Math 131BH – Honors Real Analysis II

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Spring 2021

This is math 131BH – Honors Real Analysis II, and it is instructed by Professor Visan. It's the second class in the undergrad real analysis sequence at UCLA. We meet weekly on MWF from 10:00 – 10:50 am for online lectures. Similar to 131AH, there are two textbooks associated to the course, *Principles of Mathematical Analysis* by *Rudin* and *Metric Spaces* by *Copson*. You can find the previous analysis lecture notes along with the other course notes through my github. Please email me if you notice any significant mathematical errors/typos that needs to be addressed. Thank you, and I hope you find this helpful for your study!

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§1 Lec 1: Mar 29, 2021

§1.1 Compactness

Definition 1.1 (Open Cover) — Let (X,d) be a metric space and let $A \subseteq X$. An open cover of A is a family $\{G_i\}_{i\in I}$ of open sets in X such that

$$A \subseteq \bigcup_{i\in I} G_i$$

The open cover is called finite if the cardinality of I is finite. If it's not finite, the open cover is called infinite.

Definition 1.2 (Compactness & Precompactness) — Let (X,d) be a metric space and let $K \subseteq X$.

1. We say that K is a compact set if every open cover $\{G_i\}_{i\in I}$ of K admits a finite subcover, that is,

$$\exists n \geq 1 \text{ and } \exists i_1, \ldots, i_n \in I \text{ s.t. } K \subseteq \bigcup_{j=1} G_{i_j}$$

2. We say that a set $A \subseteq X$ is precompact if $\overline{A}$ is compact.

Lemma 1.3

Let (X,d) be a metric space and let $\emptyset \neq Y \subseteq X$. We equip Y with the induced metric $d_1 : Y \times Y \to \mathbb{R}$, $d_1(y_1,y_2) = d(y_1,y_2)$. Let $K \subseteq Y \subseteq X$. The followings are equivalent:

1. K is compact in (X,d).
2. K is compact in (Y,d_1).

Proof. 1) $\implies$ 2) Assume K is compact in (X,d). Let $\{V_i\}_{i\in I}$ be a family of open sets in (Y,d_1) s.t.

$$K \subseteq \bigcup_{i\in I} V_i$$

For $i \in I$ fixed, V_i is open in $(Y,d_1) \implies \exists G_i \subseteq X$ open in (X,d) s.t.

$$V_i = G_i \cap Y$$

Then

$$\left.\begin{array}{l} K \subseteq \bigcup_{i\in I} V_i \subseteq \bigcup_{i\in I} G_i \\ K \text{ compact in } (X,d) \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists i_1,\dots,i_n \in I \text{ s.t.}$$

$$\left.\begin{array}{l} K \subseteq \bigcup_{j=1}^n G_{i_j} \\ K \subseteq Y \end{array}\right\} \implies K \subseteq \left(\bigcup_{j=1}^n G_{i_j}\right) \cap Y = \bigcup_{j=1}^n \left(G_{i_j} \cap Y\right) = \bigcup_{j=1}^n V_{i_j}$$

So K is compact in (Y, d_1).
2) $\implies$ 1) Assume K is compact in (Y, d_1). Let $\{G_i\}_{i\in I}$ be a family of open sets in (X,d) s.t.

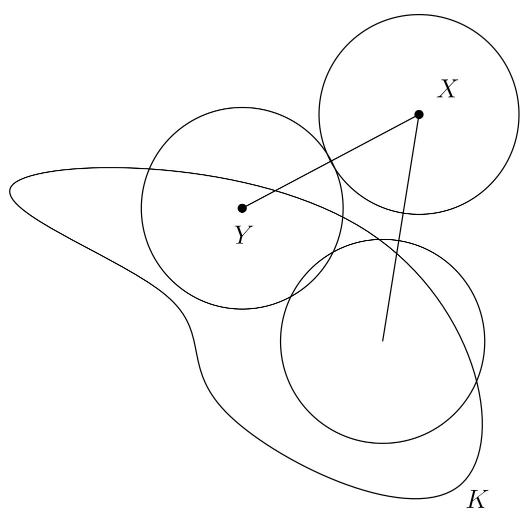
$$\left.\begin{array}{l} K \subseteq \bigcup_{i\in I} G_i \\ K \subseteq Y \end{array}\right\} \implies \left.\begin{array}{l} K \subseteq \left(\bigcup_{i\in I} G_i\right) \cap Y = \bigcup_{i\in I} \underbrace{(G_i \cap Y)}_{\text{open in } Y} \\ K \text{ is compact in } (Y,d_1) \end{array}\right\} \implies$$

$\implies \exists n \geq 1$ and $\exists i_1,\dots,i_n \in I$ s.t. $K \subseteq \bigcup_{j=1}^n \left(G_{i_j} \cap Y\right) \subseteq \bigcup_{j=1}^n G_{i_j}$. □

Proposition 1.4

Let (X,d) be a metric space and let $K \subseteq X$ be compact. Then K is closed and bounded.

Proof. Let's prove K is closed. We'll show ${}^cK$ is open.
Case 1: ${}^cK = \emptyset$. This is open.
Case 2: ${}^cK \neq \emptyset$. Let $x \in {}^cK$
For $y \in K$ let $r_y = \frac{d(x,y)}{2}$. Note $r_y > 0$ (since $x \in {}^cK$ and $y \in K$).



Note

$$\left.\begin{array}{l} K \subseteq \bigcup_{y\in K} \underbrace{B_{r_y}(y)}_{\text{open}} \\ K \text{ is compact} \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists y_1,\dots,y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^n B_{r_j}(y_j)$$

where we use the shorthand $r_j = r_{y_j}$.
Let $r = \min_{1 \le j \le n} r_j > 0$.
By construction, $B_r(x) \cap B_{r_j}(y_j) = \emptyset \quad \forall 1 \le j \le n$.

$$\implies B_r(x) \subseteq {}^cB_{r_j}(y_j) \quad \forall 1 \le j \le n$$

$$\implies B_r(x) \subseteq \bigcap_{j=1}^{n} {}^cB_{r_j}(y_j) = {}^c\left(\bigcup_{j=1}^{n} B_{r_j}(y_j)\right) \subseteq {}^cK$$

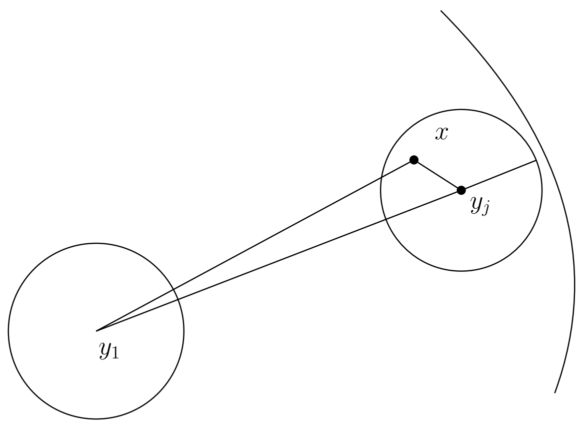
$$\implies \left.\begin{array}{l} x \in \overset{\circ}{\widehat{{}^cK}} \\ x \in {}^cK \text{ was arbitrary} \end{array}\right\} \implies {}^cK = \overset{\circ}{\widehat{{}^cK}}$$

Let's show K is bounded. Note

$$\left.\begin{array}{l} K \subseteq \bigcup_{y \in K} \underbrace{B_1(y)}_{\text{open}} \\ K \text{ compact} \end{array}\right\} \implies \exists n \ge 1 \text{ and } \exists y_1, \ldots, y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^{n} B_1(y_j)$$

For $2 \le j \le n$, let $r_j = d(y_1, y_j) + 1$.

Claim 1.1. $B_1(y_j) \subseteq B_{r_j}(y_1)$



Indeed, if $x \in B_1(y_j) \implies d(x, y_j) < 1$. By the triangle inequality

$$d(y_1, x) \le d(y_j, x) + d(y_1, y_j) < 1 + d(y_1, y_j) = r_j \implies x \in B_{r_j}(y_1)$$

So with $r = \max_{2 \le j \le n} r_j$,

$$K \subseteq \bigcup_{j=1}^{n} B_1(y_j) \subseteq B_r(y_1)$$

□

Proposition 1.5

Let (X, d) be a metric space and let $F \subseteq K \subseteq X$ such that F is closed in X and K is compact. Then F is compact.

Proof. Let $\{G_i\}_{i\in I}$ be a family of open sets in X s.t.

$$F \subseteq \bigcup_{i\in I} G_i$$

Then

$$\left.\begin{array}{l} K \subseteq F \cup {}^cF \subseteq \bigcup_{i\in I} G_i \cup \underbrace{{}^cF}_{\text{open in } X} \\ K \text{ compact} \end{array}\right\} \implies$$

$\implies \exists n \geq 1$ and $\exists i_1, \dots, i_n \in I$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j=1}^n G_{i_j} \cup {}^cF \\ F \subseteq K \end{array}\right\} \implies F = \left(\bigcup_{j=1}^n G_{i_j} \cup {}^cF\right) \cap F \subseteq \bigcup_{j=1}^n G_{i_j}$$

So F is compact. □

Corollary 1.6

Let (X, d) be a metric space and let $F \subseteq X$ be closed and let $K \subseteq X$ be compact. Then $K \cap F$ is compact.

Proof. K is compact. So

$$\left.\begin{array}{l} K \text{ closed} \\ F \text{ closed} \end{array}\right\} \implies \left.\begin{array}{l} K \cap F \text{ is closed} \\ K \cap F \subseteq K \text{ compact} \end{array}\right\} \implies K \cap F \text{ is compact}$$

□

§1.2 Sequential Compactness

Definition 1.7 (Sequential Compactness) — Let (X, d) be a metric space. A set $K \subseteq X$ is called sequentially compact if every sequence $\{x_n\}_{n\geq 1} \subseteq K$ admits a subsequence that converges in K.

§2 Lec 2: Mar 31, 2021

§2.1 Sequential Compactness (Cont'd)

Theorem 2.1 (Bolzano – Weierstrass)
Let (X, d) be a metric space and let $K \subseteq X$ be infinite. The following are equivalent:

1. K is sequentially compact.

2. For every infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

Proof. 1) $\implies$ 2) Let $A \subseteq K$ be infinite. As every infinite set has a countable subset we can find a sequence $\{a_n\}_{n\geq 1} \subseteq A$ such that $a_n \neq a_m \, \forall n \neq m$. As K is sequentially compact, $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

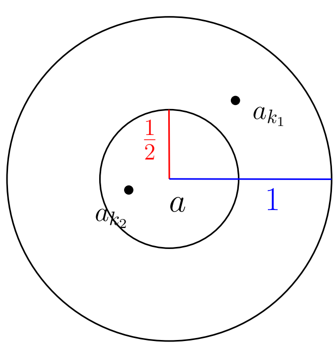
$$a_{k_n} \xrightarrow[n\to\infty]{d} a \in K$$

Claim 2.1. $a \in A' \iff \forall r > 0 \; B_r(a) \cap A \setminus \{a\} \neq \emptyset$.

Indeed, fix $r > 0$.

$$a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_r \in \mathbb{N} \text{ s.t. } d(a, a_{k_n}) < r \quad \forall n \geq n_r$$

As $a_n \neq a_m \, \forall n \neq m$, $\exists n_0 \geq n_r$ s.t. $a_{k_{n_0}} \neq a$. Then $a_{k_{n_0}} \in B_r(a) \cap A \setminus \{a\}$. We get $a \in A' \cap K$.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. We distinguish two cases:
Case 1: The sequence $\{a_n\}_{n\geq 1}$ contains a constant subsequence. That subsequence converges to an element in K.
Case 2: $\{a_n\}_{n\geq 1}$ does not contain a constant subsequence. Then $A = \{a_n : n \geq 1\}$ is infinite and $A \subseteq K$. So $A' \cap K \neq \emptyset$. Let $a \in A' \cap K$. Then $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t. $a_{k_n} \xrightarrow[n\to\infty]{d} a$.



$$B_1(a) \cap A \setminus \{a\} \neq \emptyset$$

$$B_{\min\left\{\frac{1}{2}, d(a, a_{k_1})\right\}}(a) \cap A \setminus \{a\} \neq \emptyset$$

$$k_2 > k_1$$

$\square$

Theorem 2.2

Let (X, d) be a metric space and let $K \subseteq X$ be compact. Then K is sequentially compact.

Proof. If K is finite, then any sequence $\{x_n\}_{n\geq 1} \subseteq K$ will have a constant subsequence.

Assume now K is infinite. We will use the Bolzano – Weierstrass theorem. It suffices to prove that for any infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

$$\left.\begin{array}{l}\text{Note } A \subseteq K \text{ then } A' \subseteq K' \\ K \text{ compact} \implies K \text{ closed} \implies K' \subseteq K\end{array}\right\} \implies A' \subseteq K \implies A' \cap K = A'$$

We argue by contradiction. Assume $A' = \emptyset$. Then for $x \in K$ we have $x \notin A' \implies \exists r_x > 0$ s.t. $B_{r_x}(x) \cap A \setminus \{x\} = \emptyset$. So

$$\left.\begin{array}{l}K \subseteq \bigcup_{x\in K} \underbrace{B_{r_x}(x)}_{\text{open}} \\ K \text{ compact}\end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists x_1, \ldots, x_n \in K \text{ s.t.}$$

$$K \subseteq \bigcup_{j=1}^{n} B_{r_j}(x_j) \text{ where } r_j = r_{x_j}$$

In particular,

$$\left.\begin{array}{l}A = \left(\bigcup_{j=1}^{n} B_{r_j}(x_j)\right) \cap A = \bigcup_{j=1}^{n} \left[B_{r_j}(x_j) \cap A\right] \\ \text{By construction, } B_{r_j}(x_j) \cap A \subseteq \{x_j\}\end{array}\right\} \implies \underbrace{A}_{\text{infinite}} \subseteq \underbrace{\bigcup_{j=1}^{n} \{x_j\}}_{\text{finite}}$$

– Contradiction! So $A' \neq \emptyset$. ◻

Proposition 2.3

Let (X, d) be a metric space and let $K \subseteq X$ be sequentially compact. Then K is closed and bounded.

Proof. Let's show K is closed $\iff K = \overline{K}$.

We know $K \subseteq \overline{K}$. We need to show $\overline{K} \subseteq K$. Let $x \in \overline{K} \implies \exists \{x_n\}_{n\geq 1} \subseteq K$ s.t. $x_n \xrightarrow[n\to\infty]{d} x$.

K sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

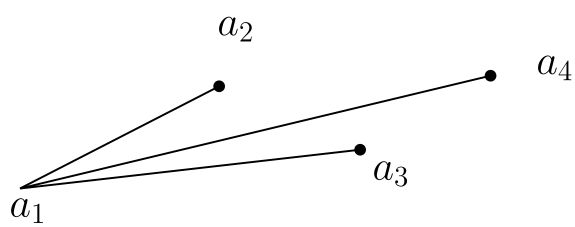
$$\left.\begin{array}{l}x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ x_n \xrightarrow[n\to\infty]{d} x \implies x_{k_n} \xrightarrow[n\to\infty]{d} x \\ \text{Limits of convergent sequences are unique}\end{array}\right\} \implies x = y \in K$$

As $x \in \overline{K}$ was arbitrary, we get $\overline{K} \subseteq K$.

Let's show K is bounded. We argue by contradiction. Assume K is not bounded. Let $a_1 \in K$.

$$\begin{aligned} K \text{ not bounded} &\implies K \nsubseteq B_1(a_1) \implies \exists a_2 \in K \text{ s.t. } d(a_1, a_2) \geq 1 \\ K \text{ not bounded} &\implies K \nsubseteq B_{1+d(a_1,a_2)}(a_1) \implies \exists a_3 \in K \text{ s.t. } d(a_1, a_3) \geq 1 + d(a_1, a_2) \end{aligned}$$

Proceeding inductively, we find a sequence $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_1, a_{n+1}) \geq 1 + d(a_1, a_n)$.



By construction,

$$|d(a_1, a_m) - d(a_1, a_n)| \geq |n - m| \quad \forall n, m \geq 1$$

By the triangle inequality,

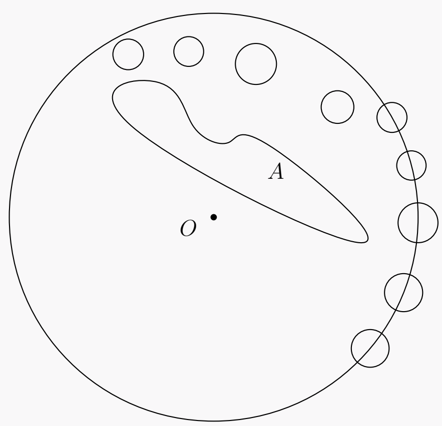
$$d(a_n, a_m) \geq |d(a_1, a_n) - d(a_1, a_m)| \geq |n - m| \quad \forall n, m \geq 1$$

This sequence cannot have a convergent (Cauchy) subsequence, thus contradiction the hypothesis that K is sequentially compact. So K is bounded. $\square$

Definition 2.4 (Totally Bounded) — Let (X, d) be a metric space. A set $A \subseteq X$ is totally bounded if for every $\epsilon > 0$, A can be covered by finitely many balls of radius ϵ.

