A GUIDED EXPLORATION OF COMPACTNESS

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Abstract. The goal of this work is to build understanding of compactness from the bottom up, by discussing open and closed intervals and starting to prove the Heine-Borel Theorem.

1. Open and closed intervals

We've often used open intervals, like $(-1, 1)$, and closed intervals, like $[0, 10]$, when writing proofs in this class. It turns out that they can interact with each other in ways that are not intuitively obvious. For one, we can form closed intervals out of open ones, and vice versa. Let's explore these fascinating connections.

Proposition 1.1. *A non-empty closed interval can be equal to an intersection of open intervals.*

Informally, we can prove the proposition by finding $(a-\delta, b+\delta)$, where δ decreases approaches 0, so the intervals become smaller and smaller, so that only $[a, b]$ "remains" after many intersections.

Proof. Consider the open interval $I_n = (-\frac{1}{n}, 1 + \frac{1}{n})$.

We claim that $\bigcap\limits_{n=1}^{\infty} I_n = [0, 1]$.

Pick $a \in [0, 1]$. For all $n \in \mathbb{Z}_{\text{pos}}$, $-\frac{1}{n} < 0$. Since $a \geq 0$, $a > -\frac{1}{n}$. Also, $1 = 1 + 0 < 1 + \frac{1}{n}$. Since $a \leq 1$, $a < 1 + \frac{1}{n}$. We showed that a is strictly greater than the lower bound of I_n and strictly less than an upper bound of I_n, regardless of the value of n.

All I_n contain a, so their intersection, $\bigcap_{n=1}^{\infty} I_n$, contains a as well. Therefore, $[0, 1] \subseteq \bigcap_{n=1}^{\infty} I_n$.

Now, pick $b \notin [0, 1]$. We know that only one of the following is true: $b > 1$ or $b < 0$.

Let $b > 1$. Note that by the Archimedean property, there exists $N \in \mathbb{Z}_{\text{pos}}$ such that $\frac{1}{N} < b-1$, allowing us to conclude that $1+\frac{1}{N} < 1+(b-1) = b$. So there exists I_N that does not contain b, since b is greater than an upper bound of I_n.

If a set does not contain element p, then the intersection over a collection containing that set does not contain p as well, so $b \notin \bigcap_{n=1}^{\infty} I_n$ for $b > 1$.

What if $b < 0$? In a similar manner, by the Archimedean property, there exists $M \in \mathbb{Z}_{\text{pos}}$ such that $\frac{1}{M} < -b$, since $-b$ is positive. Then, $-\frac{1}{M} = \frac{1}{-M} > b$. So there exists I_M that does not contain b, since b is less than a lower bound of I_M.

An intersection does not contain element p if no set in the collection over which the intersection was taken contains p, so $b \notin \bigcap_{n=1}^{\infty} I_n$ when $b < 0$.

We now know that if $b \in [0,1]^C$, then $b \in (\bigcap_{n=1}^{\infty} I_n)^C$. A contrapositive of this statement is "if $b \in \bigcap_{n=1}^{\infty} I_n$, then $b \in [0,1]$," implying that $\bigcap_{n=1}^{\infty} I_n \subseteq [0,1]$.

Since $[0,1] \subseteq \bigcap_{n=1}^{\infty} I_n$ and $\bigcap_{n=1}^{\infty} I_n \subseteq [0,1]$, we conclude that $\bigcap_{n=1}^{\infty} I_n = [0,1]$, so there exists a closed interval that is equal to am intersection of open intervals.

□

This shows us how open intervals can combine into closed ones. What about the reverse direction? Can we construct open intervals using closed ones? The next proposition addresses exactly that.

Proposition 1.2. *A non-empty open interval can be equal to a union of closed intervals.*

Imagine a "sequence" of closed intervals, whose bounds are expanding but are approaching a certain "limit" of (a,b). Let's show formally that this is possible.

Proof. Consider the closed interval $I_n = [\frac{1}{n}, 1 - \frac{1}{n}]$.

We claim that $\bigcup\limits_{n=2}^{\infty} I_n = (0,1)$.

Pick $x \in (0,1)$.

If $x = \frac{1}{2}$, then $x \in I_2 = [\frac{1}{2}, 1 - \frac{1}{2}] = [\frac{1}{2}, \frac{1}{2}] = \{\frac{1}{2}\}$.

