Universality for mathematical and physical systems

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Abstract. All physical systems in equilibrium obey the laws of thermodynamics. In other words, whatever the precise nature of the interaction between the atoms and molecules at the microscopic level, at the macroscopic level, physical systems exhibit universal behavior in the sense that they are all governed by the same laws and formulae of thermodynamics. In this paper we describe some recent history of universality ideas in physics starting with Wigner's model for the scattering of neutrons off large nuclei and show how these ideas have led mathematicians to investigate universal behavior for a variety of mathematical systems. This is true not only for systems which have a physical origin, but also for systems which arise in a purely mathematical context such as the Riemann hypothesis, and a version of the card game solitaire called patience sorting.

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1. Introduction

All physical systems in equilibrium obey the laws of thermodynamics. The first law asserts the conservation of energy. The second law has a variety of formulations, one of which is the following: Suppose that in a work cycle a heat engine extracts Q_1 units of heat from a heat reservoir at temperature T_1, performs W units of work, and then exhausts the remaining $Q_2 = Q_1 - W$ units of heat to a heat sink at temperature $T_2 < T_1$. Let $\eta = \frac{W}{Q_1}$ denote the efficiency of the conversion of heat into work. Then the second law tells us there is a maximal efficiency $\eta_{\max} = (T_1 - T_2)/T_1$, depending only on T_1 and T_2, so that for all heat engines, and all work cycles,

$$\eta \leq \eta_{\max}. \tag{1}$$

Nature is so set up that we just cannot do any better.

On the other hand, it is a very old thought, going back at least to Democritus and the Greeks, that matter, all matter, is built out of tiny constituents – atoms – obeying their own laws of interaction. The juxtaposition of these two points of view, the

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macroscopic world of tangible objects and the microscopic world of atoms, presents a fundamental, difficult and long-standing challenge to scientists; namely, how does one derive the macroscopic laws of thermodynamics from the microscopic laws of atoms? The special, salient feature of this challenge is that the *same* laws of thermodynamics should emerge no matter what the details of the atomic interaction. In other words, on the macroscopic scale, physical systems should exhibit universality[1]. Indeed, it is the very emergence of universal behavior for macroscopic systems that makes possible the existence of physical laws.

This kind of thinking, however, is not common in the world of mathematics. Mathematicians tend to think of their problems as sui generis, each with its own special, distinguishing features. Two problems are regarded as "the same" only if some isomorphism, explicit or otherwise, can be constructed between them. In recent years, however, universality in the above sense of macroscopic physics has started to emerge in a wide variety of mathematical problems, and the goal of this paper is to illustrate some of these developments. As we will see, there are problems from diverse areas, often with no discernible, mechanistic connections, all of which behave, on the appropriate scale, in precisely the same way. The list of such problems is varied, long and growing, and points to the emergence of what one might call "macroscopic mathematics."

A precedent for the kind of results that we are going to describe is given by the celebrated central limit theorem of probability theory, where one considers independent, identically distributed variables $\{x_n\}_{n\geq 1}$. The central limit theorem tells us that if we center and scale the variables, $x_n \to y_n \equiv \big(x_n - \mathbb{E}(x_n)\big)/\sqrt{\mathbb{V}(x_n)}$, then

$$\lim_{n\to\infty} \text{Prob}\left(\frac{\sum_{k=1}^{n} y_k}{\sqrt{n}} \leq t\right) = \int_{-\infty}^{t} e^{-\frac{u^2}{2}}\,\frac{du}{\sqrt{2\pi}}\,. \tag{2}$$

We see here explicitly that the Gaussian distribution on the right-hand side of (2) is *universal*, independent of the distribution for the x_n's. The proof of the central limit theorem for independent coin flips, $\text{Prob}(x_n = +1) = \text{Prob}(x_n = -1) = \frac{1}{2}$, goes back to de Moivre and Laplace in the 18th century. Of course (2) is only one of many similar universality-type results now known in probability theory.

The outline of the paper is as follows: In Section 2 we will introduce and discuss some models from random matrix theory (RMT). Various distributions associated with these models will play the same role in the problems that we discuss later on in the paper as the Gaussian does in (2). As noted above, thermodynamics reflects universality for all macroscopic systems, but there are also many universality subclasses which describe the behavior of physical systems in restricted situations. For example, many fluids, such as water and vinegar, obey the Navier–Stokes equation, but a variety of heavy oils obey the lubrication equations. In the same way we will see

[1] In physics, the term "universality" is usually used in the more limited context of scaling laws for critical phenomena. In this paper we use the term "universality" more broadly in the spirit of the preceding discussion. We trust this will cause no confusion.

that certain mathematical problems are described by so-called Unitary Ensembles of random matrices, and others by so-called Orthogonal or Symplectic Ensembles. In Section 3, we present a variety of problems from different areas of mathematics, and in Section 4 we show how these problems are described by random matrix models from Section 2. In the final Section 5 we discuss briefly some of the mathematical methods that are used to prove the results in Section 4. Here combinatorial identities, Riemann–Hilbert problems (RHP's) and the nonlinear steepest descent method of [DeiZho], as well as the classical steepest descent method, play a key role. We end the section with some speculations, suggesting how to place the results of Sections 3 and 4 in a broader mathematical framework.

2. Random matrix models

There are many ensembles of random matrices that are of interest, and we refer the reader to the classic text of Mehta [Meh] for more information (see also [Dei1]). In this paper we will consider almost exclusively (see, however, (54) et seq. below) only three kinds of ensembles:

(a) Orthogonal Ensembles (OE's) consisting of $N \times N$ real symmetric matrices M, $M = \bar{M} = M^T$.

(b) Unitary Ensembles (UE's) consisting of $N \times N$ Hermitian matrices M, $M = M^*$.

(c) Symplectic Ensembles (SE's) consisting of $2N \times 2N$ Hermitian, self-dual matrices $M = M^* = JM^TJ^T$, where J is the standard $2N \times 2N$ block diagonal symplectic matrix, $J = \mathrm{diag}(\tau, \tau, \ldots, \tau)$, $\tau = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

For reasons that will soon become clear, OE's, UE's and SE's are labeled by a parameter β, where $\beta = 1, 2$ or 4, respectively. In all three cases the ensembles are equipped with probability distributions of the form

$$P_{N,\beta}(M)\, d_\beta M = \frac{1}{Z_{N,\beta}} e^{-\mathrm{tr}\left(V_{N,\beta}(M)\right)}\, d_\beta M \tag{3}$$

where $V_{N,\beta}$ is a real-valued function on $\mathbb{R}$ such that $V_{N,\beta}(x) \to +\infty$ sufficiently rapidly as $|x| \to \infty$, $Z_{N,\beta}$ is a normalization coefficient, and $d_\beta M$ denotes Lebesgue measure on the algebraically independent entries of M. For example, in the orthogonal case, $d_{\beta=1}M = \prod_{1 \le j \le k \le N} dM_{jk}$, where $M = (M_{jk})$ (see, e.g. [Meh]). The notation "orthogonal", "unitary", and "symplectic" refers to the fact that the above ensembles with associated distributions (3) are invariant under conjugation $M \to SMS^{-1}$, where S is orthogonal, unitary, or unitary-symplectic (i.e., $S \in \mathrm{USp}(2N) = \{S : SS^* = I,\ SJS^T = J\}$) respectively. When $V_{N,\beta}(x) = x^2$, one has the so-called Gaussian Orthogonal Ensemble (GOE), the Gaussian Unitary Ensemble (GUE), and the Gaussian Symplectic Ensemble (GSE), for $\beta = 1, 2$ or 4, respectively.

The distributions (3) in turn give rise to distributions on the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ of M,

$$\hat{P}_{N,\beta}(\lambda)\, d^N\lambda = \frac{1}{\hat{Z}_{N,\beta}} e^{-\eta_\beta \sum_{i=1}^N V_{N,\beta}(\lambda_i)} \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta \, d\lambda_1 \cdots d\lambda_N \qquad (4)$$

where $\hat{Z}_{N,\beta}$ is again a normalization coefficient, and $\eta_\beta = 1$ if $\beta = 1$ or 2 and $\eta_4 = 2$ (this is because the eigenvalues for $\beta = 4$ double up). The labeling of OE's, UE's, and SE's by $\beta = 1, 2$ and 4 is now clear. In all three cases, we see that the random matrix ensembles give rise to random particle systems $\{\lambda_1, \lambda_2, \dots\}$ with *repulsion* built in: the probability that two eigenvalues are close together is small and vanishes like a power of the distance between them. This is an essential feature of random matrix ensembles, in contrast to random Poisson particle systems, say, where the particles may bunch together or exhibit large gaps.

