Math 131BH – Honors Real Analysis II

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This is math 131BH – Honors Real Analysis II, and it is instructed by Professor Visan. It's the second class in the undergrad real analysis sequence at UCLA. We meet weekly on MWF from 10:00 – 10:50 am for online lectures. Similar to 131AH, there are two textbooks associated to the course, *Principles of Mathematical Analysis* by *Rudin* and *Metric Spaces* by *Copson*. You can find the previous analysis lecture notes along with the other course notes through my github. Please email me if you notice any significant mathematical errors/typos that needs to be addressed. Thank you, and I hope you find this helpful for your study!

Contents

List of Theorems

List of Definitions

§1 Lec 1: Mar 29, 2021

§1.1 Compactness

Definition 1.1 (Open Cover) — Let (X, d) be a metric space and let $A \subseteq X$. An open cover of A is a family $\{G_i\}_{i \in I}$ of open sets in X such that

$$A \subseteq \bigcup_{i \in I} G_i$$

The open cover is called finite if the cardinality of I is finite. If it's not finite, the open cover is called infinite.

Definition 1.2 (Compactness & Precompactness) — Let (X, d) be a metric space and let $K \subseteq X$.

1. We say that K is a compact set if every open cover $\{G_i\}_{i \in I}$ of K admits a finite subcover, that is,

$$\exists n \geq 1 \text{ and } \exists i_1, \ldots, i_n \in I \text{ s.t. } K \subseteq \bigcup_{j=1} G_{i_j}$$

2. We say that a set $A \subseteq X$ is precompact if $\overline{A}$ is compact.

Lemma 1.3

Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$. We equip Y with the induced metric $d_1 : Y \times Y \to \mathbb{R}$, $d_1(y_1, y_2) = d(y_1, y_2)$. Let $K \subseteq Y \subseteq X$. The followings are equivalent:

1. K is compact in (X, d).
2. K is compact in (Y, d_1).

Proof. 1) $\implies$ 2) Assume K is compact in (X, d). Let $\{V_i\}_{i \in I}$ be a family of open sets in (Y, d_1) s.t.

$$K \subseteq \bigcup_{i \in I} V_i$$

For $i \in I$ fixed, V_i is open in $(Y, d_1) \implies \exists G_i \subseteq X$ open in (X, d) s.t.

$$V_i = G_i \cap Y$$

Then

$$\left.\begin{array}{l} K \subseteq \bigcup_{i\in I} V_i \subseteq \bigcup_{i\in I} G_i \\ K \text{ compact in } (X,d) \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists i_1,\dots,i_n \in I \text{ s.t.}$$

$$\left.\begin{array}{l} K \subseteq \bigcup_{j=1}^n G_{i_j} \\ K \subseteq Y \end{array}\right\} \implies K \subseteq \left(\bigcup_{j=1}^n G_{i_j}\right) \cap Y = \bigcup_{j=1}^n \left(G_{i_j} \cap Y\right) = \bigcup_{j=1}^n V_{i_j}$$

So K is compact in (Y,d_1).
2) $\implies$ 1) Assume K is compact in (Y,d_1). Let $\{G_i\}_{i\in I}$ be a family of open sets in (X,d) s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{i\in I} G_i \\ K \subseteq Y \end{array}\right\} \implies \left.\begin{array}{l} K \subseteq \left(\bigcup_{i\in I} G_i\right) \cap Y = \bigcup_{i\in I} \underbrace{(G_i \cap Y)}_{\text{open in } Y} \\ K \text{ is compact in } (Y,d_1) \end{array}\right\} \implies$$

