# 115B – Linear Algebra

University of California, Los Angeles

Duc Vu

Spring 2021

This is math 115B – Linear Algebra which is the second course of the undergrad linear algebra at UCLA – continuation of 115A(H). Similar to 115AH, this class is instructed by Professor Elman, and we meet weekly on MWF from 2:00 pm to 2:50 pm. There is no official textbook used for the class. You can find the previous linear algebra notes (115AH) with other course notes through my github. Any error in this note is my responsibility and please email me if you happen to notice it.

## Contents

**1 Lec 1: Mar 29, 2021** **3**
1.1 Vector Spaces . . . 3

**2 Lec 2: Mar 31, 2021** **7**
2.1 Vector Spaces (Cont'd) . . . 7
2.2 Subspaces . . . 9
2.3 Motivation . . . 9
2.4 Direct Sums . . . 10

**3 Lec 3: Apr 2, 2021** **11**
3.1 Direct Sums (Cont'd) . . . 11
3.2 Quotient Spaces . . . 15

**4 Lec 4: Apr 5, 2021** **17**
4.1 Quotient Spaces (Cont'd) . . . 17

**5 Lec 5: Apr 7, 2021** **19**
5.1 Quotient Spaces (Cont'd) . . . 19
5.2 Linear Transformation . . . 20

**6 Lec 6: Apr 9, 2021** **22**
6.1 Linear Transformation (Cont'd) . . . 22
6.2 Projections . . . 24

**7 Lec 7: Apr 12, 2021** **27**
7.1 Projection (Cont'd) . . . 27
7.2 Dual Spaces . . . 29

**8 Lec 8: Apr 14, 2021 33**
8.1 Dual Spaces (Cont'd) . . . . . 33

**9 Lec 9: Apr 16, 2021 37**
9.1 Dual Spaces (Cont'd) . . . . . 37
9.2 The Transpose . . . . . 38
9.3 Polynomials . . . . . 40

# List of Theorems

# List of Definitions

1.1 Field . . . . . 3
1.3 Ring . . . . . 4
1.6 Vector Space . . . . . 6
2.2 Subspace . . . . . 9
2.7 Span . . . . . 11
2.8 Direct Sum . . . . . 11
3.1 Independent Subspace . . . . . 11
3.5 Complementary Subspace . . . . . 14
6.5 T-invariant . . . . . 24
6.9 Projection . . . . . 25
7.9 Dual Space . . . . . 31
8.8 Annihilator . . . . . 36
9.7 Transpose . . . . . 39
9.11 Row/Column Rank . . . . . 40
9.13 Polynomial Division . . . . . 41
9.16 Polynomial Degree and Leading Coefficient . . . . . 41

# §1 Lec 1: Mar 29, 2021

## §1.1 Vector Spaces

Notation: if $\star : A \times B \to B$ is a map (= function) write $a \star b$ for $\star(a, b)$, e.g., $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $\mathbb{Z}$ = the integer.

**Definition 1.1** (Field) — A set $F$ is called a FIELD under

- Addition: $+ : F \times F \to F$
- Multiplication: $\cdot : F \times F \to F$

if $\forall a, b, c \in F$, we have

A1) $(a + b) + c = a + (b + c)$

A2) $\exists\ 0 \in F \ni a + 0 = a = 0 + a$

A3) A2) holds and $\exists x \in F \ni a + x = 0 = x + a$

A4) $a + b = b + a$

M1) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M2) A2) holds and $\exists 1 \neq 0 \in F$ s.t. $a \cdot 1 = a = 1 \cdot a$ ( 1 is unique and written 1 or $1_F$)

M3) M2) holds and $\forall 0 \neq x \in F \quad \exists y \in F \ni xy = 1 = yx$ ($y$ is seen to be unique and written $x^{-1}$)

M4) $x \cdot y = y \cdot x$

D1) $a \cdot (b + c) = a \cdot b + a \cdot c$

D2) $(a + b) \cdot c = a \cdot c + b \cdot c$

**Example 1.2**

$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are fields as is

$$\mathbb{F}_2 := \{0, 1\} \text{ with } + : \text{ given by}$$

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

| • | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

**Fact 1.1.** Let $p > 0$ be a prime number in $\mathbb{Z}$. Then $\exists$ a field $\mathbb{F}_{p^n}$ having $p^n$ elements write $|\mathbb{F}_{p^n}| = p^n \quad \forall n \in \mathbb{Z}^+$.

**Definition 1.3** (Ring) — Let $R$ be a set with

- $+ : R \times R \to R$
- $\cdot : R \times R \to R$

satisfying A1) – A4), M1), M2), D1), D2), then $R$ is called a RING.
A ring is called

i) a commutative ring if it also satisfies M4).

ii) an (integral) domain if it is a commutative ring and satisfies

$$\text{M 3')}\; a \cdot b = 0 \implies a = 0 \text{ or } b = 0$$

($0 = \{0\}$ is also called a ring – the only ring with $1 = 0$)

**Example 1.4** (Proof left as exercises) 1. $\mathbb{Z}$ is a domain and not a field.

2. Any field is a domain.
3. Let $F$ be a field
$$F[t] \coloneqq \{\text{polys coeffs in } F\}$$
with usual $+, \cdot$ of polys, is a domain but not a field. So if $f \in F[t]$
$$f = a_0 + a_1 t + \ldots + a_n t^n$$
where $a_0, \ldots, a_n \in F$.
4. $\mathbb{Q} \coloneqq \left\{ \frac{a}{b} | a, b \in \mathbb{Z},\, b \neq 0 \right\} < \mathbb{C}$ ($<$ means $\subset$ and $\neq$) with usual $+, \cdot$ of fractions. (when does $\frac{a}{b} = \frac{c}{d}$?)
5. If $F$ is a field
$$F(t) \coloneqq \left\{ \frac{f}{g} | f, g \in F[t],\, g \neq 0 \right\} \text{ (rational function)}$$
with usual $+, \cdot$ of fractions is a field.

**Example 1.5** (Cont'd from above) 6. $\mathbb{Q}[\sqrt{-1}] \coloneqq \left\{\alpha + \beta\sqrt{-1} \in \mathbb{C} | \alpha, \beta \in \mathbb{Q}\right\} < \mathbb{C}$. Then $\mathbb{Q}[\sqrt{-1}]$ is a field and

$$\begin{aligned}\mathbb{Q}(\sqrt{-1}) &\coloneqq \left\{\frac{a}{b} | a, b \in \mathbb{Q}[\sqrt{-1}],\, b \neq 0\right\} \\ &= \mathbb{Q}[\sqrt{-1}] \\ &= \left\{\frac{a}{b} | a, b \in \mathbb{Z}[\sqrt{-1}],\, b \neq 0\right\}\end{aligned}$$

where $\mathbb{Z}[\sqrt{-1}] \coloneqq \left\{\alpha + \beta\sqrt{-1} \in \mathbb{C},\, \alpha, \beta \in \mathbb{Z}\right\} < \mathbb{C}$. How to show this? – rationalize ($\mathbb{Z}[\sqrt{-1}]$ is a domain not a field, $F[t] < F(t)$ if $F$ is a field so we have to be careful).

7. $F$ a field

$$\mathbb{M}_n F \coloneqq \{n \times n \text{ matrices entries in } F\}$$

is a ring under $+, \cdot$ of matrices.

$$1_{\mathbb{M}_n F} = I_n = n \times n \text{ identity matrix } \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$0_{\mathbb{M}_n F} = 0 = 0_n = n \times n \text{ zero matrix } \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

is <u>not</u> commutative if $n > 1$.

In the same way, if $R$ is a ring we have

$$\mathbb{M}_n R = \{n \times n \text{ matrices entries in } R\}$$

e.g., if $R$ is a field $\mathbb{M}_n F[t]$.

8. Let $\emptyset \neq I \subset \mathbb{R}$ be a subset, e.g., $[\alpha, \beta], \alpha < \beta \in \mathbb{R}$. Then

$$C(I) = \{f : I \to \mathbb{R} | f \text{ continuous}\}$$

is a commutative ring and not a domain where

$$\begin{aligned}(f \dot{+} g)(x) &\coloneqq f(x) \dot{+} g(x) \\ 0(x) &= 0 \\ 1(x) &= x\end{aligned}$$

for all $x \in I$.

<u>Notation</u>: Unless stated otherwise $F$ is always a field.

**Definition 1.6** (Vector Space) — Let $F$ be a field, $V$ a set. Then $V$ is called a VECTOR SPACE OVER $F$ write $V$ is a vector space over $F$ under

- $+ : V \times V \to V$ – Addition
- $\cdot : F \times V \to V$ – Scalar multiplication

if $\forall x, y, z \in V \quad \forall \alpha, \beta \in F$.

1. $(x + y) + z = x + (y + z)$
2. $\exists 0 \in V \ni x + 0 = x = 0 + x$ (0 is seen to be unique and written 0 or $0_V$)
3. 2) holds and $\exists v \in V \ni x + v = 0 = v + x$ ($v$ is seen to be unique and written $-x$)
4. $x + y = y + x$
5. $1_F \cdot x = x$.
6. $(\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta \cdot x)$
7. $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$
8. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

**Remark 1.7.** The usual properties we learned in 115A hold for $V$ a vector space over $F$, e.g., $0_F V = 0_V$, general association law,...

# §2 Lec 2: Mar 31, 2021

## §2.1 Vector Spaces (Cont'd)

**Example 2.1**

The following are vector space over $F$

1. $F^{m\times n} \coloneqq \{m \times n$ matrices entries in $F\}$, usual $+$, scalar multiplication, i.e., if $A \in F^{m\times n}$, let $A_{ij} = ij^{\text{th}}$ entry of $A$. If $A, B \in F^{m\times n}$, then

$$\begin{aligned} (A+B)_{ij} &\coloneqq A_{ij} + B_{ij} \\ (\alpha A)_{ij} &\coloneqq \alpha A_{ij} \quad \forall \alpha \in F \end{aligned}$$

   i.e., component-wise operations.

