

Math 170E – Intro to Probability

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This is math 170E taught by Professor Nguyen. The formal name of the class is **Introduction to Probability and Statistics 1: Probability**. The textbook used for the class is *Probability & Statistical Interference* 10th by *Hogg, Tanis*. We meet weekly on MWF from 10:00 – 10:50 and on Tue at the same time frame for discussion with our TA, Jason Snyder. You can also find other lecture notes at ductuanvu.wordpress.com/notes/. Let me know through my [email](#) if you notice something mathematically wrong/concerning. Thank you!

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§1 | Lec 1: Oct 2, 2020

§1.1 Properties of Probability

Definition 1.1 — Consider the outcome of a random experiment, e.g. flipping a coin. The collection of all such outcomes, denoted by $\underbrace{S}_{\omega \text{ in other advanced prob. textbook}}$, is called the outcome space.

- A subset $A \subseteq S$ is called an event.
- If $A_1, A_2, \dots \subseteq S$ satisfy $A_i \cap A_j = \emptyset, i \neq j$ then they are called “disjoint” (mutually exclusive)
- If $A_1, A_2, \dots, A_n \subseteq S$ satisfy $\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n = S$. Then $\{A_i\}_{i=1 \dots n}$ are called exhaustive (fully comprehensive).

Example 1.2 1. Flip two coins in order. Denote H = head, T = tail.

$$S = \{HH, HT, TH, TT\}$$

$$A = \{HH\} = \{\text{both coins are head}\}$$

$A \subseteq S$ is an event.

$$B = \{HT, TH\}$$

$B \subseteq S$ is another event.

$A \cap B = \emptyset$, they are disjoint.

2. Flip 2 coins at once.

$$S = \{HH, HT, TT\}$$

$$A = \{\text{one head, one tail}\}$$

$$A = \{HT\}, \text{ is an event.}$$

Probability – A heuristic intro:

Consider an experiment and repeat n times. Let $N(A)$ = number of times A occurs. The ratio $\frac{N(A)}{n}$ is called the relative frequency of A in n repetitions of the experiment.

$$0 \leq \frac{N(A)}{n} \leq 1$$

As $n \rightarrow \infty$,

$$\frac{N(A)}{n} \rightarrow p \in [0, 1]$$

This p is called the prob. that event A occurs.

Example 1.3

(a) Flip a coin

$$S = \{H, T\}$$

$$A = \{H\}$$

What is $P(A)$?

(b) Sometimes, we can also assign prob. based on the nature of the event Pick a random point in the unit circle.

$$A = \{\text{chosen point} \in 1^{\text{st}} \text{quadrant}\}$$

$$P(A) = \frac{\text{Area of first quadrant}}{\text{Area of unit circle}} = \frac{1}{4}$$

(c) Pick a number randomly from $\{0, 1, \dots, 9\}$, $B = \{2 \text{ is picked}\}$

$$P(B) = \frac{1}{10}$$

Table 1: From example 1.3 (a)

n	$N(A)$	$\frac{N(A)}{n}$
50	37	.74
500	333	.66

It is safe to assign $P(A) = 0.66$ **Definition 1.4** — Given an outcome space S , the probability of an event $A \subseteq S$, is a number satisfying:

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $A_1, \dots, A_n \subseteq S$ are disjoint events, i.e. $A_i \cap A_j = \emptyset, i \neq j$, then

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i) = P(A_1) + \dots + P(A_n)$$

More generally, if $A_1, \dots, A_n, \dots \subseteq S$ are disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Theorem 1.5 1. Denote A' to be the complement of A in S , i.e.

$$A' \cup A = S$$

$$A' \cap A = \emptyset$$

Then

$$P(A') = 1 - P(A)$$

$$2. P(\emptyset) = 0$$

$$3. \text{ If } A \leq B \text{ then } P(A) \leq P(B)$$

$$4. P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$5. P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Note: The pattern here is add the prob. of odd event(s) and subtract the prob. of even events. (for prop (4) and (5) of theorem 1.5).

Proof.

$$P(A') = 1 - P(A)$$

Since $A' \cap A = \emptyset$ (by def of A'). By property (c),

$$P(\underbrace{A' \cup A}_S) = P(A') + P(A)$$

$$\underbrace{P(S)}_{1 \text{ (by prop.(b))}} = P(A') + P(A)$$

Thus,

$$P(A') = 1 - P(A)$$

§2 | Lec 2: Oct 5, 2020

Cont'd of Lec 1

(2)

$$\begin{aligned} P(\emptyset) &= 1 - P(S) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

(3)

$$P(A) \leq P(B)$$

$B \setminus A$ is the set s.t.

$$A \cup (B \setminus A) = B$$

$$A \cap (B \setminus A) = \emptyset$$

something here

implying

$$P(A) \leq P(B)$$

(4)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(5) Textbook Section 1.1. □

Definition 2.1 — Suppose $S = \{e_1, \dots, e_m\}$ where each e_i is a possible outcome. Denote $n(s)$ = number of outcomes = m . If each e_i has the same prob. of occurring, then they are called equally likely. In particular,

$$P(e_i) = \frac{1}{n(s)} = \frac{1}{m}$$

Moreover, if $A \subseteq S$ is an event s.t. $n(A) = k$. Then,

$$P(A) = \frac{n(A)}{n(s)} = \frac{k}{m}$$

Example 2.2

Draw one card from a deck of 52 cards.

$$P(\text{each card is drawn}) = \frac{1}{52}$$

$A = \{\text{a king is drawn}\}$, so $n(A) = 4$. Thus,

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}$$

§2.1 Method of Enumeration

Multiplication Principle:

Suppose an experiment E_1 has n_1 outcomes

- For each outcome from E_1 , a 2nd experiment E_2 has n_2 outcomes. Then the composite $E_1 E_2$ has $n_1 \cdot n_2$ outcomes.

Permutation of size n:

Definition 2.3 — Suppose there are n positions to be filled by n persons. One such arrangement is called a permutation of size n .

FACT: the total number of different such arrangements is given by “ $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$ ”

Proof. • $E_1 =$ fill the 1st position from n persons $\implies n$ outcomes for E_1 .