Loosely speaking, we say that a system is *modeled by random matrix theory (RMT) if it behaves statistically like the eigenvalues of a "large" OE, UE,...matrix*. In analyzing such systems there is something known as the *standard procedure*: Suppose we wish to compare some statistical quantities $\{a_k\}$ in the neighborhood of some point A with the eigenvalues $\{\lambda_k\}$ of some matrix in the neighborhood of some energy E, say, in the bulk of the spectrum. Then we always *center* and *scale* the a_k's and the λ_k's,

$$a_k \to \tilde{a}_k = \gamma_a(a_k - A), \quad \lambda_k \to \tilde{\lambda}_k = \gamma_\lambda(\lambda_k - E) \qquad (5)$$

so that

$$\mathbb{E}\big(\#\{\tilde{a}_k\text{'s per unit interval}\}\big) = \mathbb{E}\big(\#\{\tilde{\lambda}_k\text{'s per unit interval}\}\big) = 1. \qquad (6)$$

For energies E at the edge of the spectrum, the above procedure must be modified slightly (see below).

This procedure can be viewed as follows: A scientist wishes to investigate some statistical phenomenon. What s'he has at hand is a microscope and a handbook of matrix ensembles. The data $\{a_k\}$ are embedded on a slide which can be inserted into the microscope. The only freedom that the scientist has is to center the slide, $a_k \to a_k - A$, and then adjust the focus $a_k - A \to \tilde{a}_k = \gamma_a(a_k - A)$ so that on average one data point $\tilde{a}_k$ appears per unit length on the slide. At that point the scientist takes out his'r handbook, and then tries to match the statistics of the $\tilde{a}_k$'s with those of the eigenvalues of some ensemble. If the fit is good, the scientist then says that the system is well-modeled by RMT.

It is a remarkable fact, going back to the work of Gaudin and Mehta, and later Dyson, in the 1960s, that the key statistics for OE's, UE's, and SE's can be computed in closed form. This is true not only for finite N, but also for various scaling limits as $N \to \infty$. For GOE, GUE, and GSE we refer the reader to [Meh]. Here the Hermite polynomials, which are orthogonal with respect to the weight $e^{-x^2}\, dx$ on $\mathbb{R}$, play

a critical role, and the scaling limits as $N \to \infty$ follow from the known, classical asymptotics of the Hermite polynomials. For UE's with general potentials $V_{N,\beta=2}$, the techniques described in [Meh] for GUE go through for finite N, the role of the Hermite polynomials now being played by the polynomials orthogonal with respect to the weight $e^{-V_{N,\beta=2}(x)}\,dx$ on $\mathbb{R}$ (see, e.g. [Dei1]). For general $V_{N,\beta=2}$, however, the asymptotic behavior of these polynomials as $N \to \infty$ does not follow from classical estimates. In order to overcome this obstacle, the authors in [DKMVZ1] and [DKMVZ2] (see also [Dei1] for a pedagogical presentation) used the Riemann–Hilbert steepest-descent method introduced by Deift and Zhou [DeiZho], and further developed with Venakides [DVZ], to compute the asymptotics as $N \to \infty$ of the orthogonal polynomials for a very general class of analytic weights. In view of the preceding comments, the scaling limits of the key statistics for UE's then follow for such weights (see also [BleIts] for the special case $V_{N,\beta=2}(x) = N(x^4 - tx^2)$). For another approach to UE universality, see [PasSch]. For OE's and SE's with classical weights, such as Laguerre, Jacobi, etc., for which the asymptotics of the associated orthogonal polynomials are known, the GOE and GSE methods in [Meh] apply (see the introductions to [DeiGio1] and [DeiGio2] for a historical discussion). For general $V_{N,\beta}$, $\beta = 1$ or 4, new techniques are needed, and these were introduced, for finite N, by Tracy and Widom in [TraWid2] and [Wid]. In [DeiGio1] and [DeiGio2], the authors use the results in [TraWid2] and [Wid], together with the asymptotic estimates in [DKMVZ2], to compute the large N limits of the key statistics for OE's and SE's with general polynomial weights $V_{N,\beta}(x) = \kappa_{2m}x^{2m} + \cdots$, $\kappa_{2m} > 0$.

It turns out that not only can the statistics for OE's, UE's and SE's be computed explicitly, but in the large N limit the behavior of these systems is universal in the sense described above, as conjectured earlier by Dyson, Mehta, Wigner, and many others. It works like this: Consider $N \times N$ matrices M in a UE with potential $V_{N,2}$. Let K_N denote the finite rank operator with kernel

$$K_N(x, y) = \sum_{j=0}^{N-1} \varphi_j(x)\varphi_j(y), \quad x, y \in \mathbb{R} \tag{7}$$

where

$$\varphi_j(x) = p_j(x)e^{-\frac{1}{2}V_{N,2}(x)}, \quad j \geq 0 \tag{8}$$

and

$$p_j(x) = \gamma_j x^j + \cdots, \quad j \geq 0,\ \gamma_j > 0 \tag{9}$$

are the orthonormal polynomials with respect to the weight $e^{-V_{N,2}(x)}\,dx$,

$$\int_{\mathbb{R}} p_j(x)\,p_k(x)e^{-V_{N,2}(x)}\,dx = \delta_{jk}, \quad j, k \geq 0.$$

Then the m-point correlation functions

$$R_m(\lambda_1, \ldots, \lambda_m) \equiv \frac{N!}{(N-m)!}\int\cdots\int \hat{P}_{N,2}(\lambda_1, \ldots, \lambda_N)\,d\lambda_{m+1}\cdots d\lambda_N$$

can be expressed in terms of K_N as follows:

$$R_m(\lambda_1, \ldots, \lambda_m) = \det\big(K_N(\lambda_i, \lambda_j)\big)_{1 \le i,j \le m}. \tag{10}$$

A simple computation for the 1-point and 2-point functions, $R_1(\lambda)$ and $R_2(\lambda_1, \lambda_2)$, shows that

$$\mathbb{E}\big(\#\{\lambda_i \in B\}\big) = \int_B R_1(\lambda)\, d\lambda \tag{11}$$

for any Borel set $B \subset \mathbb{R}$, and

$$\mathbb{E}\big(\#\{\text{ordered pairs } (i, j),\ i \neq j\ :\ (\lambda_i, \lambda_j) \in \Delta\}\big) = \iint_\Delta R_2(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2 \tag{12}$$

for any Borel set $\Delta \subset \mathbb{R}^2$.

It follows in particular from (11) that, for an energy E, $R_1(E) = K_N(E, E)$ is the density of the expected number of eigenvalues in a neighborhood of E, and hence, by the standard procedure, one should take the scaling factor γ_λ in (5) to be $K_N(E, E)$. For energies E in the bulk of the spectrum, one finds for a broad class of potentials $V_{N,2}$ (see [DKMVZ1] and [DKMVZ2]) that, in the scaling limit dictated by $K_N(E, E)$, $K_N(\lambda, \lambda')$ takes on a universal form

$$\lim_{N\to\infty} \frac{1}{K_N(E, E)}\, K_N\left(E + \frac{x}{K_N(E, E)}, E + \frac{y}{K_N(E, E)}\right) = K_\infty(x - y) \tag{13}$$

where $x, y \in \mathbb{R}$ and K_∞ is the so-called *sine-kernel*,

$$K_\infty(u) = \frac{\sin(\pi u)}{\pi u}. \tag{14}$$

Inserting this information into (10) we see that the scaling limit for R_m is universal for each $m \ge 2$, and in particular for $m = 2$, we have for $x, y \in \mathbb{R}$

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{\big(K_N(E, E)\big)^2}\, R_2\left(E + \frac{x}{K_N(E, E)}, E + \frac{y}{K_N(E, E)}\right) \\ = \det\begin{pmatrix} K_\infty(0) & K_\infty(x - y) \\ K_\infty(x - y) & K_\infty(0) \end{pmatrix} \\ = 1 - \left(\frac{\sin \pi(x - y)}{\pi(x - y)}\right)^2. \end{aligned} \tag{15}$$

For a Borel set $B \subset \mathbb{R}$, let $n_B = \#\{\lambda_j\ :\ \lambda_j \in B\}$ and let

$$\mathbb{V}_B = \mathbb{E}\big(n_B - \mathbb{E}(n_B)\big)^2 \tag{16}$$

denote the number variance in B. A simple computation again shows that

$$\mathbb{V}_B = \int_B R_1(x)\, dx + \iint_{B\times B} R_2(x, y)\, dxdy - \left(\int_B R_1(x)\, dx\right)^2.$$

For an energy E in the bulk of the spectrum as above, set

$$B_N(s) = \left(E - \frac{s}{2K_N(E,E)}, E + \frac{s}{2K_N(E,E)}\right), \quad s > 0.$$

For such B, $\mathbb{V}_B$ is the number variance for an interval about E of scaled size s. Recalling that $K_N(E,E) = R_1(E)$, and using (15), we find as $N \to \infty$

$$\lim_{N\to\infty} \mathbb{V}_{B_N(s)} = \frac{1}{\pi^2}\int_0^{2\pi s} \frac{1-\cos u}{u}\,du + \frac{2s}{\pi}\int_{\pi s}^{\infty}\left(\frac{\sin u}{u}\right)^2 du. \tag{17}$$

For large s, the right-hand side has the form (see [Meh])

$$\frac{1}{\pi^2}\big(\log(2\pi s) + \gamma + 1\big) + O\Big(\frac{1}{s}\Big) \tag{18}$$

where γ is Euler's constant.

