Math 134 – Nonlinear ODE

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This is math 134 – Linear and Nonlinear System of Differential Equations taught by Professor Wink. The class lecture is prerecorded, and we have live session every Monday and Friday at 3:00 pm – 3:50 pm for Q & A. We use *Nonlinear Dynamics and Chaos* 2^{nd} by *Steven Strogatz* as our main book for the class. Other course notes can be found through my github. Any error spotted in the notes is my responsibility, and please let me know through my email at ducvu2718@ucla.edu.

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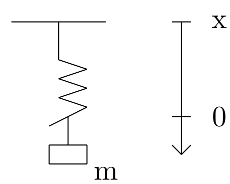
§1 Lec 1: Jan 4, 2021

§1.1 Intro to Dynamical Systems

There are two types of dynamical systems:

1. Discrete in time:
   - Difference equation
   - Iterated map: $a_{n+1} = f(a_n)$
2. Continuous in time: differential equation
   - Partial Differential Equation (PDE):
     e.g. heat equation
     $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$
     wave equation
     $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$
     where the derivatives w.r.t time and space.
   - Ordinary Differential Equation (ODE):
     i) Harmonic oscillator



m: mass
k: spring constant

$$m\dot{x} + kx = 0$$

If $\omega^2 = \frac{k}{m}$, then
$$x(t) = x_0 \cos(\omega t) + x_1 \sin(\omega t)$$

ii) Damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0, \qquad b\text{: damping constant}$$

iii) Forced, damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = F\cos(t), \qquad F\text{: force}$$

so derivatives w.r.t time only.

Definition 1.1 (Order of ODE) — Highest occurring derivative is defined as the order of the ODE.

Remark 1.2. We can always write an ODE of n^{th} order as a system of ODEs of 1^{st} order.

Trick: Consider the damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

Then,

$$\begin{aligned} \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x \\ &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{aligned}$$

i.e.,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{aligned}$$

General framework: $\dot{x} = f(t,x)$

$$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

i.e.,

$$\begin{aligned} \dot{x}_1 &= f_1(t_1, x_1, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, \ldots, x_n) \end{aligned} \tag{1}$$

which is 1^{st} order n-dimensional ODE.

Definition 1.3 (Linear ODE) — The ODE (1) is called linear if $f(t,x) = A(t) \cdot x$ for a time dependent matrix $A(t)$, otherwise we call it non-linear.

Example 1.4

The damped harmonic oscillator is linear.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Question 1.1. Why are linear equations special?

They satisfy the principle of superposition. If ϕ, ψ solve $\dot{x} = A(t)x$, then $y(t) = c \cdot \phi(t) + \psi(t), c \in \mathbb{R}$ also solves $\dot{x} = A(t)x$. This is valid because $\dot{y} = c\dot{\phi} + \dot{\psi} = cA\phi + A\psi = A(c\phi + \psi) = Ay$. For non-linear ODEs, the principle of superposition fails.

Definition 1.5 (Autonomous ODE) — The ODE (1) is called <u>autonomous</u> if f does not depend on t, i.e., $f(t, x) = f(x)$.

Example 1.6

$$m\ddot{x} + b\dot{x} + kx = F\cos(t)$$

is non-autonomous.

However, we can always consider an autonomous system instead. Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= t \end{aligned}$$

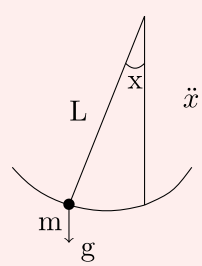
Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + F\cos(x_3) \\ \dot{x}_3 &= 1 \end{aligned}$$

We will primarily study autonomous 1$^{\text{st}}$ order system in 1 or 2 variables.

Example 1.7 (Swinging Pendulum)

Consider a swinging pendulum



$$\ddot{x} + \tfrac{g}{L}\sin(x) = 0$$

Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L}\sin(x_1) \end{aligned}$$

1$^{\text{st}}$ order, non-linear autonomous ODE in 2 variables.

Question 1.2. What can we say about the behavior of a solution $x_1(t), x_2(t)$ for larger time t? How does it depend on $\frac{g}{L}$?

Idea: Use geometric methods, without solving $\dot{x} = f(x)$ explicitly, to make qualitative statements about the long time behavior of the solution.

§2 Lec 2: Jan 6, 2021

§2.1 Phase Portraits

We want to study 1D autonomous dynamical systems

$$\dot{x} = f(x), \qquad f : \mathbb{R} \to \mathbb{R}$$

Remark 2.1. $x(t)$ is the solution to $\dot{x} = f(x)$ with $x(0) = x_0$. Find the solution $y(t)$ with $y(t_0) = x_0$.

Ans: $y(t) = x(t - t_0)$ because $y(t_0) = x(0) = x_0$ and $\dot{y}(t) = \dot{x}(t - t_0) = f(x(t - t_0)) = f(y(t))$.

Example 2.2

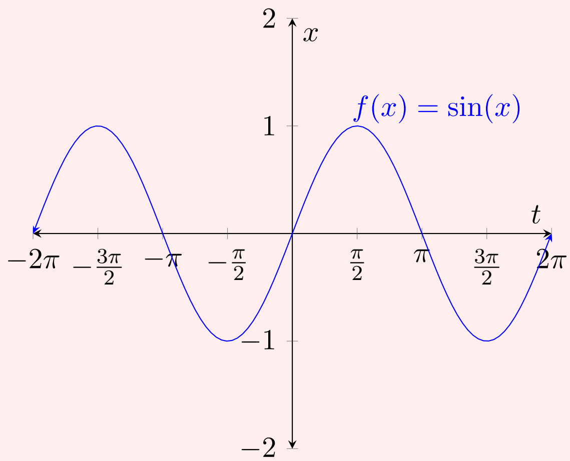
$x = \sin(x)$. Suppose $x_0 = \frac{\pi}{4}$, $x(t)$ solution with $x(0) = x_0$. Answer the followings

- Describe the long time behaviors of $x(t)$ as $t \to \infty$.
- How does the long time behavior depend on $x_0 \in \mathbb{R}$?

Attemp 1: Find explicit solution

$$\begin{aligned} \frac{dx}{dt} &= \sin(x) \\ dt &= \frac{dx}{\sin(x)} \\ t &= -\ln\left|\frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)}\right| + c \end{aligned}$$

We know $x(0) = x_0$, so $c = \ln\left|\frac{1+\cos(x_0)}{\sin(x_0)}\right|$. But what is $x(t) =$? This approach fails!
Attempt 2: Draw a phase portrait/diagram. We want to interpret the velocity $\dot{x} = f(x)$ as a vector field on the real line.

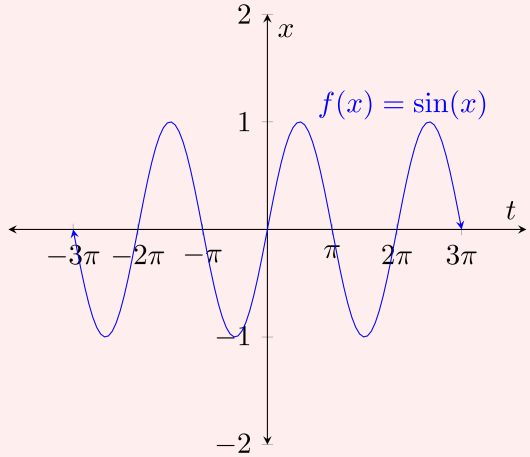


Idea:

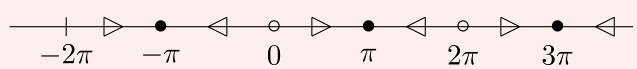
- If $f(x_0) > 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ increase near x_0.
- If $f(x_0) < 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ decrease near x_0.
- If $f(x_0) = 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ is $x(t) = x_0$ for all $t \in \mathbb{R}$, i.e., we have a fixed point/equilibrium point.

Example 2.3

$\dot{x} = f(x) = \sin(x)$



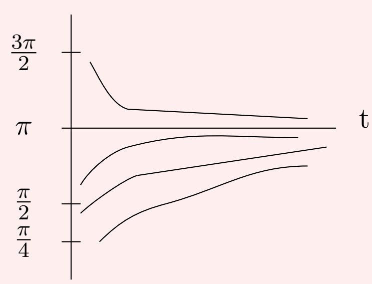
Phase portrait:



stable fixed point
(sink, attractors)

unstable fixed point
(source,)

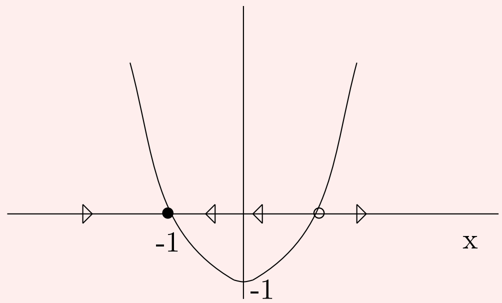
Qualitative plot of solution:



$x(t)$ is concave up
if $f(x(t))$ is increasing
$x(t)$ is concave down
if $f(x(t))$ is decreasing

Example 2.4

$\dot{x} = x^2 - 1$. Fixed points: $f(x) = x^2 - 1 = 0 \implies x = \pm 1$



<u>*Note*</u>: If $x_0 > 1$, then solution $x(t)$ with $x(0) = x_0 > 1$ is unbounded. In fact, $x(t) \to \infty$ in finite time.

§3 Lec 3: Jan 8, 2021

§3.1 Stability Types of Fixed Points

Definition 3.1 (Stability Types) — Consider the ODE $\dot{x} = f(x)$ and suppose that $f(x_*) = 0$. The fixed point x_* is called

1. Lyapunov stable if every solution $x(t)$ with $x(0) = x_0$ closed to x_* remain close to x_* for all $t \geq 0$, otherwise unstable.

2. Attracting if every solution $x(t)$ with $x(0) = x_0$ close to x_* satisfies $x(t) \to x_*$ as $t \to \infty$.

3. (asymptotically) stable if x_* is both Lyapunov stable and attracting.

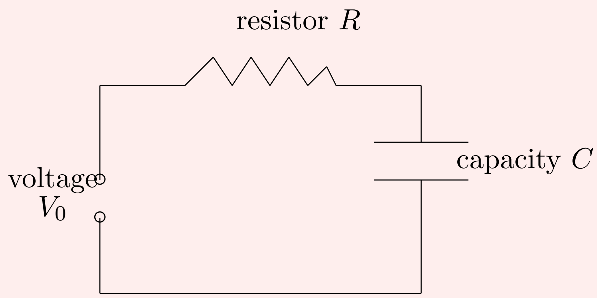
Example 3.2

Let $\alpha \in \mathbb{R}, \dot{x} = \alpha x$. General solution $x(t) = x_0 e^{\alpha t}$.

- $x_* = 0$ is always an equilibrium solution.
- $x_* = 0$ is
  1. attracting if $\alpha < 0$
  2. Lyapunov stable if $\alpha \leq 0$
  3. unstable if $\alpha > 0$

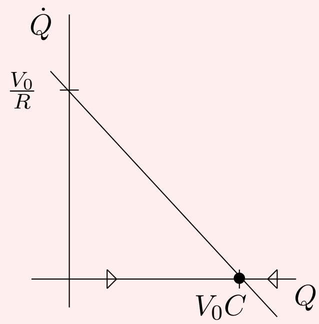
Example 3.3 (RC circuit)

We have the following circuit

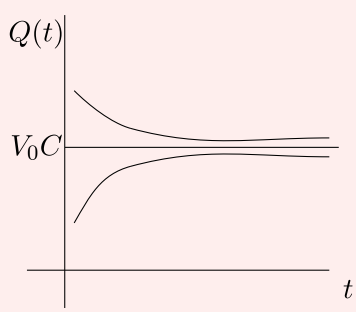


$$V_0 = RI + \frac{Q}{C}$$
$$I: \text{ current}, Q: \text{ change}$$
$$I = \dot{Q}$$
$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$$

Phase portrait

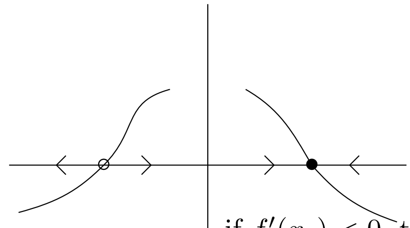


$Q_* = V_0C$ globally stable because every $Q(t)$ approaches Q_* as $t \to \infty$.



§3.2 Linear Stability Analysis

We have $\dot{x} = f(x), f(x_*) = 0$. Our task is to find an analytic criterion to decide if a fixed point x_* is stable/unstable.
Picture:



If $f'(x*) > 0$, then x_* is unstable. On the other hand, if $f'(x_*) < 0$, then x_* is a stable fixed point.

<u>The linearization</u>:

Consider: $\eta(t) = x(t) - x_*$ where $x(t)$ is the solution of $\dot{x} = f(x)$ with $x(0)$ close to x_*, $f(x_*) = 0$.

<u>*Note*</u>: $\dot{\eta}(t) = \dot{x}(t) = f(x(t)) = f(x(t) - x_* + x_*) = f(\eta(t) + x_*)$.

Taylor's Theorem:

$$f(x_* + \eta) = \underbrace{f(x_*)}_{=0} + f'(x_*)\eta + \underbrace{\mathcal{O}(\eta^2)}_{\text{error term and negligible if} f'(x_*) \neq 0 \text{ and } \eta \text{ is small}}$$

$\implies$ $\dot{\eta}(t) \approx f'(x_*)\eta(t)$ (as long as $\eta(t)$ is small) which is called the linearization of $\dot{x} = f(x)$ about x_*. The general solution is

$$\eta(t) = \eta_0 e^{f'(x_*) \cdot t}$$

In particular, η grows exponentially if $f'(x_*) > 0$ or decreases exponentially if $f'(x_*) < 0$.

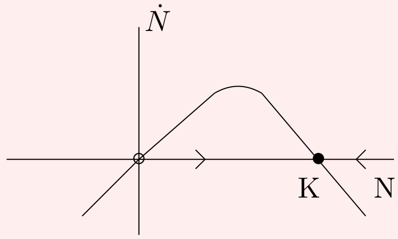
Definition 3.4 (Characteristics Time Scale) — $\frac{1}{|f'(x_*)|}$ is called the characteristics time scale.

Example 3.5 (Logstics Equation)

$N \geq 0$ population size, $r > 0$ growth rate, $K > 0$ carrying capacity

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

Fixed points: $\dot{N} = 0 \implies N_* = 0$ or $N_* = K$.
Let $f(N) = rN\left(1 - \frac{N}{K}\right) \implies f'(N) = r - 2\frac{r}{K}N$. In particular, $f'(0) = r > 0 \implies N_* = 0$ is an unstable fixed point and $f'(K) = r - 2r = -r < 0 \implies N_* = K$ is stable.
Phase portrait:



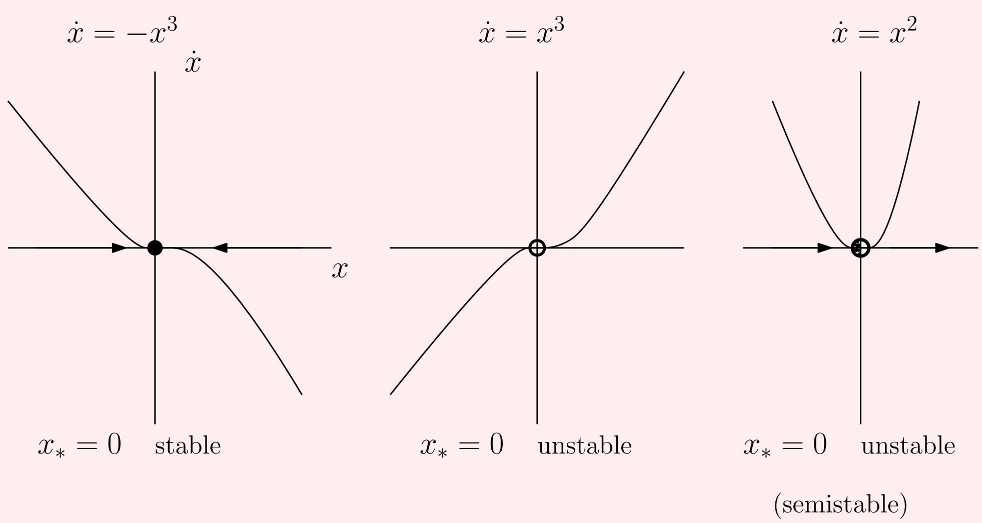
Thus, if $N(t)$ is the population with

$$N(0) = N_0 > 0 \implies N(t) \to K \text{ as } t \to \infty$$
$$N(0) = 0 \to N(t) = 0 \quad \forall t \text{ (no spontaneous outbreak)}$$

Characteristics time scale: $\frac{1}{|f'(N_*)|} = \frac{1}{r}$ for both $N_* = 0, K$.

Example 3.6

What if $f'(x_*) = 0$? Then we can't tell.

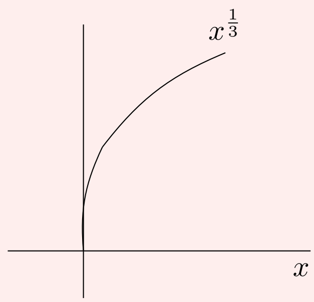


§4 Lec 4: Jan 11, 2021

§4.1 Existence and Uniqueness

Example 4.1 (Non-uniqueness)

$\dot{x} = x^{\frac{1}{3}} \implies x_1(t) \equiv 0$ (for all t) is a solution with $x_1(0) = 0$ but $x_2(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$ is also a solution with $x_2(0) = 0$



Is $x_0 = 0$ really a fixed point? No, it's unclear how it would behave (according to $x(t) = 0$ or $x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$).

Theorem 4.2 (Picard's)

Let $I = (a, b) \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ differentiable and f' continuous. Let $x_0 \in I$. Then there is $\tau > 0$ s.t. the initial value problem

$$\dot{x} = f(x), x(0) = x_0$$

has a unique solution $x : (-\tau, \tau) \to \mathbb{R}$.

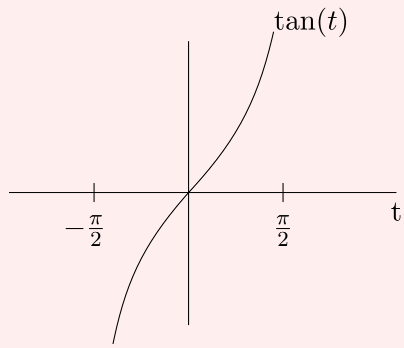
Example 4.3

(The solution might not exist for all times) Consider

$$\frac{dx}{dt} = \dot{x} = 1 + x^2, \quad x(0) = 0$$

So,

$$\begin{aligned} dt &= \frac{dx}{1+x^2} \\ t &= \int \frac{dx}{1+x^2} = \arctan x + C \\ 0 &= 0 + C \implies C = 0 \\ x(t) &= \tan(t) \end{aligned}$$



In particular,

$$x(t) \to +\infty \text{ as } t \to \frac{\pi}{2}$$
$$x(t) \to -\infty \text{ as } t \to \frac{-\pi}{2}$$

i.e., $x(t)$ reaches infinity in finite time, i.e., the solution $x(t)$ blows up in finite time.

Remark 4.4. (Hw 1) If $x_0 > 0$, then the solution to $\dot{x} = x^2, x(0) = x_0 > 0$ blows up in finite time. In fact, if $\alpha > 1$, then the solution to $\dot{x} = x^\alpha$, $x(0) = x_0 > 0$ blows up in finite time.

Theorem 4.5 (ODE Comparison)

If $x_1(t)$ solves $\dot{x} = f(x)$, $x_2(t)$ solves $\dot{x} = q(x)$ and $x_1(0) \le x_2(0)$, $f(x) < q(x)$, then $x_1(t) \le x_2(t)$ for all $t > 0$.
In particular, if $x_1(t) \to \infty$ in finite time, then $x_2(t) \to \infty$ in finite time.

Example 4.6

The solution to $\dot{x} = 1 + x^2 + x^3, x(0) = 0$ blows up in finite time.
<u>*Note*</u>: For $x \geq 0$:

$$1 + x^2 \leq 1 + x^2 + x^3$$

Recall: $\tan(t)$ solves $\dot{x} = 1 + x^2, x(0) = 0$. By comparison: the solution $x(t)$ to $\dot{x} = 1 + x^2 + x^3, x(0) = 0$ satisfies $x(t) \geq \tan(t)$. Thus, $x(t)$ blows up in finite time.
We may indeed assume that $x(t) > 0$. Since $\dot{x}(0) = 1$, it follows that $x(t) > 0$ for $t > 0$ small. In fact, $\dot{x} = 1 + x^2 + x^3 > 0$ for $x(t)$ small, i.e., whenever $x(t)$ is close to zero, it must increase $\implies x(t) > 0$ for $t > 0$.

Example 4.7 (No Oscillating Solution in 1D)

Let $f \in C^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} | f \text{ differentiable}, f' \text{ continuous}\}$. Suppose $f(x_*) = 0, x(t)$ solution of $\dot{x} = f(x)$. If $x(t_0) = x_*$ for some t_0. Then $x(t) = x_*$ for all time t. Geometrically this says that a solution can never reach/cross a fixed point (unless it is a fixed point).

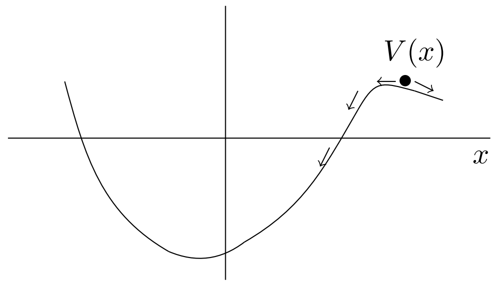
- $f(x(t)) > 0$ and $\dot{x}(t) > 0$, i.e., $x(t)$ increases.
- $f(x(t)) = 0$ and $x(t) =$ constant for all t.
- $f(x(t)) < 0$ and $\dot{x}(t) < 0$ i.e., $x(t)$ decreases.

In particular, there is no oscillating solution.

§5 Lec 5: Jan 13, 2021

§5.1 Potential

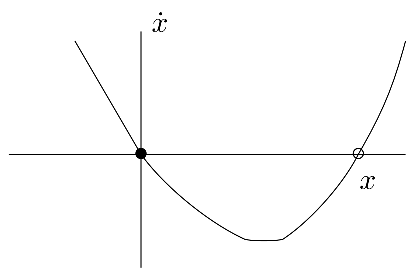
Consider the movement of a particle (with lots of friction) in a potential.



Notice:

- Particle approaches the local minimum of $V(x)$ (minimum energy level) no fixed point.
- Local minima of $V(x)$ are stable fixed points.
- Local maxima of $V(x)$ are unstable fixed points.

$\implies$ $\dot{x} = f(x) = -\frac{dV}{dx} = -V'(x)$.



Expect $t \to V(x(t))$ is non-increasing for a solution $x(t)$ of $\dot{x} = -V'(x)$.
Indeed:

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= V'(x(t))\frac{d}{dt}x(t)\\ &= V'(x(t))\left(-V'(x(t))\right)\\ &= -\left(V'(x(t))\right)^2 \leq 0\end{aligned}$$

$\implies$ particle always moves towards a lower energy level.

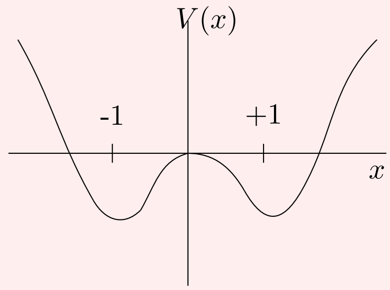
Definition 5.1 (Potential) — A function $V(x)$ s.t. $\dot{x} = f(x) = -\frac{dV}{dx}$ is called a potential.

Example 5.2

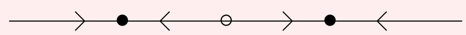
Graph potential for $\dot{x} = x - x^3$. Find/characterize equilibria (fixed points).

$$\dot{x} = f(x) = x - x^3 = -\frac{dV}{dx} \overset{\int}{\Longrightarrow} V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$$

$\Longrightarrow$ V is only defined up to a constant, we may choose any $C \in \mathbb{R}$, e.g., choose $C = 0$.



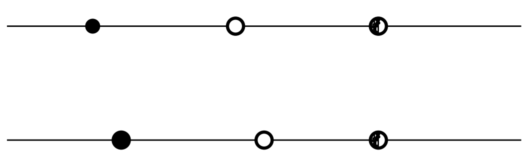
Local minima of V correspond to stable fixed points $\Longrightarrow$ $0 = -\frac{dV}{dx} = f(x) = x - x^3$, i.e., $x = \pm 1$.
Local maximum of V corresponds to an unstable fixed point at $x = 0$.
Phase portrait:



Remark 5.3. This system is often called <u>bistable</u> because it has two stable fixed points.

§5.2 Bifurcations

The qualitative behavior of 1D dynamical systems $\dot{x} = f(x)$ is determined by fixed points.



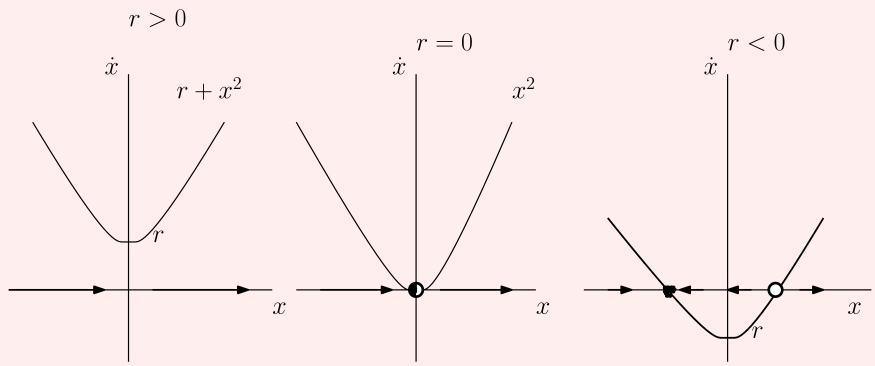
qualitative the same

(dynamics)

If $\dot{x} = f(r, x)$ depends on a parameter r, then the numbers of fixed points and their stability may change as r varies. This is called bifurcation.