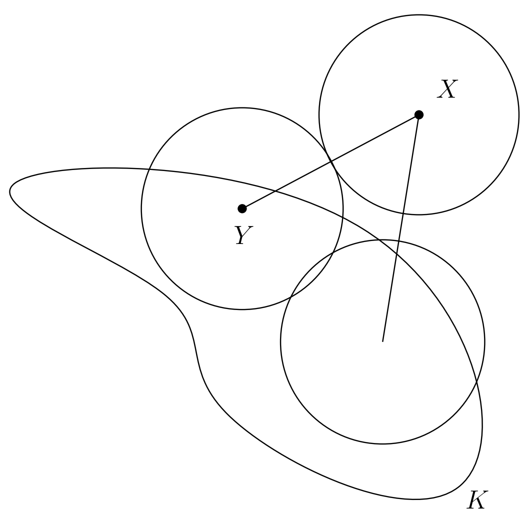
$\implies \exists n \geq 1$ and $\exists i_1,\dots,i_n \in I$ s.t. $K \subseteq \bigcup_{j=1}^n \left(G_{i_j} \cap Y\right) \subseteq \bigcup_{j=1}^n G_{i_j}$. $\square$

Proposition 1.4

Let (X,d) be a metric space and let $K \subseteq X$ be compact. Then K is closed and bounded.

Proof. Let's prove K is closed. We'll show ${}^cK$ is open. **Case 1:** ${}^cK = \emptyset$. This is open.
Case 2: ${}^cK \neq \emptyset$. Let $x \in {}^cK$

For $y \in K$ let $r_y = \frac{d(x,y)}{2}$. Note $r_y > 0$ (since $x \in {}^cK$ and $y \in K$).



Note

$$\left.\begin{array}{l} K \subseteq \bigcup_{y\in K} \underbrace{B_{r_y}(y)}_{\text{open}} \\ K \text{ is compact} \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists y_1,\dots,y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^n B_{r_j}(y_j)$$

where we use the shorthand $r_j = r_{y_j}$.
Let $r = \min_{1\le j\le n} r_j > 0$.
By construction, $B_r(x) \cap B_{r_j}(y_j) = \emptyset \quad \forall 1 \le j \le n$.

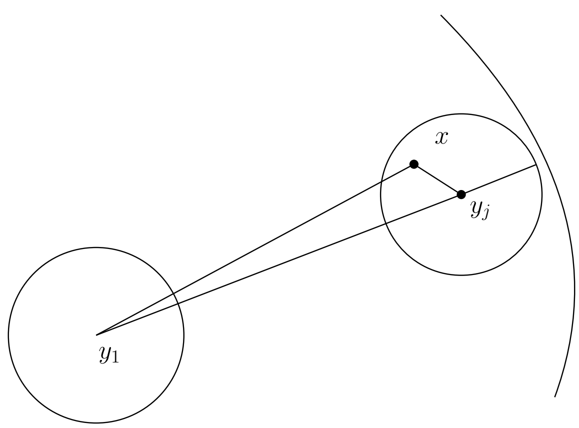
$$\begin{aligned}
&\implies B_r(x) \subseteq {}^cB_{r_j}(y_j) \quad \forall 1 \le j \le n \\
&\implies B_r(x) \subseteq \bigcap_{j=1}^{n} {}^cB_{r_j}(y_j) = {}^c\left(\bigcup_{j=1}^{n} B_{r_j}(y_j)\right) \subseteq {}^cK \\
&\implies \left.\begin{array}{l} x \in \widehat{{}^cK}^{\circ} \\ x \in {}^cK \text{ was arbitrary} \end{array}\right\} \implies {}^cK = \widehat{{}^cK}^{\circ}
\end{aligned}$$

Let's show K is bounded. Note

$$\left.\begin{array}{l} K \subseteq \bigcup_{y\in K} \underbrace{B_1(y)}_{\text{open}} \\ K \text{ compact} \end{array}\right\} \implies \exists n \ge 1 \text{ and } \exists y_1, \dots, y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^{n} B_1(y_j)$$

For $2 \le j \le n$, let $r_j = d(y_1, y_j) + 1$.

Claim 1.1. $B_1(y_j) \subseteq B_{r_j}(y_1)$



Indeed, if $x \in B_1(y_j) \implies d(x, y_j) < 1$. By the triangle inequality

$$d(y_1, x) \le d(y_j, x) + d(y_1, y_j) < 1 + d(y_1, y_j) = r_j \implies x \in B_{r_j}(y_1)$$

So with $r = \max_{2\le j\le n} r_j$,

$$K \subseteq \bigcup_{j=1}^{n} B_1(y_j) \subseteq B_r(y_1)$$

$\square$

Proposition 1.5

Let (X, d) be a metric space and let $F \subseteq K \subseteq X$ such that F is closed in X and K is compact. Then F is compact.

