Math 131AH – Honors Real Analysis I

University of California, Los Angeles

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Fall 2020

This is math 131AH – Honors Real Analysis I taught by Professor Greene, and our TA is Haiyu Huang. We meet weekly on MWF from 1:00pm – 2:00pm for lectures. There are two textbooks used for the class, *Principles of Mathematical Analysis* by *Rudin* and *Metric Spaces* by *Copson*. You can find other lecture notes at my github site. Please let me know through my email if you spot any mathematical errors/typos.

Contents

List of Theorems

List of Definitions

§1 Lec 1: Oct 2, 2020

<u>Overview</u>:

- Hmwrk: 30 %
- Midterm 1: 20 %
- Midterm 2: 20 %
- Final: 30 %

§1.1 Introduction

<u>functions</u> $\rightarrow 1, 2, 3, 4, 5, 6, 7\dots$
functions defined on $\mathbb{Q}$ with value in $\mathbb{Q}$

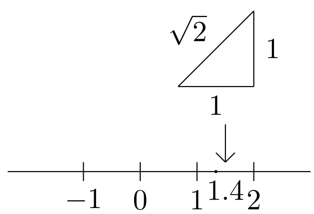
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

$a_i \in \mathbb{Q}$ $f(x) \in \mathbb{Q}$ if $x \in \mathbb{Q}$. Continuity makes sense.

$$x_0, x \quad x\text{close to } x_0 \implies f(x) \text{ close} f(x_0)$$

polynomials are continuous.
Somthing wrong: $\sqrt{2}$ is missing. What are these numbers that are not $\in \mathbb{Q}$? Choice:

1. Assume everything works and isolate what you need about "real numbers" (most of Rudin chap 1).
2. Construct the real numbers from rational numbers.



Classical argument:

$$x^2 \neq 2 \text{ if } x = \frac{p}{q} \in \mathbb{Q}$$

Proof. Suppose $\left(\frac{p}{q}\right)^2 = 2$
<u>*Note*</u>: wolog(without loss of generality)
can take $\frac{p}{q} > 0$ $p > 0$ $q > 0$

$$\left(\frac{p}{q}\right)^2 = 2$$
$$\frac{p^2}{q^2} = 2$$
$$p^2 = 2q^2$$

Now also wolog, can assume p and q are not <u>both</u> even numbers. But $p^2 = 2q^2$ means p has to be even (p^2 odd if p is odd).

$$p = 2n$$
$$p^2 = 2q^2$$
$$4n^2 = 2q^2$$

So $q^2 = 2n^2$, q is even. But it contradicts the initial assumption, p and q not both even $\square$

Related to: Why functions $\mathbb{Q}$ to $\mathbb{Q}$ not ideal for analysis?
– INFINITE DECIMAL

§2 Lec 2: Oct 5, 2020

§2.1 Mathematical Induction and More on Real Numbers

$P(n) \to 1+2+3+\ldots+n = \frac{n(n+1)}{2}$, where n is positive numbers.
Math induction: Proof by two steps:

1. Check $P(1)$ is true ✓
2. Assume $P(n)$ is true for all $n \leq N$. Check that
$$P(N+1) \text{ is true}$$

Assume $1+\ldots+N = \frac{N(N+1)}{2}$. Check

$$1+\ldots+N+(N+1) = \frac{(N+1)(N+1+1)}{2}$$

Induction on k :

$$1^k + 2^k + \ldots + n^k$$

2nd illustration:

$$1 + r + r^2 + \ldots + r^n = \frac{1-r^{n+1}}{1-r} \quad r \neq 1$$

$r = 1 \implies 1 + r = \frac{1-r^2}{1-r}$

$$\begin{aligned}
1 + r + r^2 + \ldots + r^n + r^{n+1} &= \frac{1-r^{n+1}}{1-r} + r^{n+1} \\
&= \frac{1-r^{n+1}+r^{n+1}-r^{n+2}}{1-r} \\
&= \frac{1-r^{n+2}}{1-r}
\end{aligned}$$

$$(1-r)(1+r+\ldots+r^n) = 1 - r^{n+1} \qquad \text{Inspection}$$
$$1 + r + r^2 + \ldots + r^n = \frac{1-r^{n+1}}{1-r}, \quad r \neq 1$$

$|r| < 1$ get inifite sum $\frac{1}{1-r}$

Example 2.1

Prime factors, prime = positive integers (>1) with no factors except itself and 1, $p = ab$, $a > 1$, $b > 1$

$$2 \quad 3 \quad 5 \quad 7 \quad 11 \quad 13 \quad 17 \quad 19 \quad \ldots$$

Thin out as go along

Theorem 2.2 (Fundamental Theorem of Arithmetic)

Every positive integer > 1 is a product of primes.

Proof. Induction: $P(n)$ $n = 2, 3, \ldots$

$$P(2) = 2 \checkmark$$

Assume P(n) $\ldots n \leq N$ $(N > 2)$. Every integer greater than 1 but smaller than or equal to N as a product of primes. We try to prove: $N+1$ is a product of primes.

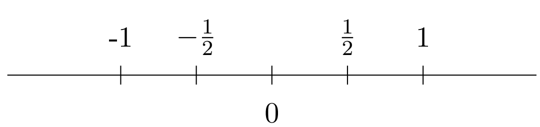
1. $N+1$ is prime: Done $N+1 = N+1$
2. $N+1$ is not a prime
$$N + 1 = a \cdot b \qquad a > 1 \;\; b > 1$$
Induction assumption ($a < N+1$ since $b > 1$), a is a product of primes $a > 1 \implies b < N+1$, b also a product of primes. So, $N+1 = ab$ is a product of primes.

$N+1 = ab$ is a product of prime. □

Why does induction work? If $P(n)$ not always true, $P(n)$ look at smallest n where $P(n)$ is false.
$n = 1$ not there $P(1)$ is supposed true (checked already). N_0 smallest one where $P(N_0)$ false $N_0 > 1$. Induction step says that $P(n)$ is true for all $n \leq \underbrace{N_0 - 1}_{>0} \implies P(N_0)$ true ($\times$
).
Let's go back to real numbers.

Last time: talked about $\sqrt{2}$ is irrational but $\sqrt{2}$ exists, so we need to enlarge our number system: $\mathbb{Q}$ rational numbers.

$$\frac{p}{q} > \frac{r}{s} \qquad ps > rq \qquad (p, q, r, s > 0)$$



x, y rational $x, y > 0$, $x + y > 0$, $xy > 0$

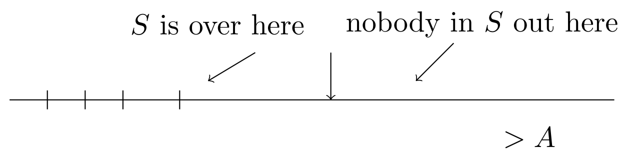
$x^2 = 2$ no answer in $\mathbb{Q}$. Enlarge number system, $\mathbb{Q} \subset \mathbb{R}$. What should $\mathbb{R}$ be like?

1. $\mathbb{R}$ ought ot have arithmetic like $\mathbb{Q}$

$$x + y \quad xy \quad \frac{x}{y} \quad 0 \quad 1$$

2. $\mathbb{Q} \subset \mathbb{R}$, arithmetic in $\mathbb{R}$ restricted to $\mathbb{Q}$, $\frac{1}{2} + \frac{1}{3}$ in $\mathbb{Q}$ ought to be $\frac{5}{6}$ in $\mathbb{R}$.
3. Order should positive in $\mathbb{Q} \implies$ in $\mathbb{R}$. $\mathbb{R}$ should have an order of its own too, $x \quad y$ positive then $x + y$ pos and xy pos.
4. want to fill in the holes in $\mathbb{Q}$. Want to have Least Upper Bound Property

   $S \subset \mathbb{R}$: An <u>upper bound</u> for S is a number A with property $A \geq x$ if $x \in S$



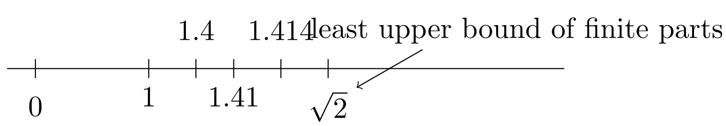
   $1, 2, 3, 4, \ldots$ have no upper bound.

   S is <u>bounded above</u> means that some upper bound A exists.

§2.2 Least Upper Bound Property

If S is bounded above ($S \neq \emptyset$) then it has a "least upper bound" where a number A_0 is called the least upper bound of S if A_0 is an upper bound for S & if A is an upper bound for S then $A_0 \leq A$.

Motivation: Think about $\sqrt{2}$



Denote: l.u.b(or supremum)(sequence) $= \sqrt{2}$
Means can define an infinite decimals: least upper bound of successive truncation.

$$0.99999\ldots \to 1.0$$

§3 Lec 3: Oct 7, 2020

§3.1 Cauchy Sequence

$\{x_n\}\ x_1, x_2, x_3, \ldots$ values $x_j \in \mathbb{Q} \quad x_j \in \mathbb{R}$
$S \quad x_1, x_i \ldots x_j \in S$

Definition 3.1 (Sequence) — A sequence with values in a set S is a function from positive integers $\{1, 2, 3\ldots\}$ into S.

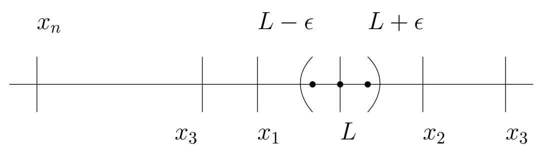
Definition 3.2 (Cauchy Sequence) — A Cauchy sequence is ($\mathbb{Q}$ valued or $\mathbb{R}$ valued) $\{x_i\}$ is sequence s.t. for every $\epsilon > 0$ there is a positive integer N_ϵ s.t.

$$|x_i - x_j| < \epsilon \quad \text{if} \quad i, j > N_\epsilon$$

ϵ rational or real (same idea).

Lemma 3.3
If $\{x_j\}$ has a finite limit then it's a Cauchy sequence.

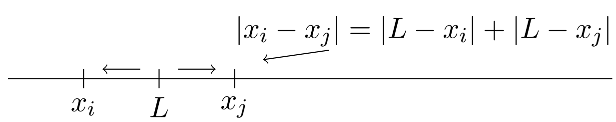
$\{x_i\}$ has L as a limit $\lim x_j = L$ means for every $\epsilon > 0$ then there is an N_ϵ such that $j \geq N_\epsilon, \quad |x_j - L| < \epsilon$



Everybody in $(L - \epsilon, L + \epsilon)$ except a finite number

Proof. Given $\epsilon > 0$, want to find N so that $i, j \geq N \implies |x_i - x_j| < \epsilon$
$|x_i - L|$ small, $|x_j - L|$ small and $\lim x_j = L$.

$$|x_i - x_j| \leq |x_i - L| + |x_j - L|$$



$i, j \geq N_{\frac{\epsilon}{2}}$:

$$|x_i - x_j| \leq \underbrace{|x_i - L|}_{<\frac{\epsilon}{2}} + \underbrace{|x_j - L|}_{<\frac{\epsilon}{2}}$$

Because $\lim x_n = L$, there is an $N_{\frac{\epsilon}{2}}$ s.t. $|L - x_n| < \frac{\epsilon}{2}$ if $n \geq N_{\frac{\epsilon}{2}}$
Get $|x_i - x_j| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ if $i, j \geq N$. Cauchy sequence: there exists number N s.t.

$$|x_i - x_j| < \epsilon \quad \text{if} \quad i, j \geq N$$

□

Cauchy sequence $\implies$ the existence of limit? Yes, for $\mathbb{R}$ valued sequences but NO for $\mathbb{Q}$ valued things.

$$\underbrace{\{x_n\}}_{\text{rational numbers}} \quad \text{can be Cauchy seq without there being a rationl number } L \quad \text{such that } \lim x_j = L$$

But allow real L then $\exists L$ s.t. $\lim x_j = L$ if $\{x_j\}$ is Cauchy sequence(no rational limit – since $\sqrt{2}$ is irrational). Because $\mathbb{Q}$ has holes in it! (intuitive idea).

Example 3.4

$1, 1.4, 1.41, 1.414, 1.4142\ldots$ (decimal approx of $\sqrt{2}$) – Cauchy sequence. No – since $\sqrt{2}$ is irrational.

§3.2 Cauchy Completeness of $\mathbb{R}$

If $\{x_j\}, x_j \in \mathbb{R}$ is Cauchy sequence, then $\exists L \in \mathbb{R}$ s.t. $\lim x_j = L$.
"$\mathbb{Q}$ is not Cauchy complete" but $\mathbb{R}$ is. Why does this work?
Need: Least upper bound property. Assume L.U.B Property proof.

Proof. (Cauchy completeness from L.U.B Property)
Hypothesis: $\{x_i\}$ Cauchy seq

1. Prove that $\{x_i\}$ bounded $\Longleftarrow$ $\exists M > 0$ s.t. $|x_i| \le M$ all i.

   Clear if take $\epsilon = 1$ in def. of Cauchy seq $\exists N$ s.t. $|x_i - x_j| < 1$ if $i, j \ge N \implies |x_N - x_j| < 1$ if $j \ge N \implies |x_j| \le |x_N| + 1 \quad j \ge N$

   So, $M = \max(|x_N| + 1, |x_1|, \ldots, |x_{N-1}|$ then $|x_j| \le M$ all j !

   Next stage is to show that a bounded sequence always has a subsequence(tricky!) with a limit. Then if a Cauchy seq has a subseq with limit L, then L is limit of whole seq. (Bolzano – Weierstrass Theorem)

□

§4 Lec 4: Oct 9, 2020

§4.1 Bolzano – Weierstrass Theorem

– implied by Least Upper Bound Property

Theorem 4.1 (Bolzano – Weierstrass)

If $\{x_n\}$ sequence $(x_1, x_2, x_3 \ldots)$ that is bounded (means: $\exists M > 0 \ni |x_n| \le M \forall\ n$), then $\exists L$ and a subsequence $\{x_{n_i}\}$ s.t. $\lim x_{n_i} = L$.

Slogan: Every bounded sequence has a convergent subsequence.

Example 4.2

$$1, 2, 1, 2, 1, 2, \ldots$$

The subsequence of the above sequence has either 1 or 2 as the limit.

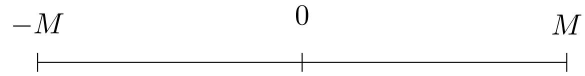
$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$$

Unbounded sequence – subsequence (limit 1, limit 2, limit 3...)

No claim of uniquneness of anything.

Proof – Summer 2008 Analysis Lec 4

Proof. So either $[-M,0]$ or $[0,M]$ (maybe both) contains x_n for infinitely many n values. If each contained x_n for only finitely many n values X.

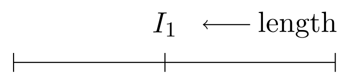


Every x_n is in $[-M,M]$ – $\{x_n\}$ is bounded

$$[-M,M]=[-M,0]\cup[0,M]$$
$$I_1=[-M,0] \quad \text{or} \quad [0,M]$$

where chosen intervalhas x_n for infinitely many n values.
Do this again!

$$I_1=[a_1,b_1] \qquad |b_1-a_1|=M$$



left half of I_1, right half of I. Let $I_2 =$ one of halves that contains x_n for infinitely many n values.

$$I_2=[a_2,b_2] \qquad a_2<b_2, \quad b_2-a_2=\frac{M}{2}$$

Continue

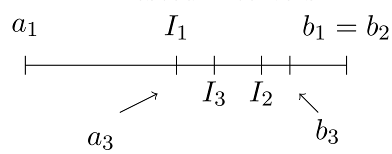
$$I_3=[a_3,b_3] \qquad a_3<b_3, \quad b_3-a_3=\frac{M}{4}$$

$$\vdots$$

$$I_k=[a_k,b_k] \qquad b_k-a_k=\frac{M}{2^{k-1}}$$

Each I_k contains x_n for infinitely many n values.

Nested Intervals



$$I_{k+1}\subset I_k\subset\ldots\subset I_1\subset[-M,M]$$
$$a_{k+1}\ge a_k\ldots \qquad b_{k+1}\le b_k\ldots$$

Claim $\bigcap_{k=1}^\infty I_k\neq\emptyset$

Reason: $\sup a_k\in\bigcap_{k=1}^\infty I_k$ where sup = sup of left hand endpoint(=greatest lower bound of bs). l.u.b of a's $\le b_k$, b_k bigger than or $\ge$ all a's.

$$\alpha = \text{lub a's}$$
$$\alpha\ge a_k \quad \forall k$$
$$\alpha\le b_k \quad \forall k$$
$$\alpha\in[a_k,b_k]$$

Goal: $\alpha\in\bigcap_{k=1}^\infty$. Find a subsequence of $\{x_n\}$ converges to α.