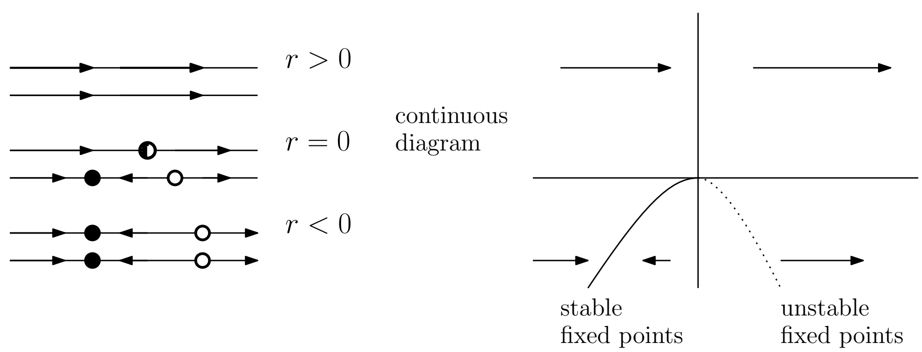
Example 5.4 (Saddle-node, blue sky bifurcation)

$\dot{x} = r + x^2, \quad r \in \mathbb{R}.$

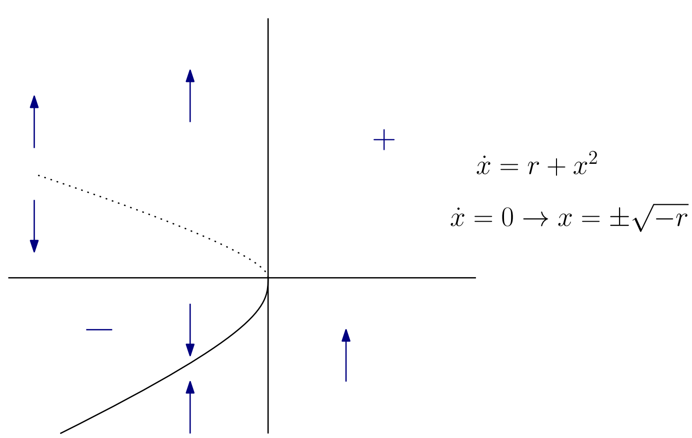


Hence, the qualitative behavior changes at $r_* = 0$, i.e., $r_* = 0$ is called a <u>bifurcation point</u>.

Ways to plot the dependence on the parameter:



Most common: <u>bifurcation diagram</u>



§6 Lec 6: Jan 15, 2021

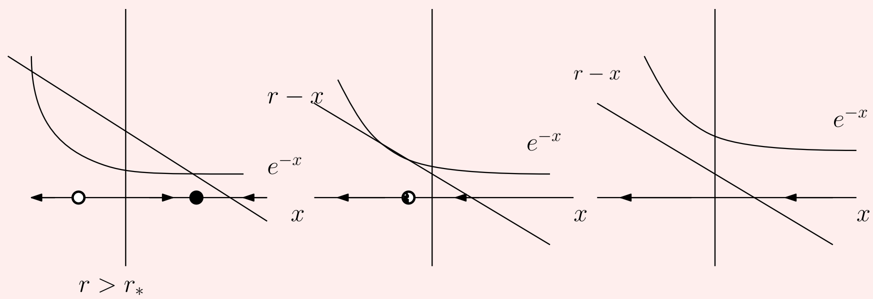
§6.1 Saddle-Node Example

Example 6.1

Argue geometrically that the ODE

$$\dot{x} = r - x - e^{-x}$$

undergoes a saddle-node bifurcation. Furthermore, find the bifurcation point.
Note: Fixed points of $\dot{x} = r - x - e^{-x}$ correspond to intersection points of the functions $r - x, e^{-x}$ because $r - x - e^{-x} = 0 \iff r - x = e^{-x}$.



Indeed we have a saddle-node bifurcation.
Note: At $r = r_*$, the graph of $r - x$ and e^{-x} intersect tangentially. Thus, for the bifurcation point we require:

$$0 = \dot{x} = r - x - e^{-x} \implies r - x = e^{-x}$$
$$0 = \frac{d}{dx}(r - x - e^{-x}) \implies \frac{d}{dx}(r - x) = \frac{d}{dx}e^{-x}$$

So,

$$\begin{aligned}
-1 &= -e^{-x} \\
e^{-x} &= 1 \\
x &= 0 \\
r_* &= x_* + e^{-x_*} = 0 + 1 = 1
\end{aligned}$$

Thus the bifurcation point is $(r_*, x_*) = (1, 0)$.

Note:

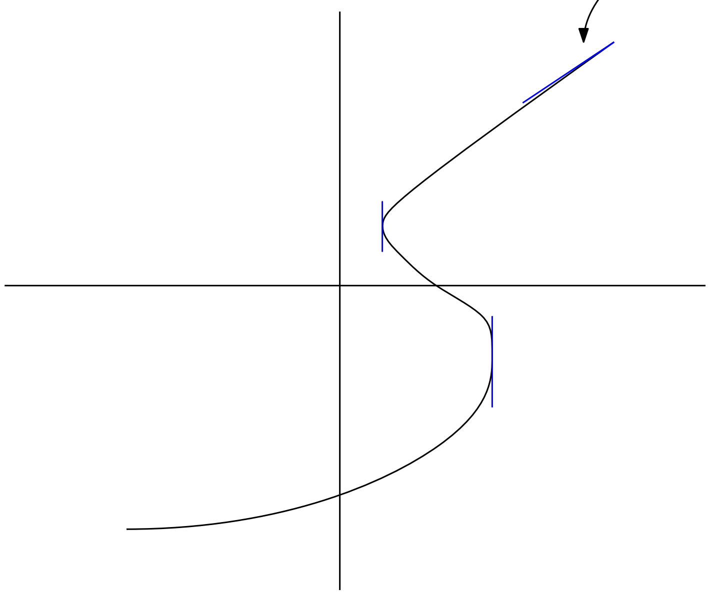
$$\begin{aligned}
\dot{x} = r - x - e^{-x} &= r - x - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right) \\
&= r - 1 - \frac{1}{2}x^2 + \frac{x^3}{6} - \dots \\
&\approx (r - 1) - \frac{1}{2}x^2 \text{ for x near } x_* = 0
\end{aligned}$$

Set $R = r - 1$, then $\dot{x} \approx R - \frac{1}{2}x^2$.
Upshot: Up to appropriate rescalings/coordinate changes, every saddle-node bifurcation looks like its normal form

$$\dot{x} = r - x^2 \qquad (\text{or } \dot{x} = r + x^2)$$

close to the bifurcation point $(r_*, x_*) = (0, 0)$.

§6.2 Normal Forms



Recall:

- Normal vector: $\begin{pmatrix} \partial_r f \\ \partial_x f \end{pmatrix}$
- Tangent vector: $\begin{pmatrix} -\partial_x f \\ \partial_r f \end{pmatrix}$

Note: Bifurcation points have vertical tangent vectors, i.e., $\partial_x f = 0, \partial_r f \neq 0$.

Theorem 6.2 (Taylor's)
Suppose $f(r_*, x_*) = 0$.

$$f(r,x) = f(r_*,x_*) + \underbrace{\frac{\partial f}{\partial r}(r_*,x_*)}_{p_1}(r - r_*) + \underbrace{\frac{\partial f}{\partial x}(r_*,x_*)}_{q_1}(x - x_*)$$
$$+\frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial r^2}(r_*,x_*)}_{p_2}(r-r_*)^2 + \underbrace{\frac{\partial^2 f}{\partial r \partial x}(r_*,x_*)}_{R}(r-r_*)(x-x_*) + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial x^2}(r_*,x_*)}_{q_2}(x-x_*)^2 + \dots$$

Remark 6.3. If $q_1 \neq 0$, then there is no bifurcation at (r_*, x_*), linear stability (sign of q_1) determines if (r_*, x_*) is (un)stable.

Theorem 6.4

Suppose that $f(r_*, x_*) = 0, q_1 = 0, p_1 \neq 0, q_2 \neq 0$, then $\dot{x} = f(r, x)$ undergoes a saddle node bifurcation at (r_*, x_*) and

$$\dot{x} = \frac{\partial f}{\partial r}(r^*, x^*)(r - r^*) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

for $|r - r_*| < \epsilon^2, \quad |x - x_*| < \epsilon$.

Remark 6.5.

i) Note that the constant $(r - r_*)(x - x_*)$ is $\mathcal{O}(\epsilon^3)$

ii) With a coordinate change $(t, x, r) \mapsto (s, y, R)$ we can arrange that ODE looks like

$$\frac{d}{ds}y = R + y^2$$

near $(0, 0) = (R(r_*), y(x_*))$

Example 6.6

$\dot{x} = e^r - x - e^{-x}$ undergoes a saddle-node bifurcation near $(r_*, x_*) = (0,0)$. Apply the theorem 6.4,

$$\begin{aligned}
f(r,x) &= e^r - x - e^{-x} \\
f(0,0) &= 1 - 0 - 1 = 0 \\
\frac{\partial f}{\partial x}(r,x) &= -1 + e^{-x} \implies \frac{\partial f}{\partial x}(0,0) = 0 \\
\frac{\partial f}{\partial r}(r,x) &= e^r \implies \frac{\partial f}{\partial r}(0,0) = 1 \neq 0 \\
\frac{\partial^2 f}{\partial x^2}(r,x) &= -e^{-x} \implies \frac{\partial^2 f}{\partial x^2}(0,0) = -1 \neq 0
\end{aligned}$$

Therefore, by theorem 6.4, $(r_*, x_*) = (0,0)$ is a bifurcation point of a saddle-node bifurcation.
Normal form near $(r_*, x_*) = (0,0)$:

$$\begin{aligned}
\dot{x} &= e^r - x - e^{-x} \\
&= 1 + r + \frac{r^2}{2} + \mathcal{O}(r^3) - x - \left(1 - x + \frac{x^2}{2} + \mathcal{O}(x^3)\right) \\
&= r + \underbrace{\frac{r^2}{2}}_{\mathcal{O}(\epsilon^4)} - \frac{x^2}{2} + \mathcal{O}(r^3) + \mathcal{O}(x^3) \\
&= \underbrace{r - \frac{x^2}{2}}_{\mathcal{O}(\epsilon^2)} + \mathcal{O}(\epsilon^3) \text{ if } |r - r_*| = |r| < \epsilon^2 \\
&\qquad\qquad \text{if } |x - x_*| = |x| < \epsilon
\end{aligned}$$

Set $y = \frac{x}{2}$, then

$$\dot{y} = \frac{1}{2}\dot{x} = \frac{r}{2} - \frac{x^2}{4} + \mathcal{O}(\epsilon^3) = \frac{r}{2} - y^2 + \mathcal{O}(\epsilon^3)$$

Set $s = -t$, then

$$\frac{d}{ds}y = -\frac{d}{dt}y = -\frac{r}{2} + y^2 + \mathcal{O}(\epsilon^3)$$

Set $R = -\frac{r}{2}$, then

$$\underbrace{\frac{d}{ds}y = R + y^2}_{\text{normal form of a saddle-node bifurcation}} \qquad + \mathcal{O}(\epsilon^3)$$

§7 Lec 7: Jan 20, 2021

§7.1 Classification of Bifurcations

Let's rewrite $\dot{x}$ in theorem 6.4 as

$$\dot{x} = p(r - r_*) + \frac{c}{2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

if $|r - r_*| < \epsilon^2, |x - x_*| < \epsilon$. After a coordinate change $(t, x, r) \mapsto (s, y, R)$ such that

$$\begin{aligned} s &= t \\ y &= \frac{c}{2}(x - x_*) \\ R &= p\frac{c}{2}(r - r_*) \end{aligned}$$

the ODE is represented by the normal form.

$$\frac{d}{ds}y = \dot{y} = R + y^2 + \mathcal{O}(\epsilon^3)$$

for $|R| < \epsilon^2, |y| < \epsilon$.
If $f(x_*, r_*) = 0$, and also $\frac{\partial f}{\partial x}(x_*, r_*) = 0 = \frac{\partial f}{\partial r}(x_*, r_*)$, then the second derivatives determines the bifurcation type.

$$\text{Hessian} \quad \text{Hess} f = \begin{pmatrix} \frac{\partial^2 f}{\partial r^2} & \frac{\partial^2 f}{\partial r \partial x} \\ \frac{\partial^2 f}{\partial r \partial x} & \frac{\partial^2 f}{\partial x^2} \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Second test: if $AC - B^2 > 0$, (r_*, x_*) is a local maximum/minimum. In particular, (r_*, x_*) is an isolated fixed point. (irrelevant case)
Practically relevant case: If $AC - B^2 < 0$: (r_*, x_*) is a saddle. If also $C \neq 0$: transcritical bifurcation.

$$\dot{y} = Ry - y^2 + \mathcal{O}(\epsilon^2)$$

for $|R| < \epsilon, |y| < \epsilon$ (after an appropriate coordinate change)

$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

If also $C = 0$: Pitchfork bifurcation

- Supercritical Pitchfork bifurcation:

$$y' = Ry - y^3 + \mathcal{O}(\epsilon^3)$$

- Subcritical Pitchfork bifurcation

$$y' = Ry + y^3 + \mathcal{O}(\epsilon^3)$$

  for $|R| < \epsilon^2, |y| < \epsilon$

Again,

$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

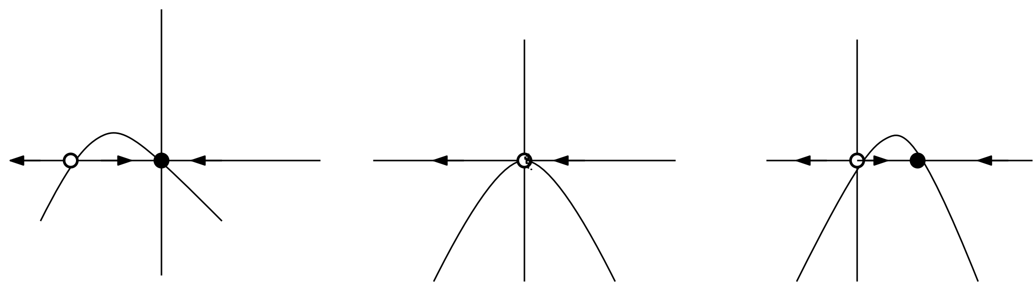
§7.2 Transcritical Bifurcation

Normal form:

$$\dot{x} = rx - x^2 = x(r - x)$$

In particular, $x_* = 0$ is always a fixed point but it changes stability.

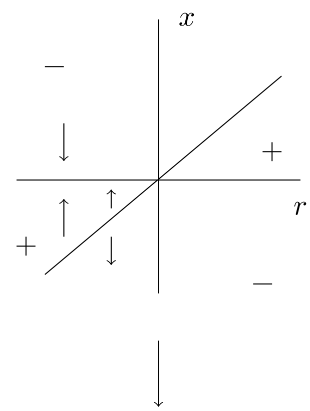
$r < 0$ $\qquad$ $r = 0$ $\qquad$ $r > 0$



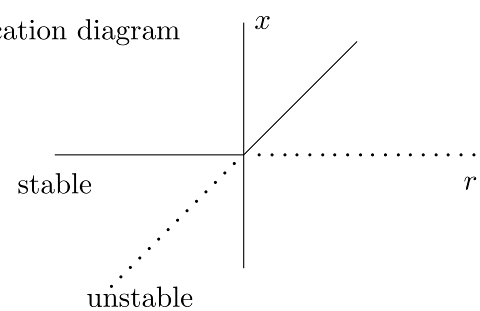
Bifurcation diagram: $\dot{x} = x(r - x) = rx - x^2 = f(x)$. Fixed points:

$$x_* = 0, \quad x_* = r \quad r \in \mathbb{R}$$

intermediate step:
draw fixed points
(without stability)



bifurcation diagram



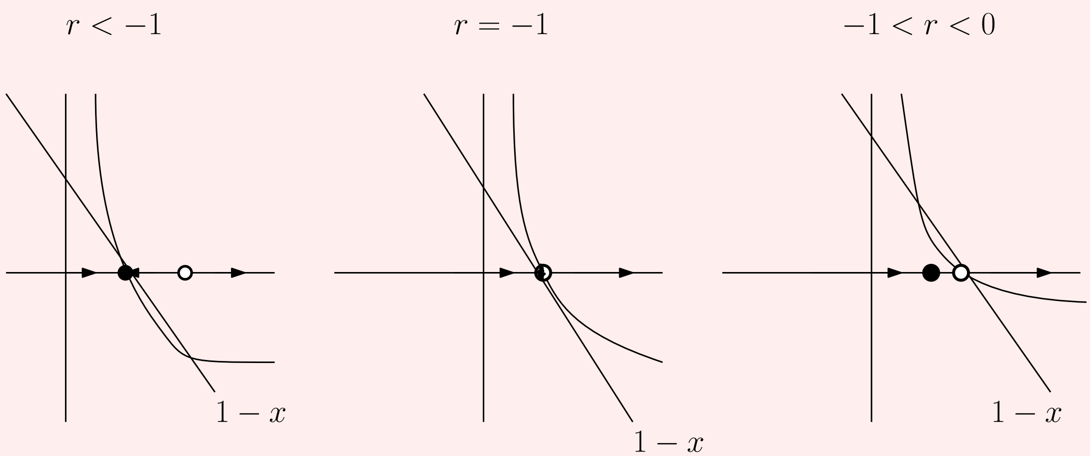
§8 Lec 8: Jan 22, 2021

§8.1 Example of Transcritical Bifurcation

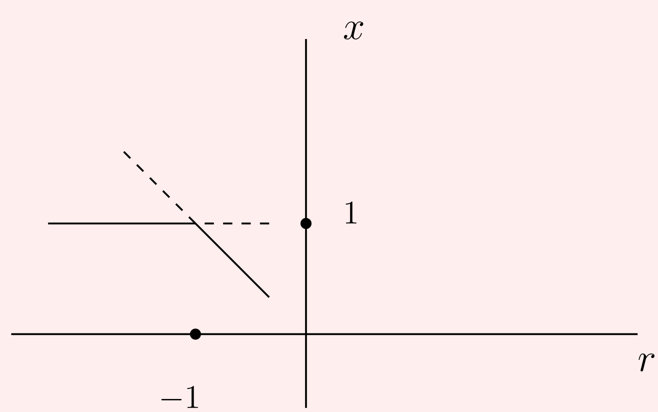
Example 8.1

$\dot{x} = r\ln(x) + x - 1$ has a transcritical bifurcation at $(r_*, x_*) = (-1, 1)$.
Geometric approach:

$$\dot{x} = 0 \iff r\ln(x) = 1 - x$$



Bifurcation near $(r_*, x_*) = (-1, 1)$



Normal form: $\dot{x} = r\ln(x) + x - 1$.

Remark 8.2. $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \quad |x| < 1$

So,

$$\begin{aligned}
\dot{x} &= r\ln(x) + x - 1 \\
&= r(x - 1 - \frac{1}{2}(x-1)^2 + \mathcal{O}((x-1)^3) + x - 1 \\
&= (r+1)(x-1) - \frac{1}{2}\left((r+1)-1\right)(x-1)^2 + \mathcal{O}\left(r(x-1)^3\right) \\
&= (r+1)(x-1) + \frac{1}{2}(x-1)^2 + \mathcal{O}(\epsilon^3)
\end{aligned}$$

if $|r-(-1)| < \epsilon$ and $|x - 1 < \epsilon|$.
Now, set $R = r+1, y = c\cdot(x-1)$. Then,

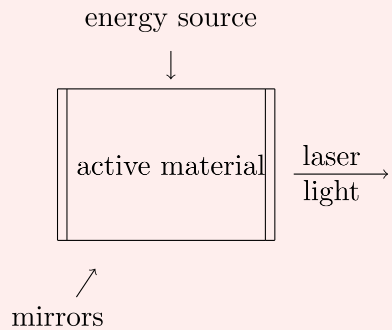
$$\begin{aligned}
\dot{y} &= c\dot{x} \\
&= (r+1)c(x-1) + \frac{1}{2}c(x-1)^2 + \mathcal{O}(\epsilon^3) \\
&= Ry + \frac{1}{2c}\left(c(x-1)\right)^2 + \mathcal{O}(\epsilon^3) \\
&= Ry + \underbrace{\frac{1}{2c}}_{=1} y^2 = Ry + y^2
\end{aligned}$$

for $c = \frac{1}{2}$.

§8.2 Application of Transcritical Bifurcations

Example 8.3 (Laser Threshold)

Consider



Simple model:

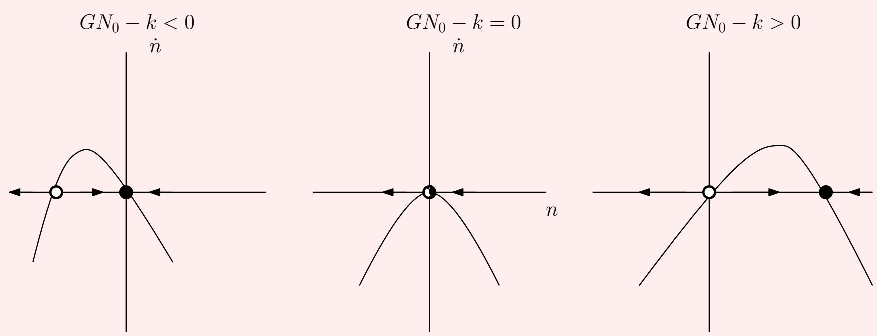
$$n = n(t) = \#\text{ photons in the laser}$$

Then

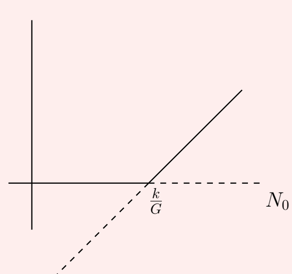
$$\begin{aligned}\dot{n} &= G \cdot \underbrace{N}_{\#\text{ excited atoms}} \cdot n - kn \\ &= N_0 - \alpha \cdot n \\ &= G(N_0 - \alpha n)n - kn \\ &= (GN_0 - k)n - \alpha G n^2\end{aligned}$$

where $G, k, \alpha > 0$. Fixed points:

$$\dot{n} = 0 \iff n = 0 \text{ or } n = \frac{GN_0 - k}{\alpha G}$$



Bifurcation diagram



transcritical bifurcation at

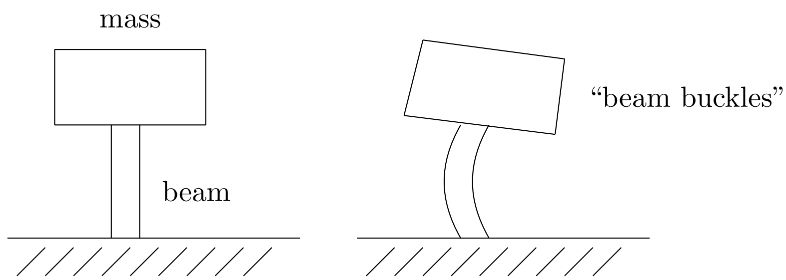
$(N, n) = (\frac{k}{G}, 0)$

$\frac{k}{G} =$ laser threshold

§9 Lec 9: Jan 25, 2021

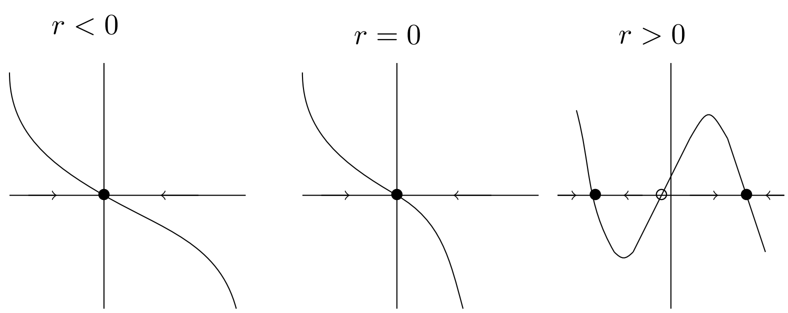
§9.1 Supercritical Pitchfork Bifurcation

Fixed points appear/disappear in symmetric pairs



Supercritical Pitchfork Bifurcation:

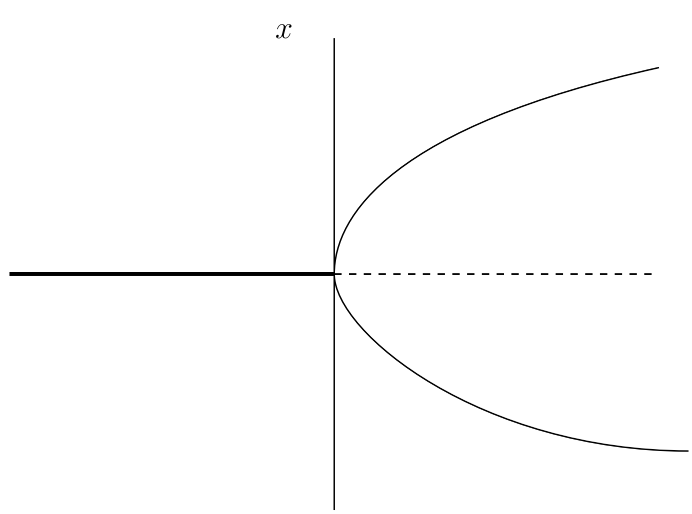
$$\dot{x} = rx - x^3$$



> **Remark 9.1.** Decay towards $x_* = 0$ is not exponential in time for $r = 0$.

