115B – Linear Algebra

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Spring 2021

This is math 115B – Linear Algebra which is the second course of the undergrad linear algebra at UCLA – continuation of 115A(H). Similar to 115AH, this class is instructed by Professor Elman, and we meet weekly on MWF from 2:00 pm to 2:50 pm. There is no official textbook used for the class. You can find the previous linear algebra notes (115AH) with other course notes through my github. Any error in this note is my responsibility and please email me if you happen to notice it.

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§1 Lec 1: Mar 29, 2021

§1.1 Vector Spaces

Notation: if $\star : A \times B \to B$ is a map (= function) write $a \star b$ for $\star(a, b)$, e.g., $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $\mathbb{Z}$ = the integer.

Definition 1.1 (Field) — A set F is called a FIELD under

- Addition: $+ : F \times F \to F$
- Multiplication: $\cdot : F \times F \to F$

if $\forall a, b, c \in F$, we have

A1) $(a + b) + c = a + (b + c)$

A2) $\exists\ 0 \in F \ni a + 0 = a = 0 + a$

A3) A2) holds and $\exists x \in F \ni a + x = 0 = x + a$

A4) $a + b = b + a$

M1) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M2) A2) holds and $\exists 1 \neq 0 \in F$ s.t. $a \cdot 1 = a = 1 \cdot a$ (1 is unique and written 1 or 1_F)

M3) M2) holds and $\forall 0 \neq x \in F \ \ \exists y \in F \ni xy = 1 = yx$ (y is seen to be unique and written x^{-1})

M4) $x \cdot y = y \cdot x$

D1) $a \cdot (b + c) = a \cdot b + a \cdot c$

D2) $(a + b) \cdot c = a \cdot c + b \cdot c$

Example 1.2

$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are fields as is

$$\mathbb{F}_2 := \{0, 1\} \text{ with } + : \text{ given by}$$

+	0	1
0	0	1
1	1	0

•	0	1
0	0	0
1	0	1

Fact 1.1. Let $p > 0$ be a prime number in $\mathbb{Z}$. Then $\exists$ a field $\mathbb{F}_{p^n}$ having p^n elements write $|\mathbb{F}_{p^n}| = p^n \quad \forall n \in \mathbb{Z}^+$.

Definition 1.3 (Ring) — Let R be a set with

- $+ : R \times R \to R$
- $\cdot : R \times R \to R$

satisfying A1) – A4), M1), M2), D1), D2), then R is called a RING.
A ring is called

i) a commutative ring if it also satisfies M4).

ii) an (integral) domain if it is a commutative ring and satisfies

$$\text{M 3')}\; a \cdot b = 0 \implies a = 0 \text{ or } b = 0$$

($0 = \{0\}$ is also called a ring – the only ring with $1 = 0$)

Example 1.4 (Proof left as exercises) 1. $\mathbb{Z}$ is a domain and not a field.

2. Any field is a domain.
3. Let F be a field
$$F[t] := \{\text{polys coeffs in } F\}$$
with usual $+, \cdot$ of polys, is a domain but not a field. So if $f \in F[t]$
$$f = a_0 + a_1 t + \ldots + a_n t^n$$
where $a_0, \ldots, a_n \in F$.
4. $\mathbb{Q} := \left\{\frac{a}{b} | a, b \in \mathbb{Z},\, b \neq 0\right\} < \mathbb{C}$ ($<$ means $\subset$ and $\neq$) with usual $+, \cdot$ of fractions. (when does $\frac{a}{b} = \frac{c}{d}$?)
5. If F is a field
$$F(t) := \left\{\frac{f}{g} | f, g \in F[t],\, g \neq 0\right\} \text{ (rational function)}$$
with usual $+, \cdot$ of fractions is a field.

Example 1.5 (Cont'd from above) 6. $\mathbb{Q}[\sqrt{-1}] \coloneqq \left\{\alpha + \beta\sqrt{-1} \in \mathbb{C} | \alpha, \beta \in \mathbb{Q}\right\} < \mathbb{C}$. Then $\mathbb{Q}[\sqrt{-1}]$ is a field and

$$\begin{aligned}\mathbb{Q}(\sqrt{-1}) &\coloneqq \left\{\frac{a}{b} | a, b \in \mathbb{Q}[\sqrt{-1}],\, b \neq 0\right\} \\ &= \mathbb{Q}[\sqrt{-1}] \\ &= \left\{\frac{a}{b} | a, b \in \mathbb{Z}[\sqrt{-1}],\, b \neq 0\right\}\end{aligned}$$

where $\mathbb{Z}[\sqrt{-1}] \coloneqq \left\{\alpha + \beta\sqrt{-1} \in \mathbb{C},\, \alpha, \beta \in \mathbb{Z}\right\} < \mathbb{C}$. How to show this? – rationalize ($\mathbb{Z}[\sqrt{-1}]$ is a domain not a field, $F[t] < F(t)$ if F is a field so we have to be careful).

7. F a field

$$\mathbb{M}_n F \coloneqq \{n \times n \text{ matrices entries in } F\}$$

is a ring under $+, \cdot$ of matrices.

$$1_{\mathbb{M}_n F} = I_n = n \times n \text{ identity matrix } \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$0_{\mathbb{M}_n F} = 0 = 0_n = n \times n \text{ zero matrix } \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

is <u>not</u> commutative if $n > 1$.

In the same way, if R is a ring we have

$$\mathbb{M}_n R = \{n \times n \text{ matrices entries in } R\}$$

e.g., if R is a field $\mathbb{M}_n F[t]$.

