Math 131BH – Honors Real Analysis II

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This is math 131BH – Honors Real Analysis II, and it is instructed by Professor Visan. It's the second class in the undergrad real analysis sequence at UCLA. We meet weekly on MWF from 10:00 – 10:50 am for online lectures. Similar to 131AH, there are two textbooks associated to the course, *Principles of Mathematical Analysis* by *Rudin* and *Metric Spaces* by *Copson*. You can find the previous analysis lecture notes along with the other course notes through my github. Please email me if you notice any significant mathematical errors/typos that needs to be addressed. Thank you, and I hope you find this helpful for your study!

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§1 Lec 1: Mar 29, 2021

§1.1 Compactness

Definition 1.1 (Open Cover) — Let (X, d) be a metric space and let $A \subseteq X$. An open cover of A is a family $\{G_i\}_{i\in I}$ of open sets in X such that

$$A \subseteq \bigcup_{i\in I} G_i$$

The open cover is called finite if the cardinality of I is finite. If it's not finite, the open cover is called infinite.

Definition 1.2 (Compactness & Precompactness) — Let (X, d) be a metric space and let $K \subseteq X$.

1. We say that K is a compact set if every open cover $\{G_i\}_{i\in I}$ of K admits a finite subcover, that is,

$$\exists n \geq 1 \text{ and } \exists i_1, \ldots, i_n \in I \text{ s.t. } K \subseteq \bigcup_{j=1} G_{i_j}$$

2. We say that a set $A \subseteq X$ is precompact if $\overline{A}$ is compact.

Lemma 1.3

Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$. We equip Y with the induced metric $d_1 : Y \times Y \to \mathbb{R}$, $d_1(y_1, y_2) = d(y_1, y_2)$. Let $K \subseteq Y \subseteq X$. The followings are equivalent:

1. K is compact in (X, d).
2. K is compact in (Y, d_1).

Proof. 1) $\implies$ 2) Assume K is compact in (X, d). Let $\{V_i\}_{i\in I}$ be a family of open sets in (Y, d_1) s.t.

$$K \subseteq \bigcup_{i\in I} V_i$$

For $i \in I$ fixed, V_i is open in $(Y, d_1) \implies \exists G_i \subseteq X$ open in (X, d) s.t.

$$V_i = G_i \cap Y$$

Then

$$\left.\begin{array}{l} K \subseteq \bigcup_{i\in I} V_i \subseteq \bigcup_{i\in I} G_i \\ K \text{ compact in } (X,d) \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists i_1,\dots,i_n \in I \text{ s.t.}$$

$$\left.\begin{array}{l} K \subseteq \bigcup_{j=1}^n G_{i_j} \\ K \subseteq Y \end{array}\right\} \implies K \subseteq \left(\bigcup_{j=1}^n G_{i_j}\right) \cap Y = \bigcup_{j=1}^n \left(G_{i_j} \cap Y\right) = \bigcup_{j=1}^n V_{i_j}$$

So K is compact in (Y,d_1).
2) $\implies$ 1) Assume K is compact in (Y,d_1). Let $\{G_i\}_{i\in I}$ be a family of open sets in (X,d) s.t.

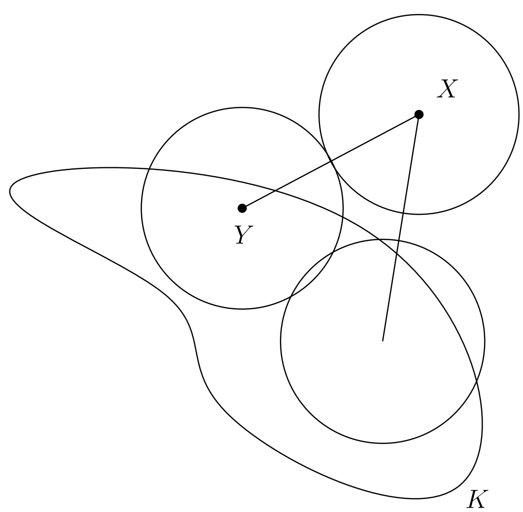
$$\left.\begin{array}{l} K \subseteq \bigcup_{i\in I} G_i \\ K \subseteq Y \end{array}\right\} \implies \left.\begin{array}{l} K \subseteq \left(\bigcup_{i\in I} G_i\right) \cap Y = \bigcup_{i\in I} \underbrace{(G_i \cap Y)}_{\text{open in } Y} \\ K \text{ is compact in } (Y,d_1) \end{array}\right\} \implies$$

$\implies \exists n \geq 1$ and $\exists i_1,\dots,i_n \in I$ s.t. $K \subseteq \bigcup_{j=1}^n \left(G_{i_j} \cap Y\right) \subseteq \bigcup_{j=1}^n G_{i_j}$. $\square$

Proposition 1.4

Let (X,d) be a metric space and let $K \subseteq X$ be compact. Then K is closed and bounded.

Proof. Let's prove K is closed. We'll show ${}^cK$ is open.
Case 1: ${}^cK = \emptyset$. This is open.
Case 2: ${}^cK \neq \emptyset$. Let $x \in {}^cK$
For $y \in K$ let $r_y = \frac{d(x,y)}{2}$. Note $r_y > 0$ (since $x \in {}^cK$ and $y \in K$).



Note

$$\left.\begin{array}{l} K \subseteq \bigcup_{y\in K} \underbrace{B_{r_y}(y)}_{\text{open}} \\ K \text{ is compact} \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists y_1,\dots,y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^n B_{r_j}(y_j)$$

where we use the shorthand $r_j = r_{y_j}$.
Let $r = \min_{1\le j\le n} r_j > 0$.
By construction, $B_r(x) \cap B_{r_j}(y_j) = \emptyset \quad \forall 1 \le j \le n$.

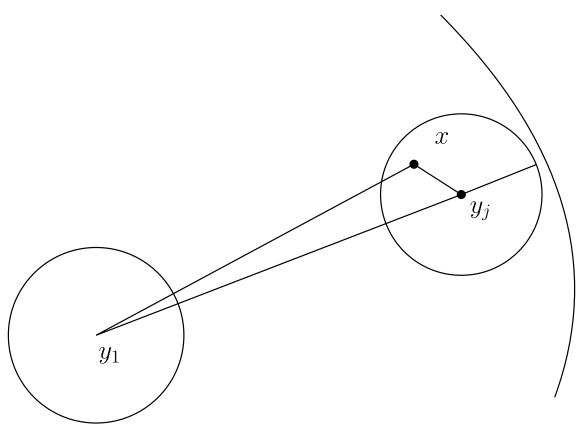
$$\implies B_r(x) \subseteq {}^c B_{r_j}(y_j) \quad \forall 1 \le j \le n$$
$$\implies B_r(x) \subseteq \bigcap_{j=1}^n {}^c B_{r_j}(y_j) = {}^c\left(\bigcup_{j=1}^n B_{r_j}(y_j)\right) \subseteq {}^c K$$
$$\implies \left.\begin{array}{l} x \in \mathring{\widehat{{}^c K}} \\ x \in {}^c K \text{ was arbitrary} \end{array}\right\} \implies {}^c K = \mathring{\widehat{{}^c K}}$$

Let's show K is bounded. Note

$$\left.\begin{array}{l} K \subseteq \bigcup_{y\in K} \underbrace{B_1(y)}_{\text{open}} \\ K \text{ compact} \end{array}\right\} \implies \exists n \ge 1 \text{ and } \exists y_1, \ldots, y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^n B_1(y_j)$$

For $2 \le j \le n$, let $r_j = d(y_1, y_j) + 1$.

Claim 1.1. $B_1(y_j) \subseteq B_{r_j}(y_1)$



Indeed, if $x \in B_1(y_j) \implies d(x, y_j) < 1$. By the triangle inequality

$$d(y_1, x) \le d(y_j, x) + d(y_1, y_j) < 1 + d(y_1, y_j) = r_j \implies x \in B_{r_j}(y_1)$$

So with $r = \max_{2\le j\le n} r_j$,

$$K \subseteq \bigcup_{j=1}^n B_1(y_j) \subseteq B_r(y_1)$$

$\square$

Proposition 1.5

Let (X, d) be a metric space and let $F \subseteq K \subseteq X$ such that F is closed in X and K is compact. Then F is compact.

Proof. Let $\{G_i\}_{i\in I}$ be a family of open sets in X s.t.

$$F \subseteq \bigcup_{i\in I} G_i$$

Then

$$\left.\begin{array}{l} K \subseteq F \cup {}^cF \subseteq \bigcup_{i\in I} G_i \cup \underbrace{{}^cF}_{\text{open in } X} \\ K \text{ compact} \end{array}\right\} \implies$$

$\implies \exists n \geq 1$ and $\exists i_1, \ldots, i_n \in I$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j=1}^n G_{i_j} \cup {}^cF \\ F \subseteq K \end{array}\right\} \implies F = \left(\bigcup_{j=1}^n G_{i_j} \cup {}^cF\right) \cap F \subseteq \bigcup_{j=1}^n G_{i_j}$$

So F is compact. □

Corollary 1.6

Let (X, d) be a metric space and let $F \subseteq X$ be closed and let $K \subseteq X$ be compact. Then $K \cap F$ is compact.

Proof. K is compact. So

$$\left.\begin{array}{l} K \text{ closed} \\ F \text{ closed} \end{array}\right\} \implies \left.\begin{array}{l} K \cap F \text{ is closed} \\ K \cap F \subseteq K \text{ compact} \end{array}\right\} \implies K \cap F \text{ is compact}$$

□

§1.2 Sequential Compactness

Definition 1.7 (Sequential Compactness) — Let (X, d) be a metric space. A set $K \subseteq X$ is called sequentially compact if every sequence $\{x_n\}_{n\geq 1} \subseteq K$ admits a subsequence that converges in K.

§2 Lec 2: Mar 31, 2021

§2.1 Sequential Compactness (Cont'd)

Theorem 2.1 (Bolzano – Weierstrass)
Let (X, d) be a metric space and let $K \subseteq X$ be infinite. The following are equivalent:

1. K is sequentially compact.
2. For every infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

Proof. 1) $\implies$ 2) Let $A \subseteq K$ be infinite. As every infinite set has a countable subset we can find a sequence $\{a_n\}_{n\geq 1} \subseteq A$ such that $a_n \neq a_m \, \forall n \neq m$. As K is sequentially compact, $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

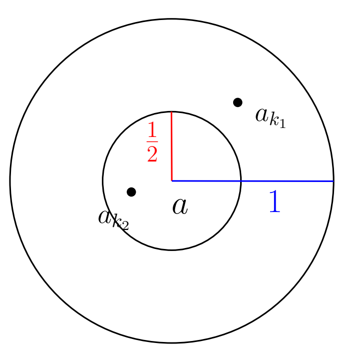
$$a_{k_n} \xrightarrow[n\to\infty]{d} a \in K$$

Claim 2.1. $a \in A' \iff \forall r > 0 \; B_r(a) \cap A \setminus \{a\} \neq \emptyset$.

Indeed, fix $r > 0$.

$$a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_r \in \mathbb{N} \text{ s.t. } d(a, a_{k_n}) < r \quad \forall n \geq n_r$$

As $a_n \neq a_m \, \forall n \neq m$, $\exists n_0 \geq n_r$ s.t. $a_{k_{n_0}} \neq a$. Then $a_{k_{n_0}} \in B_r(a) \cap A \setminus \{a\}$. We get $a \in A' \cap K$.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. We distinguish two cases:
Case 1: The sequence $\{a_n\}_{n\geq 1}$ contains a constant subsequence. That subsequence converges to an element in K.
Case 2: $\{a_n\}_{n\geq 1}$ does not contain a constant subsequence. Then $A = \{a_n : n \geq 1\}$ is infinite and $A \subseteq K$. So $A' \cap K \neq \emptyset$. Let $a \in A' \cap K$. Then $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t. $a_{k_n} \xrightarrow[n\to\infty]{d} a$.



$B_1(a) \cap A \setminus \{a\} \neq \emptyset$

$B_{\min\left\{\frac{1}{2}, d(a, a_{k_1})\right\}}(a) \cap A \setminus \{a\} \neq \emptyset$

$k_2 > k_1$

$\square$

Theorem 2.2

Let (X,d) be a metric space and let $K \subseteq X$ be compact. Then K is sequentially compact.

Proof. If K is finite, then any sequence $\{x_n\}_{n\geq 1} \subseteq K$ will have a constant subsequence.

Assume now K is infinite. We will use the Bolzano – Weierstrass theorem. It suffices to prove that for any infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

$$\left.\begin{aligned} &\text{Note } A \subseteq K \text{ then } A' \subseteq K' \\ &K \text{ compact} \implies K \text{ closed} \implies K' \subseteq K \end{aligned}\right\} \implies A' \subseteq K \implies A' \cap K = A'$$

We argue by contradiction. Assume $A' = \emptyset$. Then for $x \in K$ we have $x \notin A' \implies \exists r_x > 0$ s.t. $B_{r_x}(x) \cap A \setminus \{x\} = \emptyset$. So

$$\left.\begin{aligned} &K \subseteq \textstyle\bigcup_{x\in K} \underbrace{B_{r_x}(x)}_{\text{open}} \\ &K \text{ compact} \end{aligned}\right\} \implies \exists n \geq 1 \text{ and } \exists x_1, \dots, x_n \in K \text{ s.t.}$$

$$K \subseteq \bigcup_{j=1}^{n} B_{r_j}(x_j) \text{ where } r_j = r_{x_j}$$

In particular,

$$\left.\begin{aligned} &A = \left(\textstyle\bigcup_{j=1}^{n} B_{r_j}(x_j)\right) \cap A = \textstyle\bigcup_{j=1}^{n} \left[B_{r_j}(x_j) \cap A\right] \\ &\text{By construction, } B_{r_j}(x_j) \cap A \subseteq \{x_j\} \end{aligned}\right\} \implies \underbrace{A}_{\text{infinite}} \subseteq \underbrace{\bigcup_{j=1}^{n} \{x_j\}}_{\text{finite}}$$

– Contradiction! So $A' \neq \emptyset$. ☐

Proposition 2.3

Let (X,d) be a metric space and let $K \subseteq X$ be sequentially compact. Then K is closed and bounded.

Proof. Let's show K is closed $\iff K = \overline{K}$.

We know $K \subseteq \overline{K}$. We need to show $\overline{K} \subseteq K$. Let $x \in \overline{K} \implies \exists \{x_n\}_{n\geq 1} \subseteq K$ s.t. $x_n \xrightarrow[n\to\infty]{d} x$.

K sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

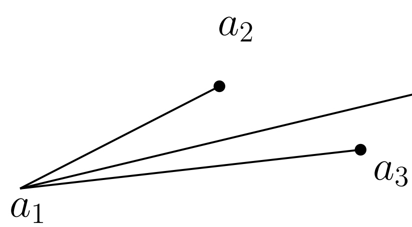
$$\left.\begin{aligned} &x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ &x_n \xrightarrow[n\to\infty]{d} x \implies x_{k_n} \xrightarrow[n\to\infty]{d} x \\ &\text{Limits of convergent sequences are unique} \end{aligned}\right\} \implies x = y \in K$$

As $x \in \overline{K}$ was arbitrary, we get $\overline{K} \subseteq K$.

Let's show K is bounded. We argue by contradiction. Assume K is not bounded. Let $a_1 \in K$.

$$\begin{aligned} K \text{ not bounded } &\implies K \nsubseteq B_1(a_1) \implies \exists a_2 \in K \text{ s.t. } d(a_1, a_2) \geq 1 \\ K \text{ not bounded } &\implies K \nsubseteq B_{1+d(a_1,a_2)}(a_1) \implies \exists a_3 \in K \text{ s.t. } d(a_1, a_3) \geq 1 + d(a_1, a_2) \end{aligned}$$

Proceeding inductively, we find a sequence $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_1, a_{n+1}) \geq 1 + d(a_1, a_n)$.



By construction,

$$|d(a_1, a_m) - d(a_1, a_n)| \geq |n - m| \quad \forall n, m \geq 1$$

By the triangle inequality,

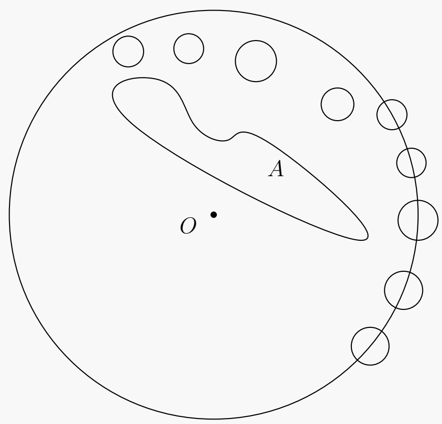
$$d(a_n, a_m) \geq |d(a_1, a_n) - d(a_1, a_m)| \geq |n - m| \quad \forall n, m \geq 1$$

This sequence cannot have a convergent (Cauchy) subsequence, thus contradiction the hypothesis that K is sequentially compact. So K is bounded. $\square$

Definition 2.4 (Totally Bounded) — Let (X, d) be a metric space. A set $A \subseteq X$ is totally bounded if for every $\epsilon > 0$, A can be covered by finitely many balls of radius ϵ.

Remark 2.5.

1. A totally bounded $\implies$ A bounded.
   Indeed, taking $\epsilon = 1$, $\exists n \geq 1$ and $\exists x_1, \ldots, x_n \in X$ s.t.
   $$A \subseteq \bigcup_{j=1}^{n} B_1(x_j) \subseteq B_r(x_1)$$
   where $r = 1 + \max_{2 \leq j \leq n} d(x_1, x_j)$.
2. A bounded $\not\Longrightarrow$ A totally bounded.
   Consider $\mathbb{N}$ equipped with the discrete metric
   $$d(n,m) = \begin{cases} 0, \; n = m \\ 1, \; n \neq m \end{cases}$$
   Then $\mathbb{N} = B_2(1)$, but $\mathbb{N}$ cannot be covered by finitely many balls of radius $\frac{1}{2}$ since $B_{\frac{1}{2}}(n) = \{n\}$.
3. On $(\mathbb{R}^n, d_2)$, A bounded $\implies$ A totally bounded. Indeed, A bounded $\implies$ $A \subseteq B_R(0)$ for some $R > 0$. $B_R(0)$ can be covered by $10^6 \left(\frac{R}{\epsilon}\right)^n$ many balls of radius ϵ.



§3 Lec 3: Apr 2, 2021

§3.1 Heine – Borel Theorem

Theorem 3.1

Let (X, d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is sequentially compact.
2. K is complete and totally bounded.

Proof. 1) $\implies$ 2) Let's show K is complete. Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence with $x_n \in K \quad \forall n \geq 1$.
K sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$\left.\begin{aligned} &x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ &\{x_n\}_{n\geq 1} \text{ is Cauchy} \end{aligned}\right\} \implies x_n \xrightarrow[n\to\infty]{d} y \in K$$

As $\{x_n\}_{n\geq 1} \subseteq K$ was arbitrary, we get that K is complete.
Let's show K is totally bounded. Fix $\epsilon > 0$ and $a_1 \in K$.

- If $K \subseteq B_\epsilon(a_1)$, then K is totally bounded.
- If $K \not\subseteq B_\epsilon(a_1)$, then $\exists a_2 \in K$ s.t. $d(a_1, a_2) \geq \epsilon$
- If $K \subseteq B_\epsilon(a_1) \cup B_\epsilon(a_2)$, then K is totally bounded.
- If $K \not\subseteq B_\epsilon(a_1) \cup B_\epsilon(a_2)$, then $\exists a_3 \in K$ s.t. $d(a_1, a_3) \geq \epsilon$ and $d(a_2, a_3) \geq \epsilon$.

We distinguish two cases:
Case 1: The process terminates in finitely many steps $\implies$ K is totally bounded.
Case 2: The process does not terminate in finitely many steps. Then we find $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_n, a_m) \geq \epsilon \quad \forall n \neq m$. This sequence does not admit a convergent subsequence, contradicting the fact that K is sequentially compact.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. K totally bounded $\implies \mathcal{J}_1$ finite and $\left\{x_j^{(1)}\right\}_{j\in\mathcal{J}_1} \subseteq X$ s.t.

$$\left.\begin{aligned} &K \subseteq \textstyle\bigcup_{j\in\mathcal{J}_1} B_1(x_j^{(1)}) \\ &\{a_n\}_{n\geq 1} \subseteq K \end{aligned}\right\} \implies \exists j_1 \in \mathcal{J}_1 \text{ s.t. } \left|\left\{n : a_n \in B_1(x_{j_1}^{(1)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(1)}\right\}_{n\geq 1}$ be the corresponding subsequence.
K totally bounded $\implies \exists \mathcal{J}_2$ finite and $\left\{x_j^{(2)}\right\}_{j\in\mathcal{J}_2} \subseteq X$ s.t.

$$\left.\begin{aligned} &K \subseteq \textstyle\bigcup_{j\in\mathcal{J}_2} B_{\frac{1}{2}}(x_j^{(2)}) \\ &\left\{a_n^{(1)}\right\}_{n\geq 1} \subseteq K \end{aligned}\right\} \implies \exists j_2 \in \mathcal{J}_2 \text{ s.t. } \left|\left\{n : a_n^{(1)} \in B_{\frac{1}{2}}(x_{j_2}^{(2)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(2)}\right\}_{n\geq 1}$ denote the corresponding subsequence.
We proceed inductively. We find that $\forall k \geq 1$

- $\left\{a_n^{(k+1)}\right\}_{n\geq 1}$ subsequence of $\left\{a_n^{(k)}\right\}_{n\geq 1}$
- $\left\{a_n^{(k)}\right\}_{n\geq 1} \subseteq B_{\frac{1}{k}}\left(x_{j_k}^{(k)}\right)$ for some $x_{j_k}^{(k)} \in X$.

We consider the subsequence $\left\{a_n^{(n)}\right\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$.

$$\begin{aligned}
\left\{a_n^{(1)}\right\}_{n\geq 1} &= \left(a_1^{(1)},\quad a_2^{(1)},\quad a_3^{(1)},\quad \ldots\right) \\
\left\{a_n^{(2)}\right\}_{n\geq 1} &= \left(\qquad a_1^{(2)},\quad a_2^{(2)},\quad a_3^{(2)},\quad \ldots\right) \\
\left\{a_n^{(3)}\right\}_{n\geq 1} &= \left(\qquad\qquad a_1^{(3)},\quad a_2^{(3)},\qquad a_3^{(3)},\quad \ldots\right)
\end{aligned}$$

For $n, m \geq k$ the $a_n^{(n)}, a_m^{(m)}$ belong to the subsequence $\left\{a_n^{(k)}\right\}_{n\geq 1}$. In particular,

$$d(a_n^{(n)}, a_m^{(m)}) \leq d(a_n^{(n)}, x_{j_k}^{(k)}) + d(a_m^{(m)}, x_{j_k}^{(k)}) < \frac{2}{k} \quad \forall n, m \geq k$$

This shows $\left\{a_n^{(n)}\right\}_{n\geq 1}$ is Cauchy and K is complete, so $a_n^{(n)} \xrightarrow[n\to\infty]{d} a \in K$. As $\{a_n\}_{n\geq 1}$ was arbitrary, we get that K is sequentially compact. □

Lemma 3.2

Let (X, d) be a sequentially compact metric space. Let $\{G_i\}_{i\in I}$ be an open cover of X. Then there exists $\epsilon > 0$ such that every ball of radius ϵ is contained in at least one G_i.

Proof. We argue by contradiction. Then

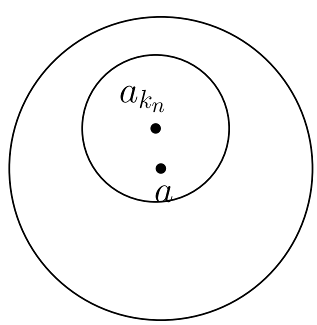
$$\forall n \geq 1 \quad \exists a_n \in X \text{ s.t. } B_{\frac{1}{n}}(a_n) \text{ is not contained in any } G_i$$

X is sequentially compact $\implies \exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

$$a_{k_n} \xrightarrow[n\to\infty]{d} a \in X = \bigcup_{i\in I} G_i \implies \exists i_0 \in I \text{ s.t. } a \in G_{i_0}$$

$$G_{i_0} \text{ open } \implies \exists r > 0 \text{ s.t. } B_r(a) \subseteq G_{i_0}$$

$$a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_1(r) \in \mathbb{N} \text{ s.t. } d(a_1, a_{k_n}) < \frac{r}{2} \forall n \geq n_1$$



Let $n_2(r)$ s.t. $n_2 > \frac{2}{r}$.

Claim 3.1. $\forall n \geq n_r = \max\{n_1, n_2\}$ we have $B_{\frac{1}{k_n}}(a_{k_n}) \subseteq B_r(a) \subseteq G_{i_0}$ thefore giving a contradiction!