Bifurcation diagram:

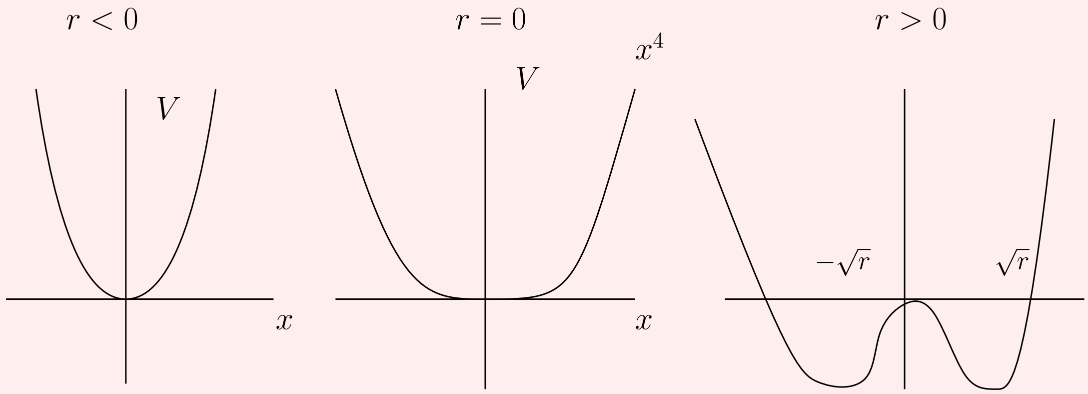
$$\dot{x} = rx - x^3 = 0$$
$$\implies x = 0, \qquad x = \pm\sqrt{r}, \quad r > 0$$



Example 9.2

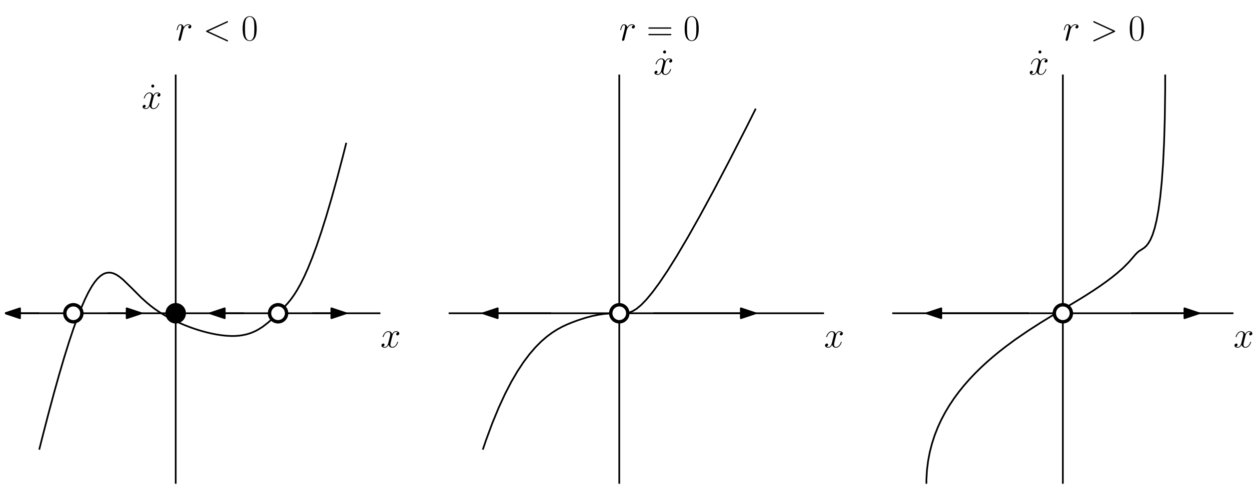
Potential for $\dot{x} = rx - x^3 = -\frac{dV}{dx}$

$$\implies V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4 + \underbrace{C}_{=0}$$



§9.2 Subcritical Pitchfork Bifurcation

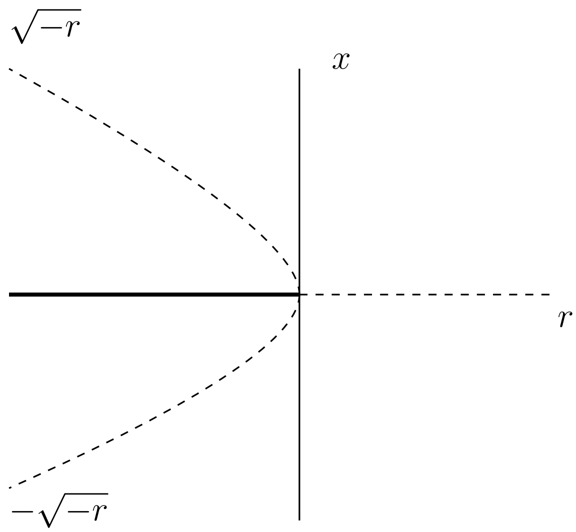
$$\dot{x} = rx + x^3$$



Fixed points:

$$\dot{x} = rx + x^3 = 0$$
$$\implies x = 0, \qquad x = \pm\sqrt{-r}, \quad r < 0$$

<u>Bifurcation Diagram</u>:



Remark 9.3. If $r > 0, x_0 > 0$, then the solution $x(t)$ with $x(0) = x_0 > 0$ blows up in finite time (cf. homework). Interpretation: $+x^3$ is destabilizing.

Physically more realistic scenario:

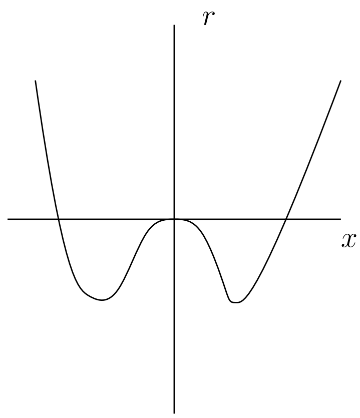
$$\dot{x} = rx + x^3 - x^5$$

where x^5 is the stabilizing higher order term.
Fixed points:

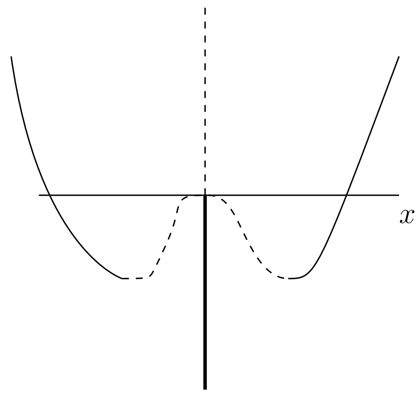
$$\dot{x} = 0 \iff x = 0, \quad r = -x^2 + x^4$$

Bifurcation diagram:

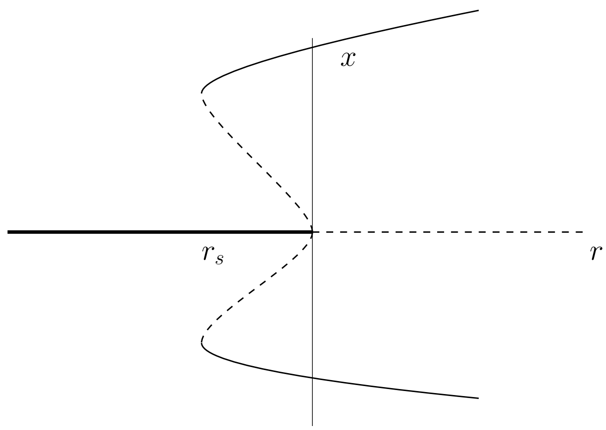
1. Intermediate step



2. Stability Types

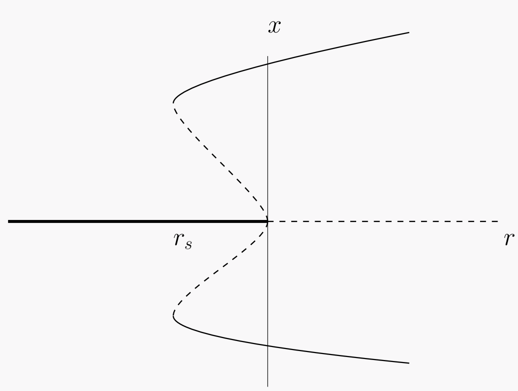


3. Change axes: bifurcation diagram



$$\dot{x} = rx + x^3 - x^5$$

Remark 9.4. i) Subcritical pitchfork bifurcation at $(r_*, x_*) = (0, 0)$ and saddle node bifurcation at $(r_s, x_*) = \left(-\frac{1}{4}, \pm\sqrt{2}\right)$.



$$\dot{x} = r(t)x + x^3 - x^5$$

ii) jump at $r_* = 0$: A small perturbation of a stable fixed point at $(0, r)$ with $r < 0$ jumps to the stable large amplitude branch as r becomes positive, but does not jump back until $r < r_s$.

This non-reversibility is called hysteresis.

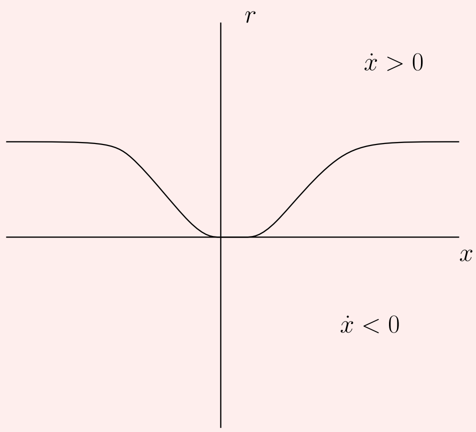
§10 Lec 10: Jan 27, 2021

§10.1 Bifurcation at Infinity

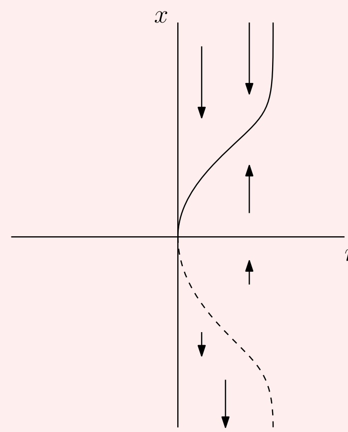
Example 10.1

$\dot{x} = r - \frac{x^2}{1+x^2}$

Fixed points: $\dot{x} = 0 \iff r = \frac{x^2}{1+x^2}$



bifurcation diagram



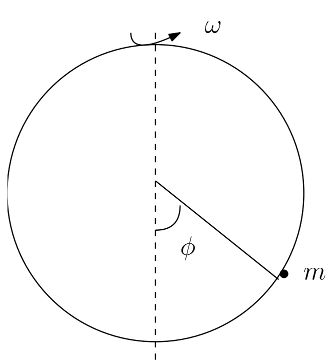
Note:

- At (r_*, x_*) we have a saddle node bifurcation.
- If $r \in (0, 1)$ we have two fixed points.
- For $r \geq 1$ we have no fixed points.

Thus, we have a bifurcation at (spatial) infinity.

§10.2 Dimensional Analysis and Scaling

Over-damped bead over a hoop:



forces: gravitation: $-mg\vec{e_2}$

centrifugal: $mr\sin\phi\omega^2\vec{e_x}$

damping: $-b\phi\vec{e_\phi}$

Physics: $mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$
Experiment: Provided ω large enough, bead slides slowly towards a fixed angle, after an initial acceleration phase.

Question 10.1. When we can neglect second order term $\ddot{\phi}$?

Problem 10.1. We're working with different dimensions, e.g.

$$[m] = kg$$
$$[b] = \frac{kg \cdot m}{s}$$

What is small – what quantity is actually small so we can neglect the second order term?

Idea: Non-dimensionalize

- small means $\ll 1$
- reduce the numbers of parameters
- no general algorithm

Quantity ω large, time scale T.
Set $\tau = \frac{t}{T} \implies d\tau = \frac{1}{T}dt$, where T is the characteristics time scale.
$\dot{\phi} = \frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d\phi}{d\tau}$
Similarly, $\ddot{\phi} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}$

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi \tag{1}$$

So

$$\implies \frac{mr}{T^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{T}\frac{d\phi}{d\tau} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi \qquad \text{(unit force)}$$
$$\implies \frac{r}{gT^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{mgT}\frac{d\phi}{d\tau} - \sin\phi + \frac{r\omega^2}{g}\sin\phi\cos\phi \qquad \text{(dimensionless)}$$

Thus 1$^{\text{st}}$ order term $\frac{d\phi}{d\tau}$ dominates $\frac{d^2\phi}{d\tau^2}$ if $\frac{r}{gT^2} \ll 1$ and $\frac{b}{mgT} \approx \mathcal{O}(1)$, i.e., $\frac{b}{mgT} = 1$ and $\epsilon = \frac{r}{gT^2}$

$$\implies T = \frac{b}{mg}$$
$$\implies \epsilon = \frac{rgm^2}{b^2} \ll 1$$

Set $\gamma = \frac{r\omega^2}{g}$. Then the non-dimensionalize equation becomes

$$\epsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma\sin\phi\cos\phi$$

Overdamped limit: $\epsilon \to 0$

$$\begin{aligned}\frac{d\phi}{d\tau} &= -\sin\phi + \gamma\sin\phi\cos\phi \\ &= \sin\phi(\gamma\cos\phi - 1)\end{aligned}$$

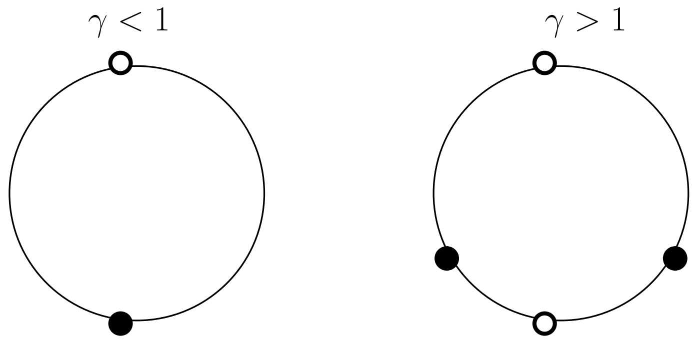
Dynamics: $\frac{d\phi}{d\tau} = 0$ (fixed points)

$$\implies \sin\phi = 0 \iff \phi = 0, \pi \text{ (bottom/top of hoop)}$$

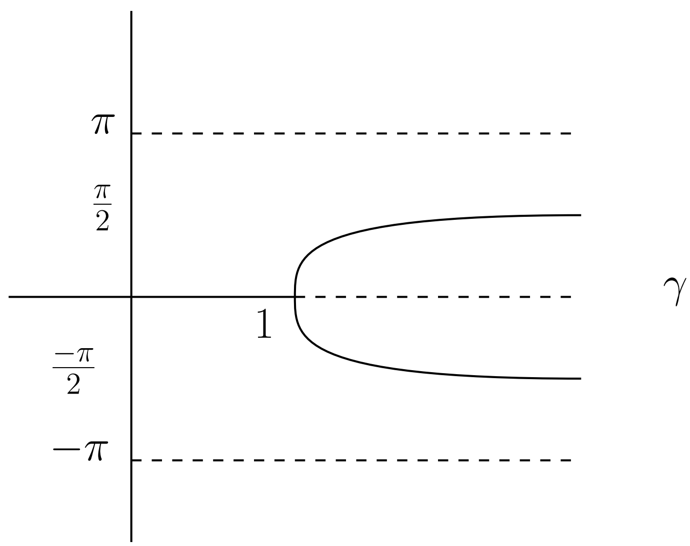
or

$$\cos\phi = \frac{1}{\gamma} \in (0, 1] \implies \gamma \geq 1$$

Fixed points:



Bifurcation Diagram:



In particular, we have a supercritical pitchfork bifurcation at $\gamma = 1$.

§11 Lec 11: Jan 29, 2021

§11.1 Imperfect Bifurcation and Catastrophes

$\dot{x} = h + rx - x^3$

- If $h = 0$: symmetry, if $x(t)$ is a solution then $-x(t)$ is also a solution (supercritical pitchfork bifurcation).
- If $h \neq 0$: imperfect parameter, breaks symmetry.

Aim: Study qualitative behavior of ODE as parameters vary.
Strategy: keep h fixed and vary r

- $h = 0$: supercritical pitchfork bifurcation

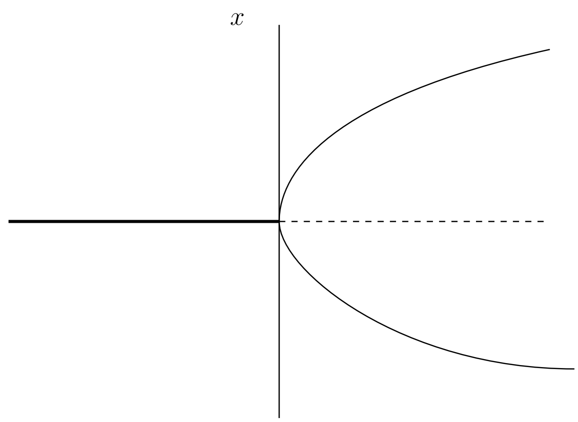


Figure 1: Bifurcation Diagram

- $h > 0$: fixed points: $\dot{x} = 0 \iff x^3 = h + rx$

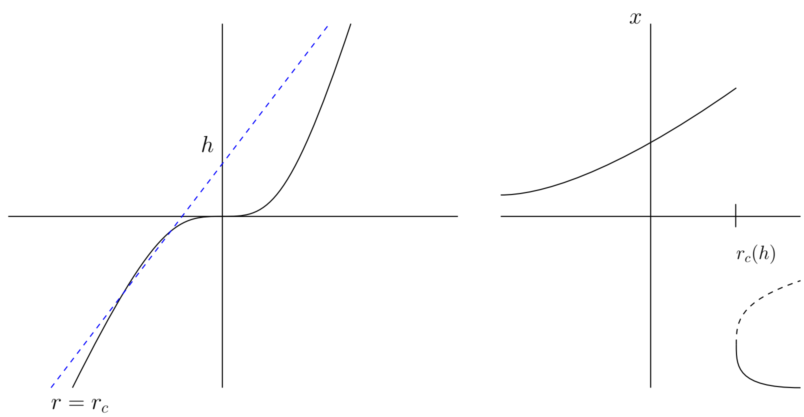


Figure 2: Bifurcation Diagram

- $h < 0$: Fixed points: $x^3 = h + rx$

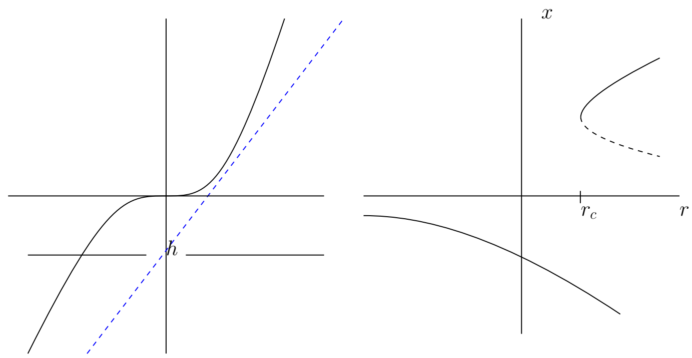


Figure 3: Bifurcation Diagram

<u>*Note*</u>:We have saddle node bifurcation at $r_c = r(h)$

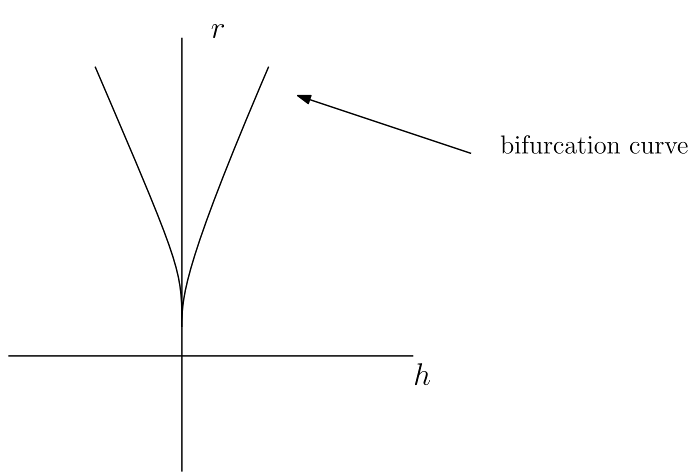
<u>Bifurcation Curves</u>

$$\left\{(h,r)|(h,r,x) \text{ solves } f=0, \frac{\partial f}{\partial x}=0\right\}$$

in our example $\dot{x} = h + rx - x^3$

$$\begin{aligned} 0 = \frac{\partial f}{\partial x} = r - 3x^2 &\implies x = \pm\sqrt{\frac{r}{3}} \\ 0 = f = h + rx - x^3 &\implies h = x^3 - rx \\ &\implies h = x^3 - rx = \pm\frac{2\sqrt{3}}{9}r^3 \\ h = h_c(r) &= \pm\frac{2\sqrt{3}}{9}r^{\frac{3}{2}} \\ &\implies r = r_c(h) = \left(\frac{9}{2\sqrt{3}}|h|\right)^{\frac{2}{3}} \end{aligned}$$

<u>Stability Diagram</u>:
Plot the bifurcation curves in the parameters space ($=(h,r)$ plane).



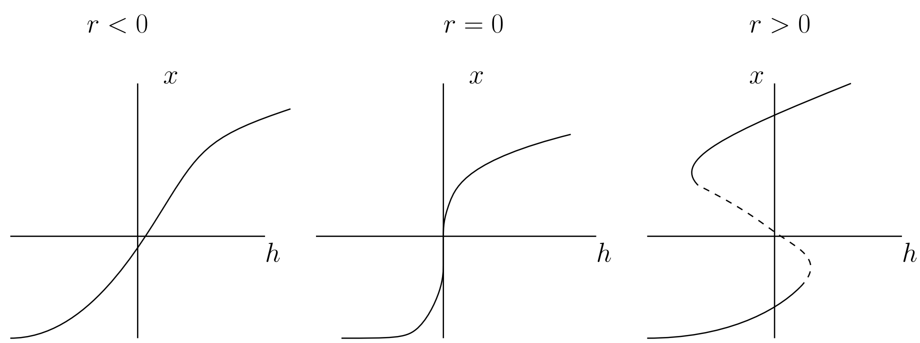
<u>*Note*</u>: qualitative behavior of ode changes as (h, r) cross bifurcation curve.
In example:

- "below" bifurcation curve: ODE has one (stable) fixed point.
- "on" bifurcation curve: two fixed points.
- "above" bifurcation curve: three fixed points.

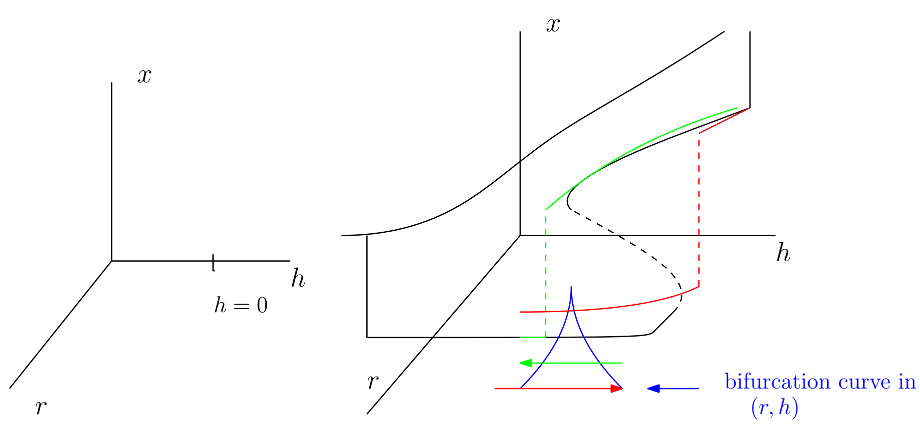
> **Remark 11.1.**
> - Saddle-node bifurcation occurs along bifurcation curve for $(h, r) \neq (0, 0)$
> - At $(h, r) = (0, 0)$, the branches $r_c(h) = \left(\frac{9}{2\sqrt{3}}|h|\right)^{\frac{2}{3}}$ for $h > 0$ and $h < 0$ meet tangentially, and we have a <u>cusp point</u> at $(h, r) = (0, 0)$. This is an example of a <u>codimension 2 bifurcation</u> (i.e., we need two parameters to model this type of bifurcation).

Bifurcation diagrams for fixed $r \in \mathbb{R}$.

$$\dot{x} = h + rx - x^3 = 0 \iff h = x^3 - rx$$



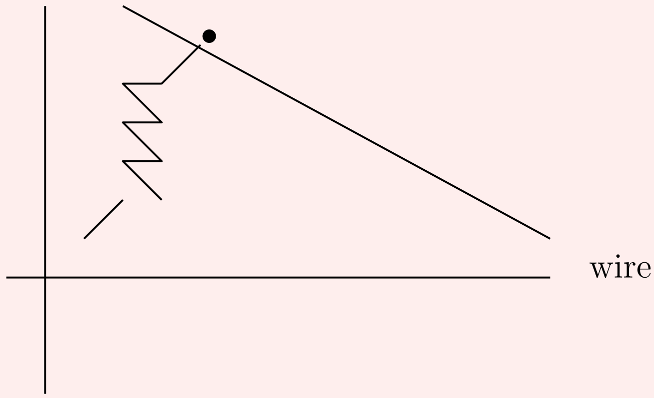
3D plot$(h, r$, fixed points $x)$



Picture/surface of cusp catastrophe solutions close to "upper" stable fixed points drop to "lower" stable fixed points as (r, h) vary (and vice versa).

Example 11.2 (practical)

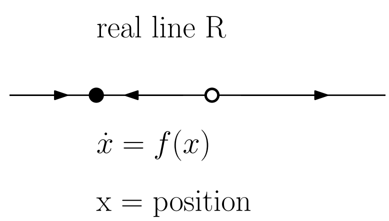
Details in the book, page 74

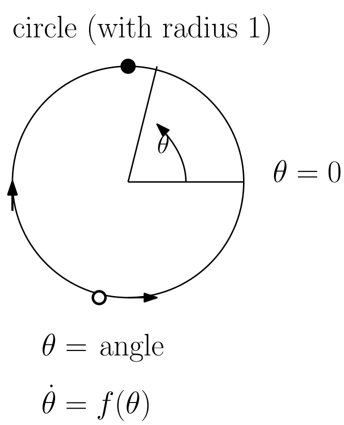


§12 Midterm 1: Feb 1, 2021

§13 Lec 12: Feb 3, 2021

§13.1 Flows on the Circle

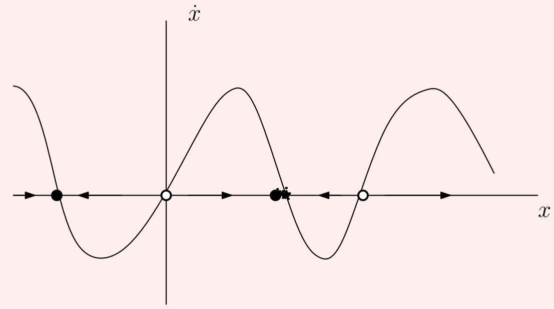




Example 13.1 i) $\dot{x} = \sin(x)$. Fixed points: $\dot{x} = 0$

$$\iff x = \ldots, -\pi, 0, \pi, 2\pi, \ldots$$

i.e., $x = k\pi, k \in \mathbb{Z}$.

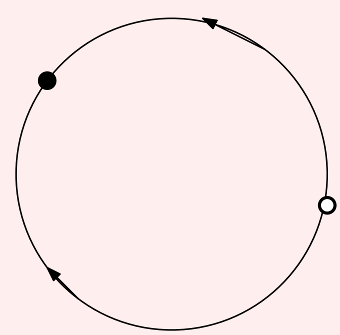


$$\dot{\theta} = \sin\theta$$
$$\dot{\theta} = 0$$
$$\iff \theta = 0 \text{ or } \theta = \pi$$
$$\underbrace{\theta = 2\pi}_{\text{same position on circle}}$$

i.e., θ is defined up to multiples of 2π.

Note: If $f(\theta) > 0$: flow is counterclockwise, and if $f(\theta) < 0$: flow is clockwise.

$\dot{\theta} = \sin(\theta)$



ii) $\dot{x} = x$ where $f(x) = x$ is not periodic.

Thus $\dot{\theta} = \theta$ does not work, because $\theta = 0, \theta = 2\pi$ describe the same position on the circle but $f(\theta) = \theta$ yields different values at $\theta = 0, 2\pi$, i.e. $f(\theta)$ is not a vector field on the circle.