For $\theta > 0$, the so-called *gap probability*

$$G_{N,2}(\theta) = \mathrm{Prob}\big(M \,:\, M \text{ has no eigenvalues in } (E-\theta, E+\theta)\big) \tag{19}$$

is given by (see [Meh], and also [Dei1])

$$G_{N,2}(\theta) = \det\big(1 - K_N\restriction_{L^2(E-\theta,E+\theta)}\big) \tag{20}$$

where $K_N\restriction_{L^2(E-\theta,E+\theta)}$ denotes the operator with kernel (7) acting on $L^2(E-\theta, E+\theta)$. In the bulk scaling limit, we find

$$\lim_{N\to\infty} G_{N,2}\left(\frac{x}{K_N(E,E)}\right) = \det\big(1 - K_\infty\restriction_{L^2(E-\theta,E+\theta)}\big), \quad x \in \mathbb{R}. \tag{21}$$

In terms of the scaled eigenvalues $\tilde{\lambda}_j = K_N(E,E)\cdot(\lambda_j - E)$, this means that for $x > 0$

$$\lim_{N\to\infty} \mathrm{Prob}\big(M \,:\, \tilde{\lambda}_j \notin (-x,x),\ 1 \le j \le N\big) = \det\big(1 - K_\infty\restriction_{L^2(-x,x)}\big). \tag{22}$$

Now consider a point E, say $E = 0$, where $K_N(E,E) = K_N(0,0) \to \infty$ as $N \to \infty$. This is true, in particular, if

$$V_{N,2}(x) = \kappa_m x^{2m} + \cdots, \quad \kappa_m > 0,\ m \ge 1, \tag{23}$$

and so for $V_{N,2}(x) = x^2$ (GUE). For such $V_{N,2}$'s, we have $K_N(0,0) \sim N^{1-\frac{1}{2m}}$ (see [DKMVZ1]). Let $t_N > 0$ be such that

$$t_N \to \infty, \quad \frac{t_N}{K_N(0,0)} \to 0. \tag{24}$$

Then

$$\hat{N} \equiv \mathbb{E}\left(\#\left\{|\lambda_j| \le \frac{t_N}{K_N(0,0)}\right\}\right) = \int_{-\frac{t_N}{K_N(0,0)}}^{\frac{t_N}{K_N(0,0)}} K_N(\lambda,\lambda)\,d\lambda \sim 2t_N \to \infty. \tag{25}$$

For $a < b$, define the Borel set $\Delta_N \subset \mathbb{R}^2$ by

$$\Delta_N = \left\{(x,y) : \frac{a}{K_N(0,0)} < x - y < \frac{b}{K_N(0,0)} \text{ and } |x|, |y| \le \frac{t_N}{K_N(0,0)}\right\}. \tag{26}$$

Then we have by (12) and (15), as $N \to \infty$,

$$\begin{aligned}
&\frac{1}{\hat{N}}\mathbb{E}\big(\#\{\text{ordered pairs } (i,j),\ i \ne j\ :\ (\lambda_i,\lambda_j) \in \Delta_N\}\big)\\
&\quad = \frac{1}{\hat{N}}\iint_{\Delta_N} R_2(\lambda_1,\lambda_2)\,d\lambda_1 d\lambda_2\\
&\quad = \frac{1}{\hat{N}}\iint_{\{(s,t):\, a<s-t<b,\ |s|,|t|<t_N\}} \frac{1}{\big(K_N(0,0)\big)^2} R_2\left(\frac{s}{K_N(0,0)}, \frac{t}{K_N(0,0)}\right) ds\,dt\\
&\quad \sim \frac{1}{\hat{N}}\iint_{\{(s,t):\, a<s-t<b,\ |s|,|t|<t_N\}} \left(1 - \Big(\frac{\sin\pi(s-t)}{\pi(s-t)}\Big)^2\right) ds\,dt\\
&\quad \sim \frac{2t_N}{\hat{N}}\int_a^b \left(1 - \Big(\frac{\sin\pi r}{\pi r}\Big)^2\right) dr\\
&\quad \sim \int_a^b \left(1 - \Big(\frac{\sin\pi r}{\pi r}\Big)^2\right) dr, \quad \text{by (25).}
\end{aligned}$$

Thus, if $\tilde{\lambda}_j \equiv K_N(0,0)\lambda_j$ are, again, the scaled eigenvalues, then for t_N as in (24)

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{\hat{N}}\mathbb{E}\big(\#\{\text{ordered pairs } (i,j), i \ne j\ :\ a < \tilde{\lambda}_i - \tilde{\lambda}_j < b,\ |\tilde{\lambda}_i|, |\tilde{\lambda}_j| \le t_N\}\big)\\
= \int_a^b \left(1 - \Big(\frac{\sin\pi r}{\pi r}\Big)^2\right) dr.
\end{aligned} \tag{27}$$

Another quantity of interest is the spacing distribution of the eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ of a random $N \times N$ matrix as $N \to \infty$. More precisely, for $s > 0$, we want to compute

$$\mathbb{E}\left(\frac{\#\{1 \le j \le N-1\ :\ \lambda_{j+1} - \lambda_j \le s\}}{N}\right)$$

as N becomes large. If we again restrict our attention to eigenvalues in a neighborhood of a bulk energy $E = 0$, say, then the eigenvalue spacing distribution exhibits universal behavior for UE's as $N \to \infty$. We have in particular the following result of Gaudin

(see [Meh], and also [Dei1]): With t_N, $\hat{N}$ and $\tilde{x}_j = K_N(0,0)x_j$ as above,

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}\left(\frac{\#\{1 \le j \le N-1 \,:\, \tilde{\lambda}_{j+1} - \tilde{\lambda}_j \le s,\ |\tilde{\lambda}_j| \le t_N\}}{\hat{N}}\right) \\ = \lim_{N\to\infty} \text{Prob(at least one eigenvalue } \tilde{\lambda}_j \text{ in } (0,s] \,|\, \text{eigenvalue at 0}) \\ = \int_0^s p(u)\,du\end{aligned} \tag{28}$$

where

$$p(u) = \frac{d^2}{du^2}\left(\det\left(1 - K_\infty\restriction_{L^2(0,u)}\right)\right). \tag{29}$$

At the upper spectral edge $E = \lambda_{\max}$, one again finds universal behavior for UE's with potentials $V_{N,2}$, in particular, of the form (23) above. For such $V_{N,2}$'s there exist constants $z_N^{(2)}$, $s_N^{(2)}$ such that for $t \in \mathbb{R}$ (see [DeiGio2] and the notes therein)

$$\lim_{N\to\infty} \text{Prob}\left(M \,:\, \frac{\lambda_{\max} - z_N^{(2)}}{s_N^{(2)}} \le t\right) = \det\left(1 - \mathcal{A}\restriction_{L^2(t,\infty)}\right). \tag{30}$$

Here $\mathcal{A}$ is the so-called *Airy operator* with kernel

$$\mathcal{A}(x,y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x-y}, \tag{31}$$

where $\text{Ai}(x)$ is the classical Airy function. For GUE, where $V_{N,2}(x) = x^2$, one has $z_N^{(2)} = \sqrt{2N}$ and $s_N^{(2)} = 2^{-\frac{1}{2}} N^{-\frac{1}{6}}$ (see Forrester [For1] and the seminal work of Tracy and Widom [TraWid1]).

It turns out that $\det\left(1 - K_\infty\restriction_{L^2(-x,x)}\right)$ in (21) and $\det\left(1 - \mathcal{A}\restriction_{L^2(t,\infty)}\right)$ can be expressed in terms of solutions of the Painlevé V and Painlevé II equations respectively. The first is a celebrated result of Jimbo, Miwa, Môri, and Sato [JMMS], and the second is an equally celebrated result of Tracy and Widom [TraWid1]. In particular for edge scaling we find

$$\lim_{N\to\infty} \text{Prob}\left(M \,:\, \frac{\lambda_{\max} - z_N^{(2)}}{s_N^{(2)}} \le t\right) = F_{\beta=2}(t) \tag{32}$$

where

$$F_{\beta=2}(t) = \det\left(1 - \mathcal{A}\restriction_{L^2(t,\infty)}\right) = e^{-\int_t^\infty (s-t)u^2(s)\,ds} \tag{33}$$

and $u(s)$ is the (unique, global) Hastings–McLeod solution of the Painlevé II equation

$$u''(s) = 2u^3(s) + su(s) \tag{34}$$

such that

$$u(s) \sim \text{Ai}(s) \quad \text{as } s \to +\infty. \tag{35}$$

$F_2(t) = F_{\beta=2}(t)$ is called the *Tracy–Widom distribution* for $\beta = 2$.