2. $F^n = F^{1\times n} \coloneqq \{(\alpha_1, \dots, \alpha_n) \,|\alpha_i \in F\}$

3. Let $V$ be a vector space over $F$, $\emptyset \neq S$ a set. Define

$$\mathcal{F}cn(S,V) \coloneqq \{f : S \to V |\, f \text{ a fcn}\}$$

   Then $\mathcal{F}cn(S,V)$ is a vector space over $F$ $\forall f, g \in \mathcal{F}cn(S,V)$, $\forall \alpha \in F$. For all $x \in S$,

$$\begin{aligned} f+g &: x \mapsto f(x) + g(x) \\ \alpha f &: x \mapsto \alpha f(x) \end{aligned}$$

   i.e.

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (\alpha f)(x) &= \alpha f(x) \end{aligned}$$

   with 0 by $0(x) = 0_V \, \forall x \in S$.

4. Let $R$ be a ring under $+, \cdot, F$ a field $\ni F \subseteq R$ with $+, \cdot$ on $F$ induced by $+, \cdot$ on $R$ and $0_F = 0_R$, $1_F = 1_R$, i.e.

$$\underbrace{+}_{\text{on } R}\Big|_{\underbrace{F\times F}_{\text{restrict dom}}} : F \times F \to F \text{ and } \underbrace{\cdot}_{\text{on } R}\Big|_{\underbrace{F\times F}_{\text{restrict dom}}} : F \times F \to F$$

   i.e. closed under the restriction of $+, \cdot$ on $R$ to $F$ and also with $0_F = 0_R$ and $1_F = 1_R$ (we call $F$ a <u>subring</u> of $R$). Then $R$ is a vector space over $F$ by restriction of scalar multiplication, i.e., same $+$ on $R$ but scalar multiplication

$$\cdot\big|_{F\times R} : F \times R \to R$$

   e.g., $\mathbb{R} \subseteq \mathbb{C}$ and $F \subseteq F[t]$.

   <u>*Note*</u>: $\mathbb{C}$ is a vector space over $\mathbb{R}$ by the above but as a vector space over $\mathbb{C}$ is different.

5. In 4) if $R$ is also a field (so $F \subseteq R$ is a subfield). Let $V$ be a vector space over $R$. Then $V$ is also a vector space over $F$ by restriction of scalars, e.g., $M_n\mathbb{C}$ is a vector space over $\mathbb{C}$ so is a vector space over $\mathbb{R}$ so is a vector space over $\mathbb{Q}$.

## §2.2 Subspaces

**Definition 2.2** (Subspace) — Let $V$ be a vector space under $+, \cdot, \emptyset \neq W \subseteq V$ a subset. We call $W$ a subspace of $V$ if $\forall w_1, w_2 \in W$, $\forall \alpha \in F$,

$$\alpha w_1,\, w_1 + w_2 \in W$$

with $0_W = 0_V$ is a vector space over $F$ under $+|_{W\times W}$ and $\cdot|_{F\times W}$ i.e., closed under the operation on $V$.

**Theorem 2.3**

Let $V$ be a vector space over $F$, $\emptyset \neq W \subseteq V$ a subset. Then $W$ is a subspace of $V$ iff $\forall \alpha \in F$, $\forall w_1, w_2 \in W$, $\alpha w_1 + w_2 \in W$.

**Example 2.4**

1. Let $\emptyset \neq I \subseteq \mathbb{R}$, $C(I)$ the commutative ring of continuous function $f : I \to \mathbb{R}$. Then $C(I)$ is a vector space over $\mathbb{R}$ and a subspace of $\mathcal{F}cn(I, \mathbb{R})$.
2. $F[t]$ is a vector space over $F$ and $n \geq 0$ in $\mathbb{Z}$.

$$F[t]_n := \{f |\, f \in F[t],\ f = 0 \text{ or } \deg f \leq d\}$$

is a subspace of $F[t]$ (it is not a ring).

Attached is a review of theorems about vector spaces from math 115A.

## §2.3 Motivation

**Problem 2.1.** Can you break down an object into simpler pieces? If yes can you do it uniquely?

**Example 2.5**

Let $n > 1$ in $\mathbb{Z}$. Then $n$ is a product of primes unique up to order.

**Example 2.6**

Let $V$ be a finite dimensional inner product space over $\mathbb{R}$ (or $\mathbb{C}$) and $T : V \to V$ a hermitian (=self adjoint) operator. Then $\exists$ an ON basis for $V$ consisting of eigenvectors for $T$. In particular, $T$ is diagonalizable. This means

$$V = E_T(\lambda_1) \perp \ldots \perp E_T(\lambda_r) \tag{*}$$

$E_T(\lambda_i) := \{v \in V \,|\, Tv = \lambda_i v\} \neq 0$ eigenspace of $\lambda_i$; $\lambda_1, \ldots, \lambda_r$ the distinct eigenvalues of $T$. So

$$T\big|_{E_T(\lambda_i)} : E_T(\lambda_i) \to E_T(\lambda_i)$$

i.e., $E_T(\lambda_i)$ is T-invariant and

$$T\big|_{E_T(\lambda_i)} = \lambda_i 1_{E_T(\lambda_i)}$$

and (*) is unique up to order.

Goal: Generalize this to $V$ any finite dimensional vector space over $F$, any $F$, and $T : V \to V$ linear. We have many problems to overcome in order to get a meaningful result, e.g.,

**Problem 2.2.**

1. $V$ may not be an inner product space.
2. $F \neq \mathbb{R}$ or $\mathbb{C}$ is possible.
3. $F \nsubseteq$ is possible, so cannot even define an inner product.
4. $V$ may not have any eigenvalues for $T : V \to V$.
5. If we prove an existence theorem, we may not have a uniqueness one.

We shall show: given $V$ a finite dimensional vector space over $F$ and $T : V \to V$ a linear operator. Then $V$ breaks up uniquely up to order into small $T$-invariant subspace that we shall show are completely determined by polys in $F[t]$ arising from $T$.

## §2.4 Direct Sums

Motivation: Generalize the concept of linear independence, Spectral Theorem Decomposition, to see how pieces are put together (if possible).

**Definition 2.7** (Span) — Let $V$ be a vector space over $F$, $W_i \subseteq V$, $i \in I$ – may not be finite, subspaces. Let

$$\sum_{i\in I} W_i = \sum_{i \in I} W_i := \left\{ v \in V | \exists w_i \in W_i,\ i \in I,\ \text{almost all } w_i = 0 \ni v = \sum_{i\in I} w_i \right\}$$

when almost all zero means only finitely many $w_i \neq 0$. Warning: In a vector space/F we can only take finite linear combination of vectors. So

$$\sum_{i\in I} W_i = \text{Span}\left(\bigcup_{i\in I} W_i\right) = \left\{ \text{finite linear combos of vectors in } \bigcup_{i\in I} W_i \right\}$$

e.g., if $I$ is finite, i.e., $|I| < \infty$, say $I = \{1, \ldots, n\}$ then

$$\sum_{i\in I} W_i = W_1 + \ldots + W_n := \{w_1 + \ldots + w_n |\, w_i \in W_i \,\forall i \in I\}$$

cf. Linear Combinations.

**Definition 2.8** (Direct Sum) — Let $V$ be a vector space over $F$, $W_i \subseteq V$, $i \in I$, subspace. Let $W \subseteq V$ be a subspace. We say that $W$ is the (internal) direct sum of the $W_i$, $i \in I$ write $W = \bigoplus_{i\in I} W_i$ if

$$\forall w \in W\, \exists!\, w_i \in W_i \text{ almost all } 0 \ni\ w = \sum_{i\in I} w_i$$

e.g., if $I = \{1, \ldots, n\}$, then

$$w \in W_1 \oplus \ldots \oplus W_n \text{ means } \exists!\, w_i \in W_i \ni\ w = w_1 + \ldots + w_n$$

Warning: It may not exist.

# §3 Lec 3: Apr 2, 2021

## §3.1 Direct Sums (Cont'd)

**Definition 3.1** (Independent Subspace) — Let $V$ be a vector space over $F, W_i \subseteq V$, $i \in I$ subspaces. We say the $W_i$, $i \in I$, are independent if whenever $w_i \in W_i$, $i \in I$, almost all $w_i = 0$, satisfy $\sum w_i = 0$, then $w_i = 0 \,\forall i \in I$.

**Theorem 3.2**

Let $V$ be a vector space over $F$, $W_i \subseteq V$, $i \in I$ subspaces, $W \subseteq V$ a subspace. Then the following are equivalent:

1. $W = \bigoplus_{i\in I} W_i$
2. $W = \sum_{i\in I} W_i$ and $\forall i$
$$W_i \cap \sum_{j\in I\setminus\{i\}} W_j = 0 := \{0\}$$
3. $W = \sum_{i\in I} W_i$ and the $W_i$, $i \in I$, are independent.

