{2}}} \\
&= \frac{\gamma_X(k+j)}{\left(\gamma_X(0)\left(\gamma_X(0) + \sigma_2^2/\beta^2\right)\right)^{\frac{1}{2}}} \\
&= \frac{\alpha^{|k+j|}}{\left(1 + \sigma_2^2\left(1-\alpha^2\right)/\left(\beta^2\sigma_1^2\right)\right)^{\frac{1}{2}}}
\end{aligned}$$

which is maximized when $k = -j$.

4. *Clickers question:* Sketch the case with $\alpha = 0.8$, $\beta = 0.4$, $\sigma_1 = \sigma_2 = 1$

and $j = 3$.



$\rho_{XY}(k)$ with $\alpha = 0.8$, $\beta = 0.4$,
$\sigma_1 = \sigma_2 = 1$, $j = 3$

5. Re-cap what you have done during this activity. What were you asked to do? Why were you asked to do this? What did you learn?
   The cross–correlation function has been explored in the special case of two interdependent stochastic processes, one of which was an $AR(1)$. *The cross–covariance and cross–correlation functions were computed in this special case, and properties of the cross–correlation explored graphically.*
   Learning outcomes encountered in the activity include:

   (a) *Define what are meant by the terms cross–covariance and cross–correlation.*

   (b) *Describe the properties of the cross–correlation function.*

   (c) *Explore the properties of a cross–correlation function in tractable special cases.*