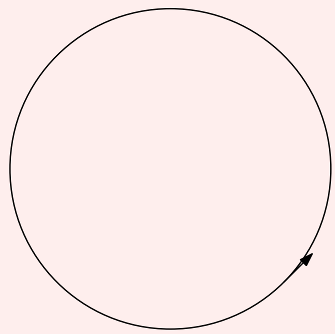
Correspondence:

$f(x)$ is 2π-periodic, i.e. $f(x + 2\pi) = f(x)$, and f is continuously differentiable $\iff f(\theta)$ defines a vector field on the circle.

Example 13.2 iii) $\dot{x} = c > 0$

$$x(t) = ct + x_0$$

$\dot{\theta} = \omega > 0$ – uniform oscillator



$$\theta(t) = \omega t + \theta_0$$

Period T:

$$\theta(T) = \theta(0) + 2\pi$$
$$\omega T + \theta_0 = \theta_0 + 2\pi$$
$$T = \frac{2\pi}{\omega}$$

In particular, periodic solutions are possible.

Example 13.3

Two runners are on a circular track, running in the same direction, with constant speed:

- Runner 1: period $T_1 = \frac{2\pi}{\omega_1}$, angle θ_1
- Runner 2: period $T_2 = \frac{2\pi}{\omega_2}$, angle θ_2

Runner 1, 2 start at the same position. Suppose $T_1 < T_2$, i.e. Runner 1 is faster than runner 2.

Question 13.1. How long does it take runner 1 to lap runner 2?

Ans: T_{lap} = time when phase difference

$$\begin{gathered} \phi = \theta_1 - \theta_2 \text{ is } 2\pi \\ \dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2,\ \phi(0) = 0 \\ \implies \phi(t) = (\omega_1 - \omega_2)t \\ \implies T_{\text{lap}} = \frac{2\pi}{\omega_1 - \omega_2} = \frac{1}{\frac{1}{T_1} - \frac{1}{T_2}} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1} \end{gathered}$$

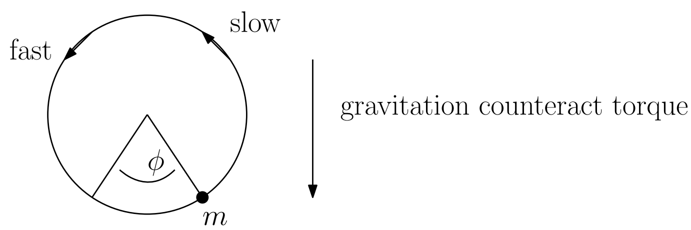
i.e. Runner 1,2 are in phase after T_{lap} again. This is called beat phenomenon.

§14 Lec 13: Feb 5, 2021

§14.1 Non-uniform Oscillator

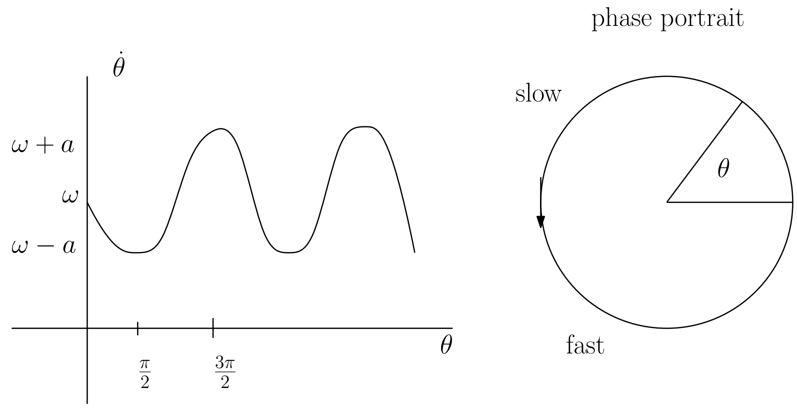
$$\dot{\theta} = \omega - a\sin\theta, \qquad \omega > 0,\ a > 0$$

Practical example: overdamped limit of pendulum driven by constant torque.



$$\dot{\phi} = \omega - a\sin\phi$$

Consider: $\dot{\theta} = \omega - a\sin\theta$
For $0 < a < \omega$:



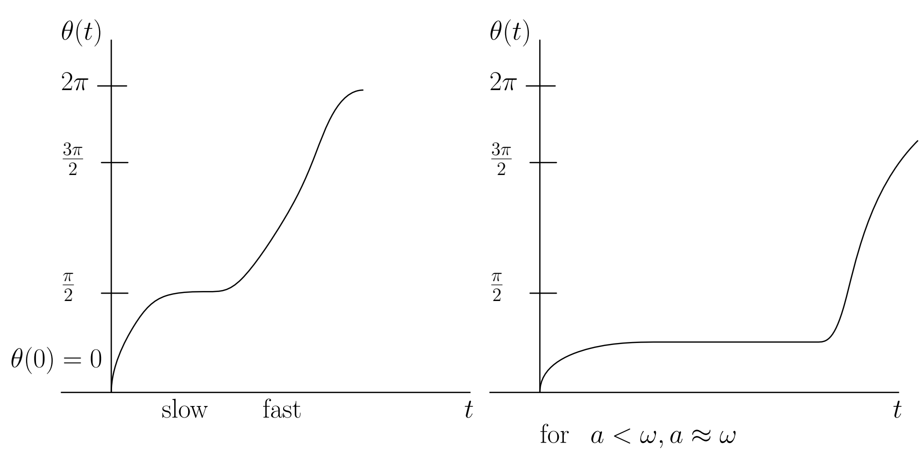
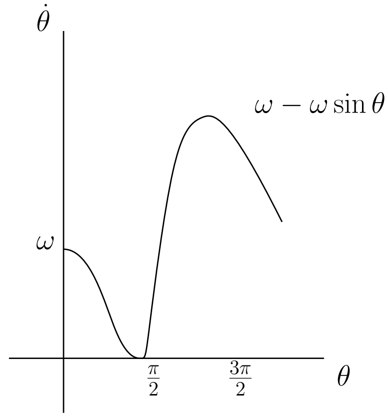
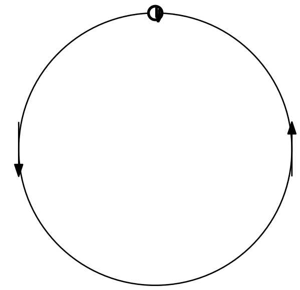


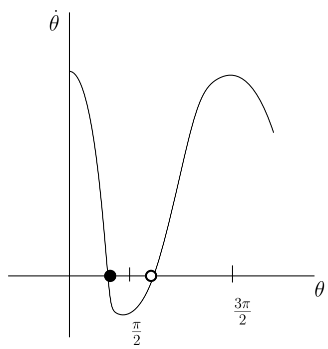
Figure 4: bottle neck remnants or "ghost" of a saddle-node bifurcation

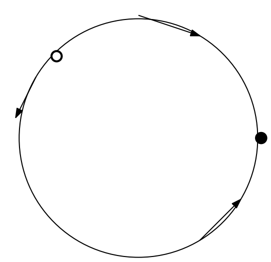
For $a = \omega$





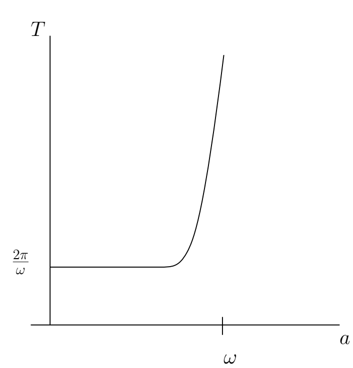
For $a > \omega$:





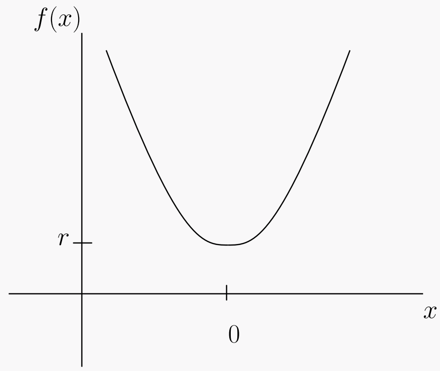
Oscillation period for $a < \omega$:

$$
\begin{aligned}
T &= \int dt = \int_0^{2\pi} \frac{dt}{d\theta} d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} \\
&= \ldots = \frac{2\pi}{\sqrt{\omega^2 - a^2}} = \frac{2\pi}{\sqrt{\omega + a}} \cdot \frac{1}{\sqrt{\omega - a}} \\
&\approx \frac{2\pi}{\sqrt{2\omega}} \cdot \underbrace{\frac{1}{\sqrt{\omega - a}}}_{\text{blow up as } a \to \omega}
\end{aligned}
$$



Remark 14.1. Bottlenecks/this scaling law are a general feature of saddle-node bifurcations:

$$\text{Normal form: } \frac{dx}{dt} = \dot{x} = r + x^2$$



$$\begin{aligned} T_{\text{bottleneck}} &\approx \int dt \\ &= \int_{-\infty}^{\infty} \frac{dt}{dx} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{r + x^2}\, dx \\ T_{\text{bottleneck}} &= \frac{\pi}{\sqrt{r}} \end{aligned}$$

blows up like $\sim r^{-\frac{1}{2}} = \frac{1}{\sqrt{r}}$ as $r \to 0$ and $r > 0$.

Example 14.2

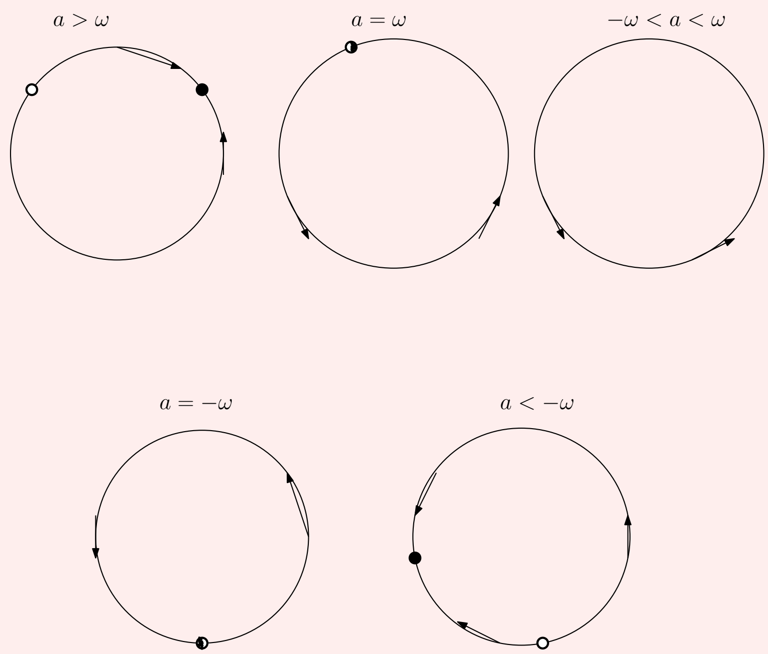
Draw all qualitatively different phase portraits of

$$\dot{\theta} = \omega - a\sin\theta \quad \text{(where } \omega > 0 \text{ fixed)}$$

Bifurcation points: $\dot{\theta} = f(\theta) = 0$, $\frac{\partial f}{\partial \theta} = 0$. Thus, $0 = -a\cos\theta \implies a = 0$ or $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

$$\text{If } a = 0 : \dot{\theta} = \omega > 0 \text{ (no bifurcation)}$$
$$\text{If } \theta = \frac{\pi}{2} : 0 = \dot{\theta} = \omega - a \implies a = \omega$$
$$\text{If } \theta = \frac{3\pi}{2} : 0 = \dot{\theta} = \omega + a \implies a = -\omega$$

Bifurcation points $(a_*, \theta_*) = \left(\omega, \frac{\pi}{2}\right), \left(-\omega, \frac{3\pi}{2}\right)$.
$\dot{\theta} = \omega - a\sin\theta$



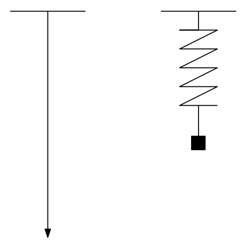
§14.2 2D Dynamical Systems

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

<u>Introduction & Linear Systems:</u>

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ i.e. } \dot{x} = Ax$$

<u>Harmonic Oscillator</u>: $m\ddot{x} + kx = 0$

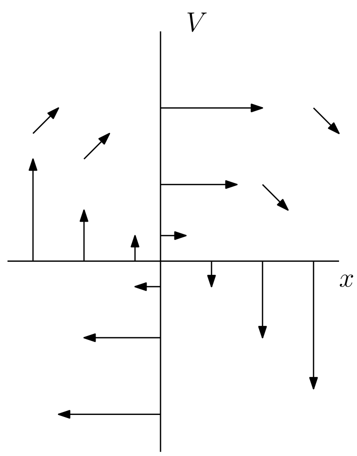


$$\ddot{x} + \omega^2 x = 0,\ \omega^2 = \frac{k}{m}$$

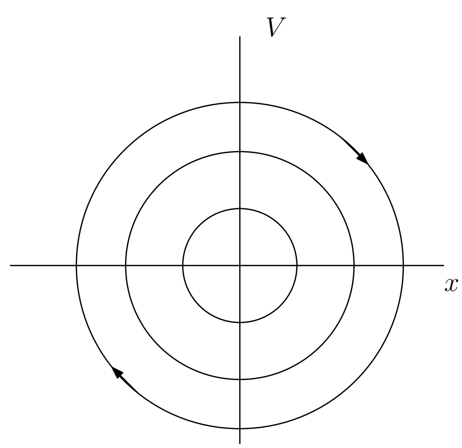
where k : spring constant and m : mass, x : position, v : velocity.

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \ddot{x} = -\omega^2 x \\ \frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2 x \end{pmatrix} \end{aligned}$$

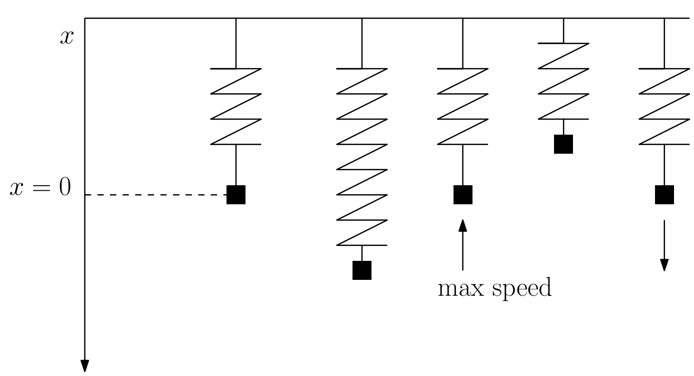
<u>*Note*</u>: the last matrix defines vector field on phase plane.



phase portrait



Harmonic oscillator:



Remark 14.3. Have:

$$\begin{aligned}\frac{d}{dt}(\omega^2 x^2 + v^2) &= 2\omega^2 x\dot{x} + 2v\dot{v} \\ &= 2\omega^2 xv - 2\omega^2 vx = 0 \\ &\implies \omega^2 x^2 + v^2 = \text{ const}\end{aligned}$$

$\implies$ trajectories $\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$ describe ellipses, in particular, they are <u>closed orbits</u> i.e. correspond to <u>periodic solutions</u>.

§15 Lec 14: Feb 8, 2021

§15.1 Classification of Linear Systems

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{i.e. } \dot{x} = Ax$$

Question 15.1. What is the stability type of $x_* = 0$?

Definition 15.1 (Eigenvector) — $v \neq 0$ is an eigenvector of A if

$$Av = \lambda v$$

for some $\lambda \in \mathbb{C}$

$\lambda \in \mathbb{C}$ is an eigenvalue

$$\begin{aligned}
\iff \Lambda_\lambda(A) &= \det(A - \lambda I) = 0 \\
&= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\
&= (a-\lambda)(d-\lambda) - bc \\
&= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) \\
&= 0 \\
\iff \lambda_{1,2} &= \frac{1}{2}\left(\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}\right)
\end{aligned}$$

3 cases:

i) $\lambda_1 \neq \lambda_2$ real valued $\iff \operatorname{tr}(A)^2 > 4\det(A)$

ii) $\lambda_1 = \lambda_2$ real valued $\iff \operatorname{tr}(A)^2 = 4\det(A)$

iii) $\lambda_1 = \overline{\lambda_2}$ complex conjugate $\iff \operatorname{tr}(A)^2 < 4\det(A)$

1. $\lambda_1 \neq \lambda_2 \implies$ there are linearly independent eigenvectors v_i :

$$Av_i = \lambda_i v_i \quad \text{for } i = 1, 2$$

   A is diagonalizable.

   Coordinate change:

$$\begin{aligned}
C &= (v_1|v_2) \\
B &= C^{-1}AC = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\
y &= C^{-1}x
\end{aligned}$$

Then $\dot{y} = C^{-1}\dot{x} = C^{-1}Ax = C^{-1}ACy = By$ i.e. $\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{pmatrix}$ i.e. the ODE decouples

$$\dot{y}_i = \lambda_i y_i \quad \text{for } i = 1, 2$$

So

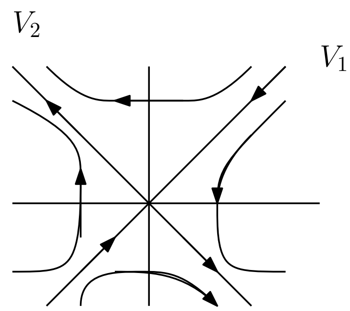
$$\begin{aligned} &\implies y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix} \\ &\implies x(t) = Cy(t) = c_1 e^{\lambda_1 t} C \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} C \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

If $\lambda_1 \neq \lambda_2$:

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

Phase portraits:

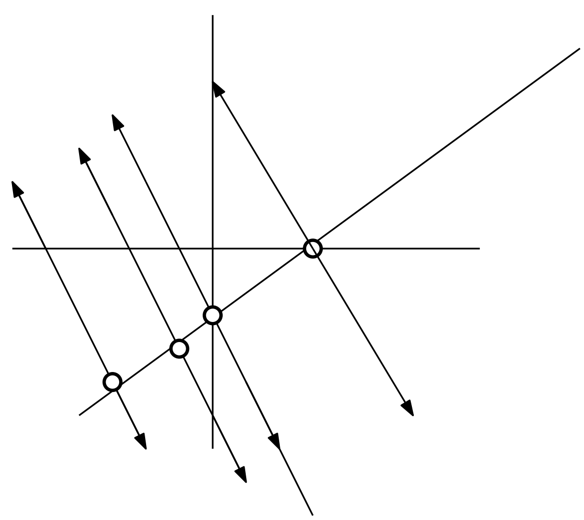
$$\lambda_1 < 0 < \lambda_2 \text{ (saddle)}$$



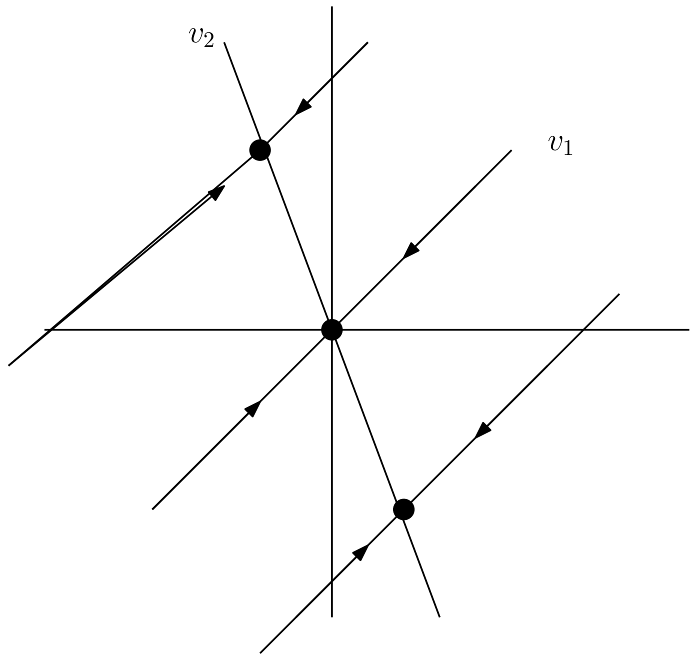
x_* is unstable

Definition 15.2 (Hyperbolic Fixed Point) — x_* is a hyperbolic fixed point if $\text{Re}(\lambda_i) \neq 0$ for $i = 1, 2$ otherwise non-hyperbolic.

$\lambda_1 = 0 < \lambda_2 : x(t) = c_1 v_1 + c_2 e^{\lambda_2 t} v_2$



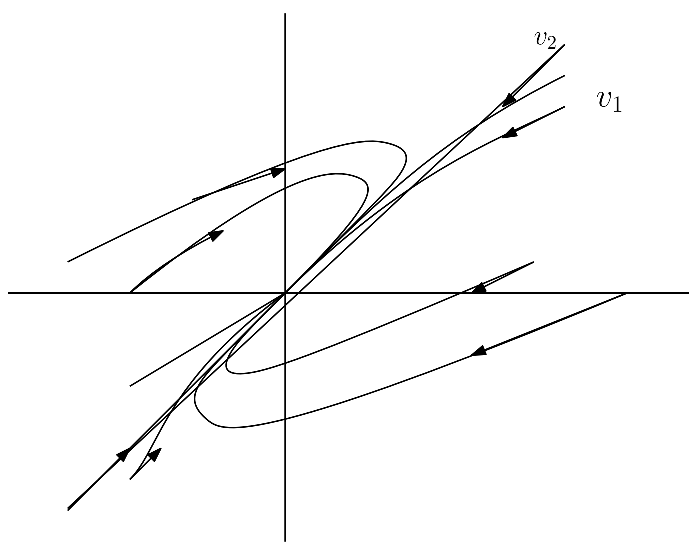
x_* is unstable and v_1 axis consists of fixed points $x_* = 0$ is a non-isolated fixed point.
$\lambda_1 < 0 = \lambda_2$: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 v_2$



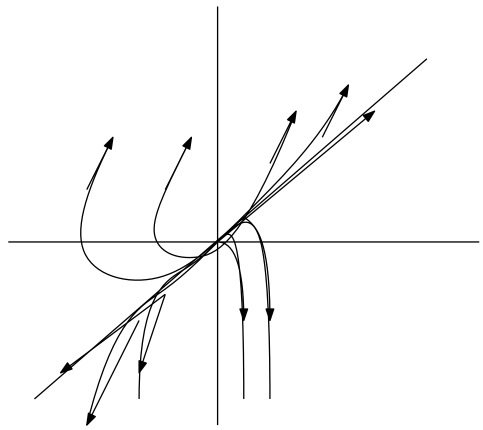
v_2 axis consists of fixed points.

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 v_2$$

$x_* = 0$ is Lypunov stable but not attracting (neutrally stable)
$\lambda_1 < \lambda_2 < 0 : \quad x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$



Trajectories approach x_* tangent to "slower" v_2 direction (note $|\lambda_1| > |\lambda_2| > 0$) – stable node.
$0 < \lambda_1 < \lambda_2$: trajectories quickly appear parallel to "faster" v_2 direction.



unstable node

Case ii) $\lambda = \lambda_1 = \lambda_2$, real valued

1. There are v_1, v_2 linearly independent eigenvectors $Av_i = \lambda v_i$ for $i = 1, 2$

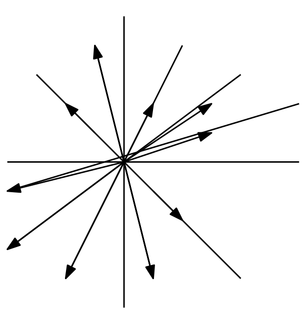
$$\implies \text{For } v \in \mathbb{R}^2 : Av = \lambda v \implies A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \lambda I$$

So, $\dot{x} = Ax$ is solved by

$$x(t) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t}$$

Phase portraits:

$\lambda > 0$



x_* is unstable

unstable star

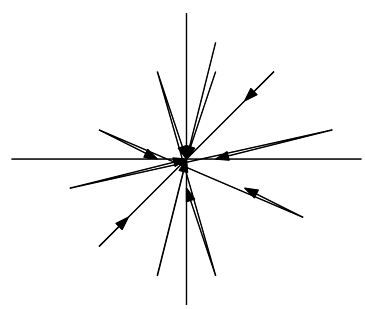
$\lambda = 0 : A = 0$

every point is a fixed point

x(t) = x(0)

($x_* = 0$
is stable non-hyperbolic,
non-isolated)

$\lambda < 0$



x_* is stable (stable star)

§16 Lec 15: Feb 10, 2021

§16.1 Lec 14 (Cont'd)

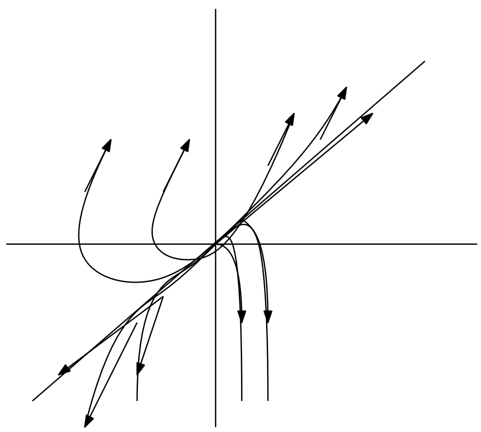
Case ii) $\lambda = \lambda_1 = \lambda_2$

2. Eigenspace $\text{Eig}_\lambda(A) = \text{span}(v)$, $v \neq 0$ A is not diagonalizable.

$$\implies x(t) = \left[(c_1 + c_2 t)v + c_2\omega\right] e^{\lambda t}$$

where λ s.t. $(A - \lambda I)\,\omega = v$. Note $\frac{x(t)}{|x(t)|} \to \frac{v}{|v|}$ as $t \to \pm\infty$ i.e. $x(t)$ tangent/parallel to v-direction as $t \to \pm\infty$.

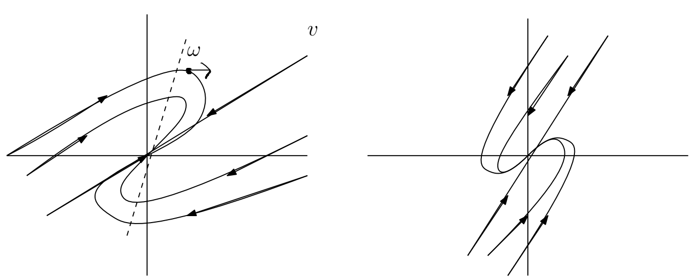
Recall: $\lambda_1 < \lambda_2 < 0$:



unstable node

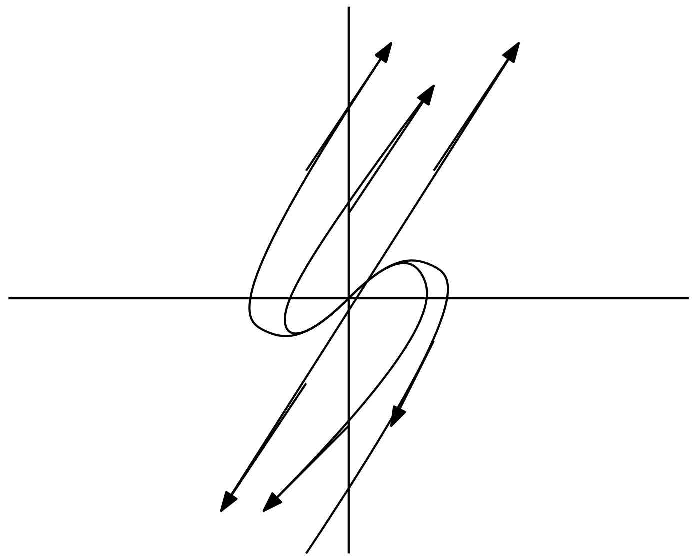
intuitively as $\lambda_1 \to \lambda_2$ and $v_1 \to v_2$.

