Activity Solution: Yule–Walker Equations

The i.i.d. sequence $Z(t)$ has mean zero and variance σ^2. Suppose we define the stochastic process $X(t)$ by

$$X(t) = 1.30X(t-1) - 0.22X(t-2) - 0.10X(t-3) + Z(t).$$

Assume that this process is stationary.

1. Multiply both sides by $X(t-k)$, for $k = 1, 2, \ldots, p$, where $p = 3$ in this case.
   Take the defining equation and multiply both sides by $X(t-k)$ *for* $k = 1, 2, 3,$ *giving:*

$$\begin{aligned}
X(t)X(t-1) &= 1.30X(t-1)^2 - 0.22X(t-2)X(t-1) \\
&\quad -0.10X(t-3)X(t-1) + Z(t)X(t-1) \\
X(t)X(t-2) &= 1.30X(t-1)X(t-2) - 0.22X(t-2)^2 \\
&\quad -0.10X(t-3)X(t-2) + Z(t)X(t-2) \\
X(t)X(t-3) &= 1.30X(t-1)X(t-3) - 0.22X(t-2)X(t-3) \\
&\quad -0.10X(t-3)^2 + Z(t)X(t-3).
\end{aligned}$$

2. Take expectations on both sides of each equation. Then divide both sides of each equation by σ_X^2, the variance of the process $X(t)$. What is the left hand side of the first equation? What about the second and third equations?
   Now taking expectations on both sides and dividing by σ_X^2 *we have*

$$\begin{aligned}
\rho(1) &= 1.30\rho(0) - 0.22\rho(1) - 0.10\rho(2) \\
\rho(2) &= 1.30\rho(1) - 0.22\rho(0) - 0.10\rho(1) \\
\rho(3) &= 1.30\rho(2) - 0.22\rho(1) - 0.10\rho(0)
\end{aligned}$$

   since $E(X(t)X(t-k))/\sigma_X^2 = \rho(k)$ *here. Since* $\rho(0) = 1$ *we have*

$$\begin{aligned}
\rho(1) &= 1.30 - 0.22\rho(1) - 0.10\rho(2) \\
\rho(2) &= 1.30\rho(1) - 0.22 - 0.10\rho(1) \\
\rho(3) &= 1.30\rho(2) - 0.22\rho(1) - 0.10.
\end{aligned}$$

3. *Clickers question:* How many equations do you have written down, in how many unknowns?
There are three linear equations in three unknowns.

4. *Clickers question:* Find $\rho(1)$ and $\rho(2)$ for $X(t)$. Comment on these values.
Since

$$\begin{aligned} 1.22\rho(1) &= 1.30 - 0.1\rho(2), \\ \rho(2) &= 1.20\rho(1) - 0.22 \end{aligned}$$

we have

$$1.22\rho(1) = 1.3 - 0.1(1.20\rho(1) - 0.22)$$

giving that $\rho(1) = 0.986$ *and therefore* $\rho(2) = 0.964$. *Note these indicate very high correlations at lags 1 and 2, indicating strong short–term dependency. We have* $\rho(3) = 0.936$.

5. The general solution of the Yule–Walker equations is of the form $A_1 d_1^{|k|} + \cdots + A_p d_p^{|k|}$ where $d_1, \ldots, d_p$ are the roots of the polynomial in D

$$D^p - \alpha_1 D^{p-1} - \cdots - \alpha_p D^0 = 0$$

and the A_i's are constants subject to the constraint

$$\sum_{i=1}^{p} A_i = 1.$$

Given that the roots of

$$D^3 - 1.3D^2 + 0.22D + 0.1 = 0$$

are $d_1 = -0.1953$, $d_2 = 0.53097$ and $d_3 = 0.96433$ (you might like to check these), write down the general solution of the Yule–Walker equations for $X(t)$. Working to 3 s.f. will suffice.

