# Math 170E – Intro to Probability

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This is math 170E taught by Professor Nguyen. The formal name of the class is **Introduction to Probability and Statistics 1: Probability**. The textbook used for the class is *Probability & Statistical Interference* $10^{th}$ by *Hogg, Tanis*. We meet weekly on MWF from 10:00 – 10:50 and on Tue at the same time frame for discussion with our TA, Jason Snyder. You can also find other lecture notes at `ductuanvu.wordpress.com/notes/`. Let me know through my email if you notice something mathematically wrong/concerning. Thank you!

## Contents

# §1 Lec 1: Oct 2, 2020

## §1.1 Properties of Probability

**Definition 1.1** — Consider the outcome of a random experiment, e.g. flipping a coin. The collection of all such outcomes, denoted by $\underbrace{S}_{\omega \text{ in other advanced prob. textbook}}$, is called the outcome space.

- A subset $A \subseteq S$ is called an event.
- If $A_1, A_2, \ldots \subseteq S$ satisfy $A_i \cap A_j = \emptyset$, $i \neq j$ then they are called "disjoint" (mutually exclusive)
- If $A_1, A_2, \ldots, A_n \subseteq$ satisfy $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = S$. Then $\{A_i\}_{i=1\ldots n}$ are called exhaustive(fully comprehensive).

**Example 1.2** 1. Flip two coins in order. Denote $H$ = head, $T$ = tail.

$$S = \{HH, HT, TH, TT\}$$
$$A = \{HH\} = \{\text{both coins are head}\}$$

$A \subseteq S$ is an event.

$$B = \{HT, TH\}$$

$B \subseteq S$ is another event.

$A \cap B = \emptyset$, they are disjoint.

2. Flip 2 coins at once.

$$S = \{HH, HT, TT\}$$
$$A = \{\text{one head, one tail}\}$$
$$A = \{HT\}\text{, is an event.}$$

<u>Probability – A heuristic intro:</u>

Consider an experiment and repeat n times. Let $N(A)$ = number of times $A$ occurs. The ratio $\frac{N(A)}{n}$ is called the relative frequency of A in n repetitions of the experiment.

$$0 \leq \frac{N(A)}{n} \leq 1$$

As $n \to \infty$,

$$\frac{N(A)}{n} \to p \in [0, 1]$$

This $p$ is called the prob. that event A occurs.

**Example 1.3**

(a) Flip a coin

$$S = \{H, T\}$$
$$A = \{H\}$$

What is $P(A)$?

(b) Sometimes, we can also assign prob. based on the nature of the event Pick a random point in the unit circle.

$$A = \left\{\text{chosen point} \in 1^{\text{st}}\text{quadrant}\right\}$$

$P(A) = \frac{\text{Area of first quadrant}}{\text{Area of unit circle}} = \frac{1}{4}$

(c) Pick a number randomly from $\{0, 1, \ldots, 9\}$, $B = \{2 \text{ is picked}\}$

$$P(B) = \frac{1}{10}$$

Table 1: From example 1.3 (a)

| $n$ | $N(A)$ | $\frac{N(A)}{n}$ |
|---|---|---|
| 50 | 37 | .74 |
| 500 | 333 | .66 |

It is safe to assign $P(A) = 0.66$

**Definition 1.4** — Given an outcome space S, the probability of an event A $A \subseteq S$, is a number satisfying:

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $A_1, \ldots, A_n \subseteq S$ are disjoint events, i.e. $A_i \cap A_j = \emptyset, i \neq j$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) = P(A_1) + \ldots + P(A_n)$$

More generally, if $A_1, \ldots, A_n, \ldots \subseteq S$ are disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

**Theorem 1.5** 1. Denote $A'$ to be the complement of A in S, i.e.

$$A' \cup A = S$$
$$A' \cap A = \emptyset$$

Then

$$P(A') = 1 - P(A)$$

2. $P(\emptyset) = 0$
3. If $A \leq B$ then $P(A) \leq P(B)$
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
5. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

*Note*: The pattern here is add the prob. of odd event(s) and substract the prob. of even events.(for prop (4) and (5) of theorem 1.5).

*Proof.*

$$P(A') = 1 - P(A)$$

Since $A' \cap A = \emptyset$ (by def of $A'$ ). By property (c),

$$P(\underbrace{A' \cup A}_{S}) = P(A') + P(A)$$
$$\underbrace{P(S)}_{1\text{(by prop.(b))}} = P(A') + P(A)$$

Thus,

$$P(A') = 1 - P(A)$$

# §2 Lec 2: Oct 5, 2020

Cont'd of Lec 1
(2)

$$\begin{aligned} P(\emptyset) &= 1 - P(S) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

(3)

$$P(A) \leq P(B)$$

$B \setminus A$ is the set s.t.

$$A \cup (B \setminus A) = B$$
$$A \cap (B \setminus A) = \emptyset$$
$$\text{something here}$$

implying

$$P(A) \le P(B)$$

(4)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(5) Textbook Section 1.1. $\square$

**Definition 2.1** — Suppose $S = \{e_1, \ldots, e_m\}$ where each $e_i$ is a possible outcome. Denote $n(s)$ = number of outcomes $= m$. If each $e_i$ has the same prob. of occurring, then they are called equally likely. In particular,

$$P(e_i) = \frac{1}{n(s)} = \frac{1}{m}$$

Moreover, if $A \subseteq S$ is an event s.t. $n(A) = k$. Then,

$$P(A) = \frac{n(A)}{n(s)} = \frac{k}{m}$$

**Example 2.2**

Draw one card from a deck of 52 cards.

$$P(\text{each card is drawn}) = \frac{1}{52}$$

$A = \{\text{a king is drawn}\}$, so $n(A) = 4$. Thus,

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}$$

## §2.1 Method of Enumeration

**Multiplication Principle**:

Suppose an experiment $E_1$ has $n_1$ outcomes

- For each outcome from $E_1$, a $2^{\text{nd}}$ experiment $E_2$ has $n_2$ outcomes. Then the composite $E_1E_2$ has $n_1 \cdot n_2$ outcomes.

**Permutation of size n**:

**Definition 2.3** — Suppose there are $n$ positions to be filled by $n$ persons. One such arrangement is called a permutation of size n.
FACT: the total number of different such arrangements is given by "$n! = 1 \cdot 2 \cdot 3 \cdot \ldots n$"

*Proof.*

- $E_1$ = fill the $1^{\text{st}}$ position from n persons $\implies$ $n$ outcomes for $E_1$.

- $E_2$ = fill the $2^{\text{nd}}$ pos. from $n-1$ persons left $\implies$ $n-1$ outcomes for $E_2$

  $\vdots$

- $E_n$ = fill the $n^{\text{th}}$ pos. from 1 person left $\implies$ 1 outcome for $E_n$

- One arrangement $= E_1E_2\dots E_n$
  Thus, total number of arrangements is $n!$. $\square$

**Permutation/Combination of n objects taken k**:

**Definition 2.4** — Given $k \le n$ and suppose there are $n$ objects. If $k$ objects are taken from n with/without order, then such a selection is called permutation/combination of size $n$ taken k.