- $E_2 =$ fill the 2nd pos. from $n - 1$ persons left $\implies n - 1$ outcomes for E_2
- \vdots
- $E_n =$ fill the n^{th} pos. from 1 person left $\implies 1$ outcome for E_n
- One arrangement $= E_1 E_2 \dots E_n$

Thus, total number of arrangements is $n!$. □

Permutation/Combination of n objects taken k :

Definition 2.4 — Given $k \leq n$ and suppose there are n objects. If k objects are taken from n **with/without** order, then such a selection is called **permutation/combination** of size n taken k .

Note: “Permutation of size n ” = “permutation of size n taken n ”.

Fact 2.1. 1. The total number of permutation n taken k (order is important here) is denoted by ${}^n P_k$ is given by

$${}^n P_k = \frac{n!}{(n - k)!}$$

2. The total numbers of combination of n taken k , denoted by ${}^n C_k$ or $\binom{n}{k}$ is given by

$${}^n C_k = \binom{n}{k} = \frac{n!}{(n - k)!k!}$$

Proof. $E_1 =$ fill 1st pos. from $n \implies n$ for E_1

\vdots

$E_k =$ fill k^{th} pos. from $n - k + 1$ persons left. Thus,

$$\text{perm}k = n \cdot \dots \cdot (n - k + 1)$$

(2) Combination of n taken k :

Start with ${}^n P_k$ as follow:

- $E_1 =$ take k from n at once, outcome $= {}^n C_k = \binom{n}{k}$
- $E_2 =$ permute k , outcomes $= k!$. Thus,

$${}^n P_k = \binom{n}{k} \cdot k!$$

implying

$$\binom{n}{k} = \frac{{}^n P_k}{k!} = \frac{n!}{(n - k)!k!}$$

□

Practice 1: https://ccle.ucla.edu/pluginfile.php/3766550/mod_resource/content/1/Practice%201.pdf

1. Consider $S = \{1, \dots, 8\}$

a)

- E_1 = filling 1st pos \implies 8 choices.
- Same for $E_2 \implies$ 8 choices.
- Likewise, E_3 has 8 choices.

Thus, the number of 3 digit numbers can be formed is 8^3

b) “3 distinct digit numbers” = “permutation of size 8 taken 3”

Thus, total such numbers is ${}_8P_3 = \frac{8!}{5!} = 8 \cdot 7 \cdot 6$

c) Considering subset where order is not taken into account

Combination of size 8 taken 3. Thus, the answer is

$$\binom{8}{3} = \frac{8!}{3!5!}$$

d) 3 digit numbers and divisible by 5

- E_1 = choose 5 for the 3rd pos, so 1 choice.
- E_2 = 8 choices
- E_3 = 8 choices

Thus, the total of choices is $8 \cdot 8 = 64$.

e) 4 element subsets of S that has one even digit.

- E_1 = choose one even digit from S , so 4 choices (2,4,6,8).
- E_2 = choose 3 digits from $\{1, 3, 5, 7\}$ without order, so $\binom{4}{3}$

Thus, total = $E_1 \cdot E_2 = 4 \cdot \binom{4}{3}$.

e') What if “at least one even digit” instead of “exactly one even”?

1. Total = exactly “one even” + “two even” + “three even” + “four even”
2. Total = “4-element subset” - “4-element subset with no even digit”

§3 | Lec 3: Oct 7, 2020

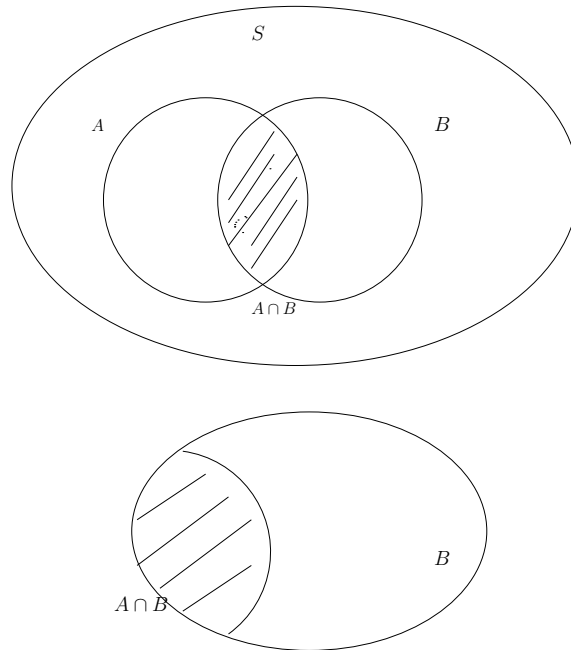
§3.1 Conditional Probability

Definition 3.1 — Let $A, B \subseteq S$ be two events. The conditional prob. of A , given that B has occurred with $P(B) > 0$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

A heuristic explanation: $A \cap B$: “the portion in B that A occurs”

$$P(A|B) = \frac{\text{“area of A in B”}}{\text{“area of B”}}$$

**Example 3.2**

Suppose my family has two kids. Given that there is at least a boy, what is the prob. my family has two boys?

$$S = \{bb, bg, gb, gg\}$$

Now, let $B = \{\text{at least a boy}\}$. So we only look at the first three outcomes from S (B). Define $A = \{\text{two boys}\}$

$$A \cap B = \{bb\}$$

Note $A = A \cap B$ since $A \subseteq B$. Thus,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Note: We can also consider the alternative outcome space without order as follows

$$S = \left\{ (b, b) - \frac{1}{4}, (b, g) - \frac{1}{2}, (g, g) - \frac{1}{4} \right\}$$

Fact 3.1. $P(A|B)$ satisfies basic properties of probability:

- $P(A|B) \geq 0$
- $P(B|B) = 1$

Moreover, if $B \leq C$ then

$$P(C|B) = 1$$

- If A_1, \dots, A_n, \dots are disjoint events,

$$P\left(\bigcup_{k=1}^{\infty} A_k | B\right) = \sum_{k=1}^{\infty} P(A_k | B)$$

Proof. (a) $P(A|B) = \frac{P(A \cap B)}{P(B)} \geq 0$

(b) $P(B|B) = \frac{P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$
 If $B \subseteq C$ then $B \cap C = B$

$$P(C|B) = \frac{P(B \cap C)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

$B \subseteq C$ means “if B occurs then C must occur”.