Remark 2.5.

1. A totally bounded $\implies$ A bounded.
   Indeed, taking $\epsilon = 1$, $\exists n \geq 1$ and $\exists x_1, \ldots, x_n \in X$ s.t.
   $$A \subseteq \bigcup_{j=1}^{n} B_1(x_j) \subseteq B_r(x_1)$$
   where $r = 1 + \max_{2 \leq j \leq n} d(x_1, x_j)$.
2. A bounded $\not\Longrightarrow$ A totally bounded.
   Consider $\mathbb{N}$ equipped with the discrete metric
   $$d(n, m) = \begin{cases} 0, \ n = m \\ 1, \ n \neq m \end{cases}$$
   Then $\mathbb{N} = B_2(1)$, but $\mathbb{N}$ cannot be covered by finitely many balls of radius $\frac{1}{2}$ since $B_{\frac{1}{2}}(n) = \{n\}$.
3. On $(\mathbb{R}^n, d_2)$, A bounded $\implies$ A totally bounded. Indeed, A bounded $\implies$ $A \subseteq B_R(0)$ for some $R > 0$. $B_R(0)$ can be covered by $10^6 \left(\frac{R}{\epsilon}\right)^n$ many balls of radius ϵ.



§3 Lec 3: Apr 2, 2021

§3.1 Heine – Borel Theorem

Theorem 3.1

Let (X, d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is sequentially compact.
2. K is complete and totally bounded.

Proof. 1) $\implies$ 2) Let's show K is complete. Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence with $x_n \in K \quad \forall n \geq 1$.
K sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$\left.\begin{array}{l} x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ \{x_n\}_{n\geq 1} \text{ is Cauchy} \end{array}\right\} \implies x_n \xrightarrow[n\to\infty]{d} y \in K$$

As $\{x_n\}_{n\geq 1} \subseteq K$ was arbitrary, we get that K is complete.
Let's show K is totally bounded. Fix $\epsilon > 0$ and $a_1 \in K$.

- If $K \subseteq B_\epsilon(a_1)$, then K is totally bounded.
- If $K \nsubseteq B_\epsilon(a_1)$, then $\exists a_2 \in K$ s.t. $d(a_1, a_2) \geq \epsilon$
- If $K \subseteq B_\epsilon(a_1) \cup B_\epsilon(a_2)$, then K is totally bounded.
- If $K \nsubseteq B_\epsilon(a_1) \cup B_\epsilon(a_2)$, then $\exists a_3 \in K$ s.t. $d(a_1, a_3) \geq \epsilon$ and $d(a_2, a_3) \geq \epsilon$.

We distinguish two cases:
Case 1: The process terminates in finitely many steps $\implies K$ is totally bounded.
Case 2: The process does not terminate in finitely many steps. Then we find $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_n, a_m) \geq \epsilon \quad \forall n \neq m$. This sequence does not admit a convergent subsequence, contradicting the fact that K is sequentially compact.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. K totally bounded $\implies \mathcal{J}_1$ finite and $\left\{x_j^{(1)}\right\}_{j\in\mathcal{J}_1} \subseteq X$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j\in\mathcal{J}_1} B_1(x_j^{(1)}) \\ \{a_n\}_{n\geq 1} \subseteq K \end{array}\right\} \implies \exists j_1 \in \mathcal{J}_1 \text{ s.t. } \left|\left\{n : a_n \in B_1(x_{j_1}^{(1)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(1)}\right\}_{n\geq 1}$ be the corresponding subsequence.
K totally bounded $\implies \exists \mathcal{J}_2$ finite and $\left\{x_j^{(2)}\right\}_{j\in\mathcal{J}_2} \subseteq X$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j\in\mathcal{J}_2} B_{\frac{1}{2}}(x_j^{(2)}) \\ \left\{a_n^{(1)}\right\}_{n\geq 1} \subseteq K \end{array}\right\} \implies \exists j_2 \in \mathcal{J}_2 \text{ s.t. } \left|\left\{n : a_n^{(1)} \in B_{\frac{1}{2}}(x_{j_2}^{(2)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(2)}\right\}_{n\geq 1}$ denote the corresponding subsequence.
We proceed inductively. We find that $\forall k \geq 1$

- $\left\{a_n^{(k+1)}\right\}_{n\geq 1}$ subsequence of $\left\{a_n^{(k)}\right\}_{n\geq 1}$
- $\left\{a_n^{(k)}\right\}_{n\geq 1} \subseteq B_{\frac{1}{k}}\left(x_{j_k}^{(k)}\right)$ for some $x_{j_k}^{(k)} \in X$.

We consider the subsequence $\left\{a_n^{(n)}\right\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$.

$$\begin{aligned}
\left\{a_n^{(1)}\right\}_{n\geq 1} &= \left(a_1^{(1)}, \quad a_2^{(1)}, \quad a_3^{(1)}, \quad \ldots\right) \\
\left\{a_n^{(2)}\right\}_{n\geq 1} &= \left(\qquad a_1^{(2)}, \quad a_2^{(2)}, \quad a_3^{(2)}, \quad \ldots\right) \\
\left\{a_n^{(3)}\right\}_{n\geq 1} &= \left(\qquad\qquad a_1^{(3)}, \quad a_2^{(3)}, \qquad a_3^{(3)}, \quad \ldots\right)
\end{aligned}$$

For $n, m \geq k$ the $a_n^{(n)}, a_m^{(m)}$ belong to the subsequence $\left\{a_n^{(k)}\right\}_{n\geq 1}$. In particular,

$$d(a_n^{(n)}, a_m^{(m)}) \leq d(a_n^{(n)}, x_{j_k}^{(k)}) + d(a_m^{(m)}, x_{j_k}^{(k)}) < \frac{2}{k} \quad \forall n, m \geq k$$

This shows $\left\{a_n^{(n)}\right\}_{n\geq 1}$ is Cauchy and K is complete, so $a_n^{(n)} \xrightarrow[n\to\infty]{d} a \in K$. As $\{a_n\}_{n\geq 1}$ was arbitrary, we get that K is sequentially compact. □

Lemma 3.2

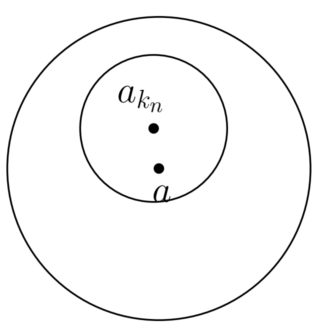
Let (X, d) be a sequentially compact metric space. Let $\{G_i\}_{i\in I}$ be an open cover of X. Then there exists $\epsilon > 0$ such that every ball of radius ϵ is contained in at least one G_i.

Proof. We argue by contradiction. Then

$$\forall n \geq 1 \quad \exists a_n \in X \text{ s.t. } B_{\frac{1}{n}}(a_n) \text{ is not contained in any } G_i$$

X is sequentially compact $\implies \exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

$$\begin{gathered}
a_{k_n} \xrightarrow[n\to\infty]{d} a \in X = \bigcup_{i\in I} G_i \implies \exists i_0 \in I \text{ s.t. } a \in G_{i_0} \\
G_{i_0} \text{ open} \implies \exists r > 0 \text{ s.t. } B_r(a) \subseteq G_{i_0} \\
a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_1(r) \in \mathbb{N} \text{ s.t. } d(a_1, a_{k_n}) < \frac{r}{2} \forall n \geq n_1
\end{gathered}$$



Let $n_2(r)$ s.t. $n_2 > \frac{2}{r}$.

Claim 3.1. $\forall n \geq n_r = \max\{n_1, n_2\}$ we have $B_{\frac{1}{k_n}}(a_{k_n}) \subseteq B_r(a) \subseteq G_{i_0}$ thefore giving a contradiction!

Fix $x \in B_{\frac{1}{k_n}}(a_{k_n})$. Then

$$d(a, x) \leq d(x, a_{k_n}) + d(a_{k_n}, a) < \frac{1}{k_n} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r$$

□

Theorem 3.3

A sequentially compact metric space (X, d) is compact.

Proof. Let $\{G_i\}_{i \in I}$ be an open cover of X. Let ϵ be given by the previous lemma. X sequentially compact $\implies$ X totally bounded $\implies$ $\exists n \geq 1$ and

$$\left.\begin{array}{l} \exists x_1, \ldots, x_n \in X \text{ s.t. } X = \bigcup_{j=1}^n B_\epsilon(x_j) \\ \forall 1 \leq j \leq n \quad \exists i_j \in I \text{ s.t. } B_\epsilon(x_j) \subseteq G_{i_j} \end{array}\right\} \implies X = \bigcup_{j=1}^n G_{i_j}$$

□

Collecting our results so far we obtain

Theorem 3.4 (Heine – Borel)

Let (X, d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is compact,
2. K is sequentially compact,
3. K is complete and totally bounded,
4. Every infinite subset of K has an accumulation point in K.

Remark 3.5. In $\mathbb{R}^n$, K is compact $\iff$ K is closed and bounded.

Definition 3.6 (Finite Intersection Property) — An infinite family $\{F_i\}_{i \in I}$ of closed sets is said to have the finite intersection property if $\forall \mathcal{J} \subseteq I$ finite we have

$$\bigcap_{j \in \mathcal{J}} F_j \neq \emptyset$$

Theorem 3.7

A metric space (X, d) is compact if and only if every infinite family $\{F_i\}_{i\in I}$ of closed sets with the finite intersection property satisfies

$$\bigcap_{i\in I} F_i \neq \emptyset$$

Proof. " $\implies$ " We argue by contradiction. Assume $\exists\{F_i\}_{i\in I}$ closed sets with the finite intersection property s.t. $\bigcap_{i\in I} F_i = \emptyset$

$$\left.\begin{aligned} & X = {}^c\left(\textstyle\bigcap_{i\in I} F_i\right) = \textstyle\bigcup_{i\in I} \underbrace{{}^cF_i}_{\text{open}} \\ & X \text{ compact} \end{aligned}\right\} \implies \exists \mathcal{J} \subseteq I \text{ finite s.t. } X = \bigcup_{j\in\mathcal{J}} {}^cF_j$$

$$\implies \emptyset = {}^c\left(\bigcup_{j\in\mathcal{J}} {}^cF_j\right) = \bigcap_{j\in\mathcal{J}} F_j \text{ -- Contradiction!}$$

" $\impliedby$ " We argue by contradiction. Assume $\exists\{G_i\}_{i\in I}$ open cover of X that does not admit a finite subcover.

So $\forall \mathcal{J} \subseteq I$ finite $X \neq \bigcup_{j\in\mathcal{J}} G_j \implies \emptyset \neq \bigcap_{j\in\mathcal{J}} \underbrace{{}^cG_j}_{\text{closed}}$. So $\{{}^cG_i\}_{i\in I}$ is a family of closed sets with the finite intersection property. Then

$$\bigcap_{i\in I} {}^cG_i \neq \emptyset \implies \bigcup_{i\in I} G_i \neq X$$

Contradiction! $\square$

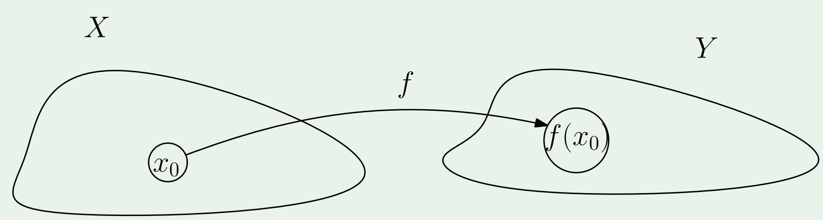
§4 Lec 4: Apr 5, 2021

§4.1 Continuity

Definition 4.1 (Continuous Function) — Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that a function $f : X \to Y$ is continuous at a point $x_0 \in X$ if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \text{ s.t. } d_X(x, x_0) < \delta \text{ then } d_Y(f(x), f(x_0)) < \epsilon$$

We say f is continuous (on X) if f is continuous at every point in X.



Remark 4.2. $f : X \to Y$ is continuous at every isolated point in X. Indeed, if $x_0 \in X$ is isolated, then $\exists \delta > 0$ s.t. $B^X_\delta(x_0) = \{x_0\}$. Then $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) = 0$

Proposition 4.3

Let $(X, d_X), (Y, d_Y)$ be two metric spaces and $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous at $x_0 \in X$.
2. For any $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $x_n \xrightarrow[n\to\infty]{d_X} x_0$ we have $f(x_n) \xrightarrow[n\to\infty]{d_Y} f(x_0)$.

Proof. 1) $\implies$ 2) Let $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $x_n \xrightarrow[n\to\infty]{d_X} x_0$.
Let $\epsilon > 0$. f continuous at $x_0 \implies \exists \delta > 0$ s.t.

$$\left.\begin{aligned} &d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon \\ &x_n \xrightarrow[n\to\infty]{d_X} x_0 \implies \exists n_\delta \in \mathbb{N} \text{ s.t. } d_X(x_n, x_0) < \delta\, \forall n \geq n_\delta \end{aligned}\right\} \implies d_X(f(x_n), f(x_0)) < \epsilon\, \forall n \geq n_\delta$$

2) $\implies$ 1) We argue by contradiction. Assume

$$\exists \epsilon_0 > 0 \text{ s.t. } \forall \delta > 0 \quad \exists x_\delta \in X \text{ s.t. } d_X(x_\delta, x_0) < \delta \text{ but } d_Y(f(x_\delta), f(x_0)) \geq \epsilon_0$$

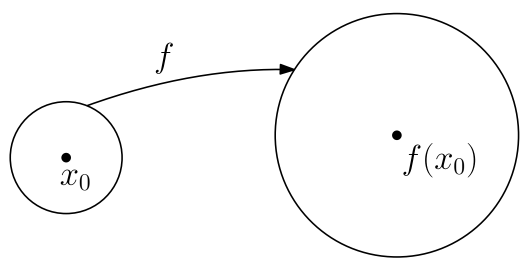
Letting $\delta = \frac{1}{n}$ we find $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $d_X(x_n, x_0) < \frac{1}{n}$ but $d_Y(f(x_n), f(x_0)) \geq \epsilon_0$ – Contradiction! $\square$

Theorem 4.4

Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous.
2. for any G open in Y, $f^{-1}(G) = \{x \in X : f(X) \in G\}$ is open in X.
3. for any F closed in Y, $f^{-1}(F)$ is closed in X.
4. for any $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.
5. for any $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

Proof. We will show 1) $\implies$ 2) $\implies$ 3) $\implies$ 4) $\implies$ 5) $\implies$ 1).
1) $\implies$ 2) Let $G \subseteq Y$ be open.



Let $x_0 \in f^{-1}(G)$

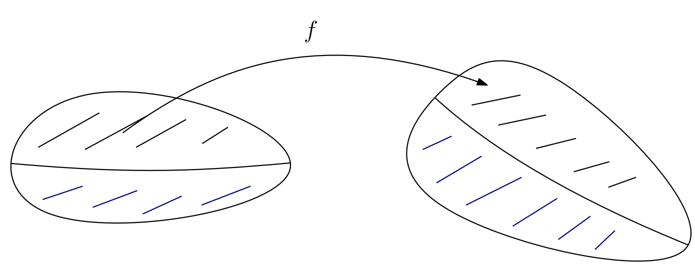
$$\implies \left.\begin{array}{l} f(x_0) \in G \\ G \text{ open in } Y \end{array}\right\} \implies \exists \epsilon > 0 \text{ s.t. } B_\epsilon^Y(f(x_0)) \subseteq G$$

f is continuous

$$\begin{aligned} &\implies \exists \delta > 0 \text{ s.t. } f\left(B_\delta^X(x_0)\right) \subseteq B_\epsilon^Y(f(x_0)) \subseteq G \\ &\implies B_\delta^X(x_0) \subseteq f^{-1}(G) \implies x_0 \in \overset{\circ}{\widehat{f^{-1}(G)}} \end{aligned}$$

So $f^{-1}(G)$ is open in X.
2) $\implies$ 3) Let $F \subseteq Y$ be closed $\implies$ ${}^cF = Y \setminus F$ is open in Y. By assumption,

$$\left.\begin{array}{l} f^{-1}\left({}^cF\right) \text{ is open in } X \\ f^{-1}\left({}^cF\right) = {}^c\left[f^{-1}(F)\right] = X \setminus f^{-1}(F) \end{array}\right\} \implies f^{-1}(F) \text{ is closed in } X$$



$$f^{-1}(Y \setminus F) = f^{-1}(Y) \setminus f^{-1}(F) = X \setminus f^{-1}(F)$$

3) $\implies$ 4) Let $B \subseteq Y \implies \overline{B}$ closed in Y. By assumption,

$$\left.\begin{aligned} &f^{-1}(\overline{B}) \text{ closed in } X \\ &f^{-1}(\overline{B}) \supseteq f^{-1}(B) \end{aligned}\right\} \implies \overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$$

4) $\implies$ 5) Let $A \subseteq X$. Use the hypothesis with $B = f(A)$. We have

$$\overline{A} \subseteq \overline{f^{-1}(f(A))} \subseteq f^{-1}\left(\overline{f(A)}\right) \implies f(\overline{A}) \subseteq \overline{f(A)}$$

5) $\implies$ 1) We argue by contradiction. Assume $\exists x_0 \in X$ s.t. f is not continuous at x_0. Then $\exists \epsilon_0 > 0$ and $\exists x_n \xrightarrow[n\to\infty]{d_X} x_0$ but $d_Y(f(x_n), f(x_0)) \geq \epsilon_0$.
Let $A = \{x_n : n \geq 1\}$. Then $x_0 \in \overline{A}$ but $f(x_0) \notin \overline{\{f(x_n) : n \geq 1\}} = \overline{f(A)}$. On the other hand, we must have

$$\left.\begin{aligned} &f(\overline{A}) \subseteq \overline{f(A)} \\ &x_0 \in \overline{A} \end{aligned}\right\} \implies f(x_0) \in \overline{f(A)}$$

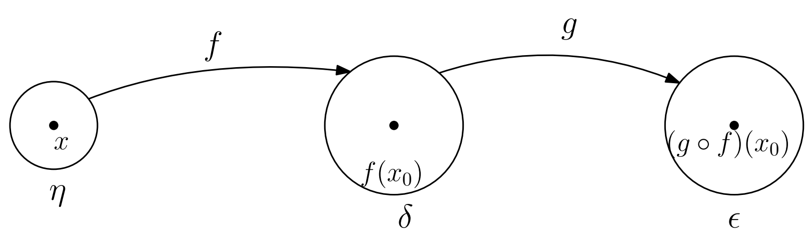
Contradiction! □

Proposition 4.5

Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces and assume $f : X \to Y$ is continuous at $x_0 \in X$ and $g : Y \to Z$ is continuous at $f(x_0) \in Y$. Then $g \circ f : X \to Z$ is continuous at x_0.