*Note*:"Permutation of size n" = "permutation of size n taken n".

**Fact 2.1.** 1. The total number of permutation $n$ taken $k$ (order is important here) is denoted by ${}^nP_k$ is given by

$$ {}^nP_k = \frac{n!}{(n-k)!} $$

2. The total numbers of combination of $n$ taken $k$, denoted by ${}^nC_k$ or $\binom{n}{k}$ is given by

$$ {}^nC_k = \binom{n}{k} = \frac{n!}{(n-k)!k!} $$

*Proof.* $E_1$ = fill $1^{\text{st}}$ pos. from n $\implies$ $n$ for $E_1$

$\vdots$

$E_k$ = fill $k^{\text{th}}$ pos. from $n-k+1$ persons left. Thus,

$$ permk = n \cdot \ldots \cdot (n-k+1) $$

(2) Combination of $n$ taken $k$ :
Start with ${}_nP_k$ as follow:

- $E_1$ = take $k$ from $n$ at once, outcome $=_n C_k = \binom{n}{k}$

- $E_2$ = permute k, outcomes $= k!$. Thus,

$$ {}^nP_k = \binom{n}{k} \cdot k! $$

  implying

$$ \binom{n}{k} = \frac{{}^nP_k}{k!} = \frac{n!}{(n-k)!k!} $$ $\square$

Practice 1: https://ccle.ucla.edu/pluginfile.php/3766550/mod_resource/content/1/Practice%201.pdf
1. Consider $S = \{1, \dots, 8\}$
a)

- $E_1$ = filling $1^{\text{st}}$ pos $\implies$ 8 choices.
- Same for $E_2 \implies$ 8 choices.
- Likewise, $E_3$ has 8 choices.

Thus, the number of 3 digit numbers can be formed is $8^3$
b) "3 distinct digit numbers" = "permutation of size 8 taken 3"
Thus, total such numbers is ${}_8P_3 = \frac{8!}{5!} = 8 \cdot 7 \cdot 6$
c) Considering subset where order is not taken into account
Combination of size 8 taken 3. Thus, the answer is

$$\binom{8}{3} = \frac{8!}{3!5!}$$

d) 3 digit numbers and divisible by 5

- $E_1$ = choose 5 for the $3^{\text{rd}}$ pos, so 1 choice.
- $E_2$ = 8 choices
- $E_3$ = 8 choices

Thus, the total of choices is $8 \cdot 8 = 64$.
e) 4 element subsets of S that has one even digit.

- $E_1$ = choose one even digit from $S$, so 4 choices (2,4,6,8).
- $E_2$ = choose 3 digits from $\{1, 3, 5, 7\}$ without order, so $\binom{4}{3}$

Thus, total $= E_1 \cdot E_2 = 4 \cdot \binom{4}{3}$.
e') What if "at least one even digit" instead of "exactly one even"?

1. Total = exactly "one even" + "two even" + "three even" + "four even"
2. Total = "4-element subset" - "4-element subset with no even digit"

# §3 Lec 3: Oct 7, 2020

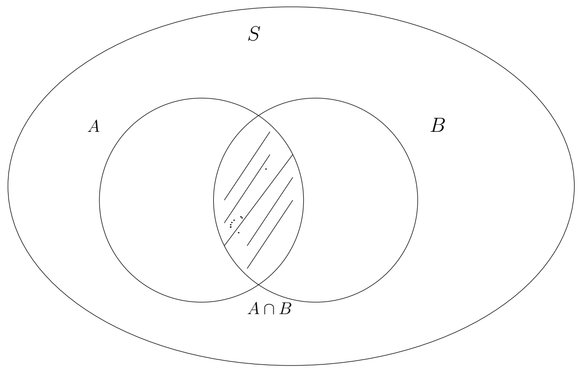
## §3.1 Conditional Probability

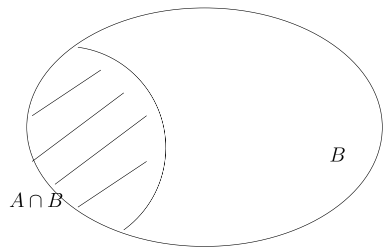
**Definition 3.1** — Let $A, B \subseteq S$ be two events. The conditional prob. of $A$, given that $B$ has occurred with $P(B) > 0$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

A heuristic explanation: $A \cap B$: "the portion in $B$ that A occurs"

$$P(A|B) = \frac{\text{"area of A in B"}}{\text{"area of B"}}$$




**Example 3.2**

Suppose my family has two kids. Given that there is at least a boy, what is the prob. my family has two boys?

$$S = \{\text{bb,bg,gb,gg}\}$$

Now, let $B = \{$ at least a boy$\}$. So we only look at the first three outcomes from $S$ $(B)$. Define $A = \{$ two boys$\}$

$$A \cap B = \{\text{bb}\}$$

Note $A = A \cap B$ since $A \subseteq B$. Thus,

$$P(A|B) = \frac{P(A \cap B}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

*Note*: We can also consider the alternative outcome space without order as follows

$$S = \left\{(b,b) - -\frac{1}{4}, (b,g) - -\frac{1}{2}, (g,g) - -\frac{1}{4}\right\}$$

**Fact 3.1.** $P(A|B)$ satisfies basic properties of probability:

- $P(A|B) \geq 0$
- $P(B|B) = 1$

  Moreover, if $B \leq C$ then

$$P(C|B) = 1$$

- If $A_1, \ldots, A_n \ldots$ are disjoint events,

$$P(\bigcup_{k=1}^{\infty} A_k|B) = \sum_{k=1}^{\infty} P(A_k|B)$$

*Proof.* (a) $P(A|B) = \frac{P(A\cap B)}{P(B)} \geq 0$
(b) $P(B|B) = \frac{P(B\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$
If $B \subseteq C$ then $B \cap C = B$

$$P(C|B) = \frac{P(B\cap C)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

$B \subseteq C$ means "if B occurs then C must occur".
(c) $P(\bigcup_{\infty}^{k=1} A_k|B) = \frac{P(\bigcup_{\infty}^{k=1} A_k \cap B}{P(B)}$. By distributive law,

$$\begin{aligned}
&= \frac{P(\bigcup_{\infty}^{k=1}(A_k\cap B))}{P(B)} \\
&= \frac{\sum_{k=1}^{\infty} P(A_k \cap B}{P(B)} \\
&= \sum_{k=1}^{\infty} P(A_k|B)
\end{aligned}$$

□

*INSERT: PRACTICE 1 #3 here*

**Theorem 3.3** 1. $P(A\cap B) = P(A|B)\cdot P(B)$ given that $P(B) > 0$

2. $P(A\cap B\cap C) = P(A)\cdot P(B|A)\cdot P(C|A\cap B)$ given $P(A), P(A\cap B) > 0$.

*Proof.* 1. By defn of cond. prob.