(c) $P(\bigcup_{k=1}^{\infty} A_k|B) = \frac{P(\bigcup_{k=1}^{\infty} A_k \cap B)}{P(B)}$. By distributive law,

$$\begin{aligned} &= \frac{P(\bigcup_{k=1}^{\infty} (A_k \cap B))}{P(B)} \\ &= \frac{\sum_{k=1}^{\infty} P(A_k \cap B)}{P(B)} \\ &= \sum_{k=1}^{\infty} P(A_k|B) \end{aligned}$$

□

INSERT: PRACTICE 1 #3 here

Theorem 3.3 1. $P(A \cap B) = P(A|B) \cdot P(B)$ given that $P(B) > 0$
 2. $P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$ given $P(A), P(A \cap B) > 0$.

Proof. 1. By defn of cond. prob.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

implying

$$P(B)P(A|B) = P(A \cap B)$$

2. $P(A \cap B \cap C) = P(C \cap (A \cap B))$. By part 1,

$$\begin{aligned} &= P(C|A \cap B)P(A \cap B)P(A \cap B) \\ &= P(C|A \cap B)P(B|A)P(A) \end{aligned}$$

□

Practice 3.1. The url: https://ccle.ucla.edu/pluginfile.php/3776692/mod_resource/content/0/Practice%202.pdf

INSERT: Look at the online notes

§4 | Lec 4: Oct 9, 2020

Cont'd (Practice)

3)

$$A = \{\text{spade}\} \quad B = \{\text{heart}\} \quad C = \{\text{diamond}\} \quad D = \{\text{club}\}$$

$P = (A \cap B \cap C \cap D = ?$ So,

$$= P(A)P(B|A)P(C|A \cap B)P(D|A \cap B \cap C)$$

(from problem 2 in practice 2)

- $P(A) = \frac{13}{52}$
- $P(B|A) =$, now restricted to outcome space {51 cards including 13 hearts} $B|A = \{\text{dealing a heart}\}$. Thus,

$$P(B|A) = \frac{13}{51}$$

- Similarly,

$$P(C|A \cap B) = \frac{13}{50}$$

(13 diamond from 50 cards left)

- $P(D|A \cap B \cap C) = \frac{13}{49}$ (13 clubs from 49 cards left).

Hence,

$$P(A \cap B \cap C \cap D) = \frac{13}{52} \frac{13}{51} \frac{13}{50} \frac{13}{49}$$

§4.1 Independent Events

Example 4.1

Flip a fair coin twice

$$S = \{HH, HT, TH, TT\}$$

$$A = \{1^{\text{st}}H\}$$

$$B = \{2^{\text{nd}}T\}$$

$$C = \{TT\}$$

$C \subseteq B$ “2 tails” \implies “2nd is T”. i.e., if C occurs then B must have occurred. Thus,

$$\begin{aligned} P(B|C) &= 1 \\ P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{\frac{1}{4}}{\frac{1}{2}} \\ &= \frac{1}{2} \\ P(A) &= \frac{1}{2} \end{aligned}$$

Thus, $P(A|B) = P(A)$, i.e., B occurring does not impact the occurrence of A.

Note also that

$$\frac{P(A \cap B)}{P(B)} = P(A|B) = P(A)$$

implying

$$P(A \cap B) = P(A)P(B)$$

Definition 4.2 — Given two events A, B which are called independent iff

$$P(A \cap B) = P(A)P(B)$$

Theorem 4.3

The following are equivalent

- A, B are independent
- $P(A|B) = P(A)$, provided $P(B) > 0$
- $P(B|A) = P(B)$, provided $P(A) > 0$

Proof. Left as an exercise. □

Theorem 4.4 1. If $P(A) = 0$ then A is independent with any event.

2. If A and B are independent then so are the following pairs:

$$A, B' \quad A', B \quad A', B'$$

Proof. 1. Let B an arbitrary event, we need to show $P(A \cap B) = P(A)P(B)$. Since $P(A) = 0$, $P(A)P(B) = 0$.

$$A \cap B \subseteq A$$

imply

$$0 \leq P(A \cap B) \leq P(A) = 0$$

thus $P(A \cap B) = 0$.

2. Textbook(section 1.5)

□

Practice 4.1. Practice 2 – Problem 4:

Let's consider C and D first

$$\begin{aligned} D &= \{ \text{sum of two rolls} = 12 \} \\ &= \{(6, 6)\} \end{aligned}$$

Thus, $D \subseteq C = \{\text{first roll is 6}\}$. Hence, C and D are dependent.

A v.s. B

$$\begin{aligned} P(A) &= \frac{5}{6} \\ B &= \{ \text{sum is even} \} \\ &= \{ \text{first and second roll are even} \} \cup \{ \text{first and second roll are odd} \} \\ P(B) &= P(\text{first even})P(\text{second even}) + P(\text{first odd})P(\text{second odd}) \\ &= \frac{3}{6} \frac{3}{6} + \frac{3}{6} \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

Now, consider $A \cap B = \{1^{\text{st}} \neq 3, \text{sum is even}\}$. So,

$$\begin{aligned} A \cap B &= \{1^{\text{st}} \neq 3, 1^{\text{st}} \text{ odd}, 2^{\text{nd}} \text{ odd}\} \cup \{1^{\text{st}} \neq 3, 1^{\text{st}} \text{ even}, 2^{\text{nd}} \text{ even}\} \\ P(A \cap B) &= P(1^{\text{st}} \neq 3, 1^{\text{st}} \text{ odd})P(2^{\text{nd}} \text{ odd}) + P(1^{\text{st}} \neq 3, 1^{\text{st}} \text{ even})P(2^{\text{nd}} \text{ even}) \\ &= \frac{2}{6} \frac{3}{6} + \frac{3}{6} \frac{3}{6} \\ &= \frac{5}{12} \end{aligned}$$

Since $P(A \cap B) = \frac{5}{12} = \frac{5}{6} \frac{1}{2} = P(A)P(B)$, A and B are independent.

§5 | Lec 5: Oct 12, 2020

§5.1 Independent Events (cont'd)

Definition 5.1 — A, B, C are called “mutually independent” if followings hold:

- pairwise independent

$$P(A \cap B) = P(A)P(B) \quad P(B \cap C) = P(B)P(C) \quad P(A \cap C) = P(A)P(C)$$

- “triple” wise independent, i.e.,

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Note: analogous defn holds for A_1, \dots, A_n, \dots in which any pairs, triple, quadruple and so on must satisfy the similarly multiplication rules. Usually, the term “mutually” is dropped but it is understood that “independence” means “mutually independence”.

Remark 5.2. In general, pairwise independence does not imply triple-wise independence.

