

Math 131AH – Honors Real Analysis I

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This is math 131AH – Honors Real Analysis I taught by Professor Greene, and our TA is Haiyu Huang. We meet weekly on MWF from 1:00pm – 2:00pm for lectures. There are two textbooks used for the class, *Principles of Mathematical Analysis* by Rudin and *Metric Spaces* by Copson. You can find other lecture notes at my [github](#) site. Please let me know through my [email](#) if you spot any mathematical errors/typos.

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§1 | Lec 1: Oct 2, 2020

Overview:

- Hmwrk: 30 %
- Midterm 1: 20 %
- Midterm 2: 20 %
- Final: 30 %

§1.1 Introduction

functions $\rightarrow 1, 2, 3, 4, 5, 6, 7 \dots$

functions defined on \mathbb{Q} with value in \mathbb{Q}

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

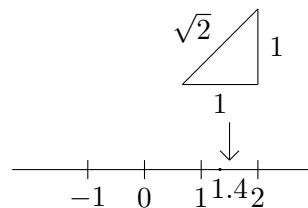
$a_i \in \mathbb{Q}$ $f(x) \in \mathbb{Q}$ if $x \in \mathbb{Q}$. Continuity makes sense.

$$x_0, x \text{ close to } x_0 \implies f(x) \text{ close } f(x_0)$$

polynomials are continuous.

Something wrong: $\sqrt{2}$ is missing. What are these numbers that are not $\in \mathbb{Q}$? Choice:

1. Assume everything works and isolate what you need about "real numbers" (most of Rudin chap 1).
2. Construct the real numbers from rational numbers.



Classical argument:

$$x^2 \neq 2 \text{ if } x = \frac{p}{q} \in \mathbb{Q}$$

Proof. Suppose $\left(\frac{p}{q}\right)^2 = 2$

Note: wolog (without loss of generality)

can take $\frac{p}{q} > 0$ $p > 0$ $q > 0$

$$\begin{aligned} \left(\frac{p}{q}\right)^2 &= 2 \\ \frac{p^2}{q^2} &= 2 \\ p^2 &= 2q^2 \end{aligned}$$

Now also wolog, can assume p and q are not both even numbers. But $p^2 = 2q^2$ means p has to be even (p^2 odd if p is odd).

$$\begin{aligned} p &= 2n \\ p^2 &= 2q^2 \\ 4n^2 &= 2q^2 \end{aligned}$$

So $q^2 = 2n^2$, q is even. But it contradicts the initial assumption, p and q not both even \square

Related to: Why functions \mathbb{Q} to \mathbb{Q} not ideal for analysis?
– INFINITE DECIMAL

§2 | Lec 2: Oct 5, 2020

§2.1 Mathematical Induction and More on Real Numbers

$P(n) \rightarrow 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$, where n is positive numbers.

Math induction: Proof by two steps:

1. Check $P(1)$ is true \checkmark
2. Assume $P(n)$ is true for all $n \leq N$. Check that

$$P(N+1) \text{ is true}$$

Assume $1 + \dots + N = \frac{N(N+1)}{2}$. Check

$$1 + \dots + N + (N+1) = \frac{(N+1)(N+1+1)}{2}$$

Induction on k :

$$1^k + 2^k + \dots + n^k$$

2nd illustration:

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad r \neq 1$$

$$r = 1 \implies 1 + r = \frac{1 - r^2}{1 - r}$$

$$\begin{aligned} 1 + r + r^2 + \dots + r^n + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \\ &= \frac{1 - r^{n+1} + r^{n+1} - r^{n+2}}{1 - r} \\ &= \frac{1 - r^{n+2}}{1 - r} \end{aligned}$$

$$(1 - r)(1 + r + \dots + r^n) = 1 - r^{n+1} \quad \text{Inspection}$$

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r}, \quad r \neq 1$$

$|r| < 1$ get infinite sum $\frac{1}{1-r}$

Example 2.1

Prime factors, prime = positive integers (> 1) with no factors except itself and 1,
 $p = ab$, $a > 1$, $b > 1$

2 3 5 7 11 13 17 19 ...

Thin out as go along

Theorem 2.2 (Fundamental Theorem of Arithmetic)

Every positive integer > 1 is a product of primes.

Proof. Induction: $P(n)$ $n = 2, 3, \dots$

$$P(2) = 2\checkmark$$

Assume $P(n) \dots n \leq N$ ($N > 2$). Every integer greater than 1 but smaller than or equal to N as a product of primes. We try to prove: $N + 1$ is a product of primes.

1. $N + 1$ is prime: Done $N + 1 = N + 1$

2. $N + 1$ is not a prime

$$N + 1 = a \cdot b \quad a > 1 \quad b > 1$$

Induction assumption ($a < N + 1$ since $b > 1$), a is a product of primes $a > 1 \implies b < N + 1$, b also a product of primes. So, $N + 1 = ab$ is a product of primes.

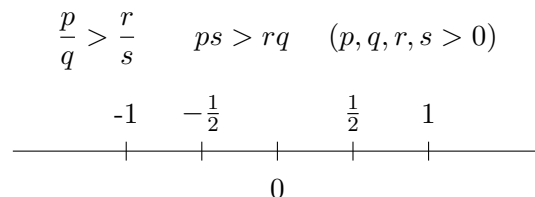
$N + 1 = ab$ is a product of prime. □

Why does induction work? If $P(n)$ not always true, $P(n)$ look at smallest n where $P(n)$ is false.

$n = 1$ not there $P(1)$ is supposed true (checked already). N_0 smallest one where $P(N_0)$ false $N_0 > 1$. Induction step says that $P(n)$ is true for all $n \leq \underbrace{N_0 - 1}_{>0} \implies P(N_0)$ true (\times).

Let's go back to real numbers.

Last time: talked about $\sqrt{2}$ is irrational but $\sqrt{2}$ exists, so we need to enlarge our number system: \mathbb{Q} rational numbers.



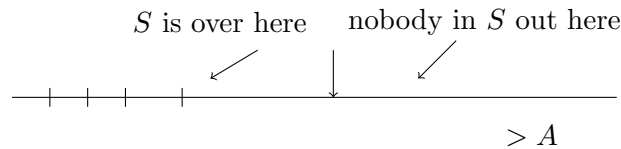
x, y rational $x, y > 0$, $x + y > 0$, $xy > 0$

$x^2 = 2$ no answer in \mathbb{Q} . Enlarge number system, $\mathbb{Q} \subset \mathbb{R}$. What should \mathbb{R} be like?

1. \mathbb{R} ought to have arithmetic like \mathbb{Q}

$$x + y \quad xy \quad \frac{x}{y} \quad 0 \quad 1$$

2. $\mathbb{Q} \subset \mathbb{R}$, arithmetic in \mathbb{R} restricted to \mathbb{Q} , $\frac{1}{2} + \frac{1}{3}$ in \mathbb{Q} ought to be $\frac{5}{6}$ in \mathbb{R} .
3. Order should positive in $\mathbb{Q} \implies$ in \mathbb{R} . \mathbb{R} should have an order of its own too, $x > y$ positive then $x + y$ pos and xy pos.
4. want to fill in the holes in \mathbb{Q} . Want to have **Least Upper Bound Property**
 $S \subset \mathbb{R}$: An upper bound for S is a number A with property $A \geq x$ if $x \in S$



$1, 2, 3, 4, \dots$ have no upper bound.

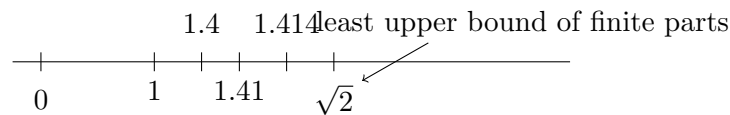
S is bounded above means that some upper bound A exists.

§2.2 Least Upper Bound Property

If S is bounded above ($S \neq \emptyset$) then it has a “least upper bound” where a number A_0 is called the least upper bound of S if A_0 is an upper bound for S & if A is an upper bound for S then $A_0 \leq A$.



Motivation: Think about $\sqrt{2}$



Denote: l.u.b(or supremum)(sequence) = $\sqrt{2}$

Means can define an infinite decimals: least upper bound of successive truncation.

$$0.99999 \dots \rightarrow 1.0$$

§3 | Lec 3: Oct 7, 2020

§3.1 Cauchy Sequence

$\{x_n\}$ x_1, x_2, x_3, \dots values $x_j \in \mathbb{Q}$ $x_j \in \mathbb{R}$
 S $x_1, x_i \dots x_j \in S$

Definition 3.1 (Sequence) — A sequence with values in a set S is a function from positive integers $\{1, 2, 3 \dots\}$ into S .

Definition 3.2 (Cauchy Sequence) — A Cauchy sequence is (\mathbb{Q} valued or \mathbb{R} valued) $\{x_i\}$ is sequence s.t. for every $\epsilon > 0$ there is a positive integer N_ϵ s.t.

$$|x_i - x_j| < \epsilon \quad \text{if } i, j > N_\epsilon$$

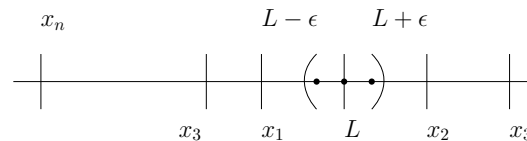


ϵ rational or real (same idea).

Lemma 3.3

If $\{x_j\}$ has a finite limit then it's a Cauchy sequence.

$\{x_i\}$ has L as a limit $\lim x_j = L$ means for every $\epsilon > 0$ then there is an N_ϵ such that $j \geq N_\epsilon, |x_j - L| < \epsilon$

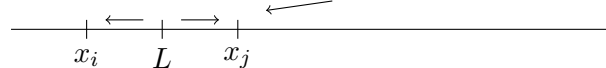


Everybody in $(L - \epsilon, L + \epsilon)$ except a finite number

Proof. Given $\epsilon > 0$, want to find N so that $i, j \geq N \implies |x_i - x_j| < \epsilon$
 $|x_i - L|$ small, $|x_j - L|$ small and $\lim x_j = L$.

$$|x_i - x_j| \leq |x_i - L| + |x_j - L|$$

$$|x_i - x_j| = |L - x_i| + |L - x_j|$$



$i, j \geq N_{\frac{\epsilon}{2}} :$

$$|x_i - x_j| \leq \underbrace{|x_i - L|}_{< \frac{\epsilon}{2}} + \underbrace{|x_j - L|}_{< \frac{\epsilon}{2}}$$

Because $\lim x_n = L$, there is an $N_{\frac{\epsilon}{2}}$ s.t. $|L - x_n| < \frac{\epsilon}{2}$ if $n \geq N_{\frac{\epsilon}{2}}$

Get $|x_i - x_j| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ if $i, j \geq N$. Cauchy sequence: there exists number N s.t.

$$|x_i - x_j| < \epsilon \quad \text{if } i, j \geq N$$

□

Cauchy sequence \implies the existence of limit? Yes, for \mathbb{R} valued sequences but NO for \mathbb{Q} valued things.

$\underbrace{\{x_n\}}_{\text{rational numbers}}$ can be Cauchy seq without there being a rational number L such that $\lim x_j = L$

But allow real L then $\exists L$ s.t. $\lim x_j = L$ if $\{x_j\}$ is Cauchy sequence (no rational limit – since $\sqrt{2}$ is irrational). Because \mathbb{Q} has holes in it! (intuitive idea).

Example 3.4

1, 1.4, 1.41, 1.414, 1.4142... (decimal approx of $\sqrt{2}$) – Cauchy sequence. No – since $\sqrt{2}$ is irrational.

§3.2 Cauchy Completeness of \mathbb{R}

If $\{x_j\}, x_j \in \mathbb{R}$ is Cauchy sequence, then $\exists L \in \mathbb{R}$ s.t. $\lim x_j = L$.

“ \mathbb{Q} is not Cauchy complete” but \mathbb{R} is. Why does this work?

Need: Least upper bound property. Assume L.U.B Property proof.