Fix $x \in B_{\frac{1}{k_n}}(a_{k_n})$. Then

$$d(a,x) \leq d(x, a_{k_n}) + d(a_{k_n}, a) < \frac{1}{k_n} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r$$

□

Theorem 3.3

A sequentially compact metric space (X, d) is compact.

Proof. Let $\{G_i\}_{i \in I}$ be an open cover of X. Let ϵ be given by the previous lemma. X sequentially compact $\implies$ X totally bounded $\implies \exists n \geq 1$ and

$$\left.\begin{array}{l} \exists x_1, \ldots, x_n \in X \text{ s.t. } X = \bigcup_{j=1}^n B_\epsilon(x_j) \\ \forall 1 \leq j \leq n \quad \exists i_j \in I \text{ s.t. } B_\epsilon(x_j) \subseteq G_{i_j} \end{array}\right\} \implies X = \bigcup_{j=1}^n G_{i_j}$$

□

Collecting our results so far we obtain

Theorem 3.4 (Heine – Borel)

Let (X, d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is compact,
2. K is sequentially compact,
3. K is complete and totally bounded,
4. Every infinite subset of K has an accumulation point in K.

Remark 3.5. In $\mathbb{R}^n$, K is compact $\iff$ K is closed and bounded.

Definition 3.6 (Finite Intersection Property) — An infinite family $\{F_i\}_{i \in I}$ of closed sets is said to have the finite intersection property if $\forall \mathcal{J} \subseteq I$ finite we have

$$\bigcap_{j \in \mathcal{J}} F_j \neq \emptyset$$

Theorem 3.7

A metric space (X, d) is compact if and only if every infinite family $\{F_i\}_{i\in I}$ of closed sets with the finite intersection property satisfies

$$\bigcap_{i\in I} F_i \neq \emptyset$$

Proof. " $\implies$ " We argue by contradiction. Assume $\exists \{F_i\}_{i\in I}$ closed sets with the finite intersection property s.t. $\bigcap_{i\in I} F_i = \emptyset$

$$\left.\begin{aligned} &X = {}^c\left(\textstyle\bigcap_{i\in I} F_i\right) = \textstyle\bigcup_{i\in I} \underbrace{{}^cF_i}_{\text{open}} \\ &X \text{ compact} \end{aligned}\right\} \implies \exists \mathcal{J} \subseteq I \text{ finite s.t. } X = \bigcup_{j\in\mathcal{J}} {}^cF_j$$

$$\implies \emptyset = {}^c\left(\bigcup_{j\in\mathcal{J}} {}^cF_j\right) = \bigcap_{j\in\mathcal{J}} F_j \text{ – Contradiction!}$$

" $\impliedby$ " We argue by contradiction. Assume $\exists \{G_i\}_{i\in I}$ open cover of X that does not admit a finite subcover.

So $\forall \mathcal{J} \subseteq I$ finite $X \neq \bigcup_{j\in\mathcal{J}} G_j \implies \emptyset \neq \bigcap_{j\in\mathcal{J}} \underbrace{{}^cG_j}_{\text{closed}}$. So $\{{}^cG_i\}_{i\in I}$ is a family of closed sets with the finite intersection property. Then

$$\bigcap_{i\in I} {}^cG_i \neq \emptyset \implies \bigcup_{i\in I} G_i \neq X$$

Contradiction! □

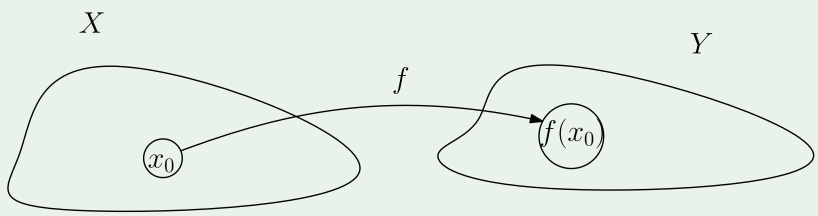
§4 Lec 4: Apr 5, 2021

§4.1 Continuity

Definition 4.1 (Continuous Function) — Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that a function $f : X \to Y$ is continuous at a point $x_0 \in X$ if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \text{ s.t. } d_X(x, x_0) < \delta \text{ then } d_Y\left(f(x), f(x_0)\right) < \epsilon$$

We say f is continuous (on X) if f is continuous at every point in X.



Remark 4.2. $f : X \to Y$ is continuous at every isolated point in X. Indeed, if $x_0 \in X$ is isolated, then $\exists \delta > 0$ s.t. $B_\delta^X(x_0) = \{x_0\}$. Then $d_X(x, x_0) < \delta \implies d_Y\left(f(x), f(x_0)\right) = 0$

Proposition 4.3

Let $(X, d_X), (Y, d_Y)$ be two metric spaces and $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous at $x_0 \in X$.
2. For any $\{x_n\}_{n\ge 1} \subseteq X$ s.t. $x_n \xrightarrow[n\to\infty]{d_X} x_0$ we have $f(x_n) \xrightarrow[n\to\infty]{d_Y} f(x_0)$.

Proof. 1) $\implies$ 2) Let $\{x_n\}_{n\ge 1} \subseteq X$ s.t. $x_n \xrightarrow[n\to\infty]{d_X} x_0$.
Let $\epsilon > 0$. f continuous at $x_0 \implies \exists \delta > 0$ s.t.

$$\left.\begin{aligned} &d_X(x, x_0) < \delta \implies d_Y\left(f(x), f(x_0)\right) < \epsilon \\ &x_n \xrightarrow[n\to\infty]{d_X} x_0 \implies \exists n_\delta \in \mathbb{N} \text{ s.t. } d_X(x_n, x_0) < \delta\, \forall n \ge n_\delta \end{aligned}\right\} \implies d_X\left(f(x_n), f(x_0)\right) < \epsilon$$

for each $n \ge n_\delta$.
2) $\implies$ 1) We argue by contradiction. Assume

$$\exists \epsilon_0 > 0 \text{ s.t. } \forall \delta > 0 \quad \exists x_\delta \in X \text{ s.t. } d_X(x_\delta, x_0) < \delta \text{ but } d_Y\left(f(x_\delta), f(x_0)\right) \ge \epsilon_0$$

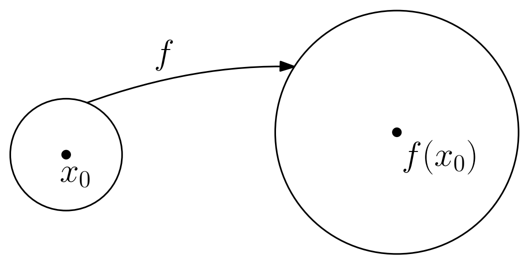
Letting $\delta = \frac{1}{n}$ we find $\{x_n\}_{n\ge 1} \subseteq X$ s.t. $d_X(x_n, x_0) < \frac{1}{n}$ but $d_Y\left(f(x_n), f(x_0)\right) \ge \epsilon_0$ – Contradiction! ☐

Theorem 4.4

Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous.
2. for any G open in Y, $f^{-1}(G) = \{x \in X : f(X) \in G\}$ is open in X.
3. for any F closed in Y, $f^{-1}(F)$ is closed in X.
4. for any $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.
5. for any $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

Proof. We will show 1) $\implies$ 2) $\implies$ 3) $\implies$ 4) $\implies$ 5) $\implies$ 1).
1) $\implies$ 2) Let $G \subseteq Y$ be open.



Let $x_0 \in f^{-1}(G)$

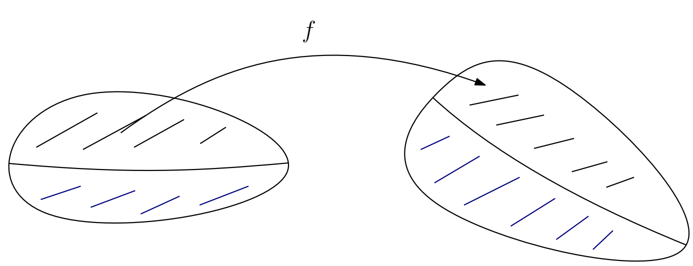
$$\implies \left.\begin{array}{l} f(x_0) \in G \\ G \text{ open in } Y \end{array}\right\} \implies \exists \epsilon > 0 \text{ s.t. } B_\epsilon^Y(f(x_0)) \subseteq G$$

f is continuous

$$\begin{aligned} &\implies \exists \delta > 0 \text{ s.t. } f\left(B_\delta^X(x_0)\right) \subseteq B_\epsilon^Y(f(x_0)) \subseteq G \\ &\implies B_\delta^X(x_0) \subseteq f^{-1}(G) \implies x_0 \in \overset{\circ}{\widehat{f^{-1}(G)}} \end{aligned}$$

So $f^{-1}(G)$ is open in X.
2) $\implies$ 3) Let $F \subseteq Y$ be closed $\implies$ ${}^cF = Y \setminus F$ is open in Y. By assumption,

$$\left.\begin{array}{l} f^{-1}\left({}^cF\right) \text{ is open in } X \\ f^{-1}\left({}^cF\right) = {}^c\left[f^{-1}(F)\right] = X \setminus f^{-1}(F) \end{array}\right\} \implies f^{-1}(F) \text{ is closed in } X$$



$$f^{-1}(Y \setminus F) = f^{-1}(Y) \setminus f^{-1}(F) = X \setminus f^{-1}(F)$$

3) $\implies$ 4) Let $B \subseteq Y \implies \overline{B}$ closed in Y. By assumption,

$$\left.\begin{aligned} &f^{-1}(\overline{B}) \text{ closed in } X \\ &f^{-1}(\overline{B}) \supseteq f^{-1}(B) \end{aligned}\right\} \implies \overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$$

4) $\implies$ 5) Let $A \subseteq X$. Use the hypothesis with $B = f(A)$. We have

$$\overline{A} \subseteq \overline{f^{-1}(f(A))} \subseteq f^{-1}\left(\overline{f(A)}\right) \implies f(\overline{A}) \subseteq \overline{f(A)}$$

5) $\implies$ 1) We argue by contradiction. Assume $\exists x_0 \in X$ s.t. f is not continuous at x_0. Then $\exists \epsilon_0 > 0$ and $\exists x_n \xrightarrow[n\to\infty]{d_X} x_0$ but $d_Y(f(x_n), f(x_0)) \geq \epsilon_0$.
Let $A = \{x_n : n \geq 1\}$. Then $x_0 \in \overline{A}$ but $f(x_0) \notin \overline{\{f(x_n) : n \geq 1\}} = \overline{f(A)}$. On the other hand, we must have

$$\left.\begin{aligned} &f(\overline{A}) \subseteq \overline{f(A)} \\ &x_0 \in \overline{A} \end{aligned}\right\} \implies f(x_0) \in \overline{f(A)}$$

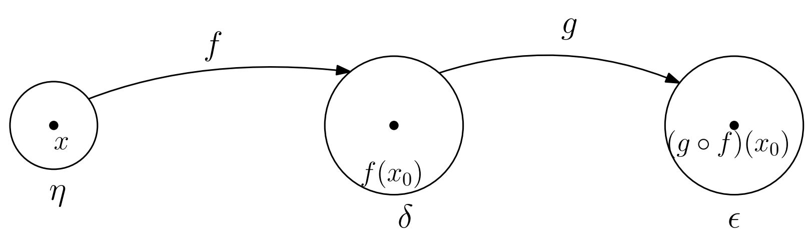
Contradiction! $\square$

Proposition 4.5

Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces and assume $f : X \to Y$ is continuous at $x_0 \in X$ and $g : Y \to Z$ is continuous at $f(x_0) \in Y$. Then $g \circ f : X \to Z$ is continuous at x_0.

Proof. Fix $\epsilon > 0$.

$$\begin{aligned} g \text{ continuous at } f(x_0) &\implies \exists \delta > 0 \text{ s.t. } d_Y(y, f(x_0)) < \delta \implies d_Z(g(y), g(f(x_0))) < \epsilon \\ f \text{ continuous at } x_0 &\implies \exists \eta > 0 \text{ s.t. } d_X(x, x_0) < \eta \implies d_Y(f(x), f(x_0)) < \delta \end{aligned}$$



So if $d_X(x, x_0) < \eta$ then $d_Z(g(f(x)), g(f(x_0))) < \epsilon$. $\square$

Exercise 4.1. Let (X, d) be a metric space and let $f, g : X \to \mathbb{R}$ be continuous at $x_0 \in X$. Then $f \pm g, f \cdot g$ are continuous at x_0. If $g(x_0) \neq 0$ then $\frac{f}{g} : X \to \mathbb{R}$ is continuous at x_0.

Exercise 4.2. Let (X, d) be a metric space and let $f_1, \ldots, f_n : X \to \mathbb{R}$. Then $f = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ is continuous at $x_0 \in X$ if and only if $f_1, \ldots, f_n$ are continuous at x_0.

Hint: $|f_i(x) - f_i(x_0)| \leq d_2(f(x), f(x_0)) = \sqrt{\sum_{j=1}^n |f_j(x) - f_j(x_0)|^2}$.

§4.2 Continuity and Compactness

Theorem 4.6
Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be continuous. If K is compact in X, then $f(K)$ is compact in Y.

Proof. Method 1: Let $\{G_i\}_{i\in I}$ be a family of open sets in Y s.t.

$$f(K) \subseteq \bigcup_{i\in I} G_i \implies K \subseteq f^{-1}\left(\bigcup_{i\in I} G_i\right) = \bigcup_{i\in I} \underbrace{f^{-1}(G_i)}_{\text{open in } X}$$

K compact $\implies \exists n \geq 1$ and $\exists i, \ldots, i_n \in I$ s.t.

$$K \subseteq \bigcup_{j=1}^{n} f^{-1}\left(G_{i_j}\right) = f^{-1}\left(\bigcup_{j=1}^{n} G_{i_j}\right) \implies f(K) \subseteq \bigcup_{j=1}^{n} G_{i_j}$$

Method 2: Let's show $f(K)$ is sequentially compact. Let $\{y_n\}_{n\geq 1} \subseteq f(K)$.

$$y_n \in f(K) \implies \exists x_n = f^{-1}(y_n) \in K$$

As K is sequentially compact, $\exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$\left.\begin{array}{l} x_{k_n} \xrightarrow[n\to\infty]{d_X} x_0 \in K \\ f \text{ is continuous} \end{array}\right\} \implies \underbrace{f\left(x_{k_n}\right)}_{=y_{k_n}} \xrightarrow[n\to\infty]{d_Y} f(x_0) \in f(K)$$

□

§5 Lec 5: Apr 7, 2021

§5.1 Continuity and Compactness (Cont'd)

Corollary 5.1

Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}^n$ be continuous. Then $f(X)$ is closed and bounded.

Corollary 5.2

Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then there exists $x_1, x_2 \in X$ s.t.

$$f(x_1) = \inf \{f(x) : x \in X\} \text{ and } f(x_2) = \sup \{f(x) : x \in X\}$$

Proof. $f(x)$ is closed and bounded.

$$\begin{aligned} \text{Boundedness} &\implies \inf f(x) \text{ and } \sup f(x) \text{ are well defined} \\ \text{Closedness} &\implies \inf f(x), \sup f(x) \in \overline{f(x)} = f(x) \end{aligned}$$

□

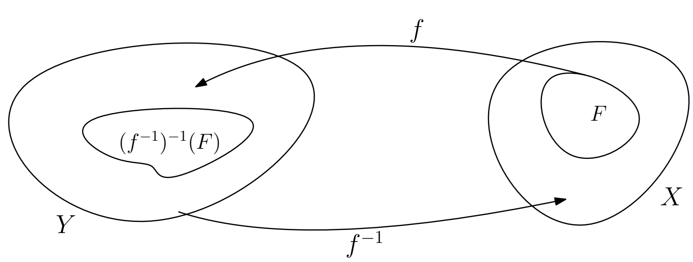
Proposition 5.3

Let $(X, d_X), (Y, d_Y)$ be metric spaces s.t. X is compact. Let $f : X \to Y$ be bijective and continuous. Then $f^{-1} : Y \to X$ is continuous.

Proof. If suffices to show that for every closed set $F \subseteq X$, we have

$$\left(f^{-1}\right)^{-1}(F) = \left\{y \in Y : f^{-1}(y) \in F\right\}$$

is closed in Y.



But $\left(f^{-1}\right)^{-1}(F) = f(F)$.

$$\left.\begin{aligned} &F \text{ closed in } X \text{ compact} \implies F\text{compact} \\ &f : X \to Y \text{ is continuous} \end{aligned}\right\} \implies f(F) \text{ is compact and closed}$$

□

Definition 5.4 (Uniform Continuity) — Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say that a function $f : X \to Y$ is uniformly continuous if

$$\forall \epsilon > 0 \quad \exists \delta = \delta(\epsilon) \text{ s.t. } d_X(x, y) < \delta \implies d_Y\left(f(x), f(y)\right) < \epsilon$$

Compare this with $g : X \to Y$ is continuous if

$$\forall x \in X \quad \forall \epsilon > 0 \quad \exists \delta = \delta(\epsilon, x) \text{ s.t. } d_X(x, y) < \delta \implies d_Y\left(f(x), f(y)\right) < \epsilon$$

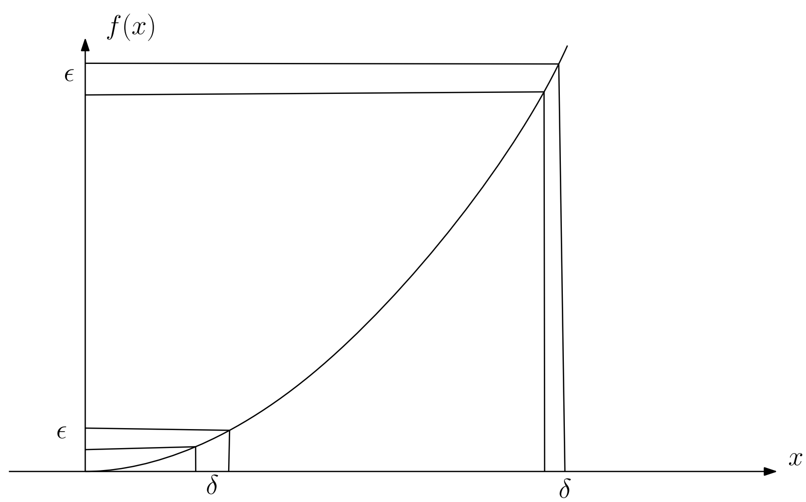
Remark 5.5.
1. Continuity is defined pointwise. Uniform continuity is a property of a function on a set.
2. Uniform continuity $\implies$ continuity.
3. There are continuous functions that are not uniformly continuous.

For example, consider

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

Let $x_n = n + \frac{1}{n}$, $y_n = n$

$$|x_n - y_n| = \frac{1}{n} \xrightarrow[n \to \infty]{} 0$$
$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} > 2$$



Theorem 5.6

Let $(X, d_X), (Y, d_Y)$ be metric spaces with X compact. Let $f : X \to Y$ continuous. Then f is uniformly continuous.

Proof. We argue by contradiction. Assume f is not uniformly continuous $\implies \exists \epsilon_0 > 0$ s.t. $\forall \delta > 0 \, \exists x_\delta, y_\delta \in X$ s.t. $d_X(x_\delta, y_\delta) < \delta$ but $d_Y(f(x_\delta), f(y_\delta)) \geq \epsilon_0$.

Let $\delta = \frac{1}{n}$ to get $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \subseteq X$ s.t. $d_X(x_n, y_n) < \frac{1}{n}$ but $d_Y(f(x_n), f(y_n)) \geq \epsilon_0$

X compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$x_{k_n} \xrightarrow[n\to\infty]{d_X} x_0 \in X$$

By the triangle inequality,

$$d(y_{k_n}, x_0) \leq \underbrace{d(x_{k_n}, y_{k_n})}_{<\frac{1}{k_n} \leq \frac{1}{n} \xrightarrow[n\to\infty]{} 0} + \underbrace{d(x_{k_n}, x_0)}_{\xrightarrow[n\to\infty]{} 0} \xrightarrow[n\to\infty]{} 0 \implies y_{k_n} \xrightarrow[n\to\infty]{d_X} x_0$$

$$f \text{ continuous} \implies \begin{cases} f(x_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0) \\ f(y_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0) \end{cases}$$

But

$$\epsilon_0 \leq d_Y(f(x_{k_n}), f(y_{k_n})) \leq \underbrace{d_Y(f(x_{k_n}), f(x_0))}_{\to 0} + \underbrace{d_Y(f(x_0), f(y_{k_n}))}_{\to 0} \xrightarrow[n\to\infty]{} 0$$

Contradiction! ☐

§5.2 Continuity and Connectedness

Theorem 5.7

Let $(X, d_X), (Y, d_Y)$ be metric spaces s.t. X is connected. Let $f : X \to Y$ be continuous. Then $f(X)$ is connected.

Proof. Method 1: Abusing notation we write $f : X \to f(x)$. It suffices to show that if $\emptyset \neq B \subseteq f(x)$ is both open and closed in $f(x)$ then $B = f(x)$.

As f is continuous, $f^{-1}(B) \neq \emptyset$ is both open and closed in X. But X is connected which implies $f^{-1}(B) = X$ and $f(x) = B$.

Method 2: Assume that $f(x)$ is not connected. Then $\exists \emptyset \neq B_1 \subseteq Y$, $\exists \emptyset \neq B_2 \subseteq Y$ s.t. $f(x) \subseteq B_1 \cup B_2$ and

$$\overline{B_1} \cap B_2 = \emptyset = B_1 \cap \overline{B_2}$$

let

$$A_1 = f^{-1}(B_1) \neq \emptyset$$
$$A_2 = f^{-1}(B_2) \neq \emptyset$$

Have

$$f(X) \subseteq B_1 \cup B_2 \implies X \subseteq f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) = A_1 \cup A_2$$
$$\overline{A_1} \cap A_2 = \overline{f^{-1}(B_1)} \cap f^{-1}(B_2) \subseteq f^{-1}(\overline{B_1}) \cap f^{-1}(B_2) = f^{-1}(\overline{B_1} \cap B_2)$$
$$= f^{-1}(\emptyset) = \emptyset$$

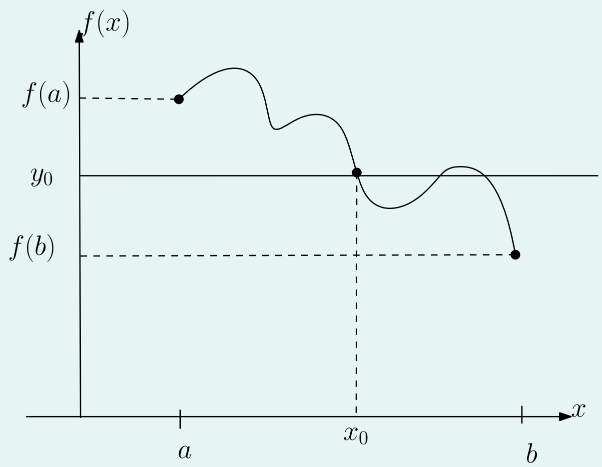
Similarly, $\overline{A_2} \cap A_1 = \emptyset$. exercise

This contradicts that X is connected. ☐

Corollary 5.8 (Darboux's Property)

Let (X, d_X) be a metric space and let $f : X \to \mathbb{R}$ be continuous. If $A \subseteq X$ is connected then $f(A)$ is an interval in $\mathbb{R}$.

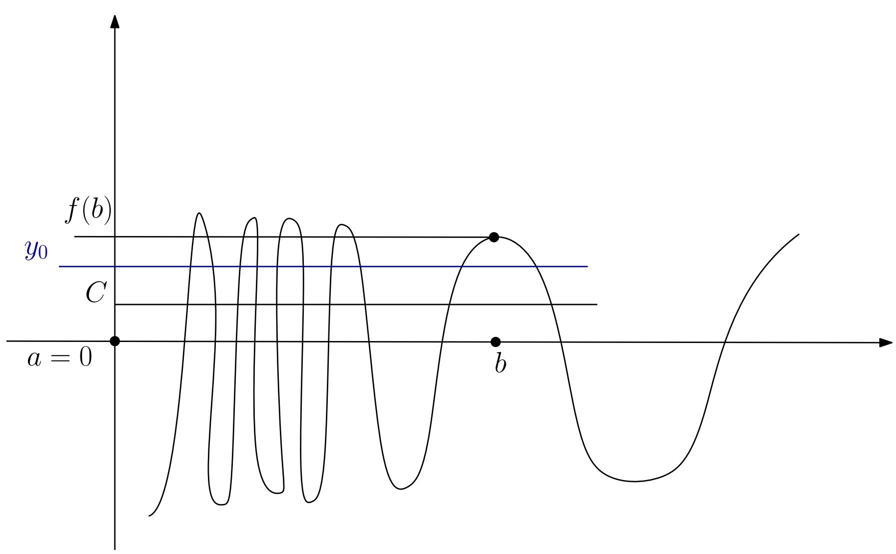
In particular, if $X = \mathbb{R}$, and $a, b \in \mathbb{R}$ s.t. $a < b$ and y_0 lies between $f(a)$ and $f(b)$, then $\exists x_0 \in (a, b)$ s.t. $f(x_0) = y_0$.