Finally we note that for OE's and SE's there are analogs for all the above results (10)–(35), and again one finds universality in the scaling limits as $N \to \infty$ for potentials $V_{N,\beta}$, $\beta = 1, 4$, of the form (23) above (see [DeiGio1] and [DeiGio2] and the historical notes therein). We note, in particular, the following results: for $V_{N,\beta}$ as above, $\beta = 1$ or 4, there exist constants $z_N^{(\beta)}$, $s_N^{(\beta)}$ such that

$$\lim_{N\to\infty} \text{Prob}\left(M \,:\, \frac{\lambda_{\max}(M) - z_N^{(\beta)}}{s_N^{(\beta)}} \le t \right) = F_\beta(t) \tag{36}$$

where

$$F_1(t) = \big(F_2(t)\big)^{\frac{1}{2}} e^{-\frac{1}{2}\int_t^\infty u(s)\,ds} \tag{37}$$

and

$$F_4(t) = \big(F_2(t)\big)^{\frac{1}{2}} \cdot \frac{e^{\frac{1}{2}\int_t^\infty u(s)\,ds} + e^{-\frac{1}{2}\int_t^\infty u(s)\,ds}}{2} \tag{38}$$

with $F_2(t)$ and $u(s)$ as above. $F_1(t)$ and $F_4(t)$ are called the *Tracy–Widom distributions* for $\beta = 1$ and 4 respectively.

3. The problems

In this section we consider seven problems. The first is from physics and is included for historical reasons that will become clear in Section 4 below; the remaining six problems are from mathematics/mathematical physics.

Problem 1. Consider the scattering of neutrons off a heavy nucleus, say uranium U^{238}. The scattering cross-section is plotted as a function of the energy E of the incoming neutrons, and one obtains a jagged graph (see [Por] and [Meh]) with many hundreds of sharp peaks $E_1 < E_2 < \cdots$ and valleys $E_1' < E_2' < \cdots$. If $E \sim E_j$ for some j, the neutron is strongly repelled from the nucleus, and if $E \sim E_j'$ for some j, then the neutron sails through the nucleus, essentially unimpeded. The E_j's are called *scattering resonances*. The challenge faced by physicists in the late 40s and early 50s was to develop an effective model to describe these resonances. One could of course write down a Schrödinger-type equation for the scattering system, but because of the high dimensionality of the problem there is clearly no hope of solving the equation for the E_j's either analytically or numerically. However, as more experiments were done on heavy nuclei, each with hundreds of E_j's, a consensus began to emerge that the "correct" theory of resonances was statistical, and here Wigner led the way. Any effective theory would have to incorporate two essential features present in the data, viz.

(i) modulo certain natural symmetry considerations, all nuclei in the same symmetry class exhibited universal behavior;

(ii) in all cases, the E_j's exhibited *repulsion*, or, more precisely, the probability that two E_j's would be close together was small.

Question 1. What theory did Wigner propose for the E_j's?

Problem 2. Here we consider the work of H. Montgomery [Mon] in the early 1970s on the zeros of the Riemann zeta function $\zeta(s)$. Assuming the Riemann hypothesis, Montgomery rescaled the imaginary parts $\gamma_1 \le \gamma_2 \le \cdots$ of the (nontrivial) zeros $\{\frac{1}{2} + i\gamma_j\}$ of $\zeta(s)$,

$$\gamma_j \to \tilde{\gamma}_j = \frac{\gamma_j \log \gamma_j}{2\pi} \tag{39}$$

to have mean spacing 1 as $T \to \infty$, i.e.

$$\lim_{T\to\infty} \frac{\#\{j \ge 1 \,:\, \tilde{\gamma}_j \le T\}}{T} = 1.$$

For any $a < b$, he then computed the two-point correlation function for the $\tilde{\gamma}_j$'s

$$\#\{\text{ordered pairs } (j_1, j_2),\, j_1 \ne j_2 \,:\, 1 \le j_1, j_2 \le N,\ \tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_2} \in (a, b)\}$$

and showed, modulo certain technical restrictions, that

$$\begin{aligned} R(a,b) \equiv \lim_{N\to\infty} \frac{1}{N} \#\{&\text{ordered pairs } (j_1, j_2),\ j_1 \ne j_2 \,: \\ &1 \le j_1, j_2 \le N,\ \tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_2} \in (a, b)\} \end{aligned} \tag{40}$$

exists and is given by a certain explicit formula.

Question 2. What formula did Montgomery obtain for $R(a, b)$?

Problem 3. Consider the solitaire card game known as *patience sorting* (see [AldDia] and [Mal]). The game is played with N cards, numbered $1, 2, \ldots, N$ for convenience. The deck is shuffled and the first card is placed face up on the table in front of the dealer. If the next card is smaller than the card on the table, it is placed face up on top of the card; if it is bigger, the card is placed face up to the right of the first card, making a new pile. If the third card in the pile is smaller than one of the cards on the table, it is placed on top of that card; if it is smaller than both cards, it is placed as far to the left as possible. If it is bigger than both cards, it is placed face up to the right of the pile(s), making a new pile. One continues in this fashion until all the cards are dealt out. Let q_N denote the number of piles obtained. Clearly q_N depends on the particular shuffle $\pi \in S_N$, the symmetric group on N numbers, and we write $q_N = q_N(\pi)$.

For example, if $N = 6$ and $\pi = 3\,4\,1\,5\,6\,2$, where 3 is the top card, 4 is the next card and so on, then patience sorting proceeds as follows:

				1			1			1				1	2				
3		3	4	3	4		3	4	5	3	4	5	6	3	4	5	6		

and $q_6(\pi) = 4$.

Question 3. Equip S_N with the uniform distribution. If each card is of unit size, how big a table does one typically need to play patience sorting with N cards? Or, more precisely, how does

$$p_{n,N} = \text{Prob}\big(\pi \ : \ q_N(\pi) \leq n\big) \tag{41}$$

behave as $N \to \infty$, $n \leq N$?

Problem 4. The city of Cuernavaca in Mexico (population about 500,000) has an extensive bus system, but there is no municipal transit authority to control the city transport. In particular there is no timetable, which gives rise to Poisson-like phenomena, with bunching and long waits between buses. Typically, the buses are owned by drivers as individual entrepreneurs, and all too often a bus arrives at a stop just as another bus is loading up. The driver then has to move on to the next stop to find his fares. In order to remedy the situation the drivers in Cuernavaca came up with a novel solution: they introduced "recorders" at specific locations along the bus routes in the city. The recorders kept track of when buses passed their locations, and then sold this information to the next driver, who could then speed up or slow down in order to optimize the distance to the preceding bus. The upshot of this ingenious scheme is that the drivers do not lose out on fares and the citizens of Cuernavaca now have a reliable and regular bus service. In the late 1990s two Czech physicists with interest in transportation problems, M. Krbálek and P. Šeba, heard about the buses in Cuernavaca and went down to Mexico to investigate. For about a month they studied the statistics of bus arrivals on Line 4 close to the city center. In particular, they studied the bus spacing distribution, and also the bus number variance measuring the fluctuations of the total number of buses arriving at a fixed location during a time interval T. Their findings are reported in [KrbSeb].

Question 4. What did Krbálek and Šeba learn about the statistics of the bus system in Cuernavaca?

Problem 5. In his investigation of wetting and melting phenomena in [Fis], Fisher introduced various "vicious" walker models. Here we will consider the so-called *random turns vicious walker model*. In this model, the walks take place on the integer lattice $\mathbb{Z}$ and initially the walkers are located at $0, 1, 2, \ldots$. The rules for a walk are as follows:

(a) at each tick of the clock, precisely one walker makes a step to the left;

(b) no two walkers can occupy the same site (hence "vicious walkers").

For example, consider the following walk from time $t = 0$ to time $t = 4$: At $t = 0$, clearly only the walker at 0 can move. At time $t = 1$, either the walker at -1 or at $+1$ can move, and so on. Let d_N be the distance traveled by the walker starting from 0. In the above example, $d_4 = 2$. For any time N, there are clearly only a finite number of possible walks of duration $t = N$. Suppose that all such walks are equally likely.

Question 5. How does d_N behave statistically as $N \to \infty$?