Proof. (Cauchy completeness from L.U.B Property)

Hypothesis: $\{x_i\}$ Cauchy seq

1. Prove that $\{x_i\}$ bounded $\iff \exists M > 0$ s.t. $|x_i| \leq M$ all i .

Clear if take $\epsilon = 1$ in def. of Cauchy seq $\exists N$ s.t. $|x_i - x_j| < 1$ if $i, j \geq N \implies |x_N - x_j| < 1$ if $j \geq N \implies |x_j| \leq |x_N| + 1 \quad j \geq N$

So, $M = \max(|x_N| + 1, |x_1|, \dots, |x_{N-1}|)$ then $|x_j| \leq M$ all j !

Next stage is to show that a bounded sequence always has a subsequence(tricky!) with a limit. Then if a Cauchy seq has a subseq with limit L , then L is limit of whole seq. (Bolzano – Weierstrass Theorem)

□

§4 | Lec 4: Oct 9, 2020**§4.1 Bolzano – Weierstrass Theorem**

– implied by Least Upper Bound Property

Theorem 4.1 (Bolzano – Weierstrass)

If $\{x_n\}$ sequence $(x_1, x_2, x_3 \dots)$ that is bounded (means: $\exists M > 0 \ni |x_n| \leq M \forall n$), then $\exists L$ and a subsequence $\{x_{n_i}\}$ s.t. $\lim x_{n_i} = L$.

Slogan: Every bounded sequence has a convergent subsequence.

Example 4.2

1, 2, 1, 2, 1, 2, ...

The subsequence of the above sequence has either 1 or 2 as the limit.

1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, ...

Unbounded sequence – subsequence (limit 1, limit 2, limit 3...)

No claim of uniqueness of anything.

Proof – Summer 2008 Analysis Lec 4

Proof. So either $[-M, 0]$ or $[0, M]$ (maybe both) contains x_n for infinitely many n values. If each contained x_n for only finitely many n values X .

$$\begin{array}{c} -M \qquad \qquad \qquad 0 \qquad \qquad \qquad M \\ | \qquad \qquad \qquad | \qquad \qquad \qquad | \\ \hline \end{array}$$

Every x_n is in $[-M, M] - \{x_n\}$ is bounded

$$[-M, M] = [-M, 0] \cup [0, M]$$

$$I_1 = [-M, 0] \quad \text{or} \quad [0, M]$$

where chosen interval has x_n for infinitely many n values.

Do this again!

$$I_1 = [a_1, b_1] \quad |b_1 - a_1| = M$$

$$I_1 \longleftarrow \text{length}$$

$$\begin{array}{c} | \qquad \qquad \qquad | \qquad \qquad \qquad | \\ \hline \end{array}$$

left half of I_1 , right half of I . Let $I_2 =$ one of halves that contains x_n for infinitely many n values.

$$I_2 = [a_2, b_2] \quad a_2 < b_2, \quad b_2 - a_2 = \frac{M}{2}$$

Continue

$$I_3 = [a_3, b_3] \quad a_3 < b_3, \quad b_3 - a_3 = \frac{M}{4}$$

$$\vdots$$

$$I_k = [a_k, b_k] \quad b_k - a_k = \frac{M}{2^{k-1}}$$

Each I_k contains x_n for infinitely many n values.

$$\begin{array}{c} \text{Nested Intervals} \\ a_1 \qquad \qquad I_1 \qquad \qquad b_1 = b_2 \\ | \qquad \qquad | \qquad | \qquad | \qquad | \\ \hline \qquad \qquad \nearrow \qquad \qquad \nwarrow \\ \qquad a_3 \qquad \qquad b_3 \\ I_{k+1} \subset I_k \subset \dots \subset I_1 \subset [-M, M] \\ a_{k+1} \geq a_k \dots \qquad b_{k+1} \leq b_k \dots \end{array}$$

Claim $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$

Reason: $\sup a_k \in \bigcap_{k=1}^{\infty} I_k$ where $\sup =$ sup of left hand endpoint (=greatest lower bound of bs). l.u.b of a 's $\leq b_k$, b_k bigger than or \geq all a 's.

$$\alpha = \text{lub } a\text{'s}$$

$$\alpha \geq a_k \quad \forall k$$

$$\alpha \leq b_k \quad \forall k$$

$$\alpha \in [a_k, b_k]$$

Goal: $\alpha \in \bigcap_{k=1}^{\infty} I_k$. Find a subsequence of $\{x_n\}$ converges to α .

Choose $x_k = x_n$ that belongs to I_k . Can also arrange successively:

$$n_1 < n_2 < n_3 < n_4$$

$x_{n_1} \in I_1$ $x_{n_2} \in I_2$ can make $n_2 > n_1$ because infinitely possible x'_n s in I_2 n value.
Continue to get subsequence, $\{x_{n_k}\}$ subsequence. Claim:

$$\lim_{k \rightarrow \infty} x_{n_k} = \alpha$$

Reason:

$$\text{dis}(x_{n_k}, \alpha) \leq \text{length of } I_k \quad \alpha \in I_k, \quad x_{n_k} \in I_k$$

which is equivalent to

$$|x_{n_k} - \alpha| \leq \frac{M}{2^{k-1}} \quad \text{given } \epsilon > 0$$

When k is large,

$$\frac{M}{2^{k-1}} < \epsilon$$

So $|x_{n_k} - \alpha| < \epsilon$

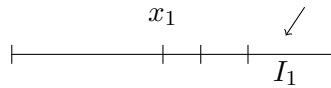
□

This argument (or a variant) shows something else:

If $\{x_n\}$ sequence in $[0, 1]$ then there's an $\alpha \in [0, 1]$ with it never happening that

$$x_n = \alpha$$

“The real numbers in $[0, 1]$ are uncountable.” (come from the least upper bound property)



I_1 one of $[0, \frac{1}{3}]$ $[\frac{1}{3}, \frac{2}{3}]$ $[\frac{2}{3}, 1]$ such that $x_1 \notin I_1$,

$$[0, \frac{1}{3}] \cap [\frac{1}{3}, \frac{2}{3}] \cap [\frac{2}{3}, 1] = \emptyset$$

$x_1 \notin I_2$ $I_2 \subset I_1$, & $x_1 \notin I_1$. Continue. Get

$$I_1 \supset I_2 \supset I_3 \supset \dots$$

length $I_k = \frac{1}{3^k}$ and I_k is such that $x_1, x_2, x_3 \dots x_k$ are none of the ?n? in I_k . Same as before

$$\exists \alpha \in \bigcap_{k=1}^{\infty} I_k$$

$\alpha = \sup$ of set of left hand endpoints of I_k . Claim α cannot be an x_N value. Clear: $x_N \notin I_N$ but $\alpha \in I_n$ $\alpha \in \bigcap_{n=1}^{\infty} I_n$. But contrast:

There is a list of rational numbers in $[0, 1]$

	$\frac{p}{q}$	$p < q$				
	2	3	4	5	6	...
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$			
2	-	$\frac{2}{3}$	$\frac{2}{4}$			
3	-	-	$\frac{3}{4}$			
\vdots	-	-	$\frac{\sqrt{2}}{2} \in [0, 1] \rightarrow$	irrational - no exist		
			$[0, 1]$	<div> <div>not</div> <div>countable</div> </div>		
Q is countable						

§5 | Lec 5: Oct 12, 2020

§5.1 Equivalence Relation

(p.10, Copson – Metric Space)

R set, relation of A and B ($A \times B$) $(a, b) \in R \implies aRb$

Functions: one b given a – exact one. ($A \rightarrow B$)

Example 5.1

$A = B = Q$

aRb or $(a, b) \in R$ if $a > b$

(mother, child)

- $(\text{Sara}, \text{Sebastian}) \in R$
- $(\text{Sara}, \text{Alita}) \in R$

Equivalence is a special kind of relation: (on a set $A; B \subseteq A \times A$)

Properties:

1. $aRa \implies A = Q$
2. $aRb \implies bRa$
3. $aRb \ \& \ bRc$ then aRc

Example: \mathbb{Z} $a \sim b$ means $a - b$ is divisible by 5

$$1 \sim 6 \quad 0 \sim 5 \dots$$

$$a \sim a \quad a - b \text{ div } 5 \implies b - a \text{ div. by } 5.$$

If $a - b$ div. by 5, and $b - c$ div by 5, then is $a - c$ div. by 5 true?

$$\text{Sure, } a - b = 5k, \quad b - c = 5l \implies a - c = 5(k + l)$$

“Equivalence classes”: set $[a] = \{ \text{all } b \text{ such that } aRb \}$

In the example above, $[a] = \{ \text{all } b \text{ such that } a - b \text{ div. by } 5 \}$

$$[2] = \{2, 7, -3, 12, -8, \dots\}$$

\mathbb{Z}_5 : integer mod 5.

1. $[a] \cap [p]$ either equal or have nothing in common.
2. $a \in [a]$ so is in some equivalence class.

A equivalence relation \sim on $A \leftrightarrow$ a partition of A into subsets which are pairwise disjoint.

\mathbb{Q} Cauchy seq. of rational numbers

$$\{x_n\} \sim \{y_n\}$$

means $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$. Equivalence relation:

1. $\{x_n\} \sim \{x_n\}$ ($\lim(x_n - x_n) = 0$)
2. $\{x_n\} \sim \{y_n\} \implies \{y_n\} \sim \{x_n\}$
3. $\{x_n\} \sim \{y_n\} \& \{y_n\} \sim \{z_n\} \implies \{x_n\} \sim \{z_n\}$

Idea: Define a real number to be a (Cauchy seq. of rationals) equivalence class.

Homework: want to check that arithmetic extends to “real numbers”

$$[\{x_n\}] + [\{z_n\}] = [\{x_n + z_n\}]$$

Check that

1. $\{x_n + z_n\}$ is a Cauchy seq.
2. Only depends on equivalence classes.

Want

$$\{x_n\} \sim \{y_n\} \quad \{z_n\} \sim \{w_n\}$$

then $\{x_n + z_n\} \sim \{y_n + w_n\}$. So,

$$[\{x_n + z_n\}] = [\{y_n + w_n\}]$$

Example 5.2

$$[2] + [11] = [2 + 11] = [13]$$

So, $[2 + 1] \sim [13]([11] = [1])$. Arithmetic (addition) in \mathbb{Z}_5 thus makes sense. How about multiplication? $\frac{[1]}{[a]} \leftarrow$ exists $[a] \neq 0$.

$$\frac{[1]}{[2]} = [3] \quad [2][3] = [6] = [1]$$

Thus, \mathbb{Z}_5 is a field.

$\frac{p}{q} \sim \frac{r}{s}$, $q, s \neq 0$ means $ps = rq$ (when talking about fractions – associate it with equivalence relation). Q = set of equivalence classes. $(\frac{p}{q})$: equivalence classes).

Last time, we proved that Cauchy seq. of real numbers have limits (lub property). Also, no sequence $\{x_n\}$ such that it hits all real numbers in $[0, 1]$ – this is important. Contrast with $Q \cap [0, 1]$, then there is a sequence that hits them all. Refer to the last figure in Lec 4 or math.ucla.edu/~greene – Summer 2008.

§6 | Lec 6: Oct 14, 2020

Bolzano - Weierstrass:

Every bounded sequence has a convergent subsequence.

And we know about the Least Upper Bound Prop.

§6.1 Continuous Functions on Closed Interval

$$f : S \rightarrow \mathbb{R}, \quad S \subset \mathbb{R}$$

Example 6.1

$$S = [a, b]$$

$$S = \mathbb{R}$$

Definition 6.2 (Continuity) — $s_0 \in S$, f is continuous at s_0 if given $\epsilon > 0$, $\exists \delta > 0$ s.t.