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

implying

$$P(B)P(A|B) = P(A\cap B)$$

2. $P(A\cap B\cap C) = P(C\cap(A\cap B)$. By part 1,

$$\begin{aligned}
&= P(C|A\cap B)P(A\cap B)P(A\cap B) \\
&= P(C|A\cap B)P(B|A)P(A)
\end{aligned}$$

□

**Practice 3.1.** The url: https://ccle.ucla.edu/pluginfile.php/3776692/mod_resource/content/0/Practice%202.pdf
*INSERT: Look at the online notes*

# §4 Lec 4: Oct 9, 2020

Cont'd (Practice)
3)

$$A = \{\text{spade}\} \quad B = \{\text{heart}\} \quad C = \{\text{diamond}\} \quad D = \{\text{club}\}$$

$P = (A \cap B \cap C \cap D =?$ So,

$$= P(A)P(B|A)P(C|A \cap B)P(D|A \cap B \cap C)$$

(from problem 2 in practice 2)

- $P(A) = \frac{13}{52}$
- $P(B|A) =$, now restricted to outcome space {51 cards in cluding 13 hearts} $B|A =$ { dealing a heart}. Thus,
$$P(B|A) = \frac{13}{51}$$
- Similarly,
$$P(C|A \cap B) = \frac{13}{50}$$
(13 diamond from 50 cards left)
- $P(D|A \cap B \cap C) = \frac{13}{49}$ (13 clubs from 49 cards left).

Hence,

$$P(A \cap B \cap C \cap D) = \frac{13}{52}\frac{13}{51}\frac{13}{50}\frac{13}{49}$$

## §4.1 Independent Events

**Example 4.1**

Flip a fair coin twice

$$\begin{aligned} S &= \{\text{ HH, HT , TH, TT}\} \\ A &= \left\{1^{\text{st}H}\right\} \\ B &= \left\{2^{\text{nd}}T\right\} \\ C &= \{\text{TT}\} \end{aligned}$$

$C \subseteq B$ "2 tails" $\implies$ "2nd is T". i.e., if C occurs then B must have occurred. Thus,

$$\begin{aligned} P(B|C) &= 1 \\ P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{\frac{1}{4}}{\frac{1}{2}} \\ &= \frac{1}{2} \\ P(A) &= \frac{1}{2} \end{aligned}$$

Thus, $P(A|B) = P(A)$, i.e., B occurring does not impact the occurrence of A.

Note also that

$$\frac{P(A \cap B)}{P(B)} = P(A|B) = P(A)$$

implying

$$P(A \cap B) = P(A)P(B)$$

**Definition 4.2** — Given two events $A, B$ which are called independents iff

$$P(A \cap B) = P(A)P(B)$$

**Theorem 4.3**

The following are equivalent

- $A, B$ are independent
- $P(A|B) = P(B)$, provided $P(B) > 0$
- $P(B|A) = P(B)$, provided $P(A) > 0$

*Proof.* Left as an excercise. □

**Theorem 4.4** 1. If $P(A) = 0$ then $A$ is independent with any event.

2. If $A$ and $B$ are independent then so are the following pairs:

$$A, B' \quad A', B \quad A', B'$$

*Proof.* 1. Let $B$ an arbitrary event, we need to show $P(A \cap B) = P(A)P(B)$. Since $P(A) = 0$, $P(A)P(B) = 0$.

$$A \cap B \subseteq A$$

imply

$$0 \le P(A \cap B) \le P(A) = 0$$

thus $P(A \cap B) = 0$.

2. Textbook(section 1.5)

$\square$

**Practice 4.1.** Practice 2 – Problem 4:
Let's consider $C$ and $D$ first

$$\begin{aligned} D &= \{ \text{ sum of two rolls } = 12\} \\ &= \{(6,6)\} \end{aligned}$$

Thus, $D \subsetneq C = \{\text{first roll is } 6\}$. Hence, $C$ and $D$ are dependent.
A v.s. B

$$\begin{aligned} P(A) &= \frac{5}{6} \\ B &= \{ \text{ sum is even}\} \\ &= \{ \text{ first and second roll are even}\} \cup \{\text{first and second roll are odd}\} \\ P(B) &= P(\text{first even})P(\text{second even}) + P(\text{first odd})P(\text{second odd}) \\ &= \frac{3}{6}\frac{3}{6} + \frac{3}{6}\frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

Now, consider $A \cap B = \left\{1^{\text{st}} \neq 3, \text{ sum is even}\right\}$. So,

$$\begin{aligned} A \cap B &= \left\{1^{\text{st}} \neq 3, 1^{\text{st}} \text{ odd}, 2^{\text{nd}} \text{ odd}\right\} \cup \left\{1^{\text{st}} \neq 3, 1^{\text{st}} \text{ even}, 2^{\text{nd}} \text{ even}\right\} \\ P(A \cap B) &= P(1^{\text{st}} \neq, 1^{\text{st}} \text{ odd})P(2^{\text{nd}} \text{ odd}) + P(1^{\text{st}} \neq 3, 1^{\text{st}} \text{ even})P(2^{\text{nd}} \text{ even}) \\ &= \frac{2}{6}\frac{3}{6} + \frac{3}{6}\frac{3}{6} \\ &= \frac{5}{12} \end{aligned}$$

Since $P(A \cap B) = \frac{5}{12} = \frac{5}{6}\frac{1}{2} = P(A)P(B)$, $A$ and $B$ are independent.

# §5 Lec 5: Oct 12, 2020

## §5.1 Independent Events (cont'd)

**Definition 5.1** — $A, B, C$ are called "mutually independent" if followings hold:

- pairwise independent

$$P(A \cap B) = P(A)P(B) \quad P(B \cap C) = P(B)P(C) \quad P(A \cap C) = P(A)P(C)$$

- "triple" wise independent, i.e.,

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

*Note*:analogous defn holds for $A_1, \dots, A_n, \dots$ in which any pairs, triple, quadruple and so on must satisfy the similarly multiplication rules. Usually, the term "mutually" is dropped but it is understood that "independence" means "mutually independence".

> **Remark 5.2.** In general, pairwise independence does not imply triple-wise independence.

**Practice 5.1.** 2 – Problem 5:

$$A = \{1, 2\}, \quad B = \{1, 3\}, \quad C = \{1, 4\}$$

$P(A) = \frac{2}{4} = P(B) = P(C)$

$$A \cap B = \{1\} = B \cap C = A \cap C$$
$$P(A \cap B) = P(B \cap C) = P(C \cap A) = \frac{1}{4}$$

Thus,

$$P(A \cap B) = \frac{1}{4} = P(A)P(B)$$

Same for $B, C$ and $A, C$ – so pairwise independent.
Triple:

$$A \cap B \cap C = \{1\}$$

$P(A \cap B \cap C) = \frac{1}{4}$; on the other hand, $P(A)P(B)P(C) = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$. They are not equal! Therefore, $A, B, C$ are not mutually independent.

## §5.2 Bayes's Theorem

**Definition 5.3** — The events $B_1, \dots, B_n$ ($n$ may be finite or $\infty$ ) are called a partition of the outcome space $S$ if followings hold

- disjoint: $B_i \cap B_k = \emptyset, i \neq k$
- exhausted: $\bigcup_n^{i=1} B_i = S$

then,

$$P(B_1) + \ldots + P(B_n) = P(S) = 1$$

**Theorem 5.4** (Law of total Probability)
Suppose $B_1, \dots, B_n$ is a partition of $S$ with $P(B_i) > 0$ for $i = 1, \dots, n$. If $A$ is an event in $S$, then

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

where $P(B_i)$ is called the prior probability.