Choose $x_k = x_n$ that belongs to I_k. Can also arrange successively:

$$n_1 < n_2 < n_3 < n_4$$

$x_{n_1} \in I_1 \quad x_{n_2} \in I_2$ can make $n_2 > n_1$ because infinitely possible $x_n's$ in I_2 n value. Continue to get subsequence, $\{x_{n_k}\}$ subsequence. Claim:

$$\lim_{k\to\infty} x_{n_k} = \infty$$

Reason:

$$\text{dis}(x_{n_k}, \alpha) \leq \text{length of } I_k \quad \alpha \in I_k, \quad x_{n_k} \in I_k$$

which is equivalent to

$$|x_{n_k} - \alpha| \leq \frac{M}{2^{k-1}} \quad \text{given } \epsilon > 0$$

When k is large,

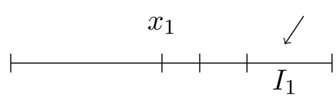
$$\frac{M}{2^{k-1}} < \epsilon$$

So $|x_{n_k} - \alpha| < \epsilon$ □

This argument (or a variant) shows something else:
If $\{x_n\}$ sequence in $[0,1]$ then there's an $\alpha \in [0,1]$ with it never happening that

$$x_n = \alpha$$

"The real numbers in $[0,1]$ are uncountable." (come from the least upper bound property)



I_1 one of $[0, \frac{1}{3}]$ $\quad [\frac{1}{3}, \frac{2}{3}]$ $\quad [\frac{2}{3}, 1]$ such that $x_1 \notin I_1$,

$$[0, \frac{1}{3}] \cap [\frac{1}{3}, \frac{2}{3}] \cap [\frac{2}{3}, 1] = \emptyset$$

$x_1 \notin I_2 \quad I_2 \subset I_1, \quad \& x_1 \notin I_1$. Continue. Get

$$I_1 \supset I_2 \supset I_3 \supset \dots$$

length $I_k = \frac{1}{3^k}$ and I_k is such that $x_1, x_2, x_3 \dots x_k$ are none of the ?n? in I_k. Same as before

$$\exists \alpha \in \bigcap_{\infty}^{k=1} I_k$$

$\alpha =$ sup of set of left hand endpoints of I_k. Claim α cannot be an x_N value. Clear: $x_N \notin I_N$ but $\alpha \in I_n \qquad \alpha \in \bigcap_{\infty}^{n=1} I_n$. But contrast:

There is a list of rational numbers in $[0,1]$

$\frac{p}{q}$ $\quad p < q$

$$\begin{array}{c|cccccc}
 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\hline
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & & & \\
2 & - & \frac{2}{3} & \frac{2}{4} & & & \\
3 & - & - & \frac{3}{4} & & & \\
\vdots & - & - & \frac{\sqrt{2}}{2} & & &
\end{array}$$

$\frac{\sqrt{2}}{2} \in [0,1] \to$ irrational - no exist

$[0,1]$ not countable

Q is countable

§5 Lec 5: Oct 12, 2020

§5.1 Equivalence Relation

(p.10, Copson – Metric Space)
R set, relation of A and B $(A \times B)$ $\quad (a, b) \in R \quad aRb$
Functions: one b given a – exact one. $(A \to B)$

Example 5.1
$A = B = Q$
aRb or $(a, b) \in R$ if $a > b$
(mother,child)

- (Sara, Sebastian) $\in R$
- (Sara, Alita) $\in R$

Equivalence is a special kind of relation: (on a set $A; B \quad A = B$)
Properties:

1. $aRa \qquad A = Q$
2. $aRb \implies bRa$
3. aRb & bRc then aRc

   Example: $\mathbb{Z} \quad a \sim b$ means $a - b$ is divisible by 5

$$1 \sim 6 \quad 0 \sim 5 \ldots$$

$a \sim a \quad a - b$ div $\implies b - a$ div. by 5.

If $a - b$ div. by 5, and $b - c$ div by 5, then is $a - c$ div. by 5 true?
Sure, $a - b = 5k, \quad b - c = 5l \implies a - c = 5(k + l)$
"Equivalence classes": set $[a] = \{$ all b such that $aRb\}$

In the example above, $[a] = \{$ all b such that $a - b$ div. by 5$\}$

$$[2] = \{2, 7, -3, 12, -8, \dots\}$$

$\mathbb{Z}_5$: integer mod 5.

1. $[a]$ $[p]$ either equal or have nothing in common.
2. $a \in [a]$ so is in some equivalence class.

A equivalence relation $\sim$ on $A \leftrightarrow$ a partition of A into subsets which are pairwise disjoint.
Q Cauchy seq. of rational numbers

$$\{x_n\} \sim \{y_n\}$$

means $\lim_{n\to\infty} |x_n - y_n| = 0$. Equivalence relation:

1. $\{x_n\} \sim \{x_n\}$ $(\lim(x_n - x_n) = 0)$
2. $\{x_n\} \sim \{y_n\} \implies \{y_n\} \sim \{x_n\}$
3. $\{x_n\} \sim \{y_n\} \,\&\, \{y_n\} \sim \{z_n\} \implies \{x_n\} \sim \{z_n\}$

Idea: Define a real number to be a (Cauchy seq. of rationals) equivalence class.
Homework: want to check that arithmetic extends to "real numbers"

$$[\{x_n\}] + [\{z_n\}] = [\{x_n + z_n\}]$$

Check that

1. $\{x_n + z_n\}$ is a Cauchy seq.
2. Only depends on equivalence classes.

Want

$$\{x_n\} \sim \{y_n\} \qquad \{z_n\} \sim \{w_n\}$$

then $\{x_n + z_n\} \sim \{y_n + w_n\}$. So,

$$[\{x_n + z_n\}] = [\{y_n + w_n\}]$$

Example 5.2

$$[2] + [11] = [2 + 11] = [13]$$

So, $[2+1] \sim [13]([11] = [1])$. Arithmetic (addition) in $\mathbb{Z}_5$ thus makes sense. How about multiplication? $\frac{[1]}{[a]} \leftarrow$ exists $[a] \neq 0$.

$$\frac{[1]}{[2]} = [3] \qquad [2][3] = [6] = [1]$$

Thus, $\mathbb{Z}_5$ is a field.

$\frac{p}{q} \sim \frac{r}{s}, \quad q, s \neq 0$ means $ps = rq$ (when talking about fractions – associate it with equivalence relation). $Q =$ set of equivalences classes. $(\frac{p}{q})$: equivalence classes).
Last time, we proved that Cauchy seq. of real numbers have limits (lub property). Also, no sequence $\{x_n\}$ such that it hits all real numbers in $[0, 1]$ – this is important. Contrast with $Q \cap [0, 1]$, then there is a sequence that hits them all. Refer to the last figure in Lec 4 or math.ucla.edu/~greene – Summer 2008.

§6 Lec 6: Oct 14, 2020

Bolzano - Weierstrass:
Every bounded sequence has a convergent subsequence.
And we know about the Least Upper Bound Prop.

§6.1 Continuous Functions on Closed Interval

$f : S \to \mathbb{R}, \quad S \subset \mathbb{R}$

Example 6.1

$$S = [a, b]$$
$$S = \mathbb{R}$$

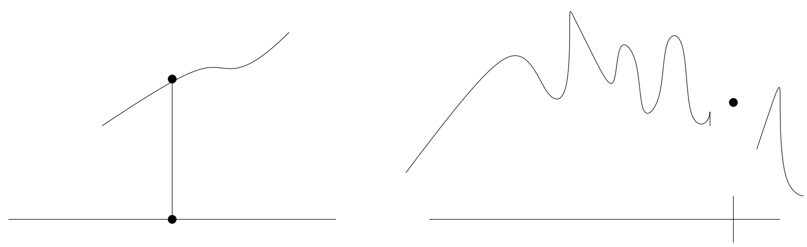
Definition 6.2 (Continuity) — $s_0 \in S, f$ is continuous at s_0 if given $\epsilon > 0, \exists \delta > 0$ s.t.

$$|s - s_0| < \delta_\epsilon \implies |f(s) - f(s_0)| < \epsilon$$

Three properties:

$$f : [a, b] \to \mathbb{R}$$
$$f \text{continuous}$$

1. f is bounded on $[a, b]$ means $\exists M$ s.t. for all $x \in [a, b], \quad |f(x)| \leq M$



not continuous at s_0

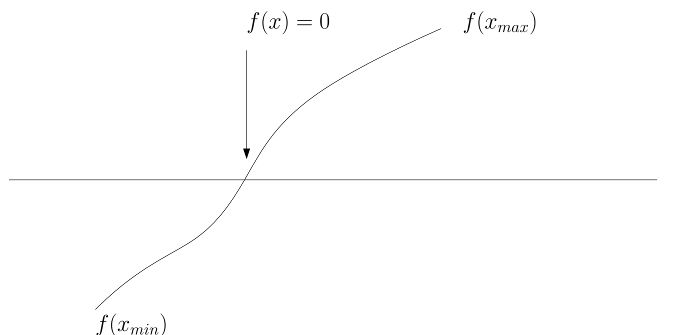
2. There exists $x_{\min}, x_{\max} \in [a, b]$ such that for all $x \in [a, b]$

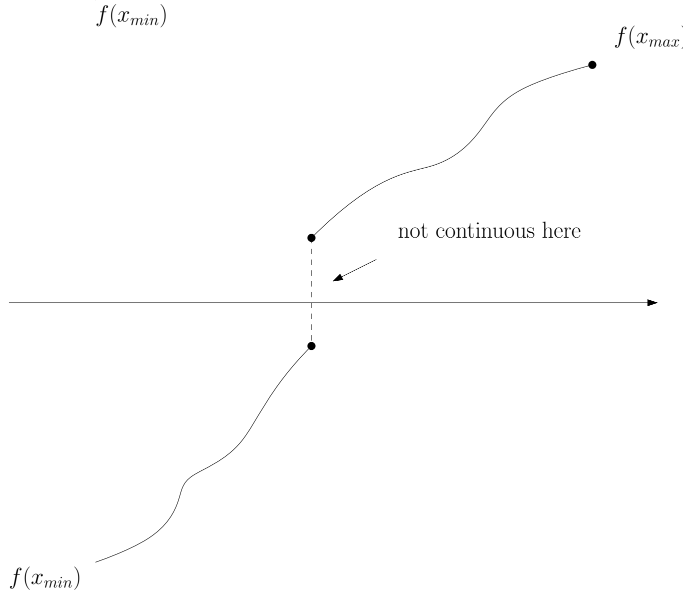
$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

Slogan: f attains its maximum and minimum.

3. If $\alpha, f(x_{\min}) < \alpha < f(x_{\max})$, then $\exists x \in S = [a, b]$ s.t. $f(X) = \alpha$.

"Intermediate Value Theorem" Need the least upper bound prop – "completeness of





real numbers"

Exercise: def of continuity $\{s_n\}$ converges to s_0 $\iff$ if $s_n \to s_0$, $s_n \in S, s_0 \in S$ then $\{f(s_n)\}$ converges to $f(s_0)$.

Example 6.3

For (3),

$$f(x) = x^2 - 2 \quad \text{on} \quad Q \cap [1,2]$$

Then $f(1) = -1$, $f(2) = 2$, but no rational $x \in [1,2]$ s.t. $f(x) = 0$.

Back to the properties:

1. f is bounded – Think about $|f| \leftarrow$ continuous if f is (exercise).

   $\exists M$ such $|f(x)| \leq M$ all $x \in [a,b]$. Suppose no such M exists.

   Try $M = 1,2,3,4,5,6,\ldots$. So $\exists x_1 \quad |f(x_1)| > 1$

   $$|f(x_2)| > 2$$
   $$\vdots$$
   $$|f(x_n)| > n$$

   But Bolzano – Weierstrass: subsequence $\{x_{n_j}\}$ that converges to x_0 say $|f(x_0)| \leftarrow$

   finite number. So $\exists N \ni \quad |f(x_0)| \leq N$.

   Now for j large enough

   $$\left|f(x_{n_j}) - f(x_0)\right| < 1$$

   x_{n_j} converges to x_0

   $$\left|f(x_{n_j})\right| < |f(x_0)| + \left|f(x_{n_j} - f(x_0)\right|$$

   So j is large enough that

   $$\underbrace{\left|f(x_{n_j})\right|}_{\geq |n_j| = n_j} \leq N + \text{ something less than 1 } \leq N$$

2. Attains max and min

   Similar: $\{f(x) : x \in [a,b]\}$ bounded set, has sup where

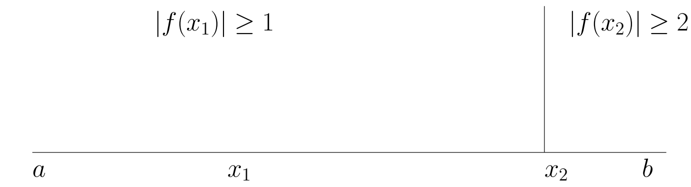
   $$\sup\{f(x) : x \in [a,b]\}$$

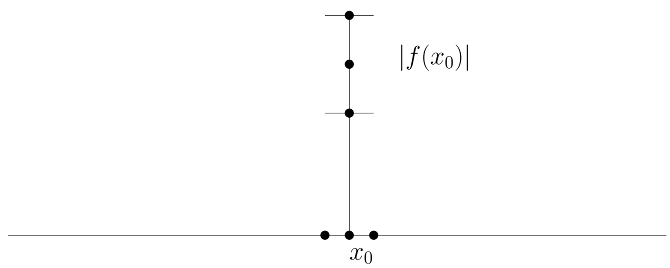
   either in the set of f-values (done if that's true), $\sup f = f(x_0)$.

   OR: $\sup f$ acutally not in the set $\{f(x) : x \in [a,b]\}$

   Now $\{x_{n_j}\}$ converges to $x_0 \in [a,b]$

   Claim 6.1. $f(x_0) = \sup\{f(x) : x \in [a,b]\}$

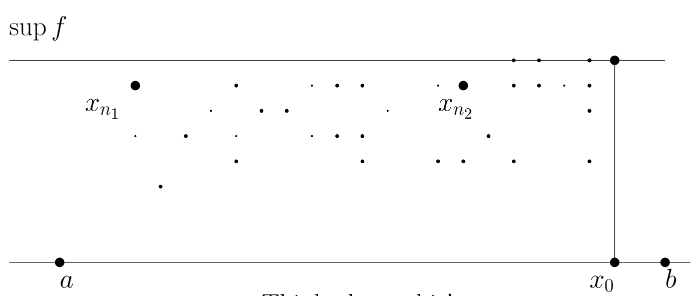




$$f(x_{n_j}) \leq \sup\{f(x) : x \in [a,b]\}$$

and $\lim f(x_{n_j}) = f(x_0) = f(\lim x_{n_j})$. So

$$f(x_0) = \sup f$$



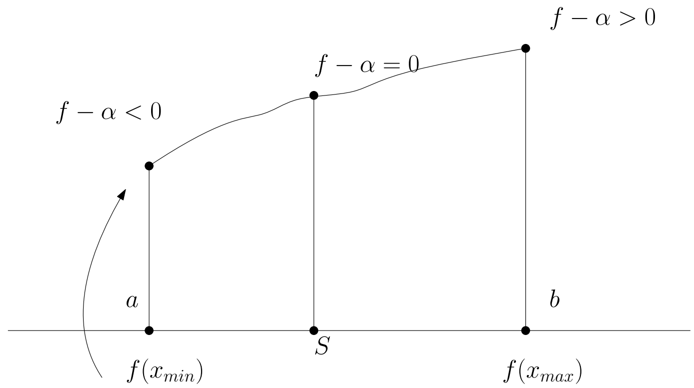
Think about this!

3. $\alpha \in [f(x_{\min}), f(x_{\max})]$ then x such that $f(x) = \alpha$.

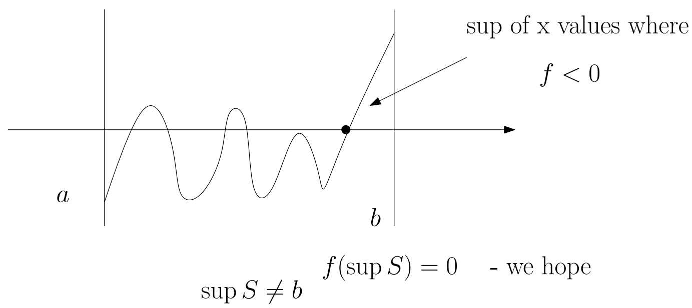
Proof. Wolog:

$$f(a) < 0 \quad \text{and} \quad f(b) > 0$$

then $\exists x \in [a,b]$ with $f(x) = 0$.

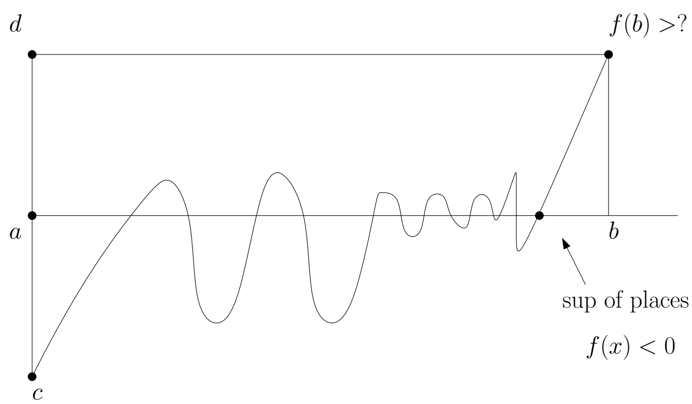


Use l.u.b: Look at S : $\{x : f(x) < 0\}$ and $S \neq \emptyset$ because $f(a) \in S$. Also, S is bounded above – $\exists$ l.u.b for S, $\sup S \in [a, b]$. Hope that $f(\sup S) = 0$.



$\sup S \neq b$ is clear because $f(b) > 0$ so $f(b - \epsilon) > 0$ for small ϵ.

So $\sup S = x_0, \quad a < x_0 < b$. What is $f(x_0)$? If it's negative, then there are slightly bigger $x \in [a_0, b] \ni f(x) < 0$ (continuity). In addition, x_0 cannot be a limit of x with $f(x) < 0$ – $x_0 =$ sup places where $f < 0$. □



f continuous on $[a, b]$ if it is

1. bounded.
2. attains max and min.
3. attains every value between max value and min value.

$f([a, b]) = [c, d]$ where c is min of f and d is max of f.

§7 Lec 7: Oct 16, 2020

§7.1 Uniform Continuity

Definition 7.1 (Uniform Continuity) — $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$. f is uniformly continuous on S if given $\epsilon > 0$ there is a $\delta > 0$ s.t. $|f(x) - f(y)| < \epsilon$ if $x, y \in S$ and $|x - y| < \delta_\epsilon$

Example 7.2

$f : S \to \mathbb{R}$, $S = \mathbb{R}$, $f(x) = x^2$. Continuous on $\mathbb{R}$ but it is not uniformly continuous on $\mathbb{R}$.

Continuity: Given fixed x, and $\epsilon > 0$ want δ so that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

$|x^2 - y^2| = |x - y||x + y|$ and want it smaller than ϵ. Assume $\delta \le 1$.

$$|x + y| \le |x| + |y|$$
$$|y| < |x| + 1 \text{ if } |x - y| < \delta (\le 1)$$

So, if $|x - y| < \delta(\le 1)$,

$$\begin{aligned} \left|x^2 - y^2\right| &= |x - y||x + y| \\ &\le |x - y|(2|x| + 1) \end{aligned}$$

Choose $\delta < \frac{\epsilon}{2|x|+1}$ (ok since x is fixed)

$$\begin{aligned} |x^2 - y^2| &< \frac{\epsilon}{2|x| + 1}(2|x| + 1) \\ &= \epsilon \text{ if } |x - y| < \min\left\{1, \frac{1}{2|x| + 1}\right\} \end{aligned}$$

Uniform continuity does not work on $\mathbb{R}$.

Claim 7.1. $\epsilon = 1 > 0$, there is no $\delta > 0$ s.t. $\left|x^2 - y^2\right| < 1 = \epsilon$ for all x, y with $|x - y| < \delta$.

