

# Math 164 – Optimization

University of California, Los Angeles

Duc Vu

Spring 2021

This is math 164 – Optimization taught by Professor Li. We meet weekly on MWF from 3:00 pm to 3:50 pm for lecture. The main textbook used for the class is *An Introduction to Optimization* 4<sup>th</sup> by Chong and Zak. Other course notes can be found at my [github](#). Please let me know through my [email](#) if you spot any typos in the note.

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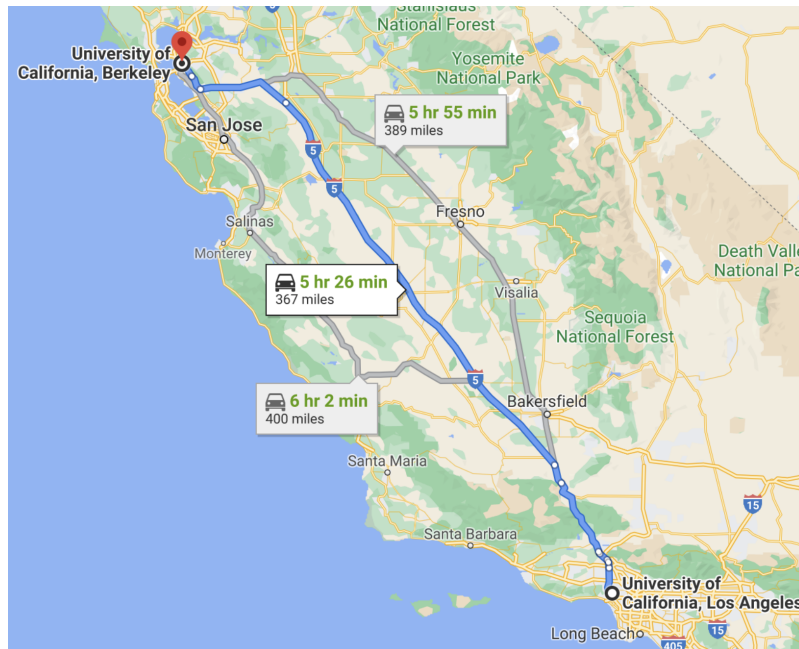
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# §1 | Lec 1: Mar 29, 2021

## §1.1 Introduction

**Question 1.1.** Why Optimization?

- Find the fastest route from  $A$  to  $B$ .
- Possible constraints: avoid tolls?



**Question 1.2.** So what is optimization?

- Optimization is an important tool in decision science and in the analysis of artificial or physical systems.
- An optimization problem involves
  - An objective, which is a scalar, quantitative measure of the performance of the system under study.
  - examples of objectives include profit, time, energy, error, loss, cost, etc.
  - The objective depends on certain characteristics of the system, called variables or unknowns or parameters.
  - Often the variables are restricted, or constrained in some way.
  - The goal of solving an optimization problem is to find values of the variables that satisfy the constraints and optimize/minimize/maximize the objective.

In general, an optimization problem can be summarized as

|           |                      |            |                          |
|-----------|----------------------|------------|--------------------------|
| Optimized | Objective(Variables) | Subject to | Constraints on variables |
|-----------|----------------------|------------|--------------------------|

Applying the optimization framework to solve problems involves three steps:

1. Modeling: identifying objective, variables, and constraints for a given problem.
2. Solving: employing an optimization algorithm to find solutions, usually with the help of a computer.
3. Analyzing: recognizing whether the problem has been successfully solved using optimality conditions.

Mathematically speaking, optimization is the minimization or maximization of a (scalar valued) function subject to constraints on its variables.

We use the following notation

- $x$  is a vector of variables/unknowns/parameters.
- $f(x)$  is the objective function, a scalar function of  $x$  that we want to maximize or minimize.
- $C_i(x)$  are constraint functions, which are scalar functions of  $x$  that define certain equations or inequalities that the unknown vector  $x$  must satisfy.