*Proof.* (sketch)

$$\begin{aligned} P(A) &= P(\bigcup_{n}^{i=1}(A \cap B_i)) \\ &= \sum_{i=1}^{n} P(A \cap B_i) \\ &= \sum_{i=1}^{n} P(A|B_i)P(B_i) \end{aligned}$$

□

**Practice 5.2.** 3 – problem 1:

$$\begin{aligned} P(I) &= .35 \\ P(II) &= .25 \\ P(III) &= .4 \end{aligned}$$

$A = \{$ a spring is defective$\}$, $P(A) =?$ We know

$$\begin{aligned} P(A|I) &= .02 \\ P(A|II) &= .01 \\ P(A|III) &= .03 \end{aligned}$$

By law of total prob:

$$\begin{aligned} P(A) &= P(A|I)P(I) + P(A|II)P(II) + P(A|III)P(III) \\ &= 0.0215 \end{aligned}$$

**Theorem 5.5** (Bayes's Theorem)
Suppose $\{B_i\}_{i=1,\dots,n}$ is a partition of $S$ with $P(B_i) > 0$. If $A$ with $P(A) > 0$, then for all $i = 1, \dots, n$

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{k=1}^{n} P(A|B_k)P(B_k)}$$

where $P(B_i|A)$ is called posterior probability.

*Proof.*

$$\begin{aligned} P(B_i|A) &= \frac{P(B_i \cap A)}{P(A)} \\ &= \frac{P(A \cap B_i)}{P(A)} \\ &= \frac{P(A|B_i)P(B_i)}{P(A)} \\ &= \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + \ldots + P(A|B_n)P(B_n)} \end{aligned}$$

□

**Practice 5.3.** 3 – problem 2: $A = \{\text{ person has disease }\}, \quad P(A) = .005.$

$$\begin{aligned} + &= \{\text{test } +\} \\ - &= \{\text{ test } -\} \\ P(+|A) &= .99 \\ P(\underbrace{+|A'}_{\text{false positive}}) &= .03 \\ P(A|+) &= ? \end{aligned}$$

By Bayes's Theorem:

$$\begin{aligned} P(A|+) &= \frac{P(+|A)P(A)}{P(+|A)P(A) + P(+|A')P(A')} \\ &= \frac{(.99)(.005)}{(.99)(.005) + (.03)(.995)} \end{aligned}$$

$\{A, A'\}$ is a partition of $S$.

# §6 Lec 6: Oct 14, 2020

**Practice 6.1.** 3 – Problem 3: Trial: know at least 1 girl

$$P(\text{GG}|\text{at least a girl}) = \frac{1}{3}$$

However, the above approach is not correct.
Intuition: The moment the girl opens the door, the first child's gender is determined – which makes the other kid's gender is now independent of the girl. Thus, $P(\text{other kid is girl}) = \frac{1}{2}$.
Correct approach:

$$\begin{aligned} A &= \{\text{ a girl opens the door}\} \\ P(GG|A) &= ? \end{aligned}$$

- $P(A|GG) = 1$
- $P(A|BB) = 0$
- $P(A|GB) = \frac{1}{2}$

- $P(A|BG) = \frac{1}{2}$

By Bayes' Theorem

$$P(GG|A) = \frac{P(A|GG)P(GG)}{P(A|GG)P(GG) + P(A|BB)P(BB) + P(A|BG)P(BG) + P(A|GB)P(GB)} = \frac{1}{2}$$

## §6.1 Random Variables with Discrete Type

**Example 6.1**

Flip a coin

$$S = \{\text{H, T}\}$$

Define

$$X : S \to \mathbb{R}$$

$$\triangle \mapsto X(s) \in \mathbb{R}$$

s.t. $X(H) = 0, \quad X(T) = 1$

$$\text{H} \xrightarrow{X} 0$$

$$\text{T} \longrightarrow 1$$

The function $X$ is called a random variable (RV). Since $S$ is discrete space, $X$ is called a RV of discrete-type.

**Definition 6.2** — Given an outcome space $S$, a function $X$ that assigns $X(s) = x \in \mathbb{R}$ for each $s \in S$ is called a random variable.
The space(range) of $X$ is the collection of real numbers, denoted by $S_x$,

$$S_x = \{x \in \mathbb{R} : \quad \exists s \in S, X(s) = x\}$$

$S_x$ is also called the "support" of $X$.
When the outcome space $S$ is discrete, then $X$ is called a discrete random variable.

Example above:

$$S_x = \{0, 1\}$$

*Note*: the space of $X$ is denoted by $S$ in the textbook. Here we will use $S_x$.

**Remark 6.3.** Under the above definition, for $x \in S_x$,

$$P(X = x) = P(\{s \in S : X(s) = x\})$$

**Example 6.4**

Roll a fair dice

$$
\begin{aligned}
S &= \{1, 2, \ldots, 6\} \\
X &: S \to \mathbb{R} \\
s &\mapsto X(s) = x \\
S_x &= \{1, 2, \ldots, 6\} \, (= S)
\end{aligned}
$$

For each $k \in S_x$,

$$P(X = k) = P(\{k\}) = \frac{1}{6}$$

Also,

$$\sum_{k \in S_x} P(X = k) = \sum_{k=1}^{6} \frac{1}{6} = 1$$

**Definition 6.5** (Probability Mass Function) — The probability mass function (pmf) $f(x)$ of a discrete random variable $X$ is a function satisfying the followings:

- $f(x) > 0, \quad x \in S_x$.
- $\sum_{x \in S_x} f(x) = 1$.
- If $A \subseteq S_x$,

$$P(X \in A) = \sum_{x \in A} f(x)$$

<u>*Note*</u>: if $x \notin S_x$, then we assign $f(x) = 0 (P(X = x) = 0)$.

**Example 6.6** (above)

the pmf of $X$ is given by $f(k) = \frac{1}{6}$ for $k = 1, \ldots, 6$

$$
\begin{aligned}
A = \{1, 2, 3\} &= \text{“}X < 4\text{”} \\
A &\subseteq S_x
\end{aligned}
$$

$$P(X \in A) = \sum_{k \in A} f(k) = \sum_{k=1}^{3} \frac{1}{6} = \frac{1}{2}$$

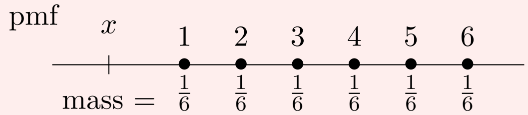
**Definition 6.7** — Cumulative distribution function (cdf) $F(x)$ of a RV $x$ is a function given by

$$F(x) = P(X \leq x), \quad -\infty < x < \infty$$

<u>*Note*</u>: $F(x)$ is usually called distribution function, "cumulative" is dropped.