*Proof.* 1) $\implies$ 2) Suppose $W = \bigoplus_{i\in I} W_i$. Certainly, $W = \sum_{i\in I} W_i$. Fix $i$ and suppose that
$$\exists x \in W_i \cap \sum_{j\in I\setminus\{i\}} W_j$$
By definition, $\exists w_i \in W_i$, $w_j \in W_j$, $j \in I \setminus \{i\}$ almost all 0 satisfying
$$w_i = x = \sum_{j\neq i} w_j$$
So
$$0_V = 0_W = w_i - \sum_{j\neq i} w_j$$
But
$$0_W = \sum_I 0_{W_k} \qquad 0_{W_k} = 0_V \; \forall k \in I$$
By uniqueness of 1), $w_i = 0$ so $x = 0$.
2) $\implies$ 3) Let $w_i \in W_i$, $i \in I$, almost all zero satisfy
$$\sum_{i\in I} w_i = 0$$
Suppose that $w_k \neq 0$. Then
$$w_k = -\sum_{i\in I\setminus\{k\}} w_i \in W_k \cap \sum_{i\neq k} w_i = 0,$$
a contradiction. So $w_i = 0 \,\forall i$
3) $\implies$ 1) Suppose $v \in \sum_{i\in I} W_i$ and $\exists w_i, w_i' \in W_i$, $i \in I$, almost all 0 $\ni$
$$\sum_{i\in I} w_i = v = \sum_{i\in I} w_i'$$
Then $\sum_{i\in I}(w_i - w_i') = 0$, $w_i - w_i' \in W_i \,\forall i$. So
$$w_i - w_i' = 0, \text{ i.e., } w_i = w_i' \quad \forall i$$
and the $w_i's$ are unique. $\square$

Warning: 2) DOES NOT SAY $W_i \cap W_j = 0$ if $i \neq j$. This is too weak. It says $W_i \cap \sum_{j\neq i} W_j = 0$.

**Corollary 3.3**

Let $V$ be a vector space over $F, W_i \subseteq V$, $i \in I$ subspaces. Suppose $I = I_1 \cup I_2$ with $I_1 \cap I_2 = \emptyset$ and $V = \bigoplus_{i \in I} W_i$. Set

$$W_{I_1} = \bigoplus_{i \in I_1} W_i \quad \text{and} \quad W_{I_2} = \bigoplus_{j \in I_2} W_j$$

Then

$$V = W_{I_1} \oplus W_{I_2}$$

*Proof.* Left as exercise – Homework. □

Notation: Let $V$ be a vector space over $F$, $v \in V$. Set

$$Fv := \{\alpha v | \alpha \in F\} = \text{Span}(v)$$

if $v \neq 0$, then $Fv$ is the line containing $v$, i.e., $Fv$ is the one dimensional vector space over $F$ with basis $\{v\}$.

**Example 3.4**

Let $V$ be a vector space over $F$

1. If $\emptyset \neq S \subseteq V$ is a subset, then
$$\sum_{v \in S} Fv = \text{Span}(S)$$
the span of $S$. So
$$\text{Span } S = \{\text{all finite linear combos of vectors in } S\}$$

2. If $\emptyset \neq S$ is linearly indep. (i.e. meaning every finite nonempty subset of $S$ is linearly indep.), then
$$\text{Span}(S) = \bigoplus_{s \in S} Fs$$

3. If $S$ is a basis for $V$, then $V = \bigoplus_{s \in S} Fs$

4. If $\exists$ a finite set $S \subseteq V \ni V = \text{Span}(S)$, then $V = \sum_{s \in S} Fs$ and $\exists$ a subset $\mathscr{B} \subseteq S$ that is a basis for $V$, i.e., $V$ is a finite dimensional vector space over $F$ and $\dim V = \dim_F V = |\mathscr{B}|$ is indep. of basis $\mathscr{B}$ for $V$.

5. Let $V$ be a vector space over $F, W_1, W_2 \subseteq V$ finite dimensional subspaces. Then $W_1 + W_2$, $W_1 \cap W_2$ are finite dimensional vector space over $F$ and
$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$
So
$$W_1 + W_2 = W_1 \oplus W_2 \iff W_1 \cap W_2 = \emptyset$$
(warning: be very careful if you wish to generalize this)

**Definition 3.5** (Complementary Subspace) — Let $V$ be a finite dimensional vector space over $F, W \subseteq V$ a subspace if
$$V = W \oplus W', \quad W' \subseteq V \text{ a subspace}$$
We call $W'$ a complementary subspace of $W$ in $V$.

**Example 3.6**

Let $\mathscr{B}_0$ be a basis of $W$. Extend $\mathscr{B}_0$ to a basis $\mathscr{B}$ for $V$ (even works if $V$ is not finite dimensional). Then

$$W' = \bigoplus_{\mathscr{B}\setminus\mathscr{B}_0} Fv \text{ is a complement of } W \text{ in } V$$

*Note*: $W'$ is not the unique complement of $W$ in $V$ – counter-example?

Consequences: Let $V$ be a finite dimensional vector space over $F, W_1, \ldots, W_n \subseteq V$ subspaces, $W_i \neq 0 \, \forall i$. Then the following are equivalent

1. $V = W_1 \oplus \ldots \oplus W_n$.
2. If $\mathscr{B}_i$ is a basis (resp., ordered basis) for $W_i \, \forall i$, then $\mathscr{B} = \mathscr{B}_1 \cup \ldots \cup \mathscr{B}_n$ is a basis (resp. ordered) – with obvious order – for $V$.

*Proof.* Left as exercise (good one)! □

Notation: Let $V$ be a vector space over $F$, $\mathscr{B}$ a basis for $V$, $x \in V$. Then, $\exists! \alpha_v \in F$, $v \in \mathscr{B}$, almost all $\alpha_v = 0$ (i.e., all but finitely many) s.t. $x = \sum_{\mathscr{B}} \alpha_v v$. Given $x \in V$,

$$x = \sum_{v \in \mathscr{B}} \alpha_v v$$

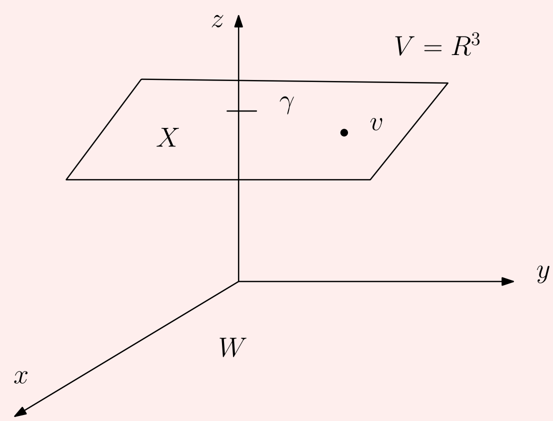
to mean $\alpha_v$ is the unique complement of $x$ on $v$ and hence $\alpha_v = 0$ for almost all $v \in \mathscr{B}$.

## §3.2 Quotient Spaces

Idea: Given a surjective map $f : X \to Y$ and "nice", can we use properties of $Y$ to obtain properties of $X$?

**Example 3.7**

Let $V = \mathbb{R}^3$, $W = X - Y$ plane. Let $X$ = plane parallel to $W$ intersecting the z-axis at $\gamma$.




So

$$\begin{aligned} X &= \{(\alpha, \beta, \gamma) | \alpha, \beta \in \mathbb{R}\} \\ &= \{(\alpha, \beta, 0) + (0, 0, \gamma) | \alpha, \beta \in \mathbb{R}\} \\ &= W + \gamma \underbrace{e_3}_{(0,0,1)} \end{aligned}$$

<u>*Note*</u>: $X$ is a vector space over $\mathbb{R} \iff \gamma = 0 \iff W = X$ (need $0_V$). Let $v \in X$. So $v = (x, y, \gamma)$ some $x, y \in \mathbb{R}$. So

$$\begin{aligned} W + v &:= \left\{ \underbrace{(\alpha, \beta, 0)}_{\text{arbitrary}} + \underbrace{(x, y, \gamma)}_{\text{fixed}} \,|\, \alpha, \beta \in \mathbb{R} \right\} \\ &= \{(\alpha + x, \beta + y, \gamma) \,|\, \alpha, \beta \in \mathbb{R}\} \\ &= W + \gamma_{e_3} \end{aligned}$$

It follows if $v, v' \in V$, then

$$W + v = W + v' \implies v - v' \in W$$

Conversely, if $v, v' \in V$ with $X = W + v$, then

$$v' \in X \implies v' = w + v \text{ some } w \in W$$

hence

$$v' - v \in W$$

So for arbitrary $v, v' \in V$, we have the <u>conclusion</u> $W + v = W + v' \iff v - v' \in W$. We can also write $W + v$ as $v + W$.

# §4 Lec 4: Apr 5, 2021

## §4.1 Quotient Spaces (Cont'd)

Recall from the last example of the last lecture, we have

$$V = \bigcup_{v \in V} W + v$$

If $v, v' \in V$, then

$$0 \neq v'' \in (W + v) \cap (W + v')$$

means

$$W + v - W + v'' = W + v'$$

This means either $W + v = W + v'$ or $W + v \cap W + v' = \emptyset$, i.e., planes parallel to the xy-plane partition $V$ into a disjoint unions of planes.

Let

$$S := \{W + v | \, v \in V\}$$

the set of these planes. We make $S$ into a vector space over $\mathbb{R}$ as follows: $\forall v, v' \in V$, $\forall \alpha \in \mathbb{R}$ define

$$(W + v) + (W + v') := W + (v + v')$$
$$\alpha \cdot (W + v) := W + \alpha v$$

We must check these two operations are well-defined and we set

$$0_S := W$$

Then $(W + v) + W = W + v = W + (W + v)$ make $S$ into a vector space over $\mathbb{R}$.
If $v \in V$ let $\gamma_v^1 =$ the $k^{\text{th}}$ component of $v$. Define

$$S \to \{(0, 0, \gamma) | \, \gamma \in \mathbb{R}\} \to \mathbb{R}$$

by

$$W + v \mapsto (0, 0, \gamma_v) \mapsto \gamma$$

both maps are bijection and, in fact, linear isomorphism. So

$$S \cong \{(0, 0, \gamma) | \, \gamma \in \mathbb{R}\} \cong \mathbb{R}$$

<u>Note</u>: $\dim V = 3$, $\dim W = 2$, $\dim S = 1$ and we also have a linear transformation

$$V \to S \text{ by } (\alpha, \beta, \gamma) \mapsto W + \gamma_{e_3}$$

a surjection.
We can now generalize this.
<u>Construction</u>: Let $V$ be a vector space over $F$, $W \subseteq V$ a subspace. Define $\equiv \mod W$ called <u>congruent mod $W$ on $V$</u> as follows: if $x, y \in V$, then

$$x \equiv y \mod W \iff x - y \in W \iff \exists w \in W \ni x = w + y$$

Then, for all $x, y, z \in V$, $\equiv \mod W$ satisfies

1. $x \equiv x \mod W$
2. $x \equiv y \mod W \implies y \equiv x \mod W$
3. $x \equiv y \mod W$ and $y \equiv z \mod W \implies x \equiv z \mod W$

We can conclude that $\equiv \mod W$ is an equivalence relation on $V$.