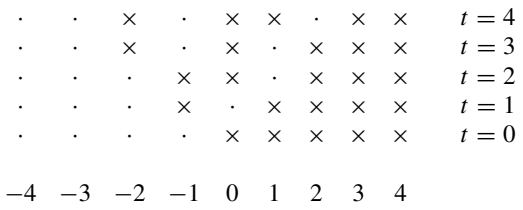

Figure 1. Random turns walk.

Problem 6. Consider tilings $\{T\}$ of the tilted square $\boldsymbol{T}_n = \{(x, y) : |x|+|y| \leq n+1\}$ in $\mathbb{R}^2$ by horizontal and vertical dominos of length 2 and width 1. For example, for $n = 3$ we have the tiling T of Figure 2. For each tiling the dominos must lie strictly

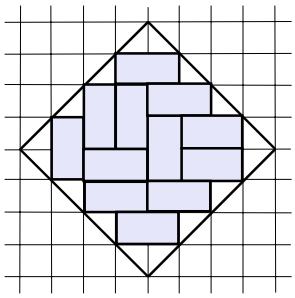

Figure 2. Aztec diamond for $n = 3$.

within $\boldsymbol{T}_n$. The tilings T are called *Aztec diamonds* because the boundary of T in $\{(x, y) : y > 0\}$, say, has the shape of a Mexican pyramid. It is a nontrivial theorem (see [EKLP]) that for any n, the number of domino tilings of $\boldsymbol{T}_n$ is $2^{\frac{n(n+1)}{2}}$. Assume that all tilings are equally likely.

Question 6. What does a typical tiling look like as $n \to \infty$?

Finally we have

Problem 7. How long does it take to board an airplane? We consider the random boarding strategy in [BBSSS] under the following simplifying assumptions:

(a) there is only 1 seat per row;

(b) the passengers are very thin compared to the distance between seats;

(c) the passengers move very quickly between seats. The main delay in boarding is the time – one unit – that it takes for the passengers to organize their luggage and seat themselves once they arrive at their assigned seats.

For the full problem with more than one seat per row, passengers who are not "very thin", etc., see [BBSSS], and also the discussion of the boarding problem in Section 4 below.

The passengers enter the airplane through a door in front and the seats are numbered $1, 2, \ldots, N$, with seat 1 closest to the door. How does boarding proceed? Consider, for example, the case $N = 6$. There are 6 passengers, each with a seating card $1, 2, \ldots, 6$. At the call to board, the passengers line up randomly at the gate. Suppose for definitiveness that the order in the line is given by

$$\pi : \ 3\,4\,1\,5\,6\,2 \tag{42}$$

with 3 nearest the gate. Now 3 can proceed to his'r seat, but 4 is blocked and must wait behind 3 while s'he puts his'r bag up into the overhead bin. However, at the same time, 1 can proceed to his'r seat, but 5, 6, and 2 are blocked. At the end of one unit of time, 3 and 1 sit down, and 4 and 2 can proceed to their seats, but 5 and 6 are blocked behind 4. After one more unit of time 4 and 2 sit down, and 5 can proceed to his'r seat, but 6 is blocked. At the end of one more unit of time 5 sits down, and finally 6 can move to his'r seat. Thus for π as above, it takes 4 units of time to board. Let $b_N = b_N(\pi)$ denote the boarding time for any $\pi \in S_N$, and assume that the π's are uniformly distributed.

Question 7. How does $b_N(\pi)$ behave statistically as $N \to \infty$?

4. Solutions and explanations

As indicated in the Introduction, the remarkable fact of the matter is that all seven problems in Section 3 are modeled by RMT.

Problem 1 (Neutron scattering). At some point in the mid-1950s, in a striking development, Wigner suggested that the statistics of the neutron scattering resonances was governed by GOE[2] (and hence, by universality [DeiGio1], by all OE's). And indeed, if one scales real scattering data for a variety of nuclei according to the standard procedure and then evaluates, in particular, the nearest neighbor distribution, one finds remarkable agreement with the OE analog of the spacing distribution (28), (29).

It is interesting, and informative, to trace the development of ideas that led Wigner to his suggestion (see [Wig1], [Wig2], [Wig3]; all three papers are reproduced in [Por]). In these papers, Wigner is guided by the fact that any model for the resonances would have to satisfy the constraints of universality and repulsion, (i) and (ii) respectively, in the description of Problem 1. In [Wig2] he recalls a paper that he had written with von Neumann in 1929 in which they showed, in particular, that in the $\frac{n(n+1)}{2}$-dimensional space of real $n \times n$ symmetric matrices, the matrices with double

[2]Here we must restrict the data to scattering for situations where the nuclear forces are time-reversal invariant. If not, the statistics of the scattering resonances should be governed by GUE.

eigenvalues form a set of codimension 2. For example, if a 2×2 real symmetric matrix has double eigenvalues, then it must be a multiple of the identity and hence it lies in a set of dimension 1 in $\mathbb{R}^3$. It follows that if one equips the space of real, symmetric matrices with a probability measure with a smooth density, the probability of a matrix M having equal eigenvalues would be zero and the eigenvalues $\lambda_1, \ldots, \lambda_n$ of M would comprise a random set with repulsion built in. So Wigner had a model, or more precisely, a class of models, which satisfied constraint (ii). But why choose GOE? This is where the universality constraint (i) comes into play. We quote from [Wig3][3]: "Let me say only one more word. It is very likely that the curve in Figure 1 is a universal function. In other words, it doesn't depend on the details of the model with which you are working. There is one particular model in which the probability of the energy levels can be written down exactly. I mentioned this distribution already in Gatlinburg. It is called the Wishart distribution. Consider a set…." So in this way Wigner introduced GOE into theoretical physics: It provided a model with repulsion (and time-reversal) built in. Furthermore, the energy level distribution could be computed explicitly. By universality, it should do the trick!

As remarkable as these developments were, even the most prophetic observer could not have predicted that, a few years down the line, these developments would make themselves felt within pure mathematics.

Problem 2 (Riemann zeta function). Soon after completing his work on the scaling limit (40) of the two-point correlation function for the zeros of zeta, Montgomery was visiting the Institute for Advanced Study in Princeton and it was suggested that he show his result to Dyson. What happened is a celebrated, and oft repeated, story in the lore of the Institute: before Montgomery could describe his hard won result to Dyson, Dyson took out a pen, wrote down a formula, and asked Montgomery "And did you get this?"

$$R(a,b) = \int_a^b 1 - \left(\frac{\sin(\pi r)}{\pi r}\right)^2 dr \tag{43}$$

Montgomery was stunned: this was exactly the formula he had obtained. Dyson explained: "If the zeros of the zeta function behaved like the eigenvalues of a random GUE matrix, then (43) would be exactly the formula for the two-point correlation function!" (See (27) above.)

More precisely, what Montgomery actually proved was that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\le i\ne j\le N} f(\tilde{\gamma}_i - \tilde{\gamma}_j) = \int_{\mathbb{R}} f(r) \left(1 - \left(\frac{\sin \pi r}{\pi r}\right)^2\right) dr \tag{44}$$

for any rapidly decaying function f whose Fourier transform $\hat{f}(\xi)$ is supported in the interval $|\xi| < 2$. Of course, if one could prove (44) for all smooth, rapidly decaying functions, one would recover the full result (43). Nevertheless, in an impressive series

[3]In the quotation that follows, "Figure 1" portrays a level spacing distribution, the "Wishart distribution" is the statisticians' name for GOE, and "Gatlinburg" is [Wig1].

of numerical computations starting in the 1980s, Odlyzko verified (43) to extraordinary accuracy (see [Odl1], [Odl2], and the references therein). In his computations, Odlyzko also considered GUE behavior for other statistics for the $\tilde{\gamma}_j$'s, such as the nearest neighbor spacing, verifying in particular the relationship

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \le j \le N-1 \,:\, \tilde{\gamma}_{j+1} - \tilde{\gamma}_j \le s\} = \int_0^s p(u)\,du, \quad s > 0, \tag{45}$$

(cf. (28), (29)) to high accuracy.

The relationship between the zeros of the zeta function and random matrix theory first discovered by Montgomery has been taken up with great virtuosity by many researchers in analytic number theory, with Rudnick and Sarnak [RudSar], and then Katz and Sarnak [KatSar], leading the way. GUE behavior for the zeros of quite general automorphic L-functions over $\mathbb{Q}$, as well as for a wide class of zeta and L-functions over finite fields, has now been established (modulo technicalities as in (44) above in the number field case). Another major development has been the discovery of a relationship between random polynomials whose roots are given by the eigenvalues of a matrix from some random ensemble, and the moments of the L-functions on the critical line $\operatorname{Re} z = \frac{1}{2}$ (see [KeaSna1], [KeaSna2]). The discovery of Montgomery/Odlyzko counts as one of the major developments in analytic number theory in many, many years.