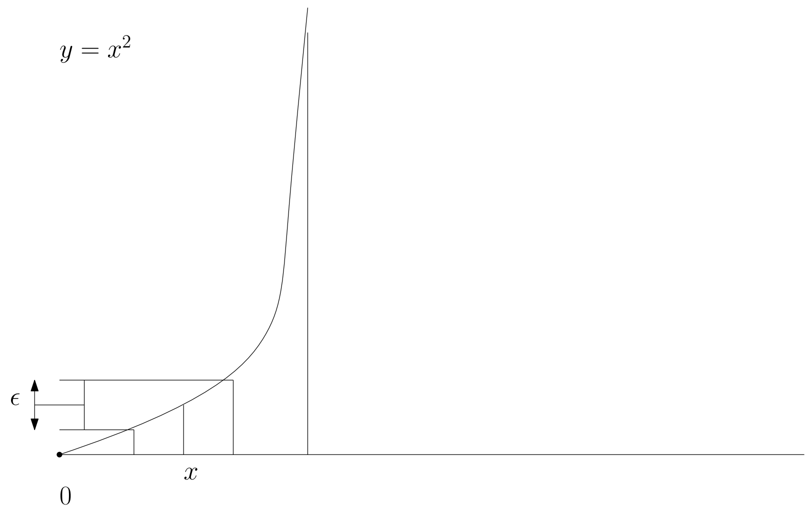
Why? Look at for $\delta > 0$, consider $y = \frac{1}{\delta} + \frac{\delta}{2}$, $x = \frac{1}{\delta}$

$$|x - y| < \delta$$

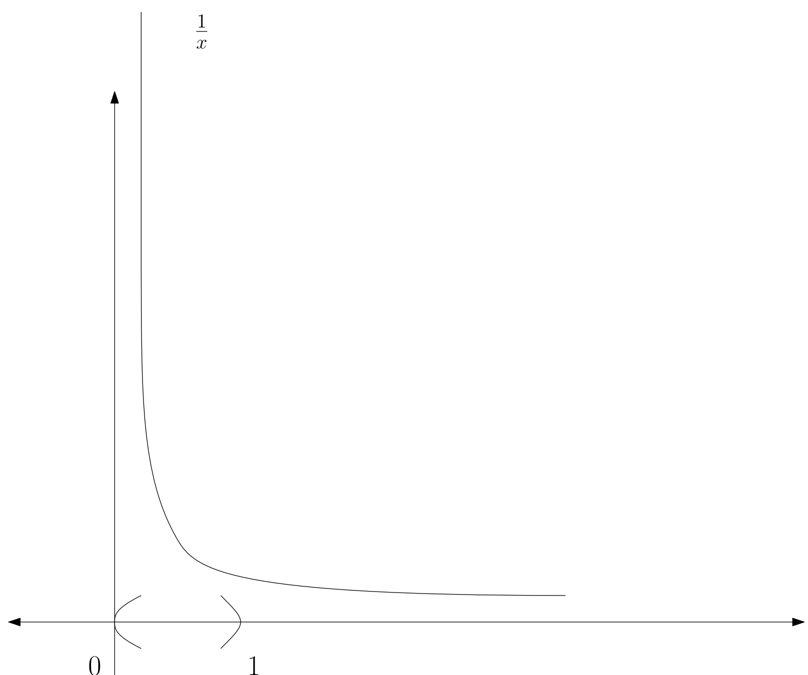
Also,

$$\begin{aligned} &\left|\left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \left(\frac{1}{\delta}\right)^2\right| \\ &= \left|\frac{1}{\delta^2} + 2\left(\frac{1}{\delta}\right)\left(\frac{\delta}{2}\right) + \left(\frac{\delta}{2}\right)^2 - \frac{1}{\delta^2}\right| \\ &= 1 + \left(\frac{\delta}{2}\right)^2 > 1 \end{aligned}$$

which is a contradiction.



Exercise 7.1. $\frac{1}{x}$ on $(0,1)$ is continuous but <u>not</u> uniformly continuous. Sugges plausibly f



continuous on $[a,b]$ then it's uniformly continuous on $[a,b]$ where a,b are finite.

Theorem 7.3 (Heine – Cantor (Uniformly Continuous))
A continuous function f on a closed interval is uniformly continuous.

Proof. (By contradiction) Suppose not. Then $\epsilon > 0$ s.t. no δ "works". In particular, $\exists \epsilon > 0$

s.t. $\delta = 1$ fails, $\delta = \frac{1}{2}$ fails, etc. So $x, y \in [a, b]$ with $|f(x_1) - (fy_1)| \geq \epsilon$ but $|x_1 - y_1| < 1$. $x_n, y_n \in [a, b]$ with $|f(x_n) - f(y_n)| \geq \epsilon$ but $|x_n - y_n| < \frac{1}{n}$. Hope this is impossible. Bolzano - Weierstrass $\implies$ $\{n_j\}$ s.t. $\left\{x_{n_j}\right\}$ has a limit

$$x_0 = \lim, \qquad x_0 \in [a, b]$$

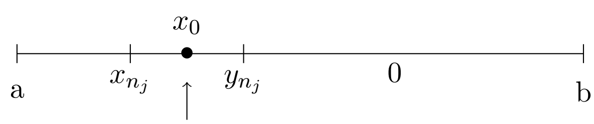
<u>Now</u>, claim $\left\{y_{n_j}\right\}$ also has limit x_0.

$$\left|x_{n_j} - y_{n_j}\right| < \frac{1}{n_j}$$

small when n_j large (j large).

$$\begin{aligned} \lim x_{n_j} &= x_0 \\ \lim y_{n_j} &= x_0 \\ \lim f(x_{n_j}) &= f(x_0) \\ \lim f(y_{n_j}) &= f(x_0) \end{aligned}$$

So, $\lim f(x_{n_j}) - f(y_{n_j}) = 0$, but it contradicts $\left|f(x_{n_j} - f(y_{n_j})\right| \geq \epsilon$ for all j. □



$$f(x_0) \leq \left|f(x_{n_j}) - f(x_0)\right| + \left|f(x_0) - f(y_{n_j})\right| \to 0$$

Ideas of continuity and uniform continuity and Bolzano - Weierstrass Theorem – all have reasons in metric spaces.

§8 Lec 8: Oct 19, 2020

§8.1 Convergence of Series

Series is "formal sum", an infinite sum

$$a_0 + a_1 + a_2 + \ldots = \sum_{j=1}^{\infty} a_j$$

A series $\iff$ sequence $a_1, a_2, a_3, \ldots$ add together. Associated to $a_1 + a_2 + a_3 + a_4 \ldots$ is a sequence of partial sum

$$S_N = \sum_{n=1}^{N} a_n, \qquad N = 1, 2, 3, 4, 5, \ldots$$

<u>number</u> valued sequence.

Definition 8.1 (Convergence of Series) — Series converges if sequence associated $\{S_N\}$ converges (has a limit).

Lots of things are defined by series such as ($x \in \mathbb{R}$),

$$e^x = \lim_{N\to\infty} \left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^N}{N!}\right)$$

Given series $a_0 + a_1 + a_2 + a_3 + \ldots$, when does it converge?

$$1 - 2 + 3 - 4 + 5 - 6 + 7 \ldots$$
$$S_1 = 1, \quad S_2 = -1, \quad S_3 = 2 \ldots$$

<u>NO LIMIT!</u> Series do not necessarily have to converge then it's okay to write

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

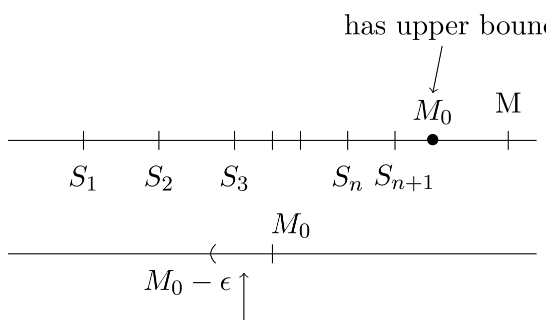
First thing to look at – Case where $a_j \geq 0$

$$S_N \leq S_{N+1}, \quad N = 1, 2, 3, \ldots$$

$S_{N+1} = S_N + a_{N+1}$ so $a_{N+1} \geq 0$ means $S_{N+1} \geq S_N$. Two cases:
Case 1: $\{S_n\}$ not bounded above.

$\lim S_N$ does not exist $\to$ Series diverges (sequences with limits are always bounded above and below).
Case 2: $\{S_n\}$ bounded above.

$\lim_{n\to\infty} S_n$ always exists. Namely, it is the least upper bound of set of values of S_n.



There is an S_{n_0} in this interval $(M_0 - \epsilon, M_0]$, M_0 is lub

From that n_0 on,

$$S_n \geq S_{n_0}, \quad S_n \leq M$$

S_n satisfies $|S_n - M_0| < \epsilon$ if $n \geq n_0$. So $\lim S_n = M_0$. This implies that S_n is a Cauchy sequence (it has a limit). Given $\epsilon > 0, \exists N_\epsilon$ s.t. $\left|\underbrace{\sum_1^{n_1} a_n}_{S_{n_1}} - \underbrace{\sum_1^{n_2} a_n}_{S_{n_2}}\right| < \epsilon$ if $n_1, n_2 \geq N_\epsilon$.

Suppose $n_1 > n_2 \geq N_\epsilon$

$$\sum_1^{n_1} a_n - \sum_1^{n_2} a_n = \sum_{n_2+1}^{n_1} a_n$$

Note: $S_7 - S_5 = a_6 + a_7$ which explains the above expression.

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} \dots$$

converges, but so does the following series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots < 2$$

This works for arbitrary choices of + or −.

Theorem 8.2 (Absolute Convergence)
If $|b_1| + |b_2| + |b_3| + \dots$ converges, then

$$b_1 + b_2 + b_3 + \dots \quad \text{converges}$$

"Absolute convergence" $\implies$ convergence (but not necessarily the same limit).

Proof. Assume $\underbrace{\{S_n^A\}}_{A \text{ for absolute}}$ for absoluted series has limit. So

$$\sum_1^\infty |b_n| \quad \text{converges}$$

$\implies \{S_n^A\}$ Cauchy sequence.
We hope it $\implies \{S_n\} = \left\{\sum_{j=1}^n b_j\right\}$ is a Cauchy sequence.

$$S_{n_1}^A - S_{n_2}^A = |b_{n_2+1}| + |b_{n_2+2}| + \dots + |b_n|$$

But

$$|b_{n_2+1} + \dots + b_n| \le |b_{n_2+1}| + \dots + |b_n| (= S_{n_1}^A - S_{n_2}^A)$$

So,

$$|S_{n_1} - S_{n_2}| \le S_{n_1}^A - S_{n_2}^A < \epsilon \quad \text{for } n_1, n_2 \ge N_\epsilon$$

Then $|S_{n_1} - S_{n_2}| < \epsilon$ for $n_1, n_2 \ge N_\epsilon$. □

This is IMPORTANT – Better understand it thoroughly.

Corollary 8.3 (Root Test)
$|b_n| \le Cr^n, 0 < r < 1$, C, r fixed, then $\sum b_n$ converges.

Reason: $\sum_{n=0}^\infty Cr^n = C\frac{1}{1-r}$ (geometric series).

Exercise 8.1. $\sum_{n=0}^N Cr^n = C\frac{r^{N+1}-1}{r-1}, 0 < r < 1$ has limit $\frac{C}{1-r}$. Prove by induction.

Detail: Hypothesis:

$$|b_n| \le Cr^n$$
$$\sum_1^\infty |b_n| \le \sum_1^\infty Cr^n < \infty$$
$$\sum_b^N |b_n| \le \sum_0^N Cr^n \le M < \infty$$

So $\sum_0^N |b_n|$ converges and bounded by Cr, and $b_1 + b_2 + \dots$ converges absolutely.

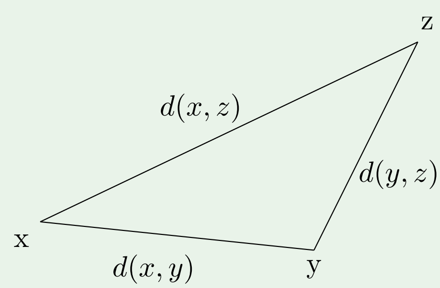
§9 Lec 9: Oct 21, 2020

§9.1 Metric Spaces

Definition 9.1 (Metric Spaces) — A set X, elements are "points", together with a function on $\underbrace{X \times X}_{\text{ordered pairs }(x,y)}$, $x \in X, y \in Y$, $\underbrace{d(x,y)}_{\text{distance}}$ with the following properties:

1. $d(x,y) \ge 0$ for all x, y.
   $d(x,y) = 0 \iff x = y$. Or $d(x,x) = 0$.
2. $d(x,y) = d(y,x)$.
3. $\triangle$ inequality:

$$d(x,y) + d(y,z) \ge d(x,z)$$
$$d(x,z) \le d(x,y) + d(y,z)$$



Example 9.2
1. X set. Can you define a $d : X \times X \to \mathbb{R}$ to make (x, d) a metric space?

   YES! Define given set X, $d(x_1, x_2) = 0$ if $x_1 = x_2$, or $d(x_1, x_2) = 1$ if $x_1 \neq x_2$. "discrete".

   - $d(x, y) \geq 0$.
   - $d(x, y) = d(y, x)$.
     $x = y$ both are 0.
     $x \neq y$ both are 1.
   - $d(x, z) \leq d(x, y) + d(y, z)$
     $x = z \implies d = 0$.
     $x \neq z \implies d(x, z) = 1$.
     If $x = y$ then $y \neq z$ so $1 \leq 0 + 1$
     ...

2. (INTERESTING) $d(x, y) = |x - y|$ for $\mathbb{R}$.
   $d(\frac{p}{q}, \frac{r}{s}) = |\frac{p}{q} - \frac{r}{s}|$ for $\mathbb{Q}$.

   Note: X is a metric space $Y \subset X$ then $\left(Y, d\Big|_{Y \times Y}\right)$ is a metric space.

<u>Motivation</u>: Stuff about $\mathbb{R}$ involving e.g., continuity and limits can be transferred to metric space.

Example 9.3
$\{x_n\}$ is a sequence in a metric space (X, d) (or X) has limit $x_0 \in X$ if for every $\epsilon > 0$, there is an N_ϵ s.t. $d(x, x_0) < \epsilon$ if $n \geq N_\epsilon$. (If $X = \mathbb{R}$, $d(x, y) = |x - y|$ same as before)

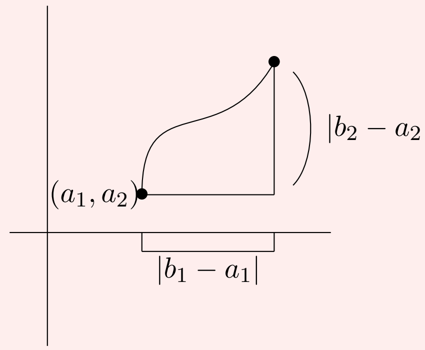
Example 9.4
Function: $f : (X, d_1) \to (Y, d_2)$. Continuity at $x_0 \in X$?
Real case: f cont at x_0 means given $\epsilon > 0$ $\exists \delta > 0$ s.t. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.
Metric space case: f cont at x_0 means given $\epsilon > 0 \exists \delta > 0$ s.t. $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \epsilon$.

More examples:

Example 9.5
$\mathbb{R}^2 = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$
$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}\}$
$\vdots$
$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}\}$

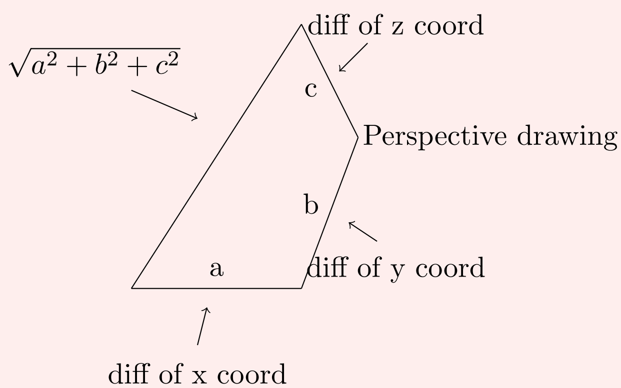
Interesting metric on $\mathbb{R}^2$ $d\left((a_1,a_2),(b_1,b_2)\right)$

$$d = \sqrt{(b_1-a_1)^2+(b_2-a_2)^2}$$



$\mathbb{R}^n (x_1, x_2, \ldots, x_n), (y_1, \ldots, y_n)$

$$d \coloneqq \sqrt{(y_1-x_1)^2+\ldots+(y_n-x_n)^2}$$



Is this function on $\mathbb{R}^n$ a metric?

1. $d(x,y) \geq 0, = 0 \iff x = y$ where $x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n)$ and

$$d(x,y) = \sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2}$$

2. $d(x,y) = d(y,x)$

3. BUT BUT BUT – $\triangle$ inequality is not so easy.

$$\sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2} \leq \sqrt{(x_1-z_1)^2+\ldots+(x_n-z_n)^2} + \sqrt{(z_1-y_1)^2+\ldots+(z_n-y_n)^2}???$$

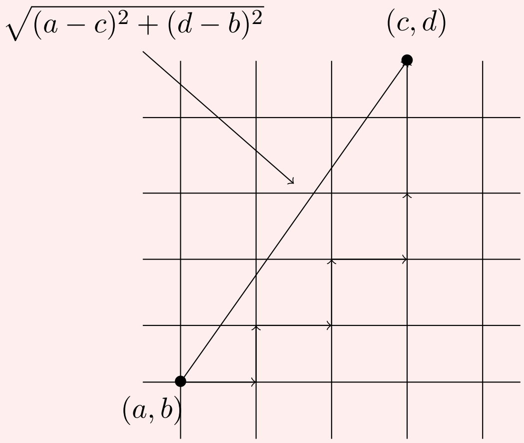
   Does $d(x,y) \leq d(x,z) + d(z,y)$ work?

   YES but proof later :(

Realize that it's okay to assume $z = (0,0,\ldots,0)$

Example 9.6

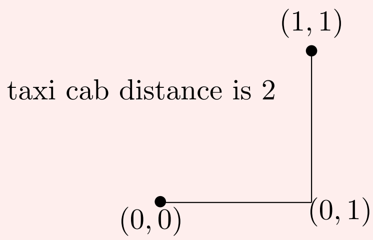
Try another metric
$\mathbb{R}^2$ – taxicab



$$|c-a| + |d-b| = d((a,b),(c,d))$$

min of length of taxi car

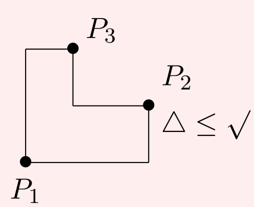
Easy to see that this d is really a metric. $\triangle$ inequality is easy!