Notation: For $x \in V$, $W \subseteq V$, let

$$\overline{x} := \{y \in V \mid y \equiv x \mod W\}$$

We can also write $\overline{x}$ as $[x]_W$ if $W$ is not understood. Also, $\overline{x} \subseteq V$ is a subset and not an element of $V$ called a coset of $V$ by $W$. We have

$$\begin{aligned}\overline{x} &= \{y \in V \mid y \equiv x \mod W\} \\ &= \{y \in V \mid y = w + x \text{ for some } w \in W\} \\ &= \{w + x \mid w \in W\} = W + x = x + W\end{aligned}$$

**Example 4.1**

$\overline{0}_V = W + 0_V = W$.

Note: $W + x$ translates every element of $W$ by $x$. By 2), 3) of $\equiv \mod W$, we have check

$$y \in \overline{x} = W + x \iff x \in \overline{y} = W + y$$

and

$$x \equiv y \mod W \iff \overline{x} = \overline{y} \iff W + x = W + y$$

and check

$$\overline{x} \cap \overline{y} = \emptyset \iff (W + x) \cap (W + y) = \emptyset \iff x \not\equiv y \mod W$$

This means the $W + x$ partition $V$, i.e.,

$$V = \bigcup_V (W + x) \text{ with } (W + x) \cap (W + y) = \emptyset \text{ if } \overline{x} = (W + x) \neq (W + y) = \overline{y}$$

Let

$$\overline{V} := V/W := \{\overline{x} \mid x \in V\} = \{W + x \mid x \in V\}$$

a collection of subsets of $V$.

# §5 Lec 5: Apr 7, 2021

## §5.1 Quotient Spaces (Cont'd)

Suppose we have $W \subseteq V$ a subspace. For $x, y, z, v \in V$

$$\begin{aligned} x &\equiv y \mod W \\ z &\equiv v \mod W \end{aligned} \tag{+}$$

Then

$$(x+z)-(y+v) = \underbrace{(x-y)}_{\in W} + \underbrace{(z-v)}_{\in W} \in W$$

So

$$x+z \mod y+v \mod W$$

and if $\alpha \in F$

$$\alpha x - \alpha y = \alpha(x-y) \in W \quad \forall x, y \in V$$

So

$$\alpha x \equiv \alpha y \mod W$$

Therefore, $\overline{V} = V/W$. If (+) holds, then for all $x, y, z, v \in V$ and $\alpha \in F$, we have

$$\begin{aligned} \overline{x+z} &= \overline{y+v} \in \overline{V} \\ \overline{\alpha x} &= \overline{\alpha y} \in \overline{V} \end{aligned}$$

Notice $\overline{V} = V/W$ satisfies all the axioms of a vector space with $0_{\overline{V}} = \overline{0_V} = \{y \in V \,|\, y \equiv 0 \mod W\} = W + 0_V = W$.

We call $\overline{V} = V/W$ the QUOTIENT SPACE of $V$ by $W$.
We also have a map

$$- : V \to \overline{V} = V/W \text{ by } x \mapsto \overline{x} = W + x$$

which satisfies

$$\alpha v + v' \overset{-}{\mapsto} \overline{\alpha u + v'} = \alpha\overline{v} + \overline{v'}$$

for all $v, v' \in V$ and $\alpha \in F$. Then

$$\begin{aligned} \dim V &= \dim \ker^{-} \\ \dim V &= \dim W + \dim V/W \\ \dim V/W &= \dim V - \dim W \end{aligned}$$

which is called the <u>codimension</u> of $W$ in $V$.

**Proposition 5.1**

Let $V$ be a vector space over $F$, $W \subseteq V$ a subspace, $\overline{V} = V/W$. Let $\mathscr{B}_0$ be a basis for $W$ and

$$\mathscr{B}_1 = \{v_i | \, i \in I, \, v_i - v_j \notin W \text{ if } i \neq j\}$$

where $\overline{v_i} \neq \overline{v_j}$ if $i \neq j$ or $w + v_i \neq w + v_j$ if $i \neq j$.

Let

$$\mathscr{C} = \{\overline{v_i} = W + v_i | \, i \in I, \, v_i \in \mathscr{B}_1\}$$

If $\mathscr{C}$ is a basis for $\overline{V} = V/W$, then $\mathscr{B}_0 \cup \mathscr{B}_1$ is a basis for $V$ (compare with the proof of the Dimension Theorem).

*Proof.* Hw 2 # 3. □

## §5.2 Linear Transformation

A review of linear of linear transformation can be found here.
Now, we consider

$$GL_n F \coloneqq \{A \in \mathbb{M}_n F | \, \det A \neq 0\}$$

The elements in $GL_n F$ in the ring $\mathbb{M}_n F$ are those having a multiplicative inverse. If $R$ is a commutative ring, determinants are still as before but

$$\begin{aligned} GL_n R &\coloneqq \{A \in \mathbb{M}_n R | \, \det A \text{ is a unit in } R\} \\ &= \left\{A \in \mathbb{M}_n R | \, A^{-1} \text{ exists}\right\} \end{aligned}$$

**Example 5.2**

Let $V$ be a vector space over $F, W \subseteq V$ a subspace. Recall

$$\overline{V} = V/W = \{\overline{v} = W + v | \, v \in V\}$$

a vector space over $F$ s.t. for all $v_1, v_2 \in F$ and $\alpha \in F$

$$\begin{aligned} 0_{\overline{V}} &= \overline{0_V} = W \\ \overline{v_1} + \overline{v_2} &= \overline{v_1 + v_2} \\ \alpha \overline{v_1} &= \overline{\alpha v_1} \end{aligned}$$

Then

$$- : V \to V/W = \overline{V} \text{ by } v \mapsto \overline{v} = W + v$$

is an epimorphism with $\ker^- = W$.

Recall from 115A(H) that the most important theorem about linear transformation is Universal Property of Vector Spaces. As a result, we can deduce the following corollary

**Corollary 5.3**

Let $V, W$ be vector space over $F$ with bases $\mathscr{B}, \mathscr{C}$ respectively. Suppose there exists a bijection $f : \mathscr{B} \to \mathscr{C}$, i.e., $|\mathscr{B}| = |\mathscr{C}|$. Then $V \cong W$.

*Proof.* There exists a unique $T : V \to W \ni T\big|_{\mathscr{B}} = f$. $T$ is monic by the Monomorphism Theorem ($T$ takes linearly indep. sets to linearly indep. sets iff it's monic) and is onto as $W = \text{Span}(\mathscr{C}) = \text{Span}\left(f(\mathscr{B})\right)$. □

# §6 Lec 6: Apr 9, 2021

## §6.1 Linear Transformation (Cont'd)

**Theorem 6.1**

Let $T : V \to W$ be linear. Then $\exists X \subseteq V$ a subspace s.t.

$$V = \ker T \oplus X \text{ with } X \cong \text{im } T$$

*Proof.* Let $\mathscr{B}_0$ be a basis for $\ker T$. Extend $\mathscr{B}_0$ to a basis $\mathscr{B}$ for $V$ by the Extension Theorem. Let $\mathscr{B}_1 = \mathscr{B} \setminus \mathscr{B}_0$, so $\mathscr{B} = \mathscr{B}_0 \vee \mathscr{B}_1$ ($\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$ and $\mathscr{B}_0 \cap \mathscr{B}_1 = \emptyset$) and let

$$X = \bigoplus_{\mathscr{B}_1} Fv$$

As $\ker T = \bigoplus_{\mathscr{B}_0} Fv$, we have

$$V = \ker T \oplus X$$

and we have to show

$$X \cong \text{im } T$$

**Claim 6.1.** $Tv$, $v \in \mathscr{B}_1$ are linearly indep.

In particular, $Tv \neq Tv'$ if $v, v' \in \mathscr{B}_1$ and $v \neq v'$. Suppose

$$\sum_{v \in \mathscr{B}} \alpha_v Tv = 0_W, \quad \alpha_v \in F \text{ almost all } \alpha_v = 0$$

Then

$$0_W = T\left(\sum_{v \in \mathscr{B}_1} \alpha_v v\right), \quad \text{i.e. } \sum_{\mathscr{B}_1} \alpha_v v \in \ker T$$

Hence

$$\sum_{\mathscr{B}_1} \alpha_v v = \sum_{\mathscr{B}_0} \beta_v v \in \ker T \text{ almost all } \beta_v \in F = 0$$

As $\sum_{\mathscr{B}_1} \alpha_v v - \sum_{\mathscr{B}_0} \beta_v v = 0$ and $\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$ is linearly indep., $\alpha_v = 0 \, \forall v$. This proves the above claim.