Proof. Let $\{G_i\}_{i\in I}$ be a family of open sets in X s.t.

$$F \subseteq \bigcup_{i\in I} G_i$$

Then

$$\left.\begin{array}{l} K \subseteq F \cup {}^cF \subseteq \bigcup_{i\in I} G_i \cup \underbrace{{}^cF}_{\text{open in } X} \\ K \text{ compact} \end{array}\right\} \implies$$

$\implies \exists n \geq 1$ and $\exists i_1, \ldots, i_n \in I$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j=1}^n G_{i_j} \cup {}^cF \\ F \subseteq K \end{array}\right\} \implies F = \left(\bigcup_{j=1}^n G_{i_j} \cup {}^cF\right) \cap F \subseteq \bigcup_{j=1}^n G_{i_j}$$

So F is compact. □

Corollary 1.6

Let (X, d) be a metric space and let $F \subseteq X$ be closed and let $K \subseteq X$ be compact. Then $K \cap F$ is compact.

Proof. K is compact. So

$$\left.\begin{array}{l} K \text{ closed} \\ F \text{ closed} \end{array}\right\} \implies \left.\begin{array}{l} K \cap F \text{ is closed} \\ K \cap F \subseteq K \text{ compact} \end{array}\right\} \implies K \cap F \text{ is compact}$$

□

§1.2 Sequential Compactness

Definition 1.7 (Sequential Compactness) — Let (X, d) be a metric space. A set $K \subseteq X$ is called sequentially compact if every sequence $\{x_n\}_{n\geq 1} \subseteq K$ admits a subsequence that converges in K.

§2 Lec 2: Mar 31, 2021

§2.1 Sequential Compactness (Cont'd)

Theorem 2.1 (Bolzano – Weierstrass)
Let (X, d) be a metric space and let $K \subseteq X$ be infinite. The following are equivalent:

1. K is sequentially compact.
2. For every infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

Proof. 1) $\implies$ 2) Let $A \subseteq K$ be infinite. As every infinite set has a countable subset we can find a sequence $\{a_n\}_{n\geq 1} \subseteq A$ such that $a_n \neq a_m \, \forall n \neq m$. As K is sequentially compact, $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

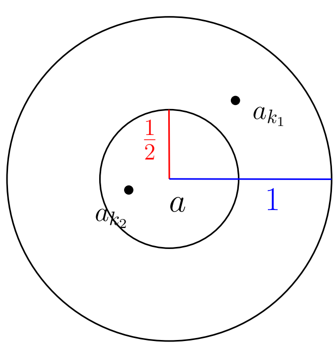
$$a_{k_n} \xrightarrow[n\to\infty]{d} a \in K$$

Claim 2.1. $a \in A' \iff \forall r > 0 \; B_r(a) \cap A \setminus \{a\} \neq \emptyset$.

Indeed, fix $r > 0$.

$$a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_r \in \mathbb{N} \text{ s.t. } d(a, a_{k_n}) < r \quad \forall n \geq n_r$$

As $a_n \neq a_m \, \forall n \neq m$, $\exists n_0 \geq n_r$ s.t. $a_{k_{n_0}} \neq a$. Then $a_{k_{n_0}} \in B_r(a) \cap A \setminus \{a\}$. We get $a \in A' \cap K$.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. We distinguish two cases:
Case 1: The sequence $\{a_n\}_{n\geq 1}$ contains a constant subsequence. That subsequence converges to an element in K.
Case 2: $\{a_n\}_{n\geq 1}$ does not contain a constant subsequence. Then $A = \{a_n : n \geq 1\}$ is infinite and $A \subseteq K$. So $A' \cap K \neq \emptyset$. Let $a \in A' \cap K$. Then $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t. $a_{k_n} \xrightarrow[n\to\infty]{d} a$.