Euclidean distance $= \sqrt{2}$

diff of x's $\le$ Euc dis

diff of y's $\le$ Euc dis

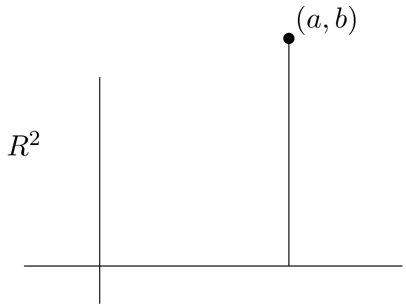


$$d(P_1, P_2) + d(P_2, P_3) \ge d(P_1, P_3)$$

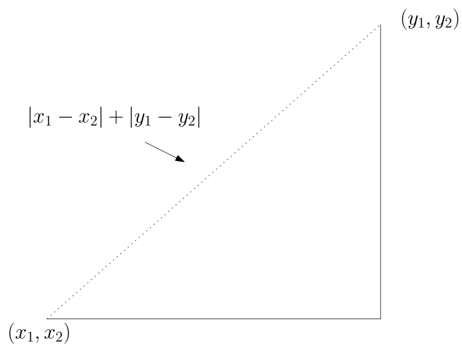
§10 Lec 10: Oct 23, 2020

§10.1 Metric on $\mathbb{R}^n$

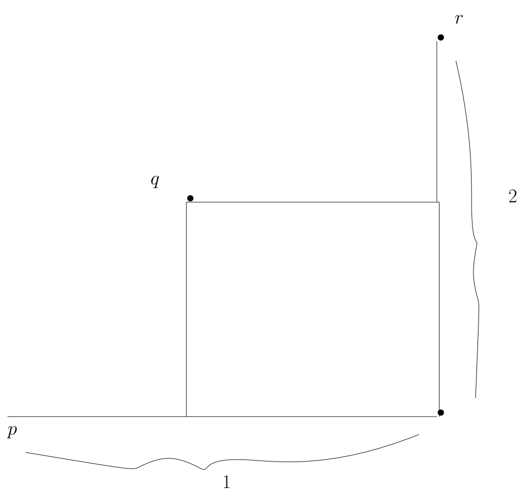
$\mathbb{R}^n : \{(x_1, \ldots, x_n) : x_j \in \mathbb{R}\}$



We want to make $\mathbb{R}^n$ a metric space. Last time, we defined "taxi cab metric", $d\left((x_1, \ldots, x_n), (y_1, \ldots, y_n)\right) = \sum_{i=1}^n |x_i - y_i|$ Verify $d(\vec{x}, \vec{y}) \geq 0$ or $= 0$ if $\vec{x} = \vec{y}$ and $\triangle$ inequality,



$$d(p, q) + d(q, r) \geq d(p, r)$$



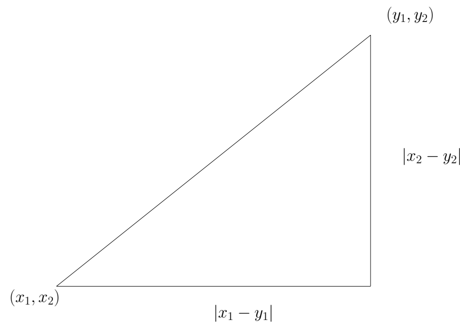
$d(p, r) = seg\,1 + seg\,2$

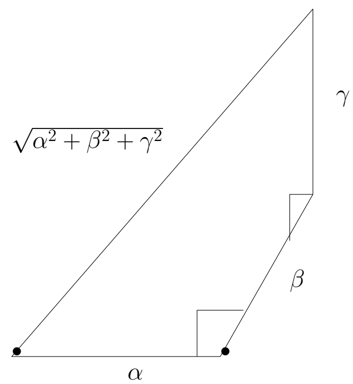
§10.2 Triangle Inequality in Euclidean Space

New idea: Euclidean distance (or Pythagorean distance)

$$d\left((x_1, x_2), (y_1, y_2)\right) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

For $\mathbb{R}^n$: $d\left((a_1, \ldots, a_n), (b_1, \ldots, b_n)\right) \coloneqq \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \ldots + (a_n - b_n)^2}$.





We need to know:

1. $d(\vec{a}, \vec{b}) \geq 0$
2. $d(\vec{a}, \vec{a}) = 0$ so $d(\vec{a}, \vec{b}) = 0 \implies \vec{a} = \vec{b}$
3. $d(\vec{a}, \vec{b}) = d(\vec{b}, \vec{a})$
4. ? $\triangle \leq 0$, $\vec{a}, \vec{b}, \vec{c}$
$$d(\vec{a}, \vec{c}) \leq d(\vec{a}, \vec{b}) + d(\vec{b}, \vec{c})$$

   For $\mathbb{R}^n$,

   $$\sqrt{(a_1-c_1)^2+\ldots+(a_n-c_n)^2} \leq \sqrt{(a_1-b_1)^2+\ldots+(a_n-b_n)^2}+\sqrt{(b_1-c_1)^2+\ldots+(b_n-c_n)^2}$$

   We certainly need proof for $\triangle$ inequality: Copson($p > 1$) – for case $p = 2$

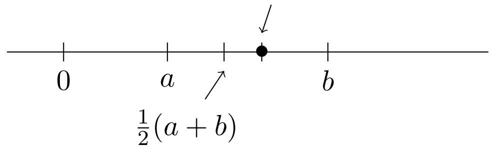
First step: $\alpha\beta \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2$ for all real α, β.
Reason:

$$\begin{aligned} 2\alpha\beta &\leq \alpha^2 + \beta^2 \\ \alpha^2 + \beta^2 - 2\alpha\beta &\geq 0 \\ (\alpha - \beta)^2 &\geq 0 \checkmark \end{aligned}$$

"Geometric mean $\leq$ Arithmetic mean"
Let $\alpha = \sqrt{a}, \beta = \sqrt{b}, a, b \geq 0$

$$\underbrace{\sqrt{ab}}_{\text{geometric mean of a,b}} \leq \frac{1}{2}(a) + \frac{1}{2}(b) = \underbrace{\frac{1}{2}(a+b)}_{\text{arithmetic mean}}$$

$$\sqrt{ab} < \tfrac{1}{2}(a+b) \text{ if } a \neq b$$



<u>Second step</u>:

$$\vec{a} = (a_1, \dots, a_n)$$
$$\vec{b} = (b_1, \dots, b_n)$$

and we know

$$a_i b_i \le \frac{1}{2}a_i^2 + \frac{1}{2}b_i^2$$

Then,

$$\sum_{i=1}^n a_i b_i \le \frac{1}{2}\sum_{i=1}^n a_i^2 + \frac{1}{2}\sum_{i=1}^n b_i^2$$

So, $\sum a_i^2 = 1$, $\sum b_i^2 = 1$, $\sum a_i b_i \le 1$

Claim 10.1.

$$\sum a_i b_i \le \left(\sum a_i^2\right)^{\frac{1}{2}} \left(\sum b_i^2\right)^{\frac{1}{2}}$$

But

$$\left|\vec{a}\cdot\vec{b}\right| \le \|\vec{a}\|\|\vec{b}\|$$

So it's okay to define θ,

$$\cos\theta = \frac{\vec{a}\vec{b}}{\|\vec{a}\|\|\vec{b}\|} \in [-1, 1]$$

Verification of claim: $\vec{a}, \vec{b} \neq \vec{0}$

$$A_i = \frac{a_i}{\sqrt{\sum a_i^2}}, \quad B_i = \frac{b_i}{\sqrt{\sum b_i^2}}$$

And $\sum A_i^2 = 1$, $\sum B_i^2 = 1$ – also $\sum_{i=1}^n A_i B_i \le 1$ which is equivalent to $\frac{\sum a_i b_i}{\sqrt{\sum a_i^2}\sqrt{\sum b_i^2}} \le 1$.
So $|\sum a_i b_i| \le \sqrt{\sum a_i^2}\sqrt{\sum b_i^2}$.
BIG DEAL: "Cauchy Schwarz inequality" What does this have to do with $\triangle$ inequality for Euclidean metric. Consider: $\vec{a}, \vec{b}$

$$\sum_{j=1}^n (a_j + b_j)^2 = \sum_{j=1}^n a_i(a_j + b_j) + \sum_{j=1}^n b_j(a_j + b_j)$$

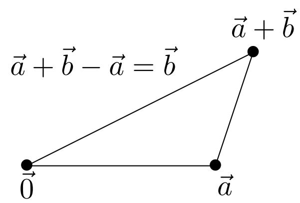
Now apply Cauchy – Schwarz

$$\begin{aligned}\sum_{j=1}^n (a_j + b_j)^2 \le &\left(\sum_{j=1}^n a_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^n (a_j + b_j)^2\right)^{\frac{1}{2}} \\ &+ \left(\sum_{j=1}^n (b_j)^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^n (a_j + b_j)^2\right)^{\frac{1}{2}}\end{aligned}$$

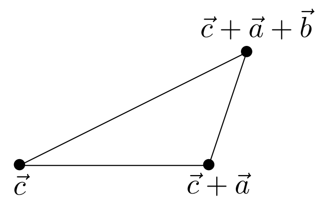
Divide through by $\left(\sum (a_j + b_j)^2\right)^{\frac{1}{2}}$

$$\left(\sum (a_j + b_j)^2\right)^{\frac{1}{2}} \le \left(\sum a_j^2\right)^{\frac{1}{2}} + \left(\sum b_j^2\right)^{\frac{1}{2}}$$

The above inequality is indeed the triangle inequality for $\vec{0}, \vec{a}, \vec{a} + \vec{b}$



But of course this gives you the triangle inequality in general.



△ inequality works in general

Last step: $\vec{p}, \vec{q}, \vec{r}$
Triangle inequality:

$$d(0, \vec{p} - \vec{r}) \le d(0, \vec{q} - \vec{p}) + d(\vec{q} - \vec{p}, \vec{r} - \vec{p})$$

Same as △ ineq for $0, \vec{q} - \vec{p}, (\vec{r} - \vec{q}) + (\vec{q} - \vec{p})$ or
$0, \vec{a}, \vec{a} + \vec{b}$ if $\vec{a} = \vec{q} - \vec{p}, b = \vec{r} - \vec{q}$.

§11 Lec 11: Oct 26, 2020

§11.1 Metric Spaces Examples

Last time, we prove △ ineq. proof, taxi-cab metric, and sup norm metric. This gives rise to same "convergence idea". Namely $x_n \in X(X, d)$ converges to $L \in X$ means

$$\lim_{n \to \infty} (x_n - L) = 0$$

In all three metrics

$$\vec{x}_j \to L \qquad \lim \vec{x}_j = L$$

means (is same as) ith coordinate of $\vec{x}_j$ converges to ith coord of L for each $i = 1, 2, \ldots, n$.
$\{x_n\}$ Cauchy if given $\epsilon > 0 \exists N_\epsilon \ni n_1, n_2 \ge N_\epsilon$

$$d(x_{n_1}, x_{n_2}) < \epsilon$$

Exercise 11.1. $\{x_n\}$ Cauchy in $\mathbb{R}^n$ (any one of three metrics – Cauchy is the same idea in all three metrics) then $\{x_n\}$ has limit L, some L.

$$\sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2} \leq \sqrt{n}\max|x_j-y_j|, j=1,\ldots,n$$

which can be derived by the followings,

$$|x_j-y_j| \leq \max|x_j-y_j|$$
$$|x_j-y_j|^2 \leq \max{}^2|x_l-y_l|, l=1,\ldots,n$$
$$(x_1-y_1)^2+\ldots+(x_n-y_n)^2 \leq n\max{}^2|x_l-y_l|$$
$$\sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2} \leq \sqrt{n}\max|x_l-y_l|$$

$l_2 : \{x_j\}$ infinite sequences $j=1,2,3,\ldots$ where $\left\{\sum_{j=1}^{\infty} x_j^2 < \infty\right\}$ which means

$$\exists M \ni \quad \sum_{j=1}^{M} x_j^2 \leq M$$

$$(1,\frac{1}{2},\frac{1}{3},\ldots) \in l_2$$
$$(1,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}},\ldots) \notin l_2$$

because $1+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+\ldots \to \infty$ $(\frac{1}{n}) \to \infty$ as $n\to\infty$.
<u>vector space</u>:

$$c\{x_j\} = \{cx_j\}$$
$$\{x_j\} \in l_2 \implies \in l_2$$
$$\sum c^2x_j^2 = c^2\sum x_j^2$$

Also,

$$\{x_j\}+\{y_j\} = \{x_j+y_j\}$$
$$(x_j+y_j)^2 \leq 2(x_j^2+y_j^2)$$
$$x_jy_j \leq \frac{1}{2}(x_j^2+y_j^2)$$

$\{x_j\},\{y_j\} \in l_2$ then

$$d(\{x_j\},\{y_j\}) = \left[\sum(x_j-y_j)^2\right]^{\frac{1}{2}}$$

makes sense. (l_2,d) is a metric space obvious except $\triangle$ ineq. It's enough to check

$$d(0,\vec{x})+d(\vec{x},\vec{x}+\vec{y}) \geq d(0,\vec{x}+\vec{y})$$

which follows by taking limits of $\triangle$ ineq. for truncation up to level N.

$$d(\vec{0},(x_1,\ldots,x_N)) + d((y_1,\ldots,y_N),(x+y) \text{ up to N}) \geq d(\vec{0},(x+y)_N)$$

<u>l_2 is metric space</u>

l_2 is complete – Cauchy sequences have some limits.

Example 11.1

$C([0,1]) :=$ cont: $R--$ valued function $[0,1]$

$$\begin{aligned} d(f,g) &= \max|f(x)-g(x)| \\ &= \sup|f(x)-g(x)| \end{aligned}$$

"sup norm" All properties clear. "L^2 norm" – distance on $C[0,1]$:

$$d_2(f,g) = \left(\int_0^1 (f(x)-g(x))^2\right)^{\frac{1}{2}}$$

where $d_2 \geq 0,\ f,g,h \in C[0,1]$.
Imitate argument for $\triangle$ ineq. on $\mathbb{R}^n$: Cauchy Schwarz ineq.

$$\int_0^1 fg \leq \left(\int f^2\right)^{\frac{1}{2}} \left(\int g^2\right)^{\frac{1}{2}}$$

So,

$$\begin{aligned} f(x)g(x) &\leq \frac{1}{2}\left(f^2(x)+g^2(x)\right) \\ \int_0^1 f(x)g(x) &\leq \frac{1}{2}\int_0^1 f^2(x) + \frac{1}{2}\int_0^1 g^2(x) \end{aligned}$$

Apply these, $F = \frac{f(x)}{\sqrt{\int_0^1 f^2}}, G = \frac{g}{\sqrt{\int_0^1 g^2}}$, $\int F^2 = 1, \int G^2 = 1$. Also, we know $\int fg \leq 1$ if $\int f^2 = 1, \int g^2 = 1$.
Remainder argument for $\triangle$ ineq. is same as before

$$\int (f+g)^2 = \int f(f+g) + \int g(f+g)$$

Apply Cauchy – Schwartz,

$$\begin{aligned} \int (f+g)^2 &\leq \left(\int f^2\right)^{\frac{1}{2}} \left(\int (f+g)^2\right)^{\frac{1}{2}} + \left(\int g^2\right)^{\frac{1}{2}} \left(\int (f+g)^2\right)^{\frac{1}{2}} \\ \left(\int (f+g)^2\right)^{\frac{1}{2}} &\leq \left(\int f^2\right)^{\frac{1}{2}} + \left(\int g^2\right)^{\frac{1}{2}} \end{aligned}$$

§11.2 A Glance at Complex Number

Special case of $\mathbb{R}^n$, Euclidean norm

$$\begin{gathered} \mathbb{R}^2 : \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = d\left((x_1,x_2),(y_1,y_2)\right) \\ \mathbb{C} : \{(a+bi)\} - \text{ Complex numbers} \end{gathered}$$

$(x_1, x_2) \leftrightarrow x_1 + ix_2$. Metric on $\mathbb{C}$, $z, w \in \mathbb{C}$

$$\begin{gathered} |z-w| = d(z,w) \quad \text{as pts in } \mathbb{R}^2 \\ z = a+bi \\ |z| = |a+bi| = \sqrt{a^2+b^2} \end{gathered}$$

We also define multiplication in $\mathbb{C}$ as follows

$$(a+bi)(c+di) \coloneqq (ac-bd) + (bc+ad)i$$

Example 11.2

$$\frac{1}{c+di} = \frac{c}{c^2+d^2} - \frac{d}{c^2+d^2}i$$

For $z = a+bi, w = c+di$ we define

$$\begin{aligned} |zw| &= |z||w| \\ &= \sqrt{a^2+b^2}\sqrt{c^2+d^2} \\ &= \sqrt{(ac-bd)^2+(bc+ad)^2} \end{aligned}$$

verify if the above step is actually equal

§12 Lec 12: Oct 28, 2020

§12.1 Midterm Announcement

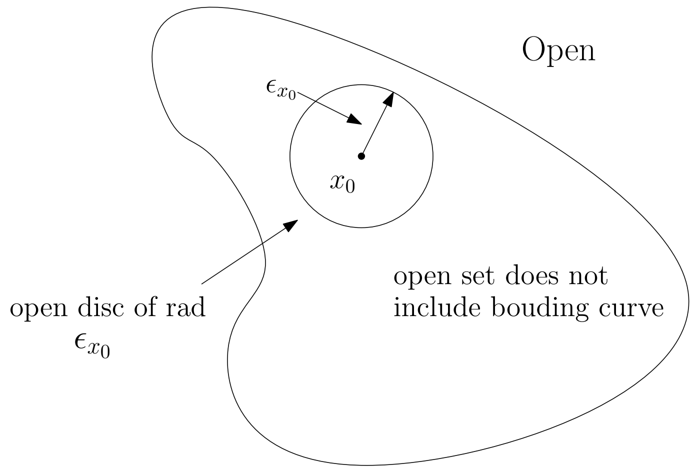
Midterm – Given out on Fri, Nov 6 at 3:00 pm. and due by Sat, Nov 7 at 11:00 pm.