Let $\mathscr{C} = \{Tv | \, v \in \mathscr{B}_1\}$. By the claim

$$\mathscr{B}_1 \to \mathscr{C} \text{ by } v \mapsto Tv \text{ is } 1-1$$

and onto as $\mathscr{C}$ is linearly indep. Lastly, we must show $\mathscr{C}$ spans im $T$. Let $w \in \text{im } T$. Then $\exists x \in V \ni Tx = w$. Then

$$\begin{aligned} w = Tx &= T\left(\sum_{\mathscr{B}_0} \alpha_v v\right) + T\left(\sum_{\mathscr{B}_1} \alpha_v v\right) \\ &= \sum_{\mathscr{B}_0} \alpha_v Tv + \sum_{\mathscr{B}_1} \alpha_v Tv = \sum_{\mathscr{B}_1} \alpha_v Tv \end{aligned}$$

lies in span $\mathscr{C}$ as needed. $\square$

**Remark 6.2.** Note that the proof is essentially the same as the proof of the Dimension Theorem.

**Corollary 6.3** (Dimension Theorem)
If $V$ is a finite dimensional vector space over $F$, $T : V \to W$ linear then

$$\dim V = \dim \ker T + \dim \ \operatorname{im} T$$

**Corollary 6.4**
If $V$ is a finite dimensional vector space over $F$, $W \subseteq V$ a subspace, then

$$\dim V = \dim W + \dim V/W$$

*Proof.* $- : V \to V/W$ by $v \mapsto \overline{v} = W + v$ is an epi. □

Important Construction: Set

$$\begin{aligned} T : V \to Z &\text{ be linear} \\ W &= \ker T \\ \overline{V} &= V/W \\ - : V \to V/W &\text{ by } v \mapsto \overline{v} = W + v \text{ linear} \end{aligned}$$

$\forall x, y \in V$ we have

$$\overline{x} = \overline{y} \in \overline{V} \iff x \equiv y \mod W \iff x - y \in W \iff T(x-y) = 0_Z$$

i.e., when $W = \ker T$

$$\overline{x} = \overline{y} \iff Tx = Ty \tag{*}$$

This means

$$\overline{T} : \overline{V} \to Z \text{ defined by } W + v = \overline{v} \mapsto Tv$$

is well-defined, i.e., via function, since if $\overline{x} = \overline{y}$, then $\overline{T}(\overline{x}) := Tx = Ty =: \overline{T}(\overline{y})$. From (*),

$$\overline{x} = \overline{y} \iff \overline{T}(\overline{x}) = T(x) = T(y) =: \overline{T}(\overline{y})$$

so

$$\overline{T} : \overline{V} \to Z \text{ is also injective}$$

As $\overline{T}$ is linear, let $\alpha \in F$, $x, y \in V$, then

$$\begin{aligned} \overline{T}\left(\alpha\overline{x} + \overline{y}\right) &= \overline{T}\left(\overline{\alpha x + y}\right) = T(\alpha x + y) \\ &= \alpha Tx + Ty = \alpha\overline{T}(\overline{x}) + \overline{T}(\overline{y}) \end{aligned}$$

as needed. Therefore,

$$\overline{T} : \overline{V} \to Z \text{ by } \overline{x} \mapsto T(x)$$

is a monomorphism, so induces an isomorphism onto im $\overline{T}$ and we recall im $\overline{T}$ = im $T$, so

$$\overline{V} \cong \text{im } \overline{T} = \text{im } T$$

and we have a commutative diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "-"'] & Z \\ V/\ker T = \overline{V} \arrow[ur, "\overline{T}"'] & \end{tikzcd}$$

This can also be written as

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "-"'] & Z \\ V/\ker T = \overline{V} \arrow[r, "\overline{T}"] & im\, T \arrow[u, bend right, "\text{inclusion map}"'] \end{tikzcd}$$

<u>Consequence</u>: Any linear transformation $T : V \to Z$ induces an isomorphism

$$\overline{T} : V/\ker T \to \text{im } T \text{ by } \overline{v} = \ker T + v \mapsto Tv$$

This is called the First Isomorphism Theorem. We also have

$$V = \ker T \oplus X \text{ with } X \subseteq V \text{ and } X \cong \text{im } T \cong V/\ker T$$

This means that all images of linear transformations from $V$ are determined, up to isomorphism, by $V$ and its subspaces. It also means, if $V$ is a finite dimensional vector space over $F$, we can try prove things by induction.

## §6.2 Projections

<u>Motivation</u>: Let $m < n$ in $\mathbb{Z}^+$ and

$$\pi : \mathbb{R}^n \to \mathbb{R}^n \text{ by } (\alpha_1, \dots, \alpha_n) \mapsto (\alpha_1, \dots, \alpha_n, 0, \dots, 0)$$

a linear operator onto $\bigoplus_{i=1}^m \Gamma e_i$ where $e_i = \left(0, \dots, \underbrace{1}_{i^m}, \dots, 0\right)$.

**Definition 6.5** (T-invariant) — Let $T : V \to V$ be linear, $W \subseteq V$ a subspace. We say $W$ is $T$-invariant if $T(W) \subseteq V$ if this is the case, then the restriction $T\big|_W$ of $T$ can be viewed as a linear operator

$$T\big|_W : W \to W$$

**Example 6.6**

Let $T : V \to V$ be linear.

1. $\ker T$ and $\operatorname{im} T$ are $T$-invariant.

2. Let $\lambda \in F$ be an eigenvalue of $T$, i.e., $\exists 0 \neq v \in V \ni Tv = \lambda v$, then any subspace of the eigenspace
$$E_T(\lambda) \coloneqq \{v \in V \,|\, Tv = \lambda v\}$$
is $T$-invariant as $T\big|_{E_T(\lambda)} = \lambda 1_{E_T(\lambda)}$

**Remark 6.7.** Let $V$ be a finite dimensional vector space over $F$, $T : V \to V$ linear. Suppose that
$$V = W_1 \oplus \ldots \oplus W_n$$
with each $W_i$ $T$-invariant, $i = 1, \ldots, n$ and $\mathscr{B}_i$ an ordered basis for $W_i$, $i = 1, \ldots, n$. Let $\mathscr{B} = \mathscr{B}_1 \cup \ldots \cup \mathscr{B}_n$ be a basis of $V$ ordered in the obvious way.
Then the matrix representation of $T$ in the $\mathscr{B}$ basis is
$$[T]_{\mathscr{B}} = \begin{pmatrix} \left[T\big|_{W_1}\right]_{\mathscr{B}_1} & & 0 \\ & \ddots & \\ 0 & & \left[T\big|_{W_n}\right]_{\mathscr{B}_n} \end{pmatrix}$$

**Example 6.8**

Suppose that $T : V \to V$ is diagonalizable, i.e., there exists a basis $\mathscr{B}$ of eigenvectors of $T$ for $V$. Then, $T : V \to V$,
$$V = \bigoplus E_T(\lambda_i)$$
each $E_T(\lambda_i)$ is $T$-invariant.
$$T\big|_{E_T(\lambda_i)} = \lambda_i 1_{E_T(\lambda_i)}$$

Goal: Let $V$ be a finite dimensional vector space over $F$, $n = \dim V$, $T : V \to V$ linear. Then $\exists W_1, \ldots, W_m \subseteq V$ all $T$-invariant subspaces with $m = m(T)$ with each $W_i$ being as small as possible with $V = W_1 \oplus \ldots \oplus W_m$. This is the theory of canonical forms.
Recall: If $V$ is a finite dimensional vector space over $F$, $T : V \to V$ linear, $\mathscr{B}$ an ordered basis for $V$, then the matrix representation $[T]_{\mathscr{B}}$ is only unique up to similarity, i.e., if $\mathscr{C}$ is an another ordered basis
$$[T]_{\mathscr{C}} = P\,[T]_{\mathscr{B}}\,P^{-1}$$
where $P = [1_V]_{\mathscr{B},\mathscr{C}} \in GL_nF$, the change of basis matrix $\mathscr{B} \to \mathscr{C}$.

**Definition 6.9** (Projection) — Let $V$ be a vector space over $F$, $P : V \to V$ linear. We call $P$ a projection if $P^2 = P \circ P = P$.

**Example 6.10**
1. $P = 0_V$ or $1_V : V \to V$, $V$ is a vector space over $F$.
2. An orthogonal projection in 115A.
3. If $P$ is a projection, so is $1_V - P$.

If $T : V \to V$ is linear, then

$$V = \ker T \oplus X \text{ with } X \cong \text{im } T$$

**Lemma 6.11**

Let $P : V \to V$ be a projection. Then

$$V = \ker P \oplus \text{im } P$$

Moreover, if $v \in \text{im } P$, then

$$Pv = v$$

i.e.

$$P\big|_{\text{im } P} : \text{im } P \to \text{im } P \text{ is } 1_{\text{im } P}$$

In particular, if $V$ is a finite dimensional vector space over $F, \mathscr{B}_1$ an ordered basis for $\ker P$, $\mathscr{B}_2$ an ordered basis for im $P$, then $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is an ordered basis for $V$ and

$$[P]_{\mathscr{B}} = \begin{pmatrix} [0]_{\mathscr{B}_1} & 0 \\ 0 & [1_{\text{im } P}]_{\mathscr{B}_2} \end{pmatrix} = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

*Proof.* Let $v \in V$, then $v - Pv \in \ker P$, since

$$P(v - Pv) = Pv - P^2 v = Pv - Pv = 0$$

Hence

$$v = (v - Pv) + Pv \in \ker P + \text{im } P$$

$\ker P \cap \text{im } P = 0$ and $P\big|_{\text{im } P} = 1_{\text{im } P}$. Let $v \in$ im P. By definition, $Pw = v$ for some $w \in V$. Therefore,

$$Pv = PPw = Pw = v$$

Hence

$$P\big|_{\text{im } P} = 1_{\text{im } P}$$

If $v \in \ker P \cap \text{im } P$, then

$$v = Pv = 0$$

□

# §7 Lec 7: Apr 12, 2021

## §7.1 Projection (Cont'd)

**Lemma 7.1**

Let $V$ be a vector space over $F$, $W, X \subseteq V$ subspaces. Suppose

$$V = W \oplus X$$

Then $\exists!\, P : V \to V$ a projection satisfying

$$\begin{aligned} W &= \ker P \\ X &= \text{im } P \end{aligned} \tag{*}$$

We say such a $P$ is the projection along $W$ onto $X$.