Problem 3 (Patience sorting). In 1999 Baik, Deift and Johansson [BDJ1] proved the following result for $q_N(\pi)$, the number of piles obtained in patience sorting starting from a shuffle π of N cards. Let $\chi_N = \frac{q_N - 2\sqrt{N}}{N^{1/6}}$. Then

$$\lim_{N\to\infty} \operatorname{Prob}\big(\chi_N \le t\big) = F_2(t) \tag{46}$$

where F_2 is the Tracy–Widom distribution (32), (33) for $\beta = 2$. Thus the number of piles, suitably centered and scaled, behaves statistically like the largest eigenvalue of a GUE matrix. In addition, the authors proved convergence of moments. For any $m = 1, 2, \ldots,$

$$\lim_{N\to\infty} \mathbb{E}(\chi_N^m) = \mathbb{E}(\chi^m) \tag{47}$$

where χ is any random variable with distribution F_2. In particular, for $m = 1, 2$ one obtains

$$\lim_{N\to\infty} \frac{\mathbb{E}(q_N) - 2\sqrt{N}}{N^{1/6}} = \int_{\mathbb{R}} t\,dF_2(t) \tag{48}$$

and

$$\lim_{N\to\infty} \frac{\mathbb{V}(q_N)}{N^{1/3}} = \int_{\mathbb{R}} t^2\,dF_2(t) - \left(\int_{\mathbb{R}} t\,dF_2(t)\right)^2. \tag{49}$$

Numerical evaluation shows that the constants on the right-hand side of (48) and (49) are given by -1.7711 and 0.8132, respectively. Thus, as $N \to \infty$,

$$\mathbb{E}(q_N) \sim 2\sqrt{N} - 1.7711 \cdot N^{1/6}$$

so that for a deck of $N = 52$ cards, one needs a table of size about 12 units on average to play the game.

Patience sorting is closely related to the problem of *longest increasing subsequences* for permutations $\pi \in S_N$. Recall that we say that $\pi(i_1), \ldots, \pi(i_k)$ is an increasing subsequence in π of length k if $i_1 < i_2 < \cdots < i_k$ and $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. Let $l_N(\pi)$ be the length of the longest increasing subsequence in π. For example, if $N = 6$ and $\pi = 3\,4\,1\,5\,6\,2$ we see that $3\,4\,5\,6$ is a longest increasing subsequence for π and hence $l_6(\pi) = 4$. Comparing with the introduction to Question 3, we see that $l_6(\pi) = q_6(\pi)$. This is no accident: for any $\pi \in S_N$, we always have $l_N(\pi) = q_N(\pi)$ (see, e.g. [AldDia]), and hence we learn from (46) that the length l_N of the longest increasing subsequence behaves statistically like the largest eigenvalue of a GUE matrix as $N \to \infty$. The relation $l_N(\pi) = q_N(\pi)$ and (48) imply in particular that

$$\lim_{N\to\infty} \frac{\mathbb{E}(l_N)}{N^{1/2}} = 2. \tag{50}$$

The claim that the limit in (50) exists, and equals 2, is known as "Ulam's problem" and has a long history (see [BDJ1]). In another direction, uniform distribution on S_N pushes forward under the Robinson–Schensted correspondence (see, e.g. [Sag]) to so-called *Plancherel measure* on Young diagrams of size N. Young diagrams are parameterized by partitions $\mu \vdash N$, $\{\mu = (\mu_1, \mu_2, \ldots, \mu_l) : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_l \geq 1, \sum_{i=1}^{l} \mu_i = N\}$, where μ_i is the number of boxes in the ith row, and it turns out that under the correspondence we have

$$\text{Prob}\big(\pi : l_N(\pi) \leq n\big) = \text{Prob}\big(\mu \vdash N : \mu_1 \leq n\big). \tag{51}$$

Consequently, the number of boxes in the first row of Plancherel-random Young diagrams behaves statistically, as $N \to \infty$, like the largest eigenvalue of a GUE matrix. In [BDJ1] the authors conjectured that the number of boxes in the first k rows of a Young diagram should behave statistically as $N \to \infty$ like the top k eigenvalues $\lambda_N \geq \lambda_{N-1} \geq \cdots \geq \lambda_{N-k+1}$ of a GUE matrix. This conjecture was proved for the 2^{nd} row in [BDJ2]. For general k, the conjecture was proved, with convergence in joint distribution, in three separate papers in rapid succession ([Oko], [BOO], [Joh1]), using very different methods. The proof in [Oko] relies on an interplay between maps on surfaces and ramified coverings of the sphere; the proof in [BOO] is based on the analysis of specific characters on $S(\infty)$, the infinite symmetric group defined as the inductive limit of the finite symmetric groups S_N under the embeddings $S_N \hookrightarrow S_{N+1}$; and the proof in [Joh1] utilizes certain discrete orthogonal polynomial ensembles arising in combinatorial probability.

One can consider the statistics of $l_N(\pi)$ for π restricted to certain distinguished subsets of S_N (see [BaiRai1]). Amongst the many results in [BaiRai1] relating combinatorics and random matrix theory, we mention the following. Let $S_N^{(\text{inv})} = \{\pi \in S_N : \pi^2 = \text{id}\}$ be the set of involutions in S_N. Then, under the Robinson–Schensted correspondence, uniform distribution on $S_N^{(\text{inv})}$ pushes forward to a new measure on

Young diagrams, different from the Plancherel measure. Denote this measure by $\mathrm{Prob}^{(\mathrm{inv})}$, and in place of (51) we have

$$\mathrm{Prob}\big(\pi \in S_N^{(\mathrm{inv})} \,:\, l_N(\pi) \le n\big) = \mathrm{Prob}^{(\mathrm{inv})}\big(\mu \vdash N \,:\, \mu_1 \le n\big).$$

In [BaiRai1] the authors show that

$$\begin{aligned}
&\lim_{N\to\infty} \mathrm{Prob}\left(\pi \in S_N^{(\mathrm{inv})} \,:\, \frac{l_N - 2\sqrt{N}}{N^{1/6}} \le x\right) \\
&\quad = \lim_{N\to\infty} \mathrm{Prob}^{(\mathrm{inv})}\left(\mu \vdash N \,:\, \frac{\mu_1 - 2\sqrt{N}}{N^{1/6}} \le x\right) \\
&\quad = F_{\beta=1}(x)
\end{aligned} \tag{52}$$

and for the second row of μ

$$\lim_{N\to\infty} \mathrm{Prob}^{(\mathrm{inv})}\left(\mu \vdash N \,:\, \frac{\mu_2 - 2\sqrt{N}}{N^{1/6}} \le x\right) = F_{\beta=4}(x). \tag{53}$$

Here $F_{\beta=1}$ and $F_{\beta=4}$ are the Tracy–Widom distributions for the largest eigenvalue of the GOE and GSE ensemble, respectively (see (36), (37), (38)). Thus all three of the basic ensembles $\beta = 1, 2$ and 4 show up in the analysis of the (general) increasing subsequence problem.

A problem which is closely related to the longest increasing subsequence problem is the random word problem. In [TraWid4] the authors consider words $\{\omega\}$ of length N in an alphabet of k letters, i.e. maps $\omega\colon \{1, 2, \ldots, N\} \to \{1, 2, \ldots, k\}$. One says that $\omega(i_1), \ldots, \omega(i_j)$ is a weakly increasing subsequence in ω of length j if $i_1 < i_2 < \cdots < i_j$ and $\omega(i_1) \le \omega(i_2) \le \cdots \le \omega(i_j)$. Let $l_N^{\mathrm{wk}}(\omega)$ denote the length of the longest weakly increasing subsequence in ω. Assuming that all words are equally likely, Tracy and Widom in [TraWid4] proved that

$$\begin{aligned}
&\lim_{N\to\infty} \mathrm{Prob}\left(\omega \,:\, \frac{l_N^{\mathrm{wk}}(\omega) - \frac{N}{k}}{\sqrt{\frac{2N}{k}}} \le s\right) \\
&\quad = \gamma_k \int_{\mathcal{L}_s} e^{-\sum_{i=1}^k x_i^2} \prod_{1\le i<j\le k} (x_i - x_j)^2 \, dx_1 \cdots dx_{k-1}
\end{aligned} \tag{54}$$

where

$$\mathcal{L}_s = \big\{(x_1, \ldots, x_k) \,:\, \max_{1\le i\le k} x_i \le s,\ x_1 + \cdots + x_k = 0\big\} \tag{55}$$

and

$$\gamma_k = \frac{\sqrt{k}\, 2^{\frac{k^2-1}{k}}}{\left(\prod_{i=1}^k i!\right)(2\pi)^{\frac{k-1}{2}}}. \tag{56}$$

It is easy to see that the right-hand side of (54) is just the distribution function for the largest eigenvalue of a $k \times k$ GUE matrix conditioned to have trace zero.