§12.2 Open sets in Metric Space

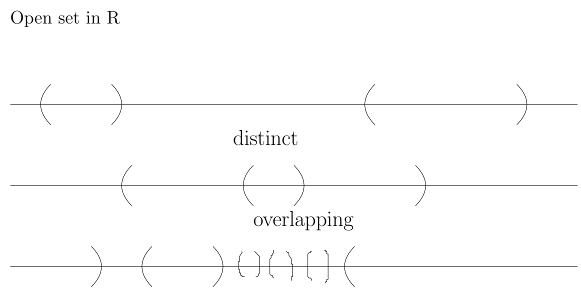
Beginning of "topology": (X, d) metric space

Definition 12.1 (Open sets) — $U \subset X$ open if for every $x_0 \in U$ there is an $\epsilon_{x_0} > 0$ s.t.

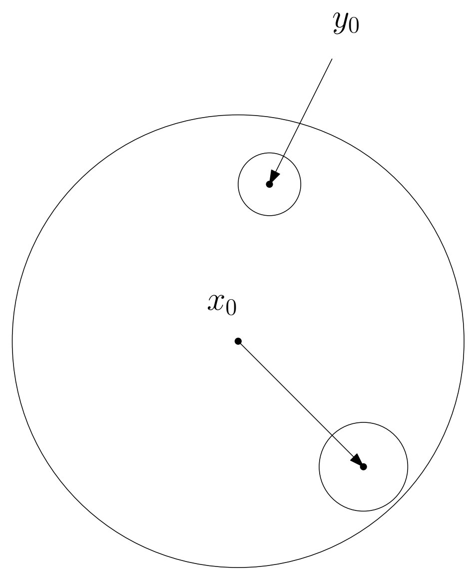
$$\underbrace{\{x \in X : d(x, x_0) < \epsilon_{x_0}\}}_{B(x_0, \epsilon_{x_0}) - \text{ open ball}} \subset U$$



open sets in R looks like unions of open intervals



> **Lemma 12.2**
> $B(x_0, \epsilon), \epsilon > 0$ open ball is open set.



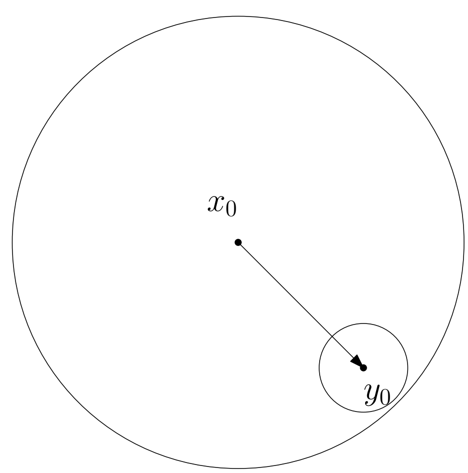
Proof. Need given $y \in B(x_0, \epsilon)$, $\lambda_y > 0$ s.t. $B(y, \lambda) \subset B(x_0, \epsilon)$.
Try $\lambda = \epsilon - d(x_0, y_0)$.
Suppose $y \in B(y_0, \epsilon) \iff d(y_0, y) < \epsilon - d(x_0, y_0)$

$$d(y_0, y) + d(x_0, y_0) < \epsilon$$

So,

$$d(x_0, y) \le d(x_0, y_0) + d(y_0, y) < \epsilon$$

So $y \in B(x_0, \epsilon)$.



□

Reason why people care about open sets:
Remember: $f(X, d) \to (Y, d)$ continuous means given $\epsilon > 0, x_0 \in X$ there exists $\delta > 0$ s.t.

$$d_X(x_0, x) < \delta \implies d_Y\left(f(x_0), f(x)\right) < \epsilon$$

$\to$ Direct transcription of number – continuity of f can be described in terms of open sets in X and in Y. For this: $f : X \to Y$ and $V \subset Y$, then $f^{-1}(V) = \{x \in X : f(x) \in V\}$ f does not need to be invertible.

Example 12.3

$$f : \underbrace{X}_{\text{people}} \to \mathbb{Z}, \quad f(x) = \text{ integer age of x}$$

$$f^{-1}\left(\{20, 21, 22\}\right) = \text{everybody that's age 20,21, or 22}$$

Theorem 12.4 (Continuity – Open Sets)
$f : (X, d_x) \to (Y, d_y)$ is continuous if and only if (in δ, ϵ sense) $f^{-1}(V)$ is open in X for every V open in Y.
Slogan: continuity means inverses of open sets are open.

$f : X \to Y$, $g : Y \to Z \to g(f(x))$ compositions of f and g.

Claim 12.1. If f, g continuous then the composition is continuous

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Proof. (of Theorem) Suppose $f^{-1}(V)$ is open when V is open. Given $x_0 \in X, \epsilon > 0$ want $\delta > 0 \ni \underbrace{x \in B(x_0, \delta)}_{d(x,x_0)<\delta} \implies d\left(f(x), f(x_0)\right) < \epsilon$

$$\{y : d(y, f(x_0) < \epsilon\} = B(f(x_0), \epsilon)$$

Know that it's open by the above lemma. So,

$$f^{-1}(B(f(x_0), \epsilon)) \text{ open}$$

and $x_0 \in (B(f(x_0), \epsilon))$. So $f^{-1}(B(f(x_0), \epsilon))$ being open

$$\implies \delta > 0 \quad B(x_0, \delta) \subset f^{-1}\left(B(f(x_0), \epsilon)\right)$$

says $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \epsilon$ ✓
Took care of f^{-1}(open) is open $\implies$ continuity. Now,
Does continuity(ϵ, δ sense) $\implies f^{-1}$ (open) is open?
This also works: Suppose V open, and $x_0 \in f^{-1}(V)$. Need $\delta > 0$ s.t. $B(x_0, \delta) \subset f^{-1}(V)$. $f(x_0) \in V$ (meaning of $x_0 \in f^{-1}(V)$) $\exists \epsilon$ s.t. $B(f(x_0), \epsilon) < V$ (V is open). Then ϵ, δ defn of continuity $\exists \delta$ s.t. $f(B(x_0, \delta)) \subset B(f(x_0), \epsilon) \subset V$. So $B(x_0, \delta) \subset f^{-1}(V)$. ✓ □

Forward continuous images of open sets are not necessarily open.

Example 12.5
$f(x) = x^2, \quad f\left((-1, 1)\right) = [0, 1)$ which is not open.

<u>*Note*</u>: A notion to help understand the concept of open sets is thinking about how a map sends a point to a point but its inverse can send a point to a set.

§13 Lec 13: Oct 30, 2020

§13.1 Open Sets (Cont'd)

Recall: U open means $\forall x \in U$, $\exists \epsilon > 0$ s.t. $B(x,\epsilon) \subset U - \{y : d(x,y) < \epsilon\}$ (open ball)
$f : X \to Y$, $f^{-1}(V)$ open in X if V open in Y $\iff$ f continuous – δ, ϵ sense (p.91, Copson)
Properties of being open: (finiteness is important)

0. ϕ, X open sets – "trivial"
1. $U_\lambda, \lambda \in \Lambda$, open for each λ, $\bigcup_{\lambda\in\Lambda} U_\lambda$ is open.
2. $U_1, \ldots, U_n$ open then
$$\bigcap_{j=1}^{n} U_j \text{ open}$$
U open does not imply $X - U$ is open (not necessarily true).
3. $U_1, U_2, U_3, \ldots$ open
$$\bigcup_{j=1}^{\infty} U_j \text{ open}$$

Example 13.1
$U_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \subset \mathbb{R}$
$$\bigcap_{n=1}^{\infty} U_n = \{0\} \text{ one point}$$
which is not open.

$U_\lambda, \lambda \in \Lambda$ open (assume). We want $\bigcup U_\lambda$ is open.

Proof. Suppose $x \in \bigcup_{\lambda\in\Lambda} U_\lambda \implies x \in U_{\lambda_1}$ open. So $\exists \epsilon > 0 \ni B(x,\epsilon) \subset U_{\lambda_1}$
$$\implies B(x,\epsilon) \subset \bigcup_{\lambda\in\Lambda} U_\lambda$$
□

$u_1, \ldots, u_n$ open (finitely many $U's$). If $x \in \bigcap_{j=1}^n U_j, x \in U_j$ for each $j = 1, \ldots, n$. So for $\epsilon_j > 0$
$$B(x,\epsilon_j) \subset U_j \quad (U_j \text{ open})$$
Let $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n) > 0$. Then $B(x,\epsilon) \subset B(x,\epsilon_j) \subset U_j$. So $B(x,\epsilon) \subset U_j$ for all j. So $B(x,\epsilon) \subset \bigcap_{j=1}^n U_j$. Therefore, $\bigcap_{j=1}^n U_j$ is open. Contrast this with $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ example.

§13.2 Topological Space

Set S with some sets specified as open with

0. ϕ, X open.
1. $\cup$ open is open.
2. $\cap$ open is open.

This is a Topological Space.
We know (X, d) with our definition of $U \subset X$ open is a topological space.

§13.3 Closed Sets

Back to metric space (but also works in topological spaces)

Definition 13.2 (Closed Sets) — $C \subset X$ is <u>closed</u> if and only if $X - C$ is open.

<u>Note</u>: Being closed does not necessarily mean the opposite of open. For example, X is both closed and open – X open and $X - X = \emptyset$ open. Also, $\emptyset$ both closed and open – $\emptyset$ open & $X - \emptyset = X$ is open.
<u>Closed sets</u>:

0. ϕ, X closed (checked already)
1. $C_\lambda, \lambda \in \Lambda$ closed then $\bigcap_{\lambda\in\Lambda} C_\lambda$ is closed
2. $C_1, \ldots, C_n$ are closed then

$$\bigcup C_j = C_1 \cup \ldots \cup C_n \text{ is closed}$$

watch out for $\left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$ $X = \mathbb{R}$, $\mathbb{R} - [-1+\frac{1}{n}, 1-\frac{1}{n}]$ which is equivalent to $(-\infty, -1+\frac{1}{n}) \cup (1-\frac{1}{n}, +\infty)$. On the other hand,

$$\bigcup_{n=1}^{\infty} \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right] = (-1, 1) \text{ not closed}$$

Proof. (1) – $\bigcap_{\lambda\in\Lambda} C_\lambda$ is it closed? Closed means $X - \bigcap_{\lambda\in\Lambda}$ open – True? According to August de Morgan

$$X - \left(\bigcap_{\lambda\in\Lambda} C_\lambda\right) = \bigcup_{\lambda\in\Lambda} (X - C_\lambda)$$

□

A notion to understand this is people - (dog owners $\cap$ cat owners) = people who do not own both a dog and cat = (people who do not own a dog) $\cup$ (people who do not own a cat) = (people - dog owners) $\cup$ (people - cat owners).

Slogan: Complements of intersections is the union of complements. Or complements of unions is the intersection of complements – De Morgan's Laws.
Now, back to the closed sets, we have $X - \cap C_\lambda$ where C_λ closed then $= \cup(X - C_\lambda)$ open because C_λ are closed. So $\cup(X - C_\lambda)$ open (by prop(1) for open sets). So $\cap C_\lambda$ closed if each C_λ is closed.
Prop (2) for closed sets

$$C_1 \cup \dots C_n$$

is closed if each $C_j j$ is closed. We need openness of $X-$ union:

$$X - (C_1 \cup \dots \cup C_n) = \bigcap_{j=1}^{n} (X - C_j)$$

which is open by C_j being closed for each j and also is the finite intersection of open sets. So it's open by prop (2) of open sets. So $C_1 \cup \dots \cup C_n$ closed (its complement is open).
Note: Continuity can be defined for functions from (S, Q_S) to (T, Q_T) : $f : S \to T$ continuous by definition if $f^{-1}(V) \forall V \subset T$ open is open in S.

§14 Lec 14: Nov 2, 2020

§14.1 Set, Tables, & Characteristics Functions

$A \subset X$, X_A is called characteristics function where

$$\begin{aligned} X_A &: X \to \{0, 1\} \\ X_A(x) &= 1 \text{ if } x \in A \\ X_A(x) &= 0 \text{ if } x \notin A \\ A &= \{x : X_A(x) = 1\} \\ X_{X-A}(x) &= 1 - X_A(x) \end{aligned}$$

X_A	X_B	$X_{A\cup B}$	$X_{A\cap B}$	X_{X-A}	X_{X-B}
0	0	0	0	1	1
1	0	1	0	0	1
0	1	1	0	1	0
1	1	1	1	0	0

$X_{(X-A)\cap(X-B)}$		$X_{X-(A\cup B)}$
1		1
0	same $\longleftrightarrow$	0
0		0
0		0

De Morgan's Law:

$$X_{(X-A)\cap(X-B)} = X_{X-(A\cup B)}$$
$$\iff (X-A)\cap(X-B) = X-(A\cup B)$$

Exercise 14.1. $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

Reason: So, $(A\cap B)\cup(A\cap C) = A\cap(B\cup C)$.

A	B	C	$A\cap(B\cup C)$	$(A\cap B)\cup(A\cap C)$
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

$$\begin{array}{lll} A_1 & 0,1 & \overset{1}{x\in A_1}\ \overset{0}{x\notin A} \\ A_2 & 0,1 & \overset{1}{x\in A_2}\ \overset{0}{x\notin A_2} \\ A_3 & & \\ A_4 & & \end{array}$$

A_1	A_2	A_3	A_4
0,1	0,1	0,1	0,1

$X_{\liminf\{A_n\}} = 1$ if only 1's some n onward.
$X_{\limsup\{A_n\}} = 1$ for x such that table for x contains infinitely many 1s. – A way to do homework.

§14.2 Closed Sets in Metric Spaces

$C\subset X$, (X,d) metric space. It is closed if $X-C$ is open.
De Morgans' Laws: $\cap C_\lambda$ closed if C_λ are closed – $C_1\cup\ldots\cup C_n$ closed if $C_1,\ldots,C_n$ are closed.

Corollary 14.1

There is a minimal closed (closure) of a set containing a given A

$$A^- = \cap C$$

C closed, $A\subset C$ closed.

We can describe the closure of A in terms of limits of sequences. A point x is a limit point of A (Copson: adherent point) if

$$\exists\{a_n\}\in A \text{ s.t. } \{a_n\} \text{ converges and } \lim a_n \text{ is the point } x$$

If $x\in A$ then x is a limit point:
$x =$ limit of sequence, $a_n = x$, for all $n = 1,2,3,\ldots$.

- Set of limit points $\supset A$.
- Set of limit points is a closed set.

In order to understand that, we have to understand the characterization of a set being closed in terms of convergence of sequence:

A set A is closed $\iff$ every limit point of A is in A.

Proof. (of characterization) $(\rightarrow)$ closed $\implies$ contains limit points
$\lim a_n = a_0$ want to know that a_0 must be in A. Suppose not: Then $X - A$ is open $\exists \epsilon > 0 B(a_0, \epsilon) \subset X - A$ which is impossible $\lim a_n = a_0$.
$(\leftarrow)$ A contains all limit points $\implies$ A closed.
Suppose $X - A$ is not open and $\exists$ some $a_0 \in X - A$ s.t. $B(a_0, \epsilon) \not\subset X - A$ for every $\epsilon > 0$. For $\epsilon = \frac{1}{n}, n = 1, 2, 3, \ldots$, $\exists x_n \in B(a_0, \frac{1}{n})$ with $x_n \in X - A$ so $x_n \in A$.

$$d(a_0, x_n) < \frac{1}{n}$$

$x_n \in A$, $\lim x_n = a_0$ where x_n is a sequence in A but $\lim \notin AX$. So $X - A$ is open. □

think carefully through this proof

Back to set of limit points of A is always closed:

$$\lim x_n = x_0$$

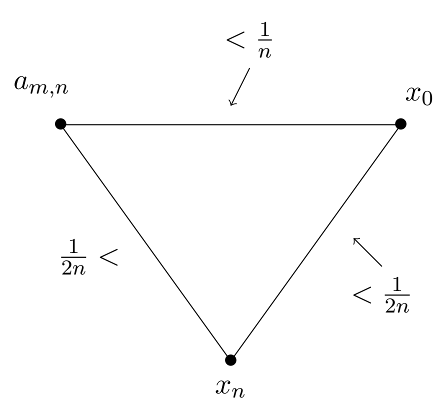
$\underbrace{\{x_n\}}_{\text{each is a limit point of A}}$. Hope x_0 is a limit point of A. To be a limit point

$$x_n = \lim_{n \to \infty} a_{m,n}$$

Passing to a subsequence, we can suppose for each n, choose $d(x_n, x_0) < \frac{1}{2n}$.
Watch out! To get from $d(x_n, x_0) \to 0$ that $d(x_n, x_0) < \frac{1}{2n}$, we need to pass to a subsequence!
For each n, there is an $x_{N(n)}$ with $d(x_N, x_0) < \frac{1}{2n}$. Relabel that as x_n, i.e., (new) $x_n =$ (old) $x_{N(n)}$. So x_0 an A-limit implies x_0 is a limit of sequence $\{x_n\}, x_n \in A$ with $d(x_n, x_0) < \frac{1}{2n}$.
Choose $a_{m,n}$ such that $d(x_n, a_{m,n}) < \frac{1}{2n}$. Consider the sequence $\{a_{m,n}\}, n = 1, 2, 3, \ldots$

$$\begin{aligned} d(x_0, a_{m,n}) &\le d(x_0, x_n) + d(x_n, a_{m,n}) \\ &< \frac{1}{2n} + \frac{1}{2n} < \frac{1}{n} \end{aligned}$$

So x_0 is a limit of seq of points in a.



subsequence of orginal $\{x_n\}$
limit point of an A - sequence

A set of limit points is closed. C closed $\supset$ A. Then limit points of A in C. $C \supset$ set of limit points of A. So set points is a closed set $\supset A$ and every closed set that contains A contains set of limit points. So $A^- =$ set of limits of A.

Example 14.2

$\mathbb{Q}^- = \mathbb{R}$

$\sqrt{2}$ is a limit point of $\mathbb{Q}$. Every real number is a limit of sequence of rationals – " $\mathbb{Q}$ is dense in $\mathbb{R}$".