Proof. Fix $\epsilon > 0$.

$$\begin{aligned} g \text{ continuous at } f(x_0) &\implies \exists \delta > 0 \text{ s.t. } d_Y(y, f(x_0)) < \delta \implies d_Z(g(y), g(f(x_0))) < \epsilon \\ f \text{ continuous at } x_0 &\implies \exists \eta > 0 \text{ s.t. } d_X(x, x_0) < \eta \implies d_Y(f(x), f(x_0)) < \delta \end{aligned}$$



So if $d_X(x, x_0) < \eta$ then $d_Z(g(f(x)), g(f(x_0))) < \epsilon$. □

Exercise 4.1. Let (X, d) be a metric space and let $f, g : X \to \mathbb{R}$ be continuous at $x_0 \in X$. Then $f \pm g, f \cdot g$ are continuous at x_0. If $g(x_0) \neq 0$ then $\frac{f}{g} : X \to \mathbb{R}$ is continuous at x_0.

Exercise 4.2. Let (X, d) be a metric space and let $f_1, \ldots, f_n : X \to \mathbb{R}$. Then $f = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ is continuous at $x_0 \in X$ if and only if $f_1, \ldots, f_n$ are continuous at x_0.

Hint: $|f_i(x) - f_i(x_0)| \leq d_2(f(x), f(x_0)) = \sqrt{\sum_{j=1}^n |f_j(x) - f_j(x_0)|^2}$.

§4.2 Continuity and Compactness

Theorem 4.6

Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be continuous. If K is compact in X, then $f(K)$ is compact in Y.

Proof. Method 1: Let $\{G_i\}_{i \in I}$ be a family of open sets in Y s.t.

$$f(K) \subseteq \bigcup_{i \in I} G_i \implies K \subseteq f^{-1}\left(\bigcup_{i \in I} G_i\right) = \bigcup_{i \in I} \underbrace{f^{-1}(G_i)}_{\text{open in } X}$$

K compact $\implies \exists n \geq 1$ and $\exists i, \ldots, i_n \in I$ s.t.

$$K \subseteq \bigcup_{j=1}^{n} f^{-1}\left(G_{i_j}\right) = f^{-1}\left(\bigcup_{j=1}^{n} G_{i_j}\right) \implies f(K) \subseteq \bigcup_{j=1}^{n} G_{i_j}$$

Method 2: Let's show $f(K)$ is sequentially compact. Let $\{y_n\}_{n \geq 1} \subseteq f(K)$.

$$y_n \in f(K) \implies \exists x_n = f^{-1}(y_n) \in K$$

As K is sequentially compact, $\exists \{x_{k_n}\}_{n \geq 1}$ subsequence of $\{x_n\}_{n \geq 1}$ s.t.

$$\left.\begin{array}{l} x_{k_n} \xrightarrow[n \to \infty]{d_X} x_0 \in K \\ f \text{ is continuous} \end{array}\right\} \implies \underbrace{f\left(x_{k_n}\right)}_{= y_{k_n}} \xrightarrow[n \to \infty]{d_Y} f(x_0) \in f(K)$$

□

§5 Lec 5: Apr 7, 2021

§5.1 Continuity and Compactness (Cont'd)

Corollary 5.1

Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}^n$ be continuous. Then $f(X)$ is closed and bounded.

Corollary 5.2

Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then there exists $x_1, x_2 \in X$ s.t.

$$f(x_1) = \inf\{f(x) : x \in X\} \text{ and } f(x_2) = \sup\{f(x) : x \in X\}$$

Proof. $f(x)$ is closed and bounded.

$$\begin{aligned} \text{Boundedness} &\implies \inf f(x) \text{ and } \sup f(x) \text{ are well defined} \\ \text{Closedness} &\implies \inf f(x), \sup f(x) \in \overline{f(x)} = f(x) \end{aligned}$$

□

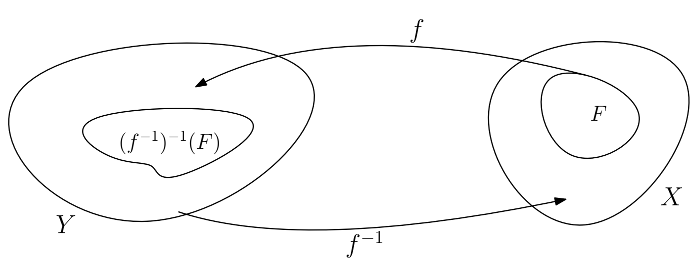
Proposition 5.3

Let $(X, d_X), (Y, d_Y)$ be metric spaces s.t. X is compact. Let $f : X \to Y$ be bijective and continuous. Then $f^{-1} : Y \to X$ is continuous.

Proof. If suffices to show that for every closed set $F \subseteq X$, we have

$$\left(f^{-1}\right)^{-1}(F) = \left\{y \in Y : f^{-1}(y) \in F\right\}$$

is closed in Y.



But $\left(f^{-1}\right)^{-1}(F) = f(F)$.

$$\left.\begin{aligned} &F \text{ closed in } X \text{ compact} \implies F\text{compact} \\ &f : X \to Y \text{ is continuous} \end{aligned}\right\} \implies f(F) \text{ is compact and closed}$$

□

Definition 5.4 (Uniform Continuity) — Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say that a function $f : X \to Y$ is uniformly continuous if

$$\forall \epsilon > 0 \quad \exists \delta = \delta(\epsilon) \text{ s.t. } d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$

Compare this with $g : X \to Y$ is continuous if

$$\forall x \in X \quad \forall \epsilon > 0 \quad \exists \delta = \delta(\epsilon, x) \text{ s.t. } d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$

Remark 5.5.
1. Continuity is defined pointwise. Uniform continuity is a property of a function on a set.
2. Uniform continuity $\implies$ continuity.
3. There are continuous functions that are not uniformly continuous.

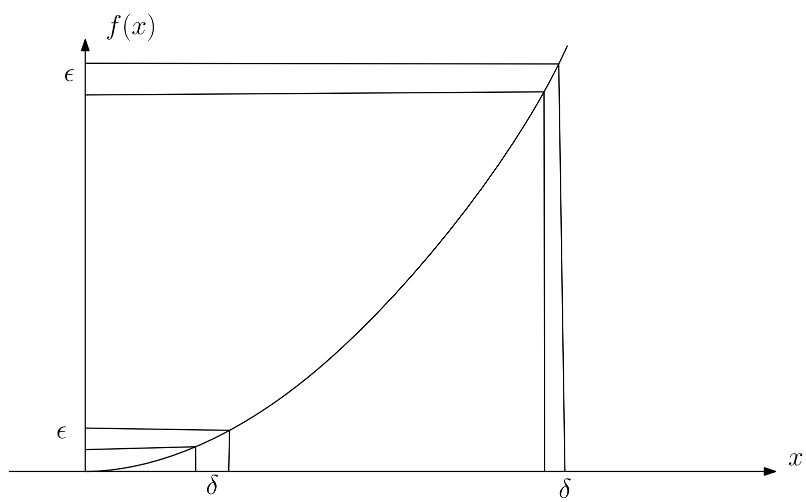
For example, consider

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

Let $x_n = n + \frac{1}{n}$, $y_n = n$

$$|x_n - y_n| = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} > 2$$



Theorem 5.6

Let $(X, d_X), (Y, d_Y)$ be metric spaces with X compact. Let $f : X \to Y$ continuous. Then f is uniformly continuous.

Proof. We argue by contradiction. Assume f is not uniformly continuous $\implies \exists \epsilon_0 > 0$ s.t. $\forall \delta > 0 \, \exists x_\delta, y_\delta \in X$ s.t. $d_X(x_\delta, y_\delta) < \delta$ but $d_Y(f(x_\delta), f(y_\delta)) \geq \epsilon_0$.

Let $\delta = \frac{1}{n}$ to get $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1} \subseteq X$ s.t. $d_X(x_n, y_n) < \frac{1}{n}$ but $d_Y(f(x_n), f(y_n)) \geq \epsilon_0$
X compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$x_{k_n} \xrightarrow[n\to\infty]{d_X} x_0 \in X$$

By the triangle inequality,

$$d(y_{k_n}, x_0) \leq \underbrace{d(x_{k_n}, y_{k_n})}_{<\frac{1}{k_n} \leq \frac{1}{n} \xrightarrow[n\to\infty]{} 0} + \underbrace{d(x_{k_n}, x_0)}_{\xrightarrow[n\to\infty]{} 0} \xrightarrow[n\to\infty]{} 0 \implies y_{k_n} \xrightarrow[n\to\infty]{d_X} x_0$$

$$f \text{ continuous } \implies \begin{cases} f(x_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0) \\ f(y_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0) \end{cases}$$

But

$$\epsilon_0 \leq d_Y(f(x_{k_n}), f(y_{k_n})) \leq \underbrace{d_Y(f(x_{k_n}), f(x_0))}_{\to 0} + \underbrace{d_Y(f(x_0), f(y_{k_n}))}_{\to 0} \xrightarrow[n\to\infty]{} 0$$

Contradiction! □

§5.2 Continuity and Connectedness

Theorem 5.7

Let $(X, d_X), (Y, d_Y)$ be metric spaces s.t. X is connected. Let $f : X \to Y$ be continuous. Then $f(X)$ is connected.

Proof. Method 1: Abusing notation we write $f : X \to f(x)$. It suffices to show that if $\emptyset \neq B \subseteq f(x)$ is both open and closed in $f(x)$ then $B = f(x)$.

As f is continuous, $f^{-1}(B) \neq \emptyset$ is both open and closed in X. But X is connected which implies $f^{-1}(B) = X$ and $f(x) = B$.

Method 2: Assume that $f(x)$ is not connected. Then $\exists \emptyset \neq B_1 \subseteq Y$, $\exists \emptyset \neq B_2 \subseteq Y$ s.t. $f(x) \subseteq B_1 \cup B_2$ and

$$\overline{B_1} \cap B_2 = \emptyset = B_1 \cap \overline{B_2}$$

let

$$A_1 = f^{-1}(B_1) \neq \emptyset$$
$$A_2 = f^{-1}(B_2) \neq \emptyset$$

Have

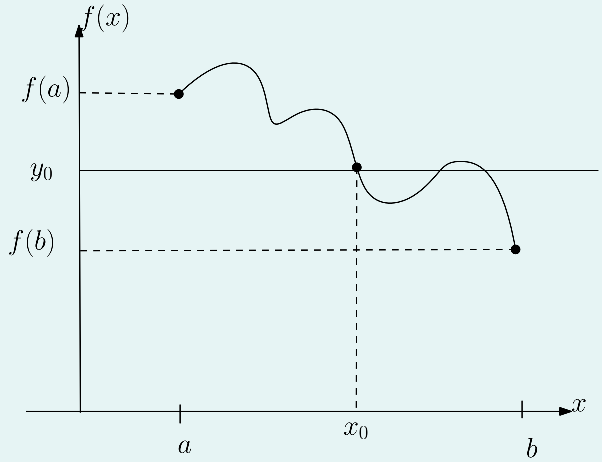
$$\begin{aligned} f(X) \subseteq B_1 \cup B_2 \implies X \subseteq f^{-1}(B_1 \cup B_2) &= f^{-1}(B_1) \cup f^{-1}(B_2) = A_1 \cup A_2 \\ \overline{A_1} \cap A_2 = \overline{f^{-1}(B_1)} \cap f^{-1}(B_2) &\subseteq f^{-1}(\overline{B_1}) \cap f^{-1}(B_2) = f^{-1}(\overline{B_1} \cap B_2) \\ &= f^{-1}(\emptyset) = \emptyset \end{aligned}$$

Similarly, $\overline{A_2} \cap A_1 = \emptyset$. exercise
This contradicts that X is connected. □

Corollary 5.8 (Darboux's Property)

Let (X, d_X) be a metric space and let $f : X \to \mathbb{R}$ be continuous. If $A \subseteq X$ is connected then $f(A)$ is an interval in $\mathbb{R}$.

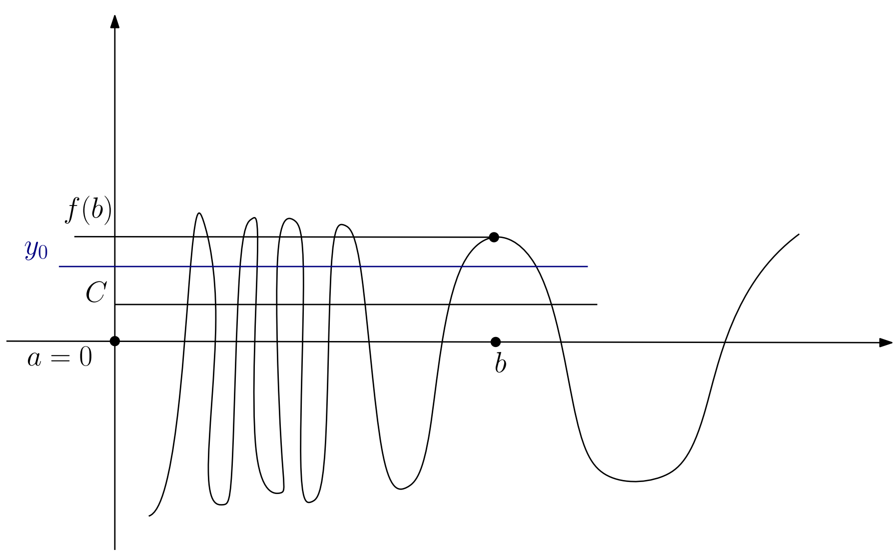
In particular, if $X = \mathbb{R}$, and $a, b \in \mathbb{R}$ s.t. $a < b$ and y_0 lies between $f(a)$ and $f(b)$, then $\exists x_0 \in (a, b)$ s.t. $f(x_0) = y_0$.



Remark 5.9. There are function that have the Darboux property, but are not continuous.

For example, consider

$$f : [0, \infty) \to \mathbb{R}, \quad f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ c, & x = 0 \end{cases} \quad \text{where } c \in [-1, 1]$$



Notice f is continuous on $(0, \infty)$ implies f has the Darboux property on $(0, \infty)$. f has the Darboux property on $[0, \infty)$, but is not continuous at $x = 0$.

§6 Lec 6: Apr 9, 2021

§6.1 Continuity and Connectedness (Cont'd)

Proposition 6.1

Let (X, d_X) and (Y, d_Y) be two connected metric spaces. Then $(X \times Y, d)$ where

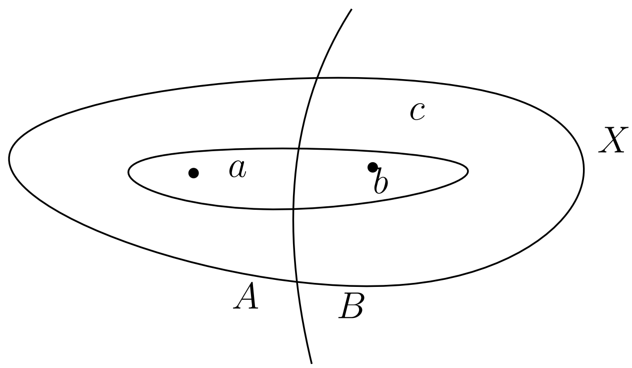
$$d\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

is a connected metric space.