General solution must be of the form

$$\rho(k) = A_1(-0.1953)^{|k|} + A_2(0.53097)^{|k|} + A_3(0.96433)^{|k|}$$

where $A_1 + A_2 + A_3 = 1$ *since* $\rho(0) = 1$. *When* $k = 1$ *we have*

$$\begin{aligned} 0.986 &= A_1(-0.195) + A_2(0.531) + (1 - A_1 - A_2)(0.964) \\ 0.964 &= A_1(-0.195)^2 + A_2(0.531)^2 + (1 - A_1 - A_2)(0.964)^2. \end{aligned}$$

So

$$\begin{aligned} 1.159A_1 &= -0.433A_2 - 0.022, \\ -0.891A_1 &= -0.647A_2 - 0.0347 \end{aligned}$$

Hence $A_1 = -0.374A_2 - 0.0189$ *and*

$$-0.333A_2 - 0.0169 = -0.647A_2 - 0.0347$$

so

$$0.314A_2 = -0.0178$$

hence $A_2 = -0.0567$. *Therefore* $A_1 = -0.374 \times (-0.0567) - 0.0189 = 2.3058 \times 10^{-3} = 0.00231$. *Then* $A_3 = 1.0543$, *and the general solution is*

$$\rho(k) = 0.00231(-0.1953)^{|k|} - 0.0567(0.53097)^{|k|} + 1.0543(0.96433)^{|k|}.$$

6. Hence find $\rho(1)$ and $\rho(2)$. Check the values are what you found earlier.
   We have

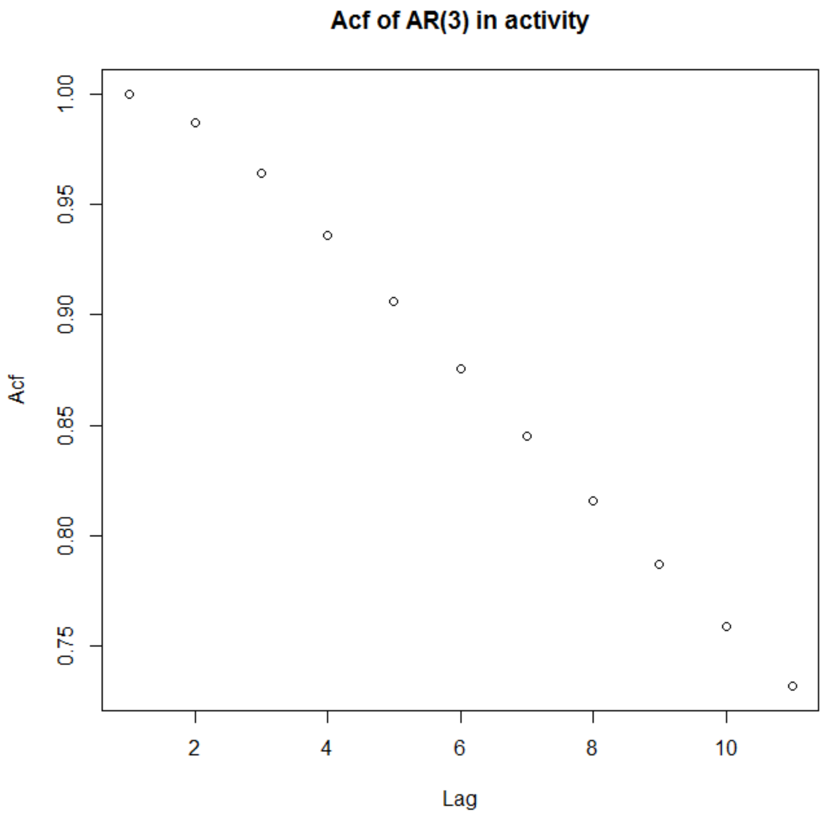
$$\begin{aligned} \rho(1) &= 0.00231(-0.1953) - 0.0567(0.53097) + 1.0543(0.96433) \\ &= 0.986 \end{aligned}$$

   as before. Further,

$$\begin{aligned} \rho(2) &= 0.00231(-0.1953)^2 - 0.0567(0.53097)^2 + 1.0543(0.96433)^2 \\ &= 0.964. \end{aligned}$$

7. If you have time, plot the acf of $X(t)$ at lags $k = 0, 1, 2, \ldots, 10$. Comment on what you observe.
   We have $\rho(3) = 0.936$ *and so on. There is a slow decay in the acf here,*

indicating strong long-term dependency.



8. Re-cap what you have done during this activity. Why do you think you were asked to do this activity? What have you learned?
   We derived a set of equations relating to a stationary $AR(3)$ process. The equations involve the acf of the process up to lag 3, but also have a general solution for arbitrary lag. These equations can be solved to find the acf of a stationary AR process exactly. This can be useful since if we had data we believed fitted an AR model, the sample acf and the theoretical acf could be compared.
   Learning outcomes met in this activity include:

   (a) *Derive the Yule–Walker equations for an $AR(p)$ process.*

   (b) *Recall the general solution to the Yule–Walker equations, and solve these equations where computationally feasible without the aid of* *a computer.*