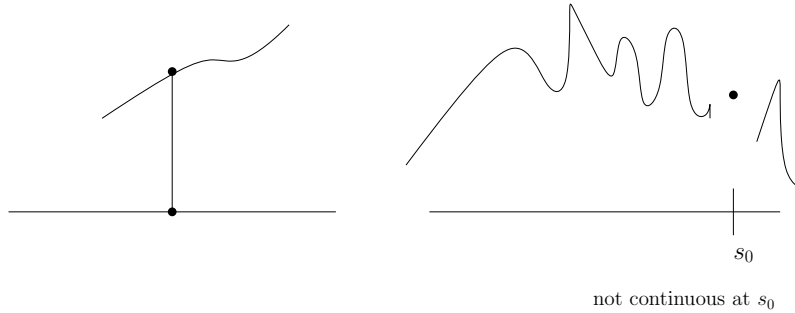
$$|s - s_0| < \delta_\epsilon \implies |f(s) - f(s_0)| < \epsilon$$

Three properties:

$$f : [a, b] \rightarrow \mathbb{R}$$

f continuous

1. f is bounded on $[a, b]$ means $\exists M$ s.t. for all $x \in [a, b]$, $|f(x)| \leq M$



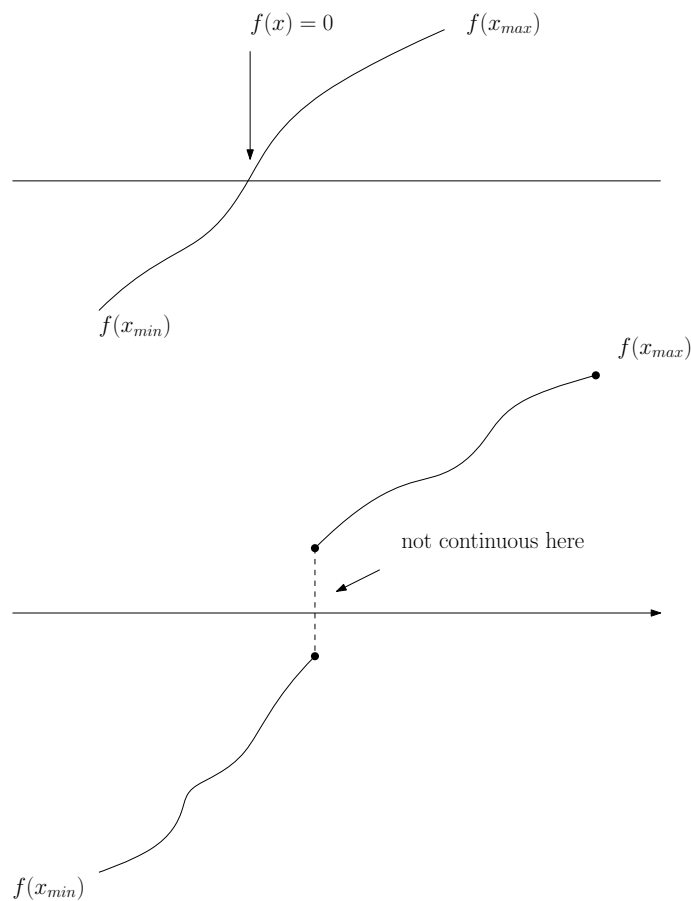
2. There exists $x_{\min}, x_{\max} \in [a, b]$ such that for all $x \in [a, b]$

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

Slogan: f attains its maximum and minimum.

3. If $\alpha, f(x_{\min}) < \alpha < f(x_{\max})$, then $\exists x \in S = [a, b]$ s.t. $f(x) = \alpha$.

“Intermediate Value Theorem” Need the least upper bound prop – “completeness of



real numbers”

Exercise: def of continuity $\{s_n\}$ converges to $s_0 \iff$ if $s_n \rightarrow s_0, s_n \in S, s_0 \in S$ then $\{f(s_n)\}$ converges to $f(s_0)$.

Example 6.3

For (3),

$$f(x) = x^2 - 2 \quad \text{on } \mathbb{Q} \cap [1, 2]$$

Then $f(1) = -1$, $f(2) = 2$, but no rational $x \in [1, 2]$ s.t. $f(x) = 0$.

Back to the properties:

1. f is bounded – Think about $|f| \leftarrow$ continuous if f is (exercise).

$\exists M$ such $|f(x)| \leq M$ all $x \in [a, b]$. Suppose no such M exists.

Try $M = 1, 2, 3, 4, 5, 6, \dots$ So $\exists x_1 \quad |f(x_1)| > 1$

$$|f(x_2)| > 2$$

$$\vdots$$

$$|f(x_n)| > n$$

But Bolzano – Weierstrass: subsequence $\{x_{n_j}\}$ that converges to x_0 say $|f(x_0)| \leftarrow$



finite number. So $\exists N \ni |f(x_0)| \leq N$.

Now for j large enough

$$|f(x_{n_j}) - f(x_0)| < 1$$

x_{n_j} converges to x_0

$$|f(x_{n_j})| < |f(x_0)| + |f(x_{n_j} - f(x_0))|$$

So j is large enough that

$$\underbrace{|f(x_{n_j})|}_{\geq |f(x_0)|} \leq N + \text{something less than } 1 \leq N$$

2. Attains max and min

Similar: $\{f(x) : x \in [a, b]\}$ bounded set, has sup where

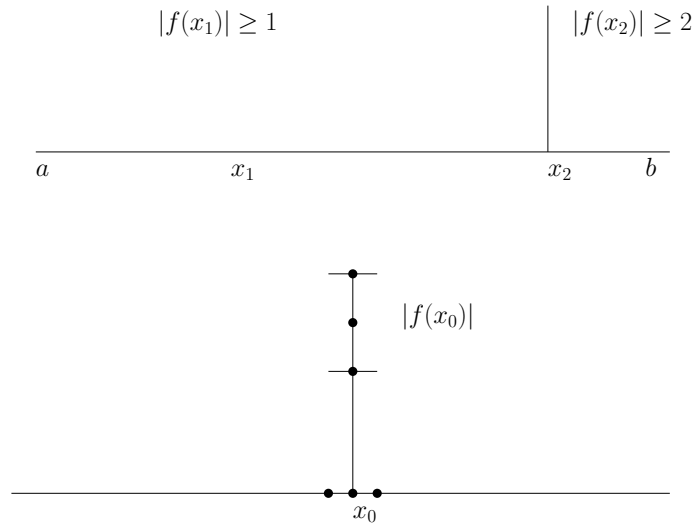
$$\sup \{f(x) : x \in [a, b]\}$$

either in the set of f -values (done if that's true), $\sup f = f(x_0)$.

OR: $\sup f$ actually not in the set $\{f(x) : x \in [a, b]\}$

Now $\{x_{n_j}\}$ converges to $x_0 \in [a, b]$

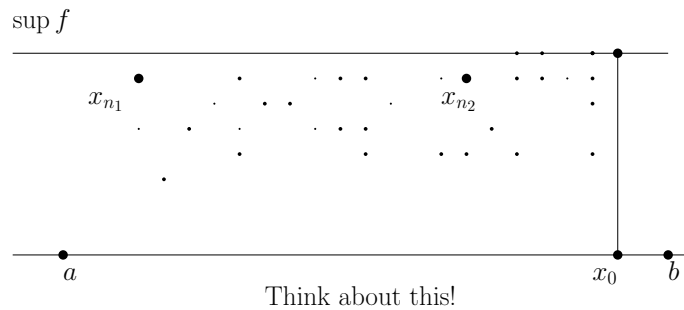
Claim 6.1. $f(x_0) = \sup \{f(x) : x \in [a, b]\}$



$$f(x_{n_j}) \leq \sup \{f(x) : x \in [a, b]\}$$

and $\lim f(x_{n_j}) = f(x_0) = f(\lim x_{n_j})$. So

$$f(x_0) = \sup f$$



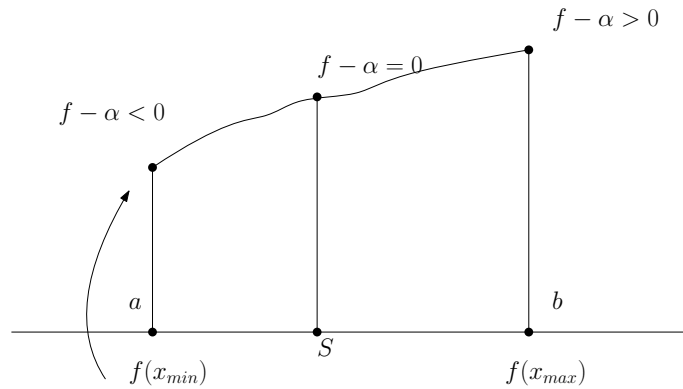
Think about this!

3. $\alpha \in [f(x_{\min}), f(x_{\max})]$ then x such that $f(x) = \alpha$.

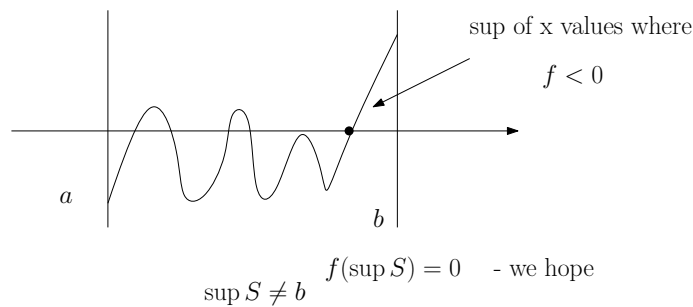
Proof. Wolog:

$$f(a) < 0 \quad \text{and} \quad f(b) > 0$$

then $\exists x \in [a, b]$ with $f(x) = 0$.

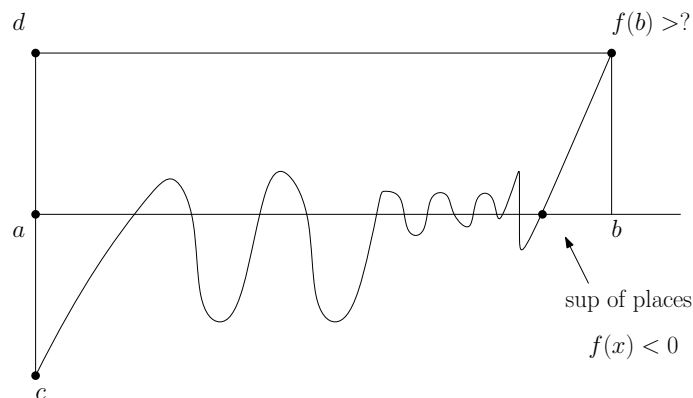


Use l.u.b: Look at $S : \{x : f(x) < 0\}$ and $S \neq \emptyset$ because $f(a) \in S$. Also, S is bounded above $\rightarrow \exists$ l.u.b for S , $\sup S \in [a, b]$. Hope that $f(\sup S) = 0$.



$\sup S \neq b$ is clear because $f(b) > 0$ so $f(b - \epsilon) > 0$ for small ϵ .

So $\sup S = x_0$, $a < x_0 < b$. What is $f(x_0)$? If it's negative, then there are slightly bigger $x \in [a_0, b] \ni f(x) < 0$ (continuity). In addition, x_0 cannot be a limit of x with $f(x) < 0 \rightarrow x_0 = \sup$ places where $f < 0$. \square



f continuous on $[a, b]$ if it is

1. bounded.
2. attains max and min.
3. attains every value between max value and min value.

$f([a, b]) = [c, d]$ where c is min of f and d is max of f .

§7 | Lec 7: Oct 16, 2020

§7.1 Uniform Continuity

Definition 7.1 (Uniform Continuity) — $S \subset \mathbb{R}$, $f : S \rightarrow \mathbb{R}$. f is uniformly continuous on S if given $\epsilon > 0$ there is a $\delta > 0$ s.t. $|f(x) - f(y)| < \epsilon$ if $x, y \in S$ and $|x - y| < \delta_\epsilon$

Example 7.2

$f : S \rightarrow \mathbb{R}$, $S = \mathbb{R}$, $f(x) = x^2$. Continuous on \mathbb{R} but it is not uniformly continuous on \mathbb{R} .

Continuity: Given fixed x , and $\epsilon > 0$ want δ so that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

$|x^2 - y^2| = |x - y||x + y|$ and want it smaller than ϵ . Assume $\delta \leq 1$.

$$\begin{aligned} |x + y| &\leq |x| + |y| \\ |y| &< |x| + 1 \quad \text{if } |x - y| < \delta (\leq 1) \end{aligned}$$

So, if $|x - y| < \delta (\leq 1)$,

$$\begin{aligned} |x^2 - y^2| &= |x - y||x + y| \\ &\leq |x - y|(2|x| + 1) \end{aligned}$$

Choose $\delta < \frac{\epsilon}{2|x|+1}$ (ok since x is fixed)

$$\begin{aligned} |x^2 - y^2| &< \frac{\epsilon}{2|x|+1}(2|x|+1) \\ &= \epsilon \quad \text{if } |x - y| < \min \left\{ 1, \frac{1}{2|x|+1} \right\} \end{aligned}$$

Uniform continuity does not work on \mathbb{R} .