If $x < \frac{1}{2}$, then, by the Archimedean property, there exists N such that $\frac{1}{N} < x < \frac{1}{2}$, so $N > 2$, which means $\frac{N-1}{N} \geq \frac{2}{3}$. Since $x < \frac{2}{3} \leq \frac{N-1}{N} = 1 - \frac{1}{N}$, we derive that $x \in I_N = [\frac{1}{N}, 1 - \frac{1}{N}]$.

If $x > \frac{1}{2}$, then, by the Archimedean property, there exists M such that $\frac{1}{M} < 1 - x < \frac{1}{2}$, implying $x < 1 - \frac{1}{M}$ and $M > 2$. Since $x > \frac{1}{2} > \frac{1}{M}$, we conclude that $x \in I_M = [\frac{1}{M}, 1 - \frac{1}{M}]$.

So, for all $x \in (0,1)$, there exists I_n such that $x \in I_n$. If a set contains an element p, then any union over a collection containing that set contains that element p as well, so all x are contained in $\bigcup_{n=2}^{\infty} I_n$, implying $(0,1) \subseteq \bigcup_{n=2}^{\infty} I_n$.

Now pick $y \notin (0,1)$. We know that only one of the following is true: $y \leq 0$ or $y \geq 1$.

Say $y \geq 1$. Then, for $n \in \mathbb{Z}_{\text{pos}}$, $y \geq 1 > 1 - \frac{1}{n}$, since $\frac{1}{n} > 0$. There is no I_n such that $y \in I_n$ because y is greater than 1, which is an upper bound on all I_n.

If $y \leq 0$, it means that, for $n \in \mathbb{Z}_{\text{pos}}$, y is less than $\frac{1}{n}$, a lower bound on all I_n, so $y \notin I_n$.

For an element to be contained in union, at least one set in the collection over which the union is taken must contain that element, so if $y \in (0,1)^C$, then $y \in (\bigcup_{n=2}^{\infty} I_n)^C$. The contrapositive is if $y \in \bigcup_{n=2}^{\infty} I_n$, then $y \in (0,1)$, so $\bigcup_{n=2}^{\infty} I_n \subseteq (0,1)$.

Having shown $(0,1) \subseteq \bigcup_{n=2}^{\infty} I_n$ and $\bigcup_{n=2}^{\infty} I_n \subseteq (0,1)$, we now see that $\bigcup_{n=2}^{\infty} I_n = (0,1)$.

□

These results show deep non-obvious connections between open and closed intervals. To explore them further, let's recall the definitions relevant to our discussion.

Definition (Open subset of $\mathbb{R}$)**.** We say a subset is open if and only if it's a union of open intervals.

Definition (Closed subset of $\mathbb{R}$)**.** We say a subset is closed if and only if its complement is open.

Why is the definition of a closed subset of $\mathbb{R}$ not analogous to that of an open interval? Our next proposition provides sheds light on thiss asymmetry.

Proposition 1.3. *Any subset S of $\mathbb{R}$ is a union of closed intervals.*

Just take a "union" of all elements of S!

Proof. Let $A = \{[s,s] : s \in S\}$. Then $\bigcup\limits_{I \in A} I = [s_1, s_1] \cup [s_2, s_2] \cup \cdots = \{s_1\} \cup \{s_2\} \cup \cdots = S$, where s_i is an element of S, since any set is equal to the union of all of its elements. □

Proposition 1.3 shows that the property of being a union of closed intervals is too general to be distinctive.

2. Compactness: special case

Open and closed intervals are relatively easy to imagine, as far as analysis concepts are concerned, since they are somewhat analogous to physically open and closed things. There is another type of intervals, though, and they are called compact intervals, which are not easily imaginable. In fact, even the definition may seem a little awkward.

To begin exploring compactness, we first need to understand the concept of open covers.

Definition (Open cover)**.** An open cover of $S \subseteq \mathbb{R}$ is any collection $\{\mathcal{O}_\alpha : \alpha \in A\}$ such that $S \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha$, where $A \subseteq \mathbb{R}$ and for each $\alpha \in A$, $\mathcal{O}_\alpha$ is an open set.

In other words, an open cover of S is a collection of open intervals, potentially infinite, that encompasses all the elements of S.

Employing open covers, we can now define what it means for a set to be compact.