8. Let $\emptyset \neq I \subset \mathbb{R}$ be a subset, e.g., $[\alpha, \beta], \alpha < \beta \in \mathbb{R}$. Then

$$C(I) = \{f : I \to \mathbb{R} | f \text{ continuous}\}$$

is a commutative ring and not a domain where

$$\begin{aligned}(f \dot{+} g)(x) &\coloneqq f(x) \dot{+} g(x) \\ 0(x) &= 0 \\ 1(x) &= x\end{aligned}$$

for all $x \in I$.

<u>Notation</u>: Unless stated otherwise F is always a field.

Definition 1.6 (Vector Space) — Let F be a field, V a set. Then V is called a VECTOR SPACE OVER F write V is a vector space over F under

- $+ : V \times V \to V$ – Addition
- $\cdot : F \times V \to V$ – Scalar multiplication

if $\forall x, y, z \in V \quad \forall \alpha, \beta \in F$.

1. $(x + y) + z = x + (y + z)$
2. $\exists 0 \in V \ni x + 0 = x = 0 + x$ (0 is seen to be unique and written 0 or 0_V)
3. 2) holds and $\exists v \in V \ni x + v = 0 = v + x$ (v is seen to be unique and written $-x$)
4. $x + y = y + x$
5. $1_F \cdot x = x$.
6. $(\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta \cdot x)$
7. $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$
8. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

Remark 1.7. The usual properties we learned in 115A hold for V a vector space over F, e.g., $0_F V = 0_V$, general association law,...

§2 Lec 2: Mar 31, 2021

§2.1 Vector Spaces (Cont'd)

Example 2.1

The following are vector space over F

1. $F^{m\times n} := \{m \times n \text{ matrices entries in } F\}$, usual $+$, scalar multiplication, i.e., if $A \in F^{m\times n}$, let $A_{ij} = ij^{\text{th}}$ entry of A. If $A, B \in F^{m\times n}$, then
$$(A+B)_{ij} := A_{ij} + B_{ij}$$
$$(\alpha A)_{ij} := \alpha A_{ij} \quad \forall \alpha \in F$$
i.e., component-wise operations.

2. $F^n = F^{1\times n} := \{(\alpha_1, \ldots, \alpha_n) \,|\alpha_i \in F\}$

3. Let V be a vector space over F, $\emptyset \neq S$ a set. Define
$$\mathcal{F}cn(S,V) := \{f : S \to V |\, f \text{ a fcn}\}$$
Then $\mathcal{F}cn(S,V)$ is a vector space over F $\forall f, g \in \mathcal{F}cn(S,V)$, $\forall \alpha \in F$. For all $x \in S$,
$$f+g : \, x \mapsto f(x) + g(x)$$
$$\alpha f : \, x \mapsto \alpha f(x)$$
i.e.
$$(f+g)(x) = f(x) + g(x)$$
$$(\alpha f)(x) = \alpha f(x)$$
with 0 by $0(x) = 0_V \,\forall x \in S$.

4. Let R be a ring under $+, \cdot, F$ a field $\ni F \subseteq R$ with $+, \cdot$ on F induced by $+, \cdot$ on R and $0_F = 0_R$, $1_F = 1_R$, i.e.
$$\underbrace{+}_{\text{on } R}\Big|_{\underbrace{F\times F}_{\text{restrict dom}}} : F \times F \to F \text{ and } \underbrace{\cdot}_{\text{on } R}\Big|_{\underbrace{F\times F}_{\text{restrict dom}}} : F \times F \to F$$
i.e. closed under the restriction of $+, \cdot$ on R to F and also with $0_F = 0_R$ and $1_F = 1_R$ (we call F a <u>subring</u> of R). Then R is a vector space over F by restriction of scalar multiplication, i.e., same $+$ on R but scalar multiplication
$$\cdot\big|_{F\times R} : F \times R \to R$$
e.g., $\mathbb{R} \subseteq \mathbb{C}$ and $F \subseteq F[t]$.

   <u>*Note*</u>: $\mathbb{C}$ is a vector space over $\mathbb{R}$ by the above but as a vector space over $\mathbb{C}$ is different.

5. In 4) if R is also a field (so $F \subseteq R$ is a subfield). Let V be a vector space over R. Then V is also a vector space over F by restriction of scalars, e.g., $M_n\mathbb{C}$ is a vector space over $\mathbb{C}$ so is a vector space over $\mathbb{R}$ so is a vector space over $\mathbb{Q}$.

§2.2 Subspaces

Definition 2.2 (Subspace) — Let V be a vector space under $+, \cdot, \emptyset \neq W \subseteq V$ a subset. We call W a subspace of V if $\forall w_1, w_2 \in W$, $\forall \alpha \in F$,

$$\alpha w_1,\, w_1 + w_2 \in W$$

with $0_W = 0_V$ is a vector space over F under $+|_{W\times W}$ and $\cdot|_{F\times W}$ i.e., closed under the operation on V.

Theorem 2.3

Let V be a vector space over F, $\emptyset \neq W \subseteq V$ a subset. Then W is a subspace of V iff $\forall \alpha \in F$, $\forall w_1, w_2 \in W$, $\alpha w_1 + w_2 \in W$.

Example 2.4

1. Let $\emptyset \neq I \subseteq \mathbb{R}$, $C(I)$ the commutative ring of continuous function $f : I \to \mathbb{R}$. Then $C(I)$ is a vector space over $\mathbb{R}$ and a subspace of $\mathcal{F}cn(I, \mathbb{R})$.
2. $F[t]$ is a vector space over F and $n \geq 0$ in $\mathbb{Z}$.

$$F[t]_n := \{f |\, f \in F[t],\, f = 0 \text{ or } \deg f \leq d\}$$

is a subspace of $F[t]$ (it is not a ring).

Attached is a review of theorems about vector spaces from math 115A.