Remark 5.9. There are function that have the Darboux property, but are not continuous.

For example, consider

$$f : [0, \infty) \to \mathbb{R}, \quad f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ c, & x = 0 \end{cases} \quad \text{where } c \in [-1, 1]$$



Notice f is continuous on $(0, \infty)$ implies f has the Darboux property on $(0, \infty)$. f has the Darboux property on $[0, \infty)$, but is not continuous at $x = 0$.

§6 Lec 6: Apr 9, 2021

§6.1 Continuity and Connectedness (Cont'd)

Proposition 6.1

Let (X, d_X) and (Y, d_Y) be two connected metric spaces. Then $(X \times Y, d)$ where

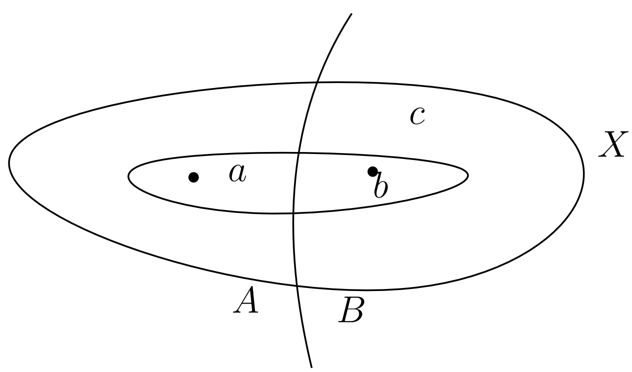
$$d\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

is a connected metric space.

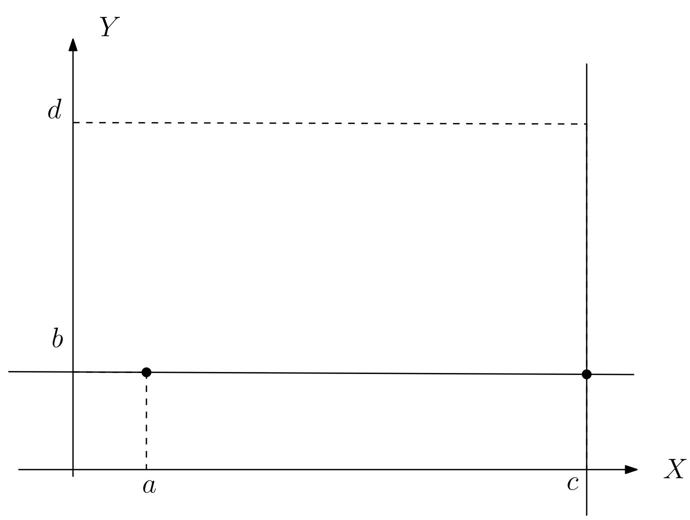
Remark 6.2. One could replace the distance d by

$$d_1\left((x_1, y_1), (x_2, y_2)\right) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$
$$d_\infty\left((x_1, y_1), (x_2, y_2)\right) = \max\left\{d_X(x_1, x_2), d_Y(y_1, y_2)\right\}$$

Proof. We will use the fact that a metric space is connected if and only if any two points are contained in a connected subset of the metric space.



So to show $X \times Y$ is connected if suffices to show that if $(a, b), (c, d) \in X \times Y$, then there exists $C \subseteq X \times Y$ connected s.t. $(a, b), (c, d) \in C$.



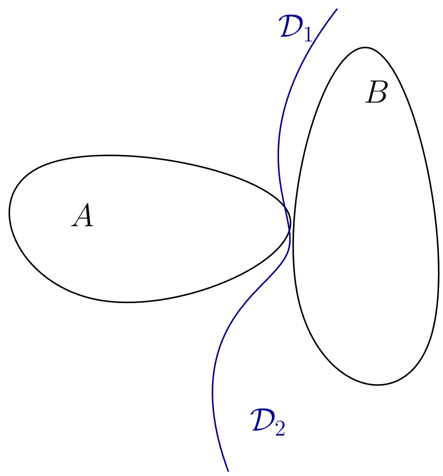
Let $f : X \to X \times Y$ where $f(x) = (x, b)$

Claim 6.1. f is continuous.

Take $\delta = \epsilon$ in the definition of continuity. As X is connected, $f(X) = X \times \{b\}$ is connected.

Similarly, $g : Y \to X \times Y$, $g(y) = (c, y)$ is continuous and since Y is connected, $g(Y) = \{c\} \times Y$ is connected.

Finally, $f(x) \cap g(y) \ni (c, b)$ and so $f(x)$, $g(y)$ are not separated. As the union of two connected not separated sets is connected we get $f(x) \cup g(y)$ is connected.



Note $(a, b), (c, d) \in f(x) \cup g(y)$. $\square$

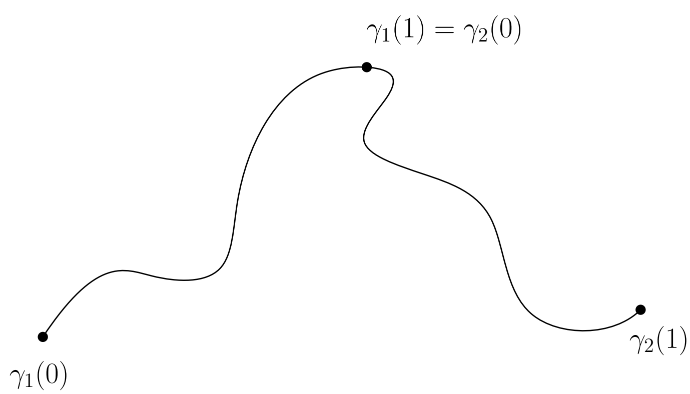
Definition 6.3 (Path) — Let (X, d) be a metric space. A path is a continuous function $\gamma : [0, 1] \to X$. $\gamma(0)$ is called the origin of the path and $\gamma(1)$ is called the end of the path.

As $[0, 1]$ is compact and connected and γ is continuous, $\gamma([0, 1])$ is compact and connected.

Given $\gamma : [0, 1] \to X$ a path, we define

$$\gamma^- : [0, 1] \to X, \qquad \gamma^-(t) = \gamma(1 - t) \text{ is a path}$$

Given $\gamma_1, \gamma_2 : [0, 1] \to X$ paths s.t. $\gamma_1(1) = \gamma_2(0)$.



We define

$$\gamma_1 \vee \gamma_2 : [0,1] \to X$$

via

$$\gamma_1 \vee \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Proposition 6.4

Let (X,d) be a metric space and let $A \subseteq X$. Then 1) $\iff$ 2) $\implies$ 3) where

1. $\exists a \in A$ s.t. $\forall x \in A \, \exists \gamma_x : [0,1] \to A$ path s.t.

$$\gamma_x(0) = a \text{ and } \gamma_x(1) = x$$

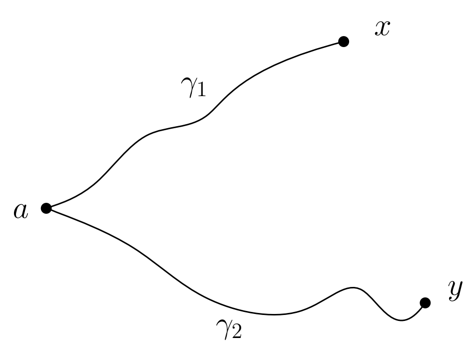
2. $\forall x, y \in A \, \exists \gamma_{x,y} : [0,1] \to A$ path s.t.

$$\gamma_{x,y}(0) = x \text{ and } \gamma_{x,y}(1) = y$$

3. A is connected.

Proof. 1) $\implies$ 2) Let $x, y \in A$. By hypothesis, $\exists \gamma_x, \gamma_y : [0,1] \to A$ paths s.t.

$$\gamma_x(0) = \gamma_y(0) = a, \quad \gamma_x(1) = x, \quad \gamma_y(1) = y$$



Then $\gamma_x^- \vee \gamma_y : [0,1] \to A$ is the desired path.
2) $\implies$ 1)Choose $a \in A$ arbitrary.
1) $\implies$ 3) Given $x \in A$, let $A_x = \gamma_x([0,1])$ connected. Note

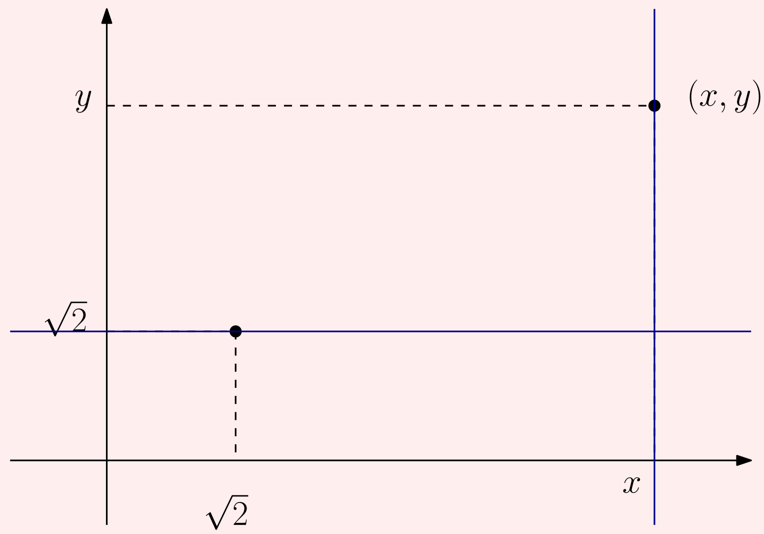
$$a \in \bigcap_{x \in A} A_x \implies \text{ no two sets } A_x,\, A_y \text{ are separated}$$

Then $A = \bigcup_{x \in A} A_x$ is connected. □

Definition 6.5 (Path Connected) — If either 1) or 2) holds in the Proposition 6.4, we say that A is <u>path connected</u>. Note A is path connected implies A is connected.

Example 6.6

$\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected.



We will show that any $(x, y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ can be joined via path in $\mathbb{R}^2 \setminus \mathbb{Q}^2$ to $\left(\sqrt{2}, \sqrt{2}\right)$.

$$(x, y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2 \implies x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q}$$

Say $x \notin \mathbb{Q}$. Then $\{x\} \times \mathbb{R} \subseteq \mathbb{R}^2 \setminus \mathbb{Q}^2$. Note also that $\mathbb{R} \times \left\{\sqrt{2}\right\} \subseteq \mathbb{R}^2 \setminus \mathbb{Q}^2$. Let $\gamma : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2$, $\gamma = \gamma_1 \vee \gamma_2$ where

$$\gamma_1 : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2,\ \gamma_1(t) = \left(\sqrt{2} + t(x - \sqrt{2}), \sqrt{2}\right) \text{ path}$$
$$\gamma_2 : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2,\ \gamma_2(t) = \left(x, \sqrt{2} + t(y - \sqrt{2})\right) \text{ path}$$

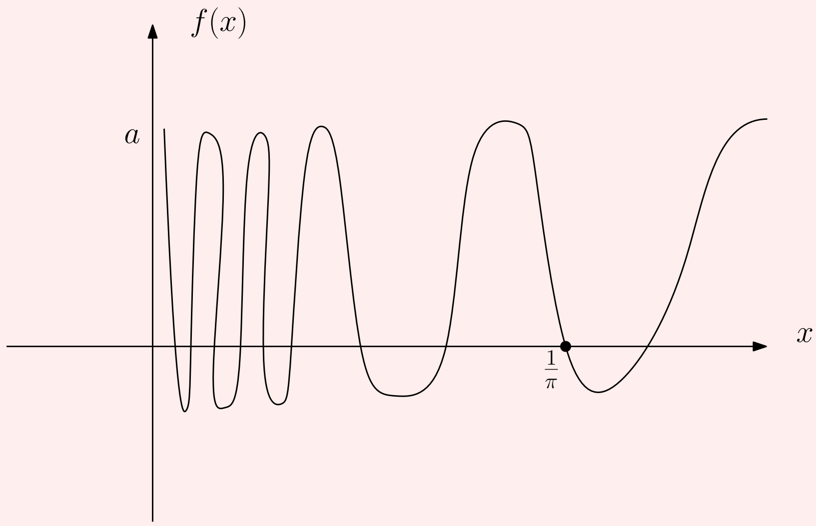
Example 6.7

A connected set which is not path connected. Let $f : [0, \infty) \to \mathbb{R}$ s.t.

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ a, & x = 0 \end{cases}$$

where $a \in [-1, 1]$ fixed.

Then $\Gamma_f = \{(x, f(x)) : x \in [0, \infty)\}$ is connected, but not path connected.



Let's show Γ_f is connected. The function $g : [0, \infty) \to \mathbb{R}^2$, $g(x) = (x, f(x))$ is continuous on $(0, \infty) \implies g((0, \infty))$ is connected.

Also, $g(\{0\}) = \{(0, a)\}$ is connected. We will show that $(0, a) \in \overline{g((0, \infty))}$ and so $\{(0, a)\}, g((0, \infty))$ are not separated. Then

$$\Gamma_f = g([0, \infty)) = g(\{0\}) \cup g((0, \infty)) \text{ is connected}$$

To see $(0, a) \in \overline{g(0, \infty)}$ we need to find $x_n \to 0$ s.t.

$$\sin\left(\frac{1}{x_n}\right) = a$$

Take $x_n = \frac{1}{\arcsin a + 2n\pi}$ where $\arcsin a \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Example 6.8 (Cont'd from above)

Now let's show Γ_f is not path connected. Assume towards a contradiction that there exists $\gamma : [0,1] \to \Gamma_f$ a path s.t.

$$\gamma(0) = (0,a), \qquad \gamma(1) = \left(\frac{1}{\Pi}, 0\right)$$

Note $\Pi_1 \circ \gamma : [0,1] \to \mathbb{R}$ is continuous

$$\left(\Pi_1 \circ \gamma\right)(0) = 0, \quad \left(\Pi_1 \circ \gamma\right)(1) = \frac{1}{\pi}$$

Let $b \in [-1,1] \setminus \{a\}$. By the Darboux property, $\exists t_n \in \left(0, \frac{1}{\pi}\right)$ s.t.

$$\left(\Pi_1 \circ \gamma\right)(t_n) = \frac{1}{\arcsin b + 2n\pi} \text{ where } \arcsin b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

As $[0,1]$ is compact, $\exists t_{k_n} \underset{n\to\infty}{\longrightarrow} t_\infty \in [0,1]$.

$$\left.\begin{aligned} &\gamma \text{ continuous } \implies \gamma\left(t_{k_n}\right) \underset{n\to\infty}{\longrightarrow} \gamma(t_\infty) \\ &\gamma\left(t_{k_n}\right) = \left(\tfrac{1}{\arcsin b + 2k_n\pi}, b\right) \underset{n\to\infty}{\longrightarrow} (0,b) \end{aligned}\right\} \implies \gamma(t_\infty) = (0,b) \notin \Gamma_f$$

§7 Lec 7: Apr 12, 2021

§7.1 Continuity and Connectedness (Cont'd)

Example 7.1

Two connected sets $A, B \subseteq [-1,1] \times [-1,1]$ s.t. $(-1,-1),(1,1) \in A$, $(-1,1),(1,-1) \in B$, $A \cap B = \emptyset$. Let $f : [-1,1] \to [-1,1]$,

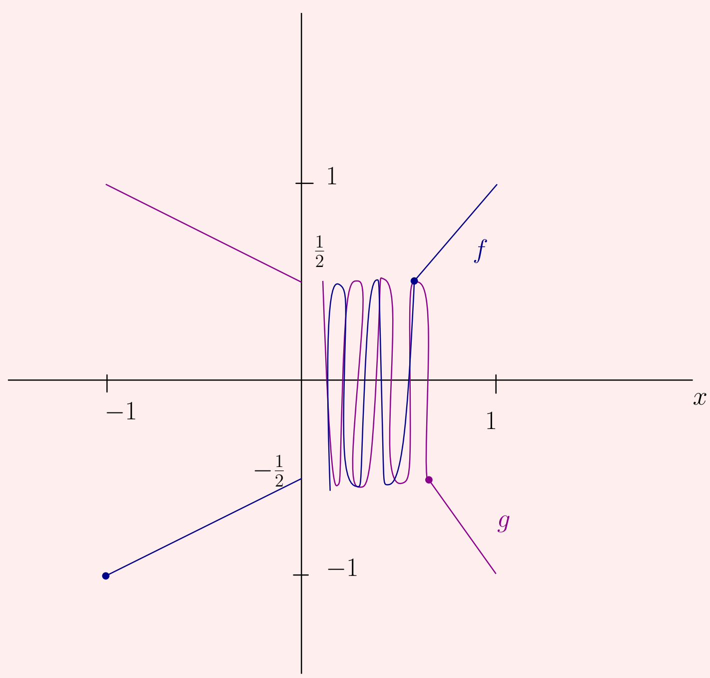
$$f(x) = \begin{cases} \frac{x-1}{2}, & -1 \le x \le 0 \\ x - \frac{1}{2}\sin\frac{\pi}{x}, & 0 < x \le \frac{1}{2} \\ x, & \frac{1}{2} \le x \le 1 \end{cases}$$

Let $g : [-1,1] \to [-1,1]$,

$$g(x) = \begin{cases} \frac{1-x}{2}, & -1 \le x \le 0 \\ -x - \frac{1}{2}\sin\frac{\pi}{x}, & 0 < x \le \frac{1}{2} \\ -x, & \frac{1}{2} \le x \le 1 \end{cases}$$

Let

$$A = \Gamma_f = \{(x_1 f(x)) : x \in [-1,1]\}$$
$$B = \Gamma_g = \{(x_1 g(x)) : x \in [-1,1]\}$$



Example 7.2 (Cont'd from above)
Let's prove $A \cap B = \emptyset$. If

$$\begin{aligned}
-1 \leq x \leq 0, &\quad f(x) = g(x) \iff \frac{x-1}{2} = \frac{1-x}{2} \iff x = 1 \\
0 < x \leq \frac{1}{2}, &\quad f(x) = g(x) \iff x = 0 \\
\frac{1}{2} \leq x \leq 1, &\quad f(x) = g(x) \iff x = 0
\end{aligned}$$

Also

$$\begin{aligned}
f(-1) = -1 &\implies (-1,-1) \in A \\
f(1) = 1 &\implies (1,1) \in A \\
g(-1) = 1 &\implies (-1,1) \in B \\
g(1) = -1 &\implies (1,-1) \in B
\end{aligned}$$

Let's show that A is connected. A similar argument can be used to prove that B is connected.

We write $A = A_1 \cup A_2$ where $A_1 = \{(x, f(x)) : -1 \leq x \leq 0\}$ and $A_2 = \{(x, f(x)) : 0 < x \leq 1\}$. Note that $h : [-1,1] \to \mathbb{R}^2$ where $h(x) = (x, f(x))$ is continuous on $[-1,0]$ and $(0,1]$.

Since $[-1,0]$ and $(0,1]$ are connected sets, we get that $h([-1,0]) = A_1$ and $h((0,1]) = A_2$ are connected.

To show that $A = A_1 \cup A_2$ is connected, it suffices to show that A_1 and A_2 are not separated. We will show $\left(0, -\frac{1}{2}\right) \in A_1 \cap \overline{A_2}$. It's clear that $f(0) = -\frac{1}{2} \implies \left(0, -\frac{1}{2}\right) \in A_1$. To show that $\left(0, -\frac{1}{2}\right) \in \overline{A_2}$ we need to find a decreasing sequence $x_n \to 0$ s.t.

$$f(x_n) = x_n - \frac{1}{2} \sin \frac{\pi}{x_n} \xrightarrow[n\to\infty]{} -\frac{1}{2}$$

We take x_n s.t. $\sin \frac{\pi}{x_n} = 1 \iff \frac{\pi}{x_n} = \frac{\pi}{2} + 2n\pi \iff x_n = \frac{2}{4n+1} \to 0$. Notice that

$$f(x_n) = \frac{2}{4n+1} - \frac{1}{2} \xrightarrow[n\to\infty]{} -\frac{1}{2}$$

§7.2 Convergent Sequences of Functions

Definition 7.3 (Pointwise Convergence) — Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\geq1}$ converges pointwise if for all $x \in X$ the sequence $\{f_n(x)\}_{n\geq1}$ converges in Y. The limit $\lim_{n\to\infty} f_n(x) = f(x)$ defines a function $f : X \to Y$.

Remark 7.4. $\{f_n\}_{n\geq1}$ converges pointwise to f if

$$\forall x \in X \quad \forall \epsilon > 0 \quad \exists n(\epsilon, x) \in \mathbb{N} \text{ s.t. } d_Y(f_n(x), f(x)) < \epsilon \quad \forall n \geq n(\epsilon, x)$$

Note that for $\epsilon > 0$ fixed, $n(\epsilon, \cdot) : X \to \mathbb{N}$ can be bounded or unbounded. If it is bounded, we get the following

Definition 7.5 (Uniform Convergence) — Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\geq 1}$ converges uniformly to a function $f : X \to Y$ if

$$\forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } d_Y(f(x), f_n(x)) < \epsilon \quad \forall n \geq n_\epsilon \, \forall x \in X$$

We denote $f_n \xrightarrow[n\to\infty]{u} f$.

Remark 7.6. Let $(X, d_X), (Y, d_Y)$ be metric spaces, $B(X, Y) = \{f : X \to Y;\ f \text{ is bounded}\}$, $d : B(X, Y) \times B(X, Y) \to \mathbb{R}$ via

$$d(f, g) = \sup_{x\in X} d_Y(f(x), g(x))$$

Exercise 7.1. Show that $(B(X, Y), d)$ is a metric space.

Note that $f_n \xrightarrow[n\to\infty]{u} f \iff M_n = d(f_n, f) \xrightarrow[n\to\infty]{} 0$.

" $\Longleftarrow$ " $\forall \epsilon > 0\ \exists n_\epsilon \in \mathbb{N}$ s.t. $M_n < \epsilon\ \forall n \geq n_\epsilon$

$$\begin{aligned} &\implies d(f_n, f) = \sup_{x\in X} d_Y(f_n(x), f(x)) < \epsilon \quad \forall n \geq n_\epsilon \\ &\implies d_Y(f_n(x), f(x)) < \epsilon \quad \forall n \geq n_\epsilon \quad \forall x \in X \end{aligned}$$

" $\Longrightarrow$ "

$$\begin{aligned} f_n \xrightarrow[n\to\infty]{u} f &\implies \forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } d_Y(f_n(x), f(x)) < \frac{\epsilon}{2} \quad \forall n \geq n_\epsilon \, \forall x \in X \\ &\implies \underbrace{\sup_{x\in X} d_Y(f_n(x), f(x))}_{d(f_n, f) = M_n} \leq \frac{\epsilon}{2} < \epsilon \quad \forall n \geq n_\epsilon \end{aligned}$$

Remark 7.7.
1. Uniform convergence $\implies$ pointwise convergence
2. Pointwise convergence $\not\Longrightarrow$ uniform convergence

$f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = x^n$

$$\{f_n\}_{n\geq 1} \text{ converges pointwise : } \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Let

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Note $f_n \underset{n\to\infty}{\overset{u}{\not\longrightarrow}} f$ since

$$d(f_n, f) = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} |x^n| = 1 \underset{n\to\infty}{\not\longrightarrow} 0$$

Theorem 7.8 (Weierstrass)
Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f_n : X \to Y$ be a sequence of functions that converges uniformly to a function $f : X \to Y$. If $\forall n \geq 1$, f_n is continuous at $x_0 \in X$ then f is continuous at x_0.

Corollary 7.9
A uniform limit of continuous functions is a continuous function.

Proof. (of theorem) Fix $\epsilon > 0$.