Claim 7.1. $\epsilon = 1 > 0$, there is no $\delta > 0$ s.t. $|x^2 - y^2| < 1 = \epsilon$ for all x, y with $|x - y| < \delta$.

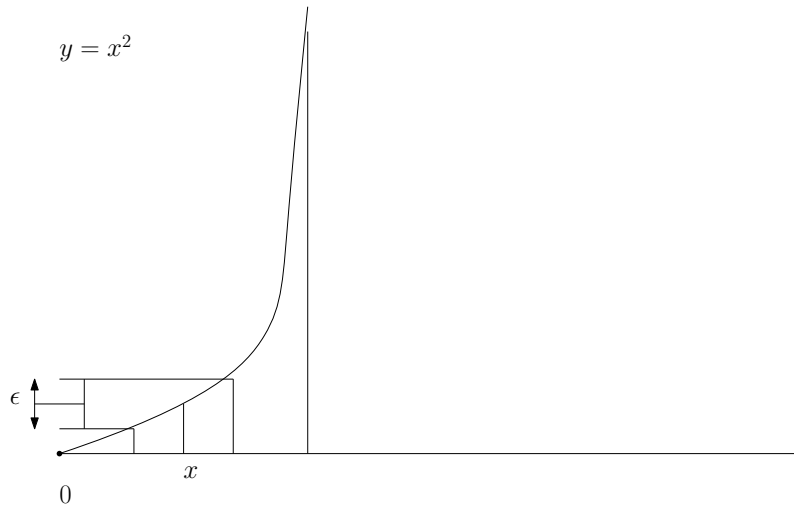
Why? Look at for $\delta > 0$, consider $y = \frac{1}{\delta} + \frac{\delta}{2}$, $x = \frac{1}{\delta}$

$$|x - y| < \delta$$

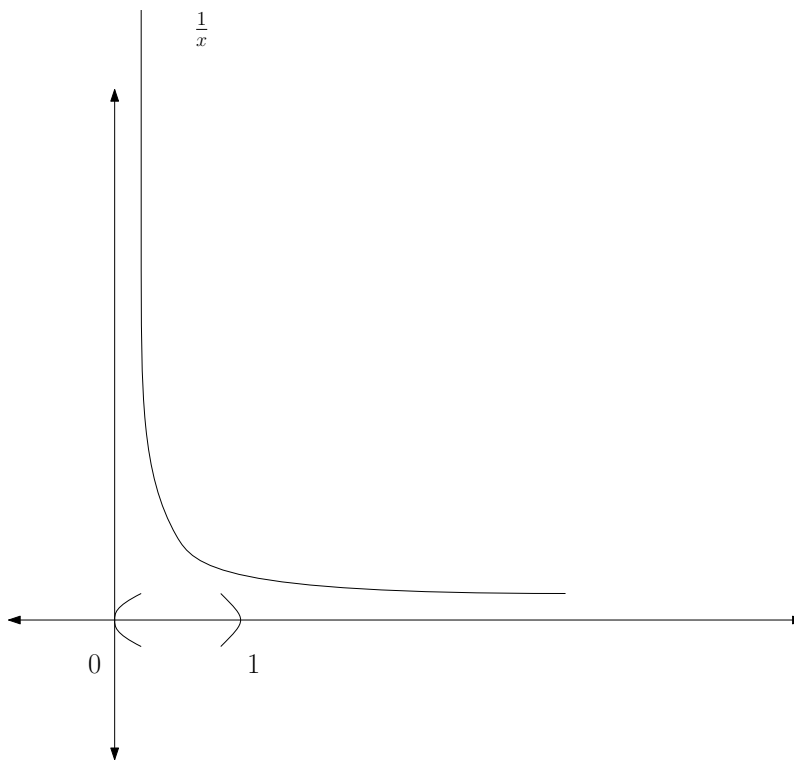
Also,

$$\begin{aligned} &\left| \left(\frac{1}{\delta} + \frac{\delta}{2} \right)^2 - \left(\frac{1}{\delta} \right)^2 \right| \\ &= \left| \frac{1}{\delta^2} + 2 \left(\frac{1}{\delta} \right) \left(\frac{\delta}{2} \right) + \left(\frac{\delta}{2} \right)^2 - \frac{1}{\delta^2} \right| \\ &= 1 + \left(\frac{\delta}{2} \right)^2 > 1 \end{aligned}$$

which is a contradiction.



Exercise 7.1. $\frac{1}{x}$ on $(0, 1)$ is continuous but not uniformly continuous. Suggest plausibly f



continuous on $[a, b]$ then it's uniformly continuous on $[a, b]$ where a, b are finite.

Theorem 7.3 (Heine – Cantor (Uniformly Continuous))

A continuous function f on a closed interval is uniformly continuous.

Proof. (By contradiction) Suppose not. Then $\epsilon > 0$ s.t. no δ “works”. In particular, $\exists \epsilon > 0$

s.t. $\delta = 1$ fails, $\delta = \frac{1}{2}$ fails, etc. So $x, y \in [a, b]$ with $|f(x_1) - (fy_1)| \geq \epsilon$ but $|x_1 - y_1| < 1$.
 $x_n, y_n \in [a, b]$ with $|f(x_n) - f(y_n)| \geq \epsilon$ but $|x_n - y_n| < \frac{1}{n}$. Hope this is impossible.
 Bolzano - Weierstrass $\implies \{n_j\}$ s.t. $\{x_{n_j}\}$ has a limit

$$x_0 = \lim, \quad x_0 \in [a, b]$$

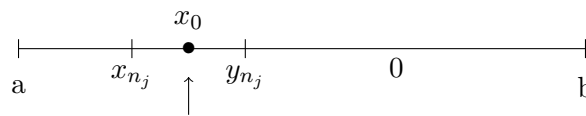
Now, claim $\{y_{n_j}\}$ also has limit x_0 .

$$|x_{n_j} - y_{n_j}| < \frac{1}{n_j}$$

small when n_j large (j large).

$$\begin{aligned} \lim x_{n_j} &= x_0 \\ \lim y_{n_j} &= x_0 \\ \lim f(x_{n_j}) &= f(x_0) \\ \lim f(y_{n_j}) &= f(x_0) \end{aligned}$$

So, $\lim f(x_{n_j}) - f(y_{n_j}) = 0$, but it contradicts $|f(x_{n_j}) - f(y_{n_j})| \geq \epsilon$ for all j . \square



$$f(x_0) \leq |f(x_{n_j}) - f(x_0)| + |f(x_0) - f(y_{n_j})| \rightarrow 0$$

Ideas of continuity and uniform continuity and Bolzano - Weierstrass Theorem – all have reasons in metric spaces.

§8 | Lec 8: Oct 19, 2020

§8.1 Convergence of Series

Series is “formal sum”, an infinite sum

$$a_0 + a_1 + a_2 + \dots = \sum_{j=1}^{\infty} a_j$$

A series \iff sequence a_1, a_2, a_3, \dots add together. Associated to $a_1 + a_2 + a_3 + a_4 \dots$ is a sequence of partial sum

$$S_N = \sum_{n=1}^N a_n, \quad N = 1, 2, 3, 4, 5, \dots$$

number valued sequence.

Definition 8.1 (Convergence of Series) — Series converges if sequence associated $\{S_N\}$ converges (has a limit).

Lots of things are defined by series such as ($x \in \mathbb{R}$),

$$e^x = \lim_{N \rightarrow \infty} \left(1 + x + \frac{x^2}{2!} + \dots + \frac{x^N}{N!} \right)$$

Given series $a_0 + a_1 + a_2 + a_3 + \dots$, when does it converge?

$$1 - 2 + 3 - 4 + 5 - 6 + 7 \dots$$

$$S_1 = 1, \quad S_2 = -1, \quad S_3 = 2 \dots$$

NO LIMIT! Series do not necessarily have to converge then it's okay to write

$$\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n$$

First thing to look at – Case where $a_j \geq 0$

$$S_N \leq S_{N+1}, \quad N = 1, 2, 3, \dots$$

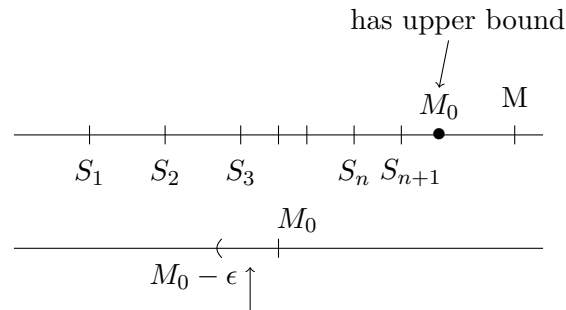
$S_{N+1} = S_N + a_{N+1}$ so $a_{N+1} \geq 0$ means $S_{N+1} \geq S_N$. Two cases:

Case 1: $\{S_n\}$ not bounded above.

$\lim S_N$ does not exist \rightarrow Series diverges (sequences with limits are always bounded above and below).

Case 2: $\{S_n\}$ bounded above.

$\lim_{n \rightarrow \infty} S_n$ always exists. Namely, it is the least upper bound of set of values of S_n .



There is an S_{n_0} in this interval $(M_0 - \epsilon, M_0]$, M_0 is lub

From that n_0 on,

$$S_n \geq S_{n_0}, \quad S_n \leq M$$

S_n satisfies $|S_n - M_0| < \epsilon$ if $n \geq n_0$. So $\lim S_n = M_0$. This implies that S_n is a Cauchy

sequence (it has a limit). Given $\epsilon > 0$, $\exists N_\epsilon$ s.t. $\left| \sum_{1}^{n_1} a_n - \sum_{1}^{n_2} a_n \right| < \epsilon$ if $n_1, n_2 \geq N_\epsilon$.

Suppose $n_1 > n_2 \geq N_\epsilon$

$$\sum_{1}^{n_1} a_n - \sum_{1}^{n_2} a_n = \sum_{n_2+1}^{n_1} a_n$$

Note: $S_7 - S_5 = a_6 + a_7$ which explains the above expression.

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} \dots$$

converges, but so does the following series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots < 2$$

This works for arbitrary choices of $+$ or $-$.

Theorem 8.2 (Absolute Convergence)

If $|b_1| + |b_2| + |b_3| + \dots$ converges, then

$$b_1 + b_2 + b_3 + \dots \text{ converges}$$

“Absolute convergence” \implies convergence (but not necessarily the same limit).

Proof. Assume $\underbrace{\{S_n^A\}}_{A \text{ for absolute}}$ for absolved series has limit. So

$$\sum_1^\infty |b_n| \text{ converges}$$

$\implies \{S_n^A\}$ Cauchy sequence.

We hope it $\implies \{S_n\} = \left\{ \sum_{j=1}^n b_j \right\}$ is a Cauchy sequence.

$$S_{n_1}^A - S_{n_2}^A = |b_{n_2+1}| + |b_{n_2+2}| + \dots + |b_{n_1}|$$

But

$$|b_{n_2+1} + \dots + b_{n_1}| \leq |b_{n_2+1}| + \dots + |b_{n_1}| (= S_{n_1}^A - S_{n_2}^A)$$

So,

$$|S_{n_1} - S_{n_2}| \leq S_{n_1}^A - S_{n_2}^A < \epsilon \text{ for } n_1, n_2 \geq N_\epsilon$$

Then $|S_{n_1} - S_{n_2}| < \epsilon$ for $n_1, n_2 \geq N_\epsilon$. □

This is IMPORTANT – Better understand it thoroughly.

Corollary 8.3 (Root Test)

$|b_n| \leq Cr^n, 0 < r < 1, C, r$ fixed, then $\sum b_n$ converges.

Reason: $\sum_{n=0}^\infty Cr^n = C \frac{1}{1-r}$ (geometric series).

Exercise 8.1. $\sum_{n=0}^N Cr^n = C \frac{r^{N+1}-1}{r-1}, 0 < r < 1$ has limit $\frac{C}{1-r}$. Prove by induction.

