Zettili Exercise 1.7

Let $M_\oplus \equiv$ Intensity of solar radiation at earth $= 1.36 \frac{kW}{m^2}$

$M_\odot \equiv$ Intensity of solar radiation at sun's surface

$R_\oplus \equiv$ Radius of earth's orbit $= 1.5 \cdot 10^{11}\,m$

$R_\odot \equiv$ Radius of sun $= 6.96 \cdot 10^{8}\,m$

In steady state, all the power emitted by the sun passes also passes through a sphere of radius $R_\oplus$:

$$P_\odot = 4\pi R_\odot^2 M_\odot = 4\pi R_\oplus^2 M_\oplus$$

(a) By the Stefan-Boltzmann law [Eq. (1.1)], $M_\odot = \sigma T_\odot^4$, where

$\sigma = 5.67 \times 10^{-8}\ W/m^2K^4$

$$T_\odot = \left[\frac{M_\odot}{\sigma}\right]^{1/4} = \left[\frac{M_\oplus}{\sigma}\cdot\frac{R_\oplus^2}{R_\odot^2}\right]^{1/4} = \left[\frac{1360\,W}{m^2}\cdot\frac{m^2\,K^4}{5.67\cdot 10^{-8}\,W}\cdot\frac{(1.5\cdot 10^{11}\,m)^2}{(6.96\,10^{8}\,m)^2}\right]^{1/4}$$

$$= \boxed{5780\,K}$$

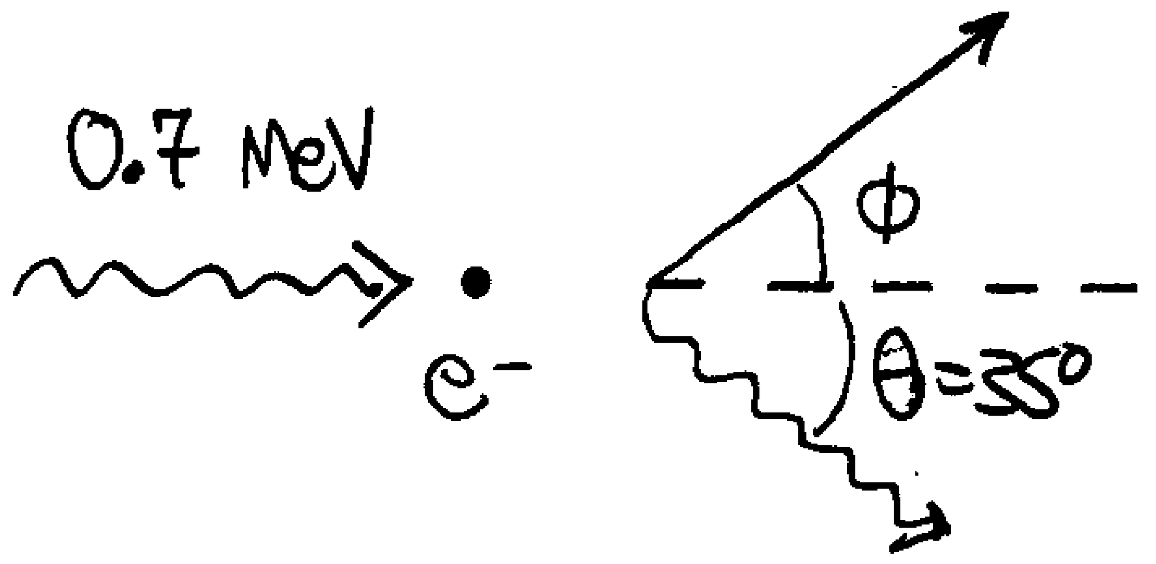
By Wien's displacement law [Eq. (1.14)],

$$\lambda_{max} = \frac{2898.9\cdot 10^{-6}\,mK}{T_\odot} = \frac{2898.9\cdot 10^{-6}\,mK}{5780\,K} = \boxed{5.017\cdot 10^{-7}\,m = 502\,nm}$$

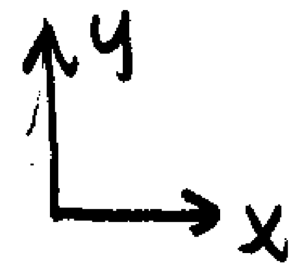
(b) $P_\odot = 4\pi R_\oplus^2 M_\oplus$

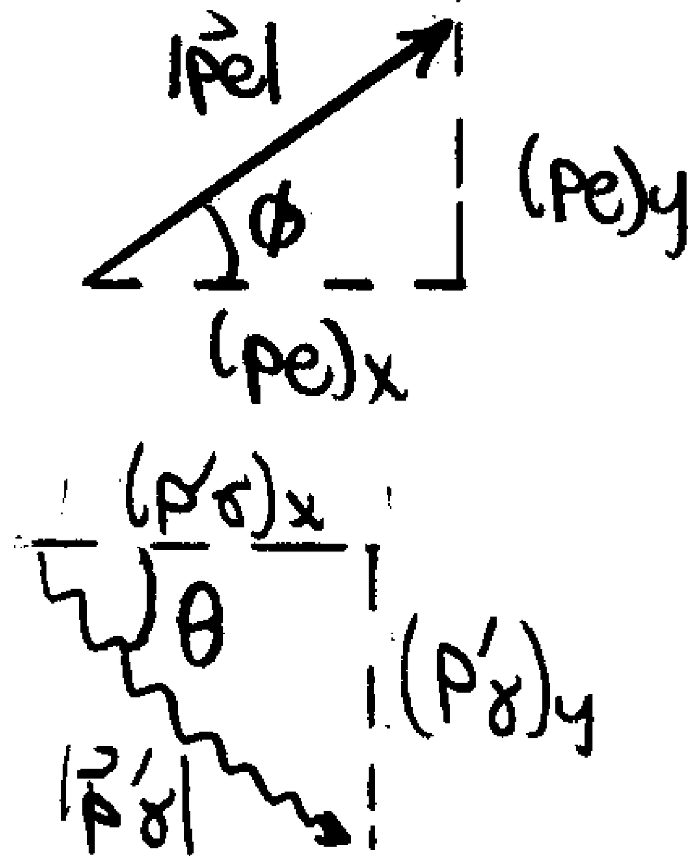
$$= 4\pi \cdot (1.5\cdot 10^{11}\,m)^2 \cdot \frac{1360\,W}{m^2} = \boxed{3.85\cdot 10^{26}\,W}$$

Zettili Exercise 1.11



Divide up the momentum components:





(a) Start with Eq. (1.35), $\Delta\lambda = \lambda' - \lambda = 4\pi \lambda\!\!\!^{-}_c \sin^2\left(\frac{\theta}{2}\right)$.

where $\lambda\!\!\!^{-}_c = \frac{\hbar}{m_e c} = 3.86 \cdot 10^{-13}\,\text{m} = 386\ \text{fm}$, $\theta = 35°$, and $E = \frac{hc}{\lambda}$, $E' = \frac{hc}{\lambda'}$.

$$\lambda' = \lambda + \Delta\lambda = \frac{hc}{E} + 4\pi\lambda\!\!\!^{-}_c \sin^2\left(\frac{\theta}{2}\right) = \frac{2\pi\hbar c}{E} + 4\pi\lambda\!\!\!^{-}_c \sin^2\frac{\theta}{2} = 2\pi\left(\frac{\hbar c}{E} + 2\lambda\!\!\!^{-}_c \sin^2\frac{\theta}{2}\right)$$

$$= 2\pi \cdot \left[\frac{197.3\ \text{MeV}\cdot\text{fm}}{0.7\ \text{MeV}} + 2\,|\,386\,\text{fm}\,|\,\sin^2(17.5°)\right] = 2209\ \text{fm} = \boxed{2.2\,\text{pm}}$$
$$= 2.2 \cdot 10^{-12}\,\text{m}$$

$$E' = \frac{hc}{\lambda'} = \frac{2\pi\hbar c}{\lambda} = \frac{2\pi\,|\,197.3\ \text{MeV}\cdot\text{fm}}{2209\ \text{fm}} = \boxed{0.56\ \text{MeV}} = 9.0 \cdot 10^{-14}\,\text{J}$$

(b) The kinetic energy of the electron, by energy conservation, is just the energy lost by the photon: $K = E - E' = 0.70\ \text{MeV} - 0.56\ \text{MeV} = \boxed{0.14\,\text{MeV}}$

(c) $\tan\phi = \frac{(p_e)_y}{(p_e)_x}$, from the diagram above

Momentum conservation gives
$$(p_\gamma)_x = (p'_\gamma)_x + (p_e)_x$$
$$0 = (p'_\gamma)_y + (p_e)_y$$

$$\Rightarrow (p_e)_x = (p_\gamma)_x - (p'_\gamma)_x = p_\gamma - p'_\gamma \cos\theta = \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta$$

$$(p_e)_y = -(p'_\gamma)_y = +p'_\gamma \sin\theta = +\frac{h}{\lambda'}\sin\theta$$

$$\Rightarrow \tan\phi = \frac{\sin\theta}{\frac{\lambda'}{\lambda} - \cos\theta}$$ [This is, for example, Zettili Eq. (1.180).] $$= \frac{\sin\theta}{\frac{E}{E'} - \cos\theta}$$

Plugging in $E = 0.70\,\text{MeV}$, $E' = 0.56\,\text{MeV}$, $\theta = 35°$, we obtain $\boxed{\phi = 53.2°}$.

Zettili Exercise 1.17

Given $V_s = 7.5\,V$ for $\lambda = 150\,nm$, $eV_s = 7.5\,eV$.

By Eq. (1.22), $V_s = \dfrac{hc}{e\lambda} - \dfrac{W}{e} \Rightarrow eV_s = \dfrac{hc}{\lambda} - W$

The work function W is the same, regardless of the wavelength of the radiation.

So $W = \dfrac{hc}{\lambda_1} - eV_{s_1} = \dfrac{hc}{\lambda_2} - eV_{s_2}$

$\Rightarrow eV_{s_2} = eV_{s_1} + hc\left(\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1}\right)$; $hc = 2\pi\hbar c = 2\pi \cdot 1973\,eV\cdot\text{Å} = 2\pi \cdot 197.3\,eV\cdot nm$

We want V_{s_2} when $V_{s_1} = 7.5\,V$, $\lambda_1 = 150\,nm$, and $\lambda_2 = 275\,nm$

$$eV_{s_2} = 7.5\,eV + 2\pi \cdot 197.3\,eV\cdot nm\left(\frac{1}{275\,nm} - \frac{1}{150\,nm}\right) = 3.74\,eV,$$

or $\boxed{V_{s_2} = 3.74\,V}$

The work function W (<u>not</u> an assigned part of the problem) is

$$\frac{hc}{\lambda_1} - eV_{s_1} = \frac{2\pi\hbar c}{\lambda_1} - eV_{s_1} = \frac{2\pi \cdot 197.3\,eV\cdot nm}{150\,nm} - 7.5\,eV = \underline{\underline{0.76\,eV}}$$

Zettili Exercise 1.20

$\nu = 7.2 \cdot 10^{14}$ Hz

(a) The photoelectric effect will occur as long as

$W < h\nu \equiv W_{max}$

so $W_{max} = h\nu = \dfrac{6.626 \cdot 10^{-34}\, \cancel{J} \cdot \cancel{s} \;\Big|\; 7.2 \cdot 10^{14} \;\Big|\; eV}{\;\Big|\; \cancel{s} \;\Big|\; 1.602 \cdot 10^{-19}\, \cancel{J}} = \boxed{2.98\ eV}$

Of the four metals, only cesium will allow electrons to be ejected.