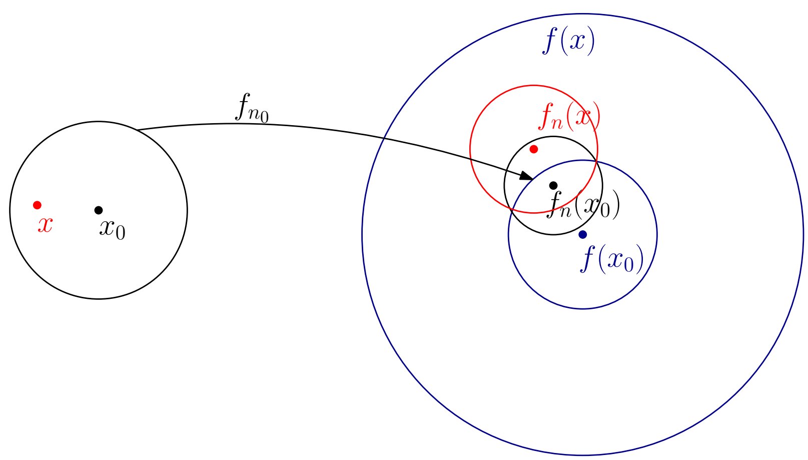
$$f_n \xrightarrow[n\to\infty]{u} f \implies \exists n_\epsilon \in \mathbb{N} \text{ s.t. } d_Y\left(f_n(x), f(x)\right) < \frac{\epsilon}{3} \quad \forall n \geq n_\epsilon \, \forall x \in X$$

Fix $n_0 \geq n_\epsilon$. f_{n_0} is continuous at x_0

$$\implies \exists \delta > 0 \text{ s.t. if } d_X(x_0, x) < \delta$$

then

$$d_Y\left(f_{n_0}(x_0), f_{n_0}(x)\right) < \frac{\epsilon}{3}$$



Then for $x \in B_\delta(x_0)$ we have

$$\begin{aligned} d_Y\left(f(x), f(x_0)\right) &\leq d_Y\left(f(x), f_{n_0}(x)\right) + d\left(f_{n_0}(x), f_{n_0}(x_0)\right) + d\left(f_{n_0}(x_0), f(x_0)\right) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

By definition, f is continuous at x_0. $\square$

§8 Lec 8: Apr 14, 2021

§8.1 Convergent Sequences of Functions (Cont'd)

Theorem 8.1 (Dini)

Let (X, d) be a compact metric space and let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions that converges pointwise to a continuous function $f : X \to \mathbb{R}$. Assume that $\{f_n\}_{n\geq 1}$ is monotone in the sense that either $\{f_n(x)\}_{n\geq 1}$ is increasing for all $x \in X$ or $\{f_n(x)\}_{n\geq 1}$ is decreasing for all $x \in X$. Then,

$$f_n \xrightarrow[n\to\infty]{u} f \text{ i.e. } d(f_n, f) = \sup_{x\in X} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$$

Proof. Assume that $\{f_n\}_{n\geq 1}$ is increasing. Then $\{f - f_n\}_{n\geq 1}$ is decreasing and for all $x \in X$ we have

$$\lim_{n\to\infty} [f(x) - f_n(x)] = \inf_{n\to\infty} [f(x) - f_n(x)] = 0$$

Then $\forall \epsilon > 0 \quad \exists n(\epsilon, x) \in \mathbb{N}$ s.t. $\forall n \geq n(\epsilon, x)$ we have

$$0 \leq f(x) - f_n(x) \leq f(x) - f_{n_{\epsilon,x}}(x) < \epsilon$$

As $f - f_{n_{\epsilon,x}}$ is continuous at x, $\exists \delta(\epsilon, x) > 0$ s.t.

$$d(x, y) < \delta_{\epsilon,x} \implies \left|\left[f(x) - f_{n_{\epsilon,x}}(x)\right] - \left[f(y) - f_{n_{\epsilon,x}}(y)\right]\right| < \epsilon$$

By the triangle inequality, we get

$$\begin{aligned} 0 \leq f(y) - f_{n_{\epsilon,x}}(y) &\leq \left|\left[f(x) - f_{n_{\epsilon,x}}(x)\right] - \left[f(y) - f_{n_{\epsilon,x}}(y)\right]\right| + f(x) - f_{n_{\epsilon,x}}(x) \\ &< \epsilon + \epsilon = 2\epsilon \end{aligned}$$

whenever $y \in B_{\delta_{\epsilon,x}}(x)$. In particular,

$$0 \leq f(y) - f_n(y) \leq f(y) - f_{n_{\epsilon,x}}(y) < 2\epsilon \quad \forall n \geq n_{\epsilon,x}, \forall y \in B_{\delta_{\epsilon,x}}(x) \tag{*}$$

Note

$$\left.\begin{array}{l} X = \bigcup_{x\in X} B_{\delta_{\epsilon,x}}(x) \\ X \text{ compact} \end{array}\right\} \implies \exists \mathcal{J} \subseteq \mathbb{N} \text{ finite and } \exists \{x_j\}_{j\in\mathcal{J}} \in X$$

s.t. $X = \bigcup_{j\in\mathcal{J}} B_{\delta_j}(x_j)$ and where $\delta_j = \delta(\epsilon, x_j)$.
Let $n_\epsilon = \max_{j\in\mathcal{J}} n(\epsilon, x_j)$. Fix $n \geq n_\epsilon$ and $x \in X$. As $x \in X = \bigcup_{j\in\mathcal{J}} B_{\delta_j}(x_j) \implies j \in \mathcal{J}$ s.t. $x \in B_{\delta_j}(x_j)$. By (*), we have

$$0 \leq f(x) - f_n(x) < 2\epsilon$$

As $x \in X$ was arbitrary we get

$$d(f, f_n) \leq 2\epsilon \qquad \forall n \geq n_\epsilon$$

□

Remark 8.2. The compactness of X is necessary in Dini's theorem.

Example 8.3

$f_n : (0,1) \to \mathbb{R}$, $f_n(x) = x^n$ continuous

$$f_{n+1}(x) \leq f_n(x) \quad \forall n \geq 1 \quad \forall x \in (0,1)$$
$$f_n(x) \underset{n\to\infty}{\longrightarrow} 0 \quad \forall x \in (0,1)$$

Let $f : (0,1) \to \mathbb{R}$, $f(x) = 0 \quad \forall x \in (0,1)$. It's continuous. But

$$d(f_n, f) = \sup_{x\in(0,1)} |x^n| = 1 \underset{n\to\infty}{\not\longrightarrow} 0 \implies f_n \overset{u}{\underset{n\to\infty}{\not\longrightarrow}} f$$

Note that $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$ continuous, $\{f_n\}_{n\geq 1}$ is decreasing and converge pointwise to $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases} \qquad \text{which is not continuous}$$

This also shows that the continuity of the limit function is necessary in Dini's theorem.

Remark 8.4. Monotonicity is necessary in Dini's theorem.

Example 8.5

$f_n : [0,1] \to \mathbb{R}$ is continuous. $\{f_n\}_{n\geq 1}$ converges pointwise to $f : [0,1] \to \mathbb{R}$, $f(x) = 0 \, \forall x \in [0,1]$ figure here f is continuous. But

$$d(f_n, f) = \sup_{x\in[0,1]} |f_n(x)| = 1 \underset{n\to\infty}{\not\longrightarrow} 0 \implies f_n \overset{u}{\underset{n\to\infty}{\not\longrightarrow}} f$$

Note that $\{f_n\}_{n\geq 1}$ is not monotone!

§8.2 Space of Functions

Fix $a, b \in \mathbb{R}$, $a < b$. We define

$$C([a,b]) = \{f : [a,b] \to \mathbb{R};\ f \text{ is continuous}\}$$

We equip $C([a,b])$ with the metric $d : C([a,b]) \times C([a,b]) \to \mathbb{R}$, given by

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

Then $(C([a,b]), d)$ is a metric space.
Completeness: Let $\{f_n\}_{n\geq 1} \subseteq C([a,b])$ be Cauchy. So $\forall \epsilon > 0\ \exists n_\epsilon \in \mathbb{N}$ s.t. $d(f_n, f_m) < \epsilon$ $\forall n, m \geq n_\epsilon$

$$\implies |f_n(x) - f_m(x)| < \epsilon \quad \forall n, m \geq n_\epsilon \quad \forall x \in [a,b]$$

So $\{f_n(x)\}_{n\geq 1}$ is Cauchy $\forall x \in [a,b]$. As $\mathbb{R}$ is complete,

$$\forall x \in [a,b] \quad f_n(x) \underset{n\to\infty}{\longrightarrow} f(x) \in \mathbb{R}$$

This defines a function $f : [a,b] \to \mathbb{R}$. Recall that for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ s.t.

$$\begin{aligned} |f_n(x) - f(x)| \leq \epsilon \quad \forall n \geq n_\epsilon \quad \forall x \in [a,b] \\ \implies d\left(f_n, f\right) \leq \epsilon \quad \forall n \geq n_\epsilon \end{aligned}$$

So $f_n \xrightarrow[n\to\infty]{u} f$. By Weierstrass, $f \in C\left([a,b]\right)$. Thus $\left(C\left([a,b]\right), d\right)$ is a complete metric space.
Compactness: Note that $\left(C\left([a,b]\right), d\right)$ is not bounded and so not compact.

Example 8.6

$f_n : [a,b] \to \mathbb{R}$, $f_n(x) = n$ for all $x \in [a,b]$.

Connectedness: $\left(C\left([a,b]\right), d\right)$ is path connected and so connected.

Let $f, g \in C\left([a,b]\right)$. Define $\gamma : [0,1] \to C\left([a,b]\right)$ via $\gamma(t) = f + t(g-f)$. Note $\forall t \in [0,1]$, $\gamma(t) \in C\left([a,b]\right)$ and

$$\gamma(0) = f, \quad \gamma(1) = g$$

To see that γ is a path we compute

$$\begin{aligned} d\left(\gamma(t), \gamma(s)\right) &= \sup_{x\in[a,b]} |\gamma(t;x) - \gamma(s;x)| \\ &= \sup_{x\in[a,b]} |t-s|\,|g(x) - f(x)| \\ &= |t-s| \underbrace{d(g,f)}_{\in\mathbb{R}} \underset{|t-s|\to 0}{\longrightarrow} 0 \end{aligned}$$

So γ is a continuous function and so a path.

§9 Lec 9: Apr 16, 2021

§9.1 Arzela–Ascoli Theorem

For $a, b \in \mathbb{R}$ with $a < b$, we define

$$C([a,b]) = \{f : [a,b] \to \mathbb{R};\ f \text{ continuous}\}$$

We equip $C([a,b])$ with the uniform metric

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

We showed that $(C([a,b]), d)$ is a complete, connected metric space, but it's not compact.

Definition 9.1 (Equicontinuity) — We say that a set $\mathcal{F} \subseteq C([a,b])$ is <u>equicontinuous</u> if

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \text{ s.t. } |f(x) - f(y)| < \epsilon \quad \forall x, y \in [a,b] \text{ with } |x-y| < \delta(\epsilon)$$

and for all $f \in \mathcal{F}$.

<u>*Note*</u>: For a fixed function $f \in \mathcal{F} \subseteq C([a,b])$, we have that f is uniformly continuous (since f is continuous on $[a,b]$ compact) which means for all $\epsilon > 0$, there exists $\delta(\epsilon, f) > 0$ s.t.

$$|f(x) - f(y)| < \epsilon \quad \forall x, y \in [a,b] \text{ with } |x-y| < \delta(\epsilon, f)$$

Note that for an equicontinuous family $\mathcal{F}$, δ_ϵ can be chosen uniformly for $f \in \mathcal{F}$.

Definition 9.2 (Uniformly Bounded) — We say that a set $\mathcal{F} \subseteq C([a,b])$ is <u>uniformly bounded</u> if $\exists M > 0$ s.t. $|f(x)| \le M\ \forall x \in [a,b]\ \forall f \in \mathcal{F}$.

<u>Note</u>: For a fixed $f \in \mathcal{F} \subseteq C[a,b]$ we have that $f([a,b])$ is bounded (since f continuous and $[a,b]$ compact which implies $f([a,b])$ is compact and so bounded). So $\exists M_f > 0$ s.t. $|f(x)| \le M_f\ \forall x \in [a,b]$. For a uniformly bounded family $\mathcal{F}$, we can choose the bound M uniformly for $f \in \mathcal{F}$.

Theorem 9.3 (Arzela-Ascoli)
Let $\mathcal{F} \subseteq C([a,b])$. The following are equivalent:

1. $\mathcal{F}$ is uniformly bounded and equicontinuous.
2. Every sequence in $\mathcal{F}$ admits a convergent subsequence.

<u>Caution</u>: We cannot guarantee that the limit of the convergent subsequence belongs to $\mathcal{F}$, unless $\mathcal{F}$ is closed in $C([a,b])$. If $\mathcal{F}$ is closed in $C([a,b])$, then the theorem becomes

$$\mathcal{F} \text{ is compact} \iff \mathcal{F} \text{ is uniformly bounded and equicontinuous}$$

Proof. 2) $\implies$ 1)

Claim 9.1. $\mathcal{F}$ is totally bounded.

Fix $\epsilon > 0$. Let $f_1 \in \mathcal{F}$.

$$\begin{aligned}
&\text{If } \mathcal{F} \subseteq B_\epsilon(f_1) \text{ then } \mathcal{F} \text{ is totally bounded}\\
&\text{If } \mathcal{F} \not\subseteq B_\epsilon(f_1) \text{ then } \exists f_2 \in \mathcal{F} \text{ s.t. } d(f_1, f_2) \geq \epsilon\\
&\text{If } \mathcal{F} \subseteq B_\epsilon(f_1) \cup B_\epsilon(f_2) \text{ then } \mathcal{F} \text{ is totally bounded}\\
&\text{If } \mathcal{F} \not\subseteq B_\epsilon(f_1) \cup B_\epsilon(f_2) \text{ then } \exists f_3 \in \mathcal{F} \text{ s.t. } \begin{cases} d(f_1, f_3) \geq \epsilon \\ d(f_2, f_3) \geq \epsilon \end{cases}
\end{aligned}$$

If the process terminates in finitely many steps, then $\mathcal{F}$ is totally bounded. Otherwise, we find $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$ s.t. $d(f_n, f_m) \geq \epsilon \, \forall n \neq m$. This sequence does not admit a convergent subsequence, leading a contradiction.

Let's show that $\mathcal{F}$ is uniformly bounded. As $\mathcal{F}$ is totally bounded, $\exists n \geq 1$ and $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t.

$$\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_1(f_j) \subseteq B_r(f_1)$$

where $r = 1 + \max_{2\leq j\leq n} d(f_1, f_j)$. In particular, for all $f \in \mathcal{F}$,

$$d(f, f_1) < r$$

f_1 is continuous on compact $[a, b] \implies \exists M_{f_1} > 0$ s.t.

$$|f_1(x)| \leq M_{f_1} \quad \forall x \in [a, b]$$

So for $f \in \mathcal{F}$

$$|f(x)| \leq |f(x) - f_1(x)| + |f_1(x)| \leq d(f, f_1) + M_{f_1} < r + M_{f_1} \quad \forall x \in [a, b]$$

So $\mathcal{F}$ is uniformly bounded.
Let's show that $\mathcal{F}$ is equicontinuous. Let $\epsilon > 0$. As $\mathcal{F}$ is totally bounded, $\exists n \geq 1$ and $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t.

$$\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_{\frac{\epsilon}{3}}(f_j)$$

For each $1 \leq j \leq n$, f_j is uniformly continuous on $[a, b]$. So $\exists \delta_j(\epsilon) > 0$ s.t.

$$|f_j(x) - f_j(y)| < \frac{\epsilon}{3} \quad \forall x, y \in [a, b] \text{ with } |x - y| < \delta_j(\epsilon)$$

Let $\delta_\epsilon = \min_{1\leq j\leq n} \delta_j(\epsilon) > 0$.
Fix $f \in \mathcal{F} \implies \exists 1 \leq j \leq n$ s.t. $f \in B_{\frac{\epsilon}{3}}(f_j)$. Then for $x, y \in [a, b]$ with $|x - y| < \delta_\epsilon$ we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)|\\
&\leq 2d(f, f_j) + |f_j(x) - f_j(y)|\\
&\leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

This shows $\mathcal{F}$ is equicontinuous.
1) $\implies$ 2) Let $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$. As $\mathcal{F}$ is uniformly bounded,

$$\exists M > 0 \text{ s.t. } |f(x)| \leq M \quad \forall x \in [a,b] \; \forall f \in \mathcal{F}$$

In particular, $|f_n(x)| \leq M \; \forall x \in [a,b] \; \forall n \geq 1$.
Let $\{r_n\}_{n\geq 1}$ denote an enumeration of the rationals in $[a,b]$. As $\{f_n(r_1)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded by M, $\exists \left\{f_n^{(1)}\right\}_{n\geq 1}$ subsequence of $\{f_n\}_{n\geq 1}$ s.t. $\left\{f_n^{(1)}(r_1)\right\}_{n\geq 1}$ converges. $\left\{f_n^{(1)}(r_2)\right\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded by $M \implies \exists \left\{f_n^{(2)}\right\}_{n\geq 1}$ subsequence of $\left\{f_n^{(1)}\right\}_{n\geq 1}$ s.t. $\left\{f_n^{(2)}(r_2)\right\}_{n\geq 1}$ converges.
Proceeding inductively we find $\forall k \geq 1$ $\left\{f_n^{(k+1)}\right\}_{n\geq 1}$ is a subsequence of $\left\{f_n^{(k)}\right\}_{n\geq 1}$ and $\left\{f_n^{(k)}(r_k)\right\}_{n\geq 1}$ converges.
We consider $\left\{f_n^{(n)}\right\}_{n\geq 1}$ subsequence of $\{f_n\}_{n\geq 1}$.
For $n, m \geq k$, $f_n^{(n)}$, $f_m^{(m)}$ are elements in $\left\{f_n^{(k)}\right\}_{n\geq 1}$. So $\left\{f_n^{(n)}\right\}_{n\geq 1}$ converges at r_k.
Caution: The convergence is not uniform in k.
Fix $\epsilon > 0$. As $\mathcal{F}$ is equicontinuous, $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\epsilon}{3} \quad \forall x, y \in [a,b] \; |x - y| < \delta, \forall f \in \mathcal{F}$$

In particular,

$$|f_n(x) - f_n(y)| < \frac{\epsilon}{3} \quad \forall x, y \in [a,b] \; |x - y| < \delta, \forall n \geq 1 \tag{*}$$

Let $r_1, \ldots, r_N \in \mathbb{Q} \cap [a,b]$ s.t. $a = r_0 < r_1 < \ldots < r_N < r_{N+1} = b$ and

$$|r_{j+1} - r_j| < \delta \qquad 0 \leq j \leq N$$

Note $N \sim \frac{|a-b|}{\delta}$. For each $1 \leq j \leq N$, $\exists n_j(\epsilon) \in \mathbb{N}$ s.t.

$$\left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| < \frac{\epsilon}{3} \qquad \forall n, m \geq n_j(\epsilon)$$

Let $n_\epsilon = \max_{1\leq j\leq N} n_j(\epsilon)$. Note

$$\left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| < \frac{\epsilon}{3} \quad \forall n, m \geq n_\epsilon \quad \forall 1 \leq j \leq N \tag{**}$$

Let $x \in [a,b] \implies \exists 1 \leq j \leq N$ s.t. $|x - r_j| < \delta$. Then

$$\left|f_n^{(n)}(x) - f_m^{(m)}(x)\right| \leq \left|f_n^{(n)}(x) - f_n^{(n)}(r_j)\right| + \left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| + \left|f_m^{(m)}(r_j) - f_m^{(m)}(x)\right|$$
$$\text{By (*) and (**)} < 2 \cdot \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \forall n, m \geq n_\epsilon$$

So $\left\{f_n^{(n)}\right\}_{n\geq 1}$ is uniformly Cauchy and so uniformly convergent. □

> **Remark 9.4.** One can replace $[a,b]$ by any other compact metric space (X,d).

§10 Lec 10: Apr 19, 2021

§10.1 Arzela-Ascoli Theorem (Cont'd)

Remark 10.1. The compactness of the set on which the functions are defined is necessary in Arzela-Ascoli.

Example 10.2

$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R};\ |f(x) - f(y)| \le |x - y|\ \forall x, y \in \mathbb{R}$ and $\sup_{x\in\mathbb{R}} |f(x)| \le 1\}$. Note $\mathcal{F}$ is equicontinuous and uniformly bounded. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$

Claim 10.1. $f \in \mathcal{F}$.

Indeed,

$$\sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+x^2} = 1$$

Moreover, for $x, y \in \mathbb{R}$

$$\begin{aligned} |f(x) - f(y)| = \left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| &= \frac{\left|x^2 - y^2\right|}{(1+x^2)(1+y^2)} \\ &= |x-y| \cdot \frac{|x+y|}{(1+x^2)(1+y^2)} \\ &\le |x-y| \, (\underbrace{\frac{|x|}{1+x^2}}_{\le \frac{1}{2}} + \underbrace{\frac{|y|}{1+y^2}}_{\le \frac{1}{2}}) \\ &\le |x-y| \end{aligned}$$

So $f \in \mathcal{F}$.
For $n \ge 1$, let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = f(x-n)$. Note $f_n \in \mathcal{F}$ since $\sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+(x-n)^2} = 1$.

$$\begin{aligned} |f_n(x) - f_n(y)| = |f(x-n) - f(y-n)| &\le |(x-n) - (y-n)| \\ &= |x-y| \end{aligned}$$

Note that $\{f_n\}_{n\ge1}$ converge pointwise to $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ since $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+(x-n)^2} = 0$. However, $\{f_n\}_{n\ge1}$ does not admit a subsequence that converges uniformly since $\forall n \ge 1$

$$d(f_n, f) = \sup_{x\in\mathbb{R}} |f_n(x)| = 1 \overset{n\to\infty}{\not\to} 0$$

Remark 10.3. Uniform boundedness is necessary in Arzela-Ascoli.

Example 10.4

$\mathcal{F} = \{f : \underbrace{[0,1]}_{\text{compact}} \to \mathbb{R};\, f \text{ is continuous and } \underbrace{\sup_{x\in[0,1]} |f(x)| \le 1}_{\text{uniformly bounded}}\}$.

Claim 10.2. $\mathcal{F}$ is not equicontinuous.

For $n \ge 1$, let $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \sin(nx)$. Note $f_n \in \mathcal{F}$. Let $x_n = \frac{3\pi}{2n}$, $y_n = \frac{\pi}{2n}$. Then $|x_n - y_n| = \frac{\pi}{n} \underset{n\to\infty}{\longrightarrow} 0$ but

$$|f_n(x_n) - f_n(y_n)| = 2$$

So $\{f_n\}_{n\ge1}$ is not equicontinuous $\implies \mathcal{F}$ is not equicontinuous.

Claim 10.3. $\{f_n\}_{n\ge1}$ does not admit a convergent subsequence.

Assume, towards a contradiction, that there exists a subsequence $\{f_{k_n}\}_{n\ge1}$ of $\{f_n\}_{n\ge1}$ that converges uniformly to $f : [0,1] \to \mathbb{R}$. By Weierstrass,

$$\left.\begin{array}{l} f \in C([0,1]) \\ \left.\begin{array}{l} f_{k_n}(0) = 0 \quad \forall n \ge 1 \\ f_{k_n}(0) \underset{n\to\infty}{\longrightarrow} f(0) \end{array}\right\} \implies f(0) = 0 \end{array}\right\} \implies \forall \epsilon > 0\, \exists \delta > 0 \text{ s.t. } |f(x)| < \epsilon\, \forall 0 < x < \delta$$

$f_{k_n} \underset{n\to\infty}{\overset{u}{\longrightarrow}} f \implies \exists n_\epsilon \in \mathbb{N}$ s.t. $d(f_{k_n}, f) < \epsilon\ \forall n \ge n_\epsilon$. In particular, for $0 < x < \delta$ and $n \ge n_\epsilon$ we have

$$|f_{k_n}(x)| \le |f_{k_n}(x) - f(x)| + |f(x)| < d(f_{k_n}, f) + \epsilon < 2\epsilon$$

Choosing $\epsilon \le \frac{1}{2}$ and N large so that $N \ge n_{\epsilon=\frac{1}{2}}$ and $\frac{\pi}{2N} < \delta_{\epsilon=\frac{1}{2}}$ we find

$$1 = \left| f_{k_N}\left(\frac{\pi}{2N}\right) \right| < 2\epsilon \le 1 \qquad \text{Contradiction!}$$

§10.2 The oscillation of a Real Function

Definition 10.5 (Oscillation of a Function) — Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. For $\emptyset \ne A \subseteq X$, the oscillation of f on A is

$$\omega(f, A) = \sup_{x\in A} f(x) - \inf_{x\in A} f(x) = \sup_{x,y\in A} [f(x) - f(y)] \ge 0$$

Note that if $A \subseteq B$ then

$$\omega(f, A) \le \omega(f, B)$$

For $x_0 \in X$, the oscillation of f at x_0 is given by

$$\omega(f, x_0) = \inf_{\delta>0} \omega(f, B_\delta(x_0))$$

Proposition 10.6

Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then f is continuous at a point $x_0 \in X$ if and only if $\omega(f, x_0) = 0$.