Definition (Compact set)**.** $S \subseteq \mathbb{R}$ is compact if whenever $S \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha$ with $\{\mathcal{O}_\alpha : \alpha \in A\}$ a collection of open sets,

$$\exists N \in \mathbb{Z}_{\text{pos}} \text{ and elements } \alpha_1, \alpha_2, \ldots, \alpha_N \in A \text{ such that } S \subseteq \bigcup_{n=1}^{N} \mathcal{O}_{\alpha_n}.$$

Informally: open covers of S can be finite or infinite. If they are finite, they are good to go. If they are infinite, there needs to be a finite subset of it which still covers the whole of S. In other words, S is *compact* if and only if we can "compress" every open cover into a "finite open cover".

Let's build our understanding of compactness by considering the following special case.

Theorem 2.1. *Suppose* $[0,1] \subseteq \bigcup\limits_{\alpha \in A} \mathcal{O}_\alpha$*, where* $A \subseteq \mathbb{R}$ *and* $\mathcal{O}_\alpha$ *is an open interval for each* $\alpha \in A$*. Then there exists* $n \in \mathbb{Z}_{\text{pos}}$ *and elements* $\alpha_1, \alpha_2, \ldots, \alpha_N \in A$ *such that* $[0,1] = \bigcup\limits_{n=1}^{N} \mathcal{O}_{\alpha_n}$*.*

Using our new terminology, Theorem 2.1 states that $[0,1]$ is compact.

Previously, we've shown that open and closed intervals can interact in non-obvious ways. What if we then replace $[0,1]$ with $(0,1)$?

Proposition 2.2. *Let Modified "Theorem" be Theorem 2.1 where* $[0,1]$ *is replaced with* $(0,1)$*. Modified "Theorem" is false.*

In short, we can construct a infinite union of open intervals to cover $(0,1)$ in a way that no finite union of the same intervals can cover $(0,1)$. Let's see how it works.

Proof. We first show that $(0,1) \subseteq \bigcup\limits_{n=1}^{\infty} I_n$, where $I_n = (\frac{1}{2} - \frac{n}{2n+1}, \frac{1}{2} + \frac{n}{2n+1})$.

Pick $x \in (0,1)$. Define $d = |x - \frac{1}{2}|$. By the Archimedean property, there exists $N \in \mathbb{Z}_{\text{pos}}$ such that $\frac{1}{N} < \frac{1-2d}{d}$, since $0 < d < \frac{1}{2}$ implies $1 - 2d > 0$. Then

$$N > \frac{d}{1-2d} \implies N(1-2d) > d \implies N > 2Nd + d \implies \frac{N}{2N+1} > d$$

Therefore, $|x - \frac{1}{2}| = d < \frac{N}{2N+1}$, which means $x \in (\frac{1}{2} - \frac{N}{2N+1}, \frac{1}{2} + \frac{N}{2N+1}) = I_N \subseteq \bigcup_{n=1}^{\infty} I_n$.

Next, consider that for any $k, l \in \mathbb{Z}_{\text{pos}}$ such that $k < l$, we have $\frac{k}{2k+1} < \frac{l}{2l+1}$. Based on this inequality, we deduce $\frac{1}{2} - \frac{l}{2l+1} < \frac{1}{2} - \frac{k}{2k+1}$ and $\frac{1}{2} + \frac{k}{2k+1} < \frac{1}{2} + \frac{l}{2l+1}$, which allows us to conclude that $I_k \subseteq I_l$.

Furthermore, $\bigcup_{n \in A} I_n = I_M$, where $M = \max\{A\}$ and A is a finite set of positive integers. Then, by the Modified "Theorem", $(0,1) \subseteq \bigcup_{n=1}^{N} \mathcal{O}_{\alpha_n} = \bigcup_{n \in A} I_n = I_M$.

But pick $y = \frac{1}{2}(\frac{1}{2} - \frac{M}{2M+1})$. Then, for any M

$$\frac{M}{2M+1} < \frac{M}{2M} = \frac{1}{2}$$
$$\frac{1}{2} - \frac{M}{2M+1} > \frac{1}{2} - \frac{1}{2} = 0$$

Therefore

$$0 < y < \frac{1}{2} - \frac{M}{2M+1}$$

So there exists a number in $(0,1)$ that is less than a lower bound of I_M, leading to the conclusion of $(0,1) \nsubseteq I_M = \bigcup_{n=1}^{N} \mathcal{O}_{\alpha_n}$. Therefore, the Modified "Theorem" is false and we can't replace $[0,1]$ with $(0,1)$ in Theorem 2.1.