(b) $K = h\nu - W = 2.98\ eV - 2.14\ eV = \boxed{0.84\ eV = 1.35 \cdot 10^{-19}\ J}$

Zettili Exercise 1.28

By Eq. (1.46), $n\lambda = 2d\sin\phi$, where ϕ is the angle of the source with the crystal plane.

$$E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{h^2 n^2}{8md^2\sin^2\phi} = \frac{(2\pi\hbar)^2 n^2}{8md^2\sin^2\phi} = \frac{(\pi n \hbar c)^2}{2mc^2 d^2 \sin^2\phi}$$

Since $E \propto n^2$, the minimum required energy has $n=1$.

$$E_{min} = \frac{1}{2mc^2}\left(\frac{\pi \hbar c}{d\sin\phi}\right)^2 =$$

$$= \frac{1}{2\cdot 5.11\cdot 10^5\,\text{eV}} \cdot \frac{\pi^2}{\sin^2(60°)} \cdot \frac{(1973\,\text{eV}\cdot\text{Å})^2}{(19\,\text{nm})^2} \cdot \frac{\text{nm}^2}{(10\,\text{Å})^2} = \boxed{1.4\cdot 10^{-3}\,\text{eV}} = 1.4\,\text{meV}$$

$$= 2.2\cdot 10^{-22}\,\text{J}$$

Zettili Exercise 1.30

By Eq. (1.81), the Balmer series corresponds to transitions from the nth state to the first excited state (n=2),

$$h\nu_B = E_n - E_2 = R\left(\frac{1}{2^2}-\frac{1}{n^2}\right), \qquad R = 13.6\,eV.$$

Similarly, the Paschen series corresponds to transitions from the nth state to the second excited state (n=3),

$$h\nu_P = E_n - E_3 = R\left(\frac{1}{3^2}-\frac{1}{n^2}\right).$$

Since $\lambda = \frac{c}{\nu}$, we have $\lambda_B = \frac{hc}{R}\left(\frac{1}{2^2}-\frac{1}{n^2}\right)^{-1}$, $\lambda_P = \frac{hc}{R}\left(\frac{1}{3^2}-\frac{1}{n^2}\right)^{-1}$

- The longest wavelength in these expressions occurs for n as small as possible, which is n=3 for the Balmer series and n=4 for the Paschen series.
- The shortest wavelength in these expressions occurs for n as large as possible, $n\to\infty$.

Starting with $\frac{hc}{R} = \frac{2\pi\hbar c}{R} = \frac{2\pi \,|\, 1973\,eV\cdot Å}{13.6\,eV} = 912\,Å = 91.2\,nm$,

the longest Balmer series wavelength is $\frac{hc}{R}\left(\frac{1}{2^2}-\frac{1}{3^2}\right)^{-1} = \frac{36}{5}\cdot 91.2\,nm = \boxed{656\,nm} = 0.656\,\mu m$

the shortest Balmer series wavelength is $\frac{hc}{R}\left(\frac{1}{2^2}-\frac{1}{\infty}\right)^{-1} = 4\cdot 91.2\,nm = \boxed{365\,nm} = 0.365\,\mu m$,

the longest Paschen series wavelength is $\frac{hc}{R}\left(\frac{1}{3^2}-\frac{1}{4^2}\right)^{-1} = \frac{144}{7}\cdot 91.2\,nm = \boxed{1875\,nm} = 1.875\,\mu m$,

the shortest Paschen series wavelength is $\frac{hc}{R}\left(\frac{1}{3^2}-\frac{1}{\infty}\right)^{-1} = 9\cdot 91.2\,nm = \boxed{820\,nm} = 0.820\,\mu m.$

Zettili Exercise 1.33

Beryllium is the element with $Z=4$.

The ground state has $n=1$, the third excited state is $n=4$.

By Eq. (1.76), $E_n = \dfrac{-Z^2}{1+m_e/M} \cdot \dfrac{R}{n^2}$, where $R = 13.6\,\text{eV}$ (1.74).

The transition from the m^{th} state to the n^{th} state requires energy

$$\Delta E = E_n - E_m = \frac{-Z^2}{1+m_e/M} \cdot R \left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

But $\Delta E = h\nu = \dfrac{hc}{\lambda}$, so $\lambda = \dfrac{hc}{\Delta E} = \dfrac{2\pi\hbar c \cdot (1+m_e/M)}{Z^2 R} \left(\dfrac{1}{m^2} - \dfrac{1}{n^2}\right)^{-1}$

The mass of the beryllium nucleus is roughly that of 8 hydrogen atoms, so

$$\frac{m_e}{M} \sim \frac{1}{8 \cdot 1836} \sim \frac{1}{15000} \Rightarrow 1 + \frac{m_e}{M} \approx 1.$$

$$\Rightarrow \lambda = \frac{2\pi}{4^2} \left| \frac{1973\,\text{eV} \cdot \text{Å}}{13.6\,\text{eV}} \right. \left(\frac{1}{1^2} - \frac{1}{4^2}\right)^{-1} = \boxed{61\,\text{Å}} = 6.1\,\text{nm}$$

Zettili Exercise 1.36

- The potential energy function $V(\vec{r})$ is related to the force $\vec{F}(\vec{r})$ by the familiar formula

$$V(\vec{r}) = -\int^{\vec{r}} \vec{F}(\vec{r}')\cdot d\vec{l} = +\int^{x} F(x')dx' \quad \text{(1-dimensional motion)} = +\int^{x} m\omega^2 x' dx'$$

$= +\frac{m\omega^2}{2}x^2 + C$. But this integration constant just defines the potential at, say, $x=0$, which we may choose to be $V(0)=0$, so $C=0$.

- The kinetic energy written in terms of momentum (since we want to use the uncertainty relation in terms of x and p) reads $K = \frac{p^2}{2m}$,

So
$$E = K + V = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \geq 0.$$

- If the uncertainty principle allowed $x=0$ (the bottom of the potential) while $p=0$ (at rest), then we would have $E=0$. However, the uncertainty principle tells us that if the position uncertainty is Δx, then the momentum uncertainty is at least $\Delta p = \frac{\hbar}{2\Delta x}$. But since the classical minimum occurs at $p=0$, we cannot infer a p less than Δp.

So
$$E \geq \frac{(\Delta p)^2}{2m} + \frac{m\omega^2(\Delta x)^2}{2} \geq \underline{\underline{\frac{1}{2m}\left(\frac{\hbar}{2\Delta x}\right)^2 + \frac{m\omega^2}{2}(\Delta x)^2}} \equiv \bar{E}.$$

We want to minimize the function $\bar{E}(\Delta x)$. When $\Delta x \to 0$ the first term grows without bound, while for $\Delta x \to \infty$ the second term grows without bound. Clearly there must be a value of Δx in the middle for which $\bar{E}(\Delta x)$ is minimal.

$$\frac{d\bar{E}}{d\Delta x} = -\frac{\hbar^2}{4m(\Delta x)^3} + m\omega^2(\Delta x) = 0 \text{ for } \Delta x = (\Delta x)_{min} \Rightarrow \boxed{(\Delta x)^2_{min} = \frac{\hbar}{2m\omega}}$$

$$E \geq \bar{E}(\Delta x_{min}) = \frac{\hbar^2}{8m}\cdot\frac{1}{(\Delta x)^2_{min}} + \frac{m\omega^2}{2}(\Delta x)^2_{min} = \frac{\hbar^2}{8m}\cdot\frac{2m\omega}{\hbar} + \frac{m\omega^2}{2}\cdot\frac{\hbar}{2m\omega}$$

$$= \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \boxed{\frac{\hbar\omega}{2}}$$

Zettili Exercise 1.50

The phase and group velocities are defined in Eq. (1.119):

$$V_{ph} \equiv \frac{\omega(k)}{k}, \quad V_g \equiv \frac{d\omega(k)}{dk}$$

Here we have $\omega = kc \Big/ \sqrt{1-\left(\frac{\pi}{bk}\right)^2}$

So immediately, $\boxed{V_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1-\left(\frac{\pi}{bk}\right)^2}}}$

V_g can be found by direct differentiation:

$$\frac{d\omega}{dk} = \frac{d}{dk}\left[\frac{ck}{\sqrt{1-\left(\frac{\pi}{bk}\right)^2}}\right] = \frac{c}{\sqrt{1-\left(\frac{\pi}{bk}\right)^2}} - \frac{1}{2}\frac{ck}{\left[1-\left(\frac{\pi}{bk}\right)^2\right]^{3/2}}(-2)\left(\frac{\pi}{bk}\right)\left(\frac{\pi}{b}\right)\left(-\frac{1}{k^2}\right)$$

$$= \frac{c}{\left[1-\left(\frac{\pi}{bk}\right)^2\right]^{3/2}}\left[1-\left(\frac{\pi}{bk}\right)^2-\left(\frac{\pi}{bk}\right)^2\right] = \boxed{\frac{c\left[1-2\left(\frac{\pi}{bk}\right)^2\right]}{\left[1-\left(\frac{\pi}{bk}\right)^2\right]^{3/2}} = V_g}$$

a tour de force of chain rule usage.

Zettili Exercise 1.53

By Eqs. (1.147) and (1.148), $\Delta x(t) = \Delta x_0 \sqrt{1 + \left[\frac{\hbar t}{2m(\Delta x_0)^2}\right]^2}$

Here, $\Delta x_0 = 1\,\text{fm} = 10^{-15}\,\text{m}$, $mc^2 = 938\,\text{MeV}$, $\Delta x(t) = 1.5\cdot 10^{11}\,\text{m}$.

Solving the above equation for t,

$$\left[\frac{\Delta x(t)}{\Delta x_0}\right]^2 = 1 + \left[\frac{\hbar t}{2m(\Delta x_0)^2}\right]^2 \Rightarrow t = \frac{2mc^2(\Delta x_0)^2}{\hbar c\cdot c}\sqrt{\left[\frac{\Delta x(t)}{\Delta x_0}\right]^2 - 1}$$

$$\Rightarrow t = \frac{2 \cdot 938\,\text{MeV} \cdot (1\,\text{fm})^2 \cdot \text{s} \cdot \text{m}}{197.3\,\text{MeV}\cdot\text{fm} \cdot 3\cdot 10^8\,\text{m} \cdot 10^{15}\,\text{fm}} \underbrace{\sqrt{\left[\frac{1.5\cdot 10^{11}\,\text{m}}{10^{-15}\,\text{m}}\right]^2 - 1}}_{1.5\cdot 10^{26}}$$

$$= 3.17\cdot 10^{-23} \times 1.5\cdot 10^{26}\,\text{s}$$

$$= \boxed{4.75\cdot 10^3\,\text{s} = 1\,\text{hr}, 19\,\text{min}}$$

Zettili Exercise 2.1

$|\psi\rangle = i|\phi_1\rangle + 3i|\phi_2\rangle - |\phi_3\rangle$, $|\chi\rangle = |\phi_1\rangle - i|\phi_2\rangle + 5i|\phi_3\rangle$,

$\langle\phi_i|\phi_j\rangle = \delta_{ij}$

(a) $\underline{\underline{\langle\psi| = -i\langle\phi_1| - 3i\langle\phi_2| - \langle\phi_3|}}$, $\underline{\underline{\langle\chi| = \langle\phi_1| + i\langle\phi_2| - 5i\langle\phi_3|}}$

$\Rightarrow \langle\psi|\psi\rangle = (-i\langle\phi_1| - 3i\langle\phi_2| - \langle\phi_3|)(i|\phi_1\rangle + 3i|\phi_2\rangle - |\phi_3\rangle)$

$= (-i)(+i)\langle\phi_1|\phi_1\rangle + (-3i)(+3i)\langle\phi_2|\phi_2\rangle + (-1)^2\langle\phi_3|\phi_3\rangle$

+ terms times $\langle\phi_i|\phi_j\rangle$ with $i \neq j$, all of which are zero by orthogonality

$= |+i|^2 + |+3i|^2 + |-1|^2 = 1 + 9 + 1 = \boxed{11}$

Note the pattern: It's just the sum of the modulus square of the coefficients of $|\psi\rangle$.

$\langle\chi|\chi\rangle = |+1|^2 + |-i|^2 + |+5i|^2 = 1 + 1 + 25 = \boxed{27}$

$\langle\psi|\chi\rangle = (-i)(+1) + (-3i)(-i) + (-1)(+5i) = \boxed{-3-6i}$

$\langle\chi|\psi\rangle = (+1)(+i) + (+i)(+3i) + (-5i)(-1) = \boxed{-3+6i} = \langle\psi|\chi\rangle^* \boxed{\neq \langle\chi|\psi\rangle}$

$\langle\psi+\chi|\psi+\chi\rangle = \langle\psi|\psi\rangle + \langle\psi|\chi\rangle + \langle\chi|\psi\rangle + \langle\chi|\chi\rangle$

$= 11 + (-3-6i) + (-3+6i) + 27 = \boxed{32}$

(b) $|\psi\rangle\langle\chi| = [+i|\phi_1\rangle + 3i|\phi_2\rangle - |\phi_3\rangle][\langle\phi_1| + i\langle\phi_2| - 5i\langle\phi_3|]$

$$\boxed{\begin{aligned} = &+i|\phi_1\rangle\langle\phi_1| - |\phi_1\rangle\langle\phi_2| + 5|\phi_1\rangle\langle\phi_3| \\ &+ 3i|\phi_2\rangle\langle\phi_1| - 3|\phi_2\rangle\langle\phi_2| + 15|\phi_2\rangle\langle\phi_3| \\ &- |\phi_3\rangle\langle\phi_1| - i|\phi_3\rangle\langle\phi_2| + 5i|\phi_3\rangle\langle\phi_3| \end{aligned}}$$

To compute $|\chi\rangle\langle\psi|$, just note that each term $(\alpha_i|\phi_i\rangle)(\beta_j\langle\phi_j|)$ is replaced with $(\beta_j^*|\phi_j\rangle)(\alpha_i^*\langle\phi_i|) \Rightarrow$ Conjugate and transpose (just like in Hermitian adjoint!)

$$\boxed{\begin{aligned} |\chi\rangle\langle\psi| = &-i|\phi_1\rangle\langle\phi_1| - 3i|\phi_1\rangle\langle\phi_2| - |\phi_1\rangle\langle\phi_3| \\ &- |\phi_2\rangle\langle\phi_1| - 3|\phi_2\rangle\langle\phi_2| + i|\phi_2\rangle\langle\phi_3| \\ &+ 5|\phi_3\rangle\langle\phi_1| + 15|\phi_3\rangle\langle\phi_2| - 5i|\phi_3\rangle\langle\phi_3| \end{aligned}}$$

Zettili Exercise 2.1 continued

The trace is defined in Eq. (2.188): $\mathrm{Tr}(\hat{A}) = \sum_n \langle \phi_n | \hat{A} | \phi_n \rangle$.