Proof. " $\implies$ " Fix $\epsilon > 0$. As f is continuous at x_0, $\exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \frac{\epsilon}{4}$ $\forall x \in B_\delta(x_0)$.

$$\begin{aligned}
&\implies |f(x) - f(y)| \le |f(x) - f(x_0)| + |f(x_0) - f(y)| < \frac{\epsilon}{2} \quad \forall x, y \in B_\delta(x_0) \\
&\implies \omega(f, B_\delta(x_0)) = \sup_{x,y \in B_\delta(x_0)} [f(x) - f(y)] \le \frac{\epsilon}{2} < \epsilon \\
&\implies \omega(f, x_0) \le \omega(f, B_\delta(x_0)) < \epsilon
\end{aligned}$$

As $\epsilon > 0$ was arbitrary, $\omega(f, x_0) = 0$.
" $\impliedby$ " Fix $\epsilon > 0$. Then $\omega(f, x_0) = 0 < \epsilon$ implies $\exists \delta > 0$ s.t. $\omega(f, B_\delta(x_0)) < \epsilon$

$$\begin{aligned}
&\implies |f(x) - f(y)| < \epsilon \qquad \forall x, y \in B_\delta(x_0) \\
&\implies |f(x) - f(x_0)| < \epsilon \qquad \forall x \in B_\delta(x_0)
\end{aligned}$$

So f is continuous at x_0. □

Lemma 10.7

Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then for any $\alpha > 0$,

$$\{x \in X : \omega(f, x) < \alpha\} \text{ is open in } X$$

Proof. Fix $\alpha > 0$ and let $A = \{x \in X : \omega(f, x) < \alpha\}$. Fix $x_0 \in A \implies \omega(f, x_0) = \inf_{\delta > 0} \omega(f, B_\delta(x_0)) < \alpha$.

$$\implies \exists \delta > 0 \text{ s.t. } \omega(f, B_\delta(x_0)) < \alpha$$

Claim 10.4. $B_\delta(x_0) \subseteq A$ (which implies $x_0 \in \mathring{A}$ and so $A = \mathring{A}$).

Let $x \in B_\delta(x_0)$. Then $r = \delta - d(x, x_0) > 0$ and $B_r(x) \subseteq B_\delta(x_0)$

$$\begin{aligned}
&\implies \omega(f, B_r(x)) \le \omega(f, B_\delta(x_0)) < \alpha \\
&\implies \omega(f, x) \le \omega(f, B_r(x)) < \alpha \implies x \in A
\end{aligned}$$

□

Remark 10.8. Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then

$$\begin{aligned}
\{x \in X : f \text{ is continuous at } x\} &= \{x \in X : \omega(f, x) = 0\} \\
&= \bigcap_{n \ge 1} \underbrace{\left\{x \in X : \omega(f, x) < \frac{1}{n}\right\}}_{= G_n}
\end{aligned}$$

By the lemma, $G_n = \mathring{G}_n$ $\forall n \ge 1$. Also, $G_{n+1} \subseteq G_n$ $\forall n \ge 1$. This observation allows us to prove that there are no functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every rational point and discontinuous at every irrational point.

§11 Lec 11: Apr 21, 2021

§11.1 Oscillation of a Function (Cont'd)

Recall from last lecture that there are no functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every rational point and discontinuous at every irrational point.

Proof. (Sketch) Assume, towards a contradiction, that $f : \mathbb{R} \to \mathbb{R}$ is such a function. Then

$$\mathbb{Q} = \{x \in \mathbb{R} : f \text{ is continuous at } x\} = \bigcap_{n \geq 1} G_n \text{ with } G_n \text{ open in } \mathbb{R}$$

Note $\forall n \geq 1$, $Q \subseteq G_n$

$$\begin{aligned} &\implies \mathbb{R} = \overline{\mathbb{Q}} \subseteq \overline{G_n} \subseteq \mathbb{R} \\ &\implies \overline{G_n} = \mathbb{R} \text{ i.e. } G_n \text{ is dense in } \mathbb{R} \end{aligned}$$

Let $\{q_n\}_{n\geq 1}$ be an enumeration of $\mathbb{Q}$. For each $n \geq 1$, let $H_n = \mathbb{R} \backslash \{q_n\} = (-\infty, q_n) \cup (q_n, \infty)$. Note H_n is open and dense ($\overline{H_n} = \mathbb{R}$) in $\mathbb{R}$. Also

$$\bigcap_{n\geq 1} H_n = \mathbb{R} \setminus \mathbb{Q}$$

So

$$\bigcap_{n\geq 1} G_n \cap \bigcap_{n\geq 1} H_n = \mathbb{Q} \cap \mathbb{R} \setminus \mathbb{Q} = \emptyset$$

This contradicts the following property of $\mathbb{R}$:

Exercise 11.1. If $\{A_n\}_{n\geq 1}$ is a countable collection of open and dense subsets of $\mathbb{R}$, then

$$\overline{\bigcap_{n\geq 1} A_n} = \mathbb{R}$$

Apply this exercise with $\{A_n : n \geq 1\} = \{G_n : n \geq 1\} \cup \{H_n : n \geq 1\}$. □

§11.2 Weierstrass Approximation Theorem

Theorem 11.1 (Weierstrass Approximation)
Fix $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ with $\deg P_n \leq n$ $\forall n \geq 1$ s.t.

$$P_n \xrightarrow[n\to\infty]{u} f \quad \text{on} \quad [a, b]$$

Proof. First, we reduce to the case when $[a, b]$ is $[0, 1]$. Let $\phi : [0, 1] \to [a, b]$, $\phi(t) = a + t(b - a)$. Note ϕ is a continuous, bijective function with the inverse

$$\phi^{-1} : [a, b] \to [0, 1], \quad \phi^{-1}(x) = \frac{x - a}{b - a} \text{ continuous}$$

As $f : [a,b] \to \mathbb{R}$ is continuous, $f \circ \phi : [0,1] \to \mathbb{R}$ is continuous.
If $\{P_n\}_{n\geq 1}$ is a sequence of polynomials with $\deg P_n \leq n$ s.t.

$$P_n \xrightarrow[n\to\infty]{u} f \circ \phi \text{ on } [0,1]$$

then $P_n \circ \phi^{-1} \xrightarrow[n\to\infty]{u} f$ on $[a,b]$. Indeed,

$$\sup_{x\in[a,b]} \left|\left(P_n \circ \phi^{-1}\right)(x) - f(x)\right| \underset{x=\phi(t)}{=} \underbrace{\sup_{t\in[0,1]} |P_n(t) - (f\circ\phi)(t)|}_{\xrightarrow[n\to\infty]{} 0}$$

Therefore, we may assume $f : [0,1] \to \mathbb{R}$ is continuous. Define the Bernstein polynomials via

$$P_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \qquad \deg P_n \leq n$$

Note that if f is a constant, say $f(x) = c \ \forall x \in [0,1]$ then

$$P_n(x) = c\sum_{k=0}^{n}\binom{n}{k} x^k (1-x)^{n-k} = c\,(x+1-x)^n = c \quad \forall x \in [0,1]\ \forall n \geq 1$$

We want to show $P_n \xrightarrow[n\to\infty]{u} f$ on $[0,1]$. Fix $x \in [0,1]$. Consider

$$\begin{aligned}
|f(x) - P_n(x)| &= \left| f(x)\sum_{k=0}^{n}\binom{n}{k} x^k(1-x)^{n-k} - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}\right| \\
&= \left|\sum_{k=0}^{n}\left[f(x) - f\left(\frac{k}{n}\right)\right]\binom{n}{k} x^k(1-x)^{n-k}\right| \\
&\leq \sum_{k=0}^{n}\left|f(x) - f\left(\frac{k}{n}\right)\right|\binom{n}{k} x^k(1-x)^{n-k}
\end{aligned}$$

To estimate the sum we use the following

- when $\frac{k}{n}$ is close to x, we use the continuity of f.
- when $\frac{k}{n}$ is far from x, we use the fact that $x \overset{g}{\mapsto} x^k(1-x)^{n-k}$ has a local maximum at $x = \frac{k}{n}$.

$$\begin{aligned}
g'(x) &= kx^{k-1}(1-x)^{n-k} - (n-k)x^k(1-x)^{n-k-1} \\
&= x^{k-1}(1-x)^{n-k-1}\{k(1-x) - (n-k)x\} \\
&= x^{k-1}(1-x)^{n-k-1}\{k - nx\} \\
&= \begin{cases} > 0 & \text{if } x < \frac{k}{n} \\ = 0 & \text{if } x = \frac{k}{n} \\ < 0 & \text{if } x > \frac{k}{n} \end{cases}
\end{aligned}$$

$f : [0, 1] \to \mathbb{R}$ is continuous $\implies$ f is uniformly continuous. Fix $\epsilon > 0$. Then $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \epsilon \quad \text{whenever} \quad x, y \in [0,1], \quad |x - y| < \delta$$

$f : [0, 1] \to \mathbb{R}$ is continuous $\implies$ f is bounded. Let $M > 0$ be s.t.

$$|f(x)| \le M \qquad \forall x \in [0,1]$$

We estimate

$$\begin{aligned}
|f(x) - P_n(x)| &\le \sum_{\substack{0 \le k \le n \\ \left|x - \frac{k}{n}\right| < \delta}} \underbrace{\left|f(x) - f\left(\frac{k}{n}\right)\right|}_{<\epsilon} \binom{n}{k} x^k (1-x)^{n-k} \\
&+ \sum_{\substack{0 \le k \le n \\ \left|x - \frac{k}{n}\right| \ge \delta}} \underbrace{\left|f(x) - f\left(\frac{k}{n}\right)\right|}_{\le 2M} \binom{n}{k} x^k (1-x)^{n-k} \\
&\le \epsilon \sum_{0 \le k \le n} \binom{n}{k} x^k (1-x)^{n-k} + 2M \sum_{0 \le k \le n} \frac{\left(x - \frac{k}{n}\right)^2}{\delta^2} \binom{n}{k} x^k (1-x)^{n-k} \\
&\le \epsilon + \frac{2M}{n^2 \delta^2} \sum_{k=0}^{n} (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k}
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{k=0}^{n} (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k} &= n^2 x^2 \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}}_{=1} \\
&- 2nx \sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} + \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} &= x \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} \\
&= nx \underbrace{\sum_{l=0}^{n-1} \frac{(n-1)!}{l!(n-1-l)!} x^l (1-x)^{n-1-l}}_{=(x+1-x)^{n-1}} \\
&= nx
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} &= nx \sum_{k=1}^{n} \frac{k(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= nx \sum_{k=1}^{n} \frac{(k-1+1)(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= n(n-1)x^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} x^{k-2}(1-x)^{n-k} \\
&+ nx \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= n(n-1)x^2 + nx
\end{aligned}$$

So

$$\begin{aligned}
\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} &= n^2x^2 - 2n^2x^2 + n(n-1)x^2 + nx \\
&= nx(1-x)
\end{aligned}$$

We get

$$\begin{aligned}
|f(x) - P_n(x)| &\le \epsilon + \frac{2M}{n^2\delta^2} \cdot nx(1-x) \\
&\le \epsilon + \frac{2M}{n\delta^2} \sup_{x\in[0,1]} x(1-x) \\
&\le \epsilon + \frac{M}{2\delta^2 n} < 2\epsilon
\end{aligned}$$

provided $n > \frac{M}{2\delta^2\epsilon}$. So $P_n \xrightarrow[n\to\infty]{u} f$ on $[0,1]$. $\square$

§12 Lec 12: Apr 23, 2021

§12.1 Weierstrass Approximation Theorem (Cont'd)

Corollary 12.1

Let $M > 0$. Then there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ s.t.

$$\begin{cases} \deg P_n \leq n & \forall n \geq 1 \\ P_n(0) = 0 & \forall n \geq 1 \\ P_n \xrightarrow[n\to\infty]{u} |x| \text{ on } [-M, M] \end{cases}$$

Proof. Let $f : [-M, M] \to \mathbb{R}$, $f(x) = |x|$. Then f is continuous and $[-M, M]$ compact. By Weierstrass Approximation, $\exists \{Q_n\}_{n\geq 1}$ sequence of polynomials s.t.

$$\begin{cases} \deg Q_n \leq n & \forall n \geq 1 \\ Q_n \xrightarrow[n\to\infty]{u} f \text{ on } [-M, M] \end{cases}$$

Note $Q_n \xrightarrow[n\to\infty]{u} f \implies Q_n(0) \xrightarrow[n\to\infty]{} f(0) = 0$.
Let $P_n(x) = Q_n(x) - Q_n(0)$. Then

$$\begin{cases} \deg P_n \leq n & \forall n \geq 1 \\ P_n(0) = 0 & \forall n \geq 1 \end{cases}$$

For $x \in [-M, M]$,

$$\begin{aligned} |P_n(x) - f(x)| &\leq |Q_n(x) - f(x)| + |Q_n(0)| \leq d(Q_n, f) + |Q_n(0)| \\ \implies d(P_n, f) &\leq d(Q_n, f) + |Q_n(0)| \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

□

§12.2 Stone-Weierstrass Theorem

Definition 12.2 (Algebra) — Let (X, d) be a metric space and let

$$\mathcal{A} \subseteq \{f : X \to \mathbb{R}(\text{or } \mathbb{C});\ f \text{ is a function}\}$$

We say that $\mathcal{A}$ is an algebra if

1. $f + g \in \mathcal{A} \qquad \forall f, g \in \mathcal{A}$.
2. $fg \in \mathcal{A} \qquad \forall f, g \in \mathcal{A}$
3. $\lambda f \in \mathcal{A} \qquad \forall f \in \mathcal{A}\ \forall \lambda \in \mathbb{R}(\text{or } \mathbb{C})$

We say that the algebra $\mathcal{A}$ separates points if whenever $x, y \in X$ with $x \neq y$ then $\exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$.
We say that the algebra $\mathcal{A}$ vanishes at no point in X if $\forall x \in X\ \exists f \in \mathcal{A}$ s.t. $f(x) \neq 0$.

Lemma 12.3

Let (X,d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra. Then its closure $\overline{\mathcal{A}}$ with respect to the uniform topology is also an algebra.

Proof. Let $f, g \in \mathcal{A}$. Then

$$\begin{cases} \exists f_n \in \mathcal{A} \text{ s.t. } f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ \exists g_n \in \mathcal{A} \text{ s.t. } g_n \xrightarrow[n\to\infty]{u} g \text{ on } X \end{cases}$$

$$\left.\begin{array}{l} d\left(f_n+g_n, f+g\right) \le d(f_n,f) + d(g_n,g) \xrightarrow[n\to\infty]{} 0 \\ f_n + g_n \in \mathcal{A} \text{ (because } \mathcal{A} \text{ is an algebra)} \end{array}\right\} \implies f+g \in \overline{\mathcal{A}}$$

Similarly, for $\lambda \in \mathbb{R}$,

$$\left.\begin{array}{l} d\left(\lambda f_n, \lambda f\right) \le |\lambda| d\left(f_n,f\right) \xrightarrow[n\to\infty]{} 0 \\ \lambda f_n \in \mathcal{A} \text{ (because } \mathcal{A} \text{ is an algebra)} \end{array}\right\} \implies \lambda f \in \overline{\mathcal{A}}$$

Then

$$\begin{aligned} d\left(f_n g_n, fg\right) &= \sup_{x\in X} |f_n(x)g_n(x) - f(x)g(x)| \\ &\le \sup_{x\in X} \left[|f_n(x)-f(x)|\,|g_n(x)| + |f(x)|\,|g_n(x)-g(x)|\right] \\ &\le d(f_n,f) \sup_{x\in X} |g_n(x)| + d(g_n,g) \sup_{x\in X} |f(x)| \end{aligned}$$

By Weierstrass,

$$\left.\begin{array}{l} f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ f_n \in C(X) \end{array}\right\} \implies \left.\begin{array}{l} f \in C(X) \\ X \text{ compact} \end{array}\right\} \implies \exists M > 0 \text{ s.t. } \sup_{x\in X} |f(x)| \le M$$

Similarly, $g \in C(X) \implies \exists M_2 > 0$ s.t. $\sup_{x\in X} |g(x)| \le M_2$

$$d(g_n,0) \le d(g_n,g) + d(g,0) \le 1 + M_2 \qquad \forall n \ge n_1$$

Let $M_3 = \max\left\{1+M_2, \underbrace{d(g_1,0)}_{<\infty}, \ldots, \underbrace{d(g_{n_1},0)}_{<\infty}\right\}$. So $d(g_n,0) \le M_3\, \forall n \ge 1$. Thus

$$\left.\begin{array}{l} d\left(f_n g_n, fg\right) \le d(f_n,f)\cdot M_3 + d(g_n,g)\cdot M_1 \xrightarrow[n\to\infty]{} 0 \\ f_n g_n \in \mathcal{A} \text{ (since } \mathcal{A} \text{ is an algebra)} \end{array}\right\} \implies f\cdot g \in \overline{\mathcal{A}}$$

□

Lemma 12.4

Let (X,d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes at no point in X. Then

$$\forall \alpha,\beta \in \mathbb{R} \quad \forall x_1,x_2 \in X \text{ s.t. } x_1 \ne x_2 \quad \exists f \in \mathcal{A} \text{ s.t. } \begin{cases} f(x_1) = \alpha \\ f(x_2) = \beta \end{cases}$$

Proof. Fix $\alpha, \beta \in \mathbb{R}$. Fix $x_1, x_2 \in X$ s.t. $x_1 \neq x_2$. We would like

$$f(x) = \alpha \cdot \frac{u(x)}{u(x_1)} + \beta \cdot \frac{v(x)}{v(x_1)}$$

for $u, v \in \mathcal{A}$ s.t.

$$\begin{aligned} u(x_1) \neq 0 \quad &\text{and} \quad u(x_2) = 0 \\ v(x_1) = 0 \quad &\text{and} \quad v(x_2) \neq 0 \end{aligned}$$

Then $f \in \mathcal{A}$ (because $\mathcal{A}$ is an algebra) is the desired function.

As $\mathcal{A}$ separates points, $\exists g \in \mathcal{A}$ s.t. $g(x_1) \neq g(x_2)$.

As $\mathcal{A}$ vanishes at no point in X,

$$\begin{cases} \exists h \in \mathcal{A} \text{ s.t } h(x_1) \neq 0 \\ \exists k \in \mathcal{A} \text{ s.t. } k(x_2) \neq 0 \end{cases}$$

Then, we define

$$\begin{aligned} u(x) &= [g(x) - g(x_2)] \cdot h(x) \in \mathcal{A} \\ v(x) &= [g(x) - g(x_1)] \cdot k(x) \in \mathcal{A} \end{aligned}$$

□

Theorem 12.5 (Stone-Weierstrass)

Let (X, d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes no point in X. Then $\mathcal{A}$ is dense in $C(X)$, i.e., $\overline{\mathcal{A}} = C(X) = \{f : X \to \mathbb{R};\ f \text{ continuous}\}$.

Proof. Want to show $\forall f \in C(X)\ \forall \epsilon > 0\ \exists g \in \mathcal{A}$ s.t. $d(f, g) < \epsilon$.

Step 1: If $f \in \overline{\mathcal{A}}$ then $|f| \in \overline{\mathcal{A}}$. Let $f \in \overline{\mathcal{A}} \implies \exists f_n \in \mathcal{A}$ s.t.

$$\left.\begin{aligned} &f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ &f_n \in C(X) \end{aligned}\right\} \implies f \in C(X)$$

As X is compact, $\exists M > 0$ s.t. $|f(x)| \leq M\ \forall x \in X$. By the previous Corollary 12.1, $\exists \{P_n\}_{n\geq 1}$ sequence of polynomials with $\deg P_n \leq n\ \forall n \geq 1$ s.t.

$$\begin{cases} P_n \xrightarrow[n\to\infty]{u} |x| \text{on } [-M, M] \\ P_n(0) = 0 \end{cases} \implies P_n(f) \xrightarrow[n\to\infty]{u} |f| \text{ on } X$$

If $P_n(x) = \sum_{k=1}^n c_k x^k$ then $P_n(f) = \sum_{k=1}^n c_k f^k \in \mathcal{A}$ which implies $|f| \in \overline{\mathcal{A}}$.

Step 2: If $f, g \in \overline{\mathcal{A}}$ then $\max\{f, g\}$, $\min\{f, g\} \in \overline{\mathcal{A}}$.

$$\begin{aligned} \max\{f, g\} &= \frac{f+g}{2} + \frac{|f-g|}{2} \in \overline{\mathcal{A}} \\ \min\{f, g\} &= \frac{f+g}{2} - \frac{|f-g|}{2} \in \overline{\mathcal{A}} \end{aligned}$$

Step 3: $\forall f \in C(X)$, $\forall x \in X$, $\forall \epsilon > 0$, $\exists g \in \overline{\mathcal{A}}$ s.t.

$$g(x) = f(x) \quad \text{and} \quad g(y) > f(y) - \epsilon \quad \forall y \in X$$

Continue in the next lecture. □

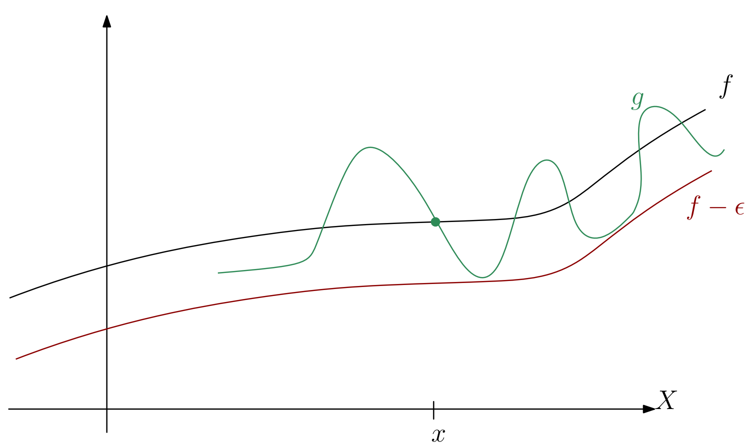
§13 Lec 13: Apr 26, 2021

§13.1 Stone-Weierstrass Theorem (Cont'd)

We continue with the proof of Stone-Weierstrass from lecture 12. Recall that we are at step 3 so far.

Proof. **Step 3**: For any $f \in C(X)$, $x \in X$, $\epsilon > 0$, there exists $g \in \overline{\mathcal{A}}$ s.t.

$$\begin{cases} g(x) = f(x) \\ g(y) > f(y) - \epsilon \qquad \forall y \in X \end{cases}$$



For any $y \in X$, there exists $h_y \in \overline{\mathcal{A}}$ s.t.

$$h_y(x) = f(x)$$
$$h_y(y) = f(y)$$

As $h_y \in \overline{\mathcal{A}}$, h_y is continuous. Thus, $h_y - f$ is continuous at y. So $\exists \delta_y > 0$ s.t. $|h_y(z) - f(z)| < \epsilon$, $\forall z \in B_{\delta_y}(y)$. In particular,

$$h_y(z) > f(z) - \epsilon \qquad \forall z \in B_{\delta_y}(y)$$

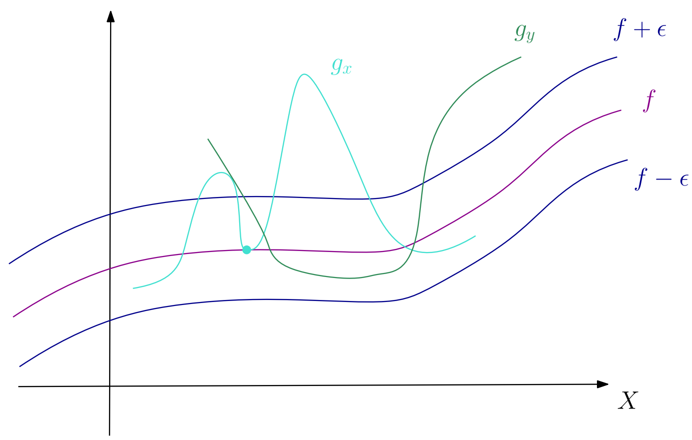
Note that

$$\left.\begin{matrix} X = \bigcup_{y \in X} B_{\delta_y}(y) \\ X \text{ compact} \end{matrix}\right\} \implies \exists N \geq 1 \text{ and } \exists y_1, \ldots, y_N \in X$$

s.t. $X = \bigcup_{n=1}^N B_{\delta_n}(y_n)$ where $\delta_n = \delta_{y_n}$.
Take $g = \max\{h_{y_1}, \ldots, h_{y_N}\}$ (by step 2). By construction, $g(x) = f(x)$. Also if $y \in X$, $\exists 1 \leq n \leq N$ s.t. $y \in B_{\delta_n}(y_n)$. So

$$g(y) \geq h_{y_n}(y) > f(y) - \epsilon$$

Step 4: For all $f \in C(X)$ and $\epsilon > 0$, $\exists g \in \overline{\mathcal{A}}$ s.t. $d(f, g) < \epsilon$. Fix $f \in C(X)$, $\epsilon > 0$



For $x \in X$, let $g_x \in \overline{\mathcal{A}}$ be the function given by step 3. In particular, $g_x(x) = f(x)$,

$$g_x(y) > f(y) - \epsilon \qquad \forall y \in X$$

As $g_x \in \overline{\mathcal{A}}$, the function $g_x - f$ is continuous at x. So $\exists \delta_x > 0$ s.t. $|g_x(y) - f(y)| < \epsilon$, $\forall y \in B_{\delta_x}(x)$. In particular,

$$g_x(y) < f(y) + \epsilon \qquad \forall y \in B_{\delta_x}(x)$$

Note

$$\left.\begin{array}{l} X = \bigcup_{x\in X} B_{\delta_x}(x) \\ X \text{ compact} \end{array}\right\} \implies \exists N \geq 1 \text{ and } \exists x_1, \ldots, x_N \in X \text{ s.t.}$$

$X = \bigcup_{n=1}^N B_{\delta_n}(x_n)$ where $\delta_n = \delta_{x_n}$.
Take $g = \min\{g_{x_1}, \ldots, g_{x_N}\} \in \overline{\mathcal{A}}$ (by step 2).
For $y \in X$, $\exists 1 \leq n \leq N$ s.t. $y \in B_{\delta_n}(x_n)$ and so

$$g(y) \leq g_{x_n}(y) < f(y) + \epsilon$$

Moreover, as $g_{x_n}(y) > f(y) - \epsilon$, $\forall y \in X$, $\forall 1 \leq n \leq N$, we have

$$g(y) > f(y) - \epsilon \qquad \forall y \in X$$

This shows $C(X) \subseteq \overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}} \subseteq C(X)$. □

§13.2 Differentiation

Definition 13.1 (Limit) — Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $\emptyset \neq A \subseteq X$, let $f : A \to Y$. For $x_0 \in A'$ and $y_0 \in Y$ we write

$$f \underset{x\to x_0}{\longrightarrow} y_0 \qquad \text{or} \qquad \lim_{x\to x_0} f(x) = y_0$$

if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(f(x), y_0) < \epsilon$ whenever $0 < d_X(x, x_0) < \delta$.
Equivalently, $\lim_{x\to x_0} f(x) = y_0$ if

$$\lim_{n\to\infty} f(x_n) = y_0 \text{ for every sequence } \{x_n\}_{n\geq 1} \subseteq A \setminus \{x_0\} \text{ s.t. } x_n \xrightarrow[n\to\infty]{d_X} x_0$$

Note also that if $x_0 \in A' \cap A$ then f is continuous at $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$.