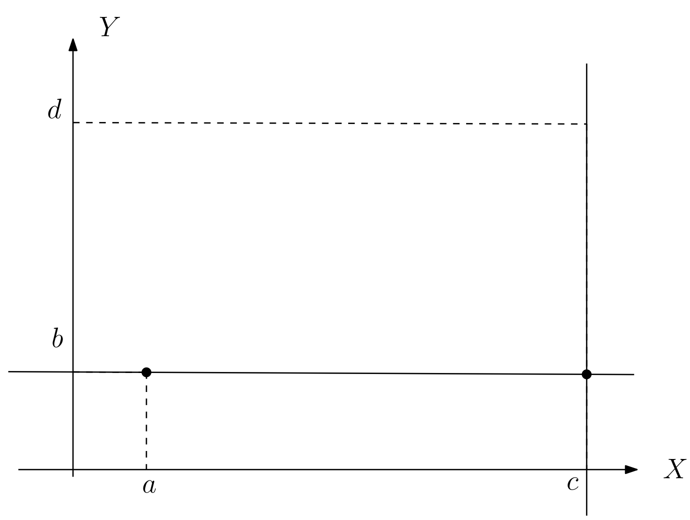
Remark 6.2. One could replace the distance d by

$$d_1\left((x_1, y_1), (x_2, y_2)\right) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$
$$d_\infty\left((x_1, y_1), (x_2, y_2)\right) = \max\left\{d_X(x_1, x_2), d_Y(y_1, y_2)\right\}$$

Proof. We will use the fact that a metric space is connected if and only if any two points are contained in a connected subset of the metric space.



So to show $X \times Y$ is connected if suffices to show that if $(a, b), (c, d) \in X \times Y$, then there exists $C \subseteq X \times Y$ connected s.t. $(a, b), (c, d) \in C$.



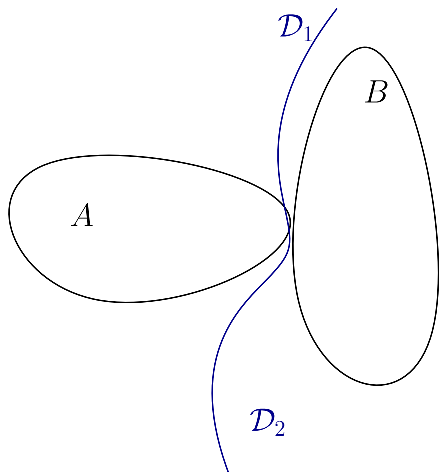
Let $f : X \to X \times Y$ where $f(x) = (x, b)$

Claim 6.1. f is continuous.

Take $\delta = \epsilon$ in the definition of continuity. As X is connected, $f(X) = X \times \{b\}$ is connected.

Similarly, $g : Y \to X \times Y$, $g(y) = (c, y)$ is continuous and since Y is connected, $g(Y) = \{c\} \times Y$ is connected.

Finally, $f(x) \cap g(y) \ni (c, b)$ and so $f(x)$, $g(y)$ are not separated. As the union of two connected not separated sets is connected we get $f(x) \cup g(y)$ is connected.



Note $(a, b), (c, d) \in f(x) \cup g(y)$. □

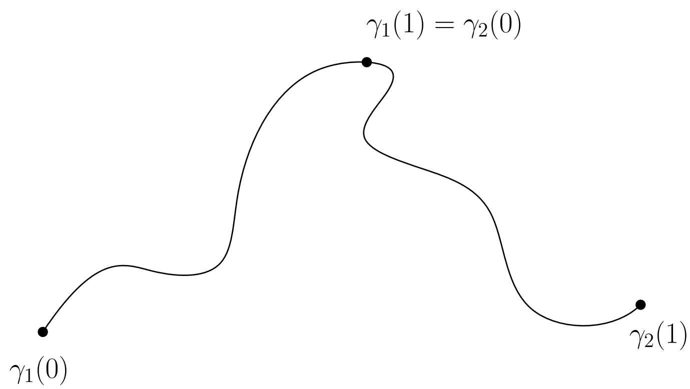
Definition 6.3 (Path) — Let (X, d) be a metric space. A path is a continuous function $\gamma : [0, 1] \to X$. $\gamma(0)$ is called the origin of the path and $\gamma(1)$ is called the end of the path.

As $[0, 1]$ is compact and connected and γ is continuous, $\gamma([0, 1])$ is compact and connected.

Given $\gamma : [0, 1] \to X$ a path, we define

$$\gamma^- : [0, 1] \to X, \qquad \gamma^-(t) = \gamma(1 - t) \text{ is a path}$$

Given $\gamma_1, \gamma_2 : [0, 1] \to X$ paths s.t. $\gamma_1(1) = \gamma_2(0)$.



We define

$$\gamma_1 \vee \gamma_2 : [0,1] \to X$$

via

$$\gamma_1 \vee \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Proposition 6.4

Let (X,d) be a metric space and let $A \subseteq X$. Then 1) $\iff$ 2) $\implies$ 3) where

1. $\exists a \in A$ s.t. $\forall x \in A\, \exists \gamma_x : [0,1] \to A$ path s.t.

$$\gamma_x(0) = a \text{ and } \gamma_x(1) = x$$

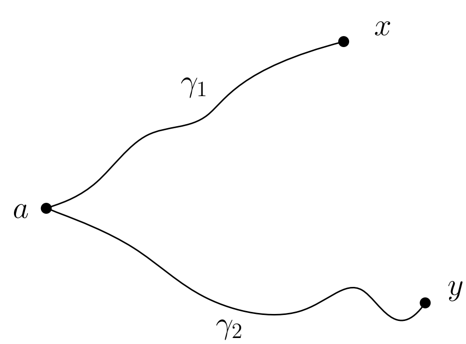
2. $\forall x, y \in A\, \exists \gamma_{x,y} : [0,1] \to A$ path s.t.

$$\gamma_{x,y}(0) = x \text{ and } \gamma_{x,y}(1) = y$$

3. A is connected.

Proof. 1) $\implies$ 2) Let $x, y \in A$. By hypothesis, $\exists \gamma_x, \gamma_y : [0,1] \to A$ paths s.t.

$$\gamma_x(0) = \gamma_y(0) = a, \quad \gamma_x(1) = x, \quad \gamma_y(1) = y$$



Then $\gamma_x^- \vee \gamma_y : [0,1] \to A$ is the desired path.
2) $\implies$ 1)Choose $a \in A$ arbitrary.
1) $\implies$ 3) Given $x \in A$, let $A_x = \gamma_x([0,1])$ connected. Note

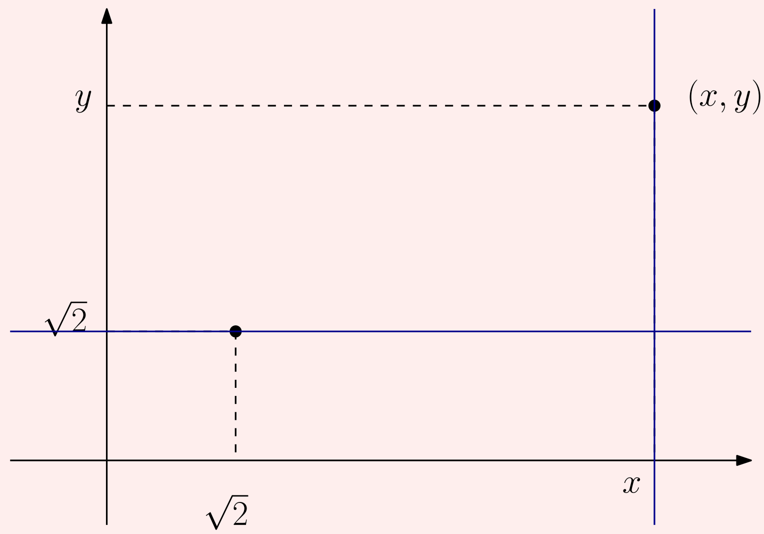
$$a \in \bigcap_{x \in A} A_x \implies \text{ no two sets } A_x,\ A_y \text{ are separated}$$

Then $A = \bigcup_{x \in A} A_x$ is connected. □

Definition 6.5 (Path Connected) — If either 1) or 2) holds in the Proposition 6.4, we say that A is path connected. Note A is path connected implies A is connected.

Example 6.6

$\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected.



We will show that any $(x, y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ can be joined via path in $\mathbb{R}^2 \setminus \mathbb{Q}^2$ to $\left(\sqrt{2}, \sqrt{2}\right)$.

$$(x, y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2 \implies x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q}$$

Say $x \notin \mathbb{Q}$. Then $\{x\} \times \mathbb{R} \subseteq \mathbb{R}^2 \setminus \mathbb{Q}^2$. Note also that $\mathbb{R} \times \left\{\sqrt{2}\right\} \subseteq \mathbb{R}^2 \setminus \mathbb{Q}^2$. Let $\gamma : [0, 1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2$, $\gamma = \gamma_1 \vee \gamma_2$ where

$$\gamma_1 : [0, 1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2,\ \gamma_1(t) = \left(\sqrt{2} + t(x - \sqrt{2}), \sqrt{2}\right) \text{ path}$$
$$\gamma_2 : [0, 1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2,\ \gamma_2(t) = \left(x, \sqrt{2} + t(y - \sqrt{2})\right) \text{ path}$$

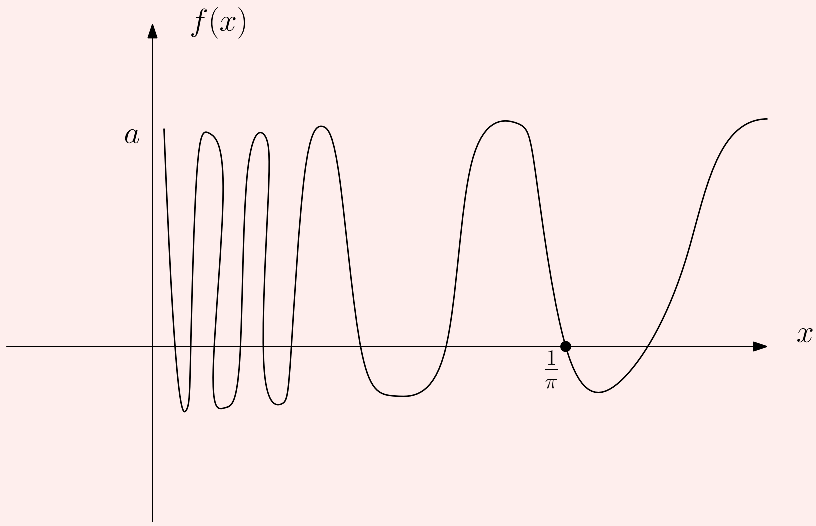
Example 6.7

A connected set which is not path connected. Let $f : [0, \infty) \to \mathbb{R}$ s.t.

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ a, & x = 0 \end{cases}$$

where $a \in [-1, 1]$ fixed.

Then $\Gamma_f = \{(x, f(x)) : x \in [0, \infty)\}$ is connected, but not path connected.



Let's show Γ_f is connected. The function $g : [0, \infty) \to \mathbb{R}^2$, $g(x) = (x, f(x))$ is continuous on $(0, \infty) \implies g((0, \infty))$ is connected.

Also, $g(\{0\}) = \{(0, a)\}$ is connected. We will show that $(0, a) \in \overline{g((0, \infty))}$ and so $\{(0, a)\}, g((0, \infty))$ are not separated. Then

$$\Gamma_f = g([0, \infty)) = g(\{0\}) \cup g((0, \infty)) \text{ is connected}$$

To see $(0, a) \in \overline{g(0, \infty)}$ we need to find $x_n \to 0$ s.t.

$$\sin\left(\frac{1}{x_n}\right) = a$$

Take $x_n = \frac{1}{\arcsin a + 2n\pi}$ where $\arcsin a \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Example 6.8 (Cont'd from above)

Now let's show Γ_f is not path connected. Assume towards a contradiction that there exists $\gamma : [0,1] \to \Gamma_f$ a path s.t.

$$\gamma(0) = (0,a), \qquad \gamma(1) = \left(\frac{1}{\Pi}, 0\right)$$

Note $\Pi_1 \circ \gamma : [0,1] \to \mathbb{R}$ is continuous

$$(\Pi_1 \circ \gamma)(0) = 0, \quad (\Pi_1 \circ \gamma)(1) = \frac{1}{\pi}$$

Let $b \in [-1,1] \setminus \{a\}$. By the Darboux property, $\exists t_n \in \left(0, \frac{1}{\pi}\right)$ s.t.

$$(\Pi_1 \circ \gamma)(t_n) = \frac{1}{\arcsin b + 2n\pi} \text{ where } \arcsin b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

As $[0,1]$ is compact, $\exists t_{k_n} \underset{n\to\infty}{\longrightarrow} t_\infty \in [0,1]$.

$$\left.\begin{aligned} &\gamma \text{ continuous} \implies \gamma\left(t_{k_n}\right) \underset{n\to\infty}{\longrightarrow} \gamma(t_\infty) \\ &\gamma\left(t_{k_n}\right) = \left(\tfrac{1}{\arcsin b + 2k_n\pi}, b\right) \underset{n\to\infty}{\longrightarrow} (0,b) \end{aligned}\right\} \implies \gamma(t_\infty) = (0,b) \notin \Gamma_f$$

§7 Lec 7: Apr 12, 2021

§7.1 Continuity and Connectedness (Cont'd)

Example 7.1

Two connected sets $A, B \subseteq [-1,1] \times [-1,1]$ s.t. $(-1,-1),(1,1) \in A$, $(-1,1),(1,-1) \in B$, $A \cap B = \emptyset$. Let $f : [-1,1] \to [-1,1]$,

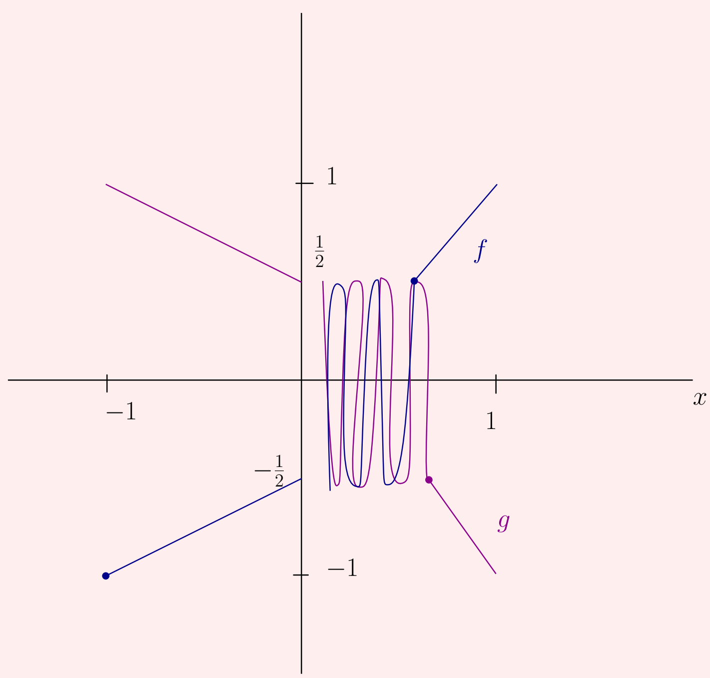
$$f(x) = \begin{cases} \frac{x-1}{2}, & -1 \le x \le 0 \\ x - \frac{1}{2}\sin\frac{\pi}{x}, & 0 < x \le \frac{1}{2} \\ x, & \frac{1}{2} \le x \le 1 \end{cases}$$

Let $g : [-1,1] \to [-1,1]$,

$$g(x) = \begin{cases} \frac{1-x}{2}, & -1 \le x \le 0 \\ -x - \frac{1}{2}\sin\frac{\pi}{x}, & 0 < x \le \frac{1}{2} \\ -x, & \frac{1}{2} \le x \le 1 \end{cases}$$

Let

$$A = \Gamma_f = \{(x_1 f(x)) : x \in [-1,1]\}$$
$$B = \Gamma_g = \{(x_1 g(x)) : x \in [-1,1]\}$$



Example 7.2 (Cont'd from above)
Let's prove $A \cap B = \emptyset$. If

$$\begin{aligned}
-1 \le x \le 0, &\quad f(x) = g(x) \iff \frac{x-1}{2} = \frac{1-x}{2} \iff x = 1 \\
0 < x \le \frac{1}{2}, &\quad f(x) = g(x) \iff x = 0 \\
\frac{1}{2} \le x \le 1, &\quad f(x) = g(x) \iff x = 0
\end{aligned}$$

Also

$$\begin{aligned}
f(-1) = -1 &\implies (-1,-1) \in A \\
f(1) = 1 &\implies (1,1) \in A \\
g(-1) = 1 &\implies (-1,1) \in B \\
g(1) = -1 &\implies (1,-1) \in B
\end{aligned}$$

Let's show that A is connected. A similar argument can be used to prove that B is connected.

We write $A = A_1 \cup A_2$ where $A_1 = \{(x, f(x)) : -1 \le x \le 0\}$ and $A_2 = \{(x, f(x)) : 0 < x \le 1\}$. Note that $h : [-1,1] \to \mathbb{R}^2$ where $h(x) = (x, f(x))$ is continuous on $[-1,0]$ and $(0,1]$.

Since $[-1,0]$ and $(0,1]$ are connected sets, we get that $h([-1,0]) = A_1$ and $h((0,1]) = A_2$ are connected.