Here, first $\hat{A} = |\psi\rangle\langle\chi|$, and then $\hat{A} = |\chi\rangle\langle\psi|$.

So $\mathrm{Tr}(|\psi\rangle\langle\chi|) = \sum_n \langle\phi_n|\psi\rangle\langle\chi|\phi_n\rangle$

Since we wrote $|\psi\rangle\langle\chi| = \sum_{i,j} a_{ij} |\phi_i\rangle\langle\phi_j|$ above, we have

$$\mathrm{Tr}(|\psi\rangle\langle\chi|) = \sum_{n,i,j} a_{ij} \underbrace{\langle\phi_n|\phi_i\rangle}_{\delta_{ni}} \underbrace{\langle\phi_j|\phi_n\rangle}_{\delta_{jn}} = \sum_{n,i} a_{in}\delta_{ni} = \sum_n a_{nn}$$

$$= (+i) + (-3) + (+5i) = \boxed{-3+6i}$$

$$\mathrm{Tr}(|\chi\rangle\langle\psi|) = (-i) + (-3) + (-5i) = \boxed{-3-6i}$$

(c) $\underline{\underline{|\psi\rangle^\dagger = \langle\psi|}}$, which we gave at the beginning of (a).

$\underline{\underline{|\chi\rangle^\dagger = \langle\chi|}}$, which we gave at the beginning of (a).

$\underline{\underline{(|\psi\rangle\langle\chi|)^\dagger = |\chi\rangle\langle\psi|}}$ and $\underline{\underline{(|\chi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\chi|}}$,

which are given in (b).

Zettili Exercise 2.13

(a) $R \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. $R^\dagger = R^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, while $R^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

To verify the form of R^{-1}, we can (i) just compute RR^{-1} and show it equals I, or (ii) use the methods of matrix inversion on pp. 105-107, or (iii) recognize that R is the two-dimensional rotation matrix in Sec. 7.1 of tutorial VAM, whose inverse is obtained by taking $\theta \to -\theta \Rightarrow \cos\theta \to \cos\theta$, $\sin\theta \to -\sin\theta$.

- In any case, we see $\underline{\underline{R^\dagger = R^{-1}\text{, making } R \text{ unitary.}}}$

(b) <u>Eigenvalues</u>: $0 = |R - \lambda I| = \begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 - \sin\theta(-\sin\theta)$

$= \lambda^2 - 2\lambda\cos\theta + \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} \Rightarrow \lambda = \cos\theta \pm \sqrt{\underbrace{\cos^2\theta - 1}_{-\sin^2\theta}} = \boxed{\cos\theta \pm i\sin\theta}$, or

$\Rightarrow \boxed{\lambda = e^{+i\theta}, e^{-i\theta}}$

<u>Eigenvectors</u>: $\underline{\lambda = e^{+i\theta}} \Rightarrow (R - \lambda I)v = 0 \Rightarrow \begin{pmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -ix + y = 0$

$\Rightarrow y = +ix \Rightarrow V_1 = N_1 \begin{pmatrix} 1 \\ +i \end{pmatrix}$. Normalize: $\langle V_1 | V_1 \rangle = |N_1|^2 (1 \ -i)\begin{pmatrix} 1 \\ +i \end{pmatrix} = 2|N_1|^2 \Rightarrow N_1 = \frac{1}{\sqrt{2}}$

$\Rightarrow \boxed{V_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +i \end{pmatrix}}$ [or equivalently $\frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$]

$\underline{\lambda = e^{-i\theta}} \Rightarrow (R - \lambda I)v = 0 \Rightarrow \begin{pmatrix} +i\sin\theta & \sin\theta \\ -\sin\theta & +i\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow +ix + y = 0 \Rightarrow y = -ix$

$\Rightarrow V_2 = N_2 \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Normalize: $\langle V_2 | V_2 \rangle = |N_2|^2 (1 \ +i)\begin{pmatrix} 1 \\ -i \end{pmatrix} = 2|N_2|^2 \Rightarrow N_2 = \frac{1}{\sqrt{2}}$

$\Rightarrow \boxed{V_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}}$ [or equivalently $\frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$].

Zettili Exercise 2.17

$\hat{A}, \hat{B}$ satisfy $[[\hat{A},\hat{B}],\hat{A}]=0 \Rightarrow [\hat{A},\hat{B}]$ commutes with <u>any</u> power $\hat{A}^m$, $m \geq 0$.

<u>Claim:</u> $[\hat{A}^m, \hat{B}] = m\hat{A}^{m-1}[\hat{A},\hat{B}]$ for $m=1,2,3,\ldots$

<u>Method 1</u>: Induction proof

For $m=1$, we have $[\hat{A}^1,\hat{B}] = 1\cdot\hat{A}^{1-1}[\hat{A},\hat{B}]$, or $[\hat{A},\hat{B}]=[\hat{A},\hat{B}]$ ✓

Assume it is true for m.

Then
$$
\begin{aligned}
[\hat{A}^{m+1},\hat{B}] &= \hat{A}^{m+1}\cdot\hat{B} - \hat{B}\cdot\hat{A}^{m+1} \\
&= \hat{A}\cdot\hat{A}^m\hat{B} - \hat{B}\cdot\hat{A}\cdot\hat{A}^m \\
&= \hat{A}\cdot([\hat{A}^m,\hat{B}] + \hat{B}\hat{A}^m) - \hat{B}\hat{A}\hat{A}^m \\
&= \hat{A}[\hat{A}^m,\hat{B}] + [\hat{A},\hat{B}]\hat{A}^m \\
&= \hat{A}\cdot m\hat{A}^{m-1}[\hat{A},\hat{B}] + [\hat{A},\hat{B}]\hat{A}^m \quad \text{(uses the result for } m) \\
&= m\hat{A}^m[\hat{A},\hat{B}] + \hat{A}^m[\hat{A},\hat{B}] \quad \text{(uses the statement on the top line)} \\
&= (m+1)\hat{A}^m[\hat{A},\hat{B}],
\end{aligned}
$$
which means the result holds for $m+1$.

<u>Method 2</u>

Using (2.87) (with typo corrected),
$$[\hat{A}^m,\hat{B}] = \sum_{j=0}^{m-1} \hat{A}^{m-j-1}[\hat{A},\hat{B}]\hat{A}^j$$

But $[[\hat{A},\hat{B}],\hat{A}]=0$ means (as we say on the top line) that $[\hat{A},\hat{B}]$ commutes with any power of $\hat{A}$, in particular $\hat{A}^j$.

So
$$[\hat{A}^m,\hat{B}] = \sum_{j=0}^{m-1}\hat{A}^{m-j-1}\hat{A}^j[\hat{A},\hat{B}] = \sum_{j=0}^{m-1}\hat{A}^{m-1}[\hat{A},\hat{B}] = \hat{A}^{m-1}[\hat{A},\hat{B}]\sum_{j=0}^{m-1}1$$
$$= m\hat{A}^{m-1}[\hat{A},\hat{B}].$$

Zettili Exercise 2.21

(a) $\mathrm{Tr}(\hat{A}\hat{B}\hat{C}) = \sum_n \langle\phi_n|(\hat{A}\hat{B}\hat{C})|\phi_n\rangle$ by Eq. (2.188), where $\{|\phi_n\rangle\}$ is any orthonormal basis for the vector space

$$= \sum_n \langle\phi_n|\hat{A}\left(\sum_p |\phi_p\rangle\langle\phi_p|\right)\hat{B}\left(\sum_q |\phi_q\rangle\langle\phi_q|\right)\hat{C}|\phi_n\rangle,$$

using the insertion of two sets of complete states, $\hat{I} = \sum_n |\phi_n\rangle\langle\phi_n|$ [(2.162)]

$$= \sum_{n,p,q} \langle\phi_n|\hat{A}|\phi_p\rangle\langle\phi_p|\hat{B}|\phi_q\rangle\langle\phi_q|\hat{C}|\phi_n\rangle$$

Each inner product is a complex number, and these commute, so we have

$$= \sum_{n,p,q} \langle\phi_q|\hat{C}|\phi_n\rangle\langle\phi_n|\hat{A}|\phi_p\rangle\langle\phi_p|\hat{B}|\phi_q\rangle$$

$$= \sum_q \langle\phi_q|\hat{C}\left(\sum_n |\phi_n\rangle\langle\phi_n|\right)\hat{A}\left(\sum_p |\phi_p\rangle\langle\phi_p|\right)\hat{B}|\phi_q\rangle$$

$$= \sum_q \langle\phi_q|\hat{C}\hat{A}\hat{B}|\phi_q\rangle = \underline{\underline{\mathrm{Tr}(\hat{C}\hat{A}\hat{B})}}$$

Similarly, we have $\sum_{n,p,q} \langle\phi_p|\hat{B}|\phi_q\rangle\langle\phi_q|\hat{C}|\phi_n\rangle\langle\phi_n|\hat{A}|\phi_p\rangle = \underline{\underline{\mathrm{Tr}(\hat{B}\hat{C}\hat{A})}}$

(b) In the $\{|\phi_n\rangle\}$ basis, $|\psi\rangle = \sum_n \langle\phi_n|\psi\rangle\,|\phi_n\rangle$, $|\phi\rangle = \sum_m \langle\phi_m|\phi\rangle\,|\phi_m\rangle$

$$\Rightarrow \mathrm{Tr}(|\psi\rangle\langle\phi|) = \sum_n \langle\phi_n|\psi\rangle\langle\phi|\phi_n\rangle = \sum_n \langle\phi|\phi_n\rangle\langle\phi_n|\psi\rangle$$

$$= \langle\phi|\left(\sum_n |\phi_n\rangle\langle\phi_n|\right)|\psi\rangle = \langle\phi|\hat{I}|\psi\rangle = \underline{\underline{\langle\phi|\psi\rangle}}.$$

Zettili Exercise 2.22

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) $A^{\dagger} = (A^{T})^{*} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A$. A is Hermitian

Eigenvalues: The characteristic equation is $0 = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = (-\lambda)[(-1-\lambda)(-\lambda) - 0^2] - 0[0(-\lambda) - 0(1)] + 1[0\cdot 0 - (-1-\lambda)\cdot 1]$

or $0 = -\lambda^2(1+\lambda) + (1+\lambda) = -(\lambda^2-1)(\lambda+1) = -(\lambda+1)^2(\lambda-1)$

$\Rightarrow$ Eigenvalues are $\boxed{\lambda = -1 \text{ (twice) and } \lambda = +1}$

Eigenvectors:

$(A - \lambda I)v = 0$

$\lambda = -1$: $A - (-1)I = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow$ $x + z = 0$, y arbitrary.

We can form two linearly independent combinations from these conditions.

For example, $x = -z = 1, y = 0$; $x = z = 0, y = 1$

$\Rightarrow V_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $V_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Normalize: $\langle v_1 | v_1 \rangle = |N_1|^2 (1\ 0\ -1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = |N_1|^2 [(+1)^2 + 0^2 + (-1)^2] = 2|N_1|^2 \Rightarrow N_1 = \frac{1}{\sqrt{2}}$

$\langle v_2 | v_2 \rangle = |N_2|^2 (0\ 1\ 0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |N_2|^2 [0^2 + 1^2 + 0^2] = |N_2|^2 \Rightarrow N_2 = 1$

$\Rightarrow \boxed{V_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

[Note: This combination is not unique. Any normalized linear combinations of V_1, V_2 also work, but still need to be made orthogonal.]