**Example 6.8**

Rolling a fair dice



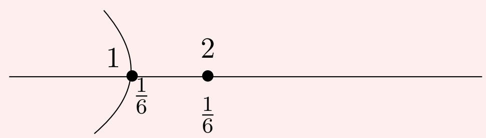

$$\begin{aligned}\text{cdf } F(x) &= P(X \le x) \\ &= \text{total mass cumulated starting from the left up to x}\end{aligned}$$

$x < 1$,

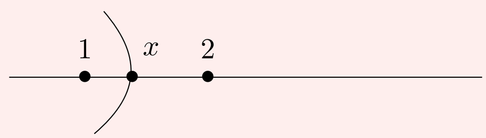
$$\begin{aligned}F(x) &= P(X \le x) \\ &= 0 \text{ (no mass up to } x < 1)\end{aligned}$$

$x = 1$,

$$F(1) = P(X \le 1)$$

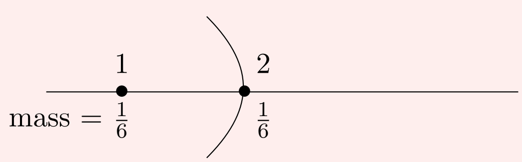



$F(1) = \frac{1}{6}$ (mass up to and including location 1).
$1 < x < 2$




$$\begin{aligned}F(x) &= P(X \le 1) \\ &= P(X = 1) \\ &= \frac{1}{6}\end{aligned}$$

$x = 2$



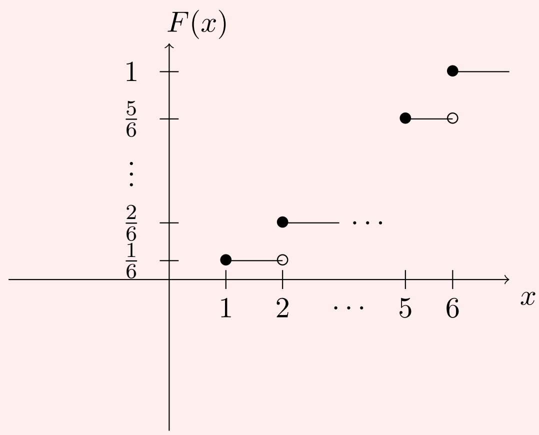

$$\begin{aligned}F(2) &= P(X \le 2) \\ &= P(X = 1) + P(X = 2) \\ &= \frac{2}{6}\end{aligned}$$

Likewise, $2 < x < 3$

$$F(x) = \frac{2}{6}$$

$$\vdots x = 6, \quad F(X) = P(X \le 6) = 1$$

$x > 6, F(x) = 1$

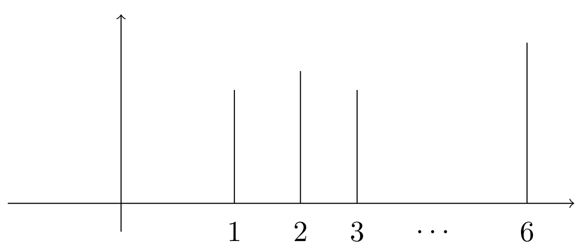



# §7 Lec 7: Oct 16, 2020

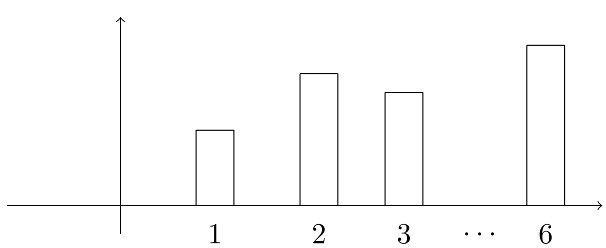
## §7.1 Lec 6 (Cont'd)

In order to graph the prob. mass function:

- Line graph




- Histogram




**Practice 7.1.** 4 – Problem 1:

$$X = \text{ max of two rolls}$$
$$S_X = \{1, 2, \ldots, 6\}$$

For $k \in S_X$. Determine $f(k) = P(X = k) = ?$

- $1^{\text{st}}$ approach:

| $1^{st}$ roll \ $2^{nd}$ roll | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | $(1,1)$ | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(1,5)$ | $(1,6)$ |
| 2 | $(2,1)$ | $(2,2)$ | $(2,3)$ | $(2,4)$ | $(2,5)$ | $(2,6)$ |
| 3 | $(3,1)$ | $(3,2)$ | $(3,3)$ | $(3,4)$ | $(3,5)$ | $(3,6)$ |
| $\vdots$ | | | | | | |
| 6 | $(6,1)$ | $(6,2)$ | $\cdots$ | | | |

| $1^{st}$ roll \ $2^{nd}$ roll | 1 | 2 | 3 | $\cdots$ | 6 |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | $\cdots$ | 6 |
| 2 | 2 | 2 | 3 | $\cdots$ | 6 |
| 3 | 3 | 3 | 3 | $\cdots$ | 6 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| 6 | 6 | 6 | 6 | $\cdots$ | 6 |

$$
\begin{aligned}
f(1) &= P(X = 1) = \frac{1}{36} \\
f(2) &= P(X = 2) = \frac{3}{36} \\
f(3) &= P(X = 3) = \frac{5}{36} \\
&\vdots \\
f(6) &= P(X = 6) = \frac{11}{36}
\end{aligned}
$$

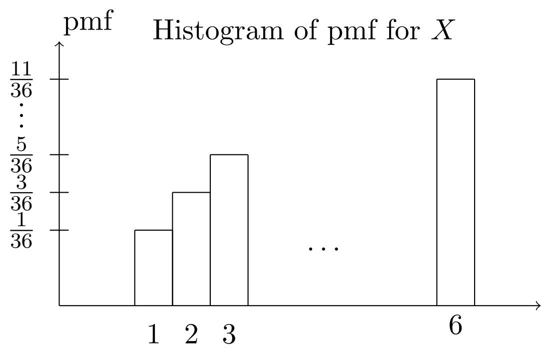
- $2^{\text{nd}}$ approach: for $k = 1, \ldots, 6$ (disjoint sub-events)

$$
\begin{aligned}
\{X = k\} &= \{\max\ = k\} \\
&= \left\{1^{\text{st}}\text{roll} = k, 2^{\text{nd}}\ < k\right\} \\
&\cup \left\{1^{\text{st}}\text{roll}\ < k, 2^{\text{nd}} = k\right\} \\
&\cup \left\{1^{\text{st}}\ \text{roll} = 2^{\text{nd}} = k\right\}
\end{aligned}
$$

Thus,

$$\begin{aligned}P(X=k) &= P(1^{\text{st}} \text{ roll } = k)P(2^{\text{nd}} < k) + P(1^{\text{st}} < k)P(2^{\text{nd}} = k) + P(1^{\text{st}} = k)P(2^{\text{nd}} = k)\\ &= \frac{1}{6}\frac{k-1}{6} + \frac{k-1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6}\\ &= \frac{2k-1}{36}\end{aligned}$$

<u>*Note*</u>: $\sum_{k=1}^{6} \frac{2k-1}{36} = 1$.




Similarly, we can calculate $Y =$ min of 2 rolls.