$\lambda < 0$: stable degenerate node

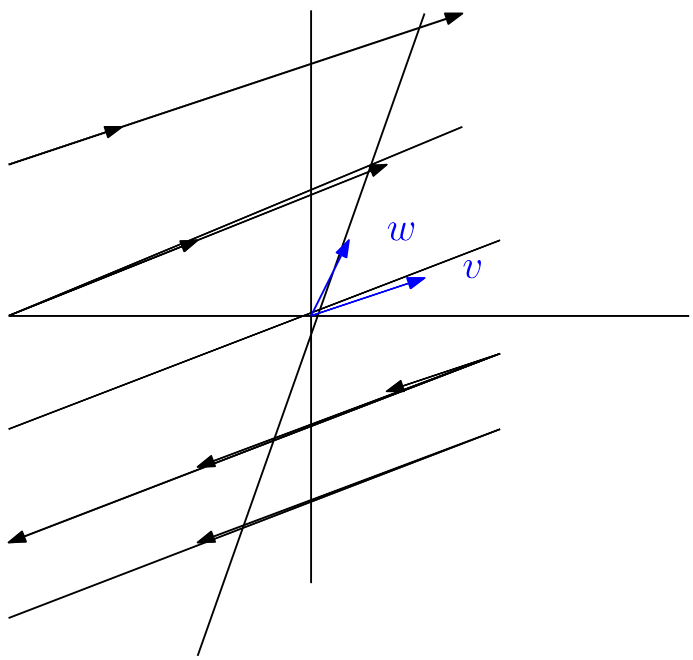


> **Remark 16.1.** Instead of solving for ω explicitly, calculate Az for some vector z to determine which way the solution "curls".

$\lambda > 0$



$\lambda = 0 : x(t) = (c_1 + c_2 t)v + c_2\omega$



Note: $x(0) = c_1 v \implies x(t) = c_1 v$ for all t i.e. the $v-$axis consists of fixed points (non-isolated fixed points, $x_* = 0$ unstable).

> **Remark 16.2.** If $\lambda = \lambda_1 = \lambda_2$, $\text{Eig}_\lambda(A) = \text{span}(v)$. Then there is ω s.t.
>
> $$\begin{aligned} &(A - \lambda I)\omega = v \\ &\implies v_1\omega \text{ lin. indep} \\ &\implies v_1\omega \text{ form a basis of } \mathbb{R}^2 \end{aligned}$$

Coordinate change:

Set

$$C = (v|w)$$
$$B = C^{-1}AC = \underbrace{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}}_{\text{Jordan normal form}}$$
$$y = C^{-1}x : \quad \dot{y} = By$$

So

$$\dot{y}_2 = \lambda y_2 \implies y_2(t) = c_2 e^{\lambda t}$$
$$\dot{y}_1 = \lambda y_1 + y_2 \implies y_1(t) = (c_1 + c_2 t)\, e^{\lambda t}$$
$$\implies x = Cy = [(c_1 + c_2 t) + c_2 \omega]\, e^{\lambda t}$$

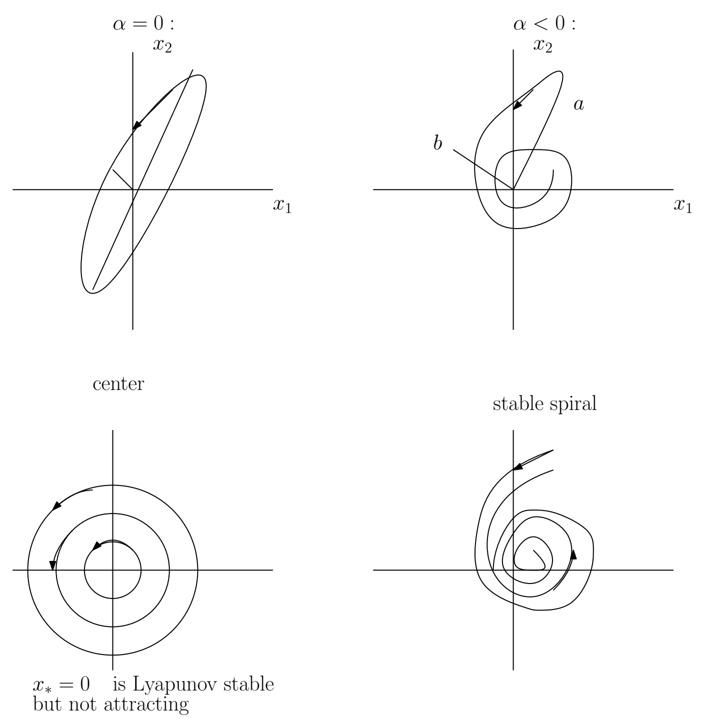
Case iii)

$$\begin{cases} \lambda_1 = \lambda = \alpha + i\beta \\ \lambda_2 = \overline{\lambda} = \alpha - i\beta \end{cases} \quad (\beta > 0)$$

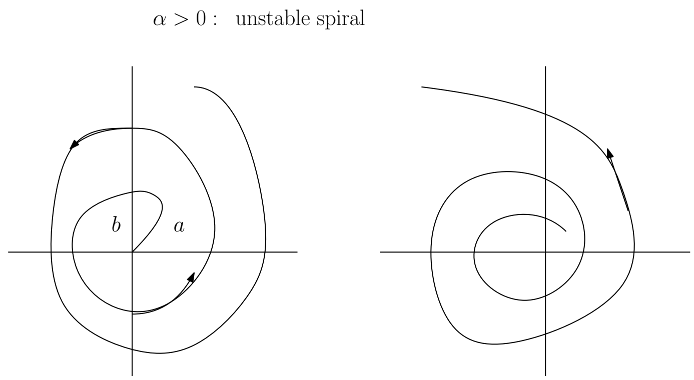
$\implies$ A is diagonalizable over $\mathbb{C}$, in particular there is $v \in \mathbb{C}^2, v \neq 0$, s.t. $Av = \lambda v$.
Let $v = a - ib$, $a, b \in \mathbb{R}^2$. Assume $a \perp b$. General solution:

$$x(t) = (a|b) \underbrace{\begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix}}_{\text{rotation} R(\beta t) \text{ period } \frac{2\pi}{\beta}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \underbrace{e^{\lambda t}}_{\text{stretching factor}}$$

In particular, $x(t) = [a \cos(\beta t) + b \sin(\beta t)]\, e^{\lambda t}$ is the solution with $x(0) = a$ and $x\left(\frac{\pi}{2\beta} = be^{\alpha t}\right)$ $\left[\text{set } \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$.
Phase portraits:



$\alpha > 0$: unstable spiral



Remark 16.3. i) If $\alpha = 0$, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies x(t) = \cos(\beta t) \cdot a + \sin(\beta t) \cdot b$. Then since $a \perp b$:

$$\begin{aligned} \frac{1}{|a|^2}\langle x(t), \frac{a}{|a|}\rangle^2 + \frac{1}{|b|^2}\langle x(t), \frac{b}{|b|}\rangle^2 &= \frac{1}{|a|^2}\left(\frac{a \cdot a}{|a|} \cdot \cos(\beta t)\right)^2 + \frac{1}{|b|^2}\left(\frac{b \cdot b}{|b|} \cdot \sin(\beta t)\right)^2 \\ &= \left(\cos(\beta t)\right)^2 + \left(\sin(\beta t)\right)^2 = 1 \end{aligned}$$

$\implies x(t)$ is on an ellipse with axes $\frac{a}{|a|}, \frac{b}{|b|}$.

ii) $\lambda = \alpha + i\beta$, $v = a - ib$. If a is not orthogonal to b, then replace v by

$$w = (\gamma + i\delta)\, v$$

with $\gamma = -2ab$

$$\delta = \left(|a|^2 - |b|^2\right) \pm \sqrt{\left(|a|^2 - |b|^2\right)^2 + 4(ab)^2}$$

Then $A\omega = \lambda\omega$ and Re $\omega \perp$ Im ω.
Assume $Av = \lambda v$, $v = a - ib$, $a \perp b$.

$$\begin{aligned} Aa - iAb = A(a - ib) = Av = \lambda v &= (\alpha + i\beta)(a - ib) \\ &= (\alpha a + \beta b) + i(\beta a - \alpha b) \end{aligned}$$

So

$$\begin{aligned} Aa &= \alpha a + \beta b \\ Ab &= -\beta a + \alpha b \end{aligned}$$

Set $C = (a|b)$. Then

$$\begin{aligned} AC &= C\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \\ B = C^{-1}AC &= \underbrace{\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}}_{\text{normal form}} \end{aligned}$$

Set $y = C^{-1}x$, $\dot{y} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} y$ with solution:

$$\begin{aligned} y(t) &= \begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\alpha t} \\ &\implies x(t) = C \cdot y(t) \end{aligned}$$

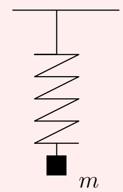
§17 Lec 16: Feb 12, 2021

§17.1 Linear Systems – Harmonic Oscillator

Example 17.1 (Harmoinc oscillator)

$$m\ddot{x} + kx = 0$$

where k : spring constant.



$\implies \ddot{x} + \omega^2 x = 0$ where $\omega^2 = \frac{k}{m}$. Set

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega^2 x_1 \end{cases}$$

i.e.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{pmatrix} \\ &= \lambda^2 + \omega^2 \end{aligned}$$

$\implies \lambda_{1,2} = \pm i\omega \implies$ center

Phase portrait:

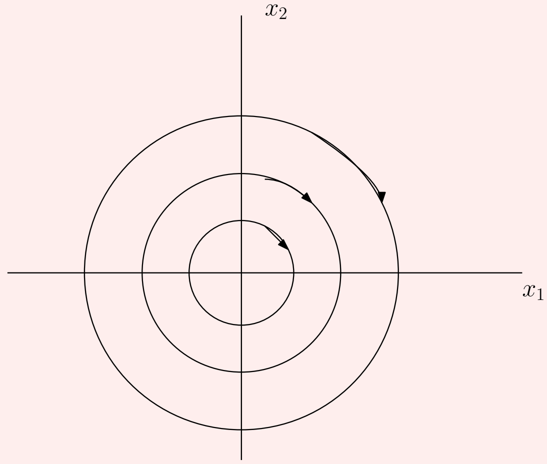
i) in practice: compute $\dot{x} = Ax$ for a specific vector to determine which way solutions turn

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} x$$

e.g. $\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\omega^2 \end{pmatrix}$.

Example 17.2 (Cont'd of example 17.1)

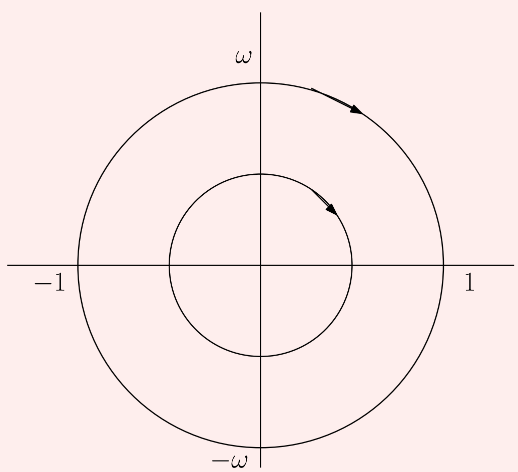
Then,



ii) more precise quantitative analysis, eigenvectors solutions of $(A - \lambda I)v = 0$

$$A - i\omega I = \begin{pmatrix} -i\omega & 1 \\ -\omega^2 & -i\omega \end{pmatrix} \to \begin{pmatrix} -i\omega & 1 \\ 0 & 0 \end{pmatrix}$$

eigenvector $v = \begin{pmatrix} -i \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ \omega \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$



Recall:

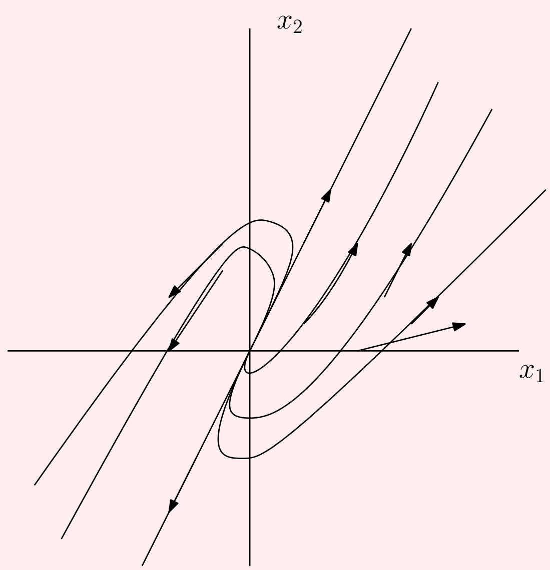
$$x(t) = C \cdot \left[\begin{pmatrix} 0 \\ \omega \end{pmatrix} \cos(\omega t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(\omega t) \right]$$

Example 17.3

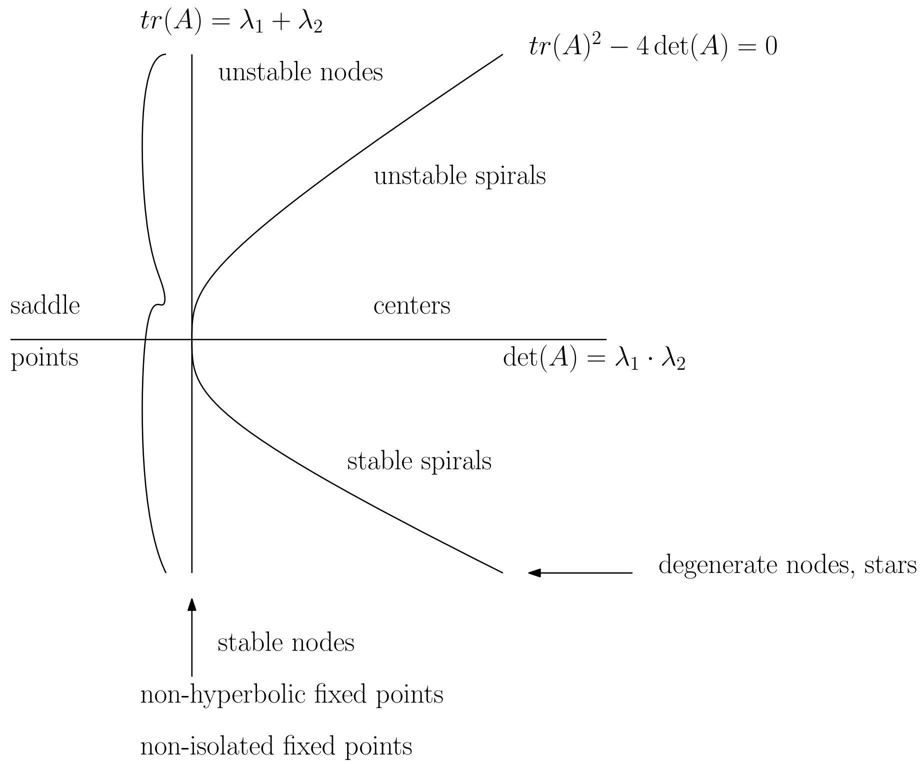
$\dot{x} = Ax \quad A = \begin{pmatrix} 8 & -1 \\ 4 & 4 \end{pmatrix}$. Eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} 8-\lambda & -1 \\ 4 & 4-\lambda \end{pmatrix} \\ &= (8-\lambda)(4-\lambda) - 4(-1) \\ &= \lambda^2 - 12\lambda + 36 = 0 \\ \implies \lambda &= 6 \end{aligned}$$

$A \neq \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$, we have an unstable degenerate node. Eigenvector: $A - \lambda I = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$, so $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector. Note $A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$.
Phase portrait:

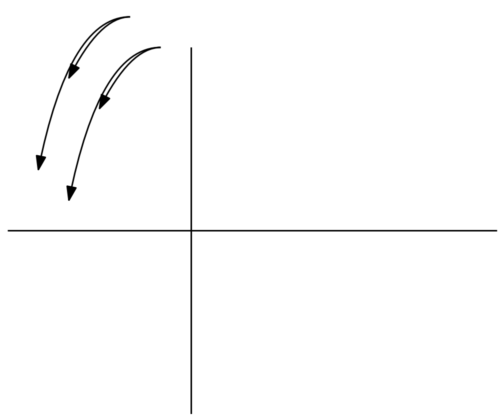


Summary:
Recall $\lambda_{1,2} = \frac{1}{2}\left(\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}\right)$



§17.2 Nonlinear Systems – Existence and Uniqueness

$$\dot{x} = f(x) \qquad \text{i.e.} \qquad \begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

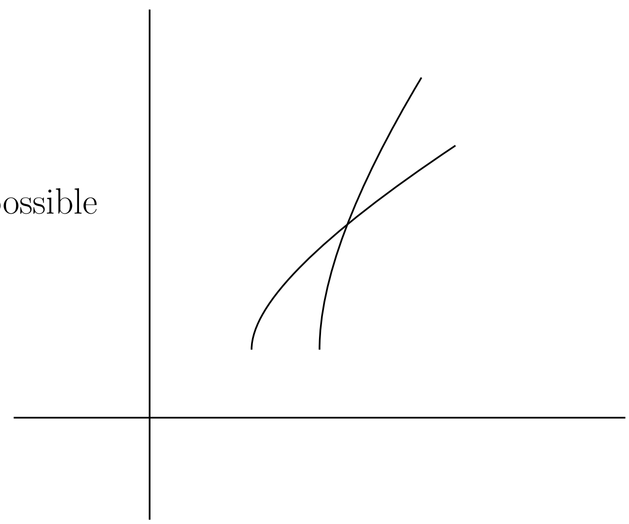


Theorem 17.4 (Existence & Uniqueness of Systems)
Let $D \subseteq \mathbb{R}^n$ be open, $f : D \to \mathbb{R}^n$ s.t. $\frac{\partial f_i}{\partial x_j}$ exist and are continuous, that is $f \in C^1(D)$. Then for every $x_0 \in D$ there $\tau > 0$ s.t. $\dot{x} = f(x)$, $x(t_0) = x_0$ has a unique solution $\phi : (t_0 - \tau, t_0 + \tau) \to \mathbb{R}^n$ i.e. $\dot{\phi}(t) = f(\phi(t))$, $\phi(t_0) = x_0$.

Remark 17.5. $f \in C^2(D)$ if $\frac{\partial^2 f_i}{\partial x_k \partial x_l}$ exist and continuous.

Consequence: Different trajectories in the phase portrait cannot intersect

impossible



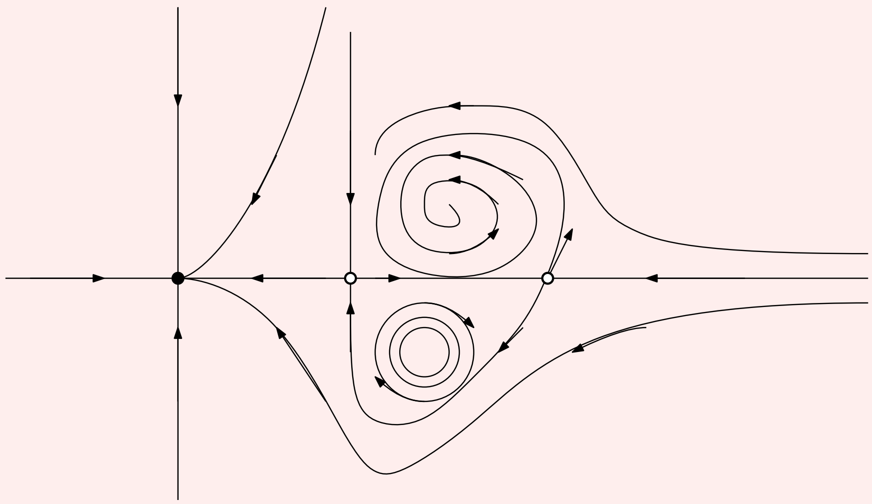
§18 Lec 17: Feb 15, 2021

§18.1 Nonlinear Systems – Nullclines

$\dot{x} = f(x)$ and $\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$

Example 18.1

Consider



Definition 18.2 (Isocline and Nullcline) — Let $c \in \mathbb{R}$. The curves $\{(x_1, x_2) | f_i(x_1, x_2) = c\}$ $i = 1, 2$ are called <u>isoclines</u>. Specifically, if $c = 0$

- $f_1(x_1, x_2) = 0$ is called <u>vertical nullcline</u>.
- $f_2(x_1, x_2) = 0$ is called <u>horizontal nullcline</u>.

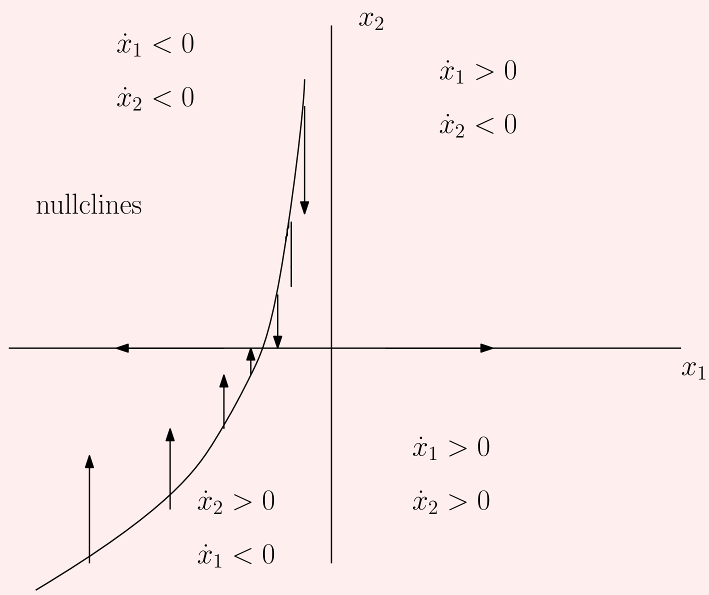
Example 18.3

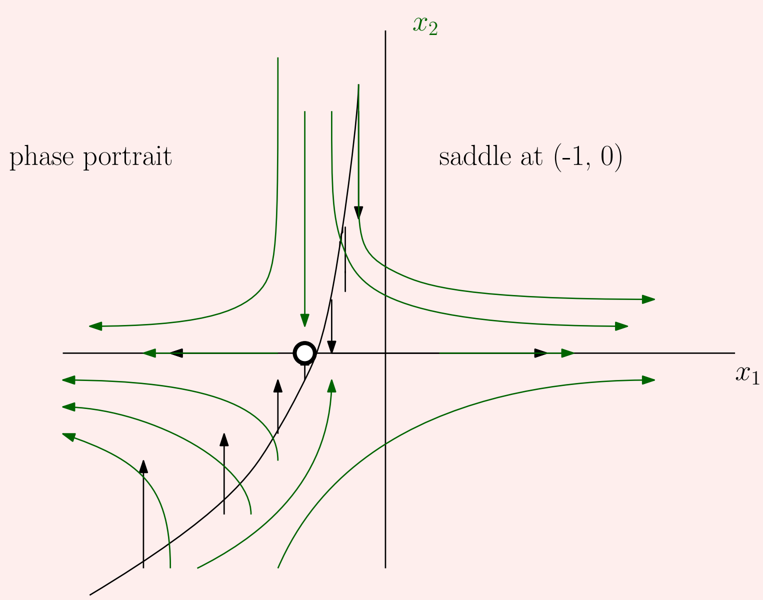
Consider:

$$\dot{x}_1 = x_1 + e^{-x_2}$$
$$\dot{x}_2 = -x_2$$

Fixed points: $\dot{x} = f(x) = 0 \iff (x_1, x_2) = (-1, 0)$.
Nullclines:

$$\dot{x}_1 = 0 : \ x_1 = -e^{-x_2} \text{ (vertical nullcline)}$$
$$\dot{x}_2 = 0 : \ x_2 = 0 \text{ (horizontal nullcline)}$$





> **Remark 18.4.** A nullclines typically are not/do not consist of trajectories. Vertical(horizontal) nullclines consist of trajectories if it is exactly vertical(horizontal.

§18.2 Principle of Linear Stability

$\dot{x} = f(x)$, $f \in C^1(D)$, $f(x_*) = 0$. We want to approximate the nonlinear DE near the fixed point.

$$\frac{d}{dt}(x - x_*) = \dot{x} = f(x) = f(x - x_* + x_*)$$
$$\overset{\text{Taylor}}{=} \underbrace{f(x_*)}_{=0} + Df(x_*)(x - x_*) + \mathcal{O}(|x - x_*|^2)$$

i.e. $y = x - x_*$ approximately solves the linear ODE

$$\dot{y} = Df(x_*)y$$

where

$$Df(x_*) \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

Let λ_1, λ_2 denote the eigenvalues of $Df(x_*)$.

Theorem 18.5 (Linear Stability)

Similar to the linear systems,

i) If $\text{Re}(\lambda_1) < 0$, $\text{Re}(\lambda_2) < 0$ then x_* is asymptotically stable, i.e. x_* is Lyapunov stable and attracting.

ii) If $\text{Re}(\lambda_i) > 0$ for $i = 1$ or $i = 2$ then x_* is unstable.