Zettili Exercise 2.22 continued

$\lambda=+1$: $A-(+1)I=\begin{pmatrix}-1&0&1\\0&-2&0\\1&0&-1\end{pmatrix} \Rightarrow \begin{pmatrix}-1&0&1\\0&-2&0\\1&0&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0 \Rightarrow \begin{matrix}-x+z=0\\-2y=0\end{matrix}$

$\Rightarrow x=z,\ y=0$

$\Rightarrow V_3=N_3\begin{pmatrix}1\\0\\1\end{pmatrix}$ Normalize: $\langle V_3|V_3\rangle=|N_3|^2(1\ 0\ 1)\begin{pmatrix}1\\0\\1\end{pmatrix}=|N_3|^2(1^2+0^2+1^2)$

$=2|N_3|^2 \Rightarrow N_3=\frac{1}{\sqrt{2}}$

$\Rightarrow \boxed{V_3=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}}$

Note that each of N_1, N_2, N_3 can be multiplied by a different phase of the form $e^{i\theta}$, θ real, without changing the normalization of $V_1, V_2,$ or V_3.

Orthogonality

$\langle V_1|V_2\rangle=\langle V_2|V_1\rangle^*=\frac{1}{\sqrt{2}}(1\ 0\ -1)\begin{pmatrix}0\\1\\0\end{pmatrix}=\frac{1}{\sqrt{2}}[1\cdot0+0\cdot1+(-1)\cdot0]=0$ ✓

$\langle V_1|V_3\rangle=\langle V_3|V_1\rangle^*=\frac{1}{\sqrt{2}}(1\ 0\ -1)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}=\frac{1}{2}[1\cdot1+0\cdot0+(-1)\cdot1]=0$ ✓

$\langle V_2|V_3\rangle=\langle V_3|V_2\rangle^*=(0\ 1\ 0)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}=\frac{1}{\sqrt{2}}[0\cdot1+1\cdot0+0\cdot1]=0$ ✓

Completeness

By (2.162), we need to compute the matrix representation of $|V_1\rangle\langle V_1|+|V_2\rangle\langle V_2|+|V_3\rangle\langle V_3|=\hat{I}$, which is

$$\left(\frac{1}{\sqrt{2}}\right)^2\begin{pmatrix}1\\0\\-1\end{pmatrix}(1\ 0\ -1)+\begin{pmatrix}0\\1\\0\end{pmatrix}(0\ 1\ 0)+\left(\frac{1}{\sqrt{2}}\right)^2\begin{pmatrix}1\\0\\1\end{pmatrix}(1\ 0\ 1)$$

$$=\frac{1}{2}\begin{pmatrix}1&0&-1\\0&0&0\\-1&0&1\end{pmatrix}+\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1&0&1\\0&0&0\\1&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},$$

which is the matrix representation of $\hat{I}$.

Zettili Exercise 2.22 continued

(b) The projection operators were already worked out in part (a)!

$$\boxed{\hat{P}_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \hat{P}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{P}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}}$$

Each of P_1, P_2, P_3 are real symmetric, and so $\hat{P}_i^\dagger = \hat{P}_i^T = \hat{P}_i \Rightarrow$ Hermitian ✓

$$\hat{P}_1^2 = \frac{1}{4}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} = P_1 \checkmark$$

$$\hat{P}_2^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = P_2 \checkmark$$

$$\hat{P}_3^2 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = P_3 \checkmark$$

You can also show this from the Dirac notation [e.g. $\hat{P}_1^2 = (|v_1\rangle\langle v_1|)(|v_1\rangle\langle v_1|)$
$= |v_1\rangle \cancel{\langle v_1|v_1\rangle}^{1} \langle v_1| = \hat{P}_1$]

You can also show that these projectors are orthogonal ($\hat{P}_i\hat{P}_j = 0$ if $i = j$).

Zettili Exercise 2.41

(a) $\hat{A} = \exp\left(b\frac{d}{dx}\right) \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\left(b\frac{d}{dx}\right)^n = \sum_{n=0}^{\infty} \frac{b^n}{n!}\frac{d^n}{dx^n}$

$\hat{A}\psi(x) = \sum_{n=0}^{\infty} \frac{b^n}{n!}\frac{d^n}{dx^n}\psi(x) = \underline{\underline{\psi(x+b)}}$ (This is just the Taylor series)

(b) If $\psi(x)$ is an eigenfunction of $\hat{A}$ for some particular b, then $\psi(x+b) = \lambda\psi(x)$, for any x. But then, $\psi(x+2b) = \hat{A}\psi(x+b) = \lambda\psi(x+b) = \lambda^2\psi(x)$, and so on.

The function $\underline{\underline{\psi(x) \text{ scales by a factor } \lambda \text{ every time increases by } b.}}$

(c) If $\lambda = +1$, then $\psi(x) = \psi(x+b) = \psi(x+2b) = \dots$

Since x is arbitrary, this means that $\underline{\underline{\psi(x) \text{ is a periodic function with period } b.}}$

Zettili Exercise 2.43

$$\hat{H} = E\left(|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2| - i|\phi_1\rangle\langle\phi_2| + i|\phi_2\rangle\langle\phi_1|\right)$$

(a) $\hat{H}^\dagger = E^*\left(|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2| + i|\phi_2\rangle\langle\phi_1| - i|\phi_1\rangle\langle\phi_2|\right)$

where each $\left(|\phi_i\rangle\langle\phi_j|\right)^\dagger = |\phi_j\rangle\langle\phi_i|$, and each number is conjugated.

But since E is real, $E^* = E$, and we see that $\hat{H}^\dagger = \hat{H} \Rightarrow$ $\hat{H}$ is Hermitian.

$$\mathrm{Tr}(\hat{H}) = \sum_n \langle\phi_n|\hat{H}|\phi_n\rangle = \langle\phi_1|\hat{H}|\phi_1\rangle + \langle\phi_2|\hat{H}|\phi_2\rangle$$

Note that, in an orthonormal basis, the term $|\phi_i\rangle\langle\phi_j|$ contributes $\langle\phi_n|\phi_i\rangle\langle\phi_j|\phi_n\rangle$

$= \delta_{ni}\delta_{jn} = \delta_{ij}\delta_{in}$.

So we have $\mathrm{Tr}\,\hat{H} = E\left[(+1) - (+1) - i(0) + i(0)\right] = \boxed{0}$

(b) $|\phi_i\rangle\langle\phi_j|$ is represented by a matrix with $+1$ in the ijth element and zero elsewhere.

$$\Rightarrow \boxed{\hat{H} \to H = E\begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}} \qquad \mathrm{Tr}\,\hat{H} = \sum_n H_{nn} = H_{11} + H_{22} = E - E = \boxed{0} \checkmark$$

Eigenvalues:

$$0 = \begin{vmatrix} E-\lambda & -iE \\ +iE & -E-\lambda \end{vmatrix} \Rightarrow \begin{array}{l} (\lambda-E)(\lambda+E) - E^2 = 0 \\ \lambda^2 - 2E^2 = 0 \end{array} \Rightarrow \boxed{\lambda = \pm\sqrt{2}\,E}$$

$$Hu = \lambda u \Rightarrow E\begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \left.\begin{array}{l} E(x-iy) = \lambda x \\ E(+ix-y) = \lambda y \end{array}\right\} \Rightarrow y = +i\left(\frac{\lambda}{E} - 1\right)x$$

$$\lambda = +\sqrt{2}E \Rightarrow y = +i(\sqrt{2}-1)x \Rightarrow u_+ = \begin{pmatrix} 1 \\ +i(\sqrt{2}-1) \end{pmatrix} \quad \text{(not normalized)}$$

$$\lambda = -\sqrt{2}E \Rightarrow y = +i(-\sqrt{2}-1)x \Rightarrow u_- = \begin{pmatrix} 1 \\ -i(\sqrt{2}+1) \end{pmatrix} \quad \text{(not normalized)}$$

Alternate equivalent forms (if we solved for x in terms of y):

$$u_+ = \begin{pmatrix} -i(\sqrt{2}+1) \\ 1 \end{pmatrix}, \quad u_- = \begin{pmatrix} +i(\sqrt{2}-1) \\ 1 \end{pmatrix} \quad \text{(again, not normalized)}$$

Zettili Exercise 2.43 continued

(c) Commutators are linear [Eq. (2.82)], so we need only work out the individual pieces:

$$[\,|\phi_1\rangle\langle\phi_1|, |\phi_1\rangle\langle\phi_1|\,] = |\phi_1\rangle\overset{1}{\cancel{\langle\phi_1|\phi_1\rangle}}\langle\phi_1| - |\phi_1\rangle\overset{1}{\cancel{\langle\phi_1|\phi_1\rangle}}\langle\phi_1|$$
$$= |\phi_1\rangle\langle\phi_1| - |\phi_1\rangle\langle\phi_1| = 0 \text{, and so on.}$$

However, it is <u>much</u> more efficient to use the matrix representations:

$$|\phi_1\rangle\langle\phi_1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad |\phi_2\rangle\langle\phi_2| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad |\phi_1\rangle\langle\phi_2| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and then

$$[\hat{H}, |\phi_1\rangle\langle\phi_1|] \leftrightarrow \left[E\begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = E\left[\begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}\right]$$

$$= E\left[\begin{pmatrix} 1 & 0 \\ +i & 0 \end{pmatrix} - \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}\right] = +iE\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \leftrightarrow \boxed{+iE\left[|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|\right]}$$

$$[\hat{H}, |\phi_2\rangle\langle\phi_2|] \leftrightarrow \left[E\begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = E\left[\begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}\right]$$

$$= E\left[\begin{pmatrix} 0 & -i \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ +i & -1 \end{pmatrix}\right] = -iE\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \leftrightarrow \boxed{-iE\left[|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|\right]}$$

$$[\hat{H}, |\phi_1\rangle\langle\phi_2|] \leftrightarrow \left[E\begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = E\left[\begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}\right]$$

$$= E\left[\begin{pmatrix} 0 & 1 \\ 0 & +i \end{pmatrix} - \begin{pmatrix} +i & -1 \\ 0 & 0 \end{pmatrix}\right] = E\begin{pmatrix} -i & 2 \\ 0 & +i \end{pmatrix}$$

$$\leftrightarrow \boxed{E\left[-i|\phi_1\rangle\langle\phi_1| + i|\phi_2\rangle\langle\phi_2| + 2|\phi_1\rangle\langle\phi_2|\right]}$$

Zettili Exercise 2.48

$\hat{A} = \left(\hat{x}\frac{d}{dx} + 2\right)$

(a) $\hat{A}|\psi\rangle = 0|\psi\rangle \Rightarrow \left(x\frac{d}{dx} + 2\right)\psi(x) = 0 \Rightarrow x\frac{d\psi}{dx} = -2\psi$

This is a separable differential equation:

$$\frac{d\psi}{\psi} = \frac{-2}{x}dx \Rightarrow \ln\psi(x) = -2\ln x + C \Rightarrow \psi(x) = \frac{e^C}{x^2} \equiv \boxed{\frac{k}{x^2}}$$

Now attempt to normalize:

$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = |k|^2 \int_{-\infty}^{\infty} \frac{1}{x^4} dx$. Since $\int \frac{dx}{x^4} = -\frac{1}{3x^3}$, which is infinite as $x \to 0$, so $\boxed{\psi(x) \text{ is not normalizable}}$.

(b) Since $\langle\phi|\hat{A}^\dagger|\psi\rangle = \langle\psi|\hat{A}|\phi\rangle^*$ for arbitrary ψ and ϕ, $\hat{A}$ being Hermitian means $\langle\phi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}|\phi\rangle^*$. Is this true for our $\hat{A}$?

$$\langle\phi|\hat{A}|\psi\rangle = \int_{-\infty}^{\infty} \phi^*(x)\left(\hat{x}\frac{d}{dx} + 2\right)\psi(x)\,dx = \int_{-\infty}^{\infty} \phi^*(x)\left(x\frac{d\psi(x)}{dx} + 2\psi(x)\right)dx$$

$$\langle\psi|\hat{A}|\phi\rangle^* = \left[\int_{-\infty}^{\infty} \psi^*(x)\left(\hat{x}\frac{d}{dx} + 2\right)\phi(x)\,dx\right]^* = \int_{-\infty}^{\infty} \psi(x)\left(x\frac{d\phi^*(x)}{dx} + 2\phi^*(x)\right)dx$$

The parts $\int_{-\infty}^{\infty} 2\phi^*(x)\psi(x)\,dx$ match in both expressions.