*Proof.* Existence: Let $v \in V$. Then

$$\exists! w \in W,\ x \in X \ni\ v = w + x$$

Define

$$P : V \to V \text{ by } v \mapsto x$$

To show $P^2 = P$ , we suppose $v \in V$ satisfies $v = w + x$, for unique $w \in W$, $x \in X$. Then

check $P$ is linear and well defined

$$Pv = Pw + Px = Px = 1_X x = x$$

so

$$P^2 v = Px = x = Pv \qquad \forall v \in V$$

hence $P^2 = P$.

Uniqueness: Any $P$ satisfying (*) takes a basis for $W$ to 0 and fix a basis of $X$. Therefore, $P$ is unique by the UPVS. ◻

> **Remark 7.2.** Compare the above to the case that $V$ is an inner product space over $F$, $W \subseteq V$ is a finite dimensional subspace and $P : V \to V$ by $v \mapsto v_W$, the orthogonal projection of $P$ onto $W$.

**Proposition 7.3**

Let $V$ be a vector space over $F$, $W, X \subseteq V$ subspaces s.t. $V = W \oplus X$, $P : V \to V$ the projection along $W$ onto $X$, and $T : V \to V$ linear. Then the following are equivalent:

1. $W$ and $X$ are both $T-$invariant.
2. $PT = TP$.

*Proof.* 2) $\implies$ 1) : $W$ is $T$-invariant: We have $W = \ker P$, so if $w \in W$, $Pw = 0$. Hence

$$PTw = TPw = T0 = 0$$

$Tw \in \ker P = W$ so $W$ is $T$-invariant.
$X$ is $T$-invariant, $X = \text{im } P$, $P|_X = 1_X$. So if $x \in X$

$$Tx = TPx = PTx \in \text{im } P = X$$

So $X$ is $T$-invariant.
1) $\implies$ 2) Let $v \in V$. Then $\exists!\, w \in W$, $x \in X$ s.t.

$$v = w + x$$

As $P|_X = 1_X$ and $P|_W = 0$, so $Pv = Px$. By 1), $W$ and $X$ are $T$-invariant, so

$$\begin{aligned} PTv &= PT(w+x) = PTw + PTx \\ &= 0 + Tx = TPx = TPw + TPx = TPv \end{aligned}$$

for all $v \in V$ and $PT = TP$. ☐

**Remark 7.4.** One can easily generalize from the case

$$V = W_1 \oplus W_2$$

that we did to the case

$$V = W_1 \oplus \ldots \oplus W_n$$

by induction on $n$ as

$$V = W_i \oplus \left( W_1 \oplus \ldots \oplus \underbrace{\hat{W}_i}_{\text{omit}} \oplus \ldots \oplus W_n \right)$$

<u>Construction</u>: Let

$$V = W_1 \oplus \ldots \oplus W_n$$

as above. Define

$$P_{W_i} : V \to V$$

to be the projection along $W_1 \oplus \ldots \oplus \hat{W}_i \oplus \ldots \oplus W_n$, i.e.

$$\ker P_{W_i} = W_1 \oplus \ldots \oplus \hat{W}_i \oplus \ldots \oplus W_n$$

and onto $W_i = \text{im } P_{W_i}$ as in the above Proposition. Then we have

a) Each $P_{W_i}$ is linear (and a projection).

b) $\ker P_{W_i} = W_1 \oplus \ldots \oplus \hat{W}_i \oplus \ldots \oplus W_n$.

c) $W_i$ is $P_{W_i}$-invariant and $P_{W_i}\big|_{W_i} = 1_{W_i}$. In particular, im $P_{W_i} = W_i$.

d) $P_{W_i} P_{W_j} = \delta_{ij} P_{W_i}$ where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

e) $1_V = P_{W_1} + \ldots + P_{W_n}$.

Moreover, if $T : V \to V$ is linear and each $W_i$ is $T$-invariant, then

$$TP_{W_i} = P_{W_i}T, \quad i = 1, \ldots, n$$

Hence

$$\begin{aligned} T = T1_V = T\left(P_{W_1} + \ldots P_{W_n}\right) &= TP_{W_1} + \ldots + TP_{W_n} \\ &= P_{W_1}T + \ldots + P_{W_n}T \end{aligned}$$

i.e., $1_V T = T 1_V$. This implies

$$T\big|_{W_i} : W_i \to W_i$$

is given by

$$T\big|_{W_i} = TP_{W_i}\big|_{W_i}$$

or $T$ is determined by what it does to each $W_i$.

> **Remark 7.5.** Compare this to the case that $T$ is diagonalizable and the $W_i$ are the eigenspaces.

**Question 7.1.** Let $V$ be a real or complex finite dimensional inner product space, $T : V \to V$ hermitian. What can you replace $\oplus$ by? What if $V$ is a complex finite dimensional inner product space and $T : V \to V$ is normal.

**Exercise 7.1.** Suppose $V$ is a vector space over $F$, $P_1, \ldots, P_n : V \to V$ linear and satisfy

i) $P_i - P_j = \delta_{ij} P_i$, $i = 1, \ldots, n$

ii) $1_V = P_1 + \ldots + P_n$

iii) $W_i = \text{im } P_i$, $i = 1, \ldots, n$

Then

$$\begin{aligned} V &= W_1 \oplus \ldots \oplus W_n \\ P_i &= P_{W_i} \qquad i = 1, \ldots, n \end{aligned}$$

## §7.2 Dual Spaces

**Question 7.2.** Let $V = \mathbb{R}^3$, $v \in V$. What is the first question that we should ask about $v$?

Motivation/Construction: Let $V$ be a vector space over $F$, $\mathscr{B}$ a basis for $V$. Fix $v_0 \in \mathscr{B}$. By the UPVS, $\exists!\, f_{v_0} : V \to F$ linear satisfying

$$f_{vv_0}(v) = \begin{cases} 1 & \text{if } v_0 = v \\ 0 & \text{if } v_0 \neq v \end{cases} = \delta_{v,v_0} \quad \forall v \in \mathscr{B}$$

**Example 7.6**

Let $\mathscr{E}_n = \{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$ and in the above $e_1 = v_0 \dots$ Then

$$f_{e_1} : \mathbb{R}^n \to \mathbb{R} \text{ satisfies}$$

If $v = (\alpha_1, \dots, \alpha_n)$ in $\mathbb{R}^n$

$$v = \sum_{i=1}^{n} \alpha_i e_i$$

so

$$\begin{aligned} f_{e_1}(v) &= f_{e_1}\left(\sum_{i=1}^{n} \alpha_i e_i\right) \\ &= \sum_{i=1}^{n} \alpha_i f_{e_1}(e_i) = \sum_{i=1}^{n} \alpha_i \delta_{ii} = \alpha_1 \end{aligned}$$

this first coordinate of $v$.

Notation: If $A \subseteq B$ are sets, we write $A < B$ if $A \neq B$.
As $v_0 \neq 0$,

$$0 < \operatorname{im} f_{v_0} \subseteq F \text{ is a subspace}$$

Notice $\dim_F F = 1$, so $\dim \operatorname{im} f_{v_0} \leq \dim F = 1$ and

$$\dim \operatorname{im} f_{v_0} = 1, \quad \text{i.e. } \operatorname{im} f_0 = F$$

So $f_{v_0} : V \to F$ is a surjective linear transformation. Since this is true for all $v_0 \in \mathscr{B}$, for each $v \in \mathscr{B}$, $\exists!\, f_v : V \to F$ s.t.

$$f_v(v') = \delta_{v,v'} = \begin{cases} 1 & \text{if } v = v' \\ 0 & \text{if } v \neq v' \end{cases} \quad \forall v' \in \mathscr{B}$$

Now suppose that $x \in V$, then

$$\exists!\, \alpha_v \in F,\ v \in \mathscr{B},\ \text{ almost all 0 s.t. } x = \sum_{\mathscr{B}} \alpha_v v$$

Hence

$$\begin{aligned} f_{v_0}(x) &= f_{v_0}\left(\sum_{v \in \mathscr{B}} \alpha_v v\right) = \sum_{\mathscr{B}} \alpha_v f_{v_0}(v) \\ &= \sum_{\mathscr{B}} \alpha_v \delta_{v,v_0} = \alpha_{v_0} \end{aligned}$$

**Example 7.7**

$\mathscr{B} = \mathscr{E}_n$ standard basis for $\mathbb{R}^n$

$$f_{e_i}(e_j) = \delta_{e_i,e_j} = \delta_{i,j} = \begin{cases} 1 & \text{if } e_i = e_j \\ 0 & \text{if } e_i \neq e_j \end{cases}$$

Then if $v = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n = V$. Then

$$f_{e_i}(v) = f_{e_i}(\alpha_1, \ldots, \alpha_n) = \alpha_i$$

So we observe in the above that if $x \in V$, then

$$x = \sum_{\mathscr{B}} f_v(x) v$$

We call $f_v$ the coordinate function on $v$ relative to $\mathscr{B}$.

**Example 7.8**

Let $V$ be a finite dimensional inner product space over $\mathbb{R}$, $\mathscr{B} = \{v_1, \ldots, v_n\}$ an orthonormal basis. Then if $x = \sum_{\mathscr{B}} \alpha_i v_i$, then

$$\alpha_i = \langle x, v_i \rangle$$

Take

$$\begin{aligned} \langle x, v_i \rangle &= \langle \sum \alpha_j v_j, v_i \rangle = \sum \alpha_j \langle v_j, v_i \rangle \\ &= \sum \alpha_j \delta_{ij} \|v_i\|^2 = \sum \alpha_j \delta_{ij} = \alpha_i \end{aligned}$$

i.e. the linear map

$$f_{v_i} \coloneqq \langle , v_i \rangle : V \to \mathbb{R} \text{ by } x \mapsto \langle x, v_i \rangle$$

is the coordinate function on vectors relative to $\mathscr{B}$.

