Activity Solution: Autoregressive Processes

Suppose $Z(t)$ is white noise with mean zero and variance σ^2. We have seen that a process $X(t)$ is said to be a moving average (MA) process of order q if

$$X(t) = \beta_0 Z(t) + \beta_1 Z(t-1) + \cdots + \beta_q Z(t-q)$$

for some constants $\beta_0, \beta_1, \ldots, \beta_q$, with usually $\beta_0 = 1$. This activity concerns a special case of the model above with $q \to \infty$, $\beta_0 = 1$ and $\beta_i = \alpha^i$ for $i \geq 1$, specifically

$$X(t) = Z(t) + \alpha Z(t-1) + \alpha^2 Z(t-2) + \cdots \tag{1}$$

1. What condition is necessary on α for the right–hand side of equation (1) to be well-defined?
   $|\alpha| < 1$, *otherwise the sum diverges.*

2. By rearranging equation (1), show that $X(t) = \alpha X(t-1) + Z(t)$.
   By rearranging equation (1) we obtain

$$\begin{aligned} Z(t) =& X(t) - \left(\alpha Z(t-1) + \alpha^2 Z(t-2) + \cdots\right) \\ =& X(t) - \alpha\left(Z(t-1) + \alpha Z(t-2) + \cdots\right) \\ =& X(t) - \alpha X(t-1). \end{aligned}$$

3. *Clickers question:* We call $X(t)$ an *autoregressive process* of order 1, denoted AR(1). Each value in the process is modelled as a constant times the previous value plus noise. Find $\mathrm{E}(X(t))$.

$$\begin{aligned} E(X(t)) =& E\left(Z(t) + \alpha Z(t-1) + \alpha^2 Z(t-2) + \cdots\right) \\ =& E(Z(t)) + E(\alpha Z(t-1)) + E\left(\alpha^2 Z(t-2)\right) + \cdots \\ =& E(Z(t)) + \alpha E(Z(t-1)) + \alpha^2 E(Z(t-2)) + \cdots \\ =& 0. \end{aligned}$$

4. *Clickers question:* Find the variance of $X(t)$. (Remember if $|r| < 1$, $\sum_{k=0}^{\infty} ar^k = a/(1-r)$.) How does this relate to your answer to question 1?

So

$$\begin{aligned} Var(X(t)) &= Var\left(Z(t) + \alpha Z(t-1) + \alpha^2 Z(t-2) + \cdots\right) \\ &= Var(Z(t)) + Var(\alpha Z(t-1)) + Var\left(\alpha^2 Z(t-2)\right) + \cdots \\ &= Var(Z(t)) + \alpha^2 Var(Z(t-1)) + \alpha^4 Var(Z(t-2)) + \cdots \\ &= \sigma^2\left(1 + \alpha^2 + \alpha^4 + \cdots\right) \end{aligned}$$

which if $|\alpha| < 1$ *(see question 1) equals*

$$\frac{\sigma^2}{1-\alpha^2}.$$

5. If $E(X(t)) = 0$ for all t, recall the acvf of $X(t)$ at lag k is

$$\gamma(k) = E(X(t)X(t+k)).$$

By substituting equation (1) into the above, for $k \geq 0$ find $\gamma(k)$ in terms of σ_X^2, the variance found in question 4. The identity given in question 4 may again be useful.

$$\begin{aligned} \gamma(k) &= E(X(t)X(t+k)) \\ &= E\left(\sum_{i=0}^{\infty} \alpha^i Z(t-i) \sum_{j=0}^{\infty} \alpha^j Z(t+k-j)\right) \\ &= \sigma^2 \sum_{i=0}^{\infty} \alpha^i \alpha^{k+i} \\ &= \frac{\alpha^k \sigma^2}{1-\alpha^2} \\ &= \alpha^k \sigma_X^2. \end{aligned}$$

6. Is $X(t)$ stationary?
Assuming $|\alpha| < 1$, *we have seen that neither the mean nor the acvf depend on* t. *Hence the process is stationary.*

7. Re-cap what you have done during this activity. Why do you think you were asked to do this activity? What have you learned?

We defined a process where each value in the process is modelled as a constant α times the previous value plus noise, $Z(t)$. We found the expectation and variance, appreciating the latter is finite only when $|\alpha| < 1$, in which case the process is also stationary. The acf for such a process, known as an autoregressive process of order 1, was looked at in general and some special cases should be explored.
Learning outcomes encountered in this activity are:

(a) *Identify an autoregressive process of order p, i.e., an $AR(p)$.*

(b) *Derive properties for an $AR(1)$, including the mean, variance, and autocorrelation function.*