However, the remaining parts are $\int_{-\infty}^{\infty} x\,\phi^*(x)\frac{d\psi}{dx}(x)\,dx$ and $\int_{-\infty}^{\infty} x\,\psi(x)\frac{d\phi^*}{dx}(x)\,dx$, which are unequal (and integration by parts does not make them equal, either).

Thus, $\boxed{\hat{A} \text{ is not Hermitian}}$.

(c) $$[\hat{A},\hat{x}]\psi(x) = \left(\hat{x}\frac{d}{dx} + 2\right)\hat{x}\psi(x) - \hat{x}\left(\hat{x}\frac{d}{dx} + 2\right)\psi(x)$$

$$= x\frac{d}{dx}(x\psi(x)) + 2x\psi(x) - x\left(x\frac{d\psi}{dx} + 2\psi(x)\right)$$

$$= x\psi + x^2\frac{d\psi}{dx} + 2x\psi - x^2\frac{d\psi}{dx} - 2x\psi$$

$$= \boxed{x\psi(x)} \Rightarrow \boxed{[\hat{A},\hat{x}] = \hat{x}}.$$

Zettili Exercise 2.49

The permutation or particle exchange operator is defined by $\hat{\pi}\psi(x,y) = \psi(y,x)$ for all ψ.

a) Linearity: Form the linear combination $\alpha_1\psi_1(x,y) + \alpha_2\psi_2(x,y) \equiv \psi(x,y)$.

Clearly, $\psi(y,x) = \alpha_1\psi_1(y,x) + \alpha_2\psi_2(y,x)$.

But $\psi(y,x) = \hat{\pi}\psi(x,y)$, $\psi_1(y,x) = \hat{\pi}\psi_1(x,y)$, $\psi_2(y,x) = \hat{\pi}\psi_2(x,y)$.

So $\hat{\pi}\psi(x,y) = \alpha_1\hat{\pi}\psi_1(x,y) + \alpha_2\hat{\pi}\psi_2(x,y)$ ✓

Hermiticity:

$\langle\psi|\hat{\pi}^{+}|\phi\rangle = \langle\phi|\hat{\pi}|\psi\rangle^*$ by Eq. (2.56).

$$\langle\phi|\hat{\pi}|\psi\rangle^* = \left[\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\,\phi^*(x,y)\,\hat{\pi}\,\psi(x,y)\right]^*$$ (Note that there are two positions, x and y, over which to integrate)

$$= \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\,\phi(x,y)\,\psi^*(y,x)$$ ($\hat{\pi}\psi(x,y) = \psi(y,x)$ and complex conjugating)

$$= \int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}dx\,\phi(y,x)\,\psi^*(x,y)$$ (Since x and y are integrated over in the same fashion, they are dummy variable names and may be switched.)

$$= \int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}dx\left[\hat{\pi}\phi(x,y)\right]\psi^*(x,y)$$

$$= \langle\psi|\hat{\pi}|\phi\rangle$$

Thus $\langle\psi|\hat{\pi}^{+}|\phi\rangle = \langle\psi|\hat{\pi}|\phi\rangle$ for any ϕ and $\psi \Rightarrow \hat{\pi}^{+} = \hat{\pi}$ ✓

(b) $\hat{\pi}^2\psi(x,y) = \hat{\pi}\psi(y,x) = \psi(x,y)$.

Since ψ is arbitrary, $\underline{\underline{\hat{\pi}^2 = \hat{I}}}$.

Eigenvector equation: $\hat{\pi}\psi(x,y) = \lambda\psi(x,y) = \psi(y,x)$ for some ψ and λ.

But then, $\psi(x,y) = \hat{I}\psi(x,y) = \hat{\pi}^2\psi(x,y) = \hat{\pi}\,\hat{\pi}\psi(x,y) = \hat{\pi}\left[\lambda\psi(x,y)\right]$

$= \lambda^2\psi(x,y) \Rightarrow \underline{\underline{\lambda^2 = 1}} \Rightarrow \boxed{\lambda = +1 \text{ or } -1}$

Zettili Exercise 2.49 continued

Now note that $\hat{\pi}\psi_\pm(x,y) = \hat{\pi}\,\frac{1}{2}[\psi(x,y) \pm \psi(y,x)] = \frac{1}{2}\hat{\pi}\,\psi(x,y) \pm \frac{1}{2}\hat{\pi}\,\psi(y,x)$

$$= \frac{1}{2}[\psi(y,x) \pm \psi(x,y)]$$

$$= \pm\frac{1}{2}[\psi(x,y) \pm \psi(y,x)]$$

$$= \pm\,\psi_\pm(x,y)$$

or

$$\boxed{\begin{aligned} \hat{\pi}\psi_+(x,y) &= (+1)\,\psi_+(x,y) \\ \hat{\pi}\psi_-(x,y) &= (-1)\,\psi_-(x,y) \end{aligned}}$$

Zettili Exercise 3.2

In general, $\psi(x,0) = \sum_{n=1}^{N} \alpha_n \psi_n(x)$. [Eq. (3.8)]

Since $\psi(x,0)$ is normalized, $\langle\psi(x,0)|\psi(x,0)\rangle = \sum_{n=1}^{N} |\alpha_n|^2 = 1$ [Eq. (3.10)]

(a) Since $P(E_1)=\frac{1}{2}$, $P(E_2)=\frac{3}{8}$, $P(E_3)=\frac{1}{8}$, and the sum of these gives unity, no other possibilities occur. $P_n = P(E_n) = |\alpha_n|^2$, so $|\alpha_1| = \frac{1}{\sqrt{2}}$, $|\alpha_2| = \sqrt{\frac{3}{8}}$, $|\alpha_3| = \frac{1}{\sqrt{8}}$.

Then $\alpha_1 = \frac{e^{i\theta_1}}{\sqrt{2}}$, $\alpha_2 = \sqrt{\frac{3}{8}}\, e^{i\theta_2}$, $\alpha_3 = \frac{1}{\sqrt{8}} e^{i\theta_3}$, where $\theta_1, \theta_2, \theta_3$ are real numbers.

Thus the most general expansion for $\psi(x,0)$ consistent with the given information is:

$$\boxed{\psi(x,0) = \frac{e^{i\theta_1}}{\sqrt{2}}\psi_1(x) + \sqrt{\frac{3}{8}}\, e^{i\theta_2}\psi_2(x) + \frac{1}{\sqrt{8}} e^{i\theta_3}\psi_3(x)}$$

(b) Each energy eigenstate $\psi_n(x)$ is a stationary state, and develops in time according to Eq. (3.65), $\psi_n(x,t) = \psi_n(x) e^{-iE_n t/\hbar}$.

Thus $$\boxed{\psi(x,t) = \frac{e^{i\theta_1}}{\sqrt{2}}\psi_1(x) e^{-iE_1 t/\hbar} + \sqrt{\frac{3}{8}}\, e^{i\theta_2}\psi_2(x) e^{-iE_2 t/\hbar} + \frac{1}{\sqrt{8}} e^{i\theta_3}\psi_3(x) e^{-iE_3 t/\hbar}}$$

(c) $\langle \hat{H} \rangle = \langle \psi(t) | \hat{H} | \psi(t) \rangle$

where of course

$$|\psi(t)\rangle = \frac{e^{i\theta_1}}{\sqrt{2}} e^{-iE_1 t/\hbar} |\psi_1\rangle + \sqrt{\frac{3}{8}}\, e^{i\theta_2} e^{-iE_2 t/\hbar} |\psi_2\rangle + \frac{1}{\sqrt{8}} e^{i\theta_3} e^{-iE_3 t/\hbar} |\psi_3\rangle$$

and then

$$\hat{H}|\psi(t)\rangle = \frac{e^{i\theta_1}}{\sqrt{2}} (E_1) e^{-iE_1 t/\hbar} |\psi_1\rangle + \sqrt{\frac{3}{8}}\, e^{i\theta_2} (E_2) e^{-iE_2 t/\hbar} |\psi_2\rangle + \frac{1}{\sqrt{8}} e^{i\theta_3} (E_3) e^{-iE_3 t/\hbar} |\psi_3\rangle$$

When we form $\langle\psi(t)|\hat{H}|\psi(t)\rangle$, we obtain inner products of the form $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ due to orthonormality.

Zettili Exercise 3.2 continued

Then $\langle \hat{H} \rangle = \left| \frac{e^{i\theta_1}}{\sqrt{2}} e^{-iE_1 t/\hbar} \right|^2 E_1 + \left| \sqrt{\frac{3}{8}} e^{i\theta_2} e^{-iE_2 t/\hbar} \right|^2 E_2 + \left| \frac{1}{\sqrt{8}} e^{i\theta_3} e^{-iE_3 t/\hbar} \right|^2 E_3$

$$\boxed{\langle \hat{H} \rangle = \frac{1}{2} E_1 + \frac{3}{8} E_2 + \frac{1}{8} E_3}$$

This is of course a constant in time.

Zettili Exercise 3.11

$$H = \varepsilon_0 \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = a_0 \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad H^2 = \varepsilon_0^2 \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) $HA = \varepsilon_0 a_0 \begin{pmatrix} -10 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = AH$, so $[H,A]=0$.

H is already diagonal in this basis, so the eigenvalues are the diagonal elements

$\underline{\underline{-2\varepsilon_0, +\varepsilon_0, +\varepsilon_0}}$. Hence $+\varepsilon_0$ is twofold degenerate, and H by itself is not a CSCO.

So we seek the eigenvalues and eigenvectors of A.

Eigenvalues: $|A - \lambda \hat{I}| = \begin{vmatrix} 5a_0-\lambda & 0 & 0 \\ 0 & -\lambda & 2a_0 \\ 0 & 2a_0 & -\lambda \end{vmatrix} = (5a_0-\lambda)\left[(-\lambda)^2 - (2a_0)^2\right] = 0$

$\Rightarrow \lambda = \underline{\underline{+5a_0, +2a_0, -2a_0}}$. Since these eigenvalues are nondegenerate, we immediately know that $\hat{A}$ by itself forms a CSCO.

Eigenvectors of A:

$$a_0 \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = +5a_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{array}{l} 5a_0 x = 5a_0 x \\ 2a_0 z = 5a_0 y \\ 2a_0 y = 5a_0 z \end{array} \Rightarrow \begin{array}{l} x \text{ arbitrary} \\ y = z = 0 \end{array} \Rightarrow \boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$$a_0 \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = +2a_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{array}{l} 5a_0 x = 2a_0 x \\ 2a_0 z = 2a_0 y \\ 2a_0 y = 2a_0 z \end{array} \Rightarrow \begin{array}{l} x = 0 \\ y = z \end{array} \Rightarrow \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$$

$$a_0 \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2a_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{array}{l} 5a_0 x = -2a_0 x \\ 2a_0 z = -2a_0 y \\ 2a_0 y = -2a_0 z \end{array} \Rightarrow \begin{array}{l} x = 0 \\ y = -z \end{array} \Rightarrow \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$$

Although this is guaranteed by the fact that $[H,A]=0$, it is good to check that these are also H eigenvectors: $H\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -2\varepsilon_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $H \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm 1 \end{pmatrix} = +\varepsilon_0 \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm 1 \end{pmatrix}$.

Zettili, Exercise 3.11 continued

The joint kets in the notation we used in class are $|+2\mathcal{E}_0, +5a_0\rangle$, $|+\mathcal{E}_0, +2a_0\rangle$, and $|+\mathcal{E}_0, -2a_0\rangle$, respectively.

(b) Note that if H is diagonal in some basis and has degenerate eigenvalues, the same holds for H^2, with eigenvalues $+4\mathcal{E}_0^2, +\mathcal{E}_0^2, +\mathcal{E}_0^2$.

Also, $[H^2, A] = H\,\underbrace{[H, A]}_{0} + \underbrace{[H, A]}_{0}\,H = 0$

So H^2 and A also have the same common basis of eigenvectors as H and A.

- In part (a), we saw that $\{H\}$ is not a CSCO because of the degeneracy; but $\{A\}$ is a CSCO because its eigenvalues are nondegenerate over the whole (three-dimensional) space: they are complete.

- Adding commuting operators to $\{A\}$ does not spoil it being a CSCO. It is still true that the eigenvectors of A form a complete and nondegenerate set. So if we form $\{A, H\}$, then specifying a particular A eigenvalue/eigenvector uniquely specifies the H eigenvalue, and similarly for $\{A, H\}$.

Thus $\{A, H\}$ and $\{A, H^2\}$ are both CSCOs.