Exercise 13.1. Let (X, d) be a metric space, $\emptyset \neq A \subseteq X$, $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions. Assume that at a point $a \in A'$ we have

$$\lim_{x \to x_0} f(x) = \alpha \quad \text{and} \quad \lim_{x \to x_0} g(x) = \beta$$

Then

1. $\lim_{x \to x_0} (\lambda f(x)) = \lambda\alpha$, $\lambda \in \mathbb{R}$
2. $\lim_{x \to x_0} (f(x) + g(x)) = \alpha + \beta$
3. $\lim_{x \to x_0} (f(x)g(x)) = \alpha \cdot \beta$
4. If $\beta \neq 0$ then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\alpha}{\beta}$

Definition 13.2 (Differentiability) — Let I be an open interval and let $f : I \to \mathbb{R}$ be a function. We say that f is differentiable at $a \in I$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists and is finite}$$

in which case we denote it $f'(a)$.

Example 13.3

Fix $n \geq 1$ and let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$. For $a \in \mathbb{R}$ and $x \neq a$

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} &= \frac{x^n - a^n}{x - a} \\ &= x^{n-1} + x^{n-2}a + \ldots + a^{n-1} \xrightarrow[x \to a]{} na^{n-1} \end{aligned}$$

So f is differentiable at a and $f'(a) = na^{n-1}$.

Theorem 13.4

Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable at $a \in I$. Then f is continuous at a.

Proof. For $x \in I \setminus \{a\}$, we write

$$f(x) = \underbrace{\frac{f(x) - f(a)}{x - a}}_{\xrightarrow[x \to a]{} f'(a)} \cdot \underbrace{(x - a)}_{\xrightarrow[x \to a]{} 0} + \underbrace{f(a)}_{\xrightarrow[x \to a]{} f(a)} \xrightarrow[x \to a]{} f(a)$$

$\square$

Theorem 13.5

Let I be an open interval and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be two functions differentiable at $a \in I$. Then

1. $\forall \lambda \in \mathbb{R}$, λf is differentiable at a and
$$(\lambda f)'(a) = \lambda f'(a)$$
2. $f + g$ is differentiable at a and
$$(f+g)'(a) = f'(a) + g'(a)$$
3. $f \cdot g$ is differentiable at a and
$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$
4. $\frac{f}{g}$ is differentiable at a if $g(a) \neq 0$ and
$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$$

Proof. For $x \neq a$

1. Consider
$$\frac{\lambda f(x) - \lambda f(a)}{x-a} = \lambda \cdot \frac{f(x)-f(a)}{x-a} \xrightarrow[x\to a]{} \lambda f'(a)$$
2. Consider
$$\frac{(f(x)+g(x)) - (f(a)+g(a))}{x-a} = \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a} \xrightarrow[x\to a]{} f'(a) + g'(a)$$
3. Consider
$$\underbrace{\frac{f(x)-f(a)}{x-a}}_{\xrightarrow[x\to a]{} f'(a)} \cdot \underbrace{g(x)}_{\xrightarrow[x\to a]{} g(a)} + \underbrace{f(a)}_{\xrightarrow[x\to a]{} f(a)} \cdot \underbrace{\frac{g(x)-g(a)}{x-a}}_{\xrightarrow[x\to a]{} g'(a)} \xrightarrow[x\to a]{} f'(a)g(a) + f(a)g'(a)$$
4. Consider
$$\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} \underbrace{\frac{f(x)-f(a)}{x-a}}_{\xrightarrow[x\to a]{} f'(a)} \cdot \underbrace{\frac{1}{g(x)}}_{\xrightarrow[x\to a]{} \frac{1}{g(a)}} + f(a) \cdot \underbrace{\frac{g(a)-g(x)}{x-a}}_{\xrightarrow[x\to a]{} -g'(a)} \cdot \underbrace{\frac{1}{g(x)}}_{\xrightarrow[x\to a]{} \frac{1}{g(a)}} \cdot \frac{1}{g(a)}$$
$$\xrightarrow[x\to a]{} \frac{f'(a)}{g(a)} - \frac{g'(a)}{g^2(a)} f(a)$$

∎

§14 Lec 14: Apr 28, 2021

§14.1 Chain Rule

Theorem 14.1 (Chain Rule)

Let I and J be two open intervals and let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions. Assume that f is differentiable at $a \in I$ and that g is differentiable at $f(a) \in J$. Then $g \circ f$ is well defined on a neighborhood of a, $g \circ f$ is differentiable at a, and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Proof. Consider:

$$\left.\begin{matrix} f(a) \in J \\ J \text{ is open} \end{matrix}\right\} \implies \exists \epsilon > 0 \text{ s.t. } (f(a) - \epsilon, f(a) + \epsilon) \subseteq J$$

f is differentiable at $a \implies f$ is continuous at $a \implies \exists \delta > 0$ s.t. $f((a-\delta, a+\delta) \cap I) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$. As $a \in I$ and I is open, shrinking δ if necessary, me may assume that $(a - \delta, a + \delta) \subseteq I$.

Then $g \circ f$ is well-defined on $(a - \delta, a + \delta)$.

$$\underbrace{(a-\delta, a+\delta)}_{\subseteq I} \xrightarrow{f} \underbrace{(f(a) - \epsilon, f(a) + \epsilon)}_{\subseteq J} \xrightarrow{g} \mathbb{R}$$

Caution: The following argument does not work

$$\frac{g(f(x)) - g(f(a))}{x - a} = \underbrace{\frac{g(f(x)) - g(f(a))}{f(x) - f(a)}}_{\xrightarrow{x \to a} g'(f(a))} \cdot \underbrace{\frac{f(x) - f(a)}{x - a}}_{\xrightarrow{x \to a} f'(a)}$$

because f is continuous at $a \implies f(x) \xrightarrow{x \to a} f(a)$

Instead, we argue as follows: Define $h : J \to \mathbb{R}$,

$$h(y) = \begin{cases} \frac{g(y) - g(f(a))}{y - f(a)}, & \text{if } y \in J \setminus \{f(a)\} \\ g'(f(a)), & \text{if } y = f(a) \end{cases}$$

As g is differentiable at $f(a)$, h is continuous at $f(a)$. Moreover, we can write

$$g(y) - g(f(a)) = h(y) \cdot (y - f(a)) \qquad \forall y \in J$$

For $x \in (a - \delta, a + \delta) \implies f(x) \in J$. So for $x \in (a - \delta, a + \delta) \setminus \{a\}$,

$$\frac{g(f(x)) - g(f(a))}{x - a} = \underbrace{h(f(x))}_{\xrightarrow{x \to a} h(f(a))} \cdot \underbrace{\frac{f(x) - f(a)}{x - a}}_{\xrightarrow{x \to a} f'(a)}$$

So $\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = h(f(a)) f'(a) = g'(f(a)) \cdot f'(a)$. ∎

Lemma 14.2

Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. If f is increasing then $f'(x) \geq 0\, \forall x \in (a, b)$ or decreasing then $f'(x) \leq 0\, \forall x \in (a, b)$.

Proof. Assume f is increasing (if f is decreasing, replace f by $-f$ in what follows). Fix $x \in (a, b)$ and let $\{x_n\}_{n\geq 1}$ be an increasing from (a, b) with $\lim_{n\to\infty} x_n = x$.
Then $f'(x) = \lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n-x} \geq 0$ where $f(x_n) - f(x) \leq 0$ and $x_n - x < 0$. □

Theorem 14.3

Let $f : (a, b) \to \mathbb{R}$ be a function. Assume that $x_0 \in (a, b)$ is a point of local maximum/minimum for f. Assume also that f is differentiable at x_0. Then $f'(x_0) = 0$.

Proof. Assume that x_0 is a point of local maximum for f (if x_0 is a point of local minimum, replace f by $-f$ in what follows).

Then $\exists \delta > 0$ s.t. $f(x) \leq f(x_0) \quad \forall x \in (x_0 - \delta, x_0 + \delta) \cap (a, b)$. For $x_n \in (x_0 - \delta, x_0) \cap (a, b)$ s.t. $x_n \underset{n\to\infty}{\longrightarrow} x_0$, we have

$$f'(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0) \leq 0}{x_n - x_0 < 0} \geq 0$$

On the other hand, for $y_n \in (x_0, x_0 + \delta) \cap (a, b)$ s.t. $y_n \underset{n\to\infty}{\longrightarrow} x_0$, we have

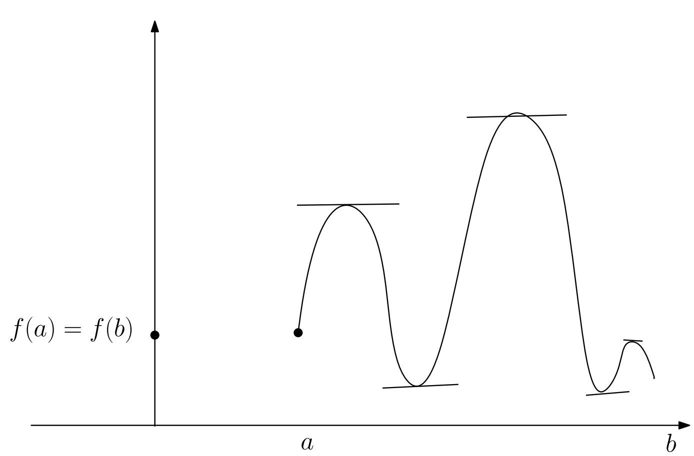
$$f'(x_0) = \lim_{n\to\infty} \frac{f(y_n) - f(x_0) \leq 0}{y_n - x_0 > 0} \leq 0$$

Thus, we get $f'(x_0) = 0$. □

§14.2 Mean Value Theorem

Theorem 14.4 (Rolle)

Let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on the $[a, b]$, differentiable on (a, b), and s.t. $f(a) = f(b)$. Then there exists (at least one) $x \in (a, b)$ s.t. $f'(x) = 0$.



Proof. Consider:

$$\left.\begin{array}{l} f:[a,b]\to\mathbb{R} \text{ continuous} \\ [a,b] \ \text{ compact}\end{array}\right\} \implies \exists x_0, y_0 \in [a,b]$$

s.t.

$$f(x_0) = \sup_{x\in[a,b]} f(x) \quad \text{and} \quad f(y_0) = \inf_{x\in[a,b]} f(x)$$

So $f(y_0) \le f(x) \le f(x_0) \quad \forall x \in [a,b]$.
Case 1: We have

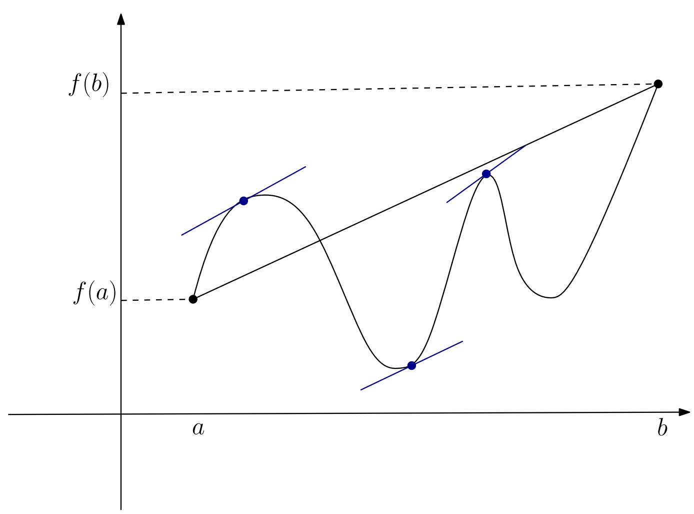
$$\left.\begin{array}{l} \{x_0,y_0\} \subseteq \{a,b\} \\ f(a)=f(b)\end{array}\right\} \implies f(x_0)=f(y_0) \implies f \text{ constant} \implies f'(x)=0\,\forall x\in(a,b)$$

Case 2: $\{x_0,y_0\} \not\subseteq \{a,b\} \implies x_0 \notin \{a,b\}$ or $y_0 \notin \{a,b\}$. Say $x_0 \notin \{a,b\} \implies x_0 \in (a,b)$. By Theorem 14.3, we get $f'(x_0)=0$. ☐

Theorem 14.5 (Mean Value)
Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists (at least one) $y\in(a,b)$ s.t.

$$f'(y) = \frac{f(b)-f(a)}{b-a}$$

Remark 14.6. The Mean Value Theorem implies Rolle's Theorem. We will see from the proof that Rolle's Theorem implies the Mean Value Theorem, so the two are equivalent.



Proof. We define $l:[a,b]\to\mathbb{R}$ where

$$l(x) = \frac{f(b)-f(a)}{b-a}(x-a)+f(a)$$

Note that l is continuous on $[a, b]$, differentiable on (a, b), and

$$l'(x) = \frac{f(b) - f(a)}{b - a} \qquad \forall x \in (a, b)$$

Let $g : [a, b] \to \mathbb{R}$, $g(x) = f(x) - l(x)$. Then g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = 0 = g(b)$. Then Rolle's implies that $\exists y \in (a, b)$ s.t.

$$g'(y) = 0 \implies f'(y) - l'(y) = 0 \implies f'(y) = \frac{f(b) - f(a)}{b - a}$$ □

Corollary 14.7
If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = 0 \, \forall x \in (a, b)$, then f is a constant.

Proof. Assume f is not a constant. Then $\exists a < x_1 < x_2 < b$ s.t.

$$f(x_1) \neq f(x_2)$$

Then f is continuous on $[x_1, x_2]$, differentiable on (x_1, x_2). By Mean Value, $\exists y \in (x_1, x_2)$ s.t.

$$f'(y) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \neq 0$$

Contradiction! □

Corollary 14.8
If $f, g : (a, b) \to \mathbb{R}$ are differentiable s.t. $f'(x) = g'(x) \, \forall x \in (a, b)$, then $\exists c \in \mathbb{R}$ s.t.

$$f(x) = g(x) + c \qquad \forall x \in (a, b)$$

§15 Lec 15: Apr 30, 2021

§15.1 Mean Value Theorem (Cont'd)

Theorem 15.1

Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists (at least one) $c \in (a, b)$ s.t.

$$f'(c)\,[g(b) - g(a)] = g'(c)\,[f(b) - f(a)]$$

Remark 15.2. Taking $g(x) = x$ we recover the Mean Value theorem. In fact, the two results are equivalent, as can be seen from the proof.

Proof. We define $h : [a, b] \to \mathbb{R}$

$$h(x) = f(x)\,[g(b) - g(a)] - g(x)\,[f(b) - f(a)]$$

Note that h is continuous on $[a, b]$ and differentiable on (a, b). Moreover,

$$\left.\begin{aligned} h(a) &= f(a)\,[g(b) - g(a)] - g(a)\,[f(b) - f(a)] = f(a)g(b) - g(a)f(b) \\ h(b) &= f(b)\,[g(b) - g(a)] - g(b)\,[f(b) - f(a)] = -f(b)g(a) + g(b)f(a) \end{aligned}\right\} \implies h(a) = h(b)$$

By Rolle's theorem, $\exists c \in (a, b)$ s.t $h'(c) = 0$. $\square$

Corollary 15.3

Let $f : (a, b) \to \mathbb{R}$ be differentiable.

1. If $f'(x) > 0\ \forall x \in (a, b)$ then f is strictly increasing.
2. If $f'(x) \geq 0\ \forall x \in (a, b)$ then f is increasing.
3. If $f'(x) < 0\ \forall x \in (a, b)$ then f is strictly decreasing.
4. If $f'(x) \leq 0\ \forall x \in (a, b)$ then f is decreasing.

Proof. We only present the details for (1).
Fix $a < x_1 < x_2 < b$. f is differentiable on $(a, b) \implies f$ is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Mean Value theorem, $\exists c \in (x_1, x_2)$ s.t.

$$0 < f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \implies f(x_1) < f(x_2)$$

As $a < x_1 < x_2 < b$ were arbitrary, f is strictly increasing. $\square$

Example 15.4

The derivative of a differentiable function need not be continuous

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f is continuous on $\mathbb{R} \setminus \{0\}$. To see that it's continuous at 0,

$$|f(x) - f(0)| = \left| x^2 \sin \frac{1}{x} \right| \leq x^2 \underset{x \to 0}{\longrightarrow} 0 \qquad (*)$$

f is differentiable on $\mathbb{R} \setminus \{0\}$. To see that it's differentiable at 0, we compute

$$x \neq 0 : \quad \frac{f(x) - f(0)}{x - 0} = x \sin \frac{1}{x} \underset{x \to 0}{\longrightarrow} 0 \quad (\text{as in } (*))$$

So $f'(0) = 0$. Thus,

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot \frac{-1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f' is continuous on $\mathbb{R} \setminus \{0\}$ (not continuous at 0). While $\lim_{x \to 0} 2x \sin \frac{1}{x} = 0$, for each $\lambda \in [-1, 1]$, there exists $x_n(\lambda) \underset{n \to \infty}{\longrightarrow} 0$ s.t. $\cos \frac{1}{x_n(\lambda)} = \lambda$. Nevertheless, the derivative of a differentiable function has the Darboux property.

Theorem 15.5 (Intermediate Value for Derivatives)

Let $f : (a, b) \to \mathbb{R}$ be differentiable. Then f' has the Darboux property, that is, if $a < x_1 < x_2 < b$ and λ lies between $f'(x_1)$ and $f'(x_2)$, then there exists $c \in (x_1, x_2)$ s.t.

$$f'(c) = \lambda$$

Proof. Let $g : (a, b) \to \mathbb{R}$, $g(x) = f(x) - \lambda x$. g is differentiable on $(a, b) \implies g$ is continuous on (a, b). Fix $a < x_1 < x_2 < b$ and assume without loss of generality

$$f'(x_1) < \lambda < f'(x_2)$$

Then

$$g'(x_1) = f'(x_1) - \lambda < 0$$
$$g'(x_2) = f'(x_2) - \lambda > 0$$

g is continuous on $[x_1, x_2]$

$$\implies \exists c \in [x_1, x_2] \text{ s.t. } g(c) = \inf_{x \in [x_1, x_2]} g(x)$$

If we can prove that $c \in (x_1, x_2)$ then $g'(c) = 0$. To see that $c \neq x_1$ we argue as follows:

$$0 > g'(x_1) = \lim_{x \to x_1} \frac{g(x) - g(x_1)}{x - x_1} \implies \exists \delta_1 > 0$$

s.t. if $0 < |x - x_1| < \delta_1$ then

$$\frac{g(x) - g(x_1)}{x - x_1} < 0$$

In particular, for $x \in (x_1, x_1 + \delta_1)$ we have

$$\frac{g(x) - g(x_1)}{\underbrace{x - x_1}_{>0}} < 0 \implies g(x) < g(x_1)$$

$$\implies g \text{ cannot attain its minimum at } x_1$$

Similarly,

$$0 < g'(x_2) = \lim_{x \to x_2} \frac{g(x) - g(x_2)}{x - x_2} \implies \exists \delta_2 > 0$$

s.t. if $0 < |x - x_2| < \delta_2$ then

$$\frac{g(x) - g(x_2)}{x - x_2} > 0$$

In particular, if $x \in (x_2 - \delta_2, x_2)$ then

$$\frac{g(x) - g(x_2)}{\underbrace{x - x_2}_{<0}} \implies g(x) < g(x_2)$$

$$\implies g \text{ cannot attain its minimum at } x_2$$

□

§15.2 Derivative of Inverse Functions

Theorem 15.6

Let I be an open interval and let $f : I \to \mathbb{R}$ be continuous and injective. Then $f(I) = J$ is an interval and $f : I \to J$ is bijective. If f is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$ then $f^{-1} : J \to I$ is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{-1}(y_0)\right)}$$

Proof. The proof uses the following two exercises:

Exercise 15.1. Let I be an interval and let $f : I \to \mathbb{R}$ be continuous and injective. Then f is strictly monotone.

Exercise 15.2. Let I be an interval and let $f : I \to \mathbb{R}$ be strictly increasing and so that $f(I)$ is an interval. Then f is continuous.

Using exercise 1, we find that f is strictly monotone. Assume f is strictly increasing $\implies f^{-1}$ is strictly increasing.
Using exercise 2 with $g = f^{-1} : J \to I$, we find that f^{-1} is continuous.

Claim 15.1. J is an open interval.

Assume, towards a contradiction, that $\inf J \in J = f(I) \implies \exists a \in I$ s.t. $f(a) = \inf J$.

$$\left.\begin{array}{l} I \text{ open } \implies \exists \delta > 0 \text{ s.t. } (a-\delta, a+\delta) \subseteq I \\ f \text{ is strictly increasing} \end{array}\right\} \implies J = f(I) \ni f\left(a - \frac{\delta}{2}\right) < f(a) = \inf J$$

Contradiction!
Similarly, one can show that $\sup J \notin J$

$$\left.\begin{array}{l} f \text{ is diff at } x_0 \implies f'(x_0) = \lim \frac{f(x)-f(x_0)}{x-x_0} \\ f'(x_0) \neq 0 \text{ and } f(x) \neq f(x_0) \quad \forall x \neq x_0 \end{array}\right\} \implies$$

$$\implies \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

$$\implies \forall \epsilon > 0 \quad \exists \delta > 0 \text{ s.t. } 0 < |x - x_0| < \delta \implies \left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \epsilon$$

f^{-1} is continuous at $y_0 \implies \exists \eta > 0$ s.t. $0 < |y - y_0| < \eta$ implies

$$0 < \left| f^{-1}(y) - f^{-1}(y_0) \right| < \delta$$

So for $0 < |y - y_0| < \eta$ we get

$$\left| \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} - \frac{1}{f'(x_0)} \right| < \epsilon$$

which implies

$$\left(f^{-1}\right)'(y_0) = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)} \qquad \square$$

§16 Lec 16: May 3, 2021

§16.1 L'Hopital Rule

Definition 16.1 (Existence of Limit) — Let $-\infty \le a < b \le \infty$ and let $f : (a,b) \to \mathbb{R}$ be a function. For $c \in (a,b) \cup \{a\}$ we write

$$\lim_{x \to c^+} f(x) = L \in \mathbb{R} \cup \{\pm\infty\}$$

if for every sequence $\{x_n\}_{n\ge 1} \subseteq (c,b)$ s.t. $\lim_{n\to\infty} x_n = c$ we have

$$\lim_{n\to\infty} f(x_n) = L$$

For $c \in (a,b) \cup \{b\}$ we write

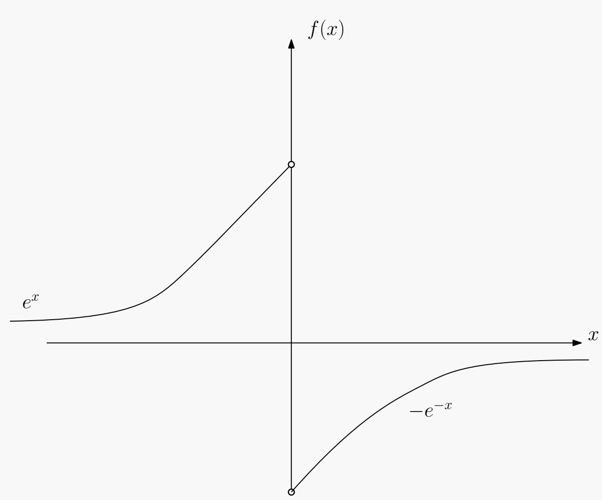
$$\lim_{x \to c^-} f(x) = M \in \mathbb{R} \cup \{\pm\infty\}$$

if for every sequence $\{x_n\}_{n\ge 1} \subseteq (a,c)$ s.t. $\lim_{n\to\infty} x_n = c$ we have

$$\lim_{n\to\infty} f(x_n) = M$$

Remark 16.2. In general, if $c \in (a,b)$ we have

$$f(c) \ne \lim_{x\to c^-} f(x) \ne \lim_{x \to c^+} f(x) \ne f(c)$$



Theorem 16.3 (L'Hopital)

Let $-\infty \leq a < b \leq \infty$ and let $f, g : (a, b) \to \mathbb{R}$ be differentiable. Assume that $g'(x) \neq 0$ $\forall x \in (a, b)$ and that

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{\pm\infty\}$$

Assume also that either

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0 \tag{1}$$

or

$$\lim_{x \to a^+} |g(x)| = \infty \tag{2}$$

Then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

Remark 16.4. $\lim_{x \to a^+}$ in the theorem can be replaced by $\lim_{x \to b^-}$ or by $\lim_{x \to c}$ for some $c \in (a, b)$.