§19 Lec 18: Feb 19, 2021

§19.1 The Stable/Unstable Manifold Theorem

$f \in C^1$, $\dot{x} = f(x)$, $f(x_*) = 0$ i.e. x_* fixed point, λ_1, λ_2 eigenvalues of $Df(x_*)$.
Let x_* be a hyperbolic fixed point and $x(t, x_0)$ be the solution of

$$\dot{x} = f(x), \quad x(0) = x_0$$

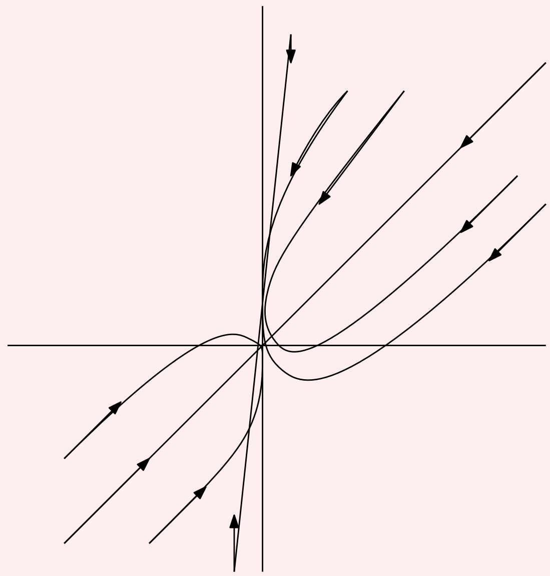
Set

$$\mathcal{M}_s \coloneqq \left\{ x_0 \in D | x(t, x_0) \text{ defined for all } t \geq 0 \text{ and } \lim_{t \to \infty} x(t, x_0) = x_* \right\} \text{ (stable manifold)}$$

$$\mathcal{M}_u \coloneqq \left\{ x_0 \in D | x(t, x_0) \text{ defined for all } t \leq 0 \text{ and } \lim_{t \to -\infty} x(t, x_0) = x_* \right\} \text{ (unstable manifold)}$$

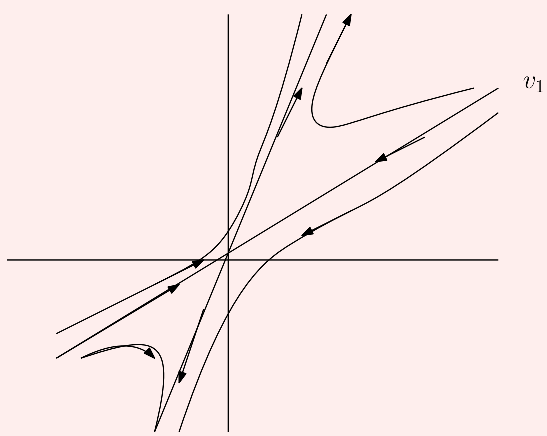
Example 19.1

Linear stable node



$$\mathcal{M}_s = \mathbb{R}^2$$
$$\mathcal{M}_u = \{x_*\} = \{0\}$$

Linear saddle



$$\begin{aligned}\mathcal{M}_s &= \operatorname{span}(v_1)\\ &= \text{ line through } v_1(\lambda_1 < 0) \;\; \text{(trajectories that approach } x_*\text{)}\\ \mathcal{M}_u &= \operatorname{span}(v_2)\\ &= \text{line through } v_2(\lambda_2 > 0) \;\; \text{(trajectories that emanate from } x_*\text{)}\end{aligned}$$

Theorem 19.2 (Stable/Unstable Manifold)

Let $f \in C^1, x_*$ is a hyperbolic fixed point.

i) If $\text{Re}(\lambda_i) < 0$ for $i = 1, 2$, then $\mathcal{M}_s$ contains an open neighborhood of x_* and $\mathcal{M}_u = \{x_*\}$.

ii) If $\text{Re}(\lambda_i) > 0$ for $i = 1, 2$, then $\mathcal{M}_s = \{x_*\}$ and $\mathcal{M}_u$ contains an open neighborhood of x_*.

iii) If $\text{Re}(\lambda_1) < 0 < \text{Re}(\lambda_2)$, then $\mathcal{M}_s, \mathcal{M}_u$ are C^1-curves through x_*. $\mathcal{M}_s$ tangent to v_1 at x_*, $Df(x_*)v_1 = \lambda_1 v_1$, and $\mathcal{M}_u$ tangent to v_2 at x_*, $Df(x_*)v_2 = \lambda_2 v_2$

Theorem 19.3

Suppose x_* is a hyperbolic fixed points of $\dot{x} = f(x)$. Then the phase portrait of $\dot{y} = Df(x_*)y$ near $y_* = 0$ gives a qualitatively accurate picture of the phase portrait of $\dot{x} = f(x)$ near x_* if

a) $f \in C^2$ i.e. $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exists and are continuous.

or

b) $f \in C^1$ and $\lambda_1 \neq \lambda_2$.

Example 19.4

Consider:

$$\begin{aligned}\dot{x}_1 &= x_1 + e^{-x_2}\\ \dot{x}_2 &= -x_2\end{aligned}$$

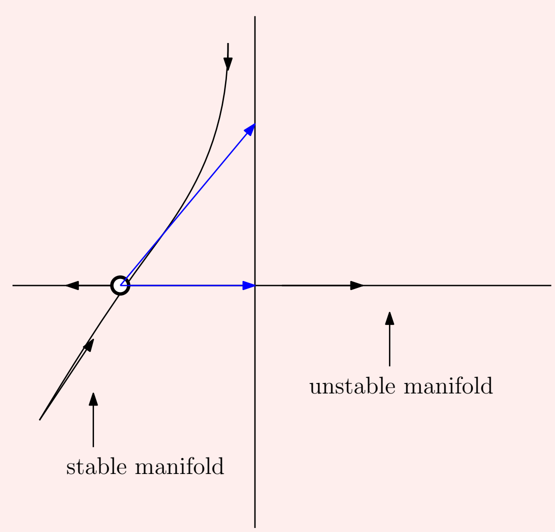
only fixed point: $(x_1, x_2) = (-1, 0)$ and note that $f(x_1, x_2) = \begin{pmatrix} x_1 + e^{-x_2} \\ -x_2 \end{pmatrix}$.

$$\begin{aligned}Df &= \begin{pmatrix} 1 & -e^{-x_2} \\ 0 & -1 \end{pmatrix}\\ Df(x_*) &= \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

Eigenvalues: $\lambda_1 = -1, \lambda_2 = 1 \implies (-1, 0)$ is unstable (by Theorem 18.5)
Eigenvectors:

$$\begin{aligned}A - (-1)I &= \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \implies v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\\ A - (1)I &= \begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \implies v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\end{aligned}$$

where v_1 is the tangent direction of stable manifold at $x_* = (-1, 0)$ and v_2 is the tangent direction of unstable manifold at $x_* = (-1, 0)$.

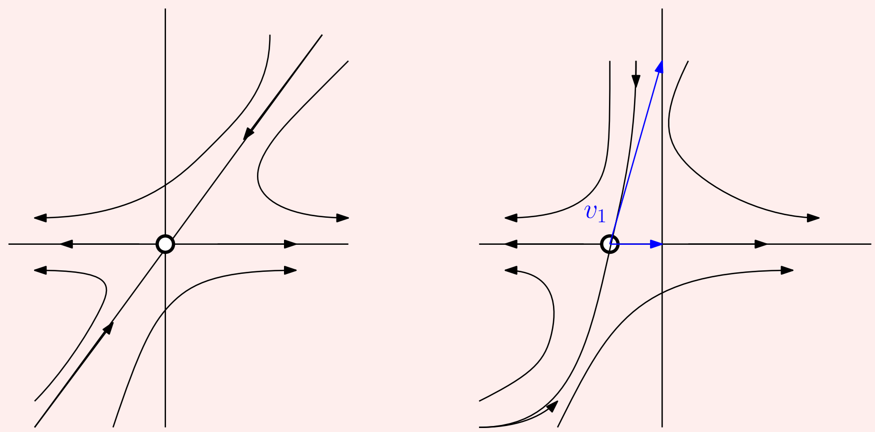


Example 19.5 (Cont'd from above)

<u>*Note*</u>: $f(x_1, x_2) = \begin{pmatrix} x_1 + e^{-x_2} \\ -x_2 \end{pmatrix}$ is infinitely often differentiable, in particular, $f \in C^2$ (or $f \in C^1$), thus the phase portrait of

$$\dot{y} = Df(x_*) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \text{ near } y_* = 0$$

is an accurate picture of the phase portrait of $\dot{x} = f(x)$ near x_*.



$\dot{x} = f(x)$ near $x_* = (-1, 0)$

where the left figure denote the approximation $\dot{y}$.

Theorem 19.6 (Hartman – Grobman)
Let $f \in C^1$, x_* a hyperbolic fixed point of $\dot{x} = f(x)$. Then the phase portrait of $\dot{x} = f(x)$ near x_* and $\dot{y} = Df(x_*)y$ near $y_* = 0$ are topologically equivalent i.e. the same up to continuous deformation (homeomorphisms).

Morally: hyperbolic fixed points are <u>structurally stable</u>.

§19.2 Lotka Volterra Model

Example 19.7 (Lotka Volterra model for competition of two species for limited resources)
Recall: logistic model

$$\dot{x} = rx\left(1 - \frac{x}{k}\right)$$

Consider:

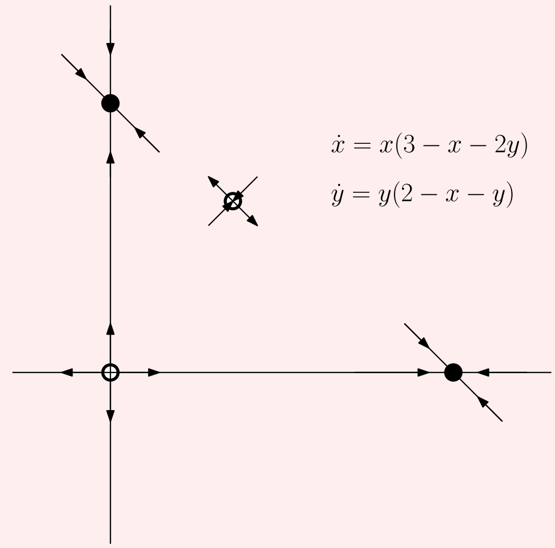
$$\begin{aligned}\dot{x} &= x(3 - x - 2y)\\ \dot{y} &= y(2 - x - y)\end{aligned}$$

fixed points (x_*, y_*)	eigenvalues/eigendirections of $Df(x_*, y_*)$			
	λ_1	v_1	λ_2	v_2
$(0,0)$	3	$\begin{pmatrix}1\\0\end{pmatrix}$	2	$\begin{pmatrix}0\\1\end{pmatrix}$
$(0,2)$	-1	$\begin{pmatrix}1\\-2\end{pmatrix}$	-2	$\begin{pmatrix}0\\1\end{pmatrix}$
$(3,0)$	-3	$\begin{pmatrix}1\\0\end{pmatrix}$	-1	$\begin{pmatrix}3\\-1\end{pmatrix}$
$(1,1)$	$-1+\sqrt{2}$	$\begin{pmatrix}\sqrt{2}\\-1\end{pmatrix}$	$-1-\sqrt{2}$	$\begin{pmatrix}\sqrt{2}\\1\end{pmatrix}$

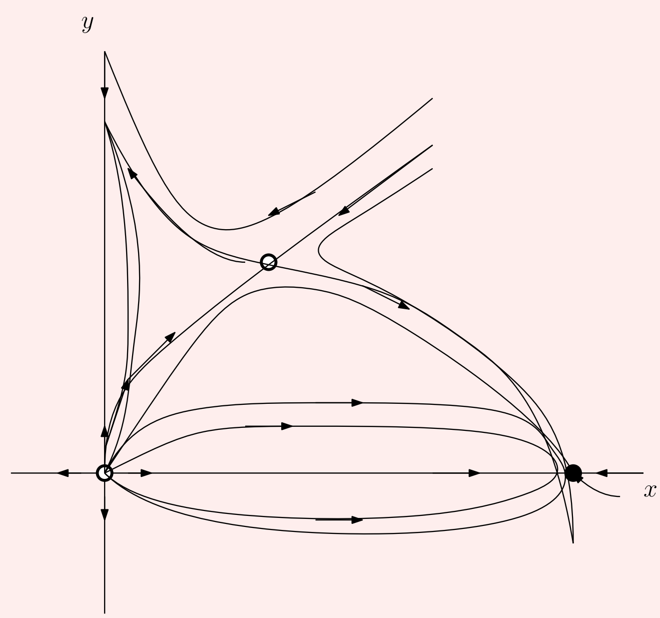
where all the fixed points above are hyperbolic fixed points.

Example 19.8 (Cont'd from above)

Phase portrait: tangent directions of stable/unstable manifolds



Phase portrait:



<u>Conclusion</u>: Only one species survives.

§20 Dis 1: Jan 7, 2021

§20.1 Fixed points and Stability

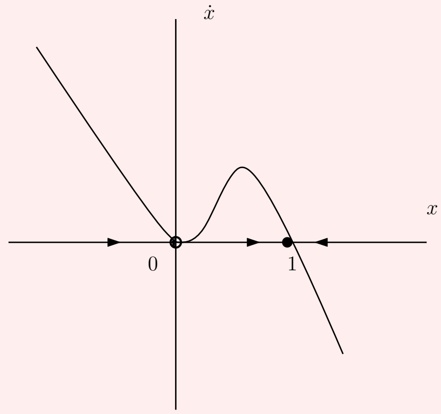
$\dot{x} = f(x)$

Example 20.1

$\dot{x} = -x^3 + x^2$

a) Sketch the vector field, classify the fixed points.

"vector fields" = x-axis with arrows



so:

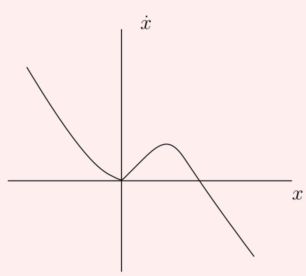
- $\dot{x} > 0 \implies x(t)$ increasing
- $\dot{x} < 0 \implies x(t)$ decreasing

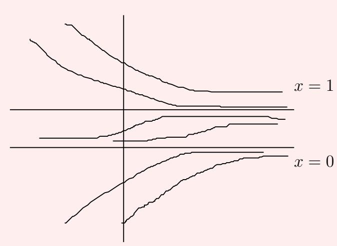
"Fixed point" $\iff$ x_* s.t. $f(x_*) = 0$ $\iff$ x_* s.t. the constant function $x(t) = x_*$ is a solution.

We have 2 fixed points:

- $x_* = 0$ is semi-stable.
- $x_* = 1$ is stable.

b) Sketch various solutions of $x(t)$.



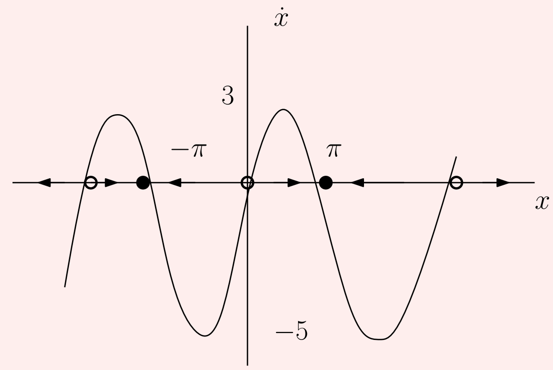


- $\dot{x} = 0$ for $x = 0, 1 \implies x(t) = 0, 1$ are solutions.
- $\dot{x} > 0$ for $x < 0 \implies x(t)$ increasing.
- $\dot{x} > 0$ for $0 < x < 1 \implies x(t)$ increasing.
- $\dot{x} < 0$ for $x > 1 \implies x(t)$ decreasing.

Example 20.2

$\dot{x} = -1 + 4\sin x$

a) Sketch vector field, classify fixed points.

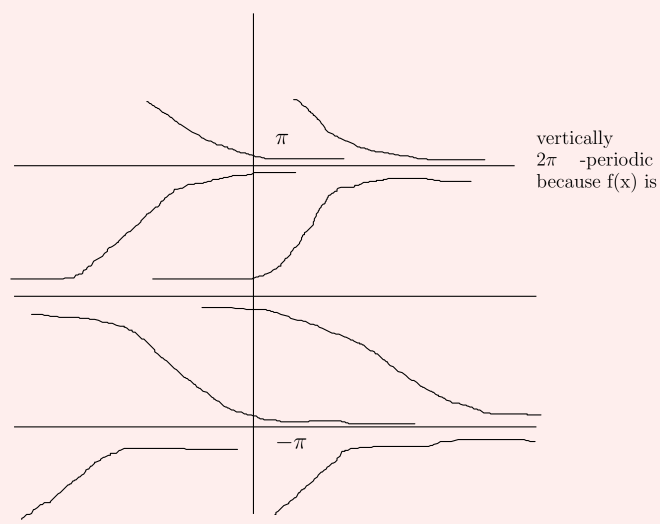


Fixed points:

$$\sin(x_*) = \frac{1}{4}$$

- $x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$ for $n = 0, \pm 1, \ldots$ are unstable.
- $x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$ for $n = 0, \pm 1, \ldots$ are stable.

b) Sketch various solutions $x(t)$.



§20.2 First Order Autonomous System

$\dot{\vec{x}} = \vec{f}(\vec{x})$ – first order and autonomous.

Example 20.3

A unit mass with displacement $x(t)$ attached to a spring with spring constant 6 obeys:

$$\ddot{x} = -6x - b(t)\dot{x}$$

where $b(t) \geq 0$ is the friction coefficient.

a) Show that this can expressed as a first order autonomous system

$$\begin{aligned} x_1 &= x, \quad x_2 = \dot{x} \\ \dot{x}_2 &= \ddot{x} = -6x_1 - b(t)x_2 \\ \vec{x} &:= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \dot{\vec{x}} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -6x_1 - b(x_3)x_2 \end{pmatrix} \end{aligned}$$

where $x_3 = t \implies \dot{x}_3 = 1$.

b) In the case $b(t) = 5$, find the explicit solution for $x(0) = x_0, \dot{x}(0) = v_0$.

$$\ddot{x} = -6x - 5\dot{x} \implies \ddot{x} + 5\dot{x} + 6x = 0$$

Try $x(t) = e^{kt}$:

$$\begin{aligned} 0 &= \ddot{x} + 5\dot{x} + 6x = k^2e^{kt} + 5ke^{kt} + 6e^{kt} \\ &= e^{kt}(k^2 + 5k + 6) \implies k = -3, -2 \end{aligned}$$

Now, $x(t) = c_1e^{-3t} + c_2e^{-2t}, c_1, c_2 \in \mathbb{R}$. Using the initial conditions, we obtain

$$x(t) = (-2x_0 - v_0)e^{-3t} + (3x_0 + v_0)e^{-2t}$$

§21 Dis 2: Jan 14, 2021

§21.1 Linearization and Potentials

Example 21.1

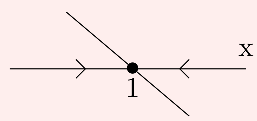
$\dot{x} = -x^3 + x^2$

a) Use linear stability analysis to classify the fixed points. If it fails, use a graphical argument.

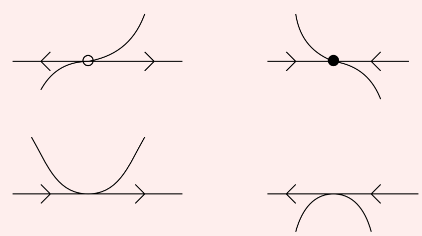
Idea: For x near a fixed point x_*, $\dot{x} = f(x) \approx f(x_*)(=0) + f'(x_*)(x - x_*) = \dots$

$$f(x) = -x^3 + x^2, \quad f'(x) = -3x^2 + 2x$$
$$0 = f(x_*) = -x_*^2(x_* - 1) \implies x_* = 0, 1$$

- $x_* = 1 : f'(1) = -1 < 0 \implies$ stable



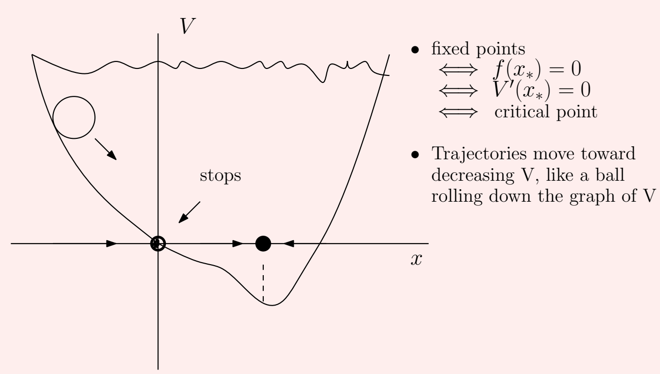
- $x_* = 0 : f'(0) = 0 :$ inconclusive



b) Find and plot a potential function.

"Potential function" $\iff V(x)$ s.t. $\dot{x} = -\frac{dV}{dx}$

$$\dot{x} = -x^3 + x^2 = -V'(x) \implies V(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 + C \text{ (choose 0)}$$



Example 21.2

$\dot{x} = 4\sin x - 1.$

a) Use linear stability analysis to classify the fixed points.

$$f(x) = 4\sin x - 1, \quad f'(x) = 4\cos x$$

Last time: Fixed points are

- $x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \dots$

$$f'(x_*) = 4\cos\left(\sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) > 0$$

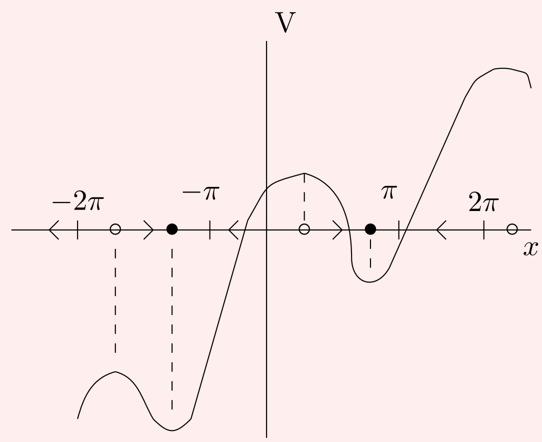
$\implies$ unstable

- $x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \dots$

$$f'(x_*) = 4\cos\left(\pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) < 0$$

$\implies$ Stable.

b) Plot potential $-1 + 4\sin x = -V'(x) \implies V(x) = x + 4\cos x$



§21.2 Existence of Solutions

Example 21.3 a) Let $a > 0$ be a constant. Show that the solution of

$$\begin{cases} \dot{x} = ax^2 \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

$$\begin{aligned} \frac{dx}{dt} = ax^2 \implies \int \frac{dx}{x^2} &= \int a\,dt \\ \implies -\frac{1}{x} &= at + c \\ \implies x(t) &= \frac{1}{c - at} \forall c \in \mathbb{R} \\ x(0) > 0 \implies c > 0 \implies \lim_{t \to T} x(t) &= +\infty \end{aligned}$$

for some $T > 0$. In fact, $c = \frac{1}{x_0}$ and so $T = \frac{c}{a} = \frac{1}{ax_0}$.

b) Let $0 < \epsilon < 1$ be a constant. Show that the solution of

$$\begin{cases} \dot{x} = x^2 \left(1 + \epsilon \sin x\right) \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

Idea: $\dot{x} \geq ax^2$ for some $a > 0$, so our solution grows faster than a function which blows up, and must blow up too.

$$\begin{aligned} |\sin x| \leq 1 &\implies 1 + \epsilon \sin x \geq 1 - \epsilon \\ &\implies \dot{x} = x^2 \left(1 + \epsilon \sin x\right) \geq \underbrace{1 - \epsilon}_{>0} x^2 \end{aligned}$$

Let $x(t)$ be the solution to

$$\begin{cases} \dot{x} = x^2 \left(1 + \epsilon \sin x\right) \\ x(0) = x_0 \end{cases}$$

Let $y(t)$ be the solution to

$$\begin{cases} \dot{x} = (1 - \epsilon) x^2 \\ x(0) = x_0 \end{cases}$$

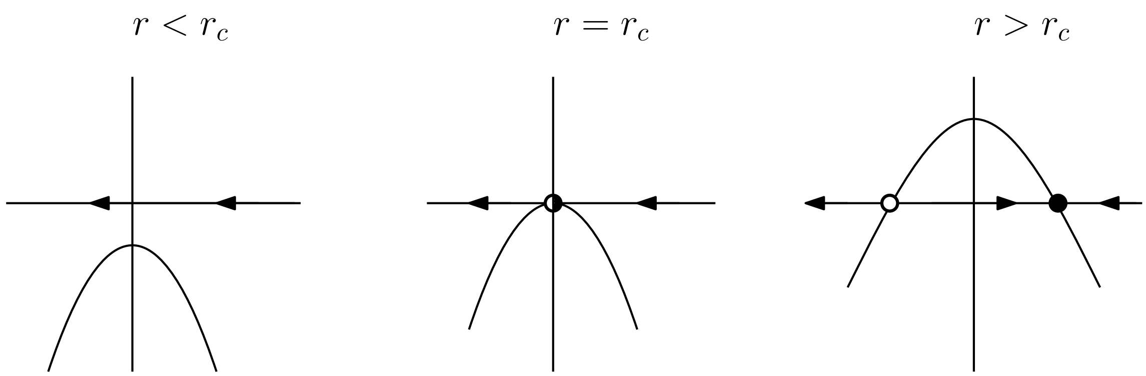
By part a), $y(t)$ blows up at some time $T > 0$. Since $x(0) = y(0)$ and $\dot{x} \geq \dot{y}$, then $x(t) \geq y(t)$ for all $t \geq 0$ (ODE Comparison Lec 4). Therefore, $x(t)$ must blow up in finite time. In fact, blow up time must be $\leq T = \frac{1}{(1-\epsilon)x_0}$.