Zettili Exercise 3.15

$$H = \varepsilon_0 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix}, \quad A = a_0 \begin{pmatrix} 0 & -i & 0 \\ i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) The possible values of energy are just the eigenvalues of H.

$$0 = |H - \lambda I| = \begin{vmatrix} -\lambda & -i\varepsilon_0 & 0 \\ i\varepsilon_0 & -\lambda & 2i\varepsilon_0 \\ 0 & -2i\varepsilon_0 & -\lambda \end{vmatrix} = -\lambda\left[(-\lambda)^2 - (2i\varepsilon_0)(-2i\varepsilon_0)\right] - (-i\varepsilon_0)\left[(i\varepsilon_0)(-\lambda)\right]$$

$$= -\lambda(\lambda^2 - 4\varepsilon_0^2) + \lambda\varepsilon_0^2$$

$$\Rightarrow 0 = -\lambda(\lambda^2 - 5\varepsilon_0^2) \Rightarrow \boxed{\lambda = 0, \pm\sqrt{5}\,\varepsilon_0}$$

(b) Measuring E and finding $+\sqrt{5}\varepsilon_0$ means that the wave function, whatever it was beforehand, collapses to the corresponding eigenvector of H.

$$\lambda = +\sqrt{5}\varepsilon_0 : \quad \varepsilon_0 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sqrt{5}\varepsilon_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{aligned} -i\varepsilon_0 y &= +\sqrt{5}\varepsilon_0 x \\ i\varepsilon_0 x + 2i\varepsilon_0 z &= +\sqrt{5}\varepsilon_0 y \\ -2i\varepsilon_0 y &= +\sqrt{5}\varepsilon_0 z \end{aligned}$$

$$\Rightarrow y = i\sqrt{5}x, \quad z = -\frac{2i}{\sqrt{5}} y = +2x$$

$$\Rightarrow \text{Eigenvector (including normalization)}: \underline{\underline{\frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ i\sqrt{5} \\ 2 \end{pmatrix}}} \rightarrow |E_{+\sqrt{5}\varepsilon_0}\rangle$$

The possible values of A are its eigenvalues:

$$0 = |A - \lambda I| = \begin{vmatrix} -\lambda & -ia_0 & 0 \\ ia_0 & a_0 - \lambda & a_0 \\ 0 & a_0 & -\lambda \end{vmatrix} = -\lambda\left[(a_0 - \lambda)(-\lambda) - a_0^2\right] - (-ia_0)\left[(ia_0)(-\lambda)\right]$$

$$= -\lambda\left[\lambda(\lambda - a_0) - a_0^2\right] + \lambda a_0^2$$

$$\Rightarrow 0 = -\lambda\left[\lambda^2 - \lambda a_0 - 2a_0^2\right] = -\lambda(\lambda + a_0)(\lambda - 2a_0) \Rightarrow \boxed{\lambda = 0, -a_0, +2a_0}$$

To compute the probability for each one, we have to project the new initial state $|E_{+\sqrt{5}\varepsilon_0}\rangle$ onto the eigenvector corresponding to each A eigenvalue.

Zettili Exercise 3.15 continued

(b) continued

Eigenvectors of A: $\lambda = 0$: $a_0 \begin{pmatrix} 0 & -i & 0 \\ i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} -iy = 0 \\ ix + y + z = 0 \\ y = 0 \end{matrix} \Rightarrow \begin{matrix} y = 0 \\ z = -ix \end{matrix}$

$$\Rightarrow \lambda = 0: \ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} \equiv |a_0\rangle$$

$\lambda = -a_0$: $a_0 \begin{pmatrix} 0 & -i & 0 \\ i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -a_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} -iy = -x \\ ix + y + z = -y \\ y = -z \end{matrix} \Rightarrow \begin{matrix} y = -ix \\ z = +ix \end{matrix} \Rightarrow |a_{-a_0}\rangle \rightarrow \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -i \\ +i \end{pmatrix}$

$\lambda = +2a_0$: $a_0 \begin{pmatrix} 0 & -i & 0 \\ i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = +2a_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} -iy = 2x \\ ix + y + z = 2y \\ y = 2z \end{matrix} \Rightarrow \begin{matrix} y = +2ix \\ z = +ix \end{matrix} \Rightarrow |a_{+2a_0}\rangle \rightarrow \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ +2i \\ +i \end{pmatrix}$

Then $P(a = -a_0) = |\langle a_{-a_0} | E_{+\sqrt{5}\varepsilon_0}\rangle|^2 = \left| \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & +i & -i \end{pmatrix} \begin{pmatrix} 1 \\ +i\sqrt{5} \\ 2 \end{pmatrix} \right|^2$

$$= \frac{1}{30} |1 - \sqrt{5} - 2i|^2 = \frac{1}{30}\left[(1-\sqrt{5})^2 + 4\right]$$

$$= \frac{1}{30}(1 - 2\sqrt{5} + 5 + 4) = \frac{1}{30}(10 - 2\sqrt{5}) = \boxed{\frac{1}{15}(5 - \sqrt{5})}$$

($\approx 18.4\%$)

$P(a = 0) = |\langle a_{-a_0} | E_{+\sqrt{5}\varepsilon_0}\rangle|^2 = \left| \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & +i \end{pmatrix} \begin{pmatrix} 1 \\ +i\sqrt{5} \\ 2 \end{pmatrix} \right|^2$

$$= \frac{1}{20} |1 + 2i|^2 = \frac{1}{20}\left[1^2 + 2^2\right] = \boxed{\frac{1}{4}}$$

$P(a = +2a_0) = |\langle a_{+2a_0} | E_{+\sqrt{5}\varepsilon_0}\rangle|^2 = \left| \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -2i & -i \end{pmatrix} \begin{pmatrix} 1 \\ +i\sqrt{5} \\ 2 \end{pmatrix} \right|^2$

$$= \frac{1}{60} |1 + 2\sqrt{5} - 2i|^2 = \frac{1}{60}\left[(1+2\sqrt{5})^2 + 4\right]$$

$$= \frac{1}{60}(1 + 4\sqrt{5} + 20 + 4) = \boxed{\frac{1}{60}(25 + 4\sqrt{5})} \ (\approx 56.6\%)$$

Note that the probabilities add up to 1, as they must.

Zettili Exercise 3.15 continued

(c) $\langle A \rangle = \sum_n P(a_n) a_n = \frac{1}{15}(5-\sqrt{5})(-a_0) + \frac{1}{4}(0) + \frac{1}{60}(25+4\sqrt{5})(+2a_0)$

$\Rightarrow \boxed{\langle A \rangle = \left(\frac{1}{2} + \frac{\sqrt{5}}{5}\right) a_0} \approx 0.95 a_0$

Zettili Exercise 3.22

$$H = \mathcal{E}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

(a) Eigenvalues: $\begin{vmatrix} -\lambda & -i\mathcal{E} \\ +i\mathcal{E} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \mathcal{E}^2 = 0 \Rightarrow \boxed{\lambda = \pm\mathcal{E}}$, so let $E_1 = +\mathcal{E}$, $E_2 = -\mathcal{E}$.

(b) $\langle\hat{H}\rangle$ in state $|\psi_0\rangle$ is $\dfrac{\langle\psi_0|\hat{H}|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle}$. $\langle\psi^0|\psi^0\rangle = (1\ 0)\begin{pmatrix}1\\0\end{pmatrix} = 1$, so $|\psi_0\rangle$ is already normalized. $\langle\psi_0|\hat{H}|\psi_0\rangle = (1\ 0)\,\mathcal{E}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = (1 0)\,\mathcal{E}\begin{pmatrix}0\\+i\end{pmatrix} = \underline{\underline{0}}$

So $\langle\hat{H}\rangle = 0$.

But we also know $\langle\hat{H}\rangle = P(E_1)E_1 + P(E_2)E_2 = P(+\mathcal{E})(+\mathcal{E}) + P(-\mathcal{E})(-\mathcal{E}) = \mathcal{E}[P(+\mathcal{E}) - P(-\mathcal{E})]$

So $\boxed{P(+\mathcal{E}) = P(-\mathcal{E}) = \tfrac{1}{2}}$, since $P(+\mathcal{E}) + P(-\mathcal{E}) = 1$.

we could also solve this part by first working out the E_1 and E_2 eigenstates and then computing $P(E_1) = |\langle\psi_1|\psi_0\rangle|^2$, $P(E_2) = |\langle\psi_2|\psi_0\rangle|^2$.

(c) $\boxed{\langle\hat{H}\rangle = 0}$ from part (b).

$$\langle\hat{H}^2\rangle = (1 0)\,\mathcal{E}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\mathcal{E}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \mathcal{E}^2(1\ 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \boxed{\mathcal{E}^2}$$

So $\Delta E = \sqrt{\langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2} = \sqrt{\mathcal{E}^2 - 0} = \boxed{\mathcal{E}}$

(d) Now we need the energy eigenstates.

$|\psi_1\rangle$: $\mathcal{E}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = +\mathcal{E}\begin{pmatrix}a\\b\end{pmatrix} \Rightarrow \begin{matrix} -ib = a \\ +ia = b \end{matrix} \Rightarrow \boxed{|\psi_1\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\+i\end{pmatrix}}$ (note the normalization)

$|\psi_2\rangle$: $\mathcal{E}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = -\mathcal{E}\begin{pmatrix}a\\b\end{pmatrix} \Rightarrow \begin{matrix} -ib = -a \\ +ia = -b \end{matrix} \Rightarrow \boxed{|\psi_2\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}}$

Noting that $|\psi_0\rangle = |\psi_1\rangle\langle\psi_1|\psi_0\rangle + |\psi_2\rangle\langle\psi_2|\psi_0\rangle$
$= \frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle$,

$$\psi(t) = \frac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1t/\hbar} + \frac{1}{\sqrt{2}}\psi_2(x)e^{-iE_2t/\hbar} = \boxed{\frac{1}{2}\left[\begin{pmatrix}1\\+i\end{pmatrix}e^{-i\mathcal{E}t/\hbar} + \begin{pmatrix}1\\-i\end{pmatrix}e^{+i\mathcal{E}t/\hbar}\right] = \begin{pmatrix}\cos(\mathcal{E}t/\hbar)\\ \sin(\mathcal{E}t/\hbar)\end{pmatrix}}$$

Zettili Exercise 3.25

By Eq. (3.88), $\frac{d\langle\hat{A}\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle + \langle\frac{\partial\hat{A}}{\partial t}\rangle$

In our case, $\hat{X}$, $\hat{P}_x$, and $\hat{H}$ have no explicit time dependence $\Rightarrow$ $\frac{\partial\hat{A}}{\partial t} = 0$ for each $\hat{A}$.

$\frac{d\langle\hat{X}\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{X},\hat{H}]\rangle$. Since $[\hat{X},\hat{X}^2]=0$, $[\hat{X},\hat{X}^3]=0$, $\hat{H} = \frac{\hat{P}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 + V_0\hat{X}^3$

gives $\frac{d\langle\hat{X}\rangle}{dt} = \frac{1}{i\hbar}\cdot\frac{1}{2m}\langle[\hat{X},\hat{P}_x^2]\rangle$.

By (2.306) [or (2.84) with $\hat{A}=\hat{X}$, $\hat{B}=\hat{C}=\hat{P}_x$], we have $[\hat{X},\hat{P}_x^2] = 2i\hbar\hat{P}_x$.

So $\boxed{\frac{d\langle\hat{X}\rangle}{dt} = \frac{1}{m}\langle\hat{P}_x\rangle}$.