Consider the representation of the number π, say, in any basis b,

$$\pi = 0.a_1a_2a_3 \ldots \times b^q, \quad q \in \mathbb{Z}. \tag{57}$$

It has long been believed that in some natural asymptotic sense the digits a_1, a_2, a_3,... are independent and identically distributed, with uniform distribution on $\{0, 1, \ldots, b-1\}$. In an attempt to formalize this notion, E. Borel (1909) introduced the idea of normality (see [Wag]): A real number x is *normal* if for any base b, any $m \geq 1$, and any m-string s,

$$\lim_{n\to\infty} \frac{\#\{\text{occurrences of } s \text{ in the first } n \text{ base-}b \text{ digits of } x\}}{n} = b^{-m}. \tag{58}$$

While it is known that non-normal numbers form a set of Lebesgue measure zero, and all numerical evidence confirms (58) to high order, no explicit examples of normal numbers are known.

Relation (54) suggests a new way to test for asymptotic randomness, as follows. Consider the first LN base-b digits $a_1a_2 \ldots a_{LN}$ of a given number x, where L and N are "large". Partition these digits into L words $\omega_j = a_{(j-1)N+1} \ldots a_{jN}$, $1 \leq j \leq L$, each of length N. For each ω_j compute $l_N^{\text{wk}}(\omega_j)$. Then if the digits $\{a_j\}$ of x are asymptotically random, we could expect that as $L, N \to \infty$, the empirical distribution

$$\frac{1}{L}\#\left\{1 \leq j \leq L \,:\, \frac{l_N^{\text{wk}}(\omega_j) - \frac{N}{b}}{\sqrt{\frac{2N}{b}}} \leq s\right\}$$

is close to the conditional GUE distribution on the right-hand side of (54). Preliminary calculations in [DeiWit] for $x = \pi$ and $b = 2$ show that for L, N "large" the empirical distribution is indeed close to the right-hand side of (54) with high accuracy. The work is in progress.

Problem 4 (Bus problem in Cuernavaca). Krbálek and Šeba found that both the bus spacing distribution and the number variance are well modeled by GUE, (28), (29) and (17) respectively (see Figures 2 and 3 in [KrbSeb]). In order to provide a plausible explanation of the observations in [KrbSeb], the authors in [BBDS] introduced a microscopic model for the bus line that leads simply and directly to GUE.

The main features of the bus system in Cuernavaca are

(a) the stop-start nature of the motion of the buses;

(b) the "repulsion" of the buses due to the presence of recorders.

To capture these features, the authors in [BBDS] introduced a model for the buses consisting of n(= # of buses) independent, rate 1 Poisson processes moving from the bus depot at time $t = 0$ to the final terminus at time T, and conditioned not to intersect

for $0 \le t \le T$. The authors then showed that at any observation point x along the route of length $N > n$, the probability distribution for the (rescaled) arrival times of the buses, $y_j = \frac{2t_j}{T} - 1 \in [-1, 1]$, $1 \le j \le n$, is given by

$$\text{const.} \prod_{j=1}^{n} w_J(y_j) \prod_{1 \le i < j \le n} (y_i - y_j)^2 \, dy_1 \cdots dy_n \tag{59}$$

where

$$w_J(y) = (1+y)^{x-1}(1-y)^{N-x-n+1}, \quad -1 < y < 1. \tag{60}$$

Formula (59) is precisely the eigenvalue distribution for the so-called Jacobi Unitary Ensemble (cf. (4) with $e^{-V_{N,2}(y)} = w_J(y) =$ weight for Jacobi polynomials on $[-1, 1]$). In the appropriate scaling limit, GUE then emerges by universality. The authors also compute the distributions of the positions $x_1, \ldots, x_n$ of the buses at any time $t \in (0, T)$. Again the statistics of the x_j's are described by a Unitary Ensemble, but now w_J is (59) is replaced by the weight for the Krawtchouk polynomials: by universality, GUE again emerges in the appropriate scaling limit.

In an intriguing recent paper, Abul-Magd [Abu] noted that drivers have a tendency "to park their cars near to each other and at the same time keep a distance sufficient for manoeuvring." He then analyzed data measuring the gaps between parked cars on four streets in central London and showed quite remarkably that the gap size distribution was well represented by the spacing distribution (28), (29) of GUE. It is an interesting challenge to develop a microscopic model for the parking problem in [Abu], analogous to the model for the bus problem in [BBDS].

Problem 5 (Random turns vicious walker model). In [BaiRai2] the authors proved that, as $N \to \infty$, d_N, the distance traveled by the walker starting from 0, behaves statistically like the largest eigenvalue of a GOE matrix. More precisely, they showed that

$$\lim_{N \to \infty} \text{Prob}\left(\frac{d_N - 2\sqrt{N}}{N^{1/6}} \le t \right) = F_1(t) \tag{61}$$

where F_1 is given by (37). In a variant of this model, [For3], the walkers again start at 0, 1, 2,..., and move to the left for a time N; thereafter they must move to the right, returning to their initial positions 0, 1, 2,...at time $2N$. Let d'_N denote the maximum excursion of the walker starting from 0. Then Forrester shows that d'_N behaves statistically like the largest eigenvalue of a GUE matrix,

$$\lim_{N \to \infty} \text{Prob}\left(\frac{d'_N - 2\sqrt{N}}{N^{1/6}} \le t \right) = F_2(t) \tag{62}$$

where F_2 is given by (33).

The proofs of (61) and (62) rely on the observation of Forrester in [For3] that, in the first case, the set of walks is in one-to-one correspondence with the set $Y^{(1)}$ of standard Young tableaux of size N (see [Sag]), whereas in the second case, the variant model,

the set of walks is in one-to-one correspondence with the set $Y^{(2)}$ of pairs (P, Q) of standard Young tableaux of size N with the same shape, $\mathrm{sh}(P) = \mathrm{sh}(Q)$. In both cases, d_N and d'_N equal the number of boxes in the first row of the corresponding standard Young tableaux. Uniform measure on $Y^{(2)}$ (resp $Y^{(1)}$) gives rise to Plancherel measure (resp. $\mathrm{Prob}^{(\mathrm{inv})}$) on Young diagrams of size N, and the proof of (62) then follows from (46), (51), and the proof of (61) follows from (52).

In [Bai], Baik proved the analogue of (61), (62) for the so-called lock step vicious walker introduced in [Fis]. The proof in [Bai] relies in part on an observation of Guttmann et al. in [GOV], which preceded [For3], that the set of path configurations for the lock step model is in one-to-one correspondence with the set of semi-standard Young tableaux (see [Sag]).

Problem 6 (Aztec diamond). After scaling by $n+1$, Jockush et al., [JPS], considered the tiling problem with dominos of size $\frac{2}{n+1} \times \frac{1}{n+1}$ in the tilted square $\boldsymbol{T}_0 = \{(u, v) : |u| + |v| \leq 1\}$. As $n \to \infty$, they found that the inscribed circle $C_0 = \{(u, v) : u^2 + v^2 = \frac{1}{2}\}$, which they called the *arctic circle*, plays a remarkable role. In the four regions of $\boldsymbol{T}_0$ outside C_0, which they call the *polar regions* and label N, E, S, W clockwise from the top, the typical tiling is *frozen*, with all the dominoes in N and S horizontal, and all the dominos in E and W vertical. In the region inside C_0, which they call the *temperate zone*, the tiling is random. (See, for example, http://www.math.wisc.edu/~propp/tiling, where a tiling with $n = 50$ is displayed.)

But more is true. In [Joh1], [Joh2], Johansson considered fluctuations of the boundary of the temperate zone about the circle C_0. More precisely, for $-1 < \alpha < 1$, $\alpha \neq 0$, let

$$(x_\alpha^+, y_\alpha^+) = \left(\tfrac{\alpha+\sqrt{1-\alpha^2}}{2}, \tfrac{\alpha-\sqrt{1-\alpha^2}}{2}\right), \quad (x_\alpha^-, y_\alpha^-) = \left(\tfrac{\alpha-\sqrt{1-\alpha^2}}{2}, \tfrac{\alpha+\sqrt{1-\alpha^2}}{2}\right)$$

denote the two points of intersection of the line $u + v = \alpha$ with $C_0 = \{u^2 + v^2 = \frac{1}{2}\}$. Then for fixed α, Johansson showed that the fluctuations of the boundary of the temperate zone along the line $u + v = \alpha$ about the points (x_α^+, y_α^+) and (x_α^-, y_α^-) were described by the Tracy–Widom distribution F_2 (see [Joh2], equation (2.72), for a precise statement). Johansson proceeds by expressing the fluctuations in terms of the Krawtchouk ensemble (cf. Problem 4), which he then evaluates asymptotically as $n \to \infty$. Such an analysis is possible because the associated Krawtchouk polynomials have an integral representation which can be evaluated asymptotically using the classical method of steepest descent. In [CLP], the authors considered tilings of hexagons of size n by unit rhombi and proved an arctic circle theorem for the tilings as $n \to \infty$ as in the case of the Aztec diamond. In [Joh1], [Joh2], Johansson again expressed the fluctuations of the arctic circle for the hexagons in terms of a random particle ensemble, but now using the Hahn polynomials rather than the Krawtchouk polynomials. The Hahn polynomials, however, do not have a convenient integral representation and their asymptotics cannot be evaluated by classical means. This obstacle was overcome by Baik et al., [BKMM], who extended the Riemann–Hilbert/steepest descent method

in [DKMVZ1] and [DKMVZ2] to a general class of discrete orthogonal polynomials. In this way they were able to compute the asymptotics of the Hahn polynomials and verify F_2-behavior for the fluctuations of the temperate zone, as in the case of the Aztec diamond.