§22 Dis 3: Jan 21, 2021

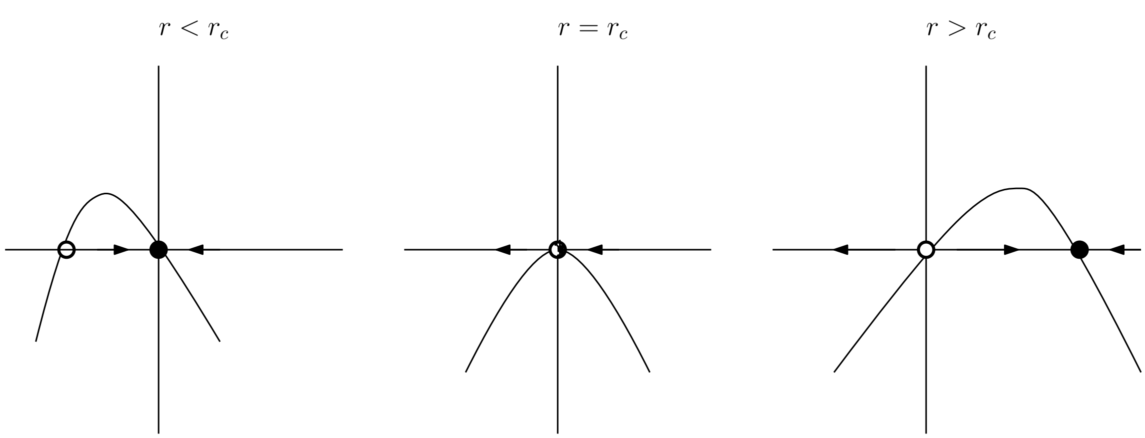
§22.1 Bifurcations

$\dot{x} = f(x, r)$, r parameter
"Bifurcation" $\iff$ change in the number or stability of fixed points.
There are two types:

- Saddle-node: $0 \to 1 \to 2$ fixed points



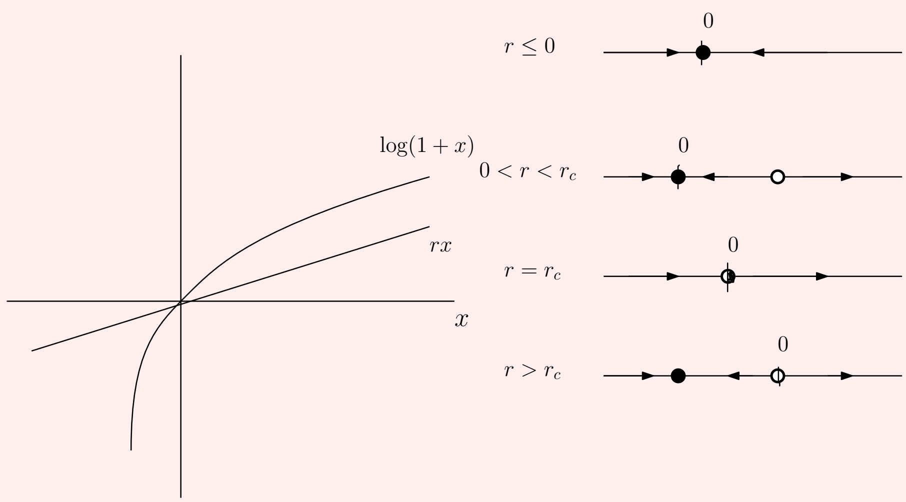
- Transcritical: $2 \to 1 \to 2$ fixed points



Example 22.1

$\dot{x} = rx - \log(1+x)$

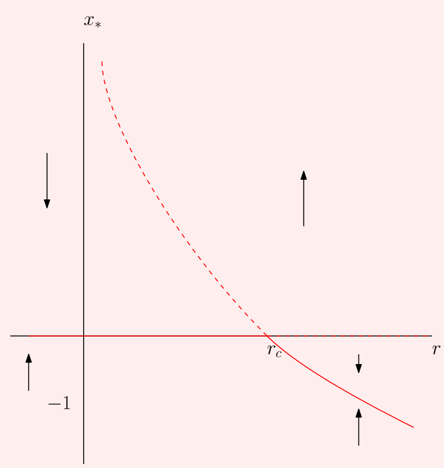
(a) Sketch all qualitatively different vector fields, sketch bifurcation diagram, find and classify bifurcations.



Transcritical bifurcation at $r = r_c$

"Bifurcation diagram" $\iff$ Plot of the fixed points x_* as a function of r.

$$0 = rx_* - \log(1+x_*)$$
$$r = \frac{1}{x_*}\log(1+x_*) \quad \text{or} \quad x_* = 0$$



- Vertical slices of constant r are vector fields
- Whole regions have same arrow direction

Bifurcation point: $(r_c, x_c) = (1, 0)$

Example 22.2 (Cont'd of example 22.1)

From the above example,

(b) Show that there is a transcritical bifurcation at $(r_c, x_c) = (1, 0)$ using normal forms. Taylor expand about $r = 1, x = 0$

$$\begin{aligned}\dot{x} &= rx - \log(1+x)\\ &= (r-1)x + x - (x - \frac{1}{2}x^2 + \mathcal{O}(x^3))\\ &= (r-1)x + \frac{1}{2}x^2 + \mathcal{O}(x^3)\end{aligned}$$

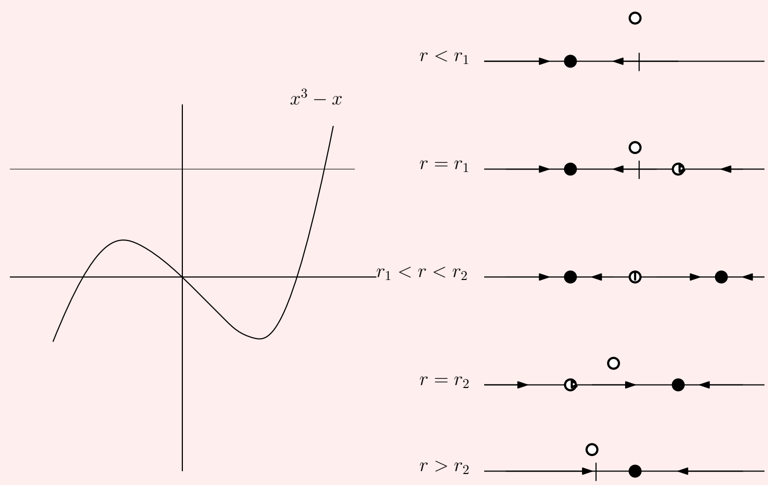
$\dot{x} = (r-1)x + \frac{1}{2}x^2 + \mathcal{O}(\epsilon^3)$ for $|x| < \epsilon, |r-1| < \epsilon$

This is normal form for transcritical bifurcation.

Example 22.3

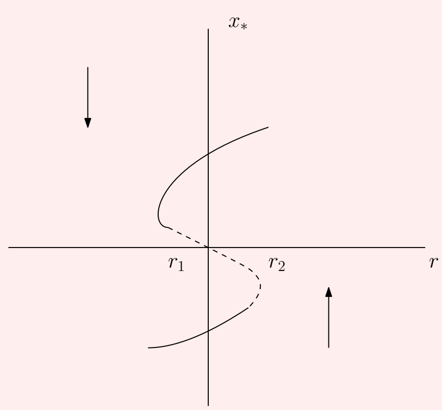
$\dot{x} = r + x - x^3$

(a) Sketch all vector field, sketch bifurcation diagram, find and classify bifurcations.



2 saddle node bifurcations at $r = r_1, r_2$

$$0 = r + x_* - x_*^3 \implies r = x_*^3 - x_*$$



Bifurcation point (x_c, r_c) satisfies:

- Fixed point: $0 = f(x_c, r_c) = r_c + x_c - x_c^3$
- $0 = \frac{\partial f}{\partial x}(x_c, r_c) = 1 - 3x_c^2$

$$0 = 1 - 3x_c^2 \implies x_c = \pm\frac{1}{\sqrt{3}}$$

$$0 = r_c + x_c - x_c^3 \implies r_1 = -\frac{2}{3\sqrt{3}}, r_2 = \pm\frac{2}{3\sqrt{3}}$$

$$(r_c, x_c) = \left(\frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Example 22.4 (Cont'd of example 22.3) (b) Show that there is a saddle-node bifurcation at $(r_c, x_c) = \left(\frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ using normal forms.

Taylor expand about $r = \frac{2}{3\sqrt{3}}, x = -\frac{1}{\sqrt{3}}$.

$$\dot{x} = r + x - x^3, \quad r = \frac{2}{3\sqrt{3}} + (r - \frac{2}{3\sqrt{3}})$$

$$x - x^3 = -\frac{2}{3\sqrt{3}} + 0(x + \frac{1}{\sqrt{3}}) + \frac{1}{2} \cdot \frac{6}{\sqrt{3}}(x + \frac{1}{\sqrt{3}})^2 + \mathcal{O}\left(\left(x + \frac{1}{\sqrt{3}}\right)^3\right)$$

Plug these in, get

$$\dot{x} = \left(r - \frac{2}{3\sqrt{3}}\right) + \sqrt{3}\left(x + \frac{1}{\sqrt{3}}\right)^2 + \mathcal{O}(\epsilon^3)$$

for $\left|r - \frac{2}{3\sqrt{3}}\right| < \epsilon^2, \left|x + \frac{1}{\sqrt{3}}\right| < \epsilon$. This is normal form for a saddle-node bifurcation ($\dot{y} = R + y^2$).

§23 Dis 4: Jan 28, 2021

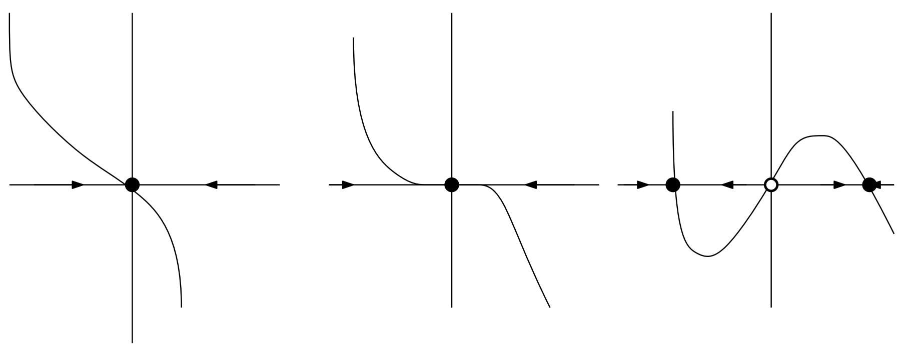
§23.1 Review of Bifurcations

We've seen 3 types of bifurcations so far

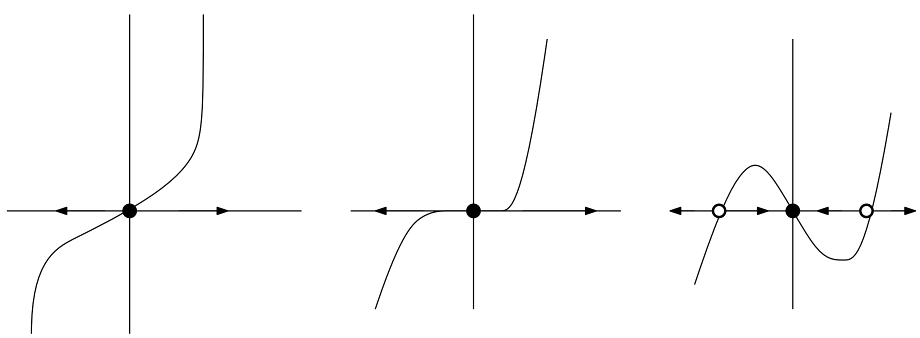
- saddle-node: $0 \to 1 \to 2$ fixed points
- transcritical: $2 \to 1 \to 2$ swap stability
- sub-/supercritical pitchfork: $1 \to 1 \to 3$

supercritical

$r < r_c$ $r = r_c$ $r > r_c$



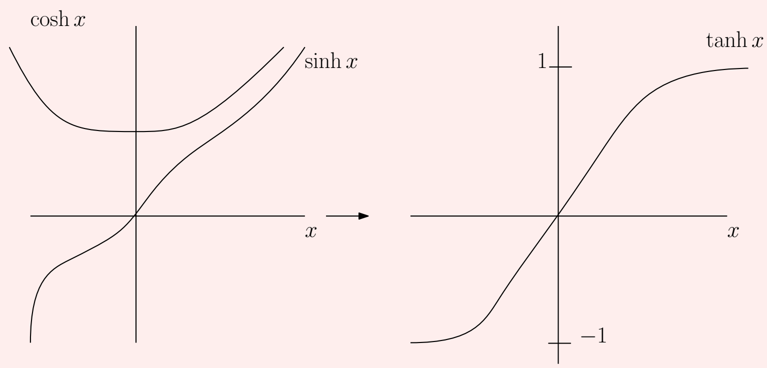
subcritical



Example 23.1

$\dot{x} = r\tanh x - x$. Sketch all qualitatively different phase portraits, sketch bifurcation diagram, find any classify bifurcations. Recall

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x}$$



$\dot{x} = r\tanh x - x$

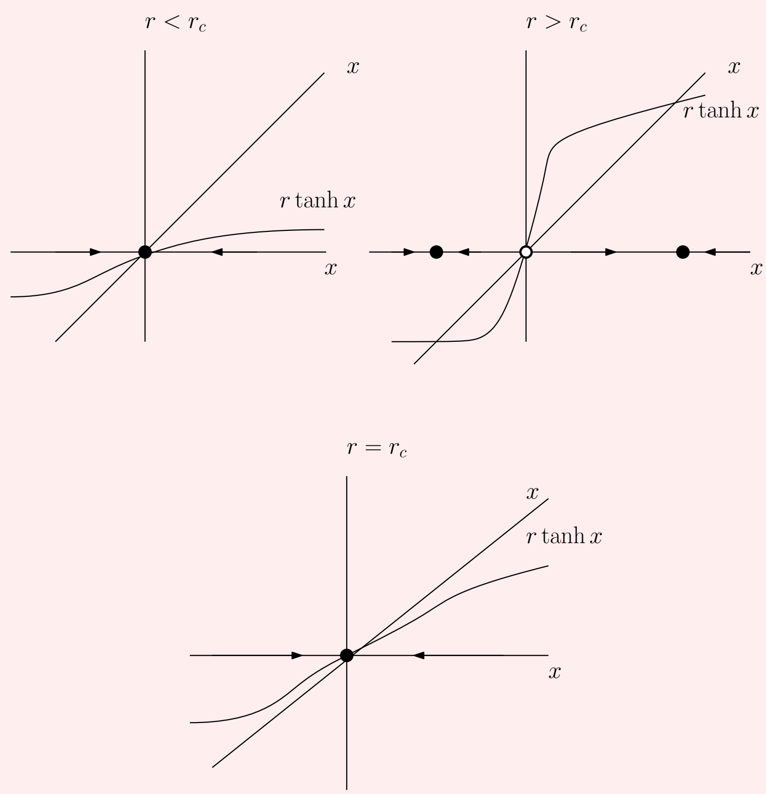
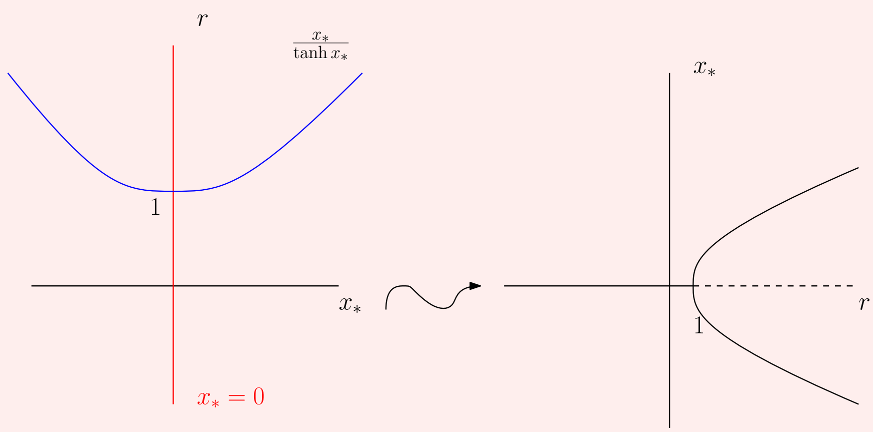


Figure 5: Supercritical pitchfork at $r = r_c$

Example 23.2 (Cont'd from above)

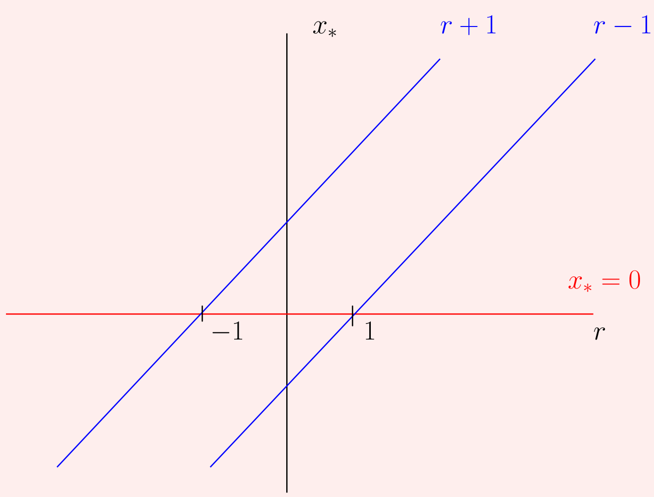
$$0 = r\tanh x_* - x_* \implies r = \frac{x_*}{\tanh x_*}, \quad x_* = 0$$



Near $x = 0, \tanh x = \frac{x+\dots}{1+\dots} = x + \dots \implies \frac{x}{\tanh x} = 1 + \dots$
Bifurcation point: $(r_c, x_c) = (1, 0)$.

Example 23.3
Same for $\dot{x} = x - x(x-r)^2$

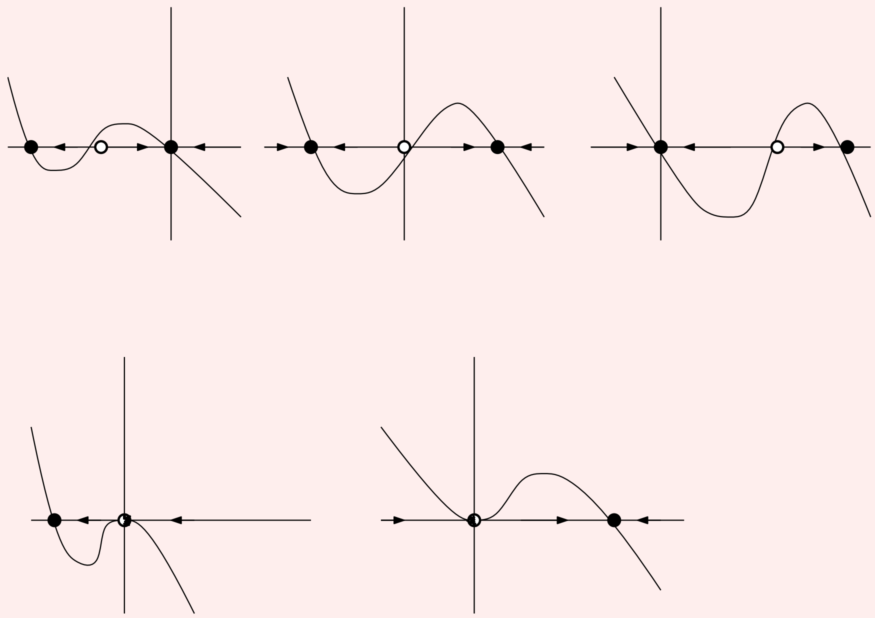
$$0 = x_*\left(1 - (x_* - r)^2\right) \implies x_* = 0, \quad x_* = r \pm 1$$



2 transcritical bifurcations at $r_c = \pm 1, x_c = 0$.

Example 23.4 (Cont'd of example 23.3)

$$\dot{x} = x\left(1-(x-r)^2\right) = -x\left(x-(r+1)\right)\left(x-(r-1)\right)$$



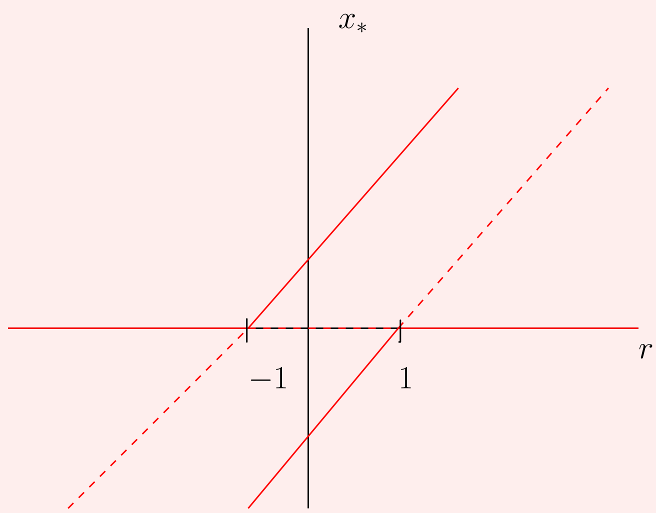
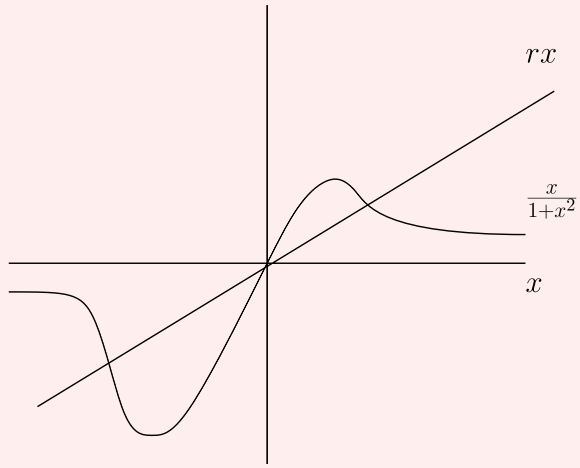
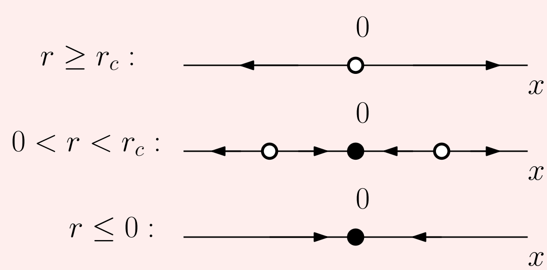


Figure 6: Bifurcation Diagram

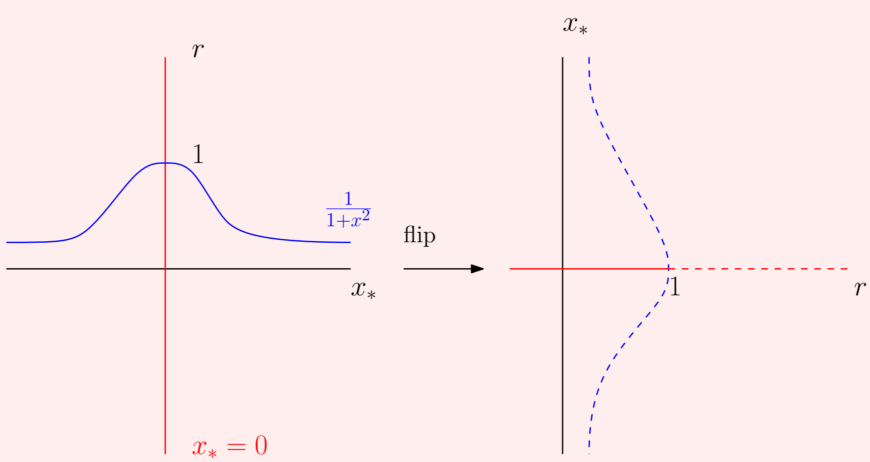
Example 23.5

Same for $\dot{x} = rx - \frac{x}{1+x^2}$





$$0 = x_* \left(r - \frac{1}{1+x_*^2} \right) \implies x_* = 0, \quad r = \frac{1}{1+x_*^2}$$

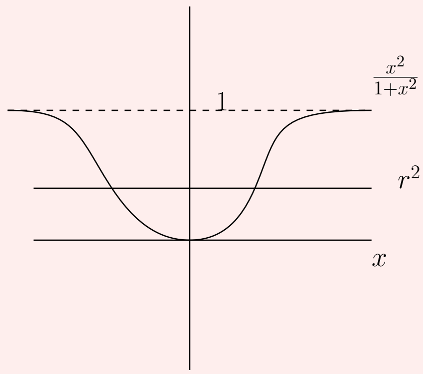


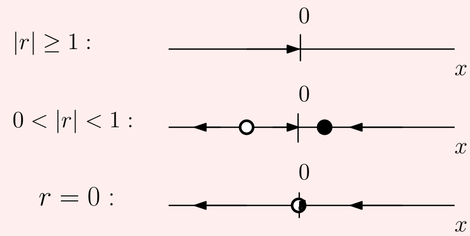
Subcritical pitchfork at $(r_c, x_c) = (1, 0)$.

§23.2 Other Bifurcations

Example 23.6

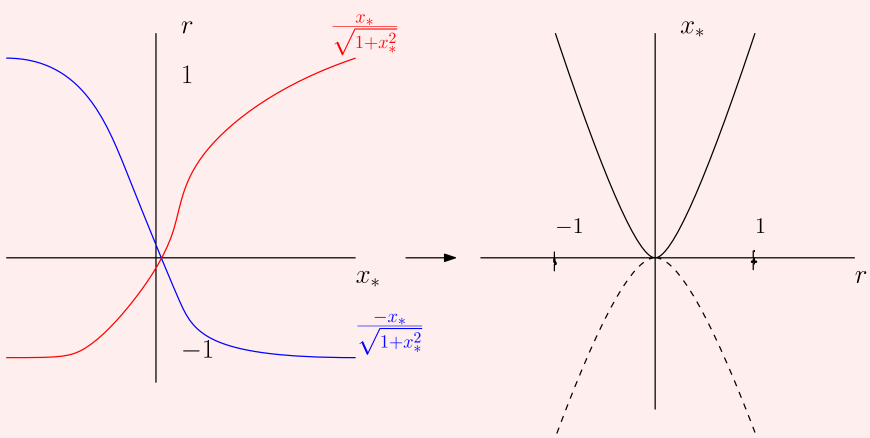
$\dot{x} = r^2 - \frac{x^2}{1+x^2}$. Sketch all qualitatively different phase portraits, sketch bifurcation diagrams, find bifurcation point.





Bifurcation point $(r_c, x_c) = (0, 0)$ and still satisfies $f(r_c, x_c) = 0$, $\frac{\partial f}{\partial x}(r_c, x_c) = 0$. Bifurcation diagram:

$$0 = r^2 - \frac{x_*^2}{1+x_*^2} \implies r = \pm\sqrt{\frac{x_*^2}{1+x_*^2}} = \pm\frac{x_*}{\sqrt{1+x_*^2}}$$



This is not one of our 3 types of bifurcations

- Graphically, it's not transcritical because both fixed points are moving.
- Analytically, we can check that the Taylor expansion at $(r, x) = (0, 0)$:

$$f(r, x) = r^2 - x^2 + \mathcal{O}(x^3)$$

doesn't match one of the 3 normal forms we know.

§24 Dis 5: Feb 4, 2021

§24.1 Bifurcations (Cont'd)

Example 24.1

$\dot{N} = rN - aN(N-b)^2$ population model, $r > 0, a > 0, b \in \mathbb{R}$ parameters. Show that this can be rewritten in the dimensionless form $\frac{dx}{d\tau} = x - x(x-c)^2$. What are x, τ, c in terms of the original quantities?

"dimensionless" $\iff$ change of variables s.t. new variables have no units

N has units pop (population) and $\dot{N} = \frac{dN}{dt}$ has units pop/time.

$$\begin{aligned} \underbrace{\dot{N}}_{\frac{\text{pop}}{\text{time}}} &= r \underbrace{N}_{\text{pop}} - a \underbrace{N^3}_{\text{pop}^3} + 2ab \underbrace{N^2}_{\text{pop}^2} - ab^2 \underbrace{N}_{\text{pop}} \\ \implies r &\sim \frac{1}{\text{time}},\ a \sim \frac{1}{\text{pop}^2 \cdot \text{time}},\ b \sim \text{pop} \end{aligned}$$

Rescale: $N = nx$, $t = T\tau$ where n, T are constant has units and τ is new variable with no units.

$$\frac{dN}{dt} = \underbrace{\frac{dN}{dx}}_{n} \frac{dx}{dt} = n \frac{dx}{d\tau} \underbrace{\frac{d\tau}{dt}}_{\frac{1}{T}} = \frac{n}{T}\frac{dx}{d\tau}$$

$$\begin{aligned} \frac{n}{T}\frac{dx}{d\tau} &= rnx - anx(nx-b)^2 \\ \frac{dx}{d\tau} &= rTx - aTx(nx-b)^2 \\ &= \underbrace{rT}_{=1}x - \underbrace{aTn^2}_{=1} x(x - \underbrace{\frac{b}{n}}_{=c})^2 \end{aligned}$$

$\implies T = \frac{1}{r}, n = \frac{1}{\sqrt{aT}} = \sqrt{\frac{r}{a}}$.