Practice 5.1. 2 – Problem 5:

$$A = \{1, 2\}, \quad B = \{1, 3\}, \quad C = \{1, 4\}$$

$$P(A) = \frac{2}{4} = P(B) = P(C)$$

$$A \cap B = \{1\} = B \cap C = A \cap C$$

$$P(A \cap B) = P(B \cap C) = P(C \cap A) = \frac{1}{4}$$

Thus,

$$P(A \cap B) = \frac{1}{4} = P(A)P(B)$$

Same for B, C and A, C – so pairwise independent.

Triple:

$$A \cap B \cap C = \{1\}$$

$P(A \cap B \cap C) = \frac{1}{4}$; on the other hand, $P(A)P(B)P(C) = \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{1}{8}$. They are not equal! Therefore, A, B, C are not mutually independent.

§5.2 Bayes’s Theorem

Definition 5.3 — The events B_1, \dots, B_n (n may be finite or ∞) are called a partition of the outcome space S if followings hold

- disjoint: $B_i \cap B_k = \emptyset, i \neq k$
- exhausted: $\bigcup_{i=1}^n B_i = S$

then,

$$P(B_1) + \dots + P(B_n) = P(S) = 1$$

Theorem 5.4 (Law of total Probability)

Suppose B_1, \dots, B_n is a partition of S with $P(B_i) > 0$ for $i = 1, \dots, n$. If A is an event in S , then

$$P(A) = \sum_{i=1}^n P(A|B_i)P(B_i)$$

where $P(B_i)$ is called the prior probability.

Proof. (sketch)

$$\begin{aligned} P(A) &= P\left(\bigcup_{i=1}^n (A \cap B_i)\right) \\ &= \sum_{i=1}^n P(A \cap B_i) \\ &= \sum_{i=1}^n P(A|B_i)P(B_i) \end{aligned} \quad \square$$

Practice 5.2. 3 – problem 1:

$$\begin{aligned} P(I) &= .35 \\ P(II) &= .25 \\ P(III) &= .4 \end{aligned}$$

$A = \{ \text{a spring is defective} \}$, $P(A) = ?$ We know

$$\begin{aligned} P(A|I) &= .02 \\ P(A|II) &= .01 \\ P(A|III) &= .03 \end{aligned}$$

By law of total prob:

$$\begin{aligned} P(A) &= P(A|I)P(I) + P(A|II)P(II) + P(A|III)P(III) \\ &= 0.0215 \end{aligned}$$

Theorem 5.5 (Bayes's Theorem)

Suppose $\{B_i\}_{i=1, \dots, n}$ is a partition of S with $P(B_i) > 0$. If A with $P(A) > 0$, then for all $i = 1, \dots, n$

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{k=1}^n P(A|B_k)P(B_k)}$$

where $P(B_i|A)$ is called posterior probability.

Proof.

$$\begin{aligned}
 P(B_i|A) &= \frac{P(B_i \cap A)}{P(A)} \\
 &= \frac{P(A \cap B_i)}{P(A)} \\
 &= \frac{P(A|B_i)P(B_i)}{P(A)} \\
 &= \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + \dots + P(A|B_n)P(B_n)} \quad \square
 \end{aligned}$$

Practice 5.3. 3 – problem 2: $A = \{ \text{person has disease} \}$, $P(A) = .005$.

$$\begin{aligned}
 + &= \{ \text{test } + \} \\
 - &= \{ \text{test } - \} \\
 P(+|A) &= .99 \\
 P(\underbrace{+|A'}_{\text{false positive}}) &= .03 \\
 P(A|+) &=?
 \end{aligned}$$

By Bayes's Theorem:

$$\begin{aligned}
 P(A|+) &= \frac{P(+|A)P(A)}{P(+|A)P(A) + P(+|A')P(A')} \\
 &= \frac{(.99)(.005)}{(.99)(.005) + (.03)(.995)}
 \end{aligned}$$

$\{A, A'\}$ is a partition of S .

§6 | Lec 6: Oct 14, 2020

Practice 6.1. 3 – Problem 3: Trial: know at least 1 girl

$$P(GG|\text{at least a girl}) = \frac{1}{3}$$

However, the above approach is not correct.

Intuition: The moment the girl opens the door, the first child's gender is determined – which makes the other kid's gender is now independent of the girl. Thus, $P(\text{other kid is girl}) = \frac{1}{2}$.

Correct approach:

$$\begin{aligned}
 A &= \{ \text{a girl opens the door} \} \\
 P(GG|A) &=?
 \end{aligned}$$

- $P(A|GG) = 1$
- $P(A|BB) = 0$
- $P(A|GB) = \frac{1}{2}$

- $P(A|BG) = \frac{1}{2}$

By Bayes' Theorem

$$\begin{aligned} P(GG|A) &= \frac{P(A|GG)P(GG)}{P(A|GG)P(GG) + P(A|BB)P(BB) + P(A|BG)P(BG) + P(A|GB)P(GB)} \\ &= \frac{1}{2} \end{aligned}$$

§6.1 Random Variables with Discrete Type

Example 6.1

Flip a coin

$$S = \{H, T\}$$

Define

$$X : S \rightarrow \mathbb{R}$$

$$\omega \mapsto X(\omega) \in \mathbb{R}$$

$$\text{s.t. } X(H) = 0, \quad X(T) = 1$$

$$H \xrightarrow{X} 0$$

$$T \xrightarrow{\quad} 1$$

The function X is called a random variable (RV). Since S is discrete space, X is called a RV of discrete-type.

Definition 6.2 — Given an outcome space S , a function X that assigns $X(s) = x \in \mathbb{R}$ for each $s \in S$ is called a random variable.

The space(range) of X is the collection of real numbers, denoted by S_x ,

$$S_x = \{x \in \mathbb{R} : \exists s \in S, X(s) = x\}$$

S_x is also called the “support” of X .

When the outcome space S is discrete, then X is called a discrete random variable.

Example above:

$$S_x = \{0, 1\}$$

Note: the space of X is denoted by S in the textbook. Here we will use S_x .