$B_1(a) \cap A \setminus \{a\} \neq \emptyset$

$B_{\min\left\{\frac{1}{2}, d(a, a_{k_1})\right\}}(a) \cap A \setminus \{a\} \neq \emptyset$

$k_2 > k_1$

$\square$

Theorem 2.2

Let (X, d) be a metric space and let $K \subseteq X$ be compact. Then K is sequentially compact.

Proof. If K is finite, then any sequence $\{x_n\}_{n\geq 1} \subseteq K$ will have a constant subsequence.

Assume now K is infinite. We will use the Bolzano – Weierstrass theorem. It suffices to prove that for any infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

$$\left.\begin{array}{l} \text{Note } A \subseteq K \text{ then } A' \subseteq K' \\ K \text{ compact} \implies K \text{ closed} \implies K' \subseteq K \end{array}\right\} \implies A' \subseteq K \implies A' \cap K = A'$$

We argue by contradiction. Assume $A' = \emptyset$. Then for $x \in K$ we have $x \notin A' \implies \exists r_x > 0$ s.t. $B_{r_x}(x) \cap A \setminus \{x\} = \emptyset$. So

$$\left.\begin{array}{l} K \subseteq \bigcup_{x\in K} \underbrace{B_{r_x}(x)}_{\text{open}} \\ K \text{ compact} \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists x_1, \ldots, x_n \in K \text{ s.t.}$$

$$K \subseteq \bigcup_{j=1}^{n} B_{r_j}(x_j) \text{ where } r_j = r_{x_j}$$

In particular,

$$\left.\begin{array}{l} A = \left(\bigcup_{j=1}^{n} B_{r_j}(x_j)\right) \cap A = \bigcup_{j=1}^{n} \left[B_{r_j}(x_j) \cap A\right] \\ \text{By construction, } B_{r_j}(x_j) \cap A \subseteq \{x_j\} \end{array}\right\} \implies \underbrace{A}_{\text{infinite}} \subseteq \underbrace{\bigcup_{j=1}^{n} \{x_j\}}_{\text{finite}}$$

– Contradiction! So $A' \neq \emptyset$. $\square$

Proposition 2.3

Let (X, d) be a metric space and let $K \subseteq X$ be sequentially compact. Then K is closed and bounded.

Proof. Let's show K is closed $\iff K = \overline{K}$.

We know $K \subseteq \overline{K}$. We need to show $\overline{K} \subseteq K$. Let $x \in \overline{K} \implies \exists \{x_n\}_{n\geq 1} \subseteq K$ s.t. $x_n \xrightarrow[n\to\infty]{d} x$.

K sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

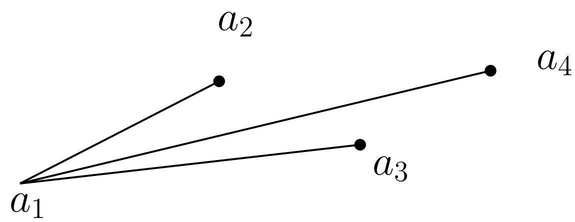
$$\left.\begin{array}{l} x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ x_n \xrightarrow[n\to\infty]{d} x \implies x_{k_n} \xrightarrow[n\to\infty]{d} x \\ \text{Limits of convergent sequences are unique} \end{array}\right\} \implies x = y \in K$$

As $x \in \overline{K}$ was arbitrary, we get $\overline{K} \subseteq K$.

Let's show K is bounded. We argue by contradiction. Assume K is not bounded. Let $a_1 \in K$.

$$\begin{aligned} K \text{ not bounded } &\implies K \not\subseteq B_1(a_1) \implies \exists a_2 \in K \text{ s.t. } d(a_1, a_2) \geq 1 \\ K \text{ not bounded } &\implies K \not\subseteq B_{1+d(a_1,a_2)}(a_1) \implies \exists a_3 \in K \text{ s.t. } d(a_1, a_3) \geq 1 + d(a_1, a_2) \end{aligned}$$

Proceeding inductively, we find a sequence $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_1, a_{n+1}) \geq 1 + d(a_1, a_n)$.