§15 Lec 15: Nov 4, 2020

§15.1 More on Open and Closed Sets

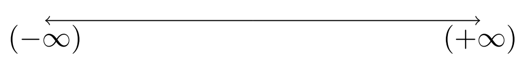
$x_n \to x_0$ where x_n is limit of a seq in A then $x_0 = A$ limit $\left[d(x_n, x_0) < \frac{1}{2n}\right]$.
Alternative view:
$x_n \to x_0$, $\lim d(x_n, x_0) = 0$. For each n, $\exists a_n \in A$ s.t. $d(a_n, x_n) < \frac{1}{n} \leftarrow$ because x_n is limit pt. So $\exists$ a seq in A converging to x_n. Then

$$\lim d(a_n, x_0) = 0$$

because $d(a_n, x_0) \le d(a_n, x_n) + d(x_n, x_0)$ as $n \to \infty$.
Closure of $A =$ set of sequence limits of sequences in A. Closed sets can be complicated. Open sets (at least in $\mathbb{R}$) seem simple. Open sets in $\mathbb{R}$:

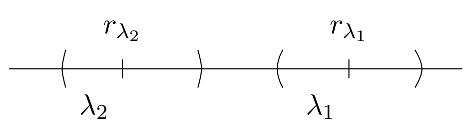
$$U \text{ open in } \mathbb{R} \iff U \text{ possibly infinite pairwise disjoint collection of } (a,b) \text{ or } (-\infty, a) \text{ or } (a, +\infty) \text{ or } \mathbb{R}$$

$$\text{maximal open interval} \subset U$$

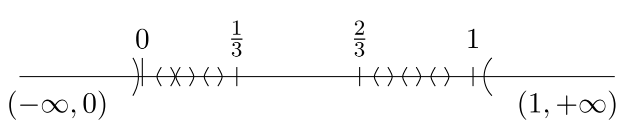


U = pairwise disjoint union of these

Number of intervals in U is countable (each contains a rational number). U_λ max open intervals $\subset U$ fixed. Pick λ_1 rational in $U_\lambda, U_{\lambda_1} \neq U_{\lambda_2}$ then $r_{\lambda_1} \neq r_{\lambda_2}$.



So rational numbers $\implies \{U_\lambda\}'s$ are countable (Λ is countable) if each max interval has one λ only.
Cantor set(closed set):



complement of $C = (+1, \infty), (-\infty, 0), (\frac{1}{3}, \frac{2}{3}), (\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9})$
At each stage, we remove open middle third of closed intervals that are left from previous stage. C is closed – complement is a union of open intervals hence open. C is not empty – $0 \in C, 1 \in C, \frac{1}{3} \in C, \frac{2}{3} \in C$. All the endpoint of $[0,1]$ and the removed open intervals $\subset C$.

<u>C inifite:</u> C contains some points that not endpoints.
Reason: set of endpoints is countable (can make a list of them) – (countable union of a finite sets). <u>But</u> C itself is uncountable. Why? We prove later a generalization of this!

$$x \in C \iff \text{a sequence } \{X_n\} \text{ in } x_n \in \{L, R\}$$

x_1 L if $x_1 \in$ down side of $[0,1) - \left(\frac{1}{3}, \frac{2}{3}\right)$, $x_1 \in$ upside of $[0,1) - \left(\frac{1}{3}, \frac{2}{3}\right)$
x_2 L if in downside – R if in upside.
Knowing x depends on a single LR valued sequence associated to unique $x \in C$ one and only $x \in C$ with that LR sequence being sequence for x. $LRL \ldots$ determined a sequence of closed intervals of successive length $\frac{1}{3^n}, n = 1, 2 \ldots$. Each is contained in previous ones "nested intervals".
Proofs earlier:

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \ldots$$

(nested intervals) of length $[a_n, b_n] \to 0$ as $n \to \infty$. $\exists$ one and one point in

$$\bigcap_{n=1}^{\infty} [a_n, b_n]$$

$$x \in C \to L, R \text{ seq}$$

L,R seq comes from exactly one $x \in C$. So to know C is uncountable just to have to know set of all L,R sequences is uncountable.

Proof. $\{L, R \text{ sequences}\}$ is countable.

1. L,R sequences no 1.
2. L,R seq no 2.

   ⋮

$\exists L, R$ sequences not in list: first element is L if first element here is R, if first element is L. 2nd: L if second element is R. R is second element of is L. New sequence is not in the list. Think about this – similar to subdivision argument to prove the accountability of $[0,1]$, powersets. □

<u>Baire Category Theorem</u>: later!
<u>Sierpinski Carpet</u>: (Check Wikipedia)

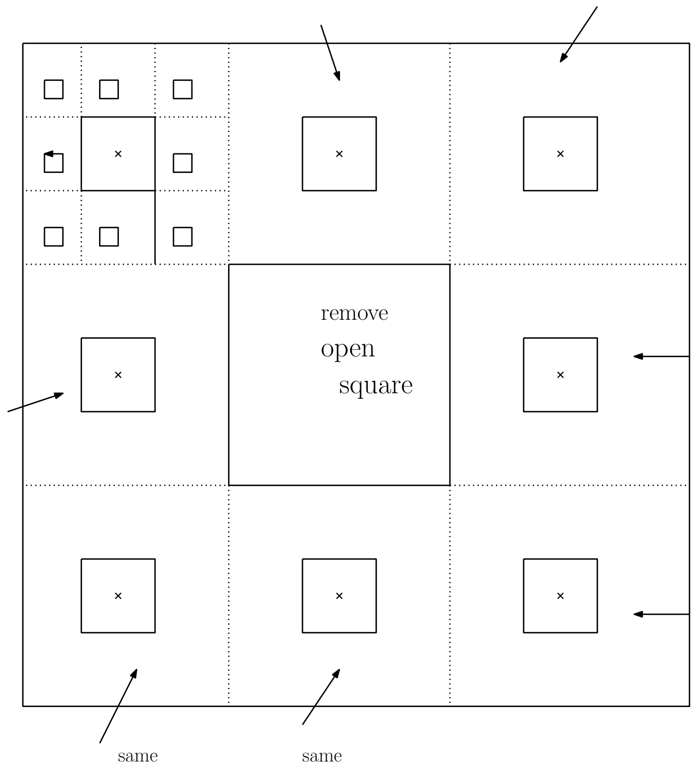


Figure 1: Sierpinski Carpet

interior (also interior of C) $= \emptyset$.
Closed uncountable set with interior $= \emptyset$.

§16 Midterm 1: Nov 6, 2020

Review for midterm 1, and it's distributed at 3:00 pm :D

§17 Lec 16: Nov 9, 2020

§17.1 Completeness (Cont'd)

Recall: (X, d) is complete means, by definition, Cauchy sequences have limits (in X).
Cauchy sequence – Given $\epsilon > 0, \exists N_\epsilon \ni n_1, n_2 \geq N_\epsilon \implies d(x_{n_1}, x_{n_2}) < \epsilon)$.
$\{x_n\} \to x_m$ means given $\epsilon > 0, \exists N_\epsilon \ni d(x_n, x_m) < \epsilon$ if $n \geq N_\epsilon$.
$\mathbb{R}$ is complete but $\mathbb{Q}$ is not.
Note: (X, d) is complete or not $\leftarrow$ depends on d as well as X. Any discrete metric space is, Cauchy $\iff$ eventually constant, complete. ✓

Example 17.1

$C\left([0,1]\right)$ is complete in $d(f,g) = \sup\left(f(x), g(x)\right)$ (sup norm) but not complete in $l^2, d(f,g) = \left(|f(x) - g(x)|^2\right)^{\frac{1}{2}}$. We will look at this later.

(X, d) metric space, $Y \subset X$, then $d\Big|_{Y\times Y}$ is a metric on Y – $(Y, d\Big|_{Y\times Y}$, Y is a subspace of X.

$$\underbrace{d_Y(y_1, y_2)}_{y_1, y_2 \in Y} = \underbrace{d_X(y_1, y_2)}_{y_1, y_2 \in X}$$

Lemma 17.2

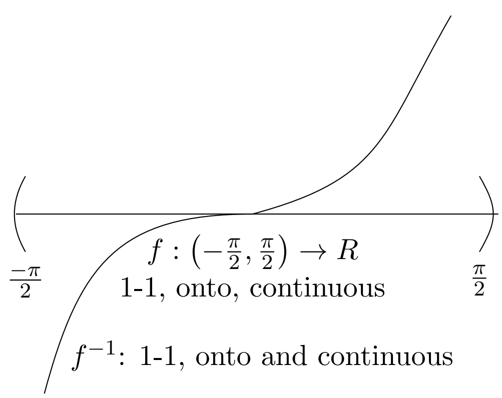
If (X, d) is complete, then $\left(Y, d\Big|_{Y\times Y}\right)$ is complete if and only if Y is closed in (X, d).

Proof. Left as exercise. □

Example 17.3

$\mathbb{R}$ is complete but not $[0,1) \subset \mathbb{R}$.

Completeness is not a topological property not determined by knowing which sets are open.



f preserves open sets – homomorphism – 1-1 and onto mapping s.t. open sets are preserved. Define a new metric on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ has usual metric $d(\alpha, \beta) - |\alpha - \beta|$ in which it is not complete.

$$d(\alpha, \beta) = d\left(f(\alpha), f(\beta)\right)$$

$$(|\alpha - \beta - \text{old}|) \overset{\text{new}}{=} d(\alpha, \beta) = |\tan\beta - \tan\alpha|$$

This new metric on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is complete.
f "isometric" by definition, distances are preserved (by f). means $(()$, new metric) and $(\mathbb{R}$, usual metric), $()$ new metric has same open sets $()$, old metric not complete. $\mathbb{R}$ usual metric is complete.
(*) Completeness is a metric property, not topological property.

Example 17.4

(Complete)

1. Discrete metric space
2. $\mathbb{R}$
3. $\mathbb{R}^n$ in any of the three metric
   - $d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$
   - sup metric $d(\vec{x}, \vec{y}) = \max|x_i - y_1|$
   - taxicab metric $d(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$

   All are complete metrics.

Proof. $d(\vec{x}_{n_1}, \vec{x}_{n_2} < \epsilon \implies$

$$|j \text{ th coord of } \vec{x}_{n_1} - j \text{ th coord of } \vec{x}_{n_2}| < \epsilon, j = 1, \ldots, n$$

Cauchy seq $\{\vec{x}_j\}$. Cauchy seq in $\mathbb{R}^n$, $l = 1, \ldots, n - l$ th coord of $\{\vec{x}_j\} \leftarrow$ numbers, is a Cauchy sequence. So it has a limit $\vec{L} \in \mathbb{R}^n, L = (L_1, \ldots, L_n)$ is the limit of $\{\vec{x}_n\}$ □

"Component-wise convergent (or Cauchyness) $\iff$ convergence or Cauchyness of vector sequences.
Issue: What l_2 infinite sequence? – Yes, Complete! Completeness of l_2 : next time.

§18 Veterans Day: Nov 11, 2020

No class :D

§19 Lec 18: Nov 13, 2020

§19.1 Completeness of l_2 and $C([0,1])$ in sup norm

Recall $l_2 : \{x_n\} \implies \sum_{j=1}^{\infty} x_j^2 < \infty, \exists M$. This means, $\exists M$ s.t. $\sum_{j=1}^{n_0} x_j^2 \le M$ for $n_0 = 1, 2, 3, \ldots$.
Also, note that vector space $\{x_n + y_n\} \in l_2$ if $\{x_n\}, \{y_n\} \in l_2$

$$d(\{x_n\}, \{y_n\}) := \left(\sum_{n=1}^{\infty} (x_n - y_n)^2\right)^{\frac{1}{2}}$$

$\triangle$ inequality : left as exercise.

Question 19.1. Is it complete?

Ans: Yes! Let us prove it

Proof. Cauchy sequence $\{x_n\}_N$, $N = 1, 2, \ldots$ number of sequences. Note

$$x_{n,N} = \text{nth component of the Nth sequence}$$

$$\begin{array}{c|ccccc}
\text{1st} & x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} & \cdots \\
\text{2nd} & x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} & \cdots \\
\vdots & \downarrow & \downarrow & \downarrow & \downarrow & \\
\hline
\downarrow \vec{L} & L_1 & L_2 & L_3 & L_4 &
\end{array}$$

limit if $\{x_n\}_N$ is Cauchy

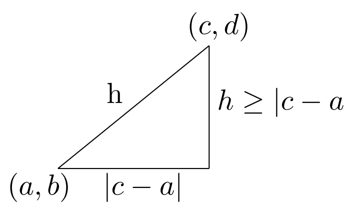
Find $\vec{L}$ given $\{x_n\}_N, N = 1, 2, 3, \ldots$ Cauchy sequence in l_2. Candidate for L : Try $L_n = \lim_{N\to\infty} x_{n,N}$. Does limit exist?

Yes: $x_{n,N}$ for $N = 1, 2, 3, \ldots$, n fixed is Cauchy sequence in $\mathbb{R}$. Cauchyness of $\{x_n\}_N, N = 1, 2, \ldots$ Given $\epsilon > 0, N_\epsilon \ni$

$$\left(\sum_{n=1}^{\infty} (x_{n,N_1} - x_{n,N_2})^2\right)^{\frac{1}{2}} < \epsilon$$

$$\geq |x_{n,N_1} - x_{n,N_2}| \checkmark$$

for all $N_1, N_2 \geq N_\epsilon$.



$$h = \sqrt{(c-a)^2 + (d-b)^2} \geq \sqrt{(c-a)^2} = |c-a|$$

So $L_n = \lim_{N\to\infty}(x_{n,N})$ exists. $\{x_{n,N}\}, N = 1, 2, 3, \ldots$ n fixed Cauchy sequence. $\vec{L} =$ candidate for $\lim \{x_{n,N}\} = \{x_n\}_N, N = 1, 2, 3, \ldots$

Exercise 19.1. $\vec{L}$ has to be l_2-limit if there is a limit in l_2.

Component-wise convergence (for fixed n – number of component, $x_{n,N} \to L_n$) does not imply (in general) l_2 convergence $(1, 0, 0, \ldots), (0, 1, 0, 0, \ldots), (0, 0, 1, 0, \ldots) \ldots$ converges component-wise to $(0, 0, \ldots, 0)$ but does not l_2 converge to $(0, 0, \ldots, 0, \ldots)$ even though everything is in l_2. $\mathbb{R}^n, l_2 = \mathbb{R}^\infty$ – saved by the fact that $\{x_n\}_N$ is a Cauchy seq in l_2. Here is a basic observation: if $\| \{x_{n,N}\} \| \leq M$ and if $\vec{L} =$ component-wise limit of $\{x_{n,N}\}$ (includes limit L exists) then $\vec{L}$ is in l_2 and

$$\|\vec{L}\| \leq M$$

□

Proof. If $L_1^2 + \ldots + L_n^2 \leq M^2$ all n, then $\sum_{n=1}^{\infty} L_n^2 < \infty$ and indeed $\leq M^2$

$$L_1^2 + \ldots + L_n^2 = \lim \left(x_{1,N}^2 + \ldots + x_{n,N}^2\right)$$

where $\left(x_{1,N}^2 + \ldots + x_{n,N}^2\right) \leq \|(x_n)_N\|^2 \leq M^2$ and $\sum_{j=1}^{\infty} L_j^2 \leq M^2$. So, $\vec{L} \in l_2$ and $\|\vec{L}\| \leq M$ ✓
Second item: $\{x_{n,N}, N = 1, 2, 3, \ldots\}$ is a Cauchy sequence then $\exists \vec{L} = (L_1, L_2, \ldots)$ component-wise limit of $\{x_{n,N}\}$ ✓ □

Proof. Proper of completeness of l_2. Want: given $\epsilon > 0, \exists N_\epsilon$ s.t. $N_1 \geq N_\epsilon \implies \| \{x_n\}_{N_1} - \vec{L}\| < \epsilon$, $\vec{L} = l_2$ limit of $\{x_n\}_N$ (know $\vec{L}$ is in l_2).
Know: (from defn of Cauchy seq), $\exists N_0$ s.t. $N_1, N_2 \geq N_\epsilon$

$$\| \{x_n\}_{N_1} - \{x_n\}_{N_2} \| < \frac{\epsilon}{2}$$

Now fix $N_1 = A \geq N_\epsilon$

$$\| \{x_n\}_A - \{x_n\}_{N_2} \| < \frac{\epsilon}{2}$$

Now let $N_2 \to +\infty$ (A fixed), $\{x_n\}_A - \{x_n\}_{N_2}$ converges (as $N_2 \to \infty$) to $\{x_n\}_A - \vec{L}$ component-wise. So by lemma:

$$\| \{x_n\}_A - \vec{L}\| \leq \frac{\epsilon}{2} < \epsilon$$

True for all $A \geq N_0$. Done! $\{x_n\}_N$ converges to $\vec{L}$ in l_2 norm. □

§20 Lec 19: Nov 16, 2020

§20.1 Lec 18 (Cont'd)

$C([0,1])$ is complete. Uniform convergence: $f_n \to f_0$, $\{f_n\}$ converges uniformly to a function f_0 if given $\epsilon > 0, \exists N_\epsilon$ s.t.

$$n \geq N_\epsilon \implies |f_n(x) - f_0(x)| < \epsilon$$

for all $x \in X([0,1])$.
Uniform convergence associated to uniform Cauchyness: $\{f_n\}$ is Cauchy if given $\epsilon > 0, \exists N_\epsilon$ such that $n_1, n_2 \geq N_\epsilon \implies$

$$|f_{n_1}(x) - f_{n_2}(x)| < \epsilon$$

for all x.

Lemma 20.1
Unif Cauchy implies $\exists f_0$ such that $\{f_n\}$ converges uniformly to f_0. (not at first known to be $C([0,1])$ actually is)

Proof. Note that for each $x, \{f_n(x)\}$ is a Cauchy sequence. Candidate for $f_0, f_0(x) = \lim f_n(x)$. Need this things:

1. $f_n \to f_0$ uniformly
2. Need (for completeness) $f_0 \in C([0,1])$

For (1): Given $\epsilon > 0, \exists N_0 \ni n_1, n_2 \geq N_0 \implies$

$$|f_{n_1}(x) - f_{n_2}(x)| < \frac{\epsilon}{2}$$

for all x. Let $n_2 \to \infty$

$$|f_{n_1}(x) - f_{n_2}(x)| \to |f_{n_1}(x) - f_0(x)|$$

as $n_2 \to \infty$. So

$$|f_{n_1}(x) - f_0(x)| \leq \frac{\epsilon}{2} < \epsilon$$

for all x and all $n_1 \geq N_\epsilon$.
2) f_0 want f_0 to be continuous. Trick three term estimate:
Given $\epsilon > 0$, choose $N_\epsilon \ni |f_n(x) - f_0(x)| < \frac{\epsilon}{3}$ if $n > N_\epsilon$ (true for all x, one choice of N_ϵ)

$$|f_0(y) - f_0(x)| \leq |f_0(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f_0(x)|$$

We can choose $n \geq N_\epsilon$ fix n ($n = N_\epsilon$ ok)

$$|f_0(y) - f_n(y)| < \frac{\epsilon}{3}$$
$$|f_n(x) - f_0(x)| < \frac{\epsilon}{3}$$

Also, we can choose (u,x fixed) $\delta > 0$ s.t. $|f_n(y) - f_n(x)| < \frac{\epsilon}{3}$ if $|x - y| < \delta$
So $|f_0(y) - f_0(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ if $|y - x| < \delta$ So f_0 is continuous at x (all $x \in [0,1]$). So

$$f_0 \in C\left([0,1]\right)$$

and $\{f_n\} \to f_0$ in $C\left([0,1]\right)$ with sup norm metric. So $C([0,1])$ complete (in sup norm metric). ☐

Everything will work (no boundedness problem) if X is sequentially compact. $f : [0,1] \to \mathbb{R}$, f, g bounded because $[0,1]$ seq compact. $C([0,1])$ complete in any sup norm.
(X, d) sequentially compact if every sequence in X $\{x_n\}$ has a subsequence which converges to a point $x_0 \in X$. Special case: $[a,b] \subset \mathbb{R}$ usual metric, Bolzano-Weierstrass – $\exists$ a convergence subsequence in $[a,b]$, $[a,b]$ closed in $\mathbb{R}$.