Units: $T \sim \frac{1}{\frac{1}{\text{time}}} = \text{time}, n \sim \sqrt{\frac{\frac{1}{\text{time}}}{\frac{1}{\text{pop}^2 \cdot \text{time}}}} = \text{pop}$ where τ no units and x no units.

$\implies \frac{dx}{d\tau} = x - x(x-c)^2$ with $x = \sqrt{\frac{a}{r}}N$, $\tau = rt$, $c = b\sqrt{\frac{a}{r}}$ which is defined since $a, r > 0$.

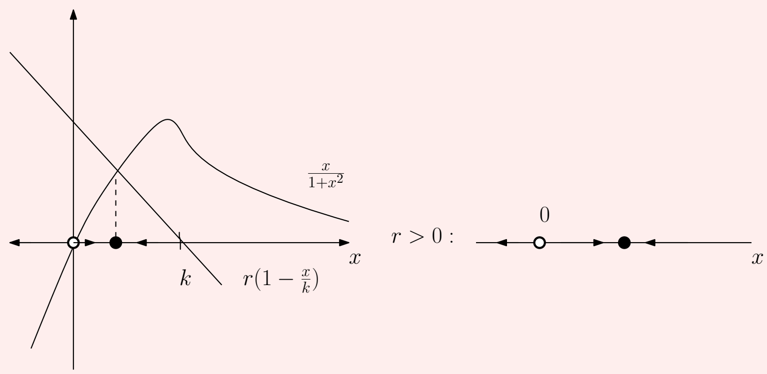
Example 24.2

Population model $\dot{x} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$ where $r, k > 0$.

(a) Sketch the 2 qualitatively different bifurcation diagrams (r, x_*) for $k \leq k_*$ and $k > k_*$.

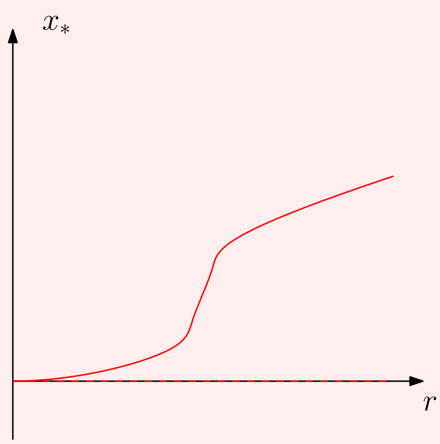
$$\dot{x} = x\left[r\left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2}\right]$$

$0 < k \leq k_*$:



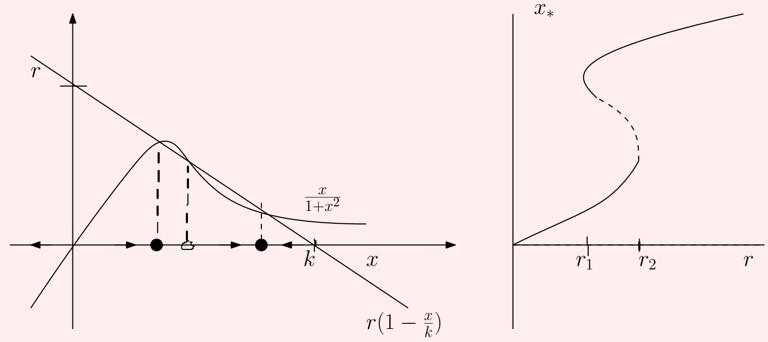
$$0 = x_*\left[r\left(1 - \frac{x_*}{k}\right) - \frac{x_*}{1+x_*^2}\right]$$

$$\implies x_* = 0 \text{ or } 0 = r\left(1 - \frac{x_*}{k}\right) - \frac{x_*}{1+x_*^2} \text{ (with 1 solution)}$$



Example 24.3 (Cont'd of example 24.2)

$k > k_*$:



$x_* = 0$ or $0 = r\left(1 - \frac{x_*}{k}\right) - \frac{x_*}{1+x_*^2}$ (with 1-3) solutions.

(b) Sketch regions in (k, r) plane with qualitatively different vector fields, classify bifurcations on boundaries.

Fixed points:

$$0 = x\left[r\left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2}\right]$$
$$\implies x = 0(\text{bifurcation}), \quad r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}(\text{includes 2 saddle-node bif}) \tag{1}$$

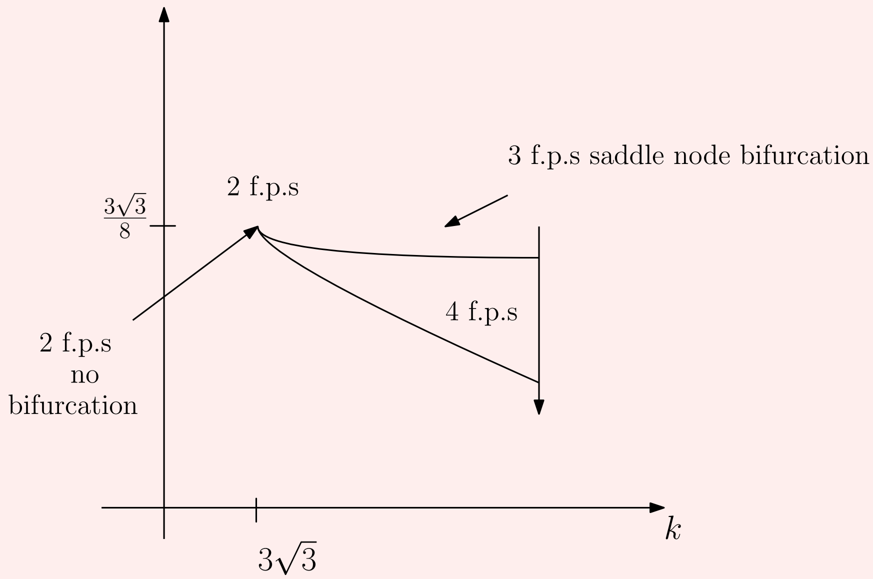
Tangent:

$$\begin{aligned}
0 &= \frac{\partial}{\partial x}\left\{x\left[r\left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2}\right]\right\} \\
&= \underbrace{r\left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2}}_{=0 \text{ by } (1)} \underbrace{x}_{\neq 0} \underbrace{\left[-\frac{r}{k} - \frac{1}{1+x^2} + \frac{2x^2}{(1+x^2)^2}\right]}_{=0} \\
&\implies -\frac{r}{k} = \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} \qquad (2) \\
&\implies \frac{x}{1+x^2} \overset{(1)}{=} r + \left(-\frac{r}{k}\right)x \overset{(2)}{=} r + \frac{x - x^3}{1+x^2} \\
&\implies r = \frac{2x^3}{(1+x^2)^2} \overset{(2)}{\implies} k = \frac{2x^3}{x^2-1}
\end{aligned}$$

for $x > 1$ (so $r, k > 0$).

Example 24.4 (Cont'd of example 24.2)

So



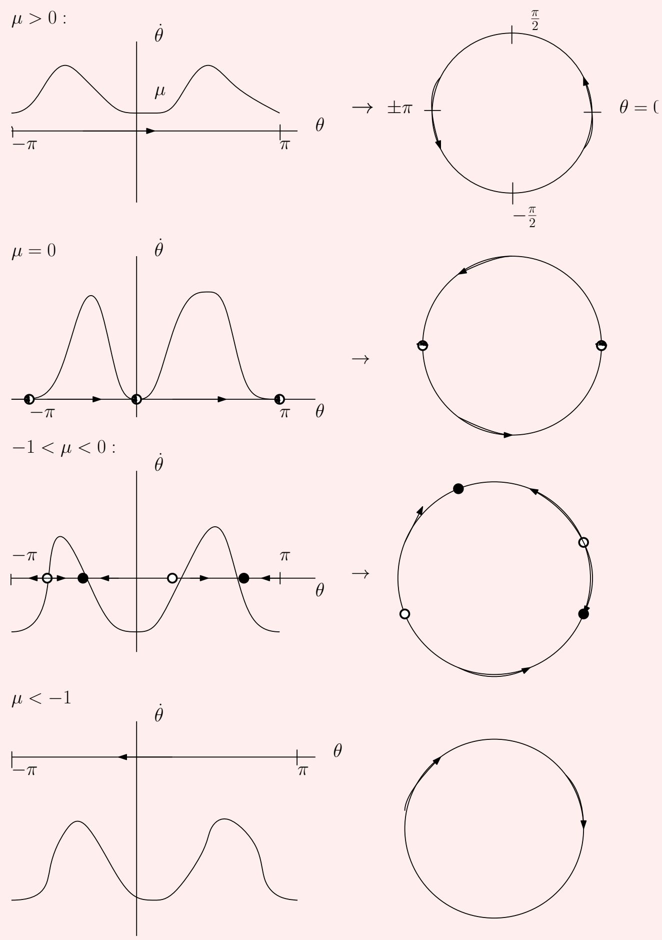
§25 Dis 6: Feb 11, 2021

§25.1 Flows on the Circle

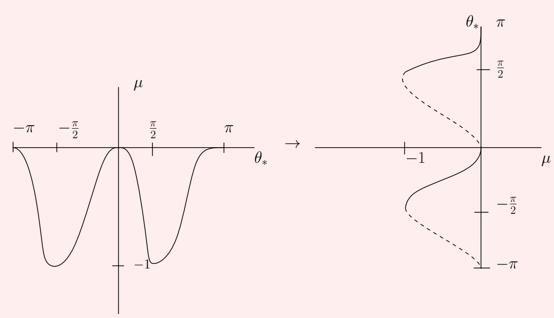
$\dot{\theta} = f(\theta)$, $\theta + 2\pi = \theta$ for all θ, so $f(\theta)$ is $2\pi-$periodic

Example 25.1

$\dot{\theta} = \mu + \sin^2\theta$. Plot all of the qualitatively different vectors fields on the circle, sketch the bifurcation diagrams, classify bifurcations.



$$0 = \mu + \sin^2(\theta_*) \implies \mu = -\sin^2\theta_*$$



4 saddle node bifurcations at: $(\mu_*, \theta_*) = (0, 0), (0, \pi), (-1, \frac{\pi}{2}), (-1, \frac{-\pi}{2})$

§25.2 2-Dim Linear Systems

$\dot{x} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = Ax$. v eigenvector (evec) with eigenvalue (eval), $\lambda : Av = \lambda v$

$$\implies x(t) = e^{\lambda t} v \text{ is a solution}$$

Example 25.2

$\dot{x} = -4x, \dot{y} = -3x - y$. Sketch phase portrait, classify fixed point $(0, 0)$.

$$\dot{x} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = Ax$$

Step 1: Find evals.

$$\begin{aligned} & Av = \lambda v \text{ for some } v \neq 0 \\ & \iff 0 = Av - \lambda v = (A - \lambda I)v \text{ for some} v \neq 0 \\ & \iff A - \lambda I \text{ not invertible} \\ & \iff \det(A - \lambda I) = 0 \\ 0 &= \det(A - \lambda I) = \det \begin{pmatrix} -4 - \lambda & 0 \\ -3 & -1 - \lambda \end{pmatrix} \\ &= (-4 - \lambda)(-1 - \lambda) - 0 = (\lambda + 1)(\lambda + 4) \end{aligned}$$

$$\implies \lambda_1 = -4, \ \lambda_2 = -1$$

Step 2: Find evecs. Want $Av_1 = \lambda v_1$ for $v_1 = \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\begin{aligned} \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= -4 \begin{pmatrix} a \\ b \end{pmatrix} \\ \implies -4a = -4b, &\quad -3a - b = -4b \\ \implies a = b & \end{aligned}$$

Any $a = b$ works, pick one $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Similarly, $Av_2 = \lambda_2 v_2 \implies \ldots \implies v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Step 3: Graph.

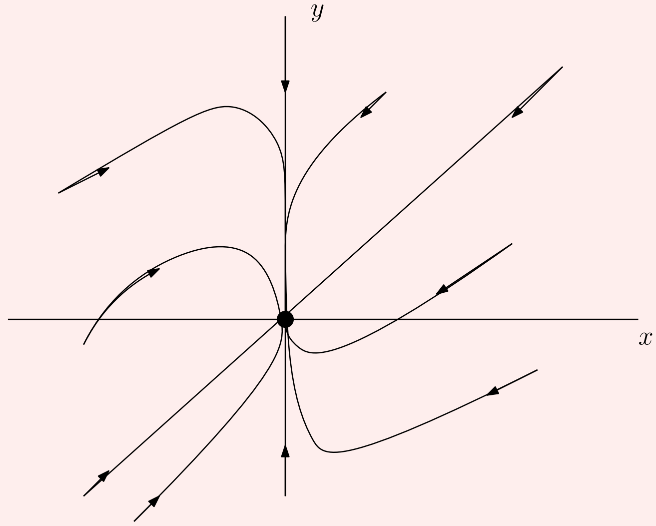
vector field $= (x, y)$-plane w/ vectors in the dimension of $\dot{x}$

phase portrait $= (x, y)$-plane w/ trajectories $(x(t), y(t))$ and arrows

General soln: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t}$ for $c_1, c_2 \in \mathbb{R}$.

Example 25.3 (Cont'd of example 25.2)

So



- $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- $c_1 e^{\lambda_1 t} v_1$ solutions straight line towards $(0,0)$.
- Solution is combination, v_1 coefficient decays faster $\implies$ soln approach v_2 as $t \to \infty$ (v_1 as $t \to -\infty$)

$x_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a stable node.

Example 25.4

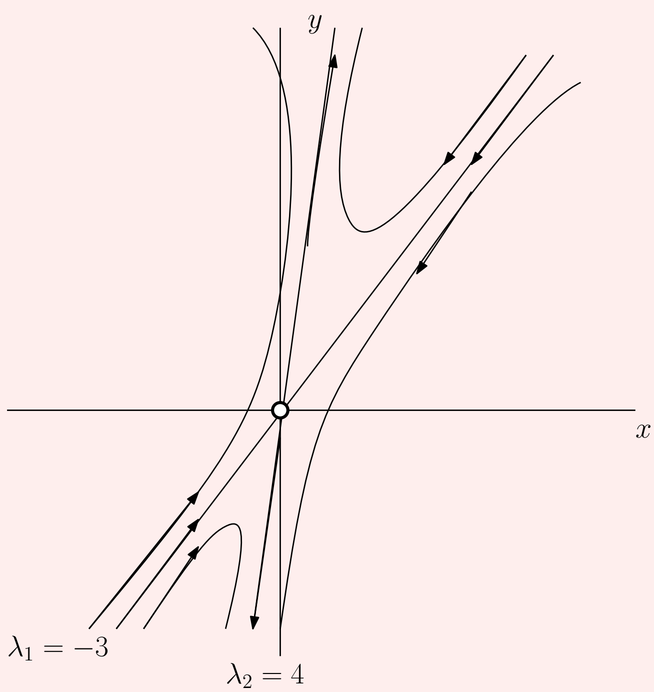
Same for $\dot{x} = -5x + 2y$ and $\dot{y} = -9x + 6y$

$$\dot{x} = Ax, \quad A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}$$

$$\lambda_1 = -3, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 4, \quad v_2 = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$$

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$



v_1 coeff decays, v_2 coeff grows.

$x_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a saddle node.

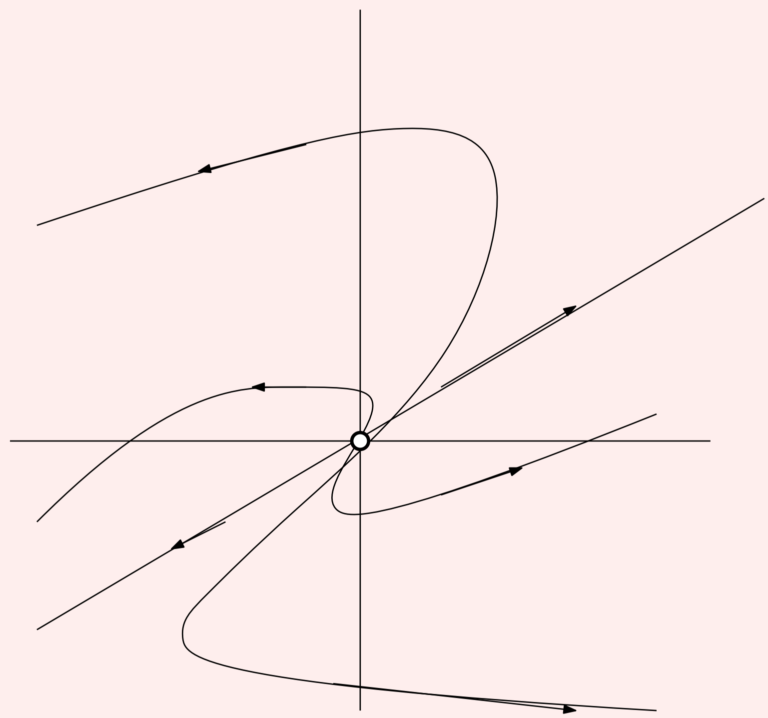
Example 25.5

Same for $\dot{x} = 3x - 4y$, $\dot{y} = x - y$.

$$\dot{x} = Ax, \quad A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

Evals: $0 = \det(A - \lambda I) = (\lambda - 1)^2 \implies \lambda = 1$

Evec: $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \implies \ldots \implies a = 2b$. Only 1 evec, take $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



- $ce^{\lambda t} v$ are solutions.
- $A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$ determines rotation direction.

$x_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an unstable degen. node.

§26 Dis 7: Feb 18, 2021

§26.1 2D – Linear System

Example 26.1

$\dot{x} = 3x - 13y$, $\dot{y} = 5x + y$. Plot phase portrait, classify the fixed point $x_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\dot{x} = Ax, \quad \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}$$

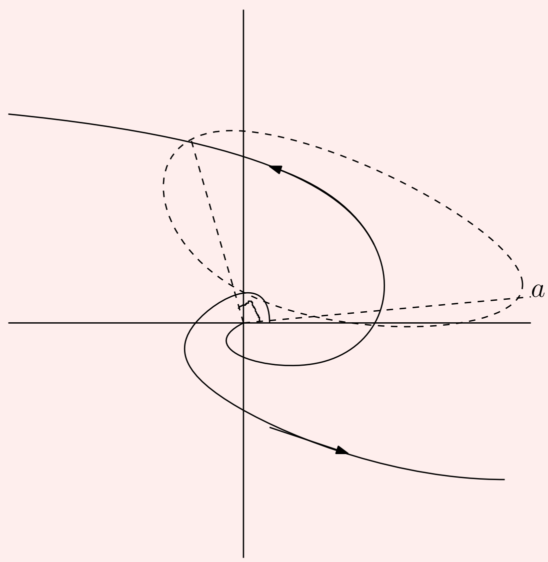
Evals: $\lambda = 2 \pm 8i$. There exists evec $v = a - ib$ for $2+8i$ s.t. a, b real are perpendicular. General solution:

$$x(t) = e^{2t} \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \cos(8t) & -\sin(8t) \\ \sin(8t) & \cos(8t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

In particular, taking $c_2 = 0$, then

$$x(t)e^{2t} \left[c_1 \cos(8t) a + c_1 \sin(8t) b \right]$$

is a solution. Rotate $a \to b \to -a \to -b$.



$x_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an unstable spiral.

To find a, b (optional)

- Find one evec:

$$Av = (2+8i)v \implies \dots \implies \text{ pick } v = \begin{pmatrix} 13 \\ 9 \end{pmatrix} - i \begin{pmatrix} -13 \\ 7 \end{pmatrix}$$

- Use Lec 15-2 Rmk (ii) to find a new evec $(\gamma + i\delta)v$ with $a \perp b$.

$$(\gamma + i\delta)v = \begin{pmatrix} 5,127.2\dots \\ 631.1 \end{pmatrix} - i \begin{pmatrix} -384.7 \\ 3,125.6 \end{pmatrix}$$

Remark 26.2.
- Not to scale: from $t = 0$ to $t = 2\pi$, Re $\lambda = 2 \implies$ grow by $e^{2\cdot 2\pi} \approx 300,000$ and Im $\lambda = 8 \implies 8$ rotations.
- For spirals on hw, just need correct stability and direction, for which we can find evals and one vector Ax.

§26.2 2-Dim Linearization

Example 26.3

$\dot{x} = x^2 - 1, \dot{y} = -y$.

a) Find fixed points.

$$\begin{aligned} 0 = \dot{x} = x^2 - 1 &\implies x = \pm 1 \\ 0 = \dot{y} = -y &\implies y = 0 \\ \implies x_* = (-1, 0), (1, 0) \end{aligned}$$

b) Find and classify linearized systems.

x_* is hyperbolic = all evals of linearized systems at x_* have Re $\lambda \neq 0$. If x_* hyperbolic $\implies$ phase portrait near x_* looks like linearized system.

$$\dot{x} = f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = \begin{pmatrix} x^2 - 1 \\ -y \end{pmatrix}$$

$$Df(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}$$

We have

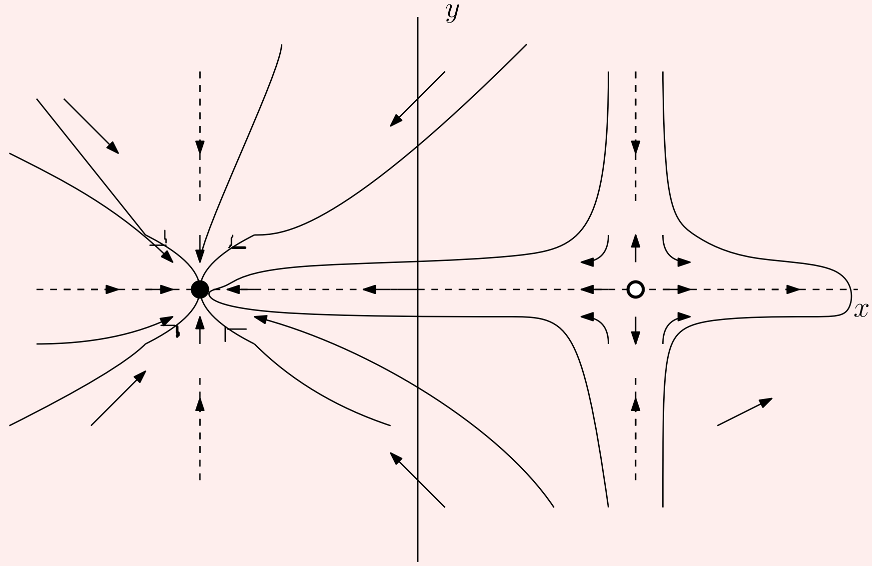
$$x_* = (-1, 0) : \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \text{ evals: } -2, -1, \text{ evecs: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

stable node, hyperbolic

$$x_* = (1, 0) : \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \text{ evals: } 2, -1, \text{ evecs: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

saddle node, hyperbolic

Example 26.4 (Cont'd from above) c) Sketch the phase portrait, determined the fixed point stability



"nullclines" = curves where vector field is horizontal or vertical

$$\text{vertical} : 0 = \dot{x} = x^2 - 1 \implies x = \pm 1$$
$$\text{horizontal} : 0 = \dot{y} = -y \implies y = 0$$

Asymptotically stable = both

- attracting: $x(0)$ near x_* $\implies$ $x(t) \to x_*$ as $t \to \infty$.
- Lyapunov stable: $x(0)$ near x_* $\implies$ $x(t)$ stays near x_*.

Unstable = neither.

$x_* = (-1, 0)$ asymptotically stable.

$x_* = (1, 0)$ unstable.

Remark 26.5. Can be either attracting or Lyapunov stable, too.

Example 26.6

$\dot{x} = x(x-y), \dot{y} = y(2x-y)$

a) Find fixed points.

$$0 = \dot{x} = x(x-y) \implies x = 0 \text{ or } y = x$$
$$0 = \dot{y} = y(2x-y) \implies y = 0 \text{ or } 2x = y$$

$\implies x_* = (0,0)$ only fixed point.

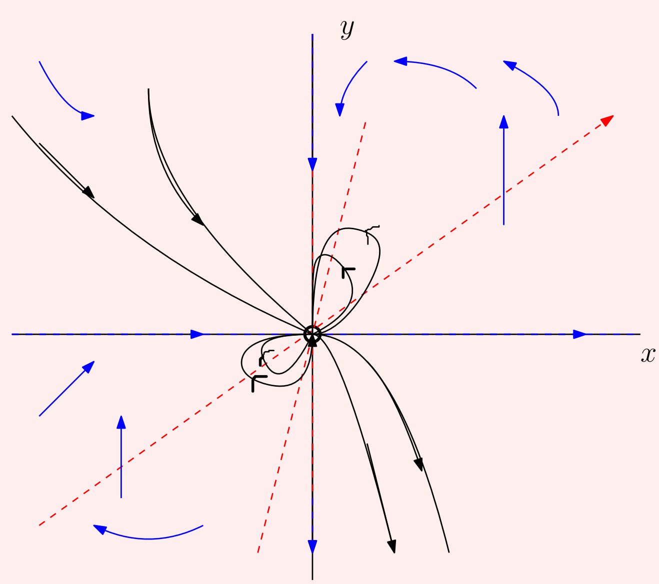
b) Find and classify linearized system.

$$f(x,y) = \begin{pmatrix} x^2 - xy \\ 2xy - y^2 \end{pmatrix}$$
$$Df(x,y) = \begin{pmatrix} 2x - y & -x \\ 2y & 2x - 2y \end{pmatrix}$$
$$Df(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Non-isolated fixed point, nonhyperbolic.

c) Sketch the phase portrait and determine fixed point stability.

$$\dot{x} = x(x-y)$$
$$\dot{y} = y(2x-y)$$



Example 26.7 (Cont'd from above)

Nullclines:

- $x = 0$:

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= -y^2 \end{aligned}$$

- $y = 0$:

$$\begin{aligned} \dot{y} &= 0 \\ \dot{x} &= x^2 \end{aligned}$$

- $y = x$:

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= x^2 \end{aligned}$$

- $y = 2x$:

$$\begin{aligned} \dot{x} &= -x^2 \\ \dot{y} &= 0 \end{aligned}$$

$x_* = (0, 0)$ is unstable.