By construction,

$$|d(a_1, a_m) - d(a_1, a_n)| \geq |n - m| \quad \forall n, m \geq 1$$

By the triangle inequality,

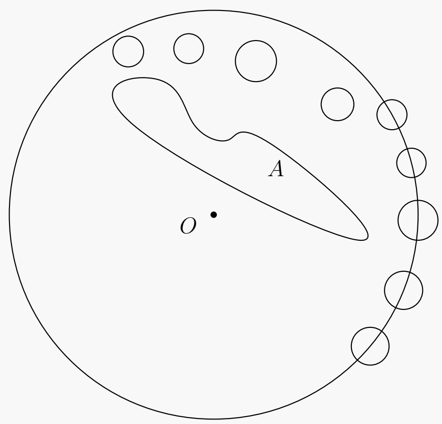
$$d(a_n, a_m) \geq |d(a_1, a_n) - d(a_1, a_m)| \geq |n - m| \quad \forall n, m \geq 1$$

This sequence cannot have a convergent (Cauchy) subsequence, thus contradiction the hypothesis that K is sequentially compact. So K is bounded. □

Definition 2.4 (Totally Bounded) — Let (X, d) be a metric space. A set $A \subseteq X$ is totally bounded if for every $\epsilon > 0$, A can be covered by finitely many balls of radius ϵ.

Remark 2.5.

1. A totally bounded $\implies$ A bounded.
   Indeed, taking $\epsilon = 1$, $\exists n \geq 1$ and $\exists x_1, \ldots, x_n \in X$ s.t.
   $$A \subseteq \bigcup_{j=1}^{n} B_1(x_j) \subseteq B_r(x_1)$$
   where $r = 1 + \max_{2 \leq j \leq n} d(x_1, x_j)$.
2. A bounded $\not\Longrightarrow$ A totally bounded.
   Consider $\mathbb{N}$ equipped with the discrete metric
   $$d(n,m) = \begin{cases} 0, \ n = m \\ 1, \ n \neq m \end{cases}$$
   Then $\mathbb{N} = B_2(1)$, but $\mathbb{N}$ cannot be covered by finitely many balls of radius $\frac{1}{2}$ since $B_{\frac{1}{2}}(n) = \{n\}$.
3. On $(\mathbb{R}^n, d_2)$, A bounded $\implies$ A totally bounded. Indeed, A bounded $\implies$ $A \subseteq B_R(0)$ for some $R > 0$. $B_R(0)$ can be covered by $10^6 \left(\frac{R}{\epsilon}\right)^n$ many balls of radius ϵ.



§3 Lec 3: Apr 2, 2021

§3.1 Heine – Borel Theorem

Theorem 3.1

Let (X, d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is sequentially compact.
2. K is complete and totally bounded.

Proof. 1) $\implies$ 2) Let's show K is complete. Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence with $x_n \in K \quad \forall n \geq 1$.
K sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$\left.\begin{array}{l} x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ \{x_n\}_{n\geq 1} \text{ is Cauchy} \end{array}\right\} \implies x_n \xrightarrow[n\to\infty]{d} y \in K$$

As $\{x_n\}_{n\geq 1} \subseteq K$ was arbitrary, we get that K is complete.
Let's show K is totally bounded. Fix $\epsilon > 0$ and $a_1 \in K$.

- If $K \subseteq B_\epsilon(a_1)$, then K is totally bounded.
- If $K \nsubseteq B_\epsilon(a_1)$, then $\exists a_2 \in K$ s.t. $d(a_1, a_2) \geq \epsilon$
- If $K \subseteq B_\epsilon(a_1) \cup B_\epsilon(a_2)$, then K is totally bounded.
- If $K \nsubseteq B_\epsilon(a_1) \cup B_\epsilon(a_2)$, then $\exists a_3 \in K$ s.t. $d(a_1, a_3) \geq \epsilon$ and $d(a_2, a_3) \geq \epsilon$.