Proof. C seq. compact. $\{x_n\} \in C$, BW $\implies x_{n_j} \to x_0 \in \mathbb{R}$. But $x_0 \in C$ because C is closed. Example: Cantor set is sequentially compact. Closed,bounded subset C of $\mathbb{R}^n$ (n finite) then C is sequentially compact. Bounded means, $x_0 \in C$

$$\underbrace{d(x_0, \vec{x})}_{\vec{x} \in C} \leq M \text{ some } M$$

$d(\vec{0}, \vec{x}) \leq M$ by triangle (exercise).
$\{x_n\}$ bdd $\exists$ subsequence s.t. 1st component converges. There exists subsequence of that subsequence such that 2nd component (and 1st one still) converges ... p times subs $\{\vec{x}_n\}$ where all p_n components converge. ☐

C seq. compact in $\mathbb{R}^p$ then C is closed and bounded. Boundedness: if it were unbounded, each n $\exists x_n \in C$ which

$$d\left(\vec{0}, \vec{x}_n\right) \geq n$$

where $\{x_n\}$ no convergent sub.
Closed: $x_n \in C$ with limit $= \vec{x}_0$ then seq. compact means $\exists \vec{y}_0$ subsequence limit in C. $\vec{y}_0 = \vec{x}_0$ but a sub limit = seq. limit if seq. limit exists.

Theorem 20.2 (Heine-Borel)
$C \subset \mathbb{R}^n$ is sequentially compact if and only if it is close in $\mathbb{R}^n$ and bounded.

Another kind of compact – covering compactness (which will be covered later). Continuous image of a compact space is compact.

$$f(X, d_X) \mapsto (Y, d_Y) \implies f(X) \subset Y \text{ is seq compact}$$

Proof. $y_j \in f(X)$ then $\exists x_j \ni f(x_j) = y_j$, for $n = 1, 2, \ldots$

$$x_j \to x_0 \in X \text{ subseq limit}$$

by cont.

$$f(x_0) = \lim f(x_{j_n})$$

$\{y_{j_n}\}$ converges to a pt in $f(X)$. ◻

§21 Dis 1: Oct 1, 2020

Notation:

$$\begin{aligned}
&\mathbb{N} = \{1, 2, 3, \ldots\} \\
&\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \\
&\mathbb{Q} = \left\{\frac{p}{q} | p, q \in \mathbb{Z}, q \neq 0\right\} \\
&\mathbb{R} = \text{real numbers} \\
&\mathbb{C} = \{a + bi, \quad a, b \in \mathbb{R}\}
\end{aligned}$$

Set theory:

- $A \subset B$ (or $A \subseteq B$) means $x \in A \implies x \in B$
- $x \in A \cap B$ means $x \in A$ and $x \in B$
- $x \in A \cup B$ means $x \in A$ or $x \in B$
- $x \in A \setminus B \iff x \in A$ and $x \notin B$
- $A = B \iff A \subset B$ and $B \subset A$

§21.1 Induction

Given a sequence of mathematical statement $P(n)$ indexed by $\mathbb{N}$. If $P(1)$ is true and $P(k) \implies P(k+1)$ is true $\forall k \in \mathbb{N}$, then $P(n)$ is true $\forall n \in \mathbb{N}$.

Example 21.1
Prove $\sum_{k=1}^{n} (2k-1) = n^2$ (*) using induction.
Base case $n = 1 : 1 = 1^2$ ✓

Induction step: assume as induction hypothesis that (*) holds

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1) &= \sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2\end{aligned}$$

Or we can prove it the following way

$$\begin{aligned} S &= 1 + 3 + 5 + \ldots + (2n-1) \\ S &= (2n-1) + (2n-3) + \ldots + 3 + 1 \\ 2S &= 2n \cdot n \\ S &= n^2 \end{aligned}$$

Example 21.2

$a_{n+1} = \sqrt{2 + a_n}$, $\quad a_1 = 1$. Prove $a_n > 0$ and a_n increasing.
$a_1 > 0$ assume $a_n > 0$, $a_{n+1} = \sqrt{2 + a_n} > 0$

$$a_2 = \sqrt{3} \approx 1.732 > 1 = a_1$$

Assume $a_n \le a_{n+1}$, want to show $a_{n+1} \le a_{n+2} \impliedby \sqrt{a_n + 2} \le \sqrt{a_{n+1} + 2} \impliedby a_n \le a_{n+1}$

Example 21.3

$(1+x)^n \ge 1 + nx$: Bernoulli Inequality
$\quad x \ge -1, \quad n \ge 0$
base case $1 \ge 1$
Assume $(1+x)^n \ge 1 + nx$

$$\begin{aligned}(1+x)^{n+1} = (1+x)^n(1+x) \ge (1+nx)(1+x) &= 1 + (n+1)x + nx^2 \\ &= 1 + (n+1)x\end{aligned}$$

Strong Induction:
If $P(1)$ true and $P(1), P(2), \ldots P(k) \implies P(k+1)$ true $\forall k \in \mathbb{N}$ then $P(n)$ holds for all $n \in \mathbb{N}$

Remark 21.4. Induction $\iff$ strong induction

Example 21.5

Every integer greater than 1 is a product of primes.
Assume $2, 3, \ldots, n$ is a product of primes. n+1 is either a prime or a composite, in which case $n + 1 = ab, \quad 1 < a, b < n + 1$.

By strong induction hypothesis, both a and b are product of primes, hence so is $n+1 = ab$.

Exercise 21.1. Every integer greater than 1 has a prime divisior.

Proof of infinitude of primes by Euclid:

Proof. Assume on the contrary there are finitely many primes $\{p_1, p_2, \ldots, p_k\}$. Define $N = p_1 \ldots p_k + 1 > 1$ and (by above exercise) let p be a prime divisior of N but $p \neq p_j$ for any $1 \leq j \leq k$ otherwise if $p = p_j$ then $p | p_2 \ldots p_k$ also $p|N \implies p|N - p_1 \ldots p_k \implies p|1$,a contradiction. (no primes divide 1) □

§22 Dis 2: Oct 8, 2020

§22.1 Number System

- $(\mathbb{N}, +, \cdot, <)$: $+ : \mathbb{N} \times \mathbb{N} = \mathbb{N}^2 \to \mathbb{N}$ satisfies commutativity and associativity. Note that 0 is the identity with respect to addition, but $\mathbb{N}$ has no additive inverse.
- $(\mathbb{Z}, +, \cdot, <)$: $(\mathbb{Z}, +)$ is a commutative group (associativity, identity, inverse). $(\mathbb{Z}, \cdot)$ satisfies commutativity, associativity with 1 as mult identity but 2 has no mult inverse.
- $(\mathbb{Q}, +, \cdot, <)$: $(\mathbb{Q}, +)$ and $(\mathbb{Q}, \cdot)$ are commutative group(i). $+$ and $\cdot$ are compatible with distributive law: $a(b+c) = ab + ac$ (ii). Both (i) and (ii) mean $(\mathbb{Q}, +, \cdot)$ is a FIELD. $(Q, <)$ is an ordered set with $<$ satisfying trichotomy and transitivity. $+, \cdot$ are compatible : $y < z \implies x + y < x + z \forall x$, $x > 0, y > 0 \implies xy > 0$. With the above compatibility, $(\mathbb{Q}, +, \cdot, <)$ is an ordered field. Even though $\mathbb{Q}$ is additivity adn multiplicatively complete, $\mathbb{Q}$ is not satisfying in that
  1. $\mathbb{Q}$ is not algebraically closed, $x^2 - 2$ is a polynomial with no root in $\mathbb{Q}$.
  2. $\mathbb{Q}$ is not complete in a metric space: there exists subsets of $\mathbb{Q}$ bounded above but with no least upper bound (supremum), e.g. $A := \left\{p \in \mathbb{Q} : p < 0 \text{ or } p^2 < 2\right\}$ and $B = \mathbb{Q} \setminus A$. A contains no largest number and B contains no smallest.
  $$\forall p \in A \exists q \in A \quad q > p$$
  Let $p \in A$. Define $q := p - \frac{p^2-2}{p+2} > p$
  $$q^2 - 2 = \left(\frac{2p+2}{p+2}\right)^2 - 2 = \frac{2(p^2-2)}{(p+2)^2} < 0 \implies q^2 < 2$$
  If A has an upper bound α, $\alpha \notin A$: then $\alpha \in B$. It follows that B is the set of all upper bounds for A. Since B contains no smallest number, A has no least upper bound in $\mathbb{Q}$.

Definition 22.1 (Least Upper Bound Property) — S has the least-upper-bound property if $\forall E \subset S$ nonempty, bounded above $\sup E \in S$.

Remark 22.2. $\mathbb{Q}$ does not satisfy the least-upper-bound property.

$(\mathbb{R}, +, \cdot, <)$ there exists an ordered field with the l.u.b property that contains an isomorphic copy of $\mathbb{Q}$.

§22.2 Equivalence Relation

An equivalence relation given $\sim$ on $A \times A$ satisfies

- $x \sim x$ reflexity
- $x \sim y \iff y \sim x$ symmetry
- $x \sim y \cdot y \sim z \implies x \sim z$ transitivy

Example 22.3

$\mathbb{Q}$ Define $\sim$ on $\{(a,b) : a, b \in \mathbb{Z}, b \neq 0\}$ by $(a,b) \sim (c,d)$ if $ad = bc$

$$A = \mathbb{Z}^2 \setminus \{(a,0) : a \in \mathbb{Z}\}$$

$$\begin{aligned} \mathbb{Q} = &\text{ the set of all equivalence classes of A write } \sim \\ &= A/\sim = \{[x] : x \in A\} \end{aligned}$$

In this construction, $\quad \mathbb{Z} \to \mathbb{Q}, \quad n \to [(n,1)]$
$+ \setminus \cdot : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$: note that $+$ and $\cdot$ need to be well-defined on $\mathbb{Q}^2$. (need to show $\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}$ if $\frac{a}{b} \sim \frac{a'}{b'}$ and $\frac{c}{d} \sim \frac{c'}{d'}$.

Example 22.4

$S' = [0,1]/0_m$

Definition 22.5 (Convergent Sequences) — $\{a_n\}_{n \geq 1} \subseteq \mathbb{R}$ is said to be convergent to l if $\forall \epsilon > 0 \quad \exists N(\epsilon) > 0$ s.t. $\forall n \geq N, \quad |a_n - l| < \epsilon$

§23 Dis 3: Oct 13, 2020

§23.1 Equivalence Relation (Cont'd)

Example 23.1

Define $\sim p$ on $\mathbb{Z}$ by $a \sim pb$ if $a - b \in p\mathbb{Z}(p|a-b)$.
$\forall a \exists! b \in \mathbb{Z}, \quad 0 \leq r < p$ s.t. $a = bp + r$.

$$F_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/\sim p = \{[0]_p, [1]_p, [2]_p, \ldots, [p-1]_p\}$$

$[a]_p + [b]_p = [a+b]_p \quad \& \quad [a]_p[b]_p = [ab]_p$

Remark 23.2. $(F_p, +, \cdot)$ is a finite field. F_p cannot be ordered: $1 > 0, 1+1 > 0, \ldots, p-1 > 0$ but $p - 1 = -1$

Example 23.3

$T = \mathbb{R}/\mathbb{Z} \quad a \sim b$ if $ab \in \mathbb{Z}$

$[0,1]/0 \sim 1$

$\forall a \in \mathbb{R}, \quad \exists b = \underbrace{\{a\}}_{\text{fractional part of } a} \in [0,1)$ s.t. $a \sim b$

§23.2 Construction of $\mathbb{R}$ via Cauchy Sequences(Cantor)

$S =$ set of rational Cauchy sequences.
$\sim$ on $S : \{x_n\} - \{y_n\}$ if $\lim(x_n - y_n) = 0$ (Q3 – Homework 2)
$Q = S/\sim = \{[\{x_n\}] : \{x_n\} \in S\}$. First we need to define arithmetic on Q.

$$\begin{aligned}
[\{p_n\}] + [\{q_n\}] &= [\{p_n + q_n\}] \\
[\{p_n\}] - [\{q_n\}] &= [\{p_n - q_n\}] \\
[\{p_n\}] \cdot [\{q_n\}] &= [\{p_n q_n\}] \\
[\{p_n\}] \,/\, [\{p_n/q_n\}] &= [\{p_n/q_n\}]\,, \quad [\{q_n\}] \neq 0, = [\{0,0,0,\ldots\}]
\end{aligned}$$

$+ : Q \times Q \to Q$. Check well-defined

- $\{x_n\} \cdot \{y_n\}$ cauchy then so is $\{x_n + y_n\}$(Q4)
- $\{x_n\} \sim \{y_n\} \quad \& \quad \{z_n\} \sim \{w_n\}$ then $\{x_n + z_n\} \sim \{y_n + w_n\}$ (Q5)
  Commutativity, assoc, identity, $(0 = [\{0,0,0,\ldots\}]$, inverse.
- Well-defined: $\{x_n\}, \{y_n\}$ so is $\{x_n y_n\}$ (Q4).
- $\{x_n\} \sim \{y_n\}$ & $\{z_n\} \sim \{w_n\}$ (Q6, Q7)
  comm, assoc, iden, $(1 = [\{1,1,\ldots,1\}]$
  mult. inverse (Q9,Q10).
  $<$: trichotomy (Q11), transitivity
  various compatibility (distributivity, etc)
  l.u.b property (Q12)

Note:All the Q used above is assumed to be Q^{hat}

Remark 23.4.

$$\begin{aligned}
Q &\to Q^{\text{hat}} \\
q &\mapsto [q^*] \\
p < q &\iff [p^*] < [q^*]
\end{aligned}$$

Sequences:

- Cauchy seq. are bounded.
- Convergent seq. is Cauchy.

  Theorem: in $\mathbb{R}$, every Cauchy seq. is convergent.

Example 23.5
$a_n = \frac{1}{n}$
$\forall \epsilon > 0 \exists N$ s.t. $\epsilon N > 1$.
$\forall n \geq N \quad \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$. □

§24 Dis 4: Oct 20, 2020

§24.1 Least Upper Bound and Its Applications

Remark 24.1 (ϵ – Principle)**.** $a, b \in \mathbb{R}, \forall \epsilon > 0,\ a \leq b + \epsilon \implies a \leq b$.

- $x, y \in \mathbb{R} \quad \forall \epsilon > 0,\ |x - y| \leq \epsilon \implies x = y$.

Supremum: $E \subset S$ bounded above. Suppose $\sup E \in S$

- $e \leq \sup E \forall e \in E$.
- $\forall \beta < \sup E, \quad \exists e \in E$ s.t. $\beta < e < \sup E$

  OR

  $\forall \epsilon > 0, \exists e \in E$ s.t. $\sup E - \epsilon < e \leq \sup E$.

Example 24.2
$\sup \left\{\frac{1}{n}\right\}_{n \geq 1} = 1, \quad \inf \left\{\frac{1}{n}\right\} = 0$.

- $0 \leq \frac{1}{n} \forall n \in \mathbb{N}$.
- $\forall \epsilon > 0, \exists n \in \mathbb{N}$ s.t. $0 \leq \frac{1}{n} < \epsilon$ by Archimedean Prop.

Theorem 24.3 (Nested Interval)
$\{I_n = [a_n, b_n]\}_{n \geq 1} \subset \mathbb{R},\ I_n \supset I_{n+1} \implies \bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Moreover, if $|I_n| \to 0$, then $\bigcap I_n$ is a singleton (a set with exactly one element).

Proof. $\sup a_n \in \bigcap I_n$. □

Theorem 24.4 ((4.1))
(Bolzano – Weierstrass): Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. $I_0 = [-M, M] \supset I_1 \supset I_2 \supset \dots$.

$$|I_n| = (2M) \cdot 2^{-n} \to 0 \quad \text{as } n \to \infty$$

From Nested Interval Thm, $\bigcap_{n=0}^{\infty} I_n = \{x\}$. Choose $x_{n_k} \in I_k, x_{n_k} \to x$. □

Remark 24.5. l.u.b property of $\mathbb{R} \implies$ Nested Interval $\implies$ Bolzano – Weierstrass $\stackrel{(*)}{\implies}$ Cauchy Completeness.

(*) Exercise: $\{x_n\}$ Cauchy. $x_{n_k} \to x \implies x_n \to x$.

Remark 24.6. In $\mathbb{R}$, to check convergence, it suffices to check Cauchyness. Useful especially when you don't have a candidate for the limit. Cauchy criterion for series $\sum_{n=1}^{\infty} a_n$ converges ($\lim_{n\to\infty} \sum_{k=0}^{n} a_k$) exists. $\iff \sum a_n$ Cauchy ($\forall \epsilon > 0 \exists N \, |\sum_{k=n}^{m} a_k| < \epsilon \quad \forall m \geq n \geq N$).

Corollary 24.7

Absolute convergence $\implies$ convergence. ($\sum |a_n|$ converges $\implies \sum a_n$ converges).

Monotone convergence theorem, $\{a_n\}$ monotone. Then $\{a_n\}$ bounded $\iff$ $\{a_n\}$ convergent. (HW 3 – Q1).

Definition 24.8 (Monotone Sequence) — $\{a_n\}$ monotone if $a_n \leq a_{n+1} \forall n$ or $a_n \geq a_{n+1} \forall n$.

Corollary 24.9

$\sum |a_n| < \infty \iff \sum |a_n|$ converges.