Remark 6.3. Under the above definition, for $x \in S_x$,

$$P(X = x) = P(\{s \in S : X(s) = x\})$$

Example 6.4

Roll a fair dice

$$\begin{aligned} S &= \{1, 2, \dots, 6\} \\ X : S &\rightarrow \mathbb{R} \\ s &\mapsto X(s) = x \\ S_x &= \{1, 2, \dots, 6\} (= S) \end{aligned}$$

For each $k \in S_x$,

$$P(X = k) = P(\{k\}) = \frac{1}{6}$$

Also,

$$\sum_{k \in S_x} P(X = k) = \sum_{k=1}^6 \frac{1}{6} = 1$$

Definition 6.5 (Probability Mass Function) — The probability mass function (pmf) $f(x)$ of a discrete random variable X is a function satisfying the followings:

- $f(x) > 0$, $x \in S_x$.
- $\sum_{x \in S_x} f(x) = 1$.
- If $A \subseteq S_x$,

$$P(X \in A) = \sum_{x \in A} f(x)$$

Note: if $x \notin S_x$, then we assign $f(x) = 0$ ($P(X = x) = 0$).

Example 6.6 (above)

the pmf of X is given by $f(k) = \frac{1}{6}$ for $k = 1, \dots, 6$

$$\begin{aligned} A &= \{1, 2, 3\} = "X < 4" \\ A &\subseteq S_x \end{aligned}$$

$$P(X \in A) = \sum_{k \in A} f(k) = \sum_{k=1}^3 \frac{1}{6} = \frac{1}{2}$$

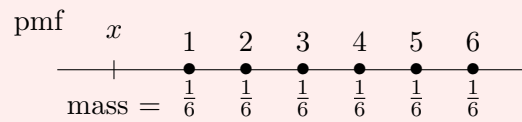
Definition 6.7 — Cumulative distribution function (cdf) $F(x)$ of a RV x is a function given by

$$F(x) = P(X \leq x), \quad -\infty < x < \infty$$

Note: $F(x)$ is usually called distribution function, “cumulative” is dropped.

Example 6.8

Rolling a fair dice



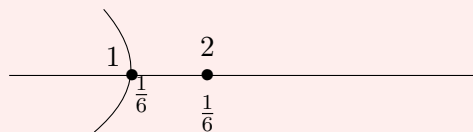
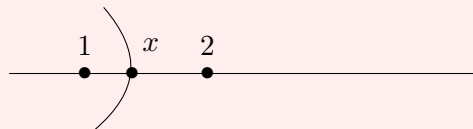
$$\text{cdf } F(x) = P(X \leq x)$$

= total mass cumulated starting from the left up to x $x < 1$,

$$F(x) = P(X \leq x)$$

= 0 (no mass up to $x < 1$) $x = 1$,

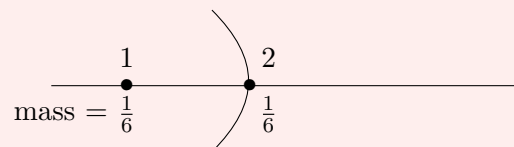
$$F(1) = P(X \leq 1)$$

 $F(1) = \frac{1}{6}$ (mass up to and including location 1). $1 < x < 2$ 

$$F(x) = P(X \leq 1)$$

$$= P(X = 1)$$

$$= \frac{1}{6}$$

 $x = 2$ 

$$F(2) = P(X \leq 2)$$

$$= P(X = 1) + P(X = 2)$$

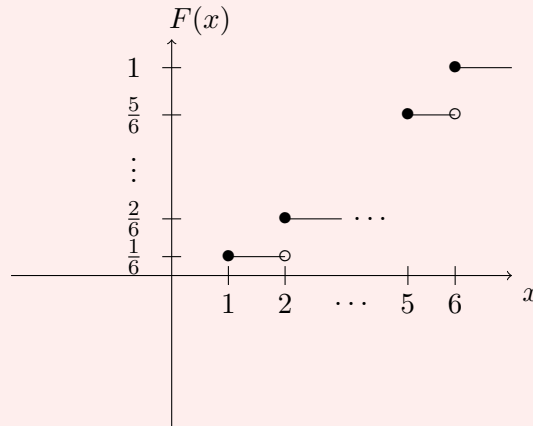
$$= \frac{2}{6}$$

Likewise, $2 < x < 3$

$$F(x) = \frac{2}{6}$$

$$\therefore x = 6, \quad F(X) = P(X \leq 6) = 1$$

$x > 6, F(x) = 1$

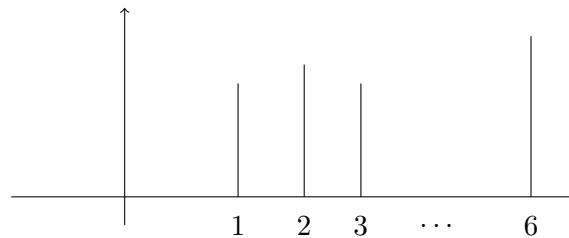


§7 | Lec 7: Oct 16, 2020

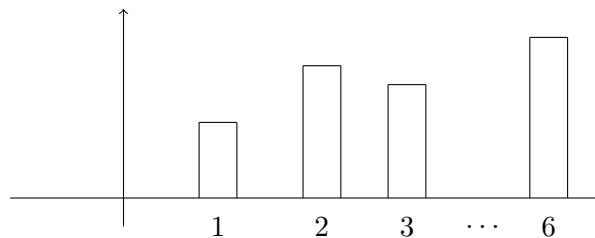
§7.1 Lec 6 (Cont'd)

In order to graph the prob. mass function:

- Line graph



- Histogram



Practice 7.1. 4 – Problem 1:

$$X = \text{max of two rolls}$$

$$S_X = \{1, 2, \dots, 6\}$$

For $k \in S_X$. Determine $f(k) = P(X = k) = ?$

- 1st approach:

$\begin{matrix} 2^{nd} \text{ roll} \\ \backslash \\ 1^{st} \text{ roll} \end{matrix}$	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
\vdots						
6	(6, 1)	(6, 2)	\dots			