§2.3 Motivation

Problem 2.1. Can you break down an object into simpler pieces? If yes can you do it uniquely?

Example 2.5

Let $n > 1$ in $\mathbb{Z}$. Then n is a product of primes unique up to order.

Example 2.6

Let V be a finite dimensional inner product space over $\mathbb{R}$ (or $\mathbb{C}$) and $T : V \to V$ a hermitian (=self adjoint) operator. Then $\exists$ an ON basis for V consisting of eigenvectors for T. In particular, T is diagonalizable. This means

$$V = E_T(\lambda_1) \perp \ldots \perp E_T(\lambda_r) \tag{*}$$

$E_T(\lambda_i) \coloneqq \{v \in V \,|\, Tv = \lambda_i v\} \neq 0$ eigenspace of λ_i; $\lambda_1, \ldots, \lambda_r$ the distinct eigenvalues of T. So

$$T\big|_{E_T(\lambda_i)} : E_T(\lambda_i) \to E_T(\lambda_i)$$

i.e., $E_T(\lambda_i)$ is T-invariant and

$$T\big|_{E_T(\lambda_i)} = \lambda_i 1_{E_T(\lambda_i)}$$

and (*) is unique up to order.

<u>Goal</u>: Generalize this to V any finite dimensional vector space over F, any F, and $T : V \to V$ linear. We have many problems to overcome in order to get a meaningful result, e.g.,

Problem 2.2.

1. V may not be an inner product space.
2. $F \neq \mathbb{R}$ or $\mathbb{C}$ is possible.
3. $F \not\subseteq$ is possible, so cannot even define an inner product.
4. V may not have any eigenvalues for $T : V \to V$.
5. If we prove an existence theorem, we may not have a uniqueness one.

We shall show: given V a finite dimensional vector space over F and $T : V \to V$ a linear operator. Then V breaks up uniquely up to order into small T-invariant subspace that we shall show are completely determined by polys in $F[t]$ arising from T.

§2.4 Direct Sums

<u>Motivation</u>: Generalize the concept of linear independence, Spectral Theorem Decomposition, to see how pieces are put together (if possible).

Definition 2.7 (Span) — Let V be a vector space over F, $W_i \subseteq V$, $i \in I$ – may not be finite, subspaces. Let

$$\sum_{i\in I} W_i = \sum_{i\in I} W_i := \left\{ v \in V | \exists w_i \in W_i,\, i \in I,\ \text{almost all } w_i = 0 \ni v = \sum_{i\in I} w_i \right\}$$

when almost all zero means only finitely many $w_i \neq 0$. Warning: In a vector space/F we can only take finite linear combination of vectors. So

$$\sum_{i\in I} W_i = \text{Span}\left(\bigcup_{i\in I} W_i\right) = \left\{ \text{finite linear combos of vectors in } \bigcup_{i\in I} W_i \right\}$$

e.g., if I is finite, i.e., $|I| < \infty$, say $I = \{1, \ldots, n\}$ then

$$\sum_{i\in I} W_i = W_1 + \ldots + W_n := \{w_1 + \ldots + w_n |\, w_i \in W_i \,\forall i \in I\}$$

cf. Linear Combinations.

Definition 2.8 (Direct Sum) — Let V be a vector space over F, $W_i \subseteq V$, $i \in I$, subspace. Let $W \subseteq V$ be a subspace. We say that W is the (internal) direct sum of the W_i, $i \in I$ write $W = \bigoplus_{i\in I} W_i$ if

$$\forall w \in W \,\exists!\, w_i \in W_i \text{ almost all } 0 \ni\ w = \sum_{i\in I} w_i$$

e.g., if $I = \{1, \ldots, n\}$, then

$$w \in W_1 \oplus \ldots \oplus W_n \text{ means } \exists!\, w_i \in W_i \ni\ w = w_1 + \ldots + w_n$$

Warning: It may not exist.

§3 Lec 3: Apr 2, 2021

§3.1 Direct Sums (Cont'd)

Definition 3.1 (Independent Subspace) — Let V be a vector space over $F, W_i \subseteq V$, $i \in I$ subspaces. We say the W_i, $i \in I$, are independent if whenever $w_i \in W_i$, $i \in I$, almost all $w_i = 0$, satisfy $\sum w_i = 0$, then $w_i = 0 \,\forall i \in I$.

Theorem 3.2

Let V be a vector space over F, $W_i \subseteq V$, $i \in I$ subspaces, $W \subseteq V$ a subspace. Then the following are equivalent:

1. $W = \bigoplus_{i \in I} W_i$
2. $W = \sum_{i \in I} W_i$ and $\forall i$
$$W_i \cap \sum_{j \in I \setminus \{i\}} W_j = 0 := \{0\}$$
3. $W = \sum_{i \in I} W_i$ and the W_i, $i \in I$, are independent.