Detail: Hypothesis:

$$|b_n| \leq Cr^n$$

$$\sum_1^\infty |b_n| \leq \sum_1^\infty Cr^n < \infty$$

$$\sum_b^N |b_n| \leq \sum_0^N Cr^n \leq M < \infty$$

So $\sum_0^N |b_n|$ converges and bounded by Cr , and $b_1 + b_2 + \dots$ converges absolutely.

§9 | Lec 9: Oct 21, 2020

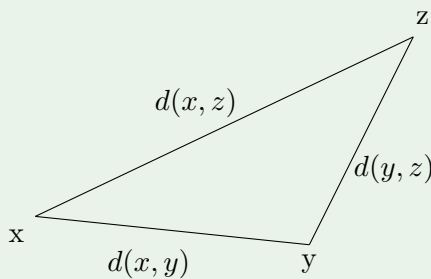
§9.1 Metric Spaces

Definition 9.1 (Metric Spaces) — A set X , elements are “points”, together with a function on $\underbrace{X \times X}_{\text{ordered pairs } (x,y)}$, $x \in X, y \in Y$, $\underbrace{d(x,y)}_{\text{distance}}$ with the following properties:

1. $d(x, y) \geq 0$ for all x, y .
 $d(x, y) = 0 \iff x = y$. Or $d(x, x) = 0$.
2. $d(x, y) = d(y, x)$.
3. \triangle inequality:

$$d(x, y) + d(y, z) \geq d(x, z)$$

$$d(x, z) \leq d(x, y) + d(y, z)$$



Example 9.2 1. X set. Can you define a $d : X \times X \rightarrow \mathbb{R}$ to make (X, d) a metric space?

YES! Define given set X , $d(x_1, x_2) = 0$ if $x_1 = x_2$, or $d(x_1, x_2) = 1$ if $x_1 \neq x_2$. “discrete”.

- $d(x, y) \geq 0$.
- $d(x, y) = d(y, x)$.
 $x = y$ both are 0.
 $x \neq y$ both are 1.
- $d(x, z) \leq d(x, y) + d(y, z)$
 $x = z \implies d = 0$.
 $x \neq z \implies d(x, z) = 1$.
 If $x = y$ then $y \neq z$ so $1 \leq 0 + 1$
 \dots

2. (INTERESTING) $d(x, y) = |x - y|$ for \mathbb{R} .

$$d\left(\frac{p}{q}, \frac{r}{s}\right) = \left|\frac{p}{q} - \frac{r}{s}\right| \text{ for } \mathbb{Q}.$$

Note: X is a metric space $Y \subset X$ then $\left(Y, d|_{Y \times Y}\right)$ is a metric space.

Motivation: Stuff about \mathbb{R} involving e.g., continuity and limits can be transferred to metric space.

Example 9.3

$\{x_n\}$ is a sequence in a metric space (X, d) (or X) has limit $x_0 \in X$ if for every $\epsilon > 0$, there is an N_ϵ s.t. $d(x, x_0) < \epsilon$ if $n \geq N_\epsilon$. (If $X = \mathbb{R}$, $d(x, y) = |x - y|$ same as before)

Example 9.4

Function: $f : (X, d_1) \rightarrow (Y, d_2)$. Continuity at $x_0 \in X$?

Real case: f cont at x_0 means given $\epsilon > 0 \exists \delta > 0$ s.t. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

Metric space case: f cont at x_0 means given $\epsilon > 0 \exists \delta > 0$ s.t. $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \epsilon$.

More examples:

Example 9.5

$$\mathbb{R}^2 = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$$

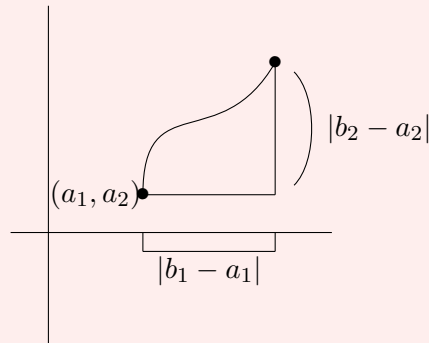
$$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}\}$$

\vdots

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \dots, x_n \in \mathbb{R}\}$$

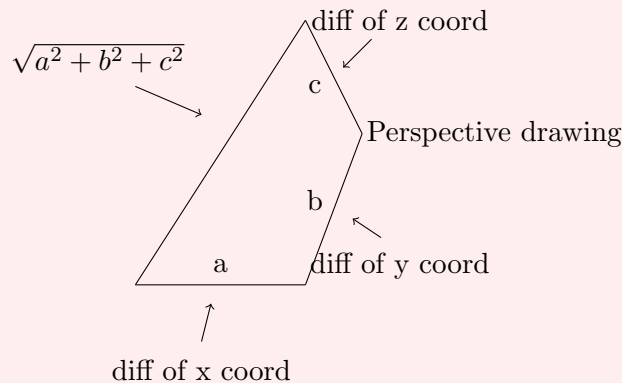
Interesting metric on \mathbb{R}^2 $d((a_1, a_2), (b_1, b_2))$

$$d = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$$



$\mathbb{R}^n(x_1, x_2, \dots, x_n), (y_1, \dots, y_n)$

$$d := \sqrt{(y_1 - x_1)^2 + \dots + (y_n - x_n)^2}$$



Is this function on \mathbb{R}^n a metric?

1. $d(x, y) \geq 0, = 0 \iff x = y$ where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ and

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

2. $d(x, y) = d(y, x)$

3. BUT BUT BUT \triangle inequality is not so easy.

$$\sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \leq \sqrt{(x_1 - z_1)^2 + \dots + (x_n - z_n)^2} + \sqrt{(z_1 - y_1)^2 + \dots + (z_n - y_n)^2}???$$

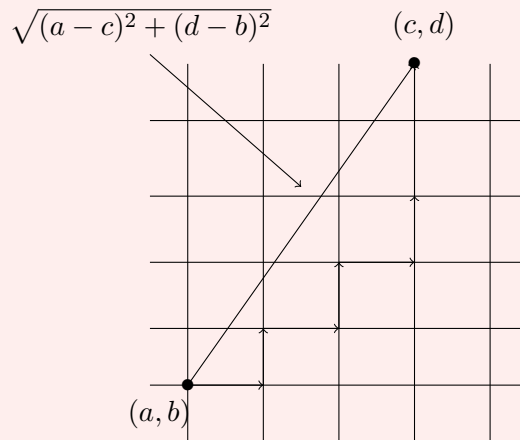
Does $d(x, y) \leq d(x, z) + d(z, y)$ work?

YES but proof later :(

Realize that it's okay to assume $z = (0, 0, \dots, 0)$

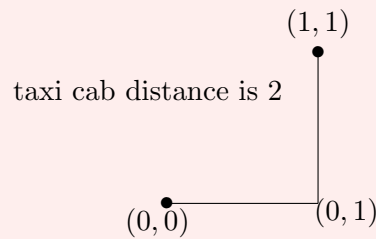
Example 9.6

Try another metric
 \mathbb{R}^2 – taxicab



← min of length of taxi car

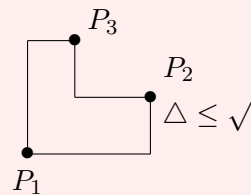
Easy to see that this d is really a metric. \triangle inequality is easy!



Euclidean distance = $\sqrt{2}$

diff of x's \leq Euc dis

diff of y's \leq Euc dis

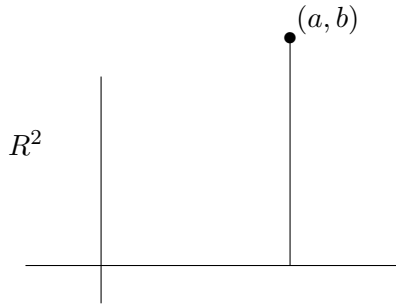


$$d(P_1, P_2) + d(P_2, P_3) \geq d(P_1, P_3)$$

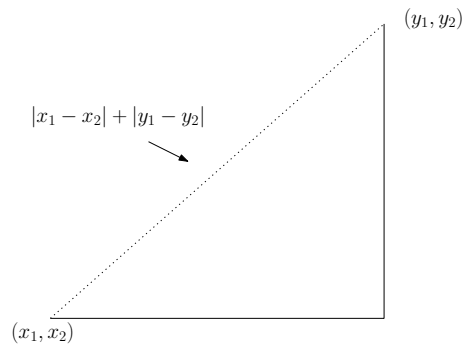
§10 | Lec 10: Oct 23, 2020

§10.1 Metric on \mathbb{R}^n

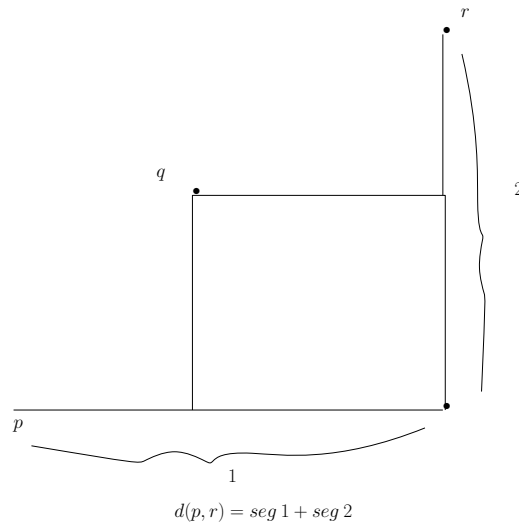
$$\mathbb{R}^n : \{(x_1, \dots, x_n) : x_j \in \mathbb{R}\}$$



We want to make \mathbb{R}^n a metric space. Last time, we defined “taxi cab metric”, $d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{i=1}^n |x_i - y_i|$. Verify $d(\vec{x}, \vec{y}) \geq 0$ or $= 0$ if $\vec{x} = \vec{y}$ and \triangle inequality,



$$d(p, q) + d(q, r) \geq d(p, r)$$

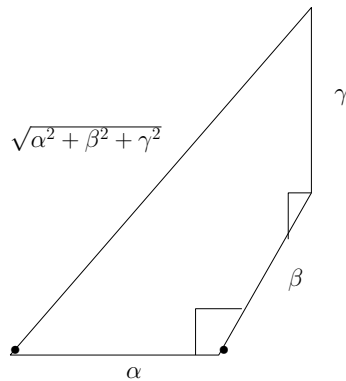
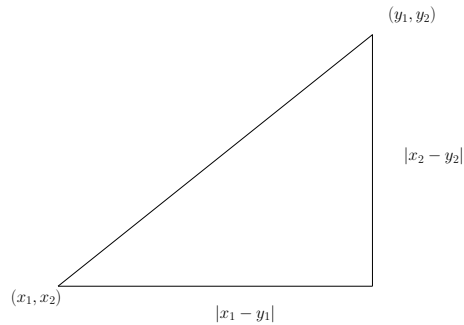


§10.2 Triangle Inequality in Euclidean Space

New idea: Euclidean distance (or Pythagorean distance)

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$\text{For } \mathbb{R}^n : d((a_1, \dots, a_n), (b_1, \dots, b_n)) := \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \dots + (a_n - b_n)^2}.$$



We need to know:

1. $d(\vec{a}, \vec{b}) \geq 0$
2. $d(\vec{a}, \vec{a}) = 0$ so $d(\vec{a}, \vec{b}) = 0 \implies \vec{a} = \vec{b}$
3. $d(\vec{a}, \vec{b}) = d(\vec{b}, \vec{a})$
4. ? $\triangle \leq 0, \vec{a}, \vec{b}, \vec{c}$

$$d(\vec{a}, \vec{c}) \leq d(\vec{a}, \vec{b}) + d(\vec{b}, \vec{c})$$

For \mathbb{R}^n ,

$$\sqrt{(a_1 - c_1)^2 + \dots + (a_n - c_n)^2} \leq \sqrt{(a_1 - b_1)^2 + \dots + (a_n - b_n)^2} + \sqrt{(b_1 - c_1)^2 + \dots + (b_n - c_n)^2}$$

We certainly need proof for \triangle inequality: Copson($p > 1$) – for case $p = 2$

First step: $\alpha\beta \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2$ for all real α, β .