$\begin{matrix} 2^{nd} \text{ roll} \\ \backslash \\ 1^{st} \text{ roll} \end{matrix}$	1	2	3	\dots	6
1	1	2	3	\dots	6
2	2	2	3	\dots	6
3	3	3	3	\dots	6
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots
6	6	6	6	\dots	6

$$f(1) = P(X = 1) = \frac{1}{36}$$

$$f(2) = P(X = 2) = \frac{3}{36}$$

$$f(3) = P(X = 3) = \frac{5}{36}$$

\vdots

$$f(6) = P(X = 6) = \frac{11}{36}$$

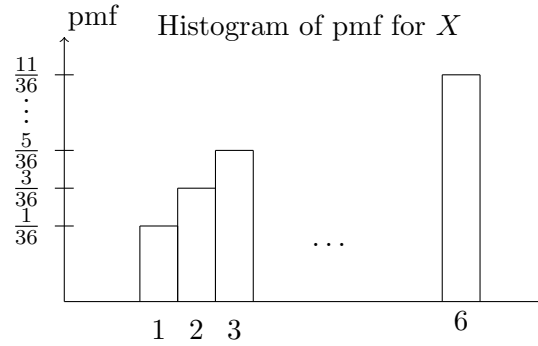
- 2nd approach: for $k = 1, \dots, 6$ (disjoint sub-events)

$$\begin{aligned} \{X = k\} &= \{\max = k\} \\ &= \left\{ 1^{\text{st}} \text{ roll} = k, 2^{\text{nd}} < k \right\} \\ &\cup \left\{ 1^{\text{st}} \text{ roll} < k, 2^{\text{nd}} = k \right\} \\ &\cup \left\{ 1^{\text{st}} \text{ roll} = 2^{\text{nd}} = k \right\} \end{aligned}$$

Thus,

$$\begin{aligned}
 P(X = k) &= P(1^{\text{st}} \text{ roll} = k)P(2^{\text{nd}} < k) + P(1^{\text{st}} < k)P(2^{\text{nd}} = k) + P(1^{\text{st}} = k)P(2^{\text{nd}} = k) \\
 &= \frac{1}{6} \frac{k-1}{6} + \frac{k-1}{6} \frac{1}{6} + \frac{1}{6} \frac{1}{6} \\
 &= \frac{2k-1}{36}
 \end{aligned}$$

Note: $\sum_{k=1}^6 \frac{2k-1}{36} = 1$.



Similarly, we can calculate $Y = \min$ of 2 rolls.

Remark 7.1. Suppose $X = \max\{U, V\}$ where U, V are 2 discrete random variables. Then pmf of X can be calculated as follows:

$$\begin{aligned}
 f(k) &= P(X = k) \\
 &= P(U = k, V < k) + P(U < k, V = k) + P(U = k, V = k)
 \end{aligned}$$

and we can often use indep. on each of the above events. On the other hand, for $Y = \min\{U, V\}$ then

$$P(Y = k) = P(U = k, V > k) + P(U > k, V = k) + P(U = k, V = k)$$

and use indep. on the above events.

§7.2 Expectation & Special Math Expectations

Definition 7.2 — Suppose X is a discrete random variable with S_X , pmf $f(x)$. Let $u(x)$ be a function, then if the sum $\sum_{x \in S_X} u(x)f(x)$ exists (finite) then the sum is mathematical expectation (expected value) of $u(X)$ and is denoted by

$$E[u(X)] := \sum_{x \in S_X} u(x)f(x)$$

Practice 7.2. 5 – Problem 1: $S_X = \{1, \dots, 6\}$. For $x \in S_X$, $u(x) = x - 3.5$

$$\begin{aligned} \text{average income} &= E[u(x)] \\ &= \sum_{x \in S_X} u(x)f(x) \\ &= \sum_{k=1}^6 (k - 3.5) \cdot \frac{1}{6} \\ &= 0 \end{aligned}$$

“After one game, on average, I do not gain/lose any money.”

Theorem 7.3

When it exists, the expectation E satisfies:

- If c is a constant, then

$$E[c] = c$$

- If c is a constant and $u(X)$ is a function, then

$$E[c \cdot u(X)] = cE[u(X)]$$

- If c_1, c_2 are constants and $u_1(X), u_2(X)$ are functions.

$$E[c_1 u_1(X) + c_2 u_2(X)] = c_1 E[u_1(X)] + c_2 E[u_2(X)]$$

Remark 7.4. Part (c) can be generalized for 2 discrete random variables X, Y .

$$E[c_1 u_1(X) + c_2 u_2(Y)] = c_1 E[u_1(X)] + c_2 E[u_2(Y)]$$

Proof. Textbook. □

Definition 7.5 — For a random variable X ,

- the mean (of X) is denoted by

$$\mu := E[X]$$

- the variance (of X) is denoted by

$$\sigma^2 := E[(X - \mu)^2]$$

- the standard deviation

$$\sigma := \sqrt{\sigma^2}$$

Example 7.6

Suppose X has pmf

x	-2	0	1
$f(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$$\text{mean} = \mu = E[x]$$

$$= \sum_{x \in S_X} x \cdot f(x)$$

$$= (-2) \frac{1}{2} + 0 \frac{1}{3} + 1 \frac{1}{6}$$

$$= -\frac{5}{6}$$

$$\text{variance} = \sigma^2 = E[(x - \mu)^2]$$

$$= \sum_{x \in S_X} (x - \mu)^2 f(x)$$

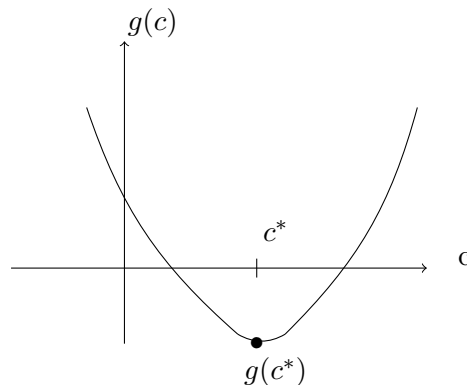
$$= (-2 - (-\frac{5}{6}))^2 \frac{1}{2} + (0 - (-\frac{5}{6}))^2 \frac{1}{3} + \dots$$

σ^2 interpretation:

For a constant $c \in \mathbb{R}$, define $g(c) := E[(x - c)^2]$. Note that

$$\begin{aligned} g(c) &= E[(X - c)^2] \\ &= E[X^2 - 2cX + c^2] \\ &= E[X^2] + E[-2cX] + E[c^2] \\ &= E[X^2] - 2cE[X] + c^2 \\ &= c^2 - 2cE[X] + E[X^2] \\ &= c^2 - 2\mu \cdot c + E[X^2] \end{aligned}$$

“ μ and $E[X^2]$ are constant with respect to c ”.