Proof. 1) $\implies$ 2) Suppose $W = \bigoplus_{i \in I} W_i$. Certainly, $W = \sum_{i \in I} W_i$. Fix i and suppose that

$$\exists x \in W_i \cap \sum_{j \in I \setminus \{i\}} W_j$$

By definition, $\exists w_i \in W_i$, $w_j \in W_j$, $j \in I \setminus \{i\}$ almost all 0 satisfying

$$w_i = x = \sum_{j \neq i} w_j$$

So

$$0_V = 0_W = w_i - \sum_{j \neq i} w_j$$

But

$$0_W = \sum_I 0_{W_k} \qquad 0_{W_k} = 0_V \; \forall k \in I$$

By uniqueness of 1), $w_i = 0$ so $x = 0$.
2) $\implies$ 3) Let $w_i \in W_i$, $i \in I$, almost all zero satisfy

$$\sum_{i \in I} w_i = 0$$

Suppose that $w_k \neq 0$. Then

$$w_k = -\sum_{i \in I \setminus \{k\}} w_i \in W_k \cap \sum_{i \neq k} w_i = 0,$$

a contradiction. So $w_i = 0 \, \forall i$
3) $\implies$ 1) Suppose $v \in \sum_{i \in I} W_i$ and $\exists w_i, w_i' \in W_i$, $i \in I$, almost all 0 $\ni$

$$\sum_{i \in I} w_i = v = \sum_{i \in I} w_i'$$

Then $\sum_{i \in I} (w_i - w_i') = 0$, $w_i - w_i' \in W_i \, \forall i$. So

$$w_i - w_i' = 0, \text{ i.e., } w_i = w_i' \quad \forall i$$

and the $w_i' s$ are unique. $\square$

Warning: 2) DOES NOT SAY $W_i \cap W_j = 0$ if $i \neq j$. This is too weak. It says $W_i \cap \sum_{j \neq i} W_j = 0$.

Corollary 3.3

Let V be a vector space over $F, W_i \subseteq V$, $i \in I$ subspaces. Suppose $I = I_1 \cup I_2$ with $I_1 \cap I_2 = \emptyset$ and $V = \bigoplus_{i \in I} W_i$. Set

$$W_{I_1} = \bigoplus_{i \in I_1} W_i \quad \text{and} \quad W_{I_2} = \bigoplus_{j \in I_2} W_j$$

Then

$$V = W_{I_1} \oplus W_{I_2}$$

Proof. Left as exercise – Homework. □

Notation: Let V be a vector space over F, $v \in V$. Set

$$Fv := \{\alpha v | \alpha \in F\} = \text{Span}(v)$$

if $v \neq 0$, then Fv is the line containing v, i.e., Fv is the one dimensional vector space over F with basis $\{v\}$.

Example 3.4

Let V be a vector space over F

1. If $\emptyset \neq S \subseteq V$ is a subset, then
$$\sum_{v \in S} Fv = \text{Span}(S)$$
the span of S. So
$$\text{Span } S = \{\text{all finite linear combos of vectors in } S\}$$

2. If $\emptyset \neq S$ is linearly indep. (i.e. meaning every finite nonempty subset of S is linearly indep.), then
$$\text{Span}(S) = \bigoplus_{s \in S} Fs$$

3. If S is a basis for V, then $V = \bigoplus_{s \in S} Fs$

4. If $\exists$ a finite set $S \subseteq V \ni V = \text{Span}(S)$, then $V = \sum_{s \in S} Fs$ and $\exists$ a subset $\mathscr{B} \subseteq S$ that is a basis for V, i.e., V is a finite dimensional vector space over F and $\dim V = \dim_F V = |\mathscr{B}|$ is indep. of basis $\mathscr{B}$ for V.

5. Let V be a vector space over $F, W_1, W_2 \subseteq V$ finite dimensional subspaces. Then $W_1 + W_2$, $W_1 \cap W_2$ are finite dimensional vector space over F and
$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$
So
$$W_1 + W_2 = W_1 \oplus W_2 \iff W_1 \cap W_2 = \emptyset$$
(warning: be very careful if you wish to generalize this)

Definition 3.5 (Complementary Subspace) — Let V be a finite dimensional vector space over $F, W \subseteq V$ a subspace if
$$V = W \oplus W', \quad W' \subseteq V \text{ a subspace}$$
We call W' a complementary subspace of W in V.

Example 3.6

Let $\mathscr{B}_0$ be a basis of W. Extend $\mathscr{B}_0$ to a basis $\mathscr{B}$ for V (even works if V is not finite dimensional). Then

$$W' = \bigoplus_{\mathscr{B} \setminus \mathscr{B}_0} Fv \text{ is a complement of } W \text{ in } V$$

Note: W' is not the unique complement of W in V – counter-example?

Consequences: Let V be a finite dimensional vector space over $F, W_1, \ldots, W_n \subseteq V$ subspaces, $W_i \neq 0 \,\forall i$. Then the following are equivalent

1. $V = W_1 \oplus \ldots \oplus W_n$.
2. If $\mathscr{B}_i$ is a basis (resp., ordered basis) for $W_i \,\forall i$, then $\mathscr{B} = \mathscr{B}_1 \cup \ldots \cup \mathscr{B}_n$ is a basis (resp. ordered) – with obvious order – for V.

Proof. Left as exercise (good one)! □

Notation: Let V be a vector space over F, $\mathscr{B}$ a basis for V, $x \in V$. Then, $\exists! \alpha_v \in F$, $v \in \mathscr{B}$, almost all $\alpha_v = 0$ (i.e., all but finitely many) s.t. $x = \sum_{\mathscr{B}} \alpha_v v$. Given $x \in V$,

$$x = \sum_{v \in \mathscr{B}} \alpha_v v$$

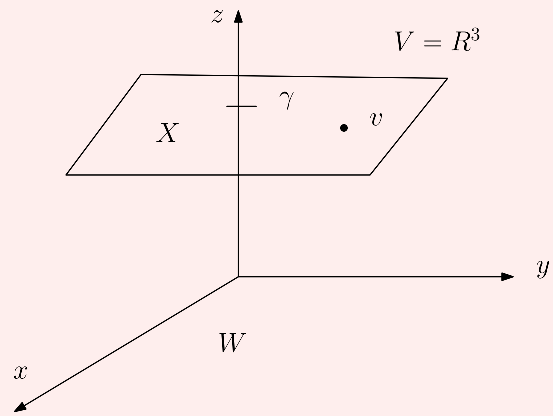
to mean α_v is the unique complement of x on v and hence $\alpha_v = 0$ for almost all $v \in \mathscr{B}$.