We distinguish two cases:
Case 1: The process terminates in finitely many steps $\implies$ K is totally bounded.
Case 2: The process does not terminate in finitely many steps. Then we find $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_n, a_m) \geq \epsilon \quad \forall n \neq m$. This sequence does not admit a convergent subsequence, contradicting the fact that K is sequentially compact.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. K totally bounded $\implies$ $\mathcal{J}_1$ finite and $\left\{x_j^{(1)}\right\}_{j\in\mathcal{J}_1} \subseteq X$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j\in\mathcal{J}_1} B_1(x_j^{(1)}) \\ \{a_n\}_{n\geq 1} \subseteq K \end{array}\right\} \implies \exists j_1 \in \mathcal{J}_1 \text{ s.t. } \left|\left\{n : a_n \in B_1(x_{j_1}^{(1)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(1)}\right\}_{n\geq 1}$ be the corresponding subsequence.
K totally bounded $\implies \exists \mathcal{J}_2$ finite and $\left\{x_j^{(2)}\right\}_{j\in\mathcal{J}_2} \subseteq X$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j\in\mathcal{J}_2} B_{\frac{1}{2}}(x_j^{(2)}) \\ \left\{a_n^{(1)}\right\}_{n\geq 1} \subseteq K \end{array}\right\} \implies \exists j_2 \in \mathcal{J}_2 \text{ s.t. } \left|\left\{n : a_n^{(1)} \in B_{\frac{1}{2}}(x_{j_2}^{(2)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(2)}\right\}_{n\geq 1}$ denote the corresponding subsequence.
We proceed inductively. We find that $\forall k \geq 1$

- $\left\{a_n^{(k+1)}\right\}_{n\geq 1}$ subsequence of $\left\{a_n^{(k)}\right\}_{n\geq 1}$
- $\left\{a_n^{(k)}\right\}_{n\geq 1} \subseteq B_{\frac{1}{k}}\left(x_{j_k}^{(k)}\right)$ for some $x_{j_k}^{(k)} \in X$.

We consider the subsequence $\left\{a_n^{(n)}\right\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$.

$$\begin{aligned}
\left\{a_n^{(1)}\right\}_{n\geq 1} &= \left(a_1^{(1)},\quad a_2^{(1)},\quad a_3^{(1)},\quad \dots\right) \\
\left\{a_n^{(2)}\right\}_{n\geq 1} &= \left(\qquad a_1^{(2)},\quad a_2^{(2)},\quad a_3^{(2)},\quad \dots\right) \\
\left\{a_n^{(3)}\right\}_{n\geq 1} &= \left(\qquad\qquad a_1^{(3)},\quad a_2^{(3)},\qquad a_3^{(3)},\quad \dots\right)
\end{aligned}$$

For $n, m \geq k$ the $a_n^{(n)}, a_m^{(m)}$ belong to the subsequence $\left\{a_n^{(k)}\right\}_{n\geq 1}$. In particular,

$$d(a_n^{(n)}, a_m^{(m)}) \leq d(a_n^{(n)}, x_{j_k}^{(k)}) + d(a_m^{(m)}, x_{j_k}^{(k)}) < \frac{2}{k} \quad \forall n, m \geq k$$

This shows $\left\{a_n^{(n)}\right\}_{n\geq 1}$ is Cauchy and K is complete, so $a_n^{(n)} \xrightarrow[n\to\infty]{d} a \in K$. As $\{a_n\}_{n\geq 1}$ was arbitrary, we get that K is sequentially compact. $\square$

Lemma 3.2

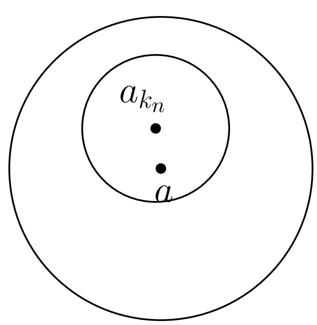
Let (X, d) be a sequentially compact metric space. Let $\{G_i\}_{i\in I}$ be an open cover of X. Then there exists $\epsilon > 0$ such that every ball of radius ϵ is contained in at least one G_i.