To show that $A = A_1 \cup A_2$ is connected, it suffices to show that A_1 and A_2 are not separated. We will show $\left(0, -\frac{1}{2}\right) \in A_1 \cap \overline{A_2}$. It's clear that $f(0) = -\frac{1}{2} \implies \left(0, -\frac{1}{2}\right) \in A_1$. To show that $\left(0, -\frac{1}{2}\right) \in \overline{A_2}$ we need to find a decreasing sequence $x_n \to 0$ s.t.

$$f(x_n) = x_n - \frac{1}{2}\sin\frac{\pi}{x_n} \xrightarrow[n\to\infty]{} -\frac{1}{2}$$

We take x_n s.t. $\sin\frac{\pi}{x_n} = 1 \iff \frac{\pi}{x_n} = \frac{\pi}{2} + 2n\pi \iff x_n = \frac{2}{4n+1} \to 0$. Notice that

$$f(x_n) = \frac{2}{4n+1} - \frac{1}{2} \xrightarrow[n\to\infty]{} -\frac{1}{2}$$

§7.2 Convergent Sequences of Functions

Definition 7.3 (Pointwise Convergence) — Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\ge1}$ <u>converges pointwise</u> if for all $x \in X$ the sequence $\{f_n(x)\}_{n\ge1}$ converges in Y. The limit $\lim_{n\to\infty} f_n(x) = f(x)$ defines a function $f : X \to Y$.

Remark 7.4. $\{f_n\}_{n\ge1}$ converges pointwise to f if

$$\forall x \in X \quad \forall \epsilon > 0 \quad \exists n(\epsilon, x) \in \mathbb{N} \text{ s.t. } d_Y(f_n(x), f(x)) < \epsilon \quad \forall n \ge n(\epsilon, x)$$

Note that for $\epsilon > 0$ fixed, $n(\epsilon, \cdot) : X \to \mathbb{N}$ can be bounded or unbounded. If it is bounded, we get the following

Definition 7.5 (Uniform Convergence) — Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\geq 1}$ converges uniformly to a function $f : X \to Y$ if

$$\forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } d_Y\left(f(x), f_n(x)\right) < \epsilon \quad \forall n \geq n_\epsilon \, \forall x \in X$$

We denote $f_n \xrightarrow[n\to\infty]{u} f$.

Remark 7.6. Let $(X, d_X), (Y, d_Y)$ be metric spaces, $B(X, Y) = \{f : X \to Y;\ f \text{ is bounded}\}$, $d:\ B(X,Y) \times B(X,Y) \to \mathbb{R}$ via

$$d(f, g) = \sup_{x\in X} d_Y\left(f(x), g(x)\right)$$

Exercise 7.1. Show that $(B(X, Y), d)$ is a metric space.

Note that $f_n \xrightarrow[n\to\infty]{u} f \iff M_n = d\left(f_n, f\right) \xrightarrow[n\to\infty]{} 0$.
" $\Longleftarrow$ " $\forall \epsilon > 0\ \exists n_\epsilon \in \mathbb{N}$ s.t. $M_n < \epsilon\ \forall n \geq n_\epsilon$

$$\begin{aligned} &\implies d(f_n, f) = \sup_{x\in X} d_Y\left(f_n(x), f(x)\right) < \epsilon \quad \forall n \geq n_\epsilon \\ &\implies d_Y\left(f_n(x), f(x)\right) < \epsilon \quad \forall n \geq n_\epsilon \quad \forall x \in X \end{aligned}$$

" $\implies$ "

$$\begin{aligned} f_n \xrightarrow[n\to\infty]{u} f &\implies \forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } d_Y\left(f_n(x), f(x)\right) < \frac{\epsilon}{2} \quad \forall n \geq n_\epsilon \, \forall x \in X \\ &\implies \underbrace{\sup_{x\in X} d_Y\left(f_n(x), f(x)\right)}_{d(f_n, f) = M_n} \leq \frac{\epsilon}{2} < \epsilon \quad \forall n \geq n_\epsilon \end{aligned}$$

Remark 7.7.
1. Uniform convergence $\implies$ pointwise convergence
2. Pointwise convergence $\not\Longrightarrow$ uniform convergence

$f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = x^n$

$$\{f_n\}_{n\geq 1} \text{ converges pointwise : } \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Let

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Note $f_n \overset{u}{\underset{n\to\infty}{\not\to}} f$ since

$$d(f_n, f) = \sup_{x\in[0,1]} \left|f_n(x) - f(x)\right| = \sup_{x\in[0,1)} \left|x^n\right| = 1 \underset{n\to\infty}{\not\to} 0$$

Theorem 7.8 (Weierstrass)
Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f_n : X \to Y$ be a sequence of functions that converges uniformly to a function $f : X \to Y$. If $\forall n \geq 1$, f_n is continuous at $x_0 \in X$ then f is continuous at x_0.

Corollary 7.9
A uniform limit of continuous functions is a continuous function.

Proof. (of theorem) Fix $\epsilon > 0$.

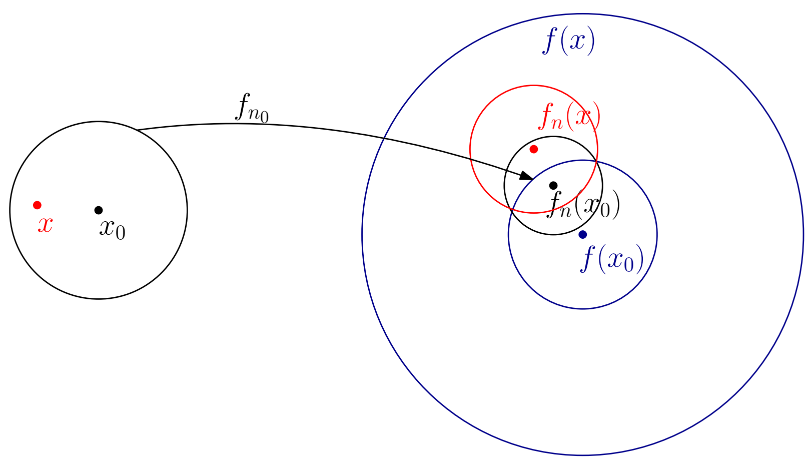
$$f_n \xrightarrow[n\to\infty]{u} f \implies \exists n_\epsilon \in \mathbb{N} \text{ s.t. } d_Y\left(f_n(x), f(x)\right) < \frac{\epsilon}{3} \quad \forall n \geq n_\epsilon \, \forall x \in X$$

Fix $n_0 \geq n_\epsilon$. f_{n_0} is continuous at x_0

$$\implies \exists \delta > 0 \text{ s.t. if } d_X(x_0, x) < \delta$$

then

$$d_Y\left(f_{n_0}(x_0), f_{n_0}(x)\right) < \frac{\epsilon}{3}$$



Then for $x \in B_\delta(x_0)$ we have

$$\begin{aligned} d_Y\left(f(x), f(x_0)\right) &\leq d_Y\left(f(x), f_{n_0}(x)\right) + d\left(f_{n_0}(x), f_{n_0}(x_0)\right) + d\left(f_{n_0}(x_0), f(x_0)\right) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

By definition, f is continuous at x_0. $\square$

§8 Lec 8: Apr 14, 2021

§8.1 Convergent Sequences of Functions (Cont'd)

Theorem 8.1 (Dini)

Let (X, d) be a compact metric space and let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions that converges pointwise to a continuous function $f : X \to \mathbb{R}$. Assume that $\{f_n\}_{n\geq 1}$ is monotone in the sense that either $\{f_n(x)\}_{n\geq 1}$ is increasing for all $x \in X$ or $\{f_n(x)\}_{n\geq 1}$ is decreasing for all $x \in X$. Then,

$$f_n \xrightarrow[n\to\infty]{u} f \text{ i.e. } d(f_n, f) = \sup_{x\in X} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$$

Proof. Assume that $\{f_n\}_{n\geq 1}$ is increasing. Then $\{f - f_n\}_{n\geq 1}$ is decreasing and for all $x \in X$ we have

$$\lim_{n\to\infty} [f(x) - f_n(x)] = \inf_{n\to\infty} [f(x) - f_n(x)] = 0$$

Then $\forall \epsilon > 0 \quad \exists n(\epsilon, x) \in \mathbb{N}$ s.t. $\forall n \geq n(\epsilon, x)$ we have

$$0 \leq f(x) - f_n(x) \leq f(x) - f_{n_{\epsilon,x}}(x) < \epsilon$$

As $f - f_{n_{\epsilon,x}}$ is continuous at x, $\exists \delta(\epsilon, x) > 0$ s.t.

$$d(x, y) < \delta_{\epsilon,x} \implies \left|\left[f(x) - f_{n_{\epsilon,x}}(x)\right] - \left[f(y) - f_{n_{\epsilon,x}}(y)\right]\right| < \epsilon$$

By the triangle inequality, we get

$$\begin{aligned} 0 \leq f(y) - f_{n_{\epsilon,x}}(y) &\leq \left|\left[f(x) - f_{n_{\epsilon,x}}(x)\right] - \left[f(y) - f_{n_{\epsilon,x}}(y)\right]\right| + f(x) - f_{n_{\epsilon,x}}(x) \\ &< \epsilon + \epsilon = 2\epsilon \end{aligned}$$

whenever $y \in B_{\delta_{\epsilon,x}}(x)$. In particular,

$$0 \leq f(y) - f_n(y) \leq f(y) - f_{n_{\epsilon,x}}(y) < 2\epsilon \quad \forall n \geq n_{\epsilon,x}, \forall y \in B_{\delta_{\epsilon,x}}(x) \tag{*}$$

Note

$$\left.\begin{array}{l} X = \bigcup_{x\in X} B_{\delta_{\epsilon,x}}(x) \\ X \text{ compact} \end{array}\right\} \implies \exists \mathcal{J} \subseteq \mathbb{N} \text{ finite and } \exists \{x_j\}_{j\in\mathcal{J}} \in X$$

s.t. $X = \bigcup_{j\in\mathcal{J}} B_{\delta_j}(x_j)$ and where $\delta_j = \delta(\epsilon, x_j)$.
Let $n_\epsilon = \max_{j\in\mathcal{J}} n(\epsilon, x_j)$. Fix $n \geq n_\epsilon$ and $x \in X$. As $x \in X = \bigcup_{j\in\mathcal{J}} B_{\delta_j}(x_j) \implies j \in \mathcal{J}$ s.t. $x \in B_{\delta_j}(x_j)$. By (*), we have

$$0 \leq f(x) - f_n(x) < 2\epsilon$$

As $x \in X$ was arbitrary we get

$$d(f, f_n) \leq 2\epsilon \qquad \forall n \geq n_\epsilon$$

□

Remark 8.2. The compactness of X is necessary in Dini's theorem.

Example 8.3

$f_n : (0,1) \to \mathbb{R}$, $f_n(x) = x^n$ continuous

$$f_{n+1}(x) \le f_n(x) \quad \forall n \ge 1 \quad \forall x \in (0,1)$$
$$f_n(x) \underset{n\to\infty}{\longrightarrow} 0 \quad \forall x \in (0,1)$$

Let $f : (0,1) \to \mathbb{R}$, $f(x) = 0 \quad \forall x \in (0,1)$. It's continuous. But

$$d(f_n, f) = \sup_{x\in(0,1)} |x^n| = 1 \underset{n\to\infty}{\not\longrightarrow} 0 \implies f_n \underset{n\to\infty}{\overset{u}{\not\longrightarrow}} f$$

Note that $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$ continuous, $\{f_n\}_{n\ge1}$ is decreasing and converge pointwise to $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} \qquad \text{which is not continuous}$$

This also shows that the continuity of the limit function is necessary in Dini's theorem.

Remark 8.4. Monotonicity is necessary in Dini's theorem.

Example 8.5

$f_n : [0,1] \to \mathbb{R}$ is continuous. $\{f_n\}_{n\ge1}$ converges pointwise to $f : [0,1] \to \mathbb{R}$, $f(x) = 0 \, \forall x \in [0,1]$ figure here f is continuous. But

$$d\left(f_n, f\right) = \sup_{x\in[0,1]} |f_n(x)| = 1 \underset{n\to\infty}{\not\longrightarrow} 0 \implies f_n \underset{n\to\infty}{\overset{u}{\not\longrightarrow}} f$$

Note that $\{f_n\}_{n\ge1}$ is not monotone!

§8.2 Space of Functions

Fix $a, b \in \mathbb{R}$, $a < b$. We define

$$C\left([a,b]\right) = \{f : [a,b] \to \mathbb{R};\ f \text{ is continuous}\}$$

We equip $C\left([a,b]\right)$ with the metric $d : C\left([a,b]\right) \times C\left([a,b]\right) \to \mathbb{R}$, given by

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

Then $\left(C\left([a,b]\right), d\right)$ is a metric space.
Completeness: Let $\{f_n\}_{n\ge1} \subseteq C\left([a,b]\right)$ be Cauchy. So $\forall \epsilon > 0\ \exists n_\epsilon \in \mathbb{N}$ s.t. $d\left(f_n, f_m\right) < \epsilon$ $\forall n, m \ge n_\epsilon$

$$\implies |f_n(x) - f_m(x)| < \epsilon \quad \forall n, m \ge n_\epsilon \quad \forall x \in [a,b]$$

So $\{f_n(x)\}_{n\geq 1}$ is Cauchy $\forall x \in [a,b]$. As $\mathbb{R}$ is complete,

$$\forall x \in [a,b] \quad f_n(x) \xrightarrow[n\to\infty]{} f(x) \in \mathbb{R}$$

This defines a function $f : [a,b] \to \mathbb{R}$. Recall that for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ s.t.

$$\begin{aligned} |f_n(x) - f(x)| \leq \epsilon \quad \forall n \geq n_\epsilon \quad \forall x \in [a,b] \\ \implies d\left(f_n, f\right) \leq \epsilon \quad \forall n \geq n_\epsilon \end{aligned}$$

So $f_n \xrightarrow[n\to\infty]{u} f$. By Weierstrass, $f \in C\left([a,b]\right)$. Thus $\left(C\left([a,b]\right), d\right)$ is a complete metric space.
Compactness: Note that $\left(C\left([a,b]\right), d\right)$ is not bounded and so not compact.

Example 8.6

$f_n : [a,b] \to \mathbb{R}$, $f_n(x) = n$ for all $x \in [a,b]$.

Connectedness: $\left(C\left([a,b]\right), d\right)$ is path connected and so connected.

Let $f, g \in C\left([a,b]\right)$. Define $\gamma : [0,1] \to C\left([a,b]\right)$ via $\gamma(t) = f + t(g-f)$. Note $\forall t \in [0,1]$, $\gamma(t) \in C\left([a,b]\right)$ and

$$\gamma(0) = f, \quad \gamma(1) = g$$

To see that γ is a path we compute

$$\begin{aligned} d\left(\gamma(t), \gamma(s)\right) &= \sup_{x\in[a,b]} |\gamma(t;x) - \gamma(s;x)| \\ &= \sup_{x\in[a,b]} |t-s|\,|g(x) - f(x)| \\ &= |t-s| \underbrace{d(g,f)}_{\in\mathbb{R}} \xrightarrow[|t-s|\to 0]{} 0 \end{aligned}$$

So γ is a continuous function and so a path.

§9 Lec 9: Apr 16, 2021

§9.1 Arzela–Ascoli Theorem

For $a, b \in \mathbb{R}$ with $a < b$, we define

$$C([a,b]) = \{f : [a,b] \to \mathbb{R};\ f \text{ continuous}\}$$

We equip $C([a,b])$ with the uniform metric

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

We showed that $(C([a,b]), d)$ is a complete, connected metric space, but it's not compact.

Definition 9.1 (Equicontinuous Set) — We say that a set $\mathcal{F} \subseteq C([a,b])$ is equicontinuous if

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \text{ s.t. } |f(x) - f(y)| < \epsilon \quad \forall x, y \in [a,b] \text{ with } |x - y| < \delta(\epsilon)$$

and for all $f \in \mathcal{F}$.

Note: For a fixed function $f \in \mathcal{F} \subseteq C([a,b])$, we have that f is uniformly continuous (since f is continuous on $[a,b]$ compact) which means for all $\epsilon > 0$, there exists $\delta(\epsilon, f) > 0$ s.t.

$$|f(x) - f(y)| < \epsilon \quad \forall x, y \in [a,b] \text{ with } |x - y| < \delta(\epsilon, f)$$

Note that for an equicontinuous family $\mathcal{F}$, δ_ϵ can be chosen uniformly for $f \in \mathcal{F}$.

Definition 9.2 (Uniformly Bounded) — We say that a set $\mathcal{F} \subseteq C([a,b])$ is uniformly bounded if $\exists M > 0$ s.t. $|f(x)| \le M\ \forall x \in [a,b]\ \forall f \in \mathcal{F}$.

Note: For a fixed $f \in \mathcal{F} \subseteq C[a,b]$ we have that $f([a,b])$ is bounded (since f continuous and $[a,b]$ compact which implies $f([a,b])$ is compact and so bounded). So $\exists M_f > 0$ s.t. $|f(x)| \le M_f\ \forall x \in [a,b]$. For a uniformly bounded family $\mathcal{F}$, we can choose the bound M uniformly for $f \in \mathcal{F}$.

Theorem 9.3 (Arzela-Ascoli)

Let $\mathcal{F} \subseteq C([a,b])$. The following are equivalent:

1. $\mathcal{F}$ is uniformly bounded and equicontinuous.
2. Every sequence in $\mathcal{F}$ admits a convergent subsequence.