§3.2 Quotient Spaces

Idea: Given a surjective map $f : X \to Y$ and "nice", can we use properties of Y to obtain properties of X?

Example 3.7

Let $V = \mathbb{R}^3$, $W = X - Y$ plane. Let X = plane parallel to W intersecting the z-axis at γ.



So

$$\begin{aligned} X &= \{(\alpha, \beta, \gamma) | \alpha, \beta \in \mathbb{R}\} \\ &= \{(\alpha, \beta, 0) + (0, 0, \gamma) | \alpha, \beta \in \mathbb{R}\} \\ &= W + \gamma \underbrace{e_3}_{(0,0,1)} \end{aligned}$$

<u>*Note*</u>: X is a vector space over $\mathbb{R} \iff \gamma = 0 \iff W = X$ (need 0_V). Let $v \in X$. So $v = (x, y, \gamma)$ some $x, y \in \mathbb{R}$. So

$$\begin{aligned} W + v &:= \left\{ \underbrace{(\alpha, \beta, 0)}_{\text{arbitrary}} + \underbrace{(x, y, \gamma)}_{\text{fixed}} \,|\, \alpha, \beta \in \mathbb{R} \right\} \\ &= \{(\alpha + x, \beta + y, \gamma) \,|\, \alpha, \beta \in \mathbb{R}\} \\ &= W + \gamma_{e_3} \end{aligned}$$

It follows if $v, v' \in V$, then

$$W + v = W + v' \implies v - v' \in W$$

Conversely, if $v, v' \in V$ with $X = W + v$, then

$$v' \in X \implies v' = w + v \text{ some } w \in W$$

hence

$$v' - v \in W$$

So for arbitrary $v, v' \in V$, we have the <u>conclusion</u> $W + v = W + v' \iff v - v' \in W$. We can also write $W + v$ as $v + W$.

§4 Lec 4: Apr 5, 2021

§4.1 Quotient Spaces (Cont'd)

Recall from the last example of the last lecture, we have

$$V = \bigcup_{v \in V} W + v$$

If $v, v' \in V$, then

$$0 \neq v'' \in (W + v) \cap (W + v')$$

means

$$W + v - W + v'' = W + v'$$

This means either $W + v = W + v'$ or $W + v \cap W + v' = \emptyset$, i.e., planes parallel to the xy-plane partition V into a disjoint unions of planes.

Let

$$S := \{W + v | \, v \in V\}$$

the set of these planes. We make S into a vector space over $\mathbb{R}$ as follows: $\forall v, v' \in V$, $\forall \alpha \in \mathbb{R}$ define

$$\begin{aligned}(W + v) + (W + v') &:= W + (v + v') \\ \alpha \cdot (W + v) &:= W + \alpha v\end{aligned}$$

We must check these two operations are well-defined and we set

$$0_S := W$$

Then $(W + v) + W = W + v = W + (W + v)$ make S into a vector space over $\mathbb{R}$.
If $v \in V$ let $\gamma_v^1 =$ the k^{th} component of v. Define

$$S \to \{(0, 0, \gamma) | \, \gamma \in \mathbb{R}\} \to \mathbb{R}$$

by

$$W + v \mapsto (0, 0, \gamma_v) \mapsto \gamma$$

both maps are bijection and, in fact, linear isomorphism. So

$$S \cong \{(0, 0, \gamma) | \, \gamma \in \mathbb{R}\} \cong \mathbb{R}$$

<u>Note</u>: $\dim V = 3$, $\dim W = 2$, $\dim S = 1$ and we also have a linear transformation

$$V \to S \text{ by } (\alpha, \beta, \gamma) \mapsto W + \gamma_{e_3}$$

a surjection.
We can now generalize this.
<u>Construction</u>: Let V be a vector space over F, $W \subseteq V$ a subspace. Define $\equiv \mod W$ called <u>congruent mod W on V</u> as follows: if $x, y \in V$, then

$$x \equiv y \mod W \iff x - y \in W \iff \exists w \in W \ni x = w + y$$

Then, for all $x, y, z \in V$, $\equiv \mod W$ satisfies

1. $x \equiv x \mod W$
2. $x \equiv y \mod W \implies y \equiv x \mod W$
3. $x \equiv y \mod W$ and $y \equiv z \mod W \implies x \equiv z \mod W$

We can conclude that $\equiv \mod W$ is an equivalence relation on V.

Notation: For $x \in V$, $W \subseteq V$, let

$$\overline{x} := \{y \in V | \, y \equiv x \mod W\}$$

We can also write $\overline{x}$ as $[x]_W$ if W is not understood. Also, $\overline{x} \subseteq V$ is a subset and not an element of V called a coset of V by W. We have

$$\begin{aligned} \overline{x} &= \{y \in V | \, y \equiv x \mod W\} \\ &= \{y \in V | \, y = w + x \text{ for some } w \in W\} \\ &= \{w + x | \, w \in W\} = W + x = x + W \end{aligned}$$

Example 4.1

$\overline{0}_V = W + 0_V = W$.

Note: $W + x$ translates every element of W by x. By 2), 3) of $\equiv \mod W$, we have check

$$y \in \overline{x} = W + x \iff x \in \overline{y} = W + y$$

and

$$x \equiv y \mod W \iff \overline{x} = \overline{y} \iff W + x = W + y$$

and check

$$\overline{x} \cap \overline{y} = \emptyset \iff (W + x) \cap (W + y) = \emptyset \iff x \not\equiv y \mod W$$

This means the $W + x$ partition V, i.e.,

$$V = \bigcup_V (W + x) \text{ with } (W + x) \cap (W + y) = \emptyset \text{ if } \overline{x} = (W + x) \neq (W + y) = \overline{y}$$

Let

$$\overline{V} := V/W := \{\overline{x} | \, x \in V\} = \{W + x | \, x \in V\}$$

a collection of subsets of V.