Next, $[\hat{P}_x,\hat{P}_x^2]=0$, and $[\hat{P}_x,\hat{X}^2] = -[\hat{X}^2,\hat{P}_x] = -2i\hbar\hat{X}$ (2.306),

$[\hat{P}_x,\hat{X}^3] = -[\hat{X}^3,\hat{P}_x] = -3i\hbar\hat{X}^2$ (2.306),

$$\frac{d\langle\hat{P}_x\rangle}{dt} = \frac{1}{2}m\omega^2\left[\frac{1}{i\hbar}(-2i\hbar)\langle\hat{X}\rangle\right] + V_0\left[\frac{1}{i\hbar}(-3i\hbar)\langle\hat{X}^2\rangle\right]$$

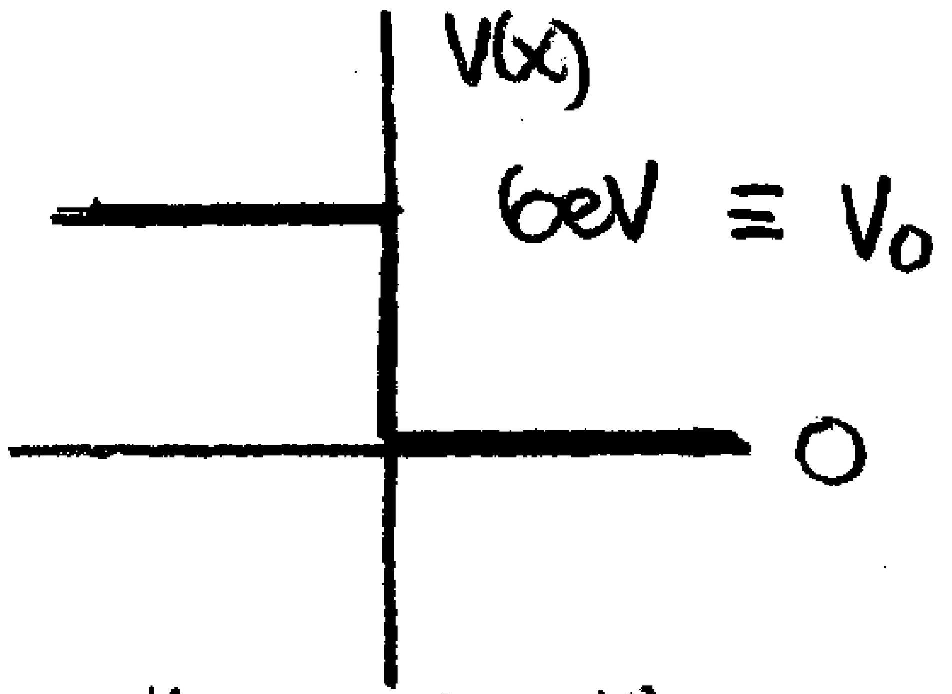
$$\boxed{\frac{d\langle\hat{P}_x\rangle}{dt} = -m\omega^2\langle\hat{X}\rangle - 3V_0\langle\hat{X}^2\rangle}$$

(More generally, using these methods we can prove $\frac{d\langle p_x\rangle}{dt} = \langle-\frac{\partial V}{\partial x}\rangle$.)

$$\boxed{\frac{d\langle\hat{H}\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{H},\hat{H}]\rangle = 0}$$

Zettili Exercise 4.8

The potential looks like this:



In part (a), the electron enters from the left with more energy than the barrier,
In part (b), the electron enters from the right with less energy than the barrier.

(a) (i) is just the transmission coefficient T, and (ii) is just the reflection coefficient R. Remember that T and R are probabilities that a given wave will transmit or reflect because they are ratios of probability currents in a given direction with respect to the probability current of the initial wave.

For $x<0$, $\frac{-\hbar^2}{2m}\psi''(x) + V_0\psi(x) = E\psi(x)$, where $\underline{\underline{E=8\,eV \text{ and } V_0 = 6eV.}}$

$$\Rightarrow \quad \psi''(x) = -\frac{2m}{\hbar^2}[E-V_0]\psi(x) \quad . \quad \text{Let } \underline{\underline{k \equiv \sqrt{\frac{2m(E-V_0)}{\hbar^2}}}} \quad (\text{since } E>V_0)$$

$$\Rightarrow \quad \underline{\underline{\psi(x) = A_k^+ e^{+ikx} + A_k^- e^{-ikx}}}$$

For $x>0$, $\frac{-\hbar^2}{2m}\psi''(x) = E\psi(x) \Rightarrow \psi''(x) = -\frac{2m}{\hbar^2}E\psi(x)$. Let $\underline{\underline{k' \equiv \sqrt{\frac{2mE}{\hbar^2}}}}$

$$\Rightarrow \quad \underline{\underline{\psi(x) = B_{k'}^+ e^{+ik'x} + \cancel{B_{k'}^- e^{-ik'x}}}}$$ Ignore, since no wave is incoming from the right

- Now we impose the boundary conditions. Since $V(x)$ is finite (although discontinuous at $x=0$), we have $\psi(x)$ and $\psi'(x)$ continuous at $x=0$.

$\underline{\psi(0)}$: $\quad A_k^+ e^{+ik(0)} + A_k^- e^{-ik(0)} = B_{k'}^+ e^{+ik'(0)}$

$$\longrightarrow \quad \underline{\underline{A_k^+ + A_k^- = B_{k'}^+}}$$

$\underline{\psi'(0)}$: $\quad +ikA_k^+ e^{+ik(0)} - ikA_k^- e^{-ik(0)} = +ik'B_{k'}^+ e^{+ik'(0)}$

$$\longrightarrow \quad \underline{\underline{+ik(A_k^+ - A_k^-) = +ik'B_{k'}^+}}$$

Zettili Exercise 4.8 continued

Solve for A_k^- and $B_{k'}^+$ in terms of the incident amplitude A_k^+:

$$A_k^- = \frac{k-k'}{k+k'} A_k^+ \,, \qquad B_{k'}^+ = \frac{2k}{k+k'} A_k^+$$

Now, we have seen that j_x for a wave of the form $\psi(x) = Ce^{\pm ikx}$ equals $\pm \frac{\hbar k}{m} |C|^2$ [or, we could compute it directly from Eq. (3.83)],

so $j_{inc} = +\frac{\hbar k}{m} |A_k^+|^2$, $j_{ref} = -\frac{\hbar k}{m} |A_k^-|^2$, $j_{trans} = +\frac{\hbar k'}{m} |B_{k'}^+|^2$,

and finally, $T = \left| \frac{j_{trans}}{j_{inc}} \right| = \frac{k'}{k} \left| \frac{B_{k'}^+}{A_k^+} \right|^2 = \frac{k'}{k} \left(\frac{2k}{k+k'} \right)^2 = \frac{4kk'}{(k+k')^2} = \frac{4\sqrt{E(E-V_0)}}{(\sqrt{E-V_0}+\sqrt{E})^2}$

Letting $\varepsilon \equiv \frac{V_0}{E} = \frac{6\,eV}{8\,eV} = \frac{3}{4}$, we have

$$T = \frac{4\sqrt{1-\varepsilon}}{(1+\sqrt{1-\varepsilon})^2} = \boxed{\frac{8}{9}}$$

while $R = \left| \frac{j_{ref}}{j_{inc}} \right| = \left| \frac{A_k^-}{A_k^+} \right|^2 = \frac{(k-k')^2}{(k+k')^2} = \frac{(\sqrt{E-V_0}-\sqrt{E})^2}{(\sqrt{E-V_0}+\sqrt{E})^2} = \frac{(1-\sqrt{1-\varepsilon})^2}{(1+\sqrt{1-\varepsilon})^2} = \boxed{\frac{1}{9}}$,

and of course $T+R=1$, as they must.

(b) Now, we have the same Schrödinger equation for $x>0$ as in part (a), but keep both terms $\psi(x) = B_{k'}^+ e^{+ik'x} + B_{k'}^- e^{-ik'x}$, where now $B_{k'}^-$ is the initial amplitude, and again $k' \equiv \sqrt{\frac{2mE}{\hbar^2}}$, but here $E = 3\,eV$.

As for $x<0$, we have the same Schrödinger equation as in part (a), but $E - V_0 = 3\,eV - 6\,eV < 0$, so we have

$$\psi''(x) = \frac{2m}{\hbar^2}[V_0 - E]\,\psi(x) \quad \text{and let} \quad k'' \equiv \sqrt{\frac{2m(V_0-E)}{\hbar^2}},$$

so that $\psi(x) = C_{k''}^+ e^{+k''x} + C_{k''}^- e^{-k''x}$

Zettili Exercise 4.8 continued

The solution with $C_{k''}^-$ blows up as $x \to -\infty$. Not only is it not normalizable, its probability density $|\psi(x)|^2$ grows without bound. So we set $C_{k''}^- = 0$.

Now we impose the boundary conditions.

$\psi(0)$:
$$C_{k''}^+ e^{+k''(0)} = B_{k'}^+ e^{+ik'(0)} + B_{k'}^- e^{-ik'(0)}$$
$$\to C_{k''}^+ = B_{k'}^+ + B_{k'}^-$$

$\psi'(0)$:
$$+k'' C_{k''}^+ e^{+k''(0)} = ik' B_{k'}^+ e^{+ik'(0)} - ik' B_{k'}^- e^{-ik'(0)}$$
$$\to k'' C_{k''}^+ = ik'(B_{k'}^+ - B_{k'}^-)$$

The solution of these in terms of initial amplitude $B_{k'}^-$ is

$$B_{k'}^+ = \frac{(ik'+k'')}{(ik'-k'')} B_{k'}^- , \quad C_{k''}^+ = \frac{2ik'}{ik'-k''} B_{k'}^-$$

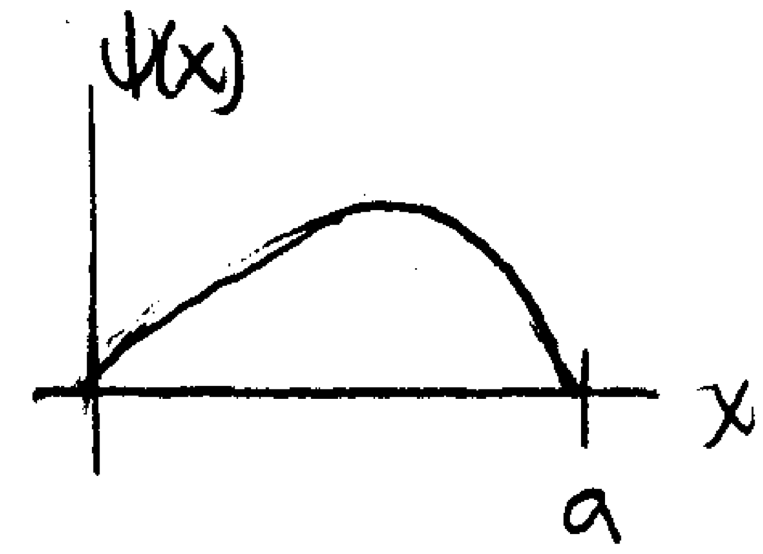
The initial wave being scaled by $B_{k'}^-$, the relative probability density for penetrating the barrier is
$$\left| \frac{C_{k''}^+ e^{k''x}}{B_{k'}^-} \right|^2 = \frac{4k'^2}{(k'^2+k''^2)} e^{-2k''x}$$

In fact, for this part we did not have to match boundary conditions. All we care about is the $e^{-2k''x}$ barrier factor which falls to e^{-1} by $x = +\frac{1}{2k''}$, a distance of $\sqrt{\frac{\hbar^2}{8m(V_0-E)}} = \frac{\hbar c}{\sqrt{8mc^2(V_0-E)}} \equiv d.$

(i) For $V_0 - E = 3\,\text{eV}$, $\hbar c = 1973\,\text{eV}\cdot\text{Å}$, $mc^2 = 511\,\text{keV}$, we have
$$d = \frac{1973\,\text{eV}\cdot\text{Å}}{\sqrt{8\cdot 511{,}000\,\text{eV}\cdot 3\,\text{eV}}} = \boxed{0.56\,\text{Å} = 5.6\cdot 10^{-11}\,\text{m} = 56\,\text{pm}}$$ (order of magnitude)

(ii) For $m = 70\,\text{kg}$, $E = \frac{mv^2}{2}$, $v = 4\frac{\text{m}}{\text{s}}$, $V_0 = 4E$, $d = \sqrt{\frac{\hbar^2}{8m\cdot 3mv^2/2}} = \frac{\hbar}{2\sqrt{3}\,mv}$
$$d = \frac{1.055\cdot 10^{-34}\,\text{J}\cdot\text{s}}{2\sqrt{3}} \cdot \frac{1}{70\,\text{kg}} \cdot \frac{\text{s}}{4\,\text{m}} \cdot \frac{\text{kg}\,\text{m}^2}{\text{J}\,\text{s}^2} = \boxed{1.1\cdot 10^{-37}\,\text{m}}$$ (again, order of magnitude)

Zettili Exercise 4.17



$$\psi(x) = \begin{cases} Ax(a^2-x^2), & 0<x<a \\ 0 & \text{elsewhere} \end{cases}$$

Since the eigenfunctions $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ for $0<x<a$ all vanish at $x=0$ and a, the fact that $\psi(0)=\psi(a)=0$ means that $\psi(x)$ can be expressed as a linear combination of them.

- To simplify the integrals, let's write $x=au$, $u=0\to1$ everywhere.