**Definition 7.9** (Dual Space) — Let $V$ be a vector space over $F$. A linear transformation $f : V \to F$ is called a <u>linear functional</u>. Set

$$V^* \coloneqq L(V, F) \coloneqq \{f : V \to F | f \text{ is linear}\}$$

is called the <u>dual space</u> of $V$.

**Proposition 7.10**

Let $V, W$ be a vector space over $F$. Then

$$L(V,W) := \{T : V \to W | T \text{ linear}\}$$

is a vector space over $F$. Moreover, if $V, W$ are finite dimensional vector spaces over $F$

$$\dim L(V,W) = \dim V \dim W$$

In particular, if $V$ is a finite dimensional vector space over $F$, then so is $V^*$ and

$$\dim V = \dim V^*$$

so

$$V \cong V^*$$

*Proof.* 115A. □

**Example 7.11**

Let $V$ be a vector space over $F$. Then the following are linear functionals

1. $0 : V \to F$
2. Let $0 \neq v_0 \in V$ then $\{v_0\}$ is a basis for $Fv_0$. Therefore, $\{v_0\}$ extends to a basis $\mathscr{B}$ for $V$. Let $fv_0 \in V^*$ be the coordinate function for $V$ on $v_0$ relative to $\mathscr{B}$. Then $fv_0 \in \mathscr{B}^* := \{fv | v \in \mathscr{B}\}$.

# §8 Lec 8: Apr 14, 2021

## §8.1 Dual Spaces (Cont'd)

**Example 8.1** (Cont'd from Lec 7) 3. trace: $\mathbb{M}_n F \to F$ by

$$A \mapsto \sum_{i=1}^{n} A_{ii}$$

4. $\alpha < \beta \in \mathbb{R}$, then

$$I : C[\alpha, \beta] \to \mathbb{R} \text{ by } f \mapsto \int_{\alpha}^{\beta} f$$

5. Fix $\gamma \in [\alpha, \beta]$, $\alpha < \beta \in \mathbb{R}$. Then the evaluation map at $\gamma$

$$e_\gamma : C[\alpha, \beta] \to \mathbb{R} \text{ by } f \mapsto f(\gamma)$$

**Lemma 8.2**

Let $V$ be a vector space over $F$, $\mathscr{B}$ a basis for $V$,

$$\mathscr{B}^* := \{fv_0 : V \to F | \text{ coordinate function on } v_0 \text{ relative to } \mathscr{B}\}$$

so

$$fv_0(v) = \delta_{v_0,v} \qquad \forall v \in \mathscr{B}$$

the set of coordinate functions relative to $\mathscr{B}$. Then $\mathscr{B}^* \subseteq V^*$ is linearly indep.

*Proof.* Suppose

$$0 = 0_{V^*} = \sum_{v \in \mathscr{B}} \beta v f v, \quad \beta v \in F \text{ almost all } 0$$

We need to show $\beta v = 0 \, \forall v \in \mathscr{B}$. Evaluation at $v_0 \in \mathscr{B}$ yields

$$\begin{aligned} 0 = 0_{V^*}(v_0) &= \left(\sum_{\mathscr{B}} \beta v f v\right)(v_0) = \sum \beta v f v(v_0) \\ &= \sum_{\mathscr{B}} \beta v f_{v,v_0} = \beta v_0 \end{aligned}$$

So $\beta v = 0 \, \forall v \in \mathscr{B}$ and the lemma follows. □

**Corollary 8.3**

Let $V$ be a vector space over $F$ with basis $\mathscr{B}$. Then the linear transformation

$$D_{\mathscr{B}} : V \to V^* \text{ induced by } \mathscr{B} \to \mathscr{B}^* \text{ by } v \mapsto fv$$

is a monomorphism.
In particular, if $V$ is a finite dimensional vector space over $F$, then $\mathscr{B}^*$ is a basis for $V^*$ and

$$D_{\mathscr{B}} : V \to V^* \text{ is an isomorphism}$$

*Proof.* By the Monomorphism Theorem, $D_{\mathscr{B}}$ is monic in view of he lemma if $V$ is a finite dimensional vectors space over $F$, then

$$\dim V = \dim V^*$$

so $V \cong V^*$ by the Isomorphism Theorem. □

**Remark 8.4.**

1. If $V = \mathbb{R}_f^\infty := \{(\alpha_1, \alpha_2, \dots) \,|\alpha_i \in \mathbb{R} \text{ almost all } 0\}$, then by HW1 # 4,

$$D\mathscr{E}_\infty : V \to V^* \text{ is not an isomorphism}$$

2. $D_{\mathscr{B}} : V \to V^*$ in the corollary depends on $\mathscr{B}$. There exists no monomorphism $V \to V^*$ that does not depend on a choice of basis. However, there exists a "nice" monomorphism, i.e., defined independent of basis.

$$L : V \to (V^*)^* =: V^{**}$$

$V^{**}$ is called the double dual of $V$. We now construct it.

**Lemma 8.5**

Let $V$ be a vector space over $F$, $v \in V$. Then

$$L_v : V^* \to F \text{ by } f \mapsto L_v(f) := f(v)$$

the evaluation map at $v$ is linear, i.e.

$$L_v \in V^{**}$$

*Proof.* For all $f, g \in V^*$, $\alpha \in F$

$$L_v(\alpha f + g) = (\alpha f + g)(v) = \alpha f(v) + g(v) = \alpha L_v f + L_v g$$ □

**Theorem 8.6**

The "natural" map

$$L : V \to V^{**} \text{ by } v \mapsto L(v) := L_v$$

is a monomorphism.

*Proof.* $L$ is linear: Let $v, w \in V$, $\alpha \in F$. Then for all $f \in V^*$, as $V^{**} = (V^*)^*$

$$\begin{aligned} L(\alpha v + w)(f) &= L_{\alpha v + w}(f) = f(\alpha v + w) \\ &= \alpha f(v) + f(w) = \alpha L_v f + L_w f = (\alpha L_v + L_w)(f) \\ &= (\alpha L(v) + L(w))(f) \end{aligned}$$

So

$$L(\alpha v + w) = \alpha L(v) + L(w)$$

$L$ is monic. Suppose $v \neq 0$. To show $L_v = L(v) \neq 0$. By example 2,

$$\exists 0 \neq f \in V^* \ni f(v) \neq 0$$

So

$$L_v f = f(v) \neq 0$$

so $L_v = L(v) \neq 0$ and $L$ is monic. □

**Corollary 8.7**

If $V$ is a finite dimensional vector space over $F$, then $L : V \to V^{**}$ is a natural isomorphism.

*Proof.* $\dim V = \dim V^* = \dim V^{**}$ and the Isomorphism Theorem. □

<u>Identification</u>: Let $V$ be a finite dimensional vector space over $F$. Then $\forall v, w \in V$

1. $v = w \iff L_v = L_w$
2. $\forall f \in V^*\, f(v) = f(w) \iff L_v f = L_w f$

Moreover, if $W$ is also a finite dimensional vector space over $F$, then if $T : V \to W$ is linear, $\exists!\, \tilde{T} : V^{**} \to W^{**}$ linear and if $\tilde{T} : V^{**} \to W^{**}$ $\exists!\, T : V \to W$ linear. In other words, $V$ and $V^{**}$ can be identified by

$$v \leftrightarrow L_v$$

because

$$L_v(f) = f(v) \qquad \forall v \in V \quad \forall f \in V^*$$

<u>Construction</u>: Let $V$ be a finite dimensional vector space over $F$ with basis $\mathscr{B} = \{v_1, \ldots, v_n\}$. Then

$$\mathscr{B}^* := \{f_1, \ldots, f_n\}$$

defined by

$$f_i(v_j) = \delta_{ij} \qquad \forall i, j$$

i.e., $f_i$ is the coordinate function on $v_i$ relative to $\mathscr{B}$. Since

$$L_{v_i}(f_j) = f_j(v_i) = \delta_{ij} \qquad \forall i, j$$

$L_{v_i} \in V^{**}$

$$\mathscr{B}^{**} := \{L_{v_1}, \ldots, L_{v_n}\}$$

is the dual basis of $\mathscr{B}^*$ for $V^{**}$. So we have if $x = \sum_{i=1}^n \alpha_i v_i \in V$, $g = \sum_{i=1}^n \beta_i f_i \in V^*$.

$$x = \sum_{i=1}^n \alpha_i v_i = \sum_{i=1}^n f_i(x) v_i$$
$$g = \sum_{i=1}^n \beta_i f_i = \sum_{i=1}^n L_{v_i}(g) f_i = \sum_{i=1}^n g(v_i) f_i$$

i.e.

$$x = \sum_{i=1}^n f_i(x) v_i \qquad \forall x \in V$$
$$g = \sum_{i=1}^n g(v_i) f_i \qquad \forall g \in V^*$$

Motivation: Let $V$ be an inner product space over $\mathbb{R}$, $\emptyset \neq S \subseteq V$ a subset. What is $S^\perp$?
*Note*: $\forall v \in V$, $\langle , v\rangle : V \to \mathbb{R}$ by $x \mapsto \langle x, v\rangle$ is a linear functional. To generalize this to an arbitrary vector space over $F$, we define the following.

**Definition 8.8** (Annihilator) — Let $V$ be a vector space over $F$, $\emptyset \neq S \subseteq V$ a subset. Define the annihilator of $S$ to be

$$\begin{aligned} S^\circ &:= \{f \in V^* \mid f(x) = 0 \,\forall x \in S\} \\ &= \{f \in V^* \mid f|_S = 0\} \subseteq V^* \end{aligned}$$

**Remark 8.9.** Many people write $\langle v, f\rangle$ for $f(v)$ in the above even though $f \notin v$.

# §9 Lec 9: Apr 16, 2021

## §9.1 Dual Spaces (Cont'd)

**Lemma 9.1**

Let $V$ be a vector space over $F$, $\emptyset \neq S \subseteq V$ a subset. Then

1. $S^\circ \subseteq V^*$ is a subspace.
2. If $V$ is a finite dimensional vector space over $F$ and we identify $V$ as $V^{**}$ (by $v \leftrightarrow L_v$), then $S \subseteq S^{\circ\circ} := (S^\circ)^\circ$.