Proof. We argue by contradiction. Then

$$\forall n \geq 1 \quad \exists a_n \in X \text{ s.t. } B_{\frac{1}{n}}(a_n) \text{ is not contained in any } G_i$$

X is sequentially compact $\implies \exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

$$\begin{gathered}
a_{k_n} \xrightarrow[n\to\infty]{d} a \in X = \bigcup_{i\in I} G_i \implies \exists i_0 \in I \text{ s.t. } a \in G_{i_0} \\
G_{i_0} \text{ open} \implies \exists r > 0 \text{ s.t. } B_r(a) \subseteq G_{i_0} \\
a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_1(r) \in \mathbb{N} \text{ s.t. } d(a_1, a_{k_n}) < \frac{r}{2} \forall n \geq n_1
\end{gathered}$$



Let $n_2(r)$ s.t. $n_2 > \frac{2}{r}$.

Claim 3.1. $\forall n \geq n_r = \max\{n_1, n_2\}$ we have $B_{\frac{1}{k_n}}(a_{k_n}) \subseteq B_r(a) \subseteq G_{i_0}$ thefore giving a contradiction!

Fix $x \in B_{\frac{1}{k_n}}(a_{k_n})$. Then

$$d(a, x) \leq d(x, a_{k_n}) + d(a_{k_n}, a) < \frac{1}{k_n} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r$$

□

Theorem 3.3

A sequentially compact metric space (X, d) is compact.

Proof. Let $\{G_i\}_{i \in I}$ be an open cover of X. Let ϵ be given by the previous lemma. X sequentially compact $\implies$ X totally bounded $\implies$ $\exists n \geq 1$ and

$$\left.\begin{array}{l} \exists x_1, \ldots, x_n \in X \text{ s.t. } X = \bigcup_{j=1}^n B_\epsilon(x_j) \\ \forall 1 \leq j \leq n \quad \exists i_j \in I \text{ s.t. } B_\epsilon(x_j) \subseteq G_{i_j} \end{array}\right\} \implies X = \bigcup_{j=1}^n G_{i_j}$$

□

Collecting our results so far we obtain

Theorem 3.4 (Heine – Borel)

Let (X, d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is compact,
2. K is sequentially compact,
3. K is complete and totally bounded,
4. Every infinite subset of K has an accumulation point in K.

Remark 3.5. In $\mathbb{R}^n$, K is compact $\iff$ K is closed and bounded.

Definition 3.6 (Finite Intersection Property) — An infinite family $\{F_i\}_{i \in I}$ of closed sets is said to have the finite intersection property if $\forall \mathcal{J} \subseteq I$ finite we have

$$\bigcap_{j \in \mathcal{J}} F_j \neq \emptyset$$

Theorem 3.7

A metric space (X,d) is compact if and only if every infinite family $\{F_i\}_{i\in I}$ of closed sets with the finite intersection property satisfies

$$\bigcap_{i\in I} F_i \neq \emptyset$$

Proof. " $\implies$ " We argue by contradiction. Assume $\exists \{F_i\}_{i\in I}$ closed sets with the finite intersection property s.t. $\bigcap_{i\in I} F_i = \emptyset$

$$\left.\begin{array}{l} X = {}^c\left(\bigcap_{i\in I} F_i\right) = \bigcup_{i\in I} \underbrace{{}^cF_i}_{\text{open}} \\ X \text{ compact} \end{array}\right\} \implies \exists \mathcal{J} \subseteq I \text{ finite s.t. } X = \bigcup_{j\in\mathcal{J}} {}^cF_j$$

$$\implies \emptyset = {}^c\left(\bigcup_{j\in\mathcal{J}} {}^cF_j\right) = \bigcap_{j\in\mathcal{J}} F_j \text{ -- Contradiction!}$$

" $\impliedby$ " We argue by contradiction. Assume $\exists \{G_i\}_{i\in I}$ open cover of X that does not admit a finite subcover.