Caution: We cannot guarantee that the limit of the convergent subsequence belongs to $\mathcal{F}$, unless $\mathcal{F}$ is closed in $C([a,b])$. If $\mathcal{F}$ is closed in $C([a,b])$, then the theorem becomes

$$\mathcal{F} \text{ is compact} \iff \mathcal{F} \text{ is uniformly bounded and equicontinuous}$$

Proof. 2) $\implies$ 1)

Claim 9.1. $\mathcal{F}$ is totally bounded.

Fix $\epsilon > 0$. Let $f_1 \in \mathcal{F}$.

$$\begin{aligned}
&\text{If } \mathcal{F} \subseteq B_\epsilon(f_1) \text{ then } \mathcal{F} \text{ is totally bounded}\\
&\text{If } \mathcal{F} \nsubseteq B_\epsilon(f_1) \text{ then } \exists f_2 \in \mathcal{F} \text{ s.t. } d\left(f_1, f_2\right) \geq \epsilon\\
&\text{If } \mathcal{F} \subseteq B_\epsilon(f_1) \cup B_\epsilon(f_2) \text{ then } \mathcal{F} \text{ is totally bounded}\\
&\text{If } \mathcal{F} \nsubseteq B_\epsilon(f_1) \cup B_\epsilon(f_2) \text{ then } \exists f_3 \in \mathcal{F} \text{ s.t. } \begin{cases} d\left(f_1, f_3\right) \geq \epsilon \\ d\left(f_2, f_3\right) \geq \epsilon \end{cases}
\end{aligned}$$

If the process terminates in finitely many steps, then $\mathcal{F}$ is totally bounded. Otherwise, we find $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$ s.t. $d\left(f_n, f_m\right) \geq \epsilon \,\forall n \neq m$. This sequence does not admit a convergent subsequence, leading a contradiction.

Let's show that $\mathcal{F}$ is uniformly bounded. As $\mathcal{F}$ is totally bounded, $\exists n \geq 1$ and $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t.

$$\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_1(f_j) \subseteq B_r(f_1)$$

where $r = 1 + \max_{2\leq j\leq n} d\left(f_1, f_j\right)$. In particular, for all $f \in \mathcal{F}$,

$$d\left(f, f_1\right) < r$$

f_1 is continuous on compact $[a, b] \implies \exists M_{f_1} > 0$ s.t.

$$|f_1(x)| \leq M_{f_1} \quad \forall x \in [a, b]$$

So for $f \in \mathcal{F}$

$$|f(x)| \leq |f(x) - f_1(x)| + |f_1(x)| \leq d(f, f_1) + M_{f_1} < r + M_{f_1} \quad \forall x \in [a, b]$$

So $\mathcal{F}$ is uniformly bounded.

Let's show that $\mathcal{F}$ is equicontinuous. Let $\epsilon > 0$. As $\mathcal{F}$ is totally bounded, $\exists n \geq 1$ and $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t.

$$\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_{\frac{\epsilon}{3}}(f_j)$$

For each $1 \leq j \leq n$, f_j is uniformly continuous on $[a, b]$. So $\exists \delta_j(\epsilon) > 0$ s.t.

$$|f_j(x) - f_j(y)| < \frac{\epsilon}{3} \quad \forall x, y \in [a, b] \text{ with } |x - y| < \delta_j(\epsilon)$$

Let $\delta_\epsilon = \min_{1\leq j\leq n} \delta_j(\epsilon) > 0$.

Fix $f \in \mathcal{F} \implies \exists 1 \leq j \leq n$ s.t. $f \in B_{\frac{\epsilon}{3}}(f_j)$. Then for $x, y \in [a, b]$ with $|x - y| < \delta_\epsilon$ we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)|\\
&\leq 2d(f, f_j) + |f_j(x) - f_j(y)|\\
&\leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

This shows $\mathcal{F}$ is equicontinuous.
1) $\implies$ 2) Let $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$. As $\mathcal{F}$ is uniformly bounded,

$$\exists M > 0 \text{ s.t. } |f(x)| \leq M \quad \forall x \in [a,b] \; \forall f \in \mathcal{F}$$

In particular, $|f_n(x)| \leq M \; \forall x \in [a,b] \; \forall n \geq 1$.
Let $\{r_n\}_{n\geq 1}$ denote an enumeration of the rationals in $[a,b]$. As $\{f_n(r_1)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded by M, $\exists \left\{f_n^{(1)}\right\}_{n\geq 1}$ subsequence of $\{f_n\}_{n\geq 1}$ s.t. $\left\{f_n^{(1)}(r_1)\right\}_{n\geq 1}$ converges. $\left\{f_n^{(1)}(r_2)\right\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded by $M \implies \exists \left\{f_n^{(2)}\right\}_{n\geq 1}$ subsequence of $\left\{f_n^{(1)}\right\}_{n\geq 1}$ s.t. $\left\{f_n^{(2)}(r_2)\right\}_{n\geq 1}$ converges.
Proceeding inductively we find $\forall k \geq 1$ $\left\{f_n^{(k+1)}\right\}_{n\geq 1}$ is a subsequence of $\left\{f_n^{(k)}\right\}_{n\geq 1}$ and $\left\{f_n^{(k)}(r_k)\right\}_{n\geq 1}$ converges.
We consider $\left\{f_n^{(n)}\right\}_{n\geq 1}$ subsequence of $\{f_n\}_{n\geq 1}$.
For $n,m \geq k$, $f_n^{(n)}$, $f_m^{(m)}$ are elements in $\left\{f_n^{(k)}\right\}_{n\geq 1}$. So $\left\{f_n^{(n)}\right\}_{n\geq 1}$ converges at r_k.
Caution: The convergence is not uniform in k.
Fix $\epsilon > 0$. As $\mathcal{F}$ is equicontinuous, $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\epsilon}{3} \quad \forall x,y \in [a,b] \; |x-y| < \delta, \forall f \in \mathcal{F}$$

In particular,

$$|f_n(x) - f_n(y)| < \frac{\epsilon}{3} \quad \forall x,y \in [a,b] \; |x-y| < \delta, \forall n \geq 1 \tag{*}$$

Let $r_1, \ldots, r_N \in \mathbb{Q} \cap [a,b]$ s.t. $a = r_0 < r_1 < \ldots < r_N < r_{N+1} = b$ and

$$|r_{j+1} - r_j| < \delta \qquad 0 \leq j \leq N$$

Note $N \sim \frac{|a-b|}{\delta}$. For each $1 \leq j \leq N$, $\exists n_j(\epsilon) \in \mathbb{N}$ s.t.

$$\left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| < \frac{\epsilon}{3} \qquad \forall n,m \geq n_j(\epsilon)$$

Let $n_\epsilon = \max_{1\leq j\leq N} n_j(\epsilon)$. Note

$$\left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| < \frac{\epsilon}{3} \quad \forall n,m \geq n_\epsilon \quad \forall 1 \leq j \leq N \tag{**}$$

Let $x \in [a,b] \implies \exists 1 \leq j \leq N$ s.t. $|x - r_j| < \delta$. Then

$$\begin{aligned}\left|f_n^{(n)}(x) - f_m^{(m)}(x)\right| &\leq \left|f_n^{(n)}(x) - f_n^{(n)}(r_j)\right| + \left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| + \left|f_m^{(m)}(r_j) - f_m^{(m)}(x)\right| \\ \text{By (*) and (**)} &< 2\cdot\frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \forall n,m \geq n_\epsilon\end{aligned}$$

So $\left\{f_n^{(n)}\right\}_{n\geq 1}$ is uniformly Cauchy and so uniformly convergent. ☐

Remark 9.4. One can replace $[a,b]$ by any other compact metric space (X,d).

§10 Lec 10: Apr 19, 2021

§10.1 Arzela-Ascoli Theorem (Cont'd)

Remark 10.1. The compactness of the set on which the functions are defined is necessary in Arzela-Ascoli.

Example 10.2

$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R};\ |f(x) - f(y)| \leq |x - y|\ \forall x, y \in \mathbb{R} \text{ and } \sup_{x\in\mathbb{R}} |f(x)| \leq 1\}$. Note $\mathcal{F}$ is equicontinuous and uniformly bounded. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$

Claim 10.1. $f \in \mathcal{F}$.

Indeed,

$$\sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+x^2} = 1$$

Moreover, for $x, y \in \mathbb{R}$

$$\begin{aligned} |f(x) - f(y)| = \left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| &= \frac{\left|x^2 - y^2\right|}{(1+x^2)(1+y^2)} \\ &= |x-y| \cdot \frac{|x+y|}{(1+x^2)(1+y^2)} \\ &\leq |x-y| \, (\underbrace{\frac{|x|}{1+x^2}}_{\leq\frac{1}{2}} + \underbrace{\frac{|y|}{1+y^2}}_{\leq\frac{1}{2}}) \\ &\leq |x-y| \end{aligned}$$

So $f \in \mathcal{F}$.
For $n \geq 1$, let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = f(x-n)$. Note $f_n \in \mathcal{F}$ since $\sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+(x-n)^2} = 1$.

$$\begin{aligned} |f_n(x) - f_n(y)| = |f(x-n) - f(y-n)| &\leq |(x-n) - (y-n)| \\ &= |x-y| \end{aligned}$$

Note that $\{f_n\}_{n\geq 1}$ converge pointwise to $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ since $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+(x-n)^2} = 0$. However, $\{f_n\}_{n\geq 1}$ does not admit a subsequence that converges uniformly since $\forall n \geq 1$

$$d(f_n, f) = \sup_{x\in\mathbb{R}} |f_n(x)| = 1 \overset{n\to\infty}{\not\to} 0$$

Remark 10.3. Uniform boundedness is necessary in Arzela-Ascoli.

Example 10.4

$\mathcal{F} = \{f : \underbrace{[0,1]}_{\text{compact}} \to \mathbb{R};\ f \text{ is continuous and } \underbrace{\sup_{x\in[0,1]} |f(x)| \le 1}_{\text{uniformly bounded}}\}$.

Claim 10.2. $\mathcal{F}$ is not equicontinuous.

For $n \ge 1$, let $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \sin(nx)$. Note $f_n \in \mathcal{F}$. Let $x_n = \frac{3\pi}{2n}$, $y_n = \frac{\pi}{2n}$. Then $|x_n - y_n| = \frac{\pi}{n} \underset{n\to\infty}{\longrightarrow} 0$ but

$$|f_n(x_n) - f_n(y_n)| = 2$$

So $\{f_n\}_{n\ge1}$ is not equicontinuous $\implies \mathcal{F}$ is not equicontinuous.

Claim 10.3. $\{f_n\}_{n\ge1}$ does not admit a convergent subsequence.

Assume, towards a contradiction, that there exists a subsequence $\{f_{k_n}\}_{n\ge1}$ of $\{f_n\}_{n\ge1}$ that converges uniformly to $f : [0,1] \to \mathbb{R}$. By Weierstrass,

$$\left.\begin{array}{l} f \in C([0,1]) \\ \left.\begin{array}{l} f_{k_n}(0) = 0 \quad \forall n \ge 1 \\ f_{k_n}(0) \underset{n\to\infty}{\longrightarrow} f(0) \end{array}\right\} \implies f(0) = 0 \end{array}\right\} \implies \forall \epsilon > 0\, \exists \delta > 0 \text{ s.t. } |f(x)| < \epsilon\, \forall 0 < x < \delta$$

$f_{k_n} \underset{n\to\infty}{\overset{u}{\longrightarrow}} f \implies \exists n_\epsilon \in \mathbb{N}$ s.t. $d(f_{k_n}, f) < \epsilon\ \forall n \ge n_\epsilon$. In particular, for $0 < x < \delta$ and $n \ge n_\epsilon$ we have

$$|f_{k_n}(x)| \le |f_{k_n}(x) - f(x)| + |f(x)| < d(f_{k_n}, f) + \epsilon < 2\epsilon$$

Choosing $\epsilon \le \frac{1}{2}$ and N large so that $N \ge n_{\epsilon=\frac{1}{2}}$ and $\frac{\pi}{2N} < \delta_{\epsilon=\frac{1}{2}}$ we find

$$1 = \left| f_{k_N}\left(\frac{\pi}{2N}\right)\right| < 2\epsilon \le 1 \qquad \text{Contradiction!}$$

§10.2 The oscillation of a Real Function

Definition 10.5 (Oscillation of a Function) — Let (X,d) be a metric space and let $f : X \to \mathbb{R}$ be a function. For $\emptyset \ne A \subseteq X$, the oscillation of f on A is

$$\omega(f, A) = \sup_{x\in A} f(x) - \inf_{x\in A} f(x) = \sup_{x,y\in A} [f(x) - f(y)] \ge 0$$

Note that if $A \subseteq B$ then

$$\omega(f, A) \le \omega(f, B)$$

For $x_0 \in X$, the oscillation of f at x_0 is given by

$$\omega(f, x_0) = \inf_{\delta>0} \omega(f, B_\delta(x_0))$$

Proposition 10.6

Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then f is continuous at a point $x_0 \in X$ if and only if $\omega(f, x_0) = 0$.

Proof. " $\implies$ " Fix $\epsilon > 0$. As f is continuous at x_0, $\exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \frac{\epsilon}{4}$ $\forall x \in B_\delta(x_0)$.

$$\implies |f(x) - f(y)| \le |f(x) - f(x_0)| + |f(x_0) - f(y)| < \frac{\epsilon}{2} \quad \forall x, y \in B_\delta(x_0)$$
$$\implies \omega(f, B_\delta(x_0)) = \sup_{x,y \in B_\delta(x_0)} [f(x) - f(y)] \le \frac{\epsilon}{2} < \epsilon$$
$$\implies \omega(f, x_0) \le \omega(f, B_\delta(x_0)) < \epsilon$$

As $\epsilon > 0$ was arbitrary, $\omega(f, x_0) = 0$.
" $\impliedby$ " Fix $\epsilon > 0$. Then $\omega(f, x_0) = 0 < \epsilon$ implies $\exists \delta > 0$ s.t. $\omega(f, B_\delta(x_0)) < \epsilon$

$$\implies |f(x) - f(y)| < \epsilon \qquad \forall x, y \in B_\delta(x_0)$$
$$\implies |f(x) - f(x_0)| < \epsilon \qquad \forall x \in B_\delta(x_0)$$

So f is continuous at x_0. $\square$

Lemma 10.7

Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then for any $\alpha > 0$,

$$\{x \in X : \omega(f, x) < \alpha\} \text{ is open in } X$$

Proof. Fix $\alpha > 0$ and let $A = \{x \in X : \omega(f, x) < \alpha\}$. Fix $x_0 \in A \implies \omega(f, x_0) = \inf_{\delta > 0} \omega(f, B_\delta(x_0)) < \alpha$.

$$\implies \exists \delta > 0 \text{ s.t. } \omega(f, B_\delta(x_0)) < \alpha$$

Claim 10.4. $B_\delta(x_0) \subseteq A$ (which implies $x_0 \in \mathring{A}$ and so $A = \mathring{A}$).

Let $x \in B_\delta(x_0)$. Then $r = \delta - d(x, x_0) > 0$ and $B_r(x) \subseteq B_\delta(x_0)$

$$\implies \omega(f, B_r(x)) \le \omega(f, B_\delta(x_0)) < \alpha$$
$$\implies \omega(f, x) \le \omega(f, B_r(x)) < \alpha \implies x \in A$$

$\square$

Remark 10.8. Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then

$$\begin{aligned}\{x \in X : f \text{ is continuous at } x\} &= \{x \in X : \omega(f, x) = 0\} \\ &= \bigcap_{n \ge 1} \underbrace{\left\{x \in X : \omega(f, x) < \frac{1}{n}\right\}}_{= G_n}\end{aligned}$$

By the lemma, $G_n = \mathring{G}_n$ $\forall n \ge 1$. Also, $G_{n+1} \subseteq G_n$ $\forall n \ge 1$. This observation allows us to prove that there are no functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every rational point and discontinuous at every irrational point.

§11 Lec 11: Apr 21, 2021

§11.1 Oscillation of a Function (Cont'd)

Recall from last lecture that there are no functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every rational point and discontinuous at every irrational point.