§5 Lec 5: Apr 7, 2021

§5.1 Quotient Spaces (Cont'd)

Suppose we have $W \subseteq V$ a subspace. For $x, y, z, v \in V$

$$\begin{aligned} x &\equiv y \mod W \\ z &\equiv v \mod W \end{aligned} \tag{+}$$

Then

$$(x+z)-(y+v) = \underbrace{(x-y)}_{\in W} + \underbrace{(z-v)}_{\in W} \in W$$

So

$$x+z \mod y+v \mod W$$

and if $\alpha \in F$

$$\alpha x - \alpha y = \alpha(x-y) \in W \quad \forall x, y \in V$$

So

$$\alpha x \equiv \alpha y \mod W$$

Therefore, $\overline{V} = V/W$. If (+) holds, then for all $x, y, z, v \in V$ and $\alpha \in F$, we have

$$\begin{aligned} \overline{x+z} &= \overline{y+v} \in \overline{V} \\ \overline{\alpha x} &= \overline{\alpha y} \in \overline{V} \end{aligned}$$

Notice $\overline{V} = V/W$ satisfies all the axioms of a vector space with $0_{\overline{V}} = \overline{0_V} = \{y \in V \,|\, y \equiv 0 \mod W\} = W + 0_V = W$.

We call $\overline{V} = V/W$ the QUOTIENT SPACE of V by W.

We also have a map

$$- : V \to \overline{V} = V/W \text{ by } x \mapsto \overline{x} = W + x$$

which satisfies

$$\alpha v + v' \overset{-}{\mapsto} \overline{\alpha u + v'} = \alpha\overline{v} + \overline{v'}$$

for all $v, v' \in V$ and $\alpha \in F$. Then

$$\begin{aligned} \dim V &= \dim \ker^{-} \\ \dim V &= \dim W + \dim V/W \\ \dim V/W &= \dim V - \dim W \end{aligned}$$

which is called the <u>codimension</u> of W in V.

Proposition 5.1

Let V be a vector space over F, $W \subseteq V$ a subspace, $\overline{V} = V/W$. Let $\mathscr{B}_0$ be a basis for W and

$$\mathscr{B}_1 = \{v_i | \, i \in I, \, v_i - v_j \notin W \text{ if } i \neq j\}$$

where $\overline{v_i} \neq \overline{v_j}$ if $i \neq j$ or $w + v_i \neq w + v_j$ if $i \neq j$.

Let

$$\mathscr{C} = \{\overline{v_i} = W + v_i | \, i \in I, \, v_i \in \mathscr{B}_1\}$$

If $\mathscr{C}$ is a basis for $\overline{V} = V/W$, then $\mathscr{B}_0 \cup \mathscr{B}_1$ is a basis for V (compare with the proof of the Dimension Theorem).

Proof. Hw 2 # 3. □

§5.2 Linear Transformation

A review of linear of linear transformation can be found here.
Now, we consider

$$GL_n F \coloneqq \{A \in \mathbb{M}_n F | \, \det A \neq 0\}$$

The elements in $GL_n F$ in the ring $\mathbb{M}_n F$ are those having a multiplicative inverse. If R is a commutative ring, determinants are still as before but

$$\begin{aligned} GL_n R &\coloneqq \{A \in \mathbb{M}_n R | \, \det A \text{ is a unit in } R\} \\ &= \left\{A \in \mathbb{M}_n R | \, A^{-1} \text{ exists}\right\} \end{aligned}$$

Example 5.2

Let V be a vector space over $F, W \subseteq V$ a subspace. Recall

$$\overline{V} = V/W = \{\overline{v} = W + v | \, v \in V\}$$

a vector space over F s.t. for all $v_1, v_2 \in F$ and $\alpha \in F$

$$\begin{aligned} 0_{\overline{V}} &= \overline{0_V} = W \\ \overline{v_1} + \overline{v_2} &= \overline{v_1 + v_2} \\ \alpha \overline{v_1} &= \overline{\alpha v_1} \end{aligned}$$

Then

$$- : V \to V/W = \overline{V} \text{ by } v \mapsto \overline{v} = W + v$$

is an epimorphism with $\ker^{-} = W$.

Recall from 115A(H) that the most important theorem about linear transformation is Universal Property of Vector Spaces. As a result, we can deduce the following corollary

Corollary 5.3

Let V, W be vector space over F with bases $\mathscr{B}, \mathscr{C}$ respectively. Suppose there exists a bijection $f : \mathscr{B} \to \mathscr{C}$, i.e., $|\mathscr{B}| = |\mathscr{C}|$. Then $V \cong W$.

Proof. There exists a unique $T : V \to W \ni T\big|_{\mathscr{B}} = f$. T is monic by the Monomorphism Theorem (T takes linearly indep. sets to linearly indep. sets iff it's monic) and is onto as $W = \text{Span}(\mathscr{C}) = \text{Span}\left(f(\mathscr{B})\right)$. □

§6 Lec 6: Apr 9, 2021

§6.1 Linear Transformation (Cont'd)

Theorem 6.1

Let $T: V \to W$ be linear. Then $\exists X \subseteq V$ a subspace s.t.

$$V = \ker T \oplus X \text{ with } X \cong \text{im } T$$

Proof. Let $\mathscr{B}_0$ be a basis for $\ker T$. Extend $\mathscr{B}_0$ to a basis $\mathscr{B}$ for V by the Extension Theorem. Let $\mathscr{B}_1 = \mathscr{B} \setminus \mathscr{B}_0$, so $\mathscr{B} = \mathscr{B}_0 \vee \mathscr{B}_1$ ($\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$ and $\mathscr{B}_0 \cap \mathscr{B}_1 = \emptyset$) and let

$$X = \bigoplus_{\mathscr{B}_1} Fv$$

As $\ker T = \bigoplus_{\mathscr{B}_0} Fv$, we have

$$V = \ker T \oplus X$$

and we have to show

$$X \cong \text{im } T$$

Claim 6.1. Tv, $v \in \mathscr{B}_1$ are linearly indep.