**Remark 7.1.** Suppose $X = \max\{U, V\}$ where $U, V$ are 2 discrete random variables. Then pmf of $X$ can be calculated as follows:

$$\begin{aligned}f(k) &= P(X=k)\\ &= P(U=k, V<k) + P(U<k, V=k) + P(U=k, V=k)\end{aligned}$$

and we can often use indep. on each of the above events. On the other hand, for $Y = \min\{U, V\}$ then

$$P(Y=k) = P(U=k, V>k) + P(U>k, V=k) + P(U=k, V=k)$$

and use indep. on the above events.

## §7.2 Expectation & Special Math Expectations

**Definition 7.2** — Suppose $X$ is a discrete random variable with $S_X$, pmf $f(x)$. Let $u(x)$ be a function, then if the sum $\sum_{x\in S_X} u(x)f(x)$ exists (finite) then the sum is mathematical expectation (expected value) of $u(X)$ and is denoted by

$$E[u(X)] \coloneqq \sum_{x\in S_X} u(x)f(x)$$

**Practice 7.2.** 5 – Problem 1: $S_X = \{1, \ldots, 6\}$. For $x \in S_X$, $u(x) = x - 3.5$

$$\begin{aligned} \text{average income} &= E[u(x)] \\ &= \sum_{x \in S_X} u(x) f(x) \\ &= \sum_{k=1}^{6} (k - 3.5) \cdot \frac{1}{6} \\ &= 0 \end{aligned}$$

"After one game, on average, I do not gain/lose any money."

**Theorem 7.3**

When it exists, the expectation $E$ satisfies:

- If $c$ is a constant, then
$$E[c] = c$$
- If $c$ is a constant and $u(X)$ is a function, then
$$E[c \cdot u(X)] = cE[u(X)]$$
- If $c_1, c_2$ are constants and $u_1(X), u_2(X)$ are functions.
$$E[c_1 u_1(X) + c_2 u_2(X)] = c_1 E[u_1(X)] + c_2 E[u_2(X)]$$

**Remark 7.4.** Part (c) can be generalized for 2 discrete random variables $X, Y$.

$$E[c_1 u_1(X) + c_2 u_2(Y)] = c_1 E[u_1(X)] + c_2 E[u_2(Y)]$$

*Proof.* Textbook. □

**Definition 7.5** — For a random variable $X$,

- the mean (of $X$ ) is denoted by
$$u := E[x]$$
- the variance (of $X$ ) is denoted by
$$\sigma^2 := E[(x - \mu)^2]$$
- the standard deviation
$$\sigma := \sqrt{\sigma^2}$$

**Example 7.6**

Suppose $X$ has pmf

| $x$ | $-2$ | $0$ | $1$ |
|---|---|---|---|
| $f(x)$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{6}$ |

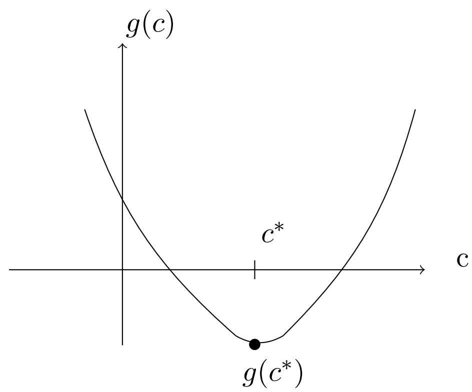
$$\begin{aligned}
\text{mean} = \mu &= E[x] \\
&= \sum_{x \in S_X} x \cdot f(x) \\
&= (-2)\frac{1}{2} + 0\frac{1}{3}1\frac{1}{6} \\
&= -\frac{5}{6} \\
\text{variance} = \sigma^2 &= E[(x-\mu)^2] \\
&= \sum_{x \in S_X} (x-\mu)^2 f(x) \\
&= (-2-(-\frac{5}{6})^2\frac{1}{2} + (0-(-\frac{5}{6}))^2\frac{1}{3} + \dots
\end{aligned}$$

$\sigma^2$ intepretation:

For a constant $c \in \mathbb{R}$, define $g(c) \coloneqq E[(x-c)^2]$. Note that

$$\begin{aligned}
g(c) &= E[(X-c)^2] \\
&= E[X^2 - 2cX + c^2] \\
&= E[X^2] + E[-2cX] + E[c^2] \\
&= E[X^2] - 2cE[X] + c^2 \\
&= c^2 - 2cE[X] \cdot + E[X^2] \\
&= c^2 - 2\mu \cdot c + E[X^2]
\end{aligned}$$

"$u$ and $E[X^2]$ are constant with respect to c".



$g(c^*) = \min g(c)$ where $c^*$ satisfies

$$\begin{aligned} g'(c^*) &= 0 \\ g'(x) &= 2c - 2\mu \end{aligned}$$

Thus

$$g'(c^*) = 0 = 2c^* - 2\mu$$

i.e., $c^* = \mu$. Hence,

$$\sigma^2 = E[(x-\mu)^2] = g(\mu)$$

minimizes $g(c) = E[(x-c)^2]$, i.e.,

$$\sigma^2 = \underbrace{\min}_{c\in\mathbb{R}} E[(x-c)^2] = E[(x-\mu)^2]$$

"$\sigma^2$ measures fluctuation of $X$ around its mean $\mu$."

# §8 Lec 8: Oct 19, 2020

## §8.1 Info about $1^{\text{st}}$ midterm

$1^{\text{st}}$ Midterm 11/2, Monday, 10am PT. Due: 10am PT – Tuesday 11/3.
$2^{\text{nd}}$ Midterm, after Thanksgiving.

## §8.2 Lec 7 (Cont'd)

Review geometric series: for $|q| < 1$,

$$\sum_{k=0}^{\infty} q^k = 1 + q + q^2 + \ldots = \frac{1}{1-q}$$

Differentiating both sides,

$$\sum_{k=1}^{\infty} kq^{k-1} = 1 + 2q + 3q^2 + \ldots = \frac{1}{(1-q)^2}$$

**Practice 8.1.** 5 – Problem 2:
$S_X = \{1, 2, \ldots\}$. The pmf $f(f) = P(X = k) = P(1^{\text{st}}$ k-1 shots are missed and k shot succcessful.
a) $E[X] = ?$

$$\begin{aligned} A_k &= \left\{k^{\text{th}} \text{ shot is successful}\right\} \\ P(A_k) &= p \\ P(A_k') &= 1 - q = P\left(\left\{k^{\text{th}} \text{ shot is missed}\right\}\right) \end{aligned}$$

$$
\begin{aligned}
P(X=k) &= P\left(\underbrace{A_1' \cap A_2' \cap \ldots \cap A_{k-1}'}_{\text{miss1}^{\text{st}}\ k-1 \text{ shots}} \cap \underbrace{A_k}_{\text{make at}k^{\text{th}} \text{ shots}}\right) \\
&\overset{\text{independence}}{=} P(A_1')P(A_2')\ldots P(A_{k-1}')P(A_k) \\
&= q \cdot q \ldots q \cdot p \\
&= q^{k-1} \cdot p
\end{aligned}
$$

for each $k = 1, 2, 3, \ldots$. Note that pmf $f(k) = P(X = k)$ indeed satisfies:

$$
\begin{aligned}
\sum_{k=1}^{\infty} f(k) &= \sum_{k=1}^{\infty} q^{k-1} \cdot p \\
&= p\left(1 + q + q^2 + \ldots\right) \\
&= p \cdot \frac{1}{1-q} \\
&= p \cdot \frac{1}{p} \\
&= 1
\end{aligned}
$$