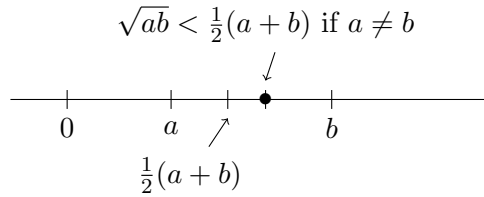
Reason:

$$\begin{aligned} 2\alpha\beta &\leq \alpha^2 + \beta^2 \\ \alpha^2 + \beta^2 - 2\alpha\beta &\geq 0 \\ (\alpha - \beta)^2 &\geq 0 \checkmark \end{aligned}$$

“Geometric mean \leq Arithmetic mean”

Let $\alpha = \sqrt{a}, \beta = \sqrt{b}, a, b \geq 0$

$$\underbrace{\sqrt{ab}}_{\text{geometric mean of a,b}} \leq \frac{1}{2}(a) + \frac{1}{2}(b) = \underbrace{\frac{1}{2}(a+b)}_{\text{arithmetic mean}}$$



Second step:

$$\vec{a} = (a_1, \dots, a_n)$$

$$\vec{b} = (b_1, \dots, b_n)$$

and we know

$$a_i b_i \leq \frac{1}{2} a_i^2 + \frac{1}{2} b_i^2$$

Then,

$$\sum_{i=1}^n a_i b_i \leq \frac{1}{2} \sum_{i=1}^n a_i^2 + \frac{1}{2} \sum_{i=1}^n b_i^2$$

So, $\sum a_i^2 = 1$, $\sum b_i^2 = 1$, $\sum a_i b_i \leq 1$

Claim 10.1.

$$\sum a_i b_i \leq \left(\sum a_i^2 \right)^{\frac{1}{2}} \left(\sum b_i^2 \right)^{\frac{1}{2}}$$

But

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$$

So it's okay to define θ ,

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \in [-1, 1]$$

Verification of claim: $\vec{a}, \vec{b} \neq \vec{0}$

$$A_i = \frac{a_i}{\sqrt{\sum a_i^2}}, \quad B_i = \frac{b_i}{\sqrt{\sum b_i^2}}$$

And $\sum A_i^2 = 1$, $\sum B_i^2 = 1$ – also $\sum_{i=1}^n A_i B_i \leq 1$ which is equivalent to $\frac{\sum a_i b_i}{\sqrt{\sum a_i^2} \sqrt{\sum b_i^2}} \leq 1$.

So $|\sum a_i b_i| \leq \sqrt{\sum a_i^2} \sqrt{\sum b_i^2}$.

BIG DEAL: “Cauchy Schwarz inequality” What does this have to do with \triangle inequality for Euclidean metric. Consider: \vec{a}, \vec{b}

$$\sum_{j=1}^n (a_j + b_j)^2 = \sum_{j=1}^n a_j (a_j + b_j) + \sum_{j=1}^n b_j (a_j + b_j)$$

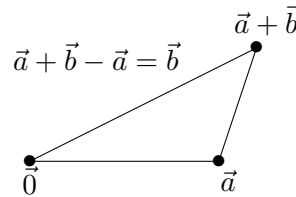
Now apply Cauchy – Schwarz

$$\begin{aligned} \sum_{j=1}^n (a_j + b_j)^2 &\leq \left(\sum_{j=1}^n a_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n (a_j + b_j)^2 \right)^{\frac{1}{2}} \\ &\quad + \left(\sum_{j=1}^n b_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n (a_j + b_j)^2 \right)^{\frac{1}{2}} \end{aligned}$$

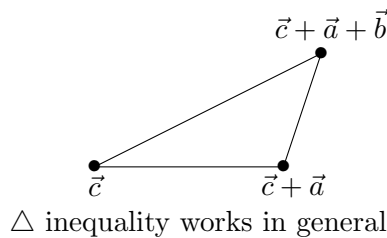
Divide through by $(\sum (a_j + b_j)^2)^{\frac{1}{2}}$

$$\left(\sum (a_j + b_j)^2\right)^{\frac{1}{2}} \leq \left(\sum a_j^2\right)^{\frac{1}{2}} + \left(\sum b_j^2\right)^{\frac{1}{2}}$$

The above inequality is indeed the triangle inequality for $\vec{0}, \vec{a}, \vec{a} + \vec{b}$



But of course this gives you the triangle inequality in general.



Last step: $\vec{p}, \vec{q}, \vec{r}$

Triangle inequality:

$$d(0, \vec{p} - \vec{r}) \leq d(0, \vec{q} - \vec{p}) + d(\vec{q} - \vec{p}, \vec{r} - \vec{p})$$

Same as \triangle ineq for $0, \vec{q} - \vec{p}, (\vec{r} - \vec{q}) + (\vec{q} - \vec{p})$ or
 $0, \vec{a}, \vec{a} + \vec{b}$ if $\vec{a} = \vec{q} - \vec{p}, \vec{b} = \vec{r} - \vec{q}$.

§11 | Lec 11: Oct 26, 2020

§11.1 Metric Spaces Examples

Last time, we prove \triangle ineq. proof, taxi-cab metric, and sup norm metric. This gives rise to same “convergence idea”. Namely $x_n \in X(X, d)$ converges to $L \in X$ means

$$\lim_{n \rightarrow \infty} (x_n - L) = 0$$

In all three metrics

$$\vec{x}_j \rightarrow L \quad \lim \vec{x}_j = L$$

means (is same as) i th coordinate of \vec{x}_j converges to i th coord of L for each $i = 1, 2, \dots, n$.
 $\{x_n\}$ Cauchy if given $\epsilon > 0 \exists N_\epsilon \ni n_1, n_2 \geq N_\epsilon$

$$d(x_{n_1}, x_{n_2}) < \epsilon$$

Exercise 11.1. $\{x_n\}$ Cauchy in \mathbb{R}^n (any one of three metrics – Cauchy is the same idea in all three metrics) then $\{x_n\}$ has limit L , some L .

$$\sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \leq \sqrt{n} \max |x_j - y_j|, j = 1, \dots, n$$

So,

$$\begin{aligned} |x_j - y_j| &\leq \max |x_l - y_l| \\ |x_j - y_j|^2 &\leq \max^2 |x_l - y_l|, l = 1, \dots, n \\ (x_1 - y_1)^2 + \dots + (x_n - y_n)^2 &\leq n \max^2 |x_l - y_l| \\ \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} &\leq \sqrt{n} \max \end{aligned}$$

$l_2 : \{x_j\}$ infinite sequences $j = 1, 2, 3, \dots$ where $\left\{ \sum_{j=1}^{\infty} x_j^2 < \infty \right\}$ which means

$$\exists M \ni \sum_{j=1}^M x_j^2 \leq M$$

$$\begin{aligned} (1, \frac{1}{2}, \frac{1}{3}, \dots) &\in l_2 \\ (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \dots) &\notin l_2 \end{aligned}$$

because $1 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \dots \rightarrow \infty$ vector space:

$$\begin{aligned} c\{x_j\} &= \{cx_j\} \\ \{x_j\} \in l_2 &\implies \in l_2 \\ \sum c^2 x_j^2 &= c^2 \sum x_j^2 \end{aligned}$$

Also,

$$\begin{aligned} \{x_j\} + \{y_j\} &= \{x_j + y_j\} \\ (x_j + y_j)^2 &\leq 2(x_j^2 + y_j^2) \\ x_j y_j &\leq \frac{1}{2}(x_j^2 + y_j^2) \end{aligned}$$

$\{x_j\}, \{y_j\} \in l_2$ then

$$d(\{x_j\}, \{y_j\}) = \left[\sum (x_j - y_j)^2 \right]^{\frac{1}{2}}$$

makes sense. (l_2, d) is a metric space obvious except \triangle ineq. It's enough to check

$$d(0, \vec{x}) + d(\vec{x}, \vec{x} + \vec{y}) \geq d(0, \vec{x} + \vec{y})$$

which follows by taking limits of \triangle ineq. for truncation up to level N .

$$d(\vec{0}, (x_1, \dots, x_N)) + d((y_1, \dots, y_N), (x + y) \text{ up to } N) \geq d(\vec{0}, (x + y)_N)$$

l_2 is metric space

l_2 is complete – Cauchy sequences have some limits.

Example 11.1

$C([0, 1]) := \text{cont: } \mathbb{R} - \text{valued function } [0, 1]$

$$\begin{aligned} d(f, g) &= \max |f(x) - g(x)| \\ &= \sup |f(x) - g(x)| \end{aligned}$$

“sup norm” All properties clear. “ L^2 norm” – distance on $C[0, 1]$:

$$d_2(f, g) = \left(\int_0^1 (f(x) - g(x))^2 \right)^{\frac{1}{2}}$$

where $d_2 \geq 0$, $f, g, h \in C[0, 1]$.

Imitate argument for \triangle ineq. on \mathbb{R}^n : Cauchy Schwarz ineq.

$$\int_0^1 fg \leq \left(\int_0^1 f^2 \right)^{\frac{1}{2}} \left(\int_0^1 g^2 \right)^{\frac{1}{2}}$$

So,

$$\begin{aligned} f(x)g(x) &\leq \frac{1}{2} (f^2(x) + g^2(x)) \\ \int_0^1 f(x)g(x) &\leq \frac{1}{2} \int_0^1 f^2(x) + \frac{1}{2} \int_0^1 g^2(x) \end{aligned}$$

Apply these, $F = \frac{f(x)}{\sqrt{\int_0^1 f^2}}$, $G = \frac{g}{\sqrt{\int_0^1 g^2}}$, $\int F^2 = 1$, $\int G^2 = 1$. Also, we know $\int fg \leq 1$ if $\int f^2 = 1$, $\int g^2 = 1$.

Remainder argument for \triangle ineq. is same as before

$$\int (f + g)^2 = \int f(f + g) + \int g(f + g)$$

Apply Cauchy – Schwartz,

$$\begin{aligned} \int (f + g)^2 &\leq \left(\int f^2 \right)^{\frac{1}{2}} \left(\int (f + g)^2 \right)^{\frac{1}{2}} + \left(\int g^2 \right)^{\frac{1}{2}} \left(\int (f + g)^2 \right)^{\frac{1}{2}} \\ \left(\int (f + g)^2 \right)^{\frac{1}{2}} &\leq \left(\int f^2 \right)^{\frac{1}{2}} + \left(\int g^2 \right)^{\frac{1}{2}} \end{aligned}$$

§11.2 A Glance at Complex Number

Special case of \mathbb{R}^n , Euclidean norm

$$\begin{aligned} \mathbb{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} &= d((x_1, x_2), (y_1, y_2)) \\ \mathbb{C} : \{(a + bi)\} &- \text{Complex numbers} \end{aligned}$$

$(x_1, x_2) \leftrightarrow x_1 + ix_2$. Metric on \mathbb{C} , $z, w \in \mathbb{C}$

$$|z - w| = d(z, w) \quad \text{as pts in } \mathbb{R}^2$$

$$z = a + bi$$

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

We also define multiplication in \mathbb{C} as follows

$$(a + bi)(c + di) := (ac - bd) + (bc + ad)i$$

Example 11.2

$$\frac{1}{c + di} = \frac{c}{c^2 + d^2} - \frac{d}{c^2 + d^2}i$$

For $z = a + bi, w = c + di$ we define

$$\begin{aligned} |zw| &= |z||w| \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} \\ &= \sqrt{(ac - bd)^2 + (bc + ad)^2} \end{aligned}$$

verify if the above step is actually equal

§12 | Lec 12: Oct 28, 2020

§12.1 Midterm Announcement

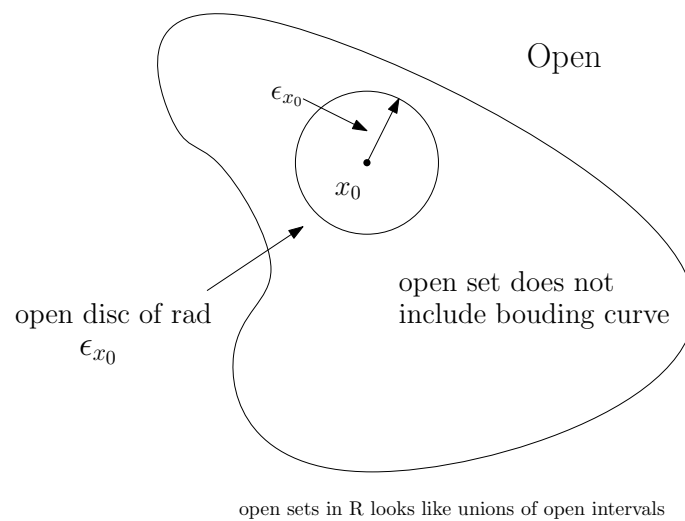
Midterm – Given out at Fri, Nov 6 at 3:00 pm. and due by Sat, Nov 7 at 11:00 pm.