$\square$

3. Proving the special case

We showed that a seemingly small change breaks a theorem — but how do we know that Theorem 2.1 itself is not false? Let's prove it!

Let's begin by examining the following helpful property of open intervals in $\mathbb{R}$.

Lemma 3.1. *Let (a,b) be an open interval and $p \in (a,b)$. Then there exists $\delta > 0$ such that $[p-\delta, p+\delta] \subseteq (a,b)$.*

Proof. Since $p \in (a,b)$, we know that $p-a$ and $b-p$ are both positive. Let $\delta = \min\{\frac{p-a}{2}, \frac{b-p}{2}\}$, which is positive. For any $x \in [p-\delta, p+\delta]$, we have $p - \delta \leq x \leq p + \delta$. When $x \leq p + \delta$, we get $x \leq p + \frac{b-p}{2} = \frac{b+p}{2} < b$. Similarly, when $x \geq p - \delta$, we get $x \geq p - \frac{p-a}{2} = \frac{p+a}{2} > a$. Therefore $a < x < b$, meaning that $x \in (a,b)$, which proves $[p-\delta, p+\delta] \subseteq (a,b)$. $\square$

This is a curious result on its own — but its main purpose is to help us prove Theorem 2.1.

Proof. Let $S = \{x \in (0,1] \mid [0,x]$, which can be covered by finitely many $\mathcal{O}_{\alpha_i}\}$, where $\mathcal{O}_{\alpha_i}$ is an open interval.

Let's show that S is nonempty by proving that it contains 0.

Since $0 \in [0,1]$ and, by the assumption of the theorem, $[0,1] \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha$, there exists $\alpha_z \in A$ such that $0 \in \mathcal{O}_{\alpha_z}$. Writing $\mathcal{O}_{\alpha_z} = (a,b)$ for some $a < 0 < b$, we apply Lemma 3.1 to obtain $\epsilon > 0$ such that $[0,\epsilon] \subseteq [-\epsilon,\epsilon] \subseteq \mathcal{O}_{\alpha_z}$. By definition of S, $\epsilon \in S$, which means that S is nonempty.

It is easy to see that S is bounded: $S \subseteq (0,1]$, so all $x \in S$ are bounded by 1 above and 0 below.

The set S is non-empty and bounded above, so there exists $c = \sup S$.

What values can c be equal to?

To begin, c cannot be greater than 1. Suppose it is: $c - 1 > 0$, which implies $c = 1 + \beta$ for some $\beta > 0$. But then consider $d = 1 + \frac{\beta}{2}$. For any $x \in (0,1]$, $d > x$, so d is an upper bound on $(0,1]$. Since $S \subseteq (0,1]$, we know that d is an upper bound on S as well. This contradicts the property of c: being less or equal to other upper bounds on S. Thus $c \leq 1$.

Also, c cannot be less than 0. Suppose $c < 0$. Since c is an upper bound, then for all $x \in S$, $x \leq c < 0$ (remember that S is non-empty, so x is a real number). However, $x \in S$ forces x to be in the interval $[0,1]$, contradicting $x < 0$. Thus $c \geq 0$.

Eliminating other options, we arrive to the fact that c has to be in $[0,1]$.

Suppose $c < 1$. Since $c \in [0,1] \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha$, there exists $\alpha_j \in A$ such that $c \in \mathcal{O}_{\alpha_j}$. Writing $\mathcal{O}_{\alpha_j} = (a_j, b_j)$ for some $a_j < c < b_j$, we apply the lemma to obtain $\delta > 0$ such that $[c-\delta, c+\delta] \subseteq \mathcal{O}_{\alpha_j}$. Recall the result of the Problem 2 of the Midterm, which implies that there exists $z \in S$ such that $c - \frac{\delta}{2} < z \leq c$. By definition of S, there exist finitely many open intervals $\mathcal{O}_{\alpha_1}, \ldots, \mathcal{O}_{\alpha_k}$, whose union covers $[0,z]$, and therefore also covers $[0, c-\frac{\delta}{2}]$.