Now,

$$
\begin{aligned}
\mu = E[x] &= \sum_{x \in S_X} x f(x) \\
&= \sum_{k=1}^{\infty} k \cdot f(k) \\
&= \sum_{k=1}^{\infty} k \cdot q^{k-1} \cdot p \\
&= p \sum_{k=1}^{\infty} k \cdot q^{k-1} \\
&= p \cdot (1 + 2q + 3q^2 + \ldots) \\
&= p \cdot \frac{1}{(1-q)^2} \\
&= p \cdot \frac{1}{p^2} \\
&= \frac{1}{p}
\end{aligned}
$$

**Definition 8.1** (moment generating function) — Given a discrete RV $X$ and $\delta_X$ and pmf $f(x)$, if $\exists$ a positive constant $h$ s.t. for all $t \in (-h, h)$, the following expectation function

$$
E[e^{tX}] = \sum_{x \in S_X} e^{tx} f(x)
$$

exists then $E[e^{tx}]$ is called the mgf of $X$ and is denoted by $M_X(t)$.

*Note*: $(-h, h)$ needs not be a symmetric interval. But it has to contain the origin 0.

**Example 8.2**

Suppose $X$ has the following pmf,

| x | $-2$ | $0$ | $1$ |
|---|---|---|---|
| $f(x)$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{6}$ |

$$\begin{aligned} E[e^{tX}] = M_X(t) &= \sum_{x \in S_X} e^{tx} f(x) \\ &= \frac{1}{2}e^{-2t} + \frac{1}{3} + \frac{1}{6}e^t \end{aligned}$$

which is finite for all $t \in \mathbb{R}$.

**Theorem 8.3**

MGF determines RV $X$, i.e., if $X$ and $Y$ are 2 RV s.t.

$$M_X(t) = M_Y(t)$$

then

$$S_X = S_y$$

and

$$\underbrace{f_X(x)}_{\text{pmf of X}} = \underbrace{f_Y(x)}_{\text{pmf of Y}} \quad \text{for } x \in S_X (= S_Y)$$

**Example 8.4** (above)

Suppose $Y$ has mgf

$$M_Y(t) = \frac{1}{2}e^{-2t} + \frac{1}{3} + \frac{1}{6}e^t$$

then

$$S_Y = \{-2, 0, 1\}$$

and $f_Y(-2) = \frac{1}{2}$, $f_Y(0) = \frac{1}{3}$, $f_Y(1) = \frac{1}{6}$. So that $X$ and $Y$ have same space and same pmf.

**Practice 8.2.** 5 – Problem 2b: $X$ has geometric distribution with parameter $p \in [0, 1]$ denoted by $X \sim \text{Geom}(P)$.

with pmf $f(k) = q^{k-1}p$ for $k = 1, 2, \dots, \quad q = 1 - p$. MGF of $X$ is given by

$$\begin{aligned} M_X(t) &= \sum_{k=1}^{\infty} e^{tk} f(k) \\ &= \sum_{k=1}^{\infty} e^{tk} q^{k-1} p \\ &= p(e^t + e^{t2}q + e^{t3}q^2 + \dots) \\ &= p \cdot e^t \left(1 + (e^t q) + (e^t q)^2 + (e^t q)^3 + \dots\right) \\ &= pe^t \frac{1}{1 - e^t q} \end{aligned}$$

which is finite for $t$,

$$\begin{aligned} 0 < e^t \cdot q < 1 \\ e^t < \frac{1}{q} \\ t < \ln\left(\frac{1}{q}\right) \end{aligned}$$

Thus,

$$M_X(t) = \frac{pe^t}{1 - qe^t}, \quad \text{with } t < \ln\left(\frac{1}{q}\right)$$

**Definition 8.5** — For each $n$ positive integer, if $E[X^n] = \sum_{x \in S_X} x^n f(x)$ exists then $E[X^n]$ is called the $n^{\text{th}}$ moment of $X$.

**Remark 8.6.** Properties of MGF $M_X(t)$

- $t = 0$, $M_X(0) = E[e^{0 \cdot X}] = E[1] = 1$.
- Derivatives of $M_X(t)$ is given by

$$\begin{aligned} \frac{d}{dt}[M_X(t)] &= \frac{d}{dt}\left[E[e^{tX}]\right] \\ &= E\left[\frac{d}{dt}e^{tX}\right] \quad \text{assume } \frac{d}{dt} \text{ and E are interchangeable} \\ M_X'(t) &= E\left[Xe^{tX}\right] \end{aligned}$$

  Thus,

$$M_X'(t)\Big|_{t=0} = E[Xe^{0 \cdot X}] = E[X], \text{ first moment of X}$$

- Similarly, 2nd derivative of $M_X(t)$ given by

$$M_X''(t) = E\left[X^2 e^{tX}\right]$$

$$M_X''(t)\Big|_{t=0} = E[x^2], \text{ second moment of X}$$

- More generally, the $n^{\text{th}}$– derivative of $M_X$ satisfies

$$M_X^(n)(t)\Big|_{t=0} = E[x^n]$$

  hence the name “mgf”.

**Example 8.7**

$X \sim \text{Geom}(p)$.

$$\begin{aligned} M_X(t) &= \frac{pe^t}{1 - qe^t}, \quad q = 1 - p \\ M_X'(t) &= \frac{pe^t}{(1 - qe^t)^2} \\ M_X'(0) &= \frac{p}{(1 - q)^2} = \frac{p}{p^2} = \frac{1}{p} = E[x] \end{aligned}$$

# §9 Dis 1: Oct 6, 2020

## §9.1 Set Theory

**Definition 9.1** — A set is a collection of items.

**Example 9.2**

$$T = \{1, 2, 3, \text{red}, \text{blue}\}$$
$$S = \{1, 3, \text{red}\}$$
$$R = \{1, 2, 4\}$$
$$S \subseteq T$$
$$S' = S^c = \{2, \text{blue}\}$$
$$R \not\subseteq T$$
$$3 \quad \underbrace{\in}_{\text{is an element of}} \quad T$$
$$\{3\} \subseteq T$$

**Example 9.3**

$$A = \{1, 3, 7\} \qquad A \cup B = \{1, 2, 3, 4, 7\}$$
$$B = \{2, 3, 4\} \qquad A \cap B = \{3\}$$
$$A \setminus B = \{1, 7\} \qquad B \setminus A = \{2, 4\}$$

De Morgan Laws:

$$(A \cup B)' = A' \cap B'$$
$$(A_1 \cup A_2 \cup \ldots \cup A_n) = A_1' \cap A_2' \cap \ldots \cap A_n'$$
$$(A \cap B)' = A' \cup B'$$