§12.2 Open sets in Metric Space

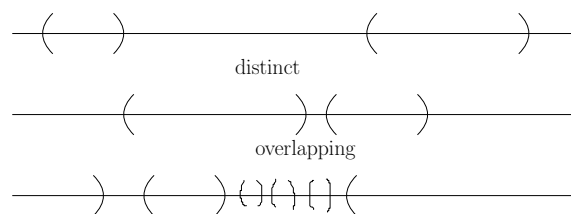
Beginning of “topology”: (X, d) metric space

Definition 12.1 (Open sets) — $U \subset X$ open if for every $x_0 \in U$ there is an $\epsilon_{x_0} > 0$ s.t.

$$\underbrace{\{x \in X : d(x, x_0) < \epsilon_{x_0}\}}_{B(x_0, \epsilon_{x_0})\text{- open ball}} \subset U$$

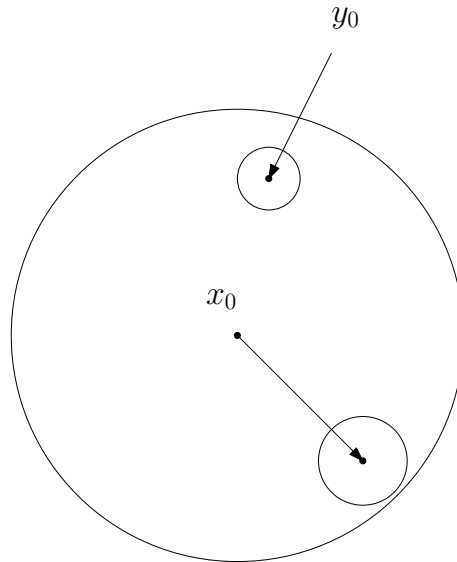


Open set in \mathbb{R}



Lemma 12.2

$B(x_0, \epsilon), \epsilon > 0$ open ball is open set.



Proof. Need given $y \in B(x_0, \epsilon)$, $\lambda_y > 0$ s.t. $B(y, \lambda) \subset B(x_0, \epsilon)$.

Try $\lambda = \epsilon - d(x_0, y_0)$.

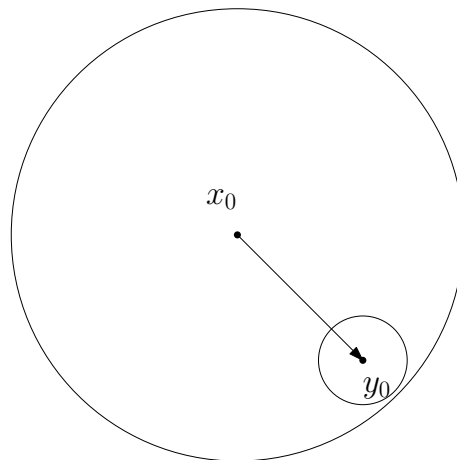
Suppose $y \in B(y_0, \epsilon) \iff d(y_0, y) < \epsilon - d(x_0, y_0)$

$$d(y_0, y) + d(x_0, y_0) < \epsilon$$

So,

$$d(x_0, y) \leq d(x_0, y_0) + d(y_0, y) < \epsilon$$

So $y \in B(x_0, \epsilon)$.



□

Reason why people care about open sets:

Remember: $f(X, d) \rightarrow (Y, d)$ continuous means given $\epsilon > 0, x_0 \in X$ there exists $\delta > 0$ s.t.

$$d_X(x_0, x) < \delta \implies d_Y(f(x_0), f(x)) < \epsilon$$

\rightarrow Direct transcription of number – continuity of f can be described in terms of open sets in X and in Y . For this: $f : X \rightarrow Y$ and $V \subset Y$, then $f^{-1}(V) = \{x \in X : f(x) \in V\}$ f does not need to be invertible.

Example 12.3

$$f : \underbrace{X}_{\text{people}} \rightarrow \mathbb{Z}, \quad f(x) = \text{integer age of } x$$

$$f^{-1}(\{20, 21, 22\}) = \text{everybody that's age 20, 21, or 22}$$

Theorem 12.4 (Continuity – Open Sets)

$f : (X, d_x) \rightarrow (Y, d_y)$ is continuous if and only if (in δ, ϵ sense) $f^{-1}(V)$ is open in X for every V open in Y .

Slogan: continuity means inverses of open sets are open.

$f : X \rightarrow Y, g : Y \rightarrow Z \rightarrow g(f(x))$ compositions of f and g .

Claim 12.1. If f, g continuous then the composition is continuous

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Proof. (of Theorem) Suppose $f^{-1}(V)$ is open when V is open. Given $x_0 \in X, \epsilon > 0$ want $\delta > 0 \ni x \in B(x_0, \delta) \implies d(f(x), f(x_0)) < \epsilon$

$$\underbrace{x \in B(x_0, \delta)}_{d(x, x_0) < \delta}$$

$$\{y : d(y, f(x_0)) < \epsilon\} = B(f(x_0), \epsilon)$$

Know that it's open by the above lemma. So,

$$f^{-1}(B(f(x_0), \epsilon)) \text{ open}$$

and $x_0 \in (B(f(x_0), \epsilon))$. So $f^{-1}(B(f(x_0), \epsilon))$ being open

$$\implies \delta > 0 \quad B(x_0, \delta) \subset f^{-1}(B(f(x_0), \epsilon))$$

says $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \epsilon \checkmark$

Took care of $f^{-1}(\text{open})$ is open \implies continuity. Now,

Does continuity (ϵ, δ sense) $\implies f^{-1}(\text{open})$ is open?

This also works: Suppose V open, and $x_0 \in f^{-1}(V)$. Need $\delta > 0$ s.t. $B(x_0, \delta) \subset f^{-1}(V)$. $f(x_0) \in V$ (meaning of $x_0 \in f^{-1}(V)$) $\exists \epsilon$ s.t. $B(f(x_0), \epsilon) \subset V$ (V is open). Then ϵ, δ defn of continuity $\exists \delta$ s.t. $f(B(x_0, \delta)) \subset B(f(x_0), \epsilon) \subset V$. So $B(x_0, \delta) \subset f^{-1}(V)$. \checkmark \square

Forward continuous images of open sets are not necessarily open.

Example 12.5

$f(x) = x^2, \quad f((-1, 1)) = [0, 1)$ which is not open.

Note: A notion to help understand the concept of open sets is thinking about how a map sends a point to a point but its inverse can send a point to a set.

§13 | Dis 1: Oct 1, 2020

Notation:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$$

$$\mathbb{R} = \text{real numbers}$$

$$\mathbb{C} = \{a + bi, \ a, b \in \mathbb{R}\}$$

Set theory:

- $A \subset B$ (or $A \subseteq B$) means $x \in A \implies x \in B$
- $x \in A \cap B$ means $x \in A$ and $x \in B$
- $x \in A \cup B$ means $x \in A$ or $x \in B$
- $x \in A \setminus B \iff x \in A$ and $x \notin B$
- $A = B \iff A \subset B$ and $B \subset A$

§13.1 Induction

Given a sequence of mathematical statement $P(n)$ indexed by \mathbb{N} . If $P(1)$ is true and $P(k) \implies P(k+1)$ is true $\forall k \in \mathbb{N}$, then $P(n)$ is true $\forall n \in \mathbb{N}$.

Example 13.1

Prove $\sum_{k=1}^n (2k-1) = n^2$ (*) using induction.

Base case $n = 1 : 1 = 1^2$ ✓

Induction step: assume as induction hypothesis that (*) holds

$$\begin{aligned} \sum_{k=1}^{n+1} (2k-1) &= \sum_{k=1}^n (2k-1) + 2(n+1) - 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

Or we can prove it the following way

$$\begin{aligned} S &= 1 + 3 + 5 + \dots + (2n-1) \\ S &= (2n-1) + (2n-3) + \dots + 3 + 1 \\ 2S &= 2n \cdot n \\ S &= n^2 \end{aligned}$$

Example 13.2

$a_{n+1} = \sqrt{2 + a_n}$, $a_1 = 1$. Prove $a_n > 0$ and a_n increasing.
 $a_1 > 0$ assume $a_n > 0$, $a_{n+1} = \sqrt{2 + a_n} > 0$

$$a_2 = \sqrt{3} \approx 1.732 > 1 = a_1$$

Assume $a_n \leq a_{n+1}$, want to show $a_{n+1} \leq a_{n+2} \iff \sqrt{a_n + 2} \leq \sqrt{a_{n+1} + 2} \iff a_n \leq a_{n+1}$

Example 13.3

$(1+x)^n \geq 1 + nx$: Bernoulli Inequality

$$x \geq -1, \quad n \geq 0$$

base case $1 \geq 1$

Assume $(1+x)^n \geq 1 + nx$

$$\begin{aligned} (1+x)^{n+1} &= (1+x)^n(1+x) \geq (1+nx)(1+x) = 1 + (n+1)x + nx^2 \\ &= 1 + (n+1)x \end{aligned}$$

Strong Induction:

If $P(1)$ true and $P(1), P(2), \dots, P(k) \implies P(k+1)$ true $\forall k \in \mathbb{N}$ then $P(n)$ holds for all $n \in \mathbb{N}$

Remark 13.4. Induction \iff strong induction

Example 13.5

Every integer greater than 1 is a product of primes.

Assume $2, 3, \dots, n$ is a product of primes. $n+1$ is either a prime or a composite, in which case $n+1 = ab$, $1 < a, b < n+1$.

By strong induction hypothesis, both a and b are product of primes, hence so is $n+1 = ab$.

Exercise 13.1. Every integer greater than 1 has a prime divisor.

Proof of infinitude of primes by Euclid:

Proof. Assume on the contrary there are finitely many primes $\{p_1, p_2, \dots, p_k\}$. Define $N = p_1 \dots p_k + 1 > 1$ and (by above exercise) let p be a prime divisor of N but $p \neq p_j$ for any $1 \leq j \leq k$ otherwise if $p = p_j$ then $p|p_2 \dots p_k$ also $p|N \implies p|N - p_1 \dots p_k \implies p|1$, a contradiction. (no primes divide 1) \square

§14.1 Number System

- $(\mathbb{N}, +, \cdot, <)$: $+$: $\mathbb{N} \times \mathbb{N} = \mathbb{N}^2 \rightarrow \mathbb{N}$ satisfies commutativity and associativity. Note that 0 is the identity with respect to addition, but \mathbb{N} has no additive inverse.
- $(\mathbb{Z}, +, \cdot, <)$: $(\mathbb{Z}, +)$ is a commutative group (associativity, identity, inverse). (\mathbb{Z}, \cdot) satisfies commutativity, associativity with 1 as mult identity but 2 has no mult inverse.
- $(\mathbb{Q}, +, \cdot, <)$: $(\mathbb{Q}, +)$ and (\mathbb{Q}, \cdot) are commutative group(i). $+$ and \cdot are compatible with distributive law: $a(b + c) = ab + ac$ (ii). Both (i) and (ii) mean $(\mathbb{Q}, +, \cdot)$ is a FIELD. $(\mathbb{Q}, <)$ is an ordered set with $<$ satisfying trichotomy and transitivity. $+$, \cdot are compatible : $y < z \implies x + y < x + z \forall x, x > 0, y > 0 \implies xy > 0$. With the above compatibility, $(\mathbb{Q}, +, \cdot, <)$ is an **ordered field**. Even though \mathbb{Q} is additivity and multiplicatively complete, \mathbb{Q} is not satisfying in that

1. \mathbb{Q} is not algebraically closed, $x^2 - 2$ is a polynomial with no root in \mathbb{Q} .
2. \mathbb{Q} is not complete in a metric space: there exists subsets of \mathbb{Q} bounded above but with no least upper bound (supremum), e.g. $A := \{p \in \mathbb{Q} : p < 0 \text{ or } p^2 < 2\}$ and $B = \mathbb{Q} \setminus A$. A contains no largest number and B contains no smallest.

$$\forall p \in A \exists q \in A \quad q > p$$

Let $p \in A$. Define $q := p - \frac{p^2 - 2}{p + 2} > p$

$$q^2 - 2 = \left(\frac{2p + 2}{p + 2} \right)^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2} < 0 \implies q^2 < 2$$

If A has an upper bound α , $\alpha \notin A$: then $\alpha \in B$. It follows that B is the set of all upper bounds for A . Since B contains no smallest number, A has no least upper bound in \mathbb{Q} .