$g(c^*) = \min g(c)$ where c^* satisfies

$$\begin{aligned} g'(c^*) &= 0 \\ g'(x) &= 2c - 2\mu \end{aligned}$$

Thus

$$g'(c^*) = 0 = 2c^* - 2\mu$$

i.e., $c^* = \mu$. Hence,

$$\sigma^2 = E[(x - \mu)^2] = g(\mu)$$

minimizes $g(c) = E[(x - c)^2]$, i.e.,

$$\sigma^2 = \underbrace{\min_{c \in \mathbb{R}} E[(x - c)^2]}_{c = \mu} = E[(x - \mu)^2]$$

“ σ^2 measures fluctuation of X around its mean μ .”

§8 | Lec 8: Oct 19, 2020

§8.1 Info about 1st midterm

1st Midterm 11/2, Monday, 10am PT. Due: 10am PT – Tuesday 11/3.
2nd Midterm, after Thanksgiving.

§8.2 Lec 7 (Cont'd)

Review geometric series: for $|q| < 1$,

$$\sum_{k=0}^{\infty} q^k = 1 + q + q^2 + \dots = \frac{1}{1 - q}$$

Differentiating both sides,

$$\sum_{k=1}^{\infty} kq^{k-1} = 1 + 2q + 3q^2 + \dots = \frac{1}{(1 - q)^2}$$

Practice 8.1. 5 – Problem 2:

$S_X = \{1, 2, \dots\}$. The pmf $f(f) = P(X = k) = P(1^{\text{st}} \text{ k-1 shots are missed and k shot successful})$.
a) $E[X] = ?$

$$\begin{aligned} A_k &= \left\{ k^{\text{th}} \text{ shot is successful} \right\} \\ P(A_k) &= p \\ P(A'_k) &= 1 - q = P\left(\left\{ k^{\text{th}} \text{ shot is missed} \right\}\right) \end{aligned}$$

$$\begin{aligned}
P(X = k) &= P\left(\underbrace{A'_1 \cap A'_2 \cap \dots \cap A'_{k-1}}_{\text{miss 1st } k-1 \text{ shots}} \cap \underbrace{A_k}_{\text{make at } k^{\text{th}} \text{ shots}}\right) \\
&\stackrel{\text{independence}}{=} P(A'_1)P(A'_2) \dots P(A'_{k-1})P(A_k) \\
&= q \cdot q \cdot \dots \cdot q \cdot p \\
&= q^{k-1} \cdot p
\end{aligned}$$

for each $k = 1, 2, 3, \dots$. Note that pmf $f(k) = P(X = k)$ indeed satisfies:

$$\begin{aligned}
\sum_{k=1}^{\infty} f(k) &= \sum_{k=1}^{\infty} q^{k-1} \cdot p \\
&= p(1 + q + q^2 + \dots) \\
&= p \cdot \frac{1}{1 - q} \\
&= p \cdot \frac{1}{p} \\
&= 1
\end{aligned}$$

Now,

$$\begin{aligned}
\mu = E[x] &= \sum_{x \in S_X} x f(x) \\
&= \sum_{k=1}^{\infty} k \cdot f(k) \\
&= \sum_{k=1}^{\infty} k \cdot q^{k-1} \cdot p \\
&= p \sum_{k=1}^{\infty} k \cdot q^{k-1} \\
&= p \cdot (1 + 2q + 3q^2 + \dots) \\
&= p \cdot \frac{1}{(1 - q)^2} \\
&= p \cdot \frac{1}{p^2} \\
&= \frac{1}{p}
\end{aligned}$$

Definition 8.1 (moment generating function) — Given a discrete RV X and δ_X and pmf $f(x)$, if \exists a positive constant h s.t. for all $t \in (-h, h)$, the following expectation function

$$E[e^{tX}] = \sum_{x \in S_X} e^{tx} f(x)$$

exists then $E[e^{tx}]$ is called the mgf of X and is denoted by $M_X(t)$.

Note: $(-h, h)$ needs not be a symmetric interval. But it has to contain the origin 0.

Example 8.2

Suppose X has the following pmf,

x	-2	0	1
$f(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$$\begin{aligned} E[e^{tX}] &= M_X(t) = \sum_{x \in S_X} e^{tx} f(x) \\ &= \frac{1}{2}e^{-2t} + \frac{1}{3} + \frac{1}{6}e^t \end{aligned}$$

which is finite for all $t \in \mathbb{R}$.

Theorem 8.3

MGF determines RV X , i.e., if X and Y are 2 RV s.t.

$$M_X(t) = M_Y(t)$$

then

$$S_X = S_Y$$

and

$$\underbrace{f_X(x)}_{\text{pmf of } X} = \underbrace{f_Y(x)}_{\text{pmf of } Y} \quad \text{for } x \in S_X (= S_Y)$$

Example 8.4 (above)

Suppose Y has mgf

$$M_Y(t) = \frac{1}{2}e^{-2t} + \frac{1}{3} + \frac{1}{6}e^t$$

then

$$S_Y = \{-2, 0, 1\}$$

and $f_Y(-2) = \frac{1}{2}$, $f_Y(0) = \frac{1}{3}$, $f_Y(1) = \frac{1}{6}$. So that X and Y have same space and same pmf.

Practice 8.2. 5 – Problem 2b: X has geometric distribution with parameter $p \in [0, 1]$ denoted by $X \sim \text{Geom}(P)$.

with pmf $f(k) = q^{k-1}p$ for $k = 1, 2, \dots$, $q = 1 - p$. MGF of X is given by

$$\begin{aligned}
 M_X(t) &= \sum_{k=1}^{\infty} e^{tk} f(k) \\
 &= \sum_{k=1}^{\infty} e^{tk} q^{k-1} p \\
 &= p(e^t + e^{t2}q + e^{t3}q^2 + \dots) \\
 &= p \cdot e^t (1 + (e^t q) + (e^t q)^2 + (e^t q)^3 + \dots) \\
 &= pe^t \frac{1}{1 - e^t q}
 \end{aligned}$$

which is finite for t ,

$$\begin{aligned}
 0 &< e^t \cdot q < 1 \\
 e^t &< \frac{1}{q} \\
 t &< \ln\left(\frac{1}{q}\right)
 \end{aligned}$$

Thus,

$$M_X(t) = \frac{pe^t}{1 - qe^t}, \quad \text{with } t < \ln\left(\frac{1}{q}\right)$$

Definition 8.5 — For each n positive integer, if $E[X^n] = \sum_{x \in S_X} x^n f(x)$ exists then $E[X^n]$ is called the n^{th} moment of X .

