Exercises 4

In the following the notation is as given in the lectures. Note that questions 8 and 9 provide a way to establish the orthogonality results (a) – (d) given in class, and are purely exercises relating to trigonometric functions.

1. Obtain the Fourier transform for the function
$$f(x) = \begin{cases} 1 & \text{if } |x| \leq c \\ 0 & \text{if } |x| > c, \end{cases}$$
for $c \geq 0$ a real constant.

2. Show that the Fourier transform of
$$h(t) = \left(-\frac{1}{3}\right)^{|t|}$$
for $t \in \mathbb{Z}$ can be expressed as
$$H(\omega) = \frac{8}{\pi\left(10 + 6\cos(\omega)\right)}.$$

3. Show that the spectral density function for the AR(1) model
$$X(t) = \alpha X(t-1) + Z(t)$$
is
$$f(\omega) = \frac{\sigma^2}{\pi\left(1 - 2\alpha\cos(\omega) + \alpha^2\right)}$$
where σ^2 is the variance of $Z(t)$. Write down the normalised spectrum in this case.

4. Derive the spectral density functions for the following MA processes:

   (i) $X(t) = Z(t) + Z(t-1) + 0.2Z(t-2)$;

   (ii) $X(t) = Z(t) + 0.8Z(t-1) - 0.3Z(t-2)$.

5. Find the spectral density for the mixed model
$$X(t) = \alpha X(t-1) + Z(t) + \beta Z(t-1).$$

6. Find the Fourier representation of the function $f(x) = x^2$, for $0 < x < 2\pi$, if $f(x)$ is periodic with period 2π. Hence deduce a series expansion for $\pi^2/6$.

7. For a random process $X(t)$ with acvf. $\gamma(k)$, define the *autocovariance generating function* to be

$$G(z) := \sum_{k=-\infty}^{\infty} \gamma(k) z^k$$

for a complex variable z. Express the spectrum of $X(t)$ in terms of $G(z)$.

8. For $N = 1, 2, \ldots,$ let

$$S_N(\omega) := \frac{\sin\left(\frac{N\omega}{2}\right)}{N\sin\left(\frac{\omega}{2}\right)},$$

and let $S_N(0) = S_N(2\pi) = 1$. Write down $S_N(\omega)$ for $N = 1, 2, 3$ and 4 in terms of cos functions, and observe that for $N > 1$,

$$S_N\left(\frac{2\pi j}{N}\right) = 0$$

for $j = \pm 1, \pm 2, \ldots, \pm(N-1)$. Hence sketch $S_N(\omega)$.

9. Expresss the summation $\sum_{t=1}^{N} e^{\mathbf{i}\omega t}$ in terms of $S_N(\omega)$ defined above. Write the following in terms of $S_N(\cdot)$:

$$\sum_{t=1}^{N} \cos(\omega t),$$

$$\sum_{t=1}^{N} \sin(\omega t),$$

$$\sum_{t=1}^{N} \cos(\omega_1 t)\cos(\omega_2 t),$$

$$\sum_{t=1}^{N} \sin(\omega_1 t)\sin(\omega_2 t),$$

$$\sum_{t=1}^{N} \sin(\omega_1 t)\cos(\omega_2 t),$$

where $\omega_1 \neq \omega_2$. Hence derive the orthogonality relations (a) – (d) stated in lectures.

10. Prove Parseval's theorem, which states that for a finite series $x(1), \ldots, x(N)$,

$$\frac{1}{N}\sum_{t=1}^{N}(x(t)-\bar{x})^2 = \frac{1}{2}\sum_{p=1}^{N/2-1} R_p^2 + a_{N/2}^2,$$

where

$$R_p = \left(a_p^2 + b_p^2\right)^{\frac{1}{2}}$$

and $\{a_p, b_p : p = 1, \ldots, N/2\}$ are the usual Fourier coefficients for the series.

11. Recall the `lynx` dataset in R gives the annual number of lynx trappings in Canada, 1821–1934. Plot the series, remark on its key features and attempt to estimate "by eye" the wavelength of any periodic component. Plot the raw periodogram for the series. Does any periodic component match the frequency you guessed by inspecting the time plot?

12. The R dataset `ldeaths` gives the monthly deaths from bronchitis, emphysema, and asthma in the UK, 1974–1979. Plot the time series, and examine the raw periodogram. Smooth the periodogram using a filter with (i) three points, (ii) three points, followed by a futher filter with five points and (iii) five points, followed by a further filter with seven points. Comment on the smoothed periodogram in each case.

13. Demonstrate that smoothing the periodogram with symmetric weights w_j say, where $\sum_j w_j = 1$, is equivalent to modifying the periodogram via a lag window with weights $\{\lambda_k : k = 0, 1, \ldots, N-1\}$. That is, show that the smoothed periodogram can be written in the form

$$\hat{f}(\omega) = \frac{1}{\pi}\left(c_0 + 2\sum_{k=1}^{N-1}\lambda_k c_k \cos(\omega k)\right)$$

and express λ_k in terms of $\{w_j\}$.

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