(a) Normalization: $\int_{-\infty}^{\infty}|\psi(x)|^2dx = \int_0^a|\psi(x)|^2dx = |A|^2\int_0^a x^2(a^2-x^2)^2dx$

$= |A|^2a^7\int_0^1 u^2(1-u^2)^2du = |A|^2a^7\int_0^1(u^2-2u^4+u^6)du = |A|^2a^7\left[\frac{u^3}{3}-\frac{2u^5}{5}+\frac{u^7}{7}\right]_0^1$

$= |A|^2a^7\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right) = \frac{8a^7}{105}|A|^2 \Rightarrow \boxed{A=\sqrt{\frac{105}{8a^7}}}$, $\psi(x)=\sqrt{\frac{105}{8a^7}}\,x(a^2-x^2)$.

(b) Noting that $E=\frac{5^2\pi^2\hbar^2}{2ma^2}$ is the infinite square well energy eigenvalue E_5 (i.e., $n=5$), we need only compute $P(E_5)=|\langle\psi_5|\psi\rangle|^2 = \left|\int_0^a\psi_5^*(x)\psi(x)\right|^2$

$= \left|\int_0^a\sqrt{\frac{2}{a}}\sin\left(\frac{5\pi x}{a}\right)\sqrt{\frac{105}{8a^7}}\,x(a^2-x^2)\,dx\right|^2.$

The integral is $\sqrt{\frac{2}{a}}\sqrt{\frac{105}{8a^7}}\,a^4\int_0^1\sin(5\pi u)\,u(1-u^2)\,du$

We can look up or derive: $\int\sin(n\pi u)\,u\,du = \frac{1}{(n\pi)^2}\left[\sin(n\pi u)-(n\pi u)\cos(n\pi u)\right]$

$\int\sin(n\pi u)\,u^3du = \frac{3}{(n\pi)^3}\left[(n\pi u)^2-2\right]\sin(n\pi u) - \frac{u}{(n\pi)^3}\left[(n\pi u)^2-6\right]\cos(n\pi u)$

The terms with $\sin(n\pi u)$ vanish at both $u=0$ and 1, and terms proportional to u vanish at $u=0$.

We have

$\int_0^1\sin(5\pi u)\,u(1-u^2)du = -\frac{1}{(5\pi)}\cos(5\pi) + \frac{1}{(5\pi)^3}\left[(5\pi)^2-6\right]\cos(5\pi)$

$= \frac{-6}{(5\pi)^3}\cos(5\pi) = \underline{\underline{+\frac{6}{(5\pi)^3}}}.$

Zettili Exercise 4.17 continued

(b) continued

$$P(E_5) = \frac{2}{a}\cdot\frac{105}{8a^7}\cdot a^8\cdot\left|\frac{6}{(5\pi)^3}\right|^2 = \boxed{\frac{189}{3125\pi^6} = 6.29\cdot 10^{-5}}$$

(c) $\Delta x = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2}$, $\Delta p = \sqrt{\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2}$.

$$\langle\hat{x}^n\rangle = \int_0^a \psi^*(x)\, x^n\, \psi(x)\, dx = \frac{105}{8a^7}\int_0^a x^{n+2}(a^2-x^2)^2\, dx = \frac{105}{8}a^n\int_0^1 u^{n+2}(1-u^2)^2\, du$$

$$\langle\hat{p}^n\rangle = \int_0^a \psi^*(x)\left(-i\hbar\frac{d}{dx}\right)^n\psi(x)\, dx = \frac{105}{8a^7}\int_0^a x(a^2-x^2)\left(-i\hbar\frac{d}{dx}\right)^n x(a^2-x^2)\, dx = \frac{105}{8a^n}(-i\hbar)^n\int_0^1 u(1-u^2)\left(\frac{d}{du}\right)^n u(1-u^2)\, du$$

$$\langle\hat{x}\rangle = \frac{105}{8}a\int_0^1 u^3(1-u^2)^2\, du = \frac{105}{8}a\cdot\frac{1}{24} = \frac{35}{64}a$$

$$\langle\hat{x}^2\rangle = \frac{105}{8}a^2\int_0^1 u^4(1-u^2)^2\, du = \frac{105}{8}a^2\cdot\frac{8}{315} = \frac{a^2}{3}$$

$$\Rightarrow \Delta x = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2} = a\sqrt{\frac{1}{3}-\left(\frac{35}{64}\right)^2} = \boxed{\frac{\sqrt{1263}}{192}a \simeq 0.185a}$$

$$\langle\hat{p}\rangle = \frac{105}{8a}(-i\hbar)\int_0^1 u(1-u^2)(1-3u^2)\, du = \boxed{0}$$

$$\langle\hat{p}^2\rangle = \frac{105}{8a^2}(-\hbar^2)\int_0^1 u(1-u^2)(-6u)\, du = \frac{105}{8a^2}\cdot(-\hbar^2)\left(-\frac{4}{5}\right) = \boxed{\frac{21}{2}\left(\frac{\hbar}{a}\right)^2}$$

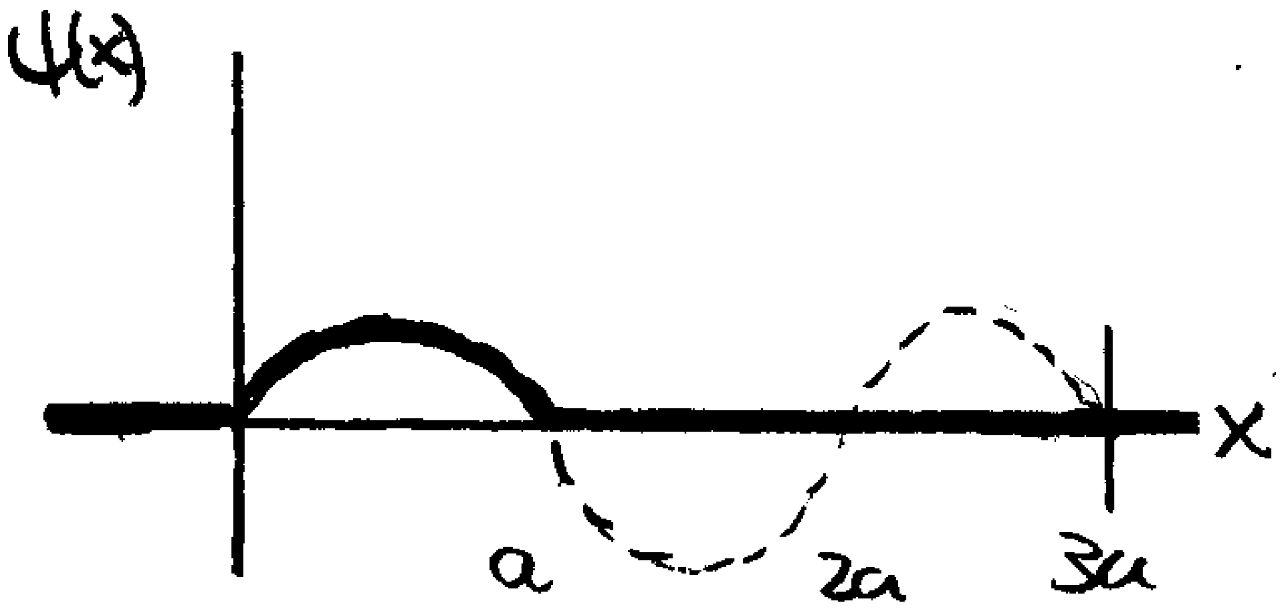
$$\Rightarrow \Delta p = \sqrt{\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2} = \boxed{\sqrt{\frac{21}{2}}\frac{\hbar}{a} \simeq 3.24\frac{\hbar}{a}}$$

$$\Delta x\cdot\Delta p = \boxed{\frac{\sqrt{5894}}{128}\hbar \simeq 0.60\hbar} \quad \left(> \frac{\hbar}{2}\right)$$

Zettili Exercise 4.26

The well instantly expands from width a to $3a$.

However, since we are not measuring anything in this process, the particle's wave function does not change or collapse to anything else.



Since the change from a to $3a$ just introduces extra space ($x=a$ to $3a$) where $\psi(x)$ is still zero, the normalization of the wave function does not change either.

Let us use lowercase $\psi_n(x)$ for the eigenfunctions of the original width a well, and uppercase $\Psi_n(x)$ for those of the width $3a$ well:

$$\psi_n(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right),\quad E_n=\frac{\pi^2\hbar^2}{2ma^2}\ ;\quad \Psi_n(x)=\sqrt{\frac{2}{3a}}\sin\left(\frac{n\pi x}{3a}\right),\quad E_n=\frac{\pi^2\hbar^2}{2m(3a)^2}.$$

The new wave function is numerically equal to $\psi_1(x)$, except in a bigger box, $x=0\to 3a$.

To emphasize this, let us use N for the quantum number of the bigger box.

We then want $P(E_N)=|\langle\Psi_N|\psi_1\rangle|^2$, with $N=1, 2, 3$ for (a), (b), and (c).

$$\langle\Psi_N|\psi_1\rangle=\int_{-\infty}^{\infty}\Psi_N^*(x)\psi_1(x)\,dx=\int_0^a\sqrt{\frac{2}{3a}}\sqrt{\frac{2}{a}}\sin\left(\frac{N\pi x}{3a}\right)\sin\left(\frac{\pi x}{a}\right)dx.$$

To perform the integral, use the identity $\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$

So
$$\langle\Psi_N|\psi_1\rangle=\frac{1}{\sqrt{3}a}\int_0^a\left(\cos\left[\left(\frac{N}{3}-1\right)\frac{\pi x}{a}\right]-\cos\left[\left(\frac{N}{3}+1\right)\frac{\pi x}{a}\right]\right)dx \qquad (*)$$

$$=\frac{1}{\sqrt{3}a}\left\{\left(\frac{a}{\pi}\right)\left[\frac{1}{\left(\frac{N}{3}-1\right)}\sin\left[\left(\frac{N}{3}-1\right)\frac{\pi x}{a}\right]-\frac{1}{\left(\frac{N}{3}+1\right)}\sin\left[\left(\frac{N}{3}+1\right)\frac{\pi x}{a}\right]\right\}_{x=0}^{a} \qquad (**)$$

Now we just plug in $N=1, 2$ for (a), (b). However, note that for $N=3$ (part c), the first integral must be treated carefully. Then we can use L'Hopital's rule on the first term in (**), or more simply, plug $N=3$ into the first integral in (*), giving in either case a.

Zettili Exercise 4.26 continued

For $N \neq 3$, (**) evaluates to

$$\langle \Psi_N | \psi_1 \rangle = \frac{1}{\sqrt{3}\pi} \left\{ \frac{1}{\left(\frac{N}{3}-1\right)} \sin\left[\left(\frac{N}{3}-1\right)\pi\right] - \frac{1}{\left(\frac{N}{3}+1\right)} \sin\left[\left(\frac{N}{3}+1\right)\pi\right] \right\}$$

(a) $N=1$: $$\langle \Psi_1 | \psi_1 \rangle = \frac{1}{\sqrt{3}\pi} \left\{ \frac{1}{(-2/3)} \sin(-2\pi/3) - \frac{1}{4/3} \sin(4\pi/3) \right\}$$

$$= \frac{1}{\sqrt{3}\pi} \left\{ \left(-\frac{3}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{3}{4}\right)\left(-\frac{\sqrt{3}}{2}\right) \right\}$$

$$= \frac{9}{8\pi}$$

$$\Rightarrow P(E_{N=1}) = |\langle \Psi_1 | \psi_1 \rangle|^2 = \boxed{\frac{81}{64\pi^2} \simeq 12.8\%}$$

(b) $N=2$: $$\langle \Psi_2 | \psi_1 \rangle = \frac{1}{\sqrt{3}\pi} \left\{ \frac{1}{(-1/3)} \sin(-\pi/3) - \frac{1}{5/3} \sin\left(\frac{5\pi}{3}\right) \right\}$$

$$= \frac{1}{\sqrt{3}\pi} \left\{ (-3)\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{3}{5}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right) \right\}$$

$$= \frac{9}{5\pi}$$

$$\Rightarrow P(E_{N=2}) = |\langle \Psi_2 | \psi_1 \rangle|^2 = \boxed{\frac{81}{25\pi^2} \simeq 32.8\%}$$

(c) $N=3$: $$\langle \Psi_3 | \psi_1 \rangle = \frac{1}{\sqrt{3}} \left\{ 1 - \frac{1}{\left(\frac{3}{3}+1\right)} \sin\left(\frac{3}{3}+1\right)\pi \right\} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow P(E_{N=3}) = |\langle \Psi_3 | \psi_1 \rangle|^2 = \boxed{\frac{1}{3} \simeq 33.3\%}$$