Proof. We'll present the details for $L \in \mathbb{R}$. We'll prove

Claim 16.1. $\forall \epsilon > 0$ $\exists \delta_1(\epsilon) > 0$ s.t.

$$\frac{f(x)}{g(x)} < L + \epsilon \qquad \forall x \in (a, a + \delta_1)$$

Claim 16.2. $\forall \epsilon > 0$ $\exists \delta_2(\epsilon) > 0$ s.t.

$$L - \epsilon < \frac{f(x)}{g(x)} \qquad \forall x \in (a, a + \delta_2)$$

Then taking $\delta(\epsilon) = \min\{\delta_1(\epsilon), \delta_2(\epsilon)\}$ we get

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon \qquad \forall x \in (a, a + \delta)$$

$\implies \lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.
Note: If $L = -\infty$ then it suffices to prove Claim 1 with $L + \epsilon$ replaced by $M < 0$.
If $L = \infty$ then it suffices to prove Claim 2 with $L - \epsilon$ replaced by $M > 0$.
By assumption, $g'(x) \neq 0$ $\forall x \in (a, b)$. As g is differentiable on (a, b), g' has the Darboux property. So either $g'(x) < 0$ $\forall x \in (a, b)$ or $g'(x) > 0$ $\forall x \in (a, b)$.
Assume $g'(x) < 0$ $\forall x \in (a, b)$ $\implies$ g strictly decreasing on (a, b). In case 1,

$$\lim_{x \to a^+} g(x) = 0$$

As g is strictly decreasing, we get

$$g(x) < 0 \qquad \forall x \in (a, b)$$

In case 2,

$$\lim_{x \to a^+} |g(x)| = \infty$$

As g is strictly decreasing, we get

$$\lim_{x\to a^+} g(x) = \infty$$

and so $\exists c \in (a,b)$ s.t. $g(x) > 0\ \forall x \in (a,c)$ (**). In particular, in both cases $g(x) \neq 0$ $\forall x \in (a,c)$. We prove claim 1:

Fix $\epsilon > 0$. As $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, $\exists \delta_1(\epsilon) > 0$ s.t.

$$\frac{f'(x)}{g'(x)} < L + \frac{\epsilon}{2} \qquad \forall x \in (a, a+\delta_1)$$

Fix $a < x < y < \min(a+\delta_1, c)$. By (an equivalent formulation of) Mean Value theorem, $\exists z \in (x,y)$ s.t.

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(z)}{g'(z)} < L + \frac{\epsilon}{2} \tag{*}$$

In case 1, take the limit $x \to a^+$ in (*) to get

$$\frac{f(y)}{g(y)} \le L + \frac{\epsilon}{2} < L + \epsilon \qquad \forall a < y < \min(a+\delta_1, c)$$

In case 2, we write

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(y)}{g(x)-g(y)} \cdot \frac{g(x)-g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

By (**) we have $g(x) > g(y) > 0 \implies \frac{g(x)-g(y)}{g(x)} > 0$. So

$$\begin{aligned}
\frac{f(x)}{g(x)} &< \left(L + \frac{\epsilon}{2}\right) \frac{g(x)-g(y)}{g(x)} + \frac{f(y)}{g(x)} \\
&= \left(L + \frac{\epsilon}{2}\right)\left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)} \\
&= L + \frac{\epsilon}{2} + \frac{f(y) - \left(L + \frac{\epsilon}{2}\right) g(y)}{g(x)}
\end{aligned}$$

For y fixed, $\lim_{x\to a^+} \frac{f(y)-\left(L+\frac{\epsilon}{2}\right)g(y)}{g(x)} = 0$

$$\implies \exists \tilde{\delta_1}(\epsilon) > 0 \text{ s.t. } \left| \frac{f(y) - \left(L + \frac{\epsilon}{2}\right) g(y)}{g(x)} \right| < \frac{\epsilon}{2} \qquad \forall x \in \left(a, a+\tilde{\delta_1}\right)$$

In particular,

$$\frac{f(x)}{g(x)} < L + \epsilon \qquad \forall a < x < \min\left\{a+\delta_1, a+\tilde{\delta_1}, c\right\}$$

Exercise 16.1. Prove claim 2. □

§16.2 Taylor's Theorem

Definition 16.5 (Taylor Expansion) — Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable of any order. For $x_0 \in I$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the Taylor expansion of f about x_0. For $n \geq 1$, we define the remainder

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Theorem 16.6 (Taylor)

Let $n \geq 1$ and assume $f : (a, b) \to \mathbb{R}$ is n times differentiable. Let $x_0 \in (a, b)$. Then for any $x \in (a, b) \setminus \{x_0\}$ there exists y between x and x_0 s.t.

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x - x_0)^n$$

In particular,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(y)}{n!}(x - x_0)^n$$

Proof. Fix $x \in (a, b) \setminus \{x_0\}$. Define $M \in \mathbb{R}$ to be the unique solution to the equation

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + M \cdot \frac{(x - x_0)^n}{n!}$$

We want to show that there exists y between x and x_0 s.t.

$$M = f^{(n)}(y)$$

Let $g : (a, b) \to \mathbb{R}$

$$g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(t - x_0)^k - M \cdot \frac{(t - x_0)^n}{n!}$$

Note g is n times differentiable. For $1 \leq l \leq n - 1$,

$$\begin{aligned} g^{(l)}(t) &= f^{(l)}(t) - \sum_{k \geq l}^{n-1} \frac{f^{(k)}(x_0)}{(k - l)!}(t - x_0)^{k-l} - M \frac{(t - x_0)^{n-l}}{(n - l)!} \\ g^{(n)}(t) &= f^{(n)}(t) - M \end{aligned}$$

In particular, if $0 \leq l \leq n - 1$,

$$g^{(l)}(x_0) = f^{(l)}(x_0) - f^{(l)}(x_0) = 0$$

Also $g(x) = 0$ by contradiction.
g is continuous on $[x, x_0]$, differentiable on (x, x_0) and

$$g(x) = g(x_0) = 0 \implies \exists x_1 \in (x, x_0) \text{ s.t. } g'(x_1) = 0$$

By Rolle's theorem,

$$\begin{aligned} \exists x_2 \in (x_1, x_0) \quad & \text{s.t.} \quad g''(x_2) = 0 \\ & \vdots \\ \exists x_n \in (x_{n-1}, x_0) \quad & \text{s.t.} \quad g^{(n)}(x_n) = 0 \end{aligned}$$

Set $y = x_n$. □

§17 Lec 17: May 5, 2021

§17.1 Taylor's Theorem (Cont'd)

Corollary 17.1

Fix $a > 0$ and let $f : (-a, a) \to \mathbb{R}$ be a function differentiable of any order. Assume that all derivatives of f are uniformly bounded on $(-a, a)$, that is,

$$\exists M > 0 \text{ s.t. } \left|f^{(n)}(x)\right| \le M \quad \forall x \in (-a, a), \quad \forall n \ge 1$$

Then

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k \xrightarrow[n\to\infty]{u} 0 \text{ on } (-a, a)$$

Proof. Fix $x \in (-a, a) \setminus \{0\}$. By Taylor, there exists y between x and 0 s.t.

$$\begin{aligned} R_n(x) &= \frac{f^{(n)}(y)}{n!} x^n \\ \implies |R_n(x)| &\le M \frac{|x|^n}{n!} \le M \frac{a^n}{n!} \\ \implies \sup_{x \in (-a,a)} |R_n(x)| &\le M \cdot \frac{a^n}{n!} \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

□

Example 17.2

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$

$$f^{(n)}(x) = \begin{cases} -\sin x, & n = 1 + 4k \\ -\cos x, & n = 2 + 4k \\ \sin x, & n = 3 + 4k \\ \cos x, & n = 4k \end{cases} \qquad \text{for } k \ge 0$$

So $\left|f^{(n)}(x)\right| \le 1$ $\forall x \in \mathbb{R}$ $\forall n \ge 0$. We get

$$f(x) = u - \lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \quad \text{on } (-a, a) \text{ for any } a > 0$$

Let $n = 2l$

$$\begin{aligned} &\implies f^{(n)}(0) = \begin{cases} -1, & \text{if } l \text{ odd} \\ 1, & \text{if } l \text{ even} \end{cases} = (-1)^l \\ &\implies f(X) = \sum_{n \ge 0} \frac{f^{(n)}(0)}{n!} x^n = \sum_{l \ge 0} \frac{(-1)^l}{(2l)!} x^{2l} \end{aligned}$$

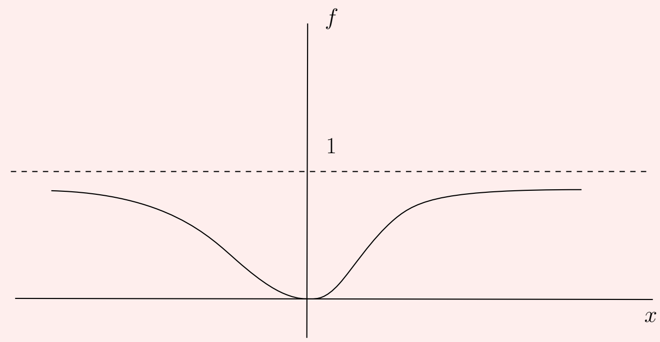
A similar argument gives

$$\sin x = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Example 17.3

$f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$



Note f is differentiable of any order on $\mathbb{R}$. Clearly, this holds on $\mathbb{R} \setminus \{0\}$. In fact, for $x \in \mathbb{R} \setminus \{0\}$,

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$$

where

$$P_n\left(\frac{1}{x}\right) = \left(\frac{2}{x^3}\right)^n + \dots$$

To see that f is differentiable at 0 we compute

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim_{t\to\infty} \frac{t}{e^{t^2}} = \lim_{t\to\infty} \frac{1}{2te^{t^2}} = 0$$

Similarly,

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{t\to -\infty} \frac{t}{e^{t^2}} = 0$$

Proceeding inductively, we can prove that f is differentiable of any order at 0 and

$$f^{(n)}(0) = 0$$

We consider

$$\lim_{x\to 0^+} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0^+} \frac{P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}}{x} \lim_{t\to\infty} \frac{tP_n(t)}{e^{t^2}} = 0$$

and

$$\lim_{x\to 0^-} \frac{f^{(n)}(x)}{x} = 0$$

Example 17.4 (Cont'd from above)
Thus,
$$\sum_{n\geq 0} \frac{f^{(n)}(0)}{n!} x^n \equiv 0$$
At leading order as $x \to 0$,
$$f^{(n)}(x) \sim 2^n \cdot \left(\frac{1}{x^2}\right)^{\frac{3n}{2}} e^{-\frac{1}{x^2}} \sim 2^n e^{-\frac{1}{x^2}+\frac{3n}{2}\ln\frac{1}{x^2}}$$
The function $g : (0, \infty) \to \mathbb{R}$, $g(t) = -t + \frac{3n}{2}\ln t$ achieves its maximum at
$$g'(t) = 0 \iff -1 + \frac{3n}{2t} = 0 \iff t = \frac{3n}{2}$$
So $f^{(n)}\left(\sqrt{\frac{2}{3n}}\right) \sim 2^n e^{-\frac{3n}{2}+\frac{3n}{2}\ln\frac{3n}{2}} \sim 2^n e^{\frac{3n}{2}\ln\left(\frac{3n}{2e}\right)} \sim 2^n \left(\frac{3n}{2e}\right)^{\frac{3n}{2}} \xrightarrow[n\to\infty]{} \infty$.

Theorem 17.5
Assume that $f_n : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Assume also that

1. $\{f_n'\}_{n\geq 1}$ converges uniformly on (a, b)
2. $\{f_n\}_{n\geq 1}$ converges at some x_0 in $[a, b]$

Then $\{f_n\}_{n\geq 1}$ converges uniformly on $[a, b]$ to some function f. Moreover, f is differentiable on (a, b) and
$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad \forall x \in (a, b)$$

Remark 17.6. We can restate the conclusion as follows:
$$\lim_{y\to x}\lim_{n\to\infty} \frac{f_n(y) - f_n(x)}{y - x} = \lim_{y\to x} \frac{f(y) - f(x)}{y - x} = f'(x) = \lim_{n\to\infty}\lim_{y\to x} \frac{f_n(y) - f_n(x)}{y - x}$$

Proof. Let's prove that $\{f_n\}_{n\geq 1}$ converges uniformly on $[a, b]$. Fix $\epsilon > 0$.
$\{f_n'\}_{n\geq 1}$ converges uniformly on (a, b) which implies $\{f_n'\}_{n\geq 1}$ is uniformly Cauchy on (a, b) which also implies $\exists n_1(\epsilon) \in \mathbb{N}$ s.t.
$$\left|f_n'(x) - f_m'(x)\right| < \epsilon \quad \forall n, m \geq n_1(\epsilon) \quad \forall x \in (a, b)$$
Also, we know that $\{f_n(x_0)\}_{n\geq 1}$ converges which means $\{f_n(x_0)\}$ is Cauchy which implies $\exists n_2(\epsilon) \in \mathbb{N}$ s.t.
$$|f_n(x_0) - f_m(x_0)| < \epsilon \quad \forall n, m \geq n_2(\epsilon)$$
For $x \in [a, b] \setminus \{x_0\}$,
$$|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]|$$

By the Mean Value theorem, there exists y between x and x_0 s.t.

$$|\,[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]\,| = \left|f_n'(y) - f_m'(y)\right| |x - x_0| < \epsilon(b-a)$$

So for $n, m \geq n(\epsilon) = \max\{n_1(\epsilon), n_2(\epsilon)\}$ we get

$$\begin{aligned} &|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + \epsilon(b-a) \leq \epsilon(1+b-a) \\ \implies\ & sup_{x\in[a,b]} |f_n(x) - f_m(x)| \leq \epsilon(1+b-a) \qquad \forall n, m \geq n(\epsilon) \end{aligned}$$

So $\{f_n\}_{n\geq 1}$ are uniformly Cauchy on $[a,b]$ and so converge to a function $f = \lim_{n\to\infty} f_n$. It remains to show that f is differentiable on (a,b) and

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

which we will prove in the next lecture. □

§18 Lec 18: May 7, 2021

§18.1 Taylor's Theorem (Cont'd)

Proof. (Cont'd from lecture 17) Fix $x \in (a,b)$. We want to show that f is differentiable at x and

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

We define

$$g : [a,b] \setminus \{x\} \to \mathbb{R}, \quad g(y) = \frac{f(y)-f(x)}{y-x}$$
$$g_n : [a,b] \setminus \{x\} \to \mathbb{R}, \quad g_n(y) = \frac{f_n(y)-f_n(x)}{y-x}$$

Since $f_n \xrightarrow[n\to\infty]{u} f$ we have

$$\lim_{n\to\infty} g_n(y) = g(y)$$

Since f_n is differentiable at x,

$$\lim_{y\to x} g_n(y) = f_n'(x)$$

Let $L(x) = \lim_{n\to\infty} f_n'(x)$. We want to show that

$$\forall \epsilon > 0 \quad \exists \delta > 0 \text{ s.t. } |g(y) - L(x)| < \epsilon \text{ whenever } 0 < |y-x| < \delta \ y \in [a,b]$$

Fix $\epsilon > 0$. By the triangle inequality,

$$|g(y) - L(x)| \le |g(y) - g_n(y)| + \left|g_n(y) - f_n'(x)\right| + \left|f_n'(x) - L(x)\right|$$

We have $\{f_n'\}_{n\ge1}$ converges uniformly on $(a,b) \implies \{f_n'\}_{n\ge1}$ is uniformly Cauchy on (a,b) $\implies \exists n_1(\epsilon) \in \mathbb{N}$ s.t.

$$\left|f_n'(z) - f_m'(z)\right| < \epsilon \qquad \forall n,m \ge n_1(\epsilon) \quad \forall z \in (a,b) \tag{1}$$

Letting $m \to \infty$ we get

$$\left|f_n'(z) - L(z)\right| \le \epsilon \qquad \forall n \ge n_1(\epsilon) \quad \forall z \in (a,b)$$

For $y \in [a,b] \setminus \{x\}$, by the Mean Value theorem, we can find a point z between x and y so that

$$\begin{aligned}|g_n(y) - g_m(y)| &= \left|\frac{f_n(y)-f_n(x)}{y-x} - \frac{f_m(y)-f_m(x)}{y-x}\right| \\ &= \frac{|[f_n(y)-f_m(y)] - [f_n(x)-f_m(x)]|}{|y-x|} \\ &= \left|f_n'(z) - f_m'(z)\right| \overset{(1)}{<} \epsilon \quad \forall n,m \ge n_1(\epsilon)\end{aligned}$$

Letting $m \to \infty$ we find

$$|g_n(y) - g(y)| \le \epsilon \qquad \forall n \ge n_1(\epsilon) \quad \forall y \in [a,b] \setminus \{x\} \tag{3}$$

Fix $n \geq n_1(\epsilon)$. As f_n is differentiable at x we find $\delta = \delta(\epsilon, n) > 0$ s.t.

$$\left|g_n(y) - f_n'(x)\right| < \epsilon \qquad \forall 0 < |y - x| < \delta \quad y \in [a,b] \tag{4}$$

Thus for this $n \geq n_1(\epsilon)$ and $0 < |y - x| < \delta$ we have

$$\begin{aligned} |g(y) - L(x)| &\leq |g(y) - g_n(y)| + \left|g_n(y) - f_n'(x)\right| + \left|f_n'(x) - L(x)\right| \\ \text{by (2), (3), (4)} &\leq 3\epsilon \end{aligned}$$

□

Example 18.1

$f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{1+nx^2}$, f_n is differentiable and

$$f_n'(x) = \frac{1}{1+nx^2} - \frac{x \cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

Now

$$f_n \xrightarrow[n\to\infty]{u} f \equiv 0$$

$$f_n'(x) \xrightarrow[n\to\infty]{} \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Note that f_n' do not converge uniformly since their limit is not continuous.

$$\lim_{n\to\infty} \lim_{y\to 0} \frac{f_n(y) - f_n(0)}{y - 0} = \lim_{n\to\infty} f_n'(0) = 1$$

but

$$\lim_{y\to 0} \lim_{n\to\infty} \frac{f_n(y) - f_n(0)}{y - 0} = \lim_{y\to 0} 0 = 0$$

§18.2 Darboux Integral

Definition 18.2 (Partition) — Let $f : [a,b] \to \mathbb{R}$ be a bounded function. If $S \subseteq [a,b]$ we denote

$$M(f;S) = \sup_{x\in S} f(x) \quad \text{and} \quad m(f;S) = \inf_{x\in S} f(x)$$

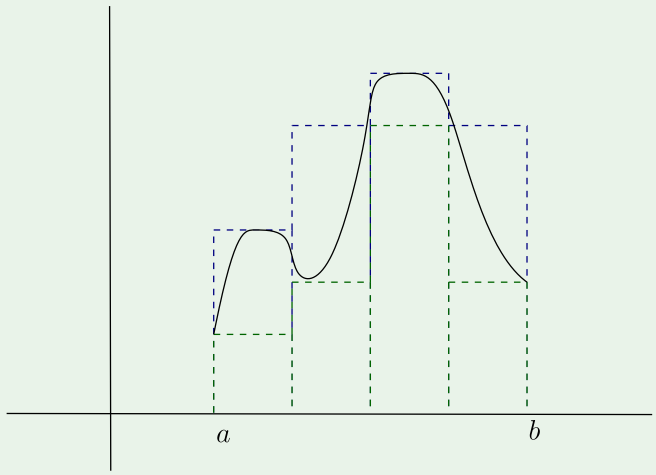
A partition of $[a,b]$ is a finite ordered set $P \subseteq [a,b]$. We write

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

for some $n \geq 1$.

Definition 18.3 (Darboux Sum) — The upper Darboux sum of f with respect to P is

$$U(f;P) = \sum_{k=1}^{n} M\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1})$$



The lower Darboux sum of f with respect to P is

$$L(f;P) = \sum_{k=1}^{n} m\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1})$$

Note that

$$m\left(f;[a,b]\right)(b-a) \le L(f;P) \le U(f;P) \le M\left(f;[a,b]\right)(b-a)$$

So

$$\begin{aligned} &\{L(f;P):\ P \text{ partition of } [a,b]\} \text{ is bounded above} \\ &\{U(f;P):\ P \text{ partition of } [a,b]\} \text{ is bounded below} \end{aligned}$$

Definition 18.4 (Darboux Integral) — The upper Darboux integral of f on $[a,b]$ is

$$U(f) = \inf\left\{U(f;P):\ P \text{ partition of } [a,b]\right\}$$

The lower Darboux integral of f on $[a,b]$ is

$$L(f) = \sup\left\{L(f;P):\ P \text{ partition of } [a,b]\right\}$$

We say that f is <u>Darboux integrable</u> on $[a,b]$ if $U(f) = L(f)$. In this case we write

$$\int_a^b f(x)\,dx = U(f) = L(f)$$

Example 18.5

Let $f : [0, M] \to \mathbb{R}$, $f(x) = x^3$. Then f is Darboux integrable.
Let $P = \{0 = t_0 < \ldots < t_n = M\}$ be a partition of $[0, M]$ and

$$\begin{aligned} U(f;P) &= \sum_{k=1}^{n} M\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} t_k^3 (t_k - t_{k-1}) \end{aligned}$$

Similarly,

$$L(f;P) = \sum_{k=1}^{n} m\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1}) = \sum_{k=1}^{n} t_{k-1}^3 (t_k - t_{k-1})$$

Take $t_k = \frac{kM}{n}$ $0 \le k \le n$. Then

$$U(f;P) = \sum_{k=1}^{n} \left(\frac{kM}{n}\right)^3 \cdot \frac{M}{n} = \frac{M^4}{n^4} \sum_{k=1}^{n} k^3 = \frac{M^4}{n^4}\left[\frac{n(n+1)^2}{2}\right] \xrightarrow[n\to\infty]{} \frac{M^4}{4}$$

$$L(f;P) = \sum_{k=1}^{n} \left(\frac{(k-1)M}{n}\right)^3 \cdot \frac{M}{n} = \frac{M^4}{n^4} \sum_{k=0}^{n-1} k^3 = \frac{M^4}{n^4}\left[\frac{n(n-1)^2}{2}\right] \xrightarrow[n\to\infty]{} \frac{M^4}{4}$$

So, $U(f) \le \frac{M^4}{4}$ and $L(f) \ge \frac{M^4}{4}$ and we will show that $L(f) \le U(f)$ which imply $U(f) = L(f) = \frac{M^4}{4}$. So f is Darboux integrable and $\int_0^M f(x)\,dx = \frac{M^4}{4}$.

Example 18.6

Given

$$f : [0,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

f is not Darboux integrable. For any partition P, $U(f;P) = 1$ and $L(f;P) = 0$ which implies $U(f) = 1$ and $L(f) = 0$.