Remark 8.6. Properties of MGF $M_X(t)$

- $t = 0$, $M_X(0) = E[e^{0 \cdot X}] = E[1] = 1$.
- Derivatives of $M_X(t)$ is given by

$$\begin{aligned}\frac{d}{dt}[M_X(t)] &= \frac{d}{dt} [E[e^{tX}]] \\ &= E \left[\frac{d}{dt} e^{tX} \right] \quad \text{assume } \frac{d}{dt} \text{ and } E \text{ are interchangeable} \\ M'_X(t) &= E[Xe^{tX}]\end{aligned}$$

Thus,

$$M'_X(t) \Big|_{t=0} = E[Xe^{0 \cdot X}] = E[X], \text{ first moment of } X$$

- Similarly, 2nd derivative of $M_X(t)$ given by

$$\begin{aligned}M''_X(t) &= E[X^2 e^{tX}] \\ M''_X(t) \Big|_{t=0} &= E[x^2], \text{ second moment of } X\end{aligned}$$

- More generally, the n^{th} - derivative of M_X satisfies

$$M_X^{(n)}(t) \Big|_{t=0} = E[x^n]$$

hence the name “mgf”.

Example 8.7

$X \sim \text{Geom}(p)$.

$$\begin{aligned}M_X(t) &= \frac{pe^t}{1 - qe^t}, \quad q = 1 - p \\ M'_X(t) &= \frac{pe^t}{(1 - qe^t)^2} \\ M'_X(0) &= \frac{p}{(1 - q)^2} = \frac{p}{p^2} = \frac{1}{p} = E[x]\end{aligned}$$

§9 | Dis 1: Oct 6, 2020

§9.1 Set Theory

Definition 9.1 — A set is a collection of items.

Example 9.2

$$T = \{1, 2, 3, \text{red}, \text{blue}\}$$

$$S = \{1, 3, \text{red}\}$$

$$R = \{1, 2, 4\}$$

$$S \subseteq T$$

$$S' = S^c = \{2, \text{blue}\}$$

$$R \not\subseteq T$$

$$\begin{array}{ccc} 3 & \underbrace{\in} & T \\ & \text{is an element of} & \\ & \{3\} \subseteq T & \end{array}$$

Example 9.3

$$A = \{1, 3, 7\} \quad A \cup B = \{1, 2, 3, 4, 7\}$$

$$B = \{2, 3, 4\} \quad A \cap B = \{3\}$$

$$A \setminus B = \{1, 7\} \quad B \setminus A = \{2, 4\}$$

De Morgan Laws:

$$(A \cup B)' = A' \cap B'$$

$$(A_1 \cup A_2 \cup \dots \cup A_n)' = A_1' \cap A_2' \cap \dots \cap A_n'$$

$$(A \cap B)' = A' \cup B'$$

If have a sample space S , and subset of S are called events. A probability function is a function \mathbb{P} that assigns a real number each event with three rules:

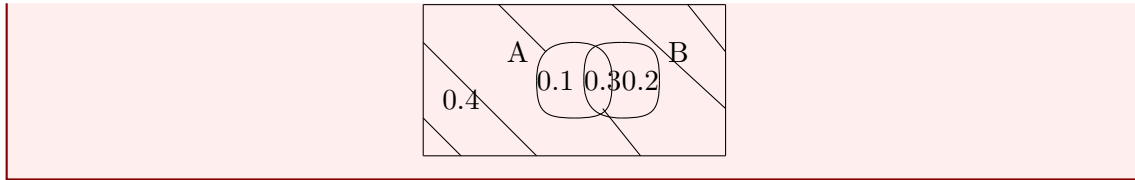
1. $P(A) \geq 0$
2. $P(S) = 1$
3. A_1, A_2, \dots, A_n with $A_i \cap A_j = \emptyset = \{\}$, then $P(A_1 \cup \dots \cup A_n) = P(A_1) + \dots + P(A_n)$

Example 9.4

1.1-6 (from the book): $P(A) = 0.4$, $P(B) = 0.5$, $P(A \cap B) = 0.3$

Find

- $P(A \cup B) = .1 + .3 + .2 = .6$
- $P(A \cap B)' = .1$
- $P(A' \cap B) = .2$



Note: (P, S) : probability space on all subsets of S

Example 9.5

1.2-5: How many four letter codes can be made from the letters in IOWA if

- Letters may not be repeat: $4! = 24$ ways.
- Letters may repeat: $4^4 = 256$ ways.

§10 | Dis 2: Oct 13, 2020

1.4.16: An urn has 5 balls. One is marked “win” and the other are marked “lose”. You and another player each take balls out one at a time until somebody picks win. You pick first.

W/o replacement: $P(\text{winning}) = \frac{1}{5} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot 1 = \frac{3}{5}$

With replacement:

$$\begin{aligned} P(\text{winning}) &= \frac{1}{5} + \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} + \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} + \dots \\ &= \frac{\frac{1}{5}}{1 - \frac{16}{25}} = \frac{5}{9} \end{aligned}$$

§10.1 Conditional Probabilities

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

or $P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$

Example 10.1

1.3.7: An urn has 4 balls. 2 are red and 2 are blue. We pull out 2 balls. We are told that at least one is red. What’s the probability that they’re both red?

$$P(\text{both red} | \text{at least one red}) = \frac{P(\text{both red and at least one red})}{P(\text{at least one red})} = \frac{P(\text{both red})}{P(\text{at least one red})} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}.$$

§10.2 Bayes’s Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Example 10.2

1.5-8: Four types of tablets: B_1, B_2, B_3, B_4 with % of sales 0.4, 0.3, 0.2, 0.1 and % tablet needs repair 0.1, 0.05, 0.03, 0.02, respectively. What is the probability that a tablet needing repair is B_i ?

$$\begin{aligned} P(B_1|\text{need repair}) &= \frac{P(\text{need repair}|B_1) \cdot P(B_1)}{P(\text{need repair})} \\ &= \frac{(0.1)(0.4)}{(0.40)(0.10) + (0.30)(0.05) + (0.20)(0.03) + (0.10)(0.02)} \\ &\approx 63.5\% \end{aligned}$$

$$P(B_2|\text{need repair}) = \frac{(0.30)(0.05)}{0.063} \approx 23.8\%$$

$$P(B_3|\text{need repair}) \approx 9.5\%$$

$$P(B_4|\text{need repair}) \approx 3.2\%$$