Proof. (Sketch) Assume, towards a contradiction, that $f : \mathbb{R} \to \mathbb{R}$ is such a function. Then

$$\mathbb{Q} = \{x \in \mathbb{R} : f \text{ is continuous at } x\} = \bigcap_{n \geq 1} G_n \text{ with } G_n \text{ open in } \mathbb{R}$$

Note $\forall n \geq 1$, $Q \subseteq G_n$

$$\begin{aligned} &\implies \mathbb{R} = \overline{\mathbb{Q}} \subseteq \overline{G_n} \subseteq \mathbb{R} \\ &\implies \overline{G_n} = \mathbb{R} \text{ i.e. } G_n \text{ is dense in } \mathbb{R} \end{aligned}$$

Let $\{q_n\}_{n\geq 1}$ be an enumeration of $\mathbb{Q}$. For each $n \geq 1$, let $H_n = \mathbb{R} \backslash \{q_n\} = (-\infty, q_n) \cup (q_n, \infty)$. Note H_n is open and dense $(\overline{H_n} = \mathbb{R})$ in $\mathbb{R}$. Also

$$\bigcap_{n\geq 1} H_n = \mathbb{R} \setminus \mathbb{Q}$$

So

$$\bigcap_{n\geq 1} G_n \cap \bigcap_{n \geq 1} H_n = \mathbb{Q} \cap \mathbb{R} \setminus \mathbb{Q} = \emptyset$$

This contradicts the following property of $\mathbb{R}$:

Exercise 11.1. If $\{A_n\}_{n\geq 1}$ is a countable collection of open and dense subsets of $\mathbb{R}$, then

$$\overline{\bigcap_{n\geq 1} A_n} = \mathbb{R}$$

Apply this exercise with $\{A_n : n \geq 1\} = \{G_n : n \geq 1\} \cup \{H_n : n \geq 1\}$. □

§11.2 Weierstrass Approximation Theorem

Theorem 11.1 (Weierstrass Approximation)

Fix $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ with $\deg P_n \leq n$ $\forall n \geq 1$ s.t.

$$P_n \xrightarrow[n\to\infty]{u} f \quad \text{on} \quad [a, b]$$

Proof. First, we reduce to the case when $[a, b]$ is $[0, 1]$. Let $\phi : [0, 1] \to [a, b]$, $\phi(t) = a + t(b - a)$. Note ϕ is a continuous, bijective function with the inverse

$$\phi^{-1} : [a, b] \to [0, 1], \quad \phi^{-1}(x) = \frac{x - a}{b - a} \text{ continuous}$$

As $f : [a, b] \to \mathbb{R}$ is continuous, $f \circ \phi : [0, 1] \to \mathbb{R}$ is continuous.
If $\{P_n\}_{n\geq 1}$ is a sequence of polynomials with $\deg P_n \leq n$ s.t.

$$P_n \xrightarrow[n\to\infty]{u} f \circ \phi \text{ on } [0, 1]$$

then $P_n \circ \phi^{-1} \xrightarrow[n\to\infty]{u} f$ on $[a, b]$. Indeed,

$$\sup_{x\in[a,b]} \left|\left(P_n \circ \phi^{-1}\right)(x) - f(x)\right| \underset{x=\phi(t)}{=} \underbrace{\sup_{t\in[0,1]} |P_n(t) - (f \circ \phi)(t)|}_{\xrightarrow[n\to\infty]{} 0}$$

Therefore, we may assume $f : [0, 1] \to \mathbb{R}$ is continuous. Define the Bernstein polynomials via

$$P_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1 - x)^{n-k} \qquad \deg P_n \leq n$$

Note that if f is a constant, say $f(x) = c\ \forall x \in [0, 1]$ then

$$P_n(x) = c\sum_{k=0}^{n}\binom{n}{k} x^k(1 - x)^{n-k} = c\,(x + 1 - x)^n = c \quad \forall x \in [0, 1]\ \forall n \geq 1$$

We want to show $P_n \xrightarrow[n\to\infty]{u} f$ on $[0, 1]$. Fix $x \in [0, 1]$. Consider

$$\begin{aligned}
|f(x) - P_n(x)| &= \left|f(x)\sum_{k=0}^{n}\binom{n}{k} x^k(1 - x)^{n-k} - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1 - x)^{n-k}\right| \\
&= \left|\sum_{k=0}^{n}\left[f(x) - f\left(\frac{k}{n}\right)\right]\binom{n}{k} x^k(1 - x)^{n-k}\right| \\
&\leq \sum_{k=0}^{n}\left|f(x) - f\left(\frac{k}{n}\right)\right|\binom{n}{k} x^k (1 - x)^{n-k}
\end{aligned}$$

To estimate the sum we use the following

- when $\frac{k}{n}$ is close to x, we use the continuity of f.
- when $\frac{k}{n}$ is far from x, we use the fact that $x \overset{g}{\mapsto} x^k(1 - x)^{n-k}$ has a local maximum at $x = \frac{k}{n}$.

$$\begin{aligned}
g'(x) &= kx^{k-1}(1 - x)^{n-k} - (n - k)x^k(1 - x)^{n-k-1} \\
&= x^{k-1}(1 - x)^{n-k-1}\{k(1 - x) - (n - k)x\} \\
&= x^{k-1}(1 - x)^{n-k-1}\{k - nx\} \\
&= \begin{cases} > 0 & \text{if } x < \frac{k}{n} \\ = 0 & \text{if } x = \frac{k}{n} \\ < 0 & \text{if } x > \frac{k}{n} \end{cases}
\end{aligned}$$

$f : [0,1] \to \mathbb{R}$ is continuous $\implies$ f is uniformly continuous. Fix $\epsilon > 0$. Then $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \epsilon \quad \text{whenever} \quad x, y \in [0,1]\,, \quad |x - y| < \delta$$

$f : [0,1] \to \mathbb{R}$ is continuous $\implies$ f is bounded. Let $M > 0$ be s.t.

$$|f(x)| \le M \qquad \forall x \in [0,1]$$

We estimate

$$\begin{aligned}
|f(x) - P_n(x)| &\le \sum_{\substack{0 \le k \le n \\ \left|x - \frac{k}{n}\right| < \delta}} \underbrace{\left| f(x) - f\left(\frac{k}{n}\right)\right|}_{<\epsilon} \binom{n}{k} x^k \left(1-x\right)^{n-k} \\
&\quad + \sum_{\substack{0 \le k \le n \\ \left|x - \frac{k}{n}\right| \ge \delta}} \underbrace{\left| f(x) - f\left(\frac{k}{n}\right)\right|}_{\le 2M} \binom{n}{k} x^k \left(1-x\right)^{n-k} \\
&\le \epsilon \sum_{0 \le k \le n} \binom{n}{k} x^k (1-x)^{n-k} + 2M \sum_{0 \le k \le n} \frac{\left(x - \frac{k}{n}\right)^2}{\delta^2} \binom{n}{k} x^k (1-x)^{n-k} \\
&\le \epsilon + \frac{2M}{n^2\delta^2} \sum_{k=0}^{n} (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k}
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} &= n^2 x^2 \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}}_{=1} \\
&\quad - 2nx \sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} + \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} &= x \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} \\
&= nx \underbrace{\sum_{l=0}^{n-1} \frac{(n-1)!}{l!(n-1-l)!} x^l (1-x)^{n-1-l}}_{=(x+1-x)^{n-1}} \\
&= nx
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} &= nx \sum_{k=1}^{n} \frac{k(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= nx \sum_{k=1}^{n} \frac{(k-1+1)(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= n(n-1)x^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} x^{k-2}(1-x)^{n-k} \\
&+ nx \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= n(n-1)x^2 + nx
\end{aligned}$$

So

$$\begin{aligned}
\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} &= n^2x^2 - 2n^2x^2 + n(n-1)x^2 + nx \\
&= nx(1-x)
\end{aligned}$$

We get

$$\begin{aligned}
|f(x) - P_n(x)| &\le \epsilon + \frac{2M}{n^2\delta^2} \cdot nx(1-x) \\
&\le \epsilon + \frac{2M}{n\delta^2} \sup_{x \in [0,1]} x(1-x) \\
&\le \epsilon + \frac{M}{2\delta^2 n} < 2\epsilon
\end{aligned}$$

provided $n > \frac{M}{2\delta^2\epsilon}$. So $P_n \xrightarrow[n\to\infty]{u} f$ on $[0,1]$. □

§12 Lec 12: Apr 23, 2021

§12.1 Weierstrass Approximation Theorem (Cont'd)

Corollary 12.1
Let $M > 0$. Then there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ s.t.

$$\begin{cases} \deg P_n \leq n & \forall n \geq 1 \\ P_n(0) = 0 & \forall n \geq 1 \\ P_n \xrightarrow[n\to\infty]{u} |x| \text{ on } [-M, M] \end{cases}$$

Proof. Let $f : [-M, M] \to \mathbb{R}$, $f(x) = |x|$. Then f is continuous and $[-M, M]$ compact. By Weierstrass Approximation, $\exists\, \{Q_n\}_{n\geq 1}$ sequence of polynomials s.t.

$$\begin{cases} \deg Q_n \leq n & \forall n \geq 1 \\ Q_n \xrightarrow[n\to\infty]{u} f \text{ on } [-M, M] \end{cases}$$

Note $Q_n \xrightarrow[n\to\infty]{u} f \implies Q_n(0) \xrightarrow[n\to\infty]{} f(0) = 0$.
Let $P_n(x) = Q_n(x) - Q_n(0)$. Then

$$\begin{cases} \deg P_n \leq n & \forall n \geq 1 \\ P_n(0) = 0 & \forall n \geq 1 \end{cases}$$

For $x \in [-M, M]$,

$$\begin{aligned} |P_n(x) - f(x)| &\leq |Q_n(x) - f(x)| + |Q_n(0)| \leq d(Q_n, f) + |Q_n(0)| \\ \implies d(P_n, f) &\leq d(Q_n, f) + |Q_n(0)| \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

□

§12.2 Stone-Weierstrass Theorem

Definition 12.2 (Algebra) — Let (X, d) be a metric space and let

$$\mathcal{A} \subseteq \left\{ f : X \to \mathbb{R}^{\mathbb{C}};\ f \text{ is a function} \right\}$$

We say that $\mathcal{A}$ is an algebra if

1. $f + g \in \mathcal{A} \quad \forall f, g \in \mathcal{A}$.
2. $fg \in \mathcal{A} \quad \forall f, g \in \mathcal{A}$
3. $\lambda f \in \mathcal{A} \quad \forall f \in \mathcal{A}\ \forall \lambda \in \mathbb{R}^{\mathbb{C}}$

We say that the algebra $\mathcal{A}$ separates points if whenever $x, y \in X$ with $x \neq y$ then $\exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$.
We say that the algebra $\mathcal{A}$ vanishes at no point in X if $\forall x \in X\ \exists f \in \mathcal{A}$ s.t. $f(x) \neq 0$.

Lemma 12.3

Let (X,d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra. Then its closure $\overline{\mathcal{A}}$ with respect to the uniform topology is also an algebra.

Proof. Let $f, g \in \mathcal{A}$. Then

$$\begin{cases} \exists f_n \in \mathcal{A} \text{ s.t. } f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ \exists g_n \in \mathcal{A} \text{ s.t. } g_n \xrightarrow[n\to\infty]{u} g \text{ on } X \end{cases}$$

$$\left.\begin{array}{l} d\left(f_n+g_n, f+g\right) \leq d(f_n,f) + d(g_n,g) \xrightarrow[n\to\infty]{} 0 \\ f_n + g_n \in \mathcal{A} \text{ (because } \mathcal{A} \text{ is an algebra)} \end{array}\right\} \implies f+g \in \overline{\mathcal{A}}$$

Similarly, for $\lambda \in \mathbb{R}$,

$$\left.\begin{array}{l} d\left(\lambda f_n, \lambda f\right) \leq |\lambda| d\left(f_n, f\right) \xrightarrow[n\to\infty]{} 0 \\ \lambda f_n \in \mathcal{A} \text{ (because } \mathcal{A} \text{ is an algebra)} \end{array}\right\} \implies \lambda f \in \overline{\mathcal{A}}$$

Then

$$\begin{aligned} d\left(f_n g_n, fg\right) &= \sup_{x\in X} |f_n(x)g_n(x) - f(x)g(x)| \\ &\leq \sup_{x\in X} \left[|f_n(x) - f(x)|\,|g_n(x)| + |f(x)|\,|g_n(x) - g(x)|\right] \\ &\leq d(f_n,f) \sup_{x\in X} |g_n(x)| + d(g_n,g) \sup_{x\in X} |f(x)| \end{aligned}$$

By Weierstrass,

$$\left.\begin{array}{l} f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ f_n \in C(X) \end{array}\right\} \implies \left.\begin{array}{l} f \in C(X) \\ X \text{ compact} \end{array}\right\} \implies \exists M > 0 \text{ s.t. } \sup_{x\in X} |f(x)| \leq M$$

Similarly, $g \in C(X) \implies \exists M_2 > 0$ s.t. $\sup_{x\in X} |g(x)| \leq M_2$

$$d(g_n, 0) \leq d(g_n, g) + d(g, 0) \leq 1 + M_2 \qquad \forall n \geq n_1$$

Let $M_3 = \max\left\{1 + M_2, \underbrace{d(g_1,0)}_{<\infty}, \dots, \underbrace{d(g_{n_1},0)}_{<\infty}\right\}$. So $d(g_n, 0) \leq M_3\, \forall n \geq 1$. Thus

$$\left.\begin{array}{l} d\left(f_n g_n, fg\right) \leq d(f_n,f)\cdot M_3 + d(g_n,g)\cdot M_1 \xrightarrow[n\to\infty]{} 0 \\ f_n g_n \in \mathcal{A} \text{ (since } \mathcal{A} \text{ is an algebra)} \end{array}\right\} \implies f\cdot g \in \overline{\mathcal{A}}$$

□

Lemma 12.4

Let (X,d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes at no point in X. Then

$$\forall \alpha, \beta \in \mathbb{R} \quad \forall x_1, x_2 \in X \text{ s.t. } x_1 \neq x_2 \quad \exists f \in \mathcal{A} \text{ s.t. } \begin{cases} f(x_1) = \alpha \\ f(x_2) = \beta \end{cases}$$

Proof. Fix $\alpha, \beta \in \mathbb{R}$. Fix $x_1, x_2 \in X$ s.t. $x_1 \neq x_2$. We would like

$$f(x) = \alpha \cdot \frac{u(x)}{u(x_1)} + \beta \cdot \frac{v(x)}{v(x_1)}$$

for $u, v \in \mathcal{A}$ s.t.

$$\begin{aligned} u(x_1) \neq 0 \quad &\text{and} \quad u(x_2) = 0 \\ v(x_1) = 0 \quad &\text{and} \quad v(x_2) \neq 0 \end{aligned}$$

Then $f \in \mathcal{A}$ (because $\mathcal{A}$ is an algebra) is the desired function.

As $\mathcal{A}$ separates points, $\exists g \in \mathcal{A}$ s.t. $g(x_1) \neq g(x_2)$.

As $\mathcal{A}$ vanishes at no point in X,

$$\begin{cases} \exists h \in \mathcal{A} \text{ s.t } h(x_1) \neq 0 \\ \exists k \in \mathcal{A} \text{ s.t. } k(x_2) \neq 0 \end{cases}$$

Then, we define

$$\begin{aligned} u(x) &= [g(x) - g(x_2)] \cdot h(x) \in \mathcal{A} \\ v(x) &= [g(x) - g(x_1)] \cdot k(x) \in \mathcal{A} \end{aligned}$$

□

Theorem 12.5 (Stone-Weierstrass)

Let (X, d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes no point in X. Then $\mathcal{A}$ is dense in $C(X)$, i.e., $\overline{\mathcal{A}} = C(X) = \{f : X \to \mathbb{R};\ f \text{ continuous}\}$.

Proof. Want to show $\forall f \in C(X)\ \forall \epsilon > 0\ \exists g \in \mathcal{A}$ s.t. $d(f, g) < \epsilon$.

Step 1: If $f \in \overline{\mathcal{A}}$ then $|f| \in \overline{\mathcal{A}}$. Let $f \in \overline{\mathcal{A}} \implies \exists f_n \in \mathcal{A}$ s.t.

$$\left.\begin{aligned} f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ f_n \in C(X) \end{aligned}\right\} \implies f \in C(X)$$

As X is compact, $\exists M > 0$ s.t. $|f(x)| \leq M\ \forall x \in X$. By the previous Corollary 12.1, $\exists \{P_n\}_{n\geq 1}$ sequence of polynomials with $\deg P_n \leq n\ \forall n \geq 1$ s.t.

$$\begin{cases} P_n \xrightarrow[n\to\infty]{u} |x| \text{on } [-M, M] \\ P_n(0) = 0 \end{cases} \implies P_n(f) \xrightarrow[n\to\infty]{u} |f| \text{ on } X$$

If $P_n(x) = \sum_{k=1}^n c_k x^k$ then $P_n(f) = \sum_{k=1}^n c_k f^k \in \mathcal{A}$ which implies $|f| \in \overline{\mathcal{A}}$.

Step 2: If $f, g \in \overline{\mathcal{A}}$ then $\max\{f, g\}$, $\min\{f, g\} \in \overline{\mathcal{A}}$.

$$\begin{aligned} \max\{f, g\} &= \frac{f+g}{2} + \frac{|f-g|}{2} \in \overline{\mathcal{A}} \\ \min\{f, g\} &= \frac{f+g}{2} - \frac{|f-g|}{2} \in \overline{\mathcal{A}} \end{aligned}$$

Step 3: $\forall f \in C(X)$, $\forall x \in X$, $\forall \epsilon > 0$, $\exists g \in \overline{\mathcal{A}}$ s.t.

$$g(x) = f(x) \quad \text{and} \quad g(y) > f(y) - \epsilon \quad \forall y \in X$$

□

§13 Dis 1: Mar 30, 2021

§13.1 Review of 131AH

Summation by parts(discrete integration by parts):

$\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$, $A_n = \sum_{k=1}^n a_k$, $A_0 = 0$. Then for $1 \leq p \leq q$,

$$\begin{aligned}\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q}(A_n - A_{n-1})b_n &= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1}b_p\end{aligned}$$

Application:

1. Dirichlet's test: $\sum a_n$ bounded, $\{b_n\}_{n\geq 1}$ decreasing and $b_n \to 0 \implies \sum a_n b_n$ converges.
2. Leibniz's Alternating series test: $|a_1| \geq |a_2| \geq \dots$ and $a_n \to 0$, $\sum (-1)^{n+1}|a_n|$ converges.
3. Kronecker's lemma: $b_n \geq 0$, $b_n \leq b_{n+1}$ $b_n \to \infty$, $A_n = \sum_{k=1}^n a_k$, and $\sum \frac{a_n}{b_n}$ converges $\implies \frac{A_n}{b_n} \to 0$.