Problem 7 (Airline boarding). In [BBSSS] the authors show that $b_N(\pi)$, the boarding time for N passengers subject to the protocol (a)(b)(c) in Problem 7, behaves statistically like the largest eigenvalue of a GUE matrix,

$$\lim_{N\to\infty} \text{Prob}\left(\frac{b_N - 2\sqrt{N}}{N^{1/6}} \le t\right) = F_2(t). \tag{63}$$

The proof of (63) in [BBSSS] relies on the description of the Robinson–Schensted correspondence in terms of Viennot diagrams (see [Sag]). We illustrate the situation with the permutation $\pi :\ 3\,4\,1\,5\,6\,2$ in S_6 (cf. Problem 3 and (42)). We say that a point (x', y') *lies in the shadow* of a point (x, y) in the plane if $x' > x$ and $y' > y$. Plot π as a graph $(1, 3), (2, 4), \ldots, (6, 2)$ in the first quadrant of $\mathbb{R}^2$. Consider all the points in the graph which are not in the shadow of any other point: in our case $(1, 3)$ and $(3, 1)$. The *first shadow line* L_1 is the boundary of the combined shadows of these two points (see Figure 3). To form the *second shadow line* L_2, one removes the points $(1, 3)$, $(3, 1)$ on L_1, and repeats the procedure, etc. Eventually one obtains $k_N(\pi) = k$ shadow lines $L_1, \ldots, L_k$ for some integer k. In our example $k = 4$, which we note is precisely $l_6(\pi)$, the length of the longest increasing subsequence for π. This is no accident: for any $\pi \in S_N$, we always have $k_N(\pi) = l_N(\pi)$ (see [Sag]).

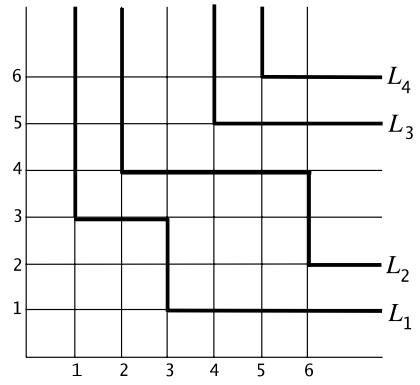

Figure 3. Shadow lines for $\pi :\ 3\,4\,1\,5\,6\,2$ in S_6.

The beautiful fact is that each shadow line describes a step in the boarding process. Indeed, noting the y-values of the L_j's, we observe that

$$\begin{aligned} L_1 &\longleftrightarrow \text{3 and 1 are seated} \\ L_2 &\longleftrightarrow \text{4 and 2 are seated} \\ L_3 &\longleftrightarrow \text{5 is seated} \\ L_4 &\longleftrightarrow \text{6 is seated} \end{aligned}$$

Thus $b_N(\pi) = k_N(\pi) = l_N(\pi) = q_N(\pi)$, and (63) follows from (46). In the language of physics, if we rotate the Viennot diagram for π counterclockwise by 45^o, we see that the shadow region of a point on the graph is simply the forward light cone based at that point (speed of light = 1). In other words, for appropriate coordinates $a,\ b$ we are dealing with the Lorentzian metric $ds^2 = dadb$. In order to incorporate more realistic features into their boarding model, such as the number of seats per row, average amount of aisle length occupied by a passenger, etc., the authors in [BBSSS] observe that it is enough simply to replace $ds^2 = dadb$ by a more general Lorentzian metric $ds^2 = 4D^2 p(a,b)(dadb + k\alpha(a,b)da^2)$ for appropriate parameters/functions D, p, k and α (see [BBSSS], equation (1)). Thus the basic phenomenon of blocking in the airline boarding problem is modeled in the general case by the forward light cone of some Lorentzian metric.

Problems 1 and 2 above, as opposed to 3–7, are purely deterministic and yet it seems that they are well described by a random model, RMT. At first blush, this might seem counterintuitive, but there is a long history of the description of deterministic systems by random models. After all, the throw of a (fair) 6-sided die through the air is completely described by Newton's laws: Nevertheless, there is no doubt that the *right* way to describe the outcome is probabilistic, with a one in six chance for each side. With this example in mind, we may say that Wigner was looking for the right stochastic model to describe the neutron scattering "die".

Problems 1–7 above are just a few of the many examples now known of mathematical/physical systems which exhibit random matrix type universal behavior. Other systems, from many different areas, can be found for example in [Meh] and the reviews [TraWid3], [For2], and [FerPra]. A particularly fruitful development has been the discovery of connections between random matrix theory and stochastic growth models in the KPZ class ([PraSpo], [FerPra]), and between random matrix theory and equilibrium crystals with short range interactions ([CerKen], [FerSpo], [OkoRes], [FerPraSpo]). In addition, for applications to principal component analysis in statistics in situations where the number of variables is comparable to the sample size, see [John] and [BBP] and the references therein. For a relatively recent review of the extensive application of RMT to quantum transport, see [Bee].

Returning to Wigner's introduction of random matrix theory into theoretical physics, we note that GOE is of course a mathematical model far removed from the laboratory of neutrons colliding with nuclei. Nevertheless, Wigner posited that

these two worlds were related: With hindsight, we recognize Wigner's insight as heralding the emergence of a scientific commonality far across the borders of physics and mathematics.

5. Comments and speculations

As is clear from the text, many different kinds of mathematics are needed to analyze Problems 3–7. These include

- combinatorial identities,
- Riemann–Hilbert methods,
- Painlevé theory,
- theory of Riemann surfaces,
- representation theory,
- classical and Riemann–Hilbert steepest descent methods

and, most importantly,

- random matrix theory.

The relevant combinatorial identities are often obtained by analyzing random particle systems conditioned not to intersect, as in Problem 4. The Riemann–Hilbert steepest descent method has its origins in the theory of integrable systems, as in [DeiZho]. There is no space in this article to describe the implementation of any of the above techniques in any detail. Instead, we refer the reader to [Dei2], which is addressed to a general mathematical audience, for a description of the proof of (46) in particular, using Gessel's formula in combinatorics [Ges], together with the Riemann–Hilbert steepest descent method. For Problem 2 the proofs are based on combinatorial facts and random matrix theory, together with techniques from the theory of L-functions, over $\mathbb{Q}$ and also (in [KatSar]) over finite fields.

Universality as described in this article poses a challenge to probability theory per se. The central limit theorem (2) above has three components: a statistical component (take independent, identically distributed random variables, centered and scaled), an algebraic component (add the variables), and an analytic component (take the limit in distribution as $n \to \infty$). The outcome of this procedure is then universal – the Gaussian distribution. The challenge to probabilists is to describe an analogous purely probabilistic procedure whose outcome is F_1, or F_2, etc. The main difficulty is to identify the algebraic component, call it operation X. Given X, if one takes i.i.d.'s, suitably centered and scaled, performs operation X on them, and then takes the limit in distribution, the outcome should be F_1, or F_2, etc. Interesting progress has been made recently (see [BodMar] and [BaiSui]) on identifying X for F_2. For a different approach to the results in [BodMar] and [BaiSui], see [Sui], where the author uses a very interesting generalized version of the Lindeberg principle due to Chatterjee [Cha1], [Cha2].

Our final comment/speculation is on the space D, say, of probability distributions. A priori, D is just a set without any "topography". But we know at least one interesting point on D, the Gaussian distribution F_G. By the central limit theorem, F_G lies in a "valley", and nearby distributions are drawn towards it. What we seem to be learning is that there are other interesting distributions, like F_1 or F_2, etc., which also lie in "valleys" and draw nearby distributions in towards them. This suggests that we equip D with some natural topological and Riemannian structure, and study the properties of D as a manifold per se.

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