In particular, $Tv \neq Tv'$ if $v, v' \in \mathscr{B}_1$ and $v \neq v'$. Suppose

$$\sum_{v \in \mathscr{B}} \alpha_v Tv = 0_W, \quad \alpha_v \in F \text{ almost all } \alpha_v = 0$$

Then

$$0_W = T\left(\sum_{v \in \mathscr{B}_1} \alpha_v v\right), \quad \text{i.e. } \sum_{\mathscr{B}_1} \alpha_v v \in \ker T$$

Hence

$$\sum_{\mathscr{B}_1} \alpha_v v = \sum_{\mathscr{B}_0} \beta_v v \in \ker T \text{ almost all } \beta_v \in F = 0$$

As $\sum_{\mathscr{B}_1} \alpha_v v - \sum_{\mathscr{B}_0} \beta_v v = 0$ and $\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$ is linearly indep., $\alpha_v = 0\, \forall v$. This proves the above claim.

Let $\mathscr{C} = \{Tv | \, v \in \mathscr{B}_1\}$. By the claim

$$\mathscr{B}_1 \to \mathscr{C} \text{ by } v \mapsto Tv \text{ is } 1-1$$

and onto as $\mathscr{C}$ is linearly indep. Lastly, we must show $\mathscr{C}$ spans im T. Let $w \in \text{im } T$. Then $\exists x \in V \ni Tx = w$. Then

$$\begin{aligned} w = Tx &= T\left(\sum_{\mathscr{B}_0} \alpha_v v\right) + T\left(\sum_{\mathscr{B}_1} \alpha_v v\right) \\ &= \sum_{\mathscr{B}_0} \alpha_v Tv + \sum_{\mathscr{B}_1} \alpha_v Tv = \sum_{\mathscr{B}_1} \alpha_v Tv \end{aligned}$$

lies in span $\mathscr{C}$ as needed. □

Remark 6.2. Note that the proof is essentially the same as the proof of the Dimension Theorem.

Corollary 6.3 (Dimension Theorem)

If V is a finite dimensional vector space over F, $T : V \to W$ linear then

$$\dim V = \dim \ker T + \dim \ \operatorname{im} \ T$$

Corollary 6.4

If V is a finite dimensional vector space over F, $W \subseteq V$ a subspace, then

$$\dim V = \dim W + \dim V/W$$

Proof. $- : V \to V/W$ by $v \mapsto \overline{v} = W + v$ is an epi. □

Important Construction: Set

$$\begin{aligned} T : V \to Z &\text{ be linear} \\ W &= \ker T \\ \overline{V} &= V/W \\ - : V \to V/W &\text{ by } v \mapsto \overline{v} = W + v \text{ linear} \end{aligned}$$

$\forall x, y \in V$ we have

$$\overline{x} = \overline{y} \in \overline{V} \iff x \equiv y \mod W \iff x - y \in W \iff T(x - y) = 0_Z$$

i.e., when $W = \ker T$

$$\overline{x} = \overline{y} \iff Tx = Ty \tag{*}$$

This means

$$\overline{T} : \overline{V} \to Z \text{ defined by } W + v = \overline{v} \mapsto Tv$$

is well-defined, i.e., via function, since if $\overline{x} = \overline{y}$, then $\overline{T}(\overline{x}) := Tx = Ty =: \overline{T}(\overline{y})$. From (*),

$$\overline{x} = \overline{y} \iff \overline{T}(\overline{x}) = T(x) = T(y) =: \overline{T}(\overline{y})$$

so

$$\overline{T} : \overline{V} \to Z \text{ is also injective}$$

As $\overline{T}$ is linear, let $\alpha \in F$, $x, y \in V$, then

$$\begin{aligned} \overline{T}\left(\alpha\overline{x} + \overline{y}\right) &= \overline{T}\left(\overline{\alpha x + y}\right) = T(\alpha x + y) \\ &= \alpha Tx + Ty = \alpha\overline{T}(\overline{x}) + \overline{T}(\overline{y}) \end{aligned}$$

as needed. Therefore,

$$\overline{T} : \overline{V} \to Z \text{ by } \overline{x} \mapsto T(x)$$

is a monomorphism, so induces an isomorphism onto im $\overline{T}$ and we recall im $\overline{T} =$ im T, so

$$\overline{V} \cong \text{im } \overline{T} = \text{im } T$$

and we have a commutative diagram

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "-"'] & Z \\
V/\ker T = \overline{V} \arrow[ur, "\overline{T}"'] &
\end{tikzcd}$$

This can also be written as

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "-"'] & Z \\
V/\ker T = \overline{V} \arrow[r, "\overline{T}"] & im\, T \arrow[u, bend right, "\text{inclusion map}"']
\end{tikzcd}$$

<u>Consequence</u>: Any linear transformation $T : V \to Z$ induces an isomorphism

$$\overline{T} : V/\ker T \to \text{im } T \text{ by } \overline{v} = \ker T + v \mapsto Tv$$

This is called the First Isomorphism Theorem. We also have

$$V = \ker T \oplus X \text{ with } X \subseteq V \text{ and } X \cong \text{im } T \cong V/\ker T$$

This means that all images of linear transformations from V are determined, up to isomorphism, by V and its subspaces. It also means, if V is a finite dimensional vector space over F, we can try prove things by induction.