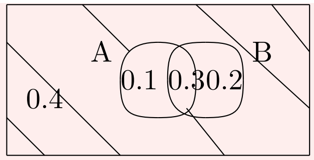
If have a sample space $S$, and subset of $S$ are called events. A probability function is a function $\mathbb{P}$ that assigns a real number each event with three rules:

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $A_1, A_2, \ldots, A_n$ with $A_i \cap A_j = \emptyset = \{\}$, then $P(A_1 \cup \ldots \cup A_n) = P(A_1) + \ldots + P(A_n)$

**Example 9.4**

1.1-6 (from the book): $P(A) = 0.4, \quad P(B) = 0.5, \quad P(A \cap B) = 0.3$
Find

- $P(A \cup B) = .1 + .3 + .2 = .6$
- $P(A \cap B)' = .1$
- $P(A' \cap B) = .2$




*Note*: $(P, S)$ : probability space on all subsets of $S$

**Example 9.5**

1.2-5: How many four letter codes can be made from the letters in IOWA if

- Letters may not be repeat: $4! = 24$ ways.
- Letters may repeat: $4^4 = 256$ ways.

# §10 Dis 2: Oct 13, 2020

1.4.16: An urn has 5 balls. One is marked "win" and the other are marked "lose". You and another player each take balls out one at a time until somebody picks win. You pick first.
W/o replacement: $P(\text{winning}) = \frac{1}{5} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot 1 = \frac{3}{5}$
With replacement:

$$\begin{aligned} P(\text{winning}) &= \frac{1}{5} + \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} + \frac{4}{5}\frac{4}{5}\frac{4}{5}\frac{4}{5}\frac{1}{5} + \dots \\ &= \frac{\frac{1}{5}}{1 - \frac{16}{25}} = \frac{5}{9} \end{aligned}$$

## §10.1 Conditional Probabilities

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

or $P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$

**Example 10.1**

1.3.7: An urn has 4 balls. 2 are red and 2 are blue. We pull out 2 balls. We are told that at least one is red. What's the probability that they're both red?

$P(\text{both red}|\text{at least one red}) = \frac{P(\text{both red and at least one red})}{P(\text{at least one red})} = \frac{P(\text{both red})}{P(\text{at least red})} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}$.

## §10.2 Bayes's Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

**Example 10.2**

1.5-8: Four types of tablets: $B_1, B_2, B_3, B_4$ with % of sales $0.4, 0.3, 0.2, 0.1$ and % tablet needs repair $0.1, 0.05, 0.03, 0.02$, respectively. What is the probability that a tablet needing repair is $B_i$?

$$\begin{aligned} P(B_1|\text{need repair}) &= \frac{P(\text{need repair}|B_1) \cdot P(B_1)}{P(\text{need repair})} \\ &= \frac{(0.1)(0.4)}{(0.40)(0.10) + (0.30)(0.05) + (0.20)(0.03) + (0.10)(0.02)} \\ &\approx 63.5\% \\ P(B_2|\text{need repair}) &= \frac{(0.30)(0.05)}{0.063} \approx 23.8\% \\ P(B_3|\text{need repair}) &\approx 9.5\% \\ P(B_4|\text{need repair}) &\approx 3.2\% \end{aligned}$$

# §11 Dis 3: Oct 20,2020

## §11.1 Recap of Terminology/Functions

We have a situation with a set of possible outcomes

- This set is called the sample space denoted $S$ or $\Omega$.
- Elements of $S$ are called outcomes.
- Subsets of $S$ are called events.
- A probability function is a function where

$$P : \{ \text{ subset of } S\} \to [0, 1]$$

**Example 11.1**

$$\begin{aligned} S &= \{HH, HT, TH, TT\} \\ A &= \{HH, HT\} \\ B &= \{HH\} \\ P(A) &= 0.5 \\ P(B) &= P(\{HH\}) = 0.25 \end{aligned}$$

A random variable, denoted $X$, is a function

$$X : \underbrace{S}_{\text{sample space}} \to \underbrace{S_X}_{\text{the space support}} \subseteq \mathbb{R}$$

"$X = a$" $\leftrightarrow \{w \in S \text{ s.t. } X(w) = a\}$.

**Example 11.2**

Define $X(w)$ to be the number of tails in the outcome w.

$$
\begin{aligned}
X(HH) &= 0 \\
X(HT) &= 1 \\
X(TH) &= 1 \\
X(TT) &= 2 \\
(X=1) &= \{HT, TH\} \\
(X=2) &= \{TT\} \\
(X=0) &= \{HH\} \\
(X=3) &= \emptyset
\end{aligned}
$$

The probability mass function or pmf of a r.v. $X$ is a function $f_x : S_X \to [0,1]$ defined by

$$
\begin{aligned}
f(x) &= P(X=x) \\
f(a) &= P(X=a)
\end{aligned}
$$

**Example 11.3**

$$
f_x(a) = \begin{cases} 0.25 & a=0 \\ 0.5 & a=1 \\ 0.25 & a=2 \end{cases}
$$

Also,

$$
\begin{aligned}
P(X=1) &= P(\{HT, TH\}) = 0.5 \\
P(X<2) &= P(\{HH, HT, TH\}) = 0.75
\end{aligned}
$$

The cumulative distribution function or cdf of a r.v. $X$ is a function $F_x : S_x \to [0,1]$ defined by

$$
F(a) = P(X \le a)
$$

**Example 11.4**

$$
F_x(a) = \begin{cases} 0.25 & a=0 \\ 0.75 & a=1 \\ 1 & a=2 \end{cases}
$$

The expectation or mean of $X$ is

$$E[x] = \sum_{a\in S_x} af(a)$$
$$E[g(x)] = \sum_{a\in S_x} g(a)f(a)$$

**Example 11.5** (above)

$$\begin{aligned} E[x] &= (0)0.25 + (1)0.5 + (2)0.25 \\ &= 1 \\ E[x^2] &= (0^2)0.25 + (1^2)0.5 + (2^2)0.5 \\ &= 2.5 \neq E[x]^2 \end{aligned}$$

The moment of generating function or mgf of $X$ is

$$M_x(t) = E[e^{tX}] = \sum_{a\in S_x} e^{ta}f(a)$$

**Example 11.6**
$M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t} = \sum_{a\in\{1,2,3\}} e^{at}f(a)$. Find mean, variance, pmf.
$S_x = \{1,2,3\}$. The pmf is

$$f_x(a) = \begin{cases} \frac{2}{5} & a = 1 \\ \frac{1}{5} & a = 2 \\ \frac{2}{5} & a = 3 \end{cases}$$

The mean is

$$E[x] = (1)\frac{2}{5} + (2)\frac{1}{5} + (3)\frac{2}{5} = 2$$

Variance is

$$\begin{aligned} \sigma^2 = \mathrm{Var}(X) &= E[x^2] - E[x]^2 \\ &= \left((1^2)\frac{2}{5} + (2^2)\frac{1}{5}(3^2)\frac{2}{5}\right) - 2^2 \\ &= \frac{4}{5} \end{aligned}$$