So $\forall \mathcal{J} \subseteq I$ finite $X \neq \bigcup_{j\in\mathcal{J}} G_j \implies \emptyset \neq \bigcap_{j\in\mathcal{J}} \underbrace{{}^cG_j}_{\text{closed}}$. So $\{{}^cG_i\}_{i\in I}$ is a family of closed sets with the finite intersection property. Then

$$\bigcap_{i\in I} {}^cG_i \neq \emptyset \implies \bigcup_{i\in I} G_i \neq X$$

Contradiction! $\square$

§4 Dis 1: Mar 30, 2021

§4.1 Review of 131AH

Summation by parts(discrete integration by parts):

$\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$, $A_n = \sum_{k=1}^{n} a_k$, $A_0 = 0$. Then for $1 \leq p \leq q$,

$$\begin{aligned}\sum_{n=p}^{q} a_n b_n &= \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\end{aligned}$$

Application:

1. Dirichlet's test: $\sum a_n$ bounded, $\{b_n\}_{n\geq 1}$ decreasing and $b_n \to 0 \implies \sum a_n b_n$ converges.
2. Leibniz's Alternating series test: $|a_1| \geq |a_2| \geq \dots$ and $a_n \to 0$, $\sum (-1)^{n+1} |a_n|$ converges.
3. Kronecker's lemma: $b_n \geq 0$, $b_n \leq b_{n+1}$ $b_n \to \infty$, $A_n = \sum_{k=1}^{n} a_k$, and $\sum \frac{a_n}{b_n}$ converges $\implies \frac{A_n}{b_n} \to 0$.

Cardinality:

$|X| \leq (= / \geq) |Y|$ to mean $\exists f : X \to Y$ injective, bijective, or surjective, respectively.

- X finite if $|X| = |\{1, \dots, n\}|$
- X countable if $|X| \leq |\mathbb{N}|$. X countably infinite if countable but not finite.
- X countably infinite $\implies |X| = |\mathbb{N}|$.
- $|X| \leq |Y| \iff |Y| \geq |X|$.
- X, Y countable $\implies X \times Y$ countable.
- A countable, X_α countable $\forall \alpha \in A \implies \bigcup_{\alpha \in A} X_\alpha$ countable.
- $|\mathbb{Z}| = |\mathbb{N}| = |\mathbb{Q}|$, $\mathbb{R}$ uncountable.

Schröder – Bernstein: $|X| \leq |Y|$, $|Y| \leq |X|$ then $|X| = |Y|$ useful for hmwrk

Metric Spaces:

Let (X, d) be a metric space, $E \subseteq X$.

- $\mathring{E} = \{x \in X : \exists r > 0, B_r(x) \subseteq E\} = \bigcup_{G \subseteq E} G$ where G is open, largest open sets contained in E.
- $\overline{E} = \{x \in X : \forall r > 0, B_r(x) \cap E \neq \emptyset\} = \bigcap_{F \supseteq E} F$ where F is closed, smallest closed sets contained in E.
- $E' = \{x \in X : \forall r > 0, (B_r(x) \setminus \{x\}) \cap E \neq \emptyset\}$

- E open if $E = \mathring{E}$
- E closed if $E = \overline{E}$ or $E \supset E'$ or $\forall \{x_n\}_{n\geq 1} \subseteq E$, $x_n \to x \implies x \in E$.

(X, d) is complete if any Cauchy sequence in X converges.

- $\mathbb{R}$ complete, $\mathbb{R}^d$ complete.
- closed subsets of a complete space is complete.
- complete subsets are closed
- completeness is not invariant under homeomorphism(continuous bijection with continuous inverse)

  $(\mathbb{R}, |\cdot|) \xrightarrow{\sim} ((0,1), |\cdot|) \leftarrow$ not complete.

(X, d) is connected if there is no disjoint open sets A, B s.t. $X = A \cup B$.

- $E \subseteq \mathbb{R}$ connected $\iff$ E is interval.
- X is connected $\iff$ its only clopen subsets are $\emptyset, X$.

Intermediate Value Theorem: $f : [a, b] \to \mathbb{R}$ continuous, then $\forall \lambda$ s.t. $f(a) < \lambda < f(b)$, $\exists c$ s.t. $f(c) = \lambda$.