Cardinality:

$|X| \leq (= / \geq) |Y|$ to mean $\exists f : X \to Y$ injective, bijective, or surjective, respectively.

- X finite if $|X| = |\{1, \dots, n\}|$
- X countable if $|X| \leq |\mathbb{N}|$. X countably infinite if countable but not finite.
- X countably infinite $\implies |X| = |\mathbb{N}|$.
- $|X| \leq |Y| \iff |Y| \geq |X|$.
- X, Y countable $\implies X \times Y$ countable.
- A countable, X_α countable $\forall \alpha \in A \implies \bigcup_{\alpha \in A} X_\alpha$ countable.
- $|\mathbb{Z}| = |\mathbb{N}| = |\mathbb{Q}|$, $\mathbb{R}$ uncountable.

Schröder – Bernstein: $|X| \leq |Y|$, $|Y| \leq |X|$ then $|X| = |Y|$ useful for hmwrk

Metric Spaces:

Let (X, d) be a metric space, $E \subseteq X$.

- $\mathring{E} = \{x \in X : \exists r > 0, B_r(x) \subseteq E\} = \bigcup_{G \subseteq E} G$ where G is open, largest open sets contained in E.
- $\overline{E} = \{x \in X : \forall r > 0, B_r(x) \cap E \neq \emptyset\} = \bigcap_{F \supseteq E} F$ where F is closed, smallest closed sets contained in E.
- $E' = \{x \in X : \forall r > 0, (B_r(x) \setminus \{x\}) \cap E \neq \emptyset\}$

- E open if $E = \mathring{E}$
- E closed if $E = \overline{E}$ or $E \supset E'$ or $\forall \{x_n\}_{n\geq 1} \subseteq E$, $x_n \to x \implies x \in E$.

(X, d) is complete if any Cauchy sequence in X converges.

- $\mathbb{R}$ complete, $\mathbb{R}^d$ complete.
- closed subsets of a complete space is complete.
- complete subsets are closed
- completeness is not invariant under homeomorphism(continuous bijection with continuous inverse)
  $(\mathbb{R}, |\cdot|) \overset{\sim}{\to} ((0,1), |\cdot|) \leftarrow$ not complete.

(X, d) is connected if there is no disjoint open sets A, B s.t. $X = A \cup B$.

- $E \subseteq \mathbb{R}$ connected $\iff$ E is interval.
- X is connected $\iff$ its only clopen subsets are $\emptyset, X$.

Intermediate Value Theorem: $f : [a, b] \to \mathbb{R}$ continuous, then $\forall \lambda$ s.t. $f(a) < \lambda < f(b)$, $\exists c$ s.t. $f(c) = \lambda$.

§14 Dis 2: Apr 6, 2021

§14.1 Compactness

Definition 14.1 — A metric space (X, d) compact if every open cover has a finite subcover.

Example 14.2

$\mathbb{Z} \subseteq \mathbb{R}$ compact?

The collection $\left\{\left(n - \frac{1}{2}, n + \frac{1}{2}\right)\right\}_{n\in\mathbb{Z}}$ open cover with no finite subcover – not compact! Note that $\mathbb{Z}$ is not bounded. An alternative is $\{(-n, n)\}_{n\in\mathbb{Z}}$

What about $\left\{\frac{1}{n}\right\}_{n\geq 1} \subseteq \mathbb{R}$?

The open cover $\left\{\left(\frac{1}{n}, 2\right)\right\}_{n\geq 1}$ is open cover with no finite subcover – not compact!

Exercise 14.1. $\left\{\frac{1}{n}\right\}_{n\geq 1} \cup \{0\}$ is compact.

Remark 14.3.

- X compact $\iff$ every $\{F_\alpha|_{\alpha\in A}\}$ closed subsets with finite intersection property satisfies $\bigcap_{\alpha\in A} F_\alpha \neq \emptyset$.
- compact subset of metric spaces are complete; complete subsets of metric spaces are closed.
- closed subset of a compact space is compact; closed subsets of complete space are complete.

Theorem 14.4

(Heine – Borel) (X, d) metric space. The following are equivalent:

1. X compact.
2. X sequential compact
3. X complete and totally bounded.
4. X limit point compact (every infinite subset of X has a limit point)

Remark 14.5.
1. In $\mathbb{R}^d(\mathbb{R}^d$ complete), closed subsets are complete. Boundedness implies totally bounded. So, closed & bounded in $\mathbb{R}^d$ implies compact.
2. $B = \{f \in l_2 : \|f\|_2 \leq 1\} \subseteq l_2$ is closed and bounded but not totally bounded. In particular, B is not compact.

   Fact 14.1. l_2 is complete and so is B.
3. totally boundedness implies separable (existence of a countable dense subset)

   homework 2

   converse is not true: $\mathbb{R}$ is separable ($\overline{\mathbb{Q}} = \mathbb{R}$), but not bounded.

Lemma 14.6

$\{f_n\}$ pointwise bounded ($\{f_n(x)\}_{n\geq 1}$ is bounded for every x) on countable set E, then $\exists$ subsequence $\{f_{n_k}\}_{n\geq 1}$ s.t. f_{n_k} converges pointwise on E.

Proof. Let $E = \{x_1, x_2, x_3, \dots\}$

$$\{f_n(x_1)\}_{n\geq 1} \text{ bounded } \overset{\text{B-W}}{\Longrightarrow} \exists \text{ subseq. } \left\{f_j^{(1)}\right\}_{j\geq 1} \text{ of } \{f_n\} \text{ s.t. } f_j^{(1)}(x_1) \to f(x_1)$$

Then
$$\left\{f_j^{(1)}(x_2)\right\} \text{ bounded } \implies \exists \left\{f_j^{(2)}\right\}_{j\geq 1} \text{ of } \left\{f_j^{(1)}\right\} \text{ s.t.} f_j^{(2)} \to f(x_2)$$

So, in general,
$$\left\{f_j^{(k)}(x_{k+1})\right\} \text{ bounded } \implies \exists \left\{f_j^{(k+1)}\right\}_{j\geq 1} \text{ of } \left\{f_j^{(k)}\right\} \text{ s.t.} f_j^{(k+1)} \to f(x_{k+1})$$

Diagonal argument

$$\begin{matrix} f_1^{(1)} & f_2^{(1)} & f_3^{(1)} \\ f_1^{(2)} & f_2^{(2)} & f_3^{(2)} \\ f_1^{(3)} & f_2^{(3)} & f_3^{(3)} \end{matrix}$$

Note that $\left\{f_k^{(k)}\right\}_{k\geq 1}$ is a subsequence of $\left\{f_j^{(n)}\right\}$ $\forall n$ except for the first $n-1$ terms. So $f_k^{(k)}(x_n) \to f(x_n)$ ∎

§14.2 Ex 7 – Hw 2

(X, d) metric space, $\mathcal{F} = \{A \subseteq X : A \text{ compact}, A \neq \emptyset\}$

$$d_H(A, B) = \max \left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

where $d(a, B) = \inf_{b\in B} d(a, b)$.

$$\sup_{a\in A} d(a, B) = \inf \{\epsilon \geq 0 : A \subseteq B^\epsilon\}$$

The distance can be rewritten as

$$\begin{aligned} d_H(A,B) &= \max\left\{\inf\left\{\epsilon:\ A\subseteq B^\epsilon\right\}, \inf\left\{\epsilon:\ B\complement A^\epsilon\right\}\right\} \\ &\overset{7b)}{=} \inf\left\{\epsilon:\ A\subseteq B^\epsilon \text{ and } B\subseteq A^\epsilon\right\} \end{aligned}$$

e.g., $d_H\left([0,1],[2,3]\right)=2$.
c) (X,d) totally bounded $\implies$ $(\mathcal{F}(X),d_H)$ totally bounded. (X,d) complete $\implies$ $(\mathcal{F}(X),d_H)$ complete.
It's easier to show (X,d) compact $\implies$ $(\mathcal{F}(X),d_H)$ complete

$$\left\{A_n\right\}_{n\geq 1} \text{ Cauchy in } d_H \quad A=\bigcap_{n\geq 1}\overline{\bigcup_{m\geq n} A_m},\ d_H(A,A_n)\to 0$$

Given $\left\{A_n\right\}_{n\geq 1}$,

$$\limsup A_n = \bigcap_{n\geq 1}\bigcup_{m\geq n} A_m = \left\{x:\ x\in A_n \text{ for infinitely many } n\right\}$$

$$\bigcap_{n\geq 1}\overline{\bigcup_{m\geq n} A_m} = \left\{x:\ \exists\left|x_{n_k}\right| \text{ s.t. } x_{n_k}\to x \text{ where } x_{n_k}\in A_{n_k},\ \left\{n_k\right\} \text{ non-decreasing } n_k\to\infty\right\}$$

§15 Dis 3: Apr 13, 2021

§15.1 Continuity

$f : X \to Y$ continuous at x if

- $(\epsilon - \delta)$: $\forall \epsilon > 0$, $\exists \delta_{\epsilon,x} > 0$, $\forall y \in X$ s.t. $|y - x| < \delta \implies |f(x) - f(y)| < \epsilon$.
- (Sequential): For each sequence $x_n \to x$, $f(x_n) \to f(x)$

$f : X \to Y$ continuous if continuous at every $x \in X$. This is equivalent to (topological): $\forall U \subseteq Y$ open, $f^{-1}(U)$ open in X.

Theorem 15.1

$f : X \to Y$ continuous. If X compact then $f(X)$ is compact. If X is connected then $f(X)$ is connected. If $Y = \mathbb{R}$, then the above statement gives the Extreme Value Theorem: $\exists x_1, x_2 \in X$ s.t.

$$f(x_1) \le f(x) \le f(x_2) \quad \forall x \in X$$

and Intermediate Value Theorem: $f(X)$ is an interval.

Proposition 15.2

X compact, $f : X \to Y$ bijective and continuous which implies f^{-1} is also continuous, i.e., f is a homeomorphism.

Example 15.3

$f : [0,1) \to S' = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$, $x \leftrightarrow (\cos 2\pi x, \sin 2\pi x)$. Is f an homeomorphism?

figure here

Remark 15.4. Completeness is not preserved under homeomorphism: $(\mathbb{R}, |\cdot|)$ is complete but $((-1,1), |\cdot|)$ is not complete.

§15.2 Uniform Continuity

$f : X \to Y$ uniformly continuous if $\forall \epsilon > 0$, $\exists \delta_\epsilon > 0$ s.t.

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Theorem 15.5

$f : X \to Y$ is continuous and X is compact. Then f is uniformly continuous.

Example 15.6

$f:\mathbb{R}\to\mathbb{R}$, $x\mapsto x^2$ is not uniformly continuous but $f\big|_{[-m,m]}$ is uniformly continuous.

Example 15.7

$x\mapsto |x|$, $x\mapsto d(x,A)=\inf_{a\in A} d(a,x)$ are uniformly continuous:

$$||x|-|y||\le |x-y|\,;\quad |d(x,A)-d(y,A)|\le d(x,y)$$

Definition 15.8 (Lipschitz Continuous) — $f:\mathbb{R}\to\mathbb{R}$ Lipschitz continuous if $\exists M>0$ s.t. $|f(x)-f(y)|\le M|x-y|$

Remark 15.9. Lipschitz continuous implies uniformly continuous. However, uniform continuity does not imply Lipschitz continuous.

Remark 15.10. For differentiable function, Lipschitz continuous $\iff$ bounded derivative.

§15.3 Ternary Expansion and Cantor Set

Every $x\in[0,1]$ has a base-3 expansion $x=\sum_{j=1}^{\infty} a_j 3^{-j}$, $a_j\in\{0,1,2\}$. Write $x=[0.a_1a_2a_3\ldots]_3$. It's unique unless $x=c3^{-k}$ for some $c,k\in\mathbb{Z}$ in which case x has 2 expansions: one with $a_j=0$ for all $j>k$ and one with $a_j=2$ for $j>k$. Assume TBA, one of the expansions will have $a_k=1$, the other will ahve $a_k\in\{0,2\}$. As convention, we always use the latter expansion, e.g. $\frac{1}{3}=0.1_3=0.022222\ldots_3$, $\frac{2}{3}=0.2_3=0.1222\ldots_3$

$$\begin{aligned}
&a_1=0 \iff x\in\left[0,\frac{1}{3}\right],\quad a_1=1\iff x\in\left(\frac{1}{3},\frac{2}{3}\right)\\
&a_1=2 \iff x\in\left[\frac{2}{3},1\right]\\
&a_1\ne 1,\, a_2=1 \iff x\in\left(\frac{1}{9},\frac{2}{9}\right)\cup\left(\frac{7}{9},\frac{8}{9}\right)
\end{aligned}$$

Cantor set $C=\left\{x\in[0,1]:\, x=\sum_{j=1}^{\infty} a_j3^{-j},\, a_j\in\{0,2\}\right\}$

$$\begin{aligned}
E_0&=[0,1]\\
E_1&=\{x:\, a_1\in\{0,2\}\}=\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]\\
E_2&=\{x\in E_1:\, a_2\in\{0,2\}\}=\left[0,\frac{1}{9}\right]\cup\left[\frac{2}{9},\frac{3}{9}\right]\cup\left[\frac{6}{9},\frac{7}{9}\right]\cup\left[\frac{8}{9},1\right]\\
&\vdots\\
E_{k+1}&=\{x\in E_k:\, a_{k+1}\in\{0,2\}\}
\end{aligned}$$

in which $E_{k+1} \subseteq E_k$.

$C = \bigcap_{k\geq 0} E_k$ compact.

- $C = C'$(C is perfect)
- $\mathring{C} = \emptyset$ (C contains no intervals)
- C is totally disconnected (the only nontrivial connected subsets are singletons)
- C is uncountable
- C is a set of length 0

$$|C| = 1 - \sum_{j=0}^{\infty} \frac{2^j}{3^{j+1}} = 1 - \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 0$$

§16 Dis 4: Apr 20, 2021

§16.1 Sequences of Functions

$f_n : X \to Y$ converges pointwise to f if $f(x) = \lim_{n\to\infty} f_n(x)$ and uniformly to f if $\forall \epsilon > 0$, $\exists N$ s.t. $|f_n(x) - f(x)| < \epsilon \ \forall n \geq N, x \in X$.

Remark 16.1.

i) Uniform convergence is metrizable. Let $B(X,Y) = \{f : X \to Y : f \text{ bounded}\}$

$$d(f,g) = \sup_{x\in X} d_Y\left(f(x), g(x)\right)$$

defines a metric on $B(X,Y)$ s.t. $f_n \to f$ uniformly $\iff$ $d(f_n, f) \to 0$ as $n \to \infty$. If $Y = \mathbb{R}$, then $\|f\|_n = \|f\|_\infty := \sup\{|f(x)| : x \in X\}$ is the uniform norm on $B(X,\mathbb{R})$ and $d(f,g) = \|f-g\|_\infty$ defines the uniform metric: $f_n \to f$ uniformly iff $\sup_{x\in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

ii) $(B(X,\mathbb{R}), \|\cdot\|_\infty)$ is a complete metric space.

iii) $(C_b(X,\mathbb{R}), \|\cdot\|_\infty)$ is a close subspace of $(B(X,\mathbb{R}), \|\cdot\|_\infty)$ (uniform limit theorem) where $C_b(X,\mathbb{R})$ is a continuous bounded function on X. If X is compact then $C_b(X,\mathbb{R}) = C(X,\mathbb{R})$.

Compactness in the space of functions

Example 16.2

$B = \{f \in C([0,1],\mathbb{R}) : \|f\|_\infty \leq 1\}$ closed (complete) & bounded in $C([0,1])$. Is B compact? No. Consider $f_n(x) = x^n$, f_n converges pointwise to

$$f(x) = \begin{cases} 0, & x < 1 \\ 1, & x = 1 \end{cases}$$

This implies no subsequence can converge uniformly by the uniform limit theorem. Note that B is complete so B not compact which implies B is not totally bounded. Compare to the closed unit ball in l^2. In fact, in a normed vector space such as $(C(X,\mathbb{R}), \|\cdot\|_\infty)$ or $\left(l^2, \|\cdot\|_2\right)$. Closed unit ball is compact iff the space is finite dimensional.

One may replace uniform boundedness by pointwise boundedness in Arzela-Ascoli.
Key steps in the proof:

a) Every compact space is separable.

b) Every pointwise bounded sequence on a countable set has a pointwise convergent subsequence.

c) Upgrade pointwise convergent subsequence on the countable dense set to uniformly convergence: $\{g_n\}_{n\geq 1} \subseteq C(X)$ on compact X. $\{g_n\}_{n\geq 1}$ equicontinuous and converges pointwise on X (or a countable dense subset), then g_n converges uniformly.

Criterion for equicontinuity: X compact, $\{f_n\} \subseteq C(X)$ if f_n converges uniformly then $\{f_n\}$ is equicontinuous.