Consider $\mathcal{O}_{\alpha_1} \cup \cdots \cup \mathcal{O}_{\alpha_k} \cup \mathcal{O}_{\alpha_j}$, which covers $[0, c+\delta]$ since $[0, c-\frac{\delta}{2}]$ is covered by $\mathcal{O}_{\alpha_1} \cup \cdots \cup \mathcal{O}_{\alpha_k}$, and $[c-\delta, c+\delta]$ is covered by $\mathcal{O}_{\alpha_j}$. These intervals overlap because $c - \delta < c - \frac{\delta}{2}$. Therefore $c + \delta \in S$, contradicting that c is the supremum of S.

Thus $c = 1$.

Assume $1 \notin S$. Then, since $1 \in [0,1] \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha$, there exists $\alpha_m \in A$ such that $1 \in \mathcal{O}_{\alpha_m}$. By Lemma 3.1, there exists $\delta > 0$ such that $[1-\delta, 1] \subseteq [1-\delta, 1+\delta] \subseteq \mathcal{O}_{\alpha_m}$. Since $1 - \frac{\delta}{2} < 1$, there exists $z \in S$ with $1 - \frac{\delta}{2} < z < 1$, and thus intervals $\mathcal{O}_{\alpha_1}, \ldots, \mathcal{O}_{\alpha_k}$ covering $[0,z]$. The set $\{\mathcal{O}_{\alpha_1}, \ldots, \mathcal{O}_{\alpha_k}, \mathcal{O}_{\alpha_m}\}$ covers $[0,1]$, so $1 \in S$. Therefore $[0,1]$ has a finite set of open intervals that are its open cover. □

4. Compactness means boundedness

Here is the first profound implication of compactness.

Proposition 4.1. *If $S \subseteq \mathbb{R}$ is compact, then S is bounded.*

The intuition behind this is that S can be covered by a finite subset of a collection of nested intervals, and the union of the intervals in that subset is equal to the largest interval, so S is bounded by the ends of that largest interval.

Proof. Consider $A = \{(-n, n) : n \in \mathbb{Z}_{\text{pos}}\}$. If $x = 0$, any $(-n, n)$ where $n \in \mathbb{Z}_{\text{pos}}$, contains x, since $-n < 0 < n$. For any $x \in \mathbb{R}, x \neq 0$, by the Archimedean property, there exists $N \in \mathbb{Z}_{\text{pos}}$ such that $\frac{1}{N} < \frac{1}{|x|}$, implying $N > |x|$.

By definition of the absolute value, $x \in (-N, N) \subseteq \bigcup_{n=1}^{\infty}(-n, n)$, since $N \in [1, \infty)$. This means A is an open cover of $\mathbb{R}$.

As $S \subseteq \mathbb{R}$, $S \subseteq \bigcup_{n=1}^{\infty}(-n, n)$. By definition of compactness, there exists a finite set B such that $B = \{(-m, m) : m \in C\}$, where C is a finite set of positive integers.

This means $S \subseteq \bigcup_{n\in C}(-n, n)$. Since C is a finite set of positive integers, there exists $M = \max\{C\}$.

Let's show that $\bigcup_{n\in C}(-n, n) = (-M, M)$.

If we pick x from $\bigcup_{n\in C}(-n, n)$, then $x \in (-k, k)$ for some $k \in C, k \leq M$. Since $k \leq M$, we have $-M \leq -k$. Therefore, $-M \leq -k < x < k \leq M$.

Conversely, if we pick x from $(-M, M)$, then $x \in (-M, M) \subseteq \bigcup_{n\in C}(-n, n)$. Since $(-M, M) \subseteq \bigcup_{n\in C}(-n, n)$ and $\bigcup_{n\in C}(-n, n) \subseteq (-M, M)$, $\bigcup_{n\in C}(-n, n) = (-M, M)$.

So, $S \subseteq \bigcup_{n\in C}(-n, n) = (-M, M)$, meaning all $s \in S$ are bounded by M above and by $-M$ below. □

5. Compact sets are closed

There is another important implication of compactness: all compact sets are closed. Through this section, assume $S \subseteq \mathbb{R}$.

Proposition 5.1. *For any $p \notin S$ and any $x \in S$, there exists $\epsilon > 0$ such that the two open intervals $(p - \epsilon, p + \epsilon)$ and $(x - \epsilon, x + \epsilon)$ are disjoint.*

This holds for an even more general case! For any pair of real numbers, it is possible to find open intervals centered at each of them such that the "radii" of the intervals are the same and the intervals are disjoint.