§19 Dis 1: Mar 30, 2021

§19.1 Review of 131AH

Summation by parts(discrete integration by parts):

$\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$, $A_n = \sum_{k=1}^n a_k$, $A_0 = 0$. Then for $1 \leq p \leq q$,

$$\begin{aligned}\sum_{n=p}^q a_n b_n = \sum_{n=p}^q (A_n - A_{n-1})b_n &= \sum_{n=p}^q A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\end{aligned}$$

Application:

1. Dirichlet's test: $\sum a_n$ bounded, $\{b_n\}_{n\geq 1}$ decreasing and $b_n \to 0 \implies \sum a_n b_n$ converges.
2. Leibniz's Alternating series test: $|a_1| \geq |a_2| \geq \ldots$ and $a_n \to 0$, $\sum (-1)^{n+1}|a_n|$ converges.
3. Kronecker's lemma: $b_n \geq 0$, $b_n \leq b_{n+1}$ $b_n \to \infty$, $A_n = \sum_{k=1}^n a_k$, and $\sum \frac{a_n}{b_n}$ converges $\implies \frac{A_n}{b_n} \to 0$.

Cardinality:

$|X| \leq (= / \geq)|Y|$ to mean $\exists f : X \to Y$ injective, bijective, or surjective, respectively.

- X finite if $|X| = |\{1, \ldots, n\}|$
- X countable if $|X| \leq |\mathbb{N}|$. X countably infinite if countable but not finite.
- X countably infinite $\implies |X| = |\mathbb{N}|$.
- $|X| \leq |Y| \iff |Y| \geq |X|$.
- X, Y countable $\implies X \times Y$ countable.
- A countable, X_α countable $\forall \alpha \in A \implies \bigcup_{\alpha \in A} X_\alpha$ countable.
- $|\mathbb{Z}| = |\mathbb{N}| = |\mathbb{Q}|$, $\mathbb{R}$ uncountable.

Schröder – Bernstein: $|X| \leq |Y|$, $|Y| \leq |X|$ then $|X| = |Y|$ useful for hmwrk

Metric Spaces:

Let (X, d) be a metric space, $E \subseteq X$.

- $\mathring{E} = \{x \in X : \exists r > 0, B_r(x) \subseteq E\} = \bigcup_{G \subseteq E} G$ where G is open, largest open sets contained in E.
- $\overline{E} = \{x \in X : \forall r > 0, B_r(x) \cap E \neq \emptyset\} = \bigcap_{F \supset E} F$ where F is closed, smallest closed sets contained in E.
- $E' = \{x \in X : \forall r > 0, (B_r(x) \setminus \{x\}) \cap E \neq \emptyset\}$

- E open if $E = \mathring{E}$
- E closed if $E = \overline{E}$ or $E \supset E'$ or $\forall \{x_n\}_{n\geq 1} \subseteq E$, $x_n \to x \implies x \in E$.

(X, d) is complete if any Cauchy sequence in X converges.

- $\mathbb{R}$ complete, $\mathbb{R}^d$ complete.
- closed subsets of a complete space is complete.
- complete subsets are closed
- completeness is not invariant under homeomorphism(continuous bijection with continuous inverse)
  $(\mathbb{R}, |\cdot|) \overset{\sim}{\to} ((0,1), |\cdot|) \leftarrow$ not complete.

(X, d) is connected if there is no disjoint open sets A, B s.t. $X = A \cup B$.

- $E \subseteq \mathbb{R}$ connected $\iff$ E is interval.
- X is connected $\iff$ its only clopen subsets are $\emptyset, X$.

Intermediate Value Theorem: $f : [a, b] \to \mathbb{R}$ continuous, then $\forall \lambda$ s.t. $f(a) < \lambda < f(b)$, $\exists c$ s.t. $f(c) = \lambda$.

§20 Dis 2: Apr 6, 2021

§20.1 Compactness

Definition 20.1 — A metric space (X, d) compact if every open cover has a finite subcover.

Example 20.2

$\mathbb{Z} \subseteq \mathbb{R}$ compact?

The collection $\left\{\left(n - \frac{1}{2}, n + \frac{1}{2}\right)\right\}_{n\in\mathbb{Z}}$ open cover with no finite subcover – not compact! Note that $\mathbb{Z}$ is not bounded. An alternative is $\{(-n, n)\}_{n\in\mathbb{Z}}$

What about $\left\{\frac{1}{n}\right\}_{n\geq 1} \subseteq \mathbb{R}$?

The open cover $\left\{\left(\frac{1}{n}, 2\right)\right\}_{n\geq 1}$ is open cover with no finite subcover – not compact!

Exercise 20.1. $\left\{\frac{1}{n}\right\}_{n\geq 1} \cup \{0\}$ is compact.

Remark 20.3.

- X compact $\iff$ every $\{F_\alpha|_{\alpha\in A}\}$ closed subsets with finite intersection property satisfies $\bigcap_{\alpha\in A} F_\alpha \neq \emptyset$.
- compact subset of metric spaces are complete; complete subsets of metric spaces are closed.
- closed subset of a compact space is compact; closed subsets of complete space are complete.

Theorem 20.4

(Heine – Borel) (X, d) metric space. The following are equivalent:

1. X compact.
2. X sequential compact
3. X complete and totally bounded.
4. X limit point compact (every infinite subset of X has a limit point)

Remark 20.5.
1. In $\mathbb{R}^d$($\mathbb{R}^d$ complete), closed subsets are complete. Boundedness implies totally bounded. So, closed & bounded in $\mathbb{R}^d$ implies compact.
2. $B = \{f \in l_2 : \|f\|_2 \le 1\} \subseteq l_2$ is closed and bounded but not totally bounded. In particular, B is not compact.

   Fact 20.1. l_2 is complete and so is B.
3. totally boundedness implies separable (existence of a countable dense subset)

   homework 2

   converse is not true: $\mathbb{R}$ is separable ($\overline{\mathbb{Q}} = \mathbb{R}$), but not bounded.

Lemma 20.6

$\{f_n\}$ pointwise bounded ($\{f_n(x)\}_{n\ge1}$ is bounded for every x) on countable set E, then $\exists$ subsequence $\{f_{n_k}\}_{n\ge1}$ s.t. f_{n_k} converges pointwise on E.

Proof. Let $E = \{x_1, x_2, x_3, \dots\}$

$$\{f_n(x_1)\}_{n\ge1} \text{ bounded } \overset{\text{B-W}}{\Longrightarrow} \exists \text{ subseq. } \left\{f_j^{(1)}\right\}_{j\ge1} \text{ of } \{f_n\} \text{ s.t. } f_j^{(1)}(x_1) \to f(x_1)$$

Then

$$\left\{f_j^{(1)}(x_2)\right\} \text{ bounded } \implies \exists \left\{f_j^{(2)}\right\}_{j\ge1} \text{ of } \left\{f_j^{(1)}\right\} \text{ s.t.} f_j^{(2)} \to f(x_2)$$

So, in general,

$$\left\{f_j^{(k)}(x_{k+1})\right\} \text{ bounded } \implies \exists \left\{f_j^{(k+1)}\right\}_{j\ge1} \text{ of } \left\{f_j^{(k)}\right\} \text{ s.t.} f_j^{(k+1)} \to f(x_{k+1})$$

Diagonal argument

$$\begin{matrix} f_1^{(1)} & f_2^{(1)} & f_3^{(1)} \\ f_1^{(2)} & f_2^{(2)} & f_3^{(2)} \\ f_1^{(3)} & f_2^{(3)} & f_3^{(3)} \end{matrix}$$

Note that $\left\{f_k^{(k)}\right\}_{k\ge1}$ is a subsequence of $\left\{f_j^{(n)}\right\}$ $\forall n$ except for the first $n-1$ terms. So $f_k^{(k)}(x_n) \to f(x_n)$ □

§20.2 Ex 7 – Hw 2

(X, d) metric space, $\mathcal{F} = \{A \subseteq X : A \text{ compact}, A \neq \emptyset\}$

$$d_H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

where $d(a, B) = \inf_{b\in B} d(a, b)$.

$$\sup_{a\in A} d(a, B) = \inf\{\epsilon \ge 0 : A \subseteq B^\epsilon\}$$

The distance can be rewritten as

$$\begin{aligned} d_H(A,B) &= \max\left\{\inf\left\{\epsilon:\ A\subseteq B^\epsilon\right\},\ \inf\left\{\epsilon:\ B\complement A^\epsilon\right\}\right\} \\ &\overset{7b)}{=} \inf\left\{\epsilon:\ A\subseteq B^\epsilon \text{ and } B\subseteq A^\epsilon\right\} \end{aligned}$$

e.g., $d_H\left([0,1],[2,3]\right)=2$.
c) (X,d) totally bounded $\implies$ $(\mathcal{F}(X),d_H)$ totally bounded. (X,d) complete $\implies$ $(\mathcal{F}(X),d_H)$ complete.
It's easier to show (X,d) compact $\implies$ $(\mathcal{F}(X),d_H)$ complete

$$\left\{A_n\right\}_{n\geq 1} \text{ Cauchy in } d_H \quad A=\bigcap_{n\geq 1}\overline{\bigcup_{m\geq n}A_m},\ d_H(A,A_n)\to 0$$

Given $\left\{A_n\right\}_{n\geq 1}$,

$$\limsup A_n=\bigcap_{n\geq 1}\bigcup_{m\geq n}A_m=\left\{x:\ x\in A_n \text{ for infinitely many } n\right\}$$

$$\bigcap_{n\geq 1}\overline{\bigcup_{m\geq n}}A_m=\left\{x:\ \exists\left|x_{n_k}\right| \text{ s.t. } x_{n_k}\to x \text{ where } x_{n_k}\in A_{n_k},\ \left\{n_k\right\} \text{ non-decreasing } n_k\to\infty\right\}$$

§21 Dis 3: Apr 13, 2021

§21.1 Continuity

$f : X \to Y$ continuous at x if

- $(\epsilon - \delta)$: $\forall \epsilon > 0$, $\exists \delta_{\epsilon,x} > 0$, $\forall y \in X$ s.t. $|y - x| < \delta \implies |f(x) - f(y)| < \epsilon$.
- (Sequential): For each sequence $x_n \to x$, $f(x_n) \to f(x)$

$f : X \to Y$ continuous if continuous at every $x \in X$. This is equivalent to (topological): $\forall U \subseteq Y$ open, $f^{-1}(U)$ open in X.

Theorem 21.1

$f : X \to Y$ continuous. If X compact then $f(X)$ is compact. If X is connected then $f(X)$ is connected. If $Y = \mathbb{R}$, then the above statement gives the Extreme Value Theorem: $\exists x_1, x_2 \in X$ s.t.

$$f(x_1) \le f(x) \le f(x_2) \quad \forall x \in X$$

and Intermediate Value Theorem: $f(X)$ is an interval.

Proposition 21.2

X compact, $f : X \to Y$ bijective and continuous which implies f^{-1} is also continuous, i.e., f is a homeomorphism.

Example 21.3

$f : [0, 1) \to S' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, $x \leftrightarrow (\cos 2\pi x, \sin 2\pi x)$. Is f an homeomorphism?

Remark 21.4. Completeness is not preserved under homeomorphism: $(\mathbb{R}, |\cdot|)$ is complete but $((-1, 1), |\cdot|)$ is not complete.

§21.2 Uniform Continuity

$f : X \to Y$ uniformly continuous if $\forall \epsilon > 0$, $\exists \delta_\epsilon > 0$ s.t.

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Theorem 21.5

$f : X \to Y$ is continuous and X is compact. Then f is uniformly continuous.

Example 21.6

$f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is not uniformly continuous but $f\big|_{[-m,m]}$ is uniformly continuous.

Example 21.7

$x \mapsto |x|$, $x \mapsto d(x, A) = \inf_{a \in A} d(a, x)$ are uniformly continuous:

$$||x| - |y|| \leq |x - y|\,; \quad |d(x, A) - d(y, A)| \leq d(x, y)$$

Definition 21.8 (Lipschitz Continuous) — $f : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous if $\exists M > 0$ s.t. $|f(x) - f(y)| \leq M|x - y|$

Remark 21.9. Lipschitz continuous implies uniformly continuous. However, uniform continuity does not imply Lipschitz continuous.

Remark 21.10. For differentiable function, Lipschitz continuous $\iff$ bounded derivative.

§21.3 Ternary Expansion and Cantor Set

Every $x \in [0, 1]$ has a base-3 expansion $x = \sum_{j=1}^{\infty} a_j 3^{-j}$, $a_j \in \{0, 1, 2\}$. Write $x = [0.a_1a_2a_3 \ldots]_3$. It's unique unless $x = c3^{-k}$ for some $c, k \in \mathbb{Z}$ in which case x has 2 expansions: one with $a_j = 0$ for all $j > k$ and one with $a_j = 2$ for $j > k$. Assume TBA, one of the expansions will have $a_k = 1$, the other will ahve $a_k \in \{0, 2\}$. As convention, we always use the latter expansion, e.g. $\frac{1}{3} = 0.1_3 = 0.022222\ldots_3$, $\frac{2}{3} = 0.2_3 = 0.1222\ldots_3$

$$\begin{aligned}
&a_1 = 0 \iff x \in \left[0, \frac{1}{3}\right], \quad a_1 = 1 \iff x \in \left(\frac{1}{3}, \frac{2}{3}\right) \\
&a_1 = 2 \iff x \in \left[\frac{2}{3}, 1\right] \\
&a_1 \neq 1,\, a_2 = 1 \iff x \in \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)
\end{aligned}$$

Cantor set $C = \left\{x \in [0, 1] : x = \sum_{j=1}^{\infty} a_j 3^{-j},\, a_j \in \{0, 2\}\right\}$

$$\begin{aligned}
E_0 &= [0, 1] \\
E_1 &= \{x : a_1 \in \{0, 2\}\} = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
E_2 &= \{x \in E_1 : a_2 \in \{0, 2\}\} = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\
&\vdots \\
E_{k+1} &= \{x \in E_k : a_{k+1} \in \{0, 2\}\}
\end{aligned}$$

in which $E_{k+1} \subseteq E_k$.

$C = \bigcap_{k\geq 0} E_k$ compact.

- $C = C'$(C is perfect)
- $\mathring{C} = \emptyset$ (C contains no intervals)
- C is totally disconnected (the only nontrivial connected subsets are singletons)
- C is uncountable
- C is a set of length 0

$$|C| = 1 - \sum_{j=0}^{\infty} \frac{2^j}{3^{j+1}} = 1 - \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 0$$

§22 Dis 4: Apr 20, 2021

§22.1 Sequences of Functions

$f_n : X \to Y$ converges pointwise to f if $f(x) = \lim_{n\to\infty} f_n(x)$ and uniformly to f if $\forall \epsilon > 0$, $\exists N$ s.t. $|f_n(x) - f(x)| < \epsilon \; \forall n \geq N$, $x \in X$.

Remark 22.1.
i) Uniform convergence is metrizable.
Let $B(X,Y) = \{f : X \to Y : f \text{ bounded}\}$

$$d(f,g) = \sup_{x\in X} d_Y\left(f(x), g(x)\right)$$

defines a metric on $B(X,Y)$ s.t. $f_n \to f$ uniformly $\iff$ $d(f_n, f) \to 0$ as $n \to \infty$. If $Y = \mathbb{R}$, then $\|f\|_n = \|f\|_\infty := \sup\{|f(x)| : x \in X\}$ is the uniform norm on $B(X,\mathbb{R})$ and $d(f,g) = \|f - g\|_\infty$ defines the uniform metric: $f_n \to f$ uniformly iff $\sup_{x\in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

ii) $(B(X,\mathbb{R}), \|\cdot\|_\infty)$ is a complete metric space.

iii) $(C_b(X,\mathbb{R}), \|\cdot\|_\infty)$ is a close subspace of $(B(X,\mathbb{R}), \|\cdot\|_\infty)$ (uniform limit theorem) where $C_b(X,\mathbb{R})$ is a continuous bounded function on X. If X is compact then $C_b(X,\mathbb{R}) = C(X,\mathbb{R})$.

Compactness in the space of functions

Example 22.2
$B = \{f \in C([0,1],\mathbb{R}) : \|f\|_\infty \leq 1\}$ closed (complete) & bounded in $C([0,1])$. Is B compact? No. Consider $f_n(x) = x^n$, f_n converges pointwise to

$$f(x) = \begin{cases} 0, & x < 1 \\ 1, & x = 1 \end{cases}$$

This implies no subsequence can converge uniformly by the uniform limit theorem. Note that B is complete so B not compact which implies B is not totally bounded. Compare to the closed unit ball in l^2. In fact, in a normed vector space such as $(C(X,\mathbb{R}), \|\cdot\|_\infty)$ or $\left(l^2, \|\cdot\|_2\right)$. Closed unit ball is compact iff the space is finite dimensional.

One may replace uniform boundedness by pointwise boundedness in Arzela-Ascoli.
Key steps in the proof:

a) Every compact space is separable.

b) Every pointwise bounded sequence on a countable set has a pointwise convergent subsequence.

c) Upgrade pointwise convergent subsequence on the countable dense set to uniformly convergence: $\{g_n\}_{n\geq 1} \subseteq C(X)$ on compact X. $\{g_n\}_{n\geq 1}$ equicontinuous and converges pointwise on X (or a countable dense subset), then g_n converges uniformly.

Criterion for equicontinuity: X compact, $\{f_n\} \subseteq C(X)$ if f_n converges uniformly then $\{f_n\}$ is equicontinuous.

§23 Dis 5: Apr 27, 2021

§23.1 Stone-Weierstrass

Definition 23.1 — $\mathcal{A} \subseteq C(X, \mathbb{R}/\mathbb{C})$ is an algebra if $\mathcal{A}$ is a vector subspace of $C(X)$ s.t. $fg \in \mathcal{A}$; separates points if $\forall x \neq y$, $\exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$; vanish nowhere if $\forall x \in X$, $\exists f \in \mathcal{A}$ s.t. $f(x) \neq 0$.

Remark 23.2. (of Stone-Weierstrass)

i) Separating points and varnishing nowhere is necessary for density.

ii) What if A vanishes somewhere? If $x_0 \in X$ s.t. $f(x_0) = 0$ $\forall f \in \mathcal{A}$, then $\overline{\mathcal{A}} = \{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$ assuming $\mathcal{A}$ still separates points.

iii) Why useful? It is often easier to prove properties/estimates for a "nice" dense subclass and extend by continuity.

iv) The Stone-Weierstrass fails for complex-valued function; e.g., $f(z) = \overline{z}$ cannot be uniformly approximated by polynomials on the unit circle $S' = \left\{e^{it} : t \in a\pi\right\}$. If $P(z) = \sum_{j=0}^{n} a_j z^j$, then

$$\int_0^{2\pi} \overline{f}(e^{it}) P(e^{it})\, dt = \int_0^{2\pi} e^{it} \sum_{j=0}^{n} a_j e^{i+j} = \sum_{j=0}^{n} a_j \int_0^{2\pi} e^{i(j+1)t}\, dt$$

$$\begin{aligned} 2\pi = \left| \int_0^{2\pi} f\overline{f}\, dt \right| &\leq \left| \int_0^{2\pi} (f-p)\overline{f}\, dt \right| + \left| \int_0^{2\pi} fp\, dt \right| \\ &\leq \int_0^{2\pi} |f-p|\ dt \leq 2\pi \|f-p\| \end{aligned}$$

$\implies \|f - p\| \geq 1$ for all polynomial p.

However, with the additional assumption of self-adjointness ($f \in \mathcal{A} \implies \overline{f} \in \mathcal{A}$) then the complex Stone-Weierstrass holds.

Application: (X, d) compact metric space, $|X| \geq 2$ then $C(X, \mathbb{R})$ is separable.

Proof. Let D be a countable dense subset of X. $D = \{x_i\}_{i \geq 1}$, $P_{x_i}(y) = d(x_i, y)$. Let $\mathcal{A} =$ subalgebra of $C(X, \mathbb{R})$ generated by $\{p_{x_i}\}_{i\geq 1}$ over

$$\mathbb{R} = \left\{ \sum_{j=1}^{n} \lambda_j P_{x_1}^{j} \dots P_{x_{k_j}}^{j} \cdot \lambda_j \in \mathbb{R},\, n, k_j \in \mathbb{N} \right\}$$

$\mathcal{A}$ separates point: $x \neq y$, $p_{x_j}(x) \neq p_{x_j}(y) \impliedby p_{x_i}(x) \subseteq p_{x_i}(y)$
$\mathcal{A}$ vanishes nowhere $x \in X$. Pick $x_i \neq x$ $p_{x_i}(x) > 0$. By Stone-Weierstrass, $\overline{\mathcal{A}} = C(X, \mathbb{R})$. Now let $\mathcal{A}_{\mathbb{R}}$ be the algebra generated by $\{p_{x_i}\}_{i\geq 1}$ over $\mathbb{Q}$. $\overline{\mathcal{A}_{\mathbb{Q}}} \supset \mathcal{A} \implies \overline{\mathcal{A}_{\mathbb{Q}}} \supset \overline{\mathcal{A}} = C(X, \mathbb{R})$, so $\overline{\mathcal{A}_{\mathbb{Q}}} = C(X, \mathbb{R})$ and check that $\mathcal{A}_{\mathbb{Q}}$ is countable. $\square$

§24 Dis 6: May 4, 2021

§24.1 Differentiation

Fermat's theorem on stationary points: $f : (a, b) \to \mathbb{R}$, $f'(x_0)$ exists and x_0 is a local extremum then $f'(x_0) = 0$.
MVT: $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$, differentiable on (a, b) then $\exists y \in (a, b)$ s.t. $f'(y) = \frac{f(b)-f(a)}{b-a}$.
$f : (a, b) \to \mathbb{R}$ differentiable

- $f' = 0 \iff f$ constant
- $f' \geq 0 \iff f$ increasing
- $f' \leq \iff f$ decreasing
- $f' > 0 \implies f$ strictly increasing
  Converse not true, e.g., $x \mapsto x^3$
- $f' < 0 \implies f$ strictly decreasing

$f : (a, b) \to \mathbb{R}$ differentiable $f'(x_0) = 0$

- if f' increasing or $f'' \geq 0$ (if exists) then x_0 is local min
- if f' decreasing or $f'' \leq 0$ (if exists) then x_0 is local max
- if f' $\nearrow\searrow$ or $\searrow\nearrow$, then x_0 is a saddle point.

$f : (a, b) \to \mathbb{R}$ differentiable then f' has the Darboux property.
e.g., any f' with f differentiable but not in C' is an example that has the Darboux property but not continuous

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

I open interval, $f : I \to \mathbb{R}$ continuous injective, $f : I \to f(I)$ is bijective. f differentiable at x_0, $f'(x_0) \neq 0 \implies f^{-1}$ differentiable at $f(x_0) = y_0$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{-1}(y_0)\right)}$$

L'Hopital Rule: f, g differentiable on (a, b), $g'(x) \neq 0$ $\forall (a, b)$, $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Suppose $\frac{f'(x)}{g'(x)} \to A$ as $x \to a^+$. If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or $g(x) \to +\infty$ as $x \to a^+$, then $\frac{f(x)}{g(x)} \to A$ as $x \to a^+$.

Example 24.1

f, g on $(0,1)$, $f(x) = x$, $g(x) = x + x^2 e^{\frac{1}{x^2}}$

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{1 + xe^{\frac{i}{x^2}}} = 1 \text{ as } \left|e^{it}\right| = 1 \quad \forall t \in \mathbb{R}$$
$$g'(x) = 1 + \left(2x - \frac{2i}{x}\right) e^{\frac{i}{x^2}}$$
$$\left|g'(x)\right| \geq \left|2x - \frac{2i}{x}\right| - 1 \geq \frac{2}{x} - 1$$

Hence

$$\left|\frac{f'(x)}{g'(x)}\right| \leq \frac{1}{\frac{2}{x} - 1} = \frac{x}{2-x}$$

$\implies \lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = 0$, note that $g'(x) \neq 0$ on $(0,1)$. L'Hopital's Rule doesn't hold for complex-valued function.

f differentiable then f is Lipschitz iff f has bounded derivative

$$\implies \left|\frac{f(x) - f(y)}{x - y}\right| \leq K \text{ for some } K > 0\,\forall x \neq y \implies \left|f'(x)\right| \leq K\,\forall x$$
$$\impliedby |f(x) - f(y)| \leq \left|f'(s)\right| |x - y| \leq M\,|x - y| \text{ if } \left|f'(x)\right| \leq M\,\forall x$$

Example 24.2

$f(x) = \sqrt{x}$ on $[0,\infty)$ not Lipschitz because $f'(x) = \frac{1}{2\sqrt{x}} \to \infty$ as $x \to 0$ but $x \mapsto \sqrt{x}$ is uniformly continuous on $[0,\infty)$.

$$\left|\sqrt{x} - \sqrt{y}\right| \leq \frac{|x-y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x-y|}{2} \text{ if } x, y \geq 1$$

So Lipschitz continuous on $[1,\infty)$. Also, $x \mapsto \sqrt{x}$ is uniformly continuous on $[0,2]$ by Heine-Borel.
For all $\epsilon > 0$, $\delta = \min\{\delta_1, \delta_2, 1\}$. Similarly, any $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{x\to\pm\infty} f(x) = 0 \implies f \text{ uniformly continuous}$$