§24.2 Continuity

Definition 24.10 ((6.2)) — $f : X \to \mathbb{R}$ is continuous at x (local prop) if

1. ($\epsilon - \delta$ def) $\forall \epsilon > 0, \exists \delta(\epsilon, x) > 0$ s.t. $\forall y \in X$, $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.
2. (Sequential def) $\forall \{x_n\} \subset X, x_n \to x \implies f(x_n) \to f(x)$ (f preserves sequential convergence).
3. $\lim_{y\to x} f(y) = f(x)$

$f : X \to \mathbb{R}$ is continuous if f is continuous at all $x \in X$.

Definition 24.11 ((7.1)) — f is uniformly continuous on X (global prop) if

1. ($\epsilon - \delta$) $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ s.t. $\forall x, y \in X$ $|x-y| < \delta \implies |f(x) - f(y)| < \epsilon$.
2. (Sequential) $\forall \{x_n\} \subset X$, $\{x_n\}_{n\geq 1}$ Cauchy $\implies \{f(x_n)\}_{n\geq 1}$ Cauchy. (f preserves Cauchy seq).

Remark 24.12. Uniform continuity $\implies$ continuity.

Example 24.13
$f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is continuous.

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy} < \frac{|x-y|}{x \cdot \frac{x}{2}} = |x-y| \cdot 2x^{-2} < \epsilon$$

$\delta = \min\left\{\frac{x}{2}, \frac{\epsilon x^2}{2}\right\}$.

Remark 24.14. $x \mapsto \frac{1}{x}$ is uniformly continuous on $(a, \infty) \forall a > 0$.
$x \mapsto \frac{1}{x}$ is NOT uniformly continuous on $(0, \infty)$.

- $x_n = \frac{1}{n}, y_n = \frac{1}{n+1} \quad |x_n - y_n| \to 0$ but $|\frac{1}{x_n} - \frac{1}{y_n} = 1 \forall n$.
- $\left\{\frac{1}{n}\right\}_{n\geq 1}$ Cauchy but $\{n\}$ is not.

§25 Dis 5: Oct 27, 2020

§25.1 Metric Spaces

Definition 25.1 ((9.1)) — A metric on a set X is a function $d : X \times X \to [0, \infty]$ s.t.

- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

Thus (X, d) is called a metric space.

Example 25.2
- $(X, d), A \subset X$. $d\Big|_{A\times A}$ is a metric on A.

- (Discrete metric) Given any set X, define

$$d(x,y) = \begin{cases} 1, x \neq y \\ 0, x = y \end{cases}$$

Check d is a metric on X.

Remark 25.3 (norm)**.** Given a vector space X. A norm on X is a function $\|\cdot\| : x \to [0,\infty)$ s.t.

- $\|x\| = 0 \iff x = 0$
- $\|\alpha x\| = |\alpha|\|x\|$
- $\|x+y\| \leq \|x\| + \|y\|$

Then $d(x,y) = \|x-y\|$ is a metric on X.

Example 25.4

- $\mathbb{R}^d, |\cdot| = \|\cdot\|_2$ where $|x| = \|x\|_2 = \sqrt{\sum_{i=1}^d |x_i|^2}$
- On $\mathbb{R}^d$, define $\|x\|_p = \left(\sum_{i=1}^d \|x_i\|^p\right)^{\frac{1}{p}}, 1 \leq p < \infty$

Inequalities:

- Young's Inequality:
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, a, b \geq 0, \frac{1}{p} + \frac{1}{q} = 1$$
- Holden's Inequality:
$$\|xy\|_1 \leq \|x\|_p \|y\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty$$
- Minkowski's Inequality (triangle inequality for $\|\cdot\|_p$)
$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Define $\|x\|_\infty = \max_{i=1}^d |x_j|$. Then

$$\|xy\|_1 \leq \|x\|_1 \|y\|_\infty$$
$$\|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$$

Hence $(\mathbb{R}^d, \|\cdot\|_p)$ is a metric space $\forall 1 \leq p \leq \infty$. *Note*:

- $p = 1:$ taxicab / Manhattan metric
- $p = 2:$ Euclidean metric
- $p = \infty:$ sup metric

Notation: $\mathbb{R}^N = \{(x_i)_{i \geq 1} : x_i \in \mathbb{R}\} = \{f : \mathbb{N} \to \mathbb{R}\}$

Definition 25.5 — Given $x \in \mathbb{R}^N$, $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}, 1 \le p < \infty$. $\|x\|_\infty = \sup |x_i|$

Example 25.6
$l^p(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{R}, \|f\|_p < \infty\}$, $1 \le p \le \infty$. So $(l^p, \|\cdot\|_p)$ is a metric space and a vector space.

Definition 25.7 (Completeness of Metric Space) — A metric space (X, d) is complete if every Cauchy sequence with respect to d is convergent with respect to d.

Example 25.8

- $(\mathbb{Q}, |\cdot|)$ is not complete; $(\mathbb{R}, |\cdot|)$ is complete.
- $(\mathbb{R}^d, \|\cdot\|_p)$ is complete.
- $(l^p(\mathbb{N}), \|\cdot\|_p)$ is complete $(1 \le p \le \infty)$.
- $([0,1], \mathbb{R}) = \{f : [0,1] \to \mathbb{R}\}$ continuous

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)| \to \|f - g\| = \sup_{x \in [0,1]} |f(x) - g(x)|$$

$(C([0,1]), \|\cdot\|_\infty)$ is a complete metric space.

Special structure when $p = 2$
Inner product space:
Given vector space $X/\mathbb{R}$ a real inner product on X is $\langle \cdot, \cdot \rangle : x \succ x \to [0, \infty]$ s.t.

- $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle, \forall a, b \in \mathbb{R}, x, y, z \in X$.
- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x, x \rangle \in (0, \infty)$ and is 0 $\iff$ $x = 0$.

With the inner product: $\|x\| = \sqrt{(x, x)}$ is a norm, then $(X, \|\cdot\|)$ is a metric space.

Example 25.9
$\mathbb{R}^d : \langle x, y \rangle = x \cdot y = \sum x_i y_i$
also, $\|x\|_2 = \sqrt{\sum x_i^2} = \sqrt{\langle x, x \rangle}$

Example 25.10
$l^2 : \langle f, g \rangle = \sum_{i=1}^{\infty} f(i)g(i)$ and $\|f\|_2 = \sqrt{\langle f, f \rangle} = \sqrt{\sum_{i=1}^{\infty} |f(i)|^2}$

Definition 25.11 (Orthogonality) — $x \perp y \iff \langle x, y \rangle = 0$

Theorem 25.12 (Cauchy – Schwarz)
$|\langle x, y \rangle \leq \|x\| \cdot \|y\||$ and equality holds $\iff$ x, y are linearly dependent.

$\forall x, y \in X, \alpha \in \mathbb{R}$

$$\langle x - \alpha y \cdot x - \alpha y \rangle = \|x - \alpha y\|^2 \geq 0$$

Goal: find α that minimize $\|x - \alpha y\|$
The intuition here is $\|x - \alpha y\|$ is shortest when $x - \alpha y \perp y$.

$$\langle x - \alpha y \cdot x - \alpha y \rangle = \|x\|^2 + \alpha^2 \|y\|^2 - 2\alpha \langle x, y \rangle$$

is minimal when $\alpha = \frac{\langle x, y \rangle}{\|y\|^2}$. Let us set α to such value, so

$$\begin{aligned} &= \|x\|^2 + \frac{|\langle x, y \rangle|^2}{\|y\|^2} - \frac{2\,|\langle x, y \rangle|^2}{\|y\|^2} \\ &= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \geq 0 \end{aligned}$$

§26 Dis 6: Nov 3, 2020

§26.1 Basic Topology – Metric Space

(X, d) metric space. If $x \in X$, the (open) ball of radius r about x is denoted $B_r(x) = B(r, x) = \{y \in X : d(x, y) < r\}$ where r is radius and x is the center.

Definition 26.1 (Open/Closed Sets) — $E \subset X$ open if $\forall x \in E \exists r > 0$ s.t. $B(r, x) \subset E$. E is closed if $E^c = X \setminus E$ is open.

Example 26.2
$B(r, x)$ is open: $\forall y \in B(r, x), B(r - d(x, y), y) \subset B(r, x)$

Example 26.3
$X, \emptyset$ is both open and closed, also known as clopen.

Example 26.4
Subsets of $\mathbb{R}$

	open	closed
$[0,1]$	$\times$	✓
$(0,1)$	✓	$\times$
$(0,1]$	$\times$	$\times$
$\mathbb{Z}$	$\times$	✓
$\left\{\frac{1}{n}\right\}_{n\geq 1}$	$\times$	$\times$

We can observe for the last case, $\left\{\frac{1}{n}\right\}_{n\geq 1}$ is not closed since any neighborhood around 0 intersects $\left\{\frac{1}{n}\right\}_{n\geq 1} \implies \left\{\frac{1}{n}\right\}^c_{n\geq 1}$ is not open.

Example 26.5

Subset of $\mathbb{R}^2$

	open	closed
$\left\{x^2+y^2<1\right\} = B(1,0)$	✓	$\times$
$\left\{x^2+y^2\leq 1\right\}$	$\times$	✓
A where $\lvert A\rvert < \infty$	$\times$	✓
$\{(x,y) : x = 1\}$	$\times$	✓
$(0,1) = \{(x,0) : x \in (0,1)\}$	$\times$	$\times$

Remark 26.6. Open/Closed is relative: $(0,1)$ open in $\mathbb{R}$ but not open in $\mathbb{R}^2$.

- $\{V_\alpha\}_{\alpha\in A}$ open $\implies \bigcup_{\alpha\in A} V_\alpha$ is open
  $\{F_\alpha\}_{\alpha\in A}$ closed $\implies \bigcap_{\alpha\in A} F_\alpha$ is closed.
- $V_1,\dots,V_n$ open $\implies \bigcap_{i=1}^n V_i$ is open
  $F_1,\dots,F_n$ closed $\implies \bigcup_{j=1}^m F_j$ is closed.
- Infinite intersection (union) of open (closed) sets need <u>not</u> be open (closed, respectively).

$$\bigcap_{n\geq 1}\left(-\frac{1}{n},\frac{1}{n}\right) = \{0\} \quad \bigcup_{n\geq 1}\left[\frac{1}{n}, 1-\frac{1}{n}\right] = (0,1)$$

Theorem 26.7

f is continuous $(X_1,d_1)\to(X_2,d_2) \iff f^{-1}(U)$ is open in $X_1 \forall U$ open in X_2.

Remember to prove this

Definition 26.8 (Boundedness) — Diameter of E : diam $E = \sup\{d(x,y) : x,y\in E\}$.
E is bounded if diam $E < \infty$.
An alternative definition: E is bounded if $\exists x\in E, R>0$ s.t. $E\subset B_R(x)$

Definition 26.9 (Closure) — $E \subset X$. The closure of E in X is denoted $\overline{E} = \bigcap_{E \subset F, F \text{closed}}^{F}$. Note $\overline{E}$ is closed.

The interior of E in X is denoted

$$\mathring{E} = \bigcup_{E \supset G, G \text{ open}} G \qquad \mathring{E} \text{ is open}$$

Remark 26.10. E closed $\iff E = \overline{E}$. E open $\iff E = \mathring{E}$.

Theorem 26.11

The followings are equivalent

1. $x \in \overline{E}$
2. $\forall r > 0, B_r(x) \cap E \neq \emptyset$
3. $\exists \{x_n\}_{n \geq 1} \subset E$ s.t. $x_n \to x$

Proof. (1) $\iff$ (2) $\equiv x \notin \overline{E} \iff r > 0 B_r(x) \cap E = \emptyset \iff \exists r > 0, B_r(x) \subset E^c$. So this implies $x \in \left(\overline{E}\right)^c$. $\exists r > 0, B_r(x) \subset \left(\overline{E}\right)^c \subset E^c$

$$\Longleftarrow \exists r > 0, B_r(x) \subset E^c \iff E \subset B_r(x)^c \implies \overline{E} \subset B_r(x)^c \implies x \notin \overline{E}$$

Note: above argument shows $\left(\overline{E}\right)^c = \left(\mathring{E^c}\right)$

(2) $\iff$ (3) – obvious. ☐

Definition 26.12 (Limit Point) —

$$\begin{aligned} E' &= \{x \in X : \exists r > 0 \, (B(r,x) \setminus \{x\}) \cap E \neq \emptyset\} \\ &:= \{x \in X : \exists \{x_n\} \subset E \setminus \{x\} \ni x_n \to x\} \end{aligned}$$

Example 26.13

$$
\begin{aligned}
E &= \left\{\frac{1}{n}\right\}_{n\geq 1} \\
E' &= \{0\} \\
\overline{E} &= \left\{\frac{1}{n}\right\}_{n\geq 1} \cup \{0\} \\
\left(\overline{E}\right)' &= Ex \\
\left(E'\right)' &= Ex
\end{aligned}
$$

Remark 26.14. $\overline{E} = E \cup E'$.

Theorem 26.15

The followings are equivalent

1. E closed (E^c is open).
2. $\overline{E} \subset E(\iff E = \overline{E})$
3. $E' \subset E$ – Rudin Definition
4. $\underbrace{\forall \{x_n\} \subset E \text{ if } x_n \to x}_{x\in\overline{E}}$ then $x \in E$.

§27 Dis 7: Nov 10, 2020

§27.1 Some Nice Theorems

Theorem 27.1 (Extreme Value)

$f : [a, b] \to \mathbb{R}$ continuous $\implies \exists x_n, x_m \in [a, b]$ s.t. $f(x_n) \leq f(x) \leq f(x_m) \forall x \in [a, b]$.

Remark 27.2.

1. $f : X \to \mathbb{R}$ continuous and if X is sequentially compact then f attains its extrema in X.

   Proof. Suppose $f(x_n) \to \sup_{x\in X} f(x)$ (allowing infinity), then by sequential compactness of X, $\exists x_{n_k} \to x \in X$. By continuity, $f(x_{n_k}) \to f(x)$ but $f(x_{n_k}) \to \sup_{x\in X} f(x)$ as well. By uniqueness of limit, $f(x) = \sup_{y\in Y} f(y) < \infty$. $\square$

2. $[a, b]$ is sequentially compact (HW)
3. Sequential compactness $\implies$ closed and bounded (HW).

   In $\mathbb{R}^n$, the converse is true by (high-dimensional) Bolzano-Weierstrass. So, in $\mathbb{R}^n$, sequential compactness $\iff$ closed and bounded.

Theorem 27.3 (Intermediate Value)
$f : [a, b] \to \mathbb{R}$ continuous. For every λ between $f(a), f(b)$, then $\exists c \in (a, b)$ s.t. $f(c) = \lambda$.

Remark 27.4. Image of connected set under continuous mapping is connected (later).

Example 27.5
- $\exists \alpha \in \mathbb{R} \ni \alpha^2 = 2$.
- Every odd polynomial $p(x)$ has a root in $\mathbb{R}$. Note: all polynomials are continuous.
- $f : [0, 1] \to [0, 1]$ continuous has a fixed point x s.t. $f(x) = x$. Show $g(x) = f(x) - x$ has a root. Note that g is also continuous $g(0) = f(0) = 0 : g(1) = f(1) - 1 \le 0$.

  If $f(0) = 0$ or $f(1) = 1$, we have the fixed point; if not, $g(0) > 0, g(1) < 0$ so IVT $\implies \exists c \in (0, 1)$ s.t. $g(1) = f(1) - c = 0$.

Theorem 27.6 (Heine – Cantor)
$f : [a, b] \to \mathbb{R}$ continuous $\implies$ f is uniformly continuous.

Remark 27.7. This also generalizes to any sequentially compact space.

Example 27.8
$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), x \neq 0 \\ c, x = 0 \end{cases}$$

f is not continuous: $x_n = \frac{1}{\frac{\pi}{2} + n\pi}$ so that $\sin\left(\frac{1}{x_n}\right) = (-1)^n$. $x_n \to 0$ but $\sin\left(\frac{1}{x_n}\right) = (-1)^n$ does not converge. For any $c \in \mathbb{R}$, f is not continuous. So there exists no continuous extension of $\sin\left(\frac{1}{x}\right)$ to the origin.

Remark 27.9. $f : [a, b] \to \mathbb{R}$ uniformly continuous. Then $\exists! \mathscr{F} : [a, b] \to \mathbb{R}$ continuous s.t. $\mathscr{F}\big|_{(a,b)} = f$. Exercise!

§27.2 Completeness

- $A \subset B \implies \overline{A} \subset \overline{B} : A \subset B \subset \overline{B} \implies \overline{A} \subset \overline{B}$
- $A \subset B \implies A^\circ \subset B^\circ$
- $\overline{A \cup B} = \overline{A} \cup \overline{B} \iff (A \cap B)^\circ = A^\circ \cap B^\circ$ $(\overline{X}^c = \mathring{X^c})$

  $\supset$: $\overline{A} \subset \overline{A \cup B}, \overline{B} \subset \overline{A \cup B} \implies \overline{A} \cup \overline{B} \subset \overline{A \cup B}$

  $\subset$: $A \cup B \subset \overline{A} \cup \overline{B} \implies \overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

- $\bigcup_{k=1}^{\infty} \overline{A_k} \subset \overline{\bigcap_{n=1}^{\infty} A_k}$, however, let $A_k = \{q_k\}$ where $Q = \{q_k\}_{k\geq 1}$ is enumeration of Q.

  $\bigcup_{k=1}^{\infty} \overline{A_k} = \bigcup_{k=1}^{\infty} \{q_k\} = Q \subsetneq \overline{\bigcup_{k=1}^{\infty} A_k} = \overline{Q} = \mathbb{R}$

  Similarly, $(\bigcap_{k=1}^{\infty} A_k)^\circ \subset \bigcap_{k=1}^{\infty} A_k^\circ$ but in general not equal.

- $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$; however,

  $$\emptyset = \overline{\emptyset} = \overline{(-1,0) \cap (0,1)} \subsetneq \overline{(-1,0)} \cap \overline{(0,1)} = [-1,0] \cap [0,1] = \{0\}$$

  Similarly, $A^\circ \cup B^\circ \subset (A \cup B)^\circ$ but in general not equal.

Definition 27.10 (Dense Set) — $A \subset X$ is dense if $\overline{A} = X$ ($x \in X = \overline{A} \implies \forall r > 0, B_r(x) \cap A \neq \emptyset$).

Example 27.11

$\overline{\mathbb{Q}} = \mathbb{R}$.