Proof. Pick $x, p \in \mathbb{R}$ such that $p \neq x$.

Without loss of generality, let $x < p$.

Assume that $(p-\epsilon, p+\epsilon)$ and $(x-\epsilon, x+\epsilon)$ are not disjoint, i.e. $(p-\epsilon, p+\epsilon) \cap (x-\epsilon, x+\epsilon) \neq \varnothing$. Then there exists $a \in \mathbb{R}$ such that $a \in (p - \epsilon, p + \epsilon)$ and $a \in (x - \epsilon, x + \epsilon)$. We have $a < x + \epsilon$

and $a > p - \epsilon$, which leads to $x + \epsilon > p - \epsilon$, implying $p - x < 2\epsilon$. But let $\epsilon = \frac{p-x}{2}$. Then $p - x < 2(\frac{p-x}{2}) < p - x$, which is a contradiction.

If $x > p$, replace all x with p and all p with x in the paragraph above, and the contradiction still arises. Therefore, our assumption was wrong. For any $x \in R$ and $p \in \mathbb{R}$, including any $x \in S$ and any $p \notin S$, there exists $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon)$ and $(x - \epsilon, x + \epsilon)$ are disjoint. □

Proposition 5.2. *For any point $p \notin S$, there exists $\delta > 0$ such that the open interval $(p - \delta, p + \delta)$ is disjoint from S. Equivalently, there exists $\delta > 0$ such that $(p - \delta, p + \delta) \subseteq S^c$.*

The intuition behind the proof is the following: construct a collection of increasing nested intervals such that they eventually cover all of S but never cover p. Then there is a finite subset of that collection, and all points in the sets of that collection "stay away" from p by at least some positive distance δ.

Proof. Suppose that for every $\delta > 0$ we have $(p - \delta, p + \delta) \cap S \neq \varnothing$.

Consider the collection of open sets $U = \{(-\infty, p - \frac{1}{n}) \cup (p + \frac{1}{n}, \infty) : n \in \mathbb{Z}_{\text{pos}}\}$.

Let's prove that U is an open cover of S. For any $x \in S$, since $p \notin S$, we have $|x - p| > 0$, and by the Archimedean property, there exists $N \in \mathbb{Z}_{\text{pos}}$ such that $\frac{1}{N} < |x - p|$, implying $x > p + \frac{1}{N}$ and $x < p - \frac{1}{N}$. Consequently, x is contained in the open set $(-\infty, p - \frac{1}{N}) \cup (p + \frac{1}{N}, \infty)$, which means that collection U is an open cover of S.

By the compactness of S, there exists a finite set of $V \subseteq U$ that is also an open cover of S. Thus, $S \subseteq \bigcup_{n \in C} [(-\infty, p - \frac{1}{n}) \cup (p + \frac{1}{n}, \infty)]$, where C is a finite subset of $\mathbb{Z}_{\text{pos}}$. Let $N = \max\{C\}$, which exists since C is a finite set of positive integers.

Observe that $\frac{1}{N} \leq \frac{1}{k}$ for all $k \leq N$. This means $(-\infty, p - \frac{1}{k}) \subseteq (-\infty, p - \frac{1}{N})$ and $(p + \frac{1}{k}, \infty) \subseteq (p + \frac{1}{N}, \infty)$. Every interval in V is contained within $(-\infty, p - \frac{1}{N}) \cup (p + \frac{1}{N}, \infty)$. Since V is an open cover of S, we can conclude that $S \subseteq (-\infty, p - \frac{1}{N}) \cup (p + \frac{1}{N}, \infty)$.

But let $\delta = \frac{1}{N}$. Then $(p - \delta, p + \delta) \cap (\bigcup_{I \in V} I) = \varnothing$. Since $S \subseteq V$, we have $(p - \delta, p + \delta) \cap S = \varnothing$. This is the opposite of our initial assumption, thus proving the proposition by contradiction. □

We are now ready to prove closeness.

Proposition 5.3. *S is closed.*

Proof. Pick $x \in S^C$. By Proposition 5.2, there exists $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap S = \varnothing$, which is equivalent to $(x - \delta_x, x + \delta_x) \subseteq S^C$.

Therefore, $S^C = \bigcup_{x \in S^C} (x - \delta_x, x + \delta_x)$, so S^C is a union of open intervals. Thus S^C is open, making S closed by definition. □