Definition 14.1 (Least Upper Bound Property) — S has the least-upper-bound property if $\forall E \subset S$ nonempty, bounded above $\sup E \in S$.

Remark 14.2. \mathbb{Q} does not satisfy the least-upper-bound property.

$(\mathbb{R}, +, \cdot, <)$ there exists an ordered field with the l.u.b property that contains an isomorphic copy of \mathbb{Q} .

§14.2 Equivalence Relation

An equivalence relation given \sim on $A \times A$ satisfies

- $x \sim x$ reflexivity
- $x \sim y \iff y \sim x$ symmetry
- $x \sim y \cdot y \sim z \implies x \sim z$ transitivity

Example 14.3

\mathbb{Q} Define \sim on $\{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$ by $(a, b) \sim (c, d)$ if $ad = bc$

$$A = \mathbb{Z}^2 \setminus \{(a, 0) : a \in \mathbb{Z}\}$$

$\mathbb{Q} =$ the set of all equivalence classes of A write \sim
 $= A / \sim = \{[x] : x \in A\}$

In this construction, $\mathbb{Z} \rightarrow \mathbb{Q}, \quad n \rightarrow [(n, 1)]$
 $+$ and $\cdot : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$: note that $+$ and \cdot need to be well-defined on \mathbb{Q}^2 . (need to show
 $\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}$ if $\frac{a}{b} \sim \frac{a'}{b'}$ and $\frac{c}{d} \sim \frac{c'}{d'}$).

Example 14.4

$$S' = [0, 1] / 0_m$$

Definition 14.5 (Convergent Sequences) — $\{a_n\}_{n \geq 1} \subseteq \mathbb{R}$ is said to be convergent to l
 if $\forall \epsilon > 0 \quad \exists N(\epsilon) > 0$ s.t. $\forall n \geq N, \quad |a_n - l| < \epsilon$

§15 | Dis 3: Oct 13, 2020

§15.1 Equivalence Relation (Cont'd)

Example 15.1

Define $\sim p$ on \mathbb{Z} by $a \sim pb$ if $a - b \in p\mathbb{Z}$ ($p|a - b$).

$$\forall a \exists ! b \in \mathbb{Z}, \quad 0 \leq r < p \text{ s.t. } a = bp + r.$$

$$F_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z} / \sim p = \{[0]_p, [1]_p, [2]_p, \dots, [p-1]_p\}$$

$$[a]_p + [b]_p = [a + b]_p \quad \& \quad [a]_p [b]_p = [ab]_p$$

Remark 15.2. $(F_p, +, \cdot)$ is a finite field. F_p cannot be ordered: $1 > 0, 1 + 1 > 0, \dots, p - 1 > 0$
 but $p - 1 = -1$

Example 15.3

$$T = \mathbb{R}/\mathbb{Z} \quad a \sim b \text{ if } ab \in \mathbb{Z}$$

$$[0, 1] / 0 \sim 1$$

$$\forall a \in \mathbb{R}, \quad \exists b = \underbrace{\{a\}}_{\text{fractional part of } a} \in [0, 1) \text{ s.t. } a \sim b$$

§15.2 Construction of \mathbb{R} via Cauchy Sequences (Cantor)

S = set of rational Cauchy sequences.

\sim on S : $\{x_n\} \sim \{y_n\}$ if $\lim(x_n - y_n) = 0$ (Q3 – Homework 2)

$Q = S / \sim = \{[\{x_n\}] : \{x_n\} \in S\}$. First we need to define arithmetic on Q .

$$\begin{aligned} [\{p_n\}] + [\{q_n\}] &= [\{p_n + q_n\}] \\ [\{p_n\}] - [\{q_n\}] &= [\{p_n - q_n\}] \\ [\{p_n\}] \cdot [\{q_n\}] &= [\{p_n q_n\}] \\ [\{p_n\}] / [\{p_n/q_n\}] &= [\{p_n/q_n\}], \quad [\{q_n\}] \neq 0, = [\{0, 0, 0, \dots\}] \end{aligned}$$

$+$: $Q \times Q \rightarrow Q$. Check well-defined

- $\{x_n\} \cdot \{y_n\}$ cauchy then so is $\{x_n + y_n\}$ (Q4)
- $\{x_n\} \sim \{y_n\}$ & $\{z_n\} \sim \{w_n\}$ then $\{x_n + z_n\} \sim \{y_n + w_n\}$ (Q5)
Commutativity, assoc, identity, ($0 = [\{0, 0, 0, \dots\}]$), inverse.
- Well-defined: $\{x_n\}, \{y_n\}$ so is $\{x_n y_n\}$ (Q4).
- $\{x_n\} \sim \{y_n\}$ & $\{z_n\} \sim \{w_n\}$ (Q6, Q7)
comm, assoc, iden, ($1 = [\{1, 1, \dots, 1\}]$)
mult. inverse (Q9, Q10).
<: trichotomy (Q11), transitivity
various compatibility (distributivity, etc)
l.u.b property (Q12)

Note: All the Q used above is assumed to be Q^{hat}

Remark 15.4.

$$\begin{aligned} Q &\rightarrow Q^{\text{hat}} \\ q &\mapsto [q^*] \\ p < q &\iff [p^*] < [q^*] \end{aligned}$$

Sequences:

- Cauchy seq. are bounded.
- Convergent seq. is Cauchy.

Theorem: in \mathbb{R} , every Cauchy seq. is convergent.

Example 15.5

$$\begin{aligned} a_n &= \frac{1}{n} \\ \forall \epsilon > 0 \exists N \text{ s.t. } \epsilon N > 1. \\ \forall n \geq N \quad \left| \frac{1}{n} - 0 \right| &= \frac{1}{n} \leq \frac{1}{N} < \epsilon. \end{aligned}$$

□

§16 | Dis 4: Oct 20, 2020

§16.1 Least Upper Bound and Its Applications

Remark 16.1 (ϵ - Principle). $a, b \in \mathbb{R}, \forall \epsilon > 0, a \leq b + \epsilon \implies a \leq b$.

- $x, y \in \mathbb{R} \quad \forall \epsilon > 0, |x - y| \leq \epsilon \implies x = y$.

Supremum: $E \subset S$ bounded above. Suppose $\sup E \in S$

- $e \leq \sup E \forall e \in E$.
- $\forall \beta < \sup E, \exists e \in E$ s.t. $\beta < e < \sup E$

OR

$\forall \epsilon > 0, \exists e \in E$ s.t. $\sup E - \epsilon < e \leq \sup E$.

Example 16.2

$\sup \left\{ \frac{1}{n} \right\}_{n \geq 1} = 1, \quad \inf \left\{ \frac{1}{n} \right\} = 0$.

- $0 \leq \frac{1}{n} \forall n \in \mathbb{N}$.
- $\forall \epsilon > 0, \exists n \in \mathbb{N}$ s.t. $0 \leq \frac{1}{n} < \epsilon$ by Archimedean Prop.

Theorem 16.3 (Nested Interval)

$\{I_n = [a_n, b_n]\}_{n \geq 1} \subset \mathbb{R}, I_n \supset I_{n+1} \implies \bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Moreover, if $|I_n| \rightarrow 0$, then $\bigcap I_n$ is a singleton (a set with exactly one element).

Proof. $\sup a_n \in \bigcap I_n$. □

Theorem 16.4 ((4.1))

(Bolzano – Weierstrass): Every bounded sequence in \mathbb{R} has a convergent subsequence.

Proof. $I_0 = [-M, M] \supset I_1 \supset I_2 \supset \dots$

$$|I_n| = (2M) \cdot 2^{-n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

From Nested Interval Thm, $\bigcap_{n=0}^{\infty} I_n = \{x\}$. Choose $x_{n_k} \in I_k, x_{n_k} \rightarrow x$. □

Remark 16.5. l.u.b property of $\mathbb{R} \implies$ Nested Interval \implies Bolzano – Weierstrass $\xRightarrow{(*)}$ Cauchy Completeness.

(*) Exercise: $\{x_n\}$ Cauchy. $x_{n_k} \rightarrow x \implies x_n \rightarrow x$.

Remark 16.6. In \mathbb{R} , to check convergence, it suffices to check Cauchyness. Useful especially when you don't have a candidate for the limit. Cauchy criterion for series $\sum_{n=1}^{\infty} a_n$ converges $(\lim_{n \rightarrow \infty} \sum_{k=0}^n a_k)$ exists. $\iff \sum a_n$ Cauchy $(\forall \epsilon > 0 \exists N |\sum_{k=n}^m a_k| < \epsilon \quad \forall m \geq n \geq N)$.

Corollary 16.7

Absolute convergence \implies convergence. $(\sum |a_n| \text{ converges} \implies \sum a_n \text{ converges})$.

Monotone convergence theorem, $\{a_n\}$ monotone. Then $\{a_n\}$ bounded $\iff \{a_n\}$ convergent. (HW 3 – Q1).

Definition 16.8 (Monotone Sequence) — $\{a_n\}$ monotone if $a_n \leq a_{n+1} \forall n$ or $a_n \geq a_{n+1} \forall n$.

Corollary 16.9

$\sum |a_n| < \infty \iff \sum |a_n| \text{ converges}$.

§16.2 Continuity

Definition 16.10 ((6.2)) — $f : X \rightarrow \mathbb{R}$ is continuous at x (local prop) if

1. ($\epsilon - \delta$ def) $\forall \epsilon > 0, \exists \delta(\epsilon, x) > 0$ s.t. $\forall y \in X, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.
2. (Sequential def) $\forall \{x_n\} \subset X, x_n \rightarrow x \implies f(x_n) \rightarrow f(x)$ (f preserves sequential convergence).
3. $\lim_{y \rightarrow x} f(y) = f(x)$

$f : X \rightarrow \mathbb{R}$ is continuous if f is continuous at all $x \in X$.

Definition 16.11 ((7.1)) — f is uniformly continuous on X (global prop) if

1. ($\epsilon - \delta$) $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ s.t. $\forall x, y \in X, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.
2. (Sequential) $\forall \{x_n\} \subset X, \{x_n\}_{n \geq 1} \text{ Cauchy} \implies \{f(x_n)\}_{n \geq 1} \text{ Cauchy}$. (f preserves Cauchy seq).

Remark 16.12. Uniform continuity \implies continuity.

Example 16.13

$f : (0, \infty) \rightarrow \mathbb{R}$, $f(x) = \frac{1}{x}$ is continuous.

$$\left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x - y|}{xy} < \frac{|x - y|}{x \cdot \frac{x}{2}} = |x - y| \cdot 2x^{-2} < \epsilon$$

$$\delta = \min \left\{ \frac{x}{2}, \frac{\epsilon x^2}{2} \right\}.$$

Remark 16.14. $x \mapsto \frac{1}{x}$ is uniformly continuous on $(a, \infty) \forall a > 0$.
 $x \mapsto \frac{1}{x}$ is NOT uniformly continuous on $(0, \infty)$.

- $x_n = \frac{1}{n}, y_n = \frac{1}{n+1}$ $|x_n - y_n| \rightarrow 0$ but $|\frac{1}{x_n} - \frac{1}{y_n}| = 1 \forall n$.
- $\{\frac{1}{n}\}_{n \geq 1}$ Cauchy but $\{n\}$ is not.