*Proof.* 1. For all $f, g \in S^\circ$, $\alpha \in F$, we have

$$(\alpha f + g)(x) = \alpha f(x) + g(x) = 0 \qquad \forall x \in S$$

Hence $\alpha f + g \in S^\circ$ and $S^\circ \subseteq V^*$ is a subspace.

2. Let $x \in S$. Then $\forall f \in S^\circ$, we have

$$0 = f(x) = L_x f, \quad \text{so } L_x \in (S^\circ)^\circ = S^{\circ\circ}$$

□

**Theorem 9.2**

Let $V$ be a finite dimensional vector space over $F$, $S \subseteq V$ a subspace. Then

$$\dim V = \dim S + \dim S^\circ$$

*Proof.* Let $\mathscr{B}_0 = \{v_1, \ldots, v_k\}$ be a basis for $S$. Extend this to

$$\begin{aligned} \mathscr{B} &= \{v_1, \ldots, v_n\} \text{ a basis for } V \\ \mathscr{B}_0 &= \{f_1, \ldots, f_n\} \text{ the dual basis of } \mathscr{B} \end{aligned}$$

**Claim 9.1.** $\mathscr{C} := \{f_{k+1}, \ldots, f_n\}$ is a basis for $S^\circ$.

If we show this, the theorem follows. Let $f \in S^\circ$. Then

$$\begin{aligned} f &= \sum_{i=1}^n L_{v_i}(f) f_i = \sum_{i=1}^n f(v_i) f_i \\ &= \sum_{i=1}^k f(v_i) f_i + \sum_{i=k+1}^n f(v_i) f_i = \sum_{i=k+1}^n f(v_i) f_i \end{aligned}$$

lies in span $\mathscr{C}$ so $\mathscr{C}$ spans. As $\mathscr{C} \subseteq \mathscr{B}^*$ which is linearly indep., so is $\mathscr{C}$. This proves the claim. □

**Corollary 9.3**

Let $V$ be a finite dimensional vector space over $F$, $S \subseteq V$ a subspace. Then $S = S^{\circ\circ}$.

*Proof.* As $S \subseteq S^{\circ\circ}$, it suffices to show $\dim S = \dim S^{\circ\circ}$. By the theorem, we have

$$\begin{aligned} \dim V &= \dim S + \dim S^{\circ} \\ \dim V^* &= \dim S^{\circ} + \dim S^{\circ\circ} \end{aligned}$$

where $\dim V = \dim V^*$. So $\dim S = \dim S^{\circ\circ}$. □

**Remark 9.4.** If $V$ is an inner product space over $\mathbb{R}$, compare all this to $\emptyset \neq S \subseteq V$ a subset and $S^{\perp}, S^{\perp\perp}$.

## §9.2 The Transpose

<u>Construction</u>: Fix $T : V \to W$ linear. For every $S : W \to X$, we have a composition

$$S \circ T : V \to X \text{ is linear}$$

So $T :\to W$ linear induces a map

$$T^{\star} : L(W, X) \to L(V, X)$$

by

$$S \mapsto S \circ T$$

**Proposition 9.5**

Let $V, W, X$ be vector spaces over $F$, $T : V \to W$ linear. Then

$$T^{\star} : L(W, X) \to L(V, X)$$

is linear.

*Proof.* Let $S_1, S_2 \in L(W, X)$, $\alpha \in F$. Then

$$\begin{aligned} T^{\star}(\alpha S_1 + S_2) &= (\alpha S_1 + S_2) \circ T \\ &= \alpha S_1 \circ T + S_2 \circ T = \alpha T^{\star} S_1 + T^{\star} S_2 \end{aligned}$$

□

**Corollary 9.6**

Let $T : V \to W$ be linear. Then

$$T^* : W^* \to V^* \text{ by } f \mapsto f \circ T$$

is linear.

*Proof.* Let $X = F$ in the proposition. □

**Definition 9.7** (Transpose) — Let $T : V \to W$ be linear. The linear map $T^\star : W^* \to V^*$ in the corollary is called the transpose of $T$ and denoted by $T^\top$.

Note: The transpose "turns thing around"

$$V \xrightarrow{T} W$$
$$V^* \xleftarrow{T^\top} W^*$$

**Lemma 9.8**
Let $T : V \to W$ be linear. Then

$$\ker T^\top = (\text{im T})^\circ \in W^*$$

*Proof.* $g \in \ker T^\top \iff T^\top g = 0 \iff (T^\top g)(v) = 0\ \forall v \in V \iff (g \circ T)(v) = 0\ \forall v \in V \iff g(Tv) = 0\ \forall v \in V \iff g \in (\text{im } T)^\circ$. □

**Theorem 9.9**
Let $V, W$ be finite dimensional vector space over $F$, $T : V \to W$ linear. Then

$$\dim \text{im } T = \dim \text{im } T^\top$$

*Proof.* Consider:

$$\dim W^* = \dim \ker T^\top + \dim \text{im } T^\top$$
$$\dim W = \dim \text{im } T + \dim (\text{im } T)^\circ$$

Notice that $\dim W^* = \dim W$. By the lemma, $\dim \text{im } T = \dim \text{im } T^\top$. □

Computation: Let $V, W$ be finite dimensional vector space over $F$.

$$\mathscr{B},\ \mathscr{B}^* \text{ ordered dual bases for } V,\ V^*$$
$$\mathscr{C},\ \mathscr{C}^* \text{ ordered dual bases for } W,\ W^*$$

Suppose

$$\mathscr{B} = \{v_1, \ldots, v_n\}, \quad \mathscr{B}^* = \{f_1, \ldots, f_n\}$$
$$f_i(v_j) = \delta_{ij} \quad \forall i, j$$

So

$$\mathscr{C} = \{w_1, \ldots, w_n\}, \quad \mathscr{C}^* = \{g_1, \ldots, g_n\}$$
$$g_i(w_j) = \delta_{ij} \quad \forall i, j$$

Let

$$A = [T]_{\mathscr{B},\mathscr{C}}, \qquad B = \left[T^\top\right]_{\mathscr{C}^*,\mathscr{B}^*}$$

be the matrix representation of $T, T^\top$ in the ordered bases $\mathscr{B},\mathscr{C}$ and $\mathscr{C}^*,\mathscr{C}^*$ respectively. By definition of $A$ and $B$, we have

$$Tv_k = \sum_{i=1}^m A_{ik} w_i \qquad k = 1,\ldots,n$$

$$T^\top g_j = \sum_{i=1}^n B_{ij} f_i \qquad j = 1,\ldots,m$$

So

$$B_{kj} = A_{jk} \qquad \forall j,k$$

So we just proved...

**Theorem 9.10**

Let $V, W$ be finite dimensional vector space over $F$, $T : V \to W$ linear, $\mathscr{B},\mathscr{B}^*$ ordered dual bases for $V, V^*$ and $\mathscr{C},\mathscr{C}^*$ ordered dual bases for $W, W^*$. Then

$$\left[T^\top\right]_{\mathscr{C}^*,\mathscr{B}^*} = \left([T]_{\mathscr{B},\mathscr{C}}\right)^\top$$

**Definition 9.11** (Row/Column Rank) — Let $A \in F^{m\times n}$. The row (column) rank of $A$ is the dimension of the span of the rows (columns) of $A$.

We know if $A \in F^{m\times n}$, we can view

$$A : F^{n\times 1} \to F^{m\times 1} \text{ by } v \mapsto A \cdot v$$

a linear transformation and the matrix representation of $A$ is

$$A = [A]_{\mathscr{E}_{n,1},\mathscr{E}_{m,1}}$$

where $\mathscr{E}_{n,1}$, $\mathscr{E}_{m,1}$ are the standard bases for $F^{n\times 1}$ and $F^{m\times 1}$ respectively.

**Corollary 9.12**

Let $A \in F^{m\times n}$. Then

$$\text{row rank } A = \text{column rank } A$$

and we call this common number the rank of $A$.

## §9.3 Polynomials

**Definition 9.13** (Polynomial Division) — Let $f, g \in F[t]$, $f \neq 0$. We say that $f$ divides $g \in F[t]$ write $f|g$ if $\exists h \in F[t]$ s.t. $g = fh$, i.e. $g$ is multiple of $f$, e.g. $t+1|t^2-1$.

**Lemma 9.14**

If $f|g$ and $f|h$ in $F[t]$, then $f|gk+hl$ in $F[t]$ for all $k, l \in F[t]$.

*Proof.* By definition,

$$g = fg_1, \quad h = fh_1, \quad g_1, h_1 \in F[t]$$

So

$$gk + hl = fg_1k + fh_1l = f(g_1k + h_1l)$$

in $F[t]$. □

**Remark 9.15.** If $f|g \in F[t]$ and $0 \neq a \in F$, then $af|g$ and $f|ag$.

**Definition 9.16** (Polynomial Degree and Leading Coefficient) — Let

$$0 \neq f = at^n + a_{n-1}t^{n-1} + \ldots + a_1t + a_0 \in F[t]$$

with $a, a_0, \ldots, a_{n-1} \in F$ and $a \neq 0$. We call $n$ the degree of $f$ write $\deg f = n$ and $a$ the leading coefficient of $F$ write lead $f = a$. If $a = 1$, we say $f$ is monic.

We can define the degree of $0 \in F[t]$ to be the symbol $-\infty$ or just do not define it at all.

**Remark 9.17.** Let $f, g \in F[t] \setminus \{0\}$. Then

$$\operatorname{lead}(fg) = \operatorname{lead}(f) + \operatorname{lead}(g) \neq 0 \in F$$

So

$$\deg fg = \deg f + \deg g$$