§6.2 Projections

<u>Motivation</u>: Let $m < n$ in $\mathbb{Z}^+$ and

$$\pi : \mathbb{R}^n \to \mathbb{R}^n \text{ by } (\alpha_1, \dots, \alpha_n) \mapsto (\alpha_1, \dots, \alpha_n, 0, \dots, 0)$$

a linear operator onto $\bigoplus_{i=1}^m \Gamma e_i$ where $e_i = \left(0, \dots, \underbrace{1}_{i^m}, \dots, 0\right)$.

Definition 6.5 (T-invariant) — Let $T : V \to V$ be linear, $W \subseteq V$ a subspace. We say W is T-invariant if $T(W) \subseteq V$ if this is the case, then the restriction $T\big|_W$ of T can be viewed as a linear operator

$$T\big|_W : W \to W$$

Example 6.6

Let $T : V \to V$ be linear.

1. $\ker T$ and $\operatorname{im} T$ are T-invariant.
2. Let $\lambda \in F$ be an eigenvalue of T, i.e., $\exists 0 \neq v \in V \ni Tv = \lambda v$, then any subspace of the eigenspace
$$E_T(\lambda) := \{v \in V \,|\, Tv = \lambda v\}$$
is T-invariant as $T\big|_{E_T(\lambda)} = \lambda 1_{E_T(\lambda)}$

Remark 6.7. Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Suppose that
$$V = W_1 \oplus \ldots \oplus W_n$$
with each W_i T-invariant, $i = 1, \ldots, n$ and $\mathscr{B}_i$ an ordered basis for W_i, $i = 1, \ldots, n$. Let $\mathscr{B} = \mathscr{B}_1 \cup \ldots \cup \mathscr{B}_n$ be a basis of V ordered in the obvious way.
Then the matrix representation of T in the $\mathscr{B}$ basis is
$$[T]_{\mathscr{B}} = \begin{pmatrix} \left[T\big|_{W_1}\right]_{\mathscr{B}_1} & & 0 \\ & \ddots & \\ 0 & & \left[T\big|_{W_n}\right]_{\mathscr{B}_n} \end{pmatrix}$$

Example 6.8

Suppose that $T : V \to V$ is diagonalizable, i.e., there exists a basis $\mathscr{B}$ of eigenvectors of T for V. Then, $T : V \to V$,
$$V = \bigoplus E_T(\lambda_i)$$
each $E_T(\lambda_i)$ is T-invariant.
$$T\big|_{E_T(\lambda_i)} = \lambda_i 1_{E_T(\lambda_i)}$$

<u>Goal</u>: Let V be a finite dimensional vector space over F, $n = \dim V$, $T : V \to V$ linear. Then $\exists W_1, \ldots, W_m \subseteq V$ all T-invariant subspaces with $m = m(T)$ with each W_i being as small as possible with $V = W_1 \oplus \ldots \oplus W_m$. This is the theory of canonical forms.
<u>Recall</u>: If V is a finite dimensional vector space over F, $T : V \to V$ linear, $\mathscr{B}$ an ordered basis for V, then the matrix representation $[T]_{\mathscr{B}}$ is only unique up to <u>similarity</u>, i.e., if $\mathscr{C}$ is an another ordered basis
$$[T]_{\mathscr{C}} = P\,[T]_{\mathscr{B}}\,P^{-1}$$
where $P = [1_V]_{\mathscr{B},\mathscr{C}} \in GL_n F$, the change of basis matrix $\mathscr{B} \to \mathscr{C}$.

Definition 6.9 (Projection) — Let V be a vector space over F, $P : V \to V$ linear. We call P a <u>projection</u> if $P^2 = P \circ P = P$.

Example 6.10 1. $P = 0_V$ or $1_V : V \to V$, V is a vector space over F.

2. An orthogonal projection in 115A.

3. If P is a projection, so is $1_V - P$.

If $T : V \to V$ is linear, then

$$V = \ker T \oplus X \text{ with } X \cong \text{im } T$$

Lemma 6.11

Let $P : V \to V$ be a projection. Then

$$V = \ker P \oplus \text{im } P$$

Moreover, if $v \in \text{im } P$, then

$$Pv = v$$

i.e.

$$P\big|_{\text{im } P} : \text{im } P \to \text{im } P \text{ is } 1_{\text{im } P}$$

In particular, if V is a finite dimensional vector space over $F, \mathscr{B}_1$ an ordered basis for $\ker P$, $\mathscr{B}_2$ an ordered basis for im P, then $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is an ordered basis for V and

$$[P]_{\mathscr{B}} = \begin{pmatrix} [0]_{\mathscr{B}_1} & 0 \\ 0 & [1_{\text{im } P}]_{\mathscr{B}_2} \end{pmatrix} = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

Proof. Let $v \in V$, then $v - Pv \in \ker P$, since

$$P(v - Pv) = Pv - P^2 v = Pv - Pv = 0$$

Hence

$$v = (v - Pv) + Pv \in \ker P + \text{im } P$$

$\ker P \cap \text{im } P = 0$ and $P\big|_{\text{im } P} = 1_{\text{im } P}$. Let $v \in$ im P. By definition, $Pw = v$ for some $w \in V$. Therefore,

$$Pv = PPw = Pw = v$$

Hence

$$P\big|_{\text{im } P} = 1_{\text{im } P}$$

If $v \in \ker P \cap \text{im } P$, then

$$v = Pv = 0$$

□