Using this notation, the optimization problem is

$$\boxed{\text{minimize } \underbrace{f(x)}_{\text{objective}} \text{ with } \underbrace{x \in \mathbb{R}^n}_{\text{variables}} \text{ subject to } \begin{cases} c_i(x) = 0, & i \in \underbrace{\mathcal{E}}_{\text{equality}} \\ c_i(x) \geq 0, & i \in \underbrace{\mathcal{I}}_{\text{inequality}} \end{cases}}$$

The set of variables that satisfies all constraints, i.e.,

$$\Omega = \{x \in \mathbb{R}^n : c_i(x) = 0, i \in \mathcal{E}, c_i(x) \geq 0, i \in \mathcal{I}\}$$

is called the feasible region/set. So the optimization can also be written in an abstract manner as

$$\boxed{\text{minimize } f(x) \text{ with } x \in \mathbb{R}^n \text{ subject to } \underbrace{x \in \Omega}_{\text{feasible/constraint set}}}$$

## §1.2 Some Examples

**Example 1.1**

Consider the problem

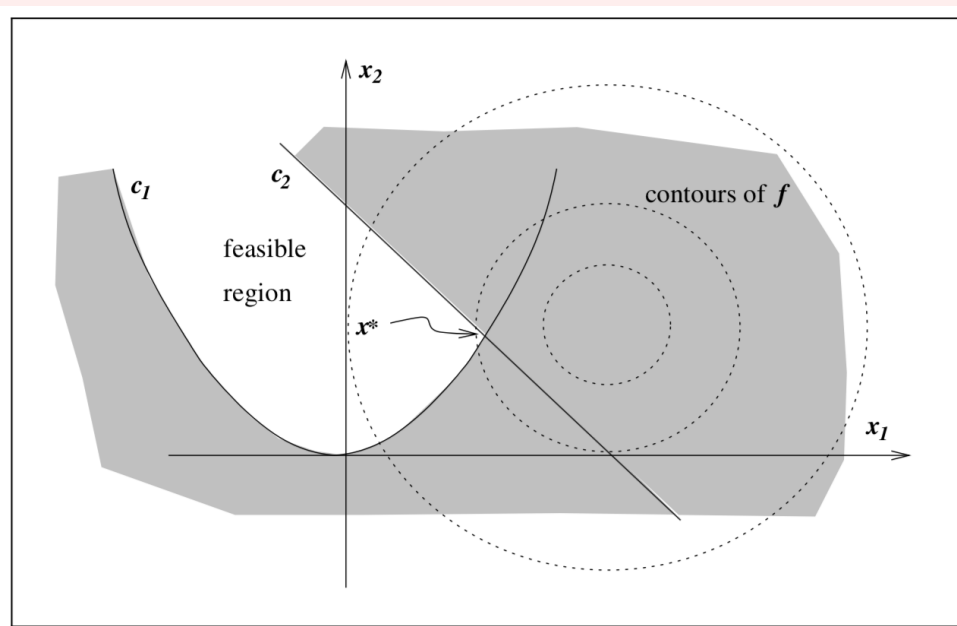
$$\begin{aligned} &\text{minimize } (x_1 - 2)^2 + (x_2 - 1)^2 \text{ subject to} \\ &\quad x_1^2 - x_2 \leq 0 \\ &\quad x_1 + x_2 \leq 2 \end{aligned}$$

We identify

- the optimization variable  $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- the objective(cost) function  $f(x) = (x_1 - 2)^2 + (x_2 - 1)^2$
- the constraint  $c(x) = \begin{bmatrix} c_1(x) \\ c_2(x) \end{bmatrix} = \begin{bmatrix} x_2 - x_1^2 \\ 2 - x_1 - x_2 \end{bmatrix}$ ,  $\mathcal{I} = \{1, 2\}$ ,  $\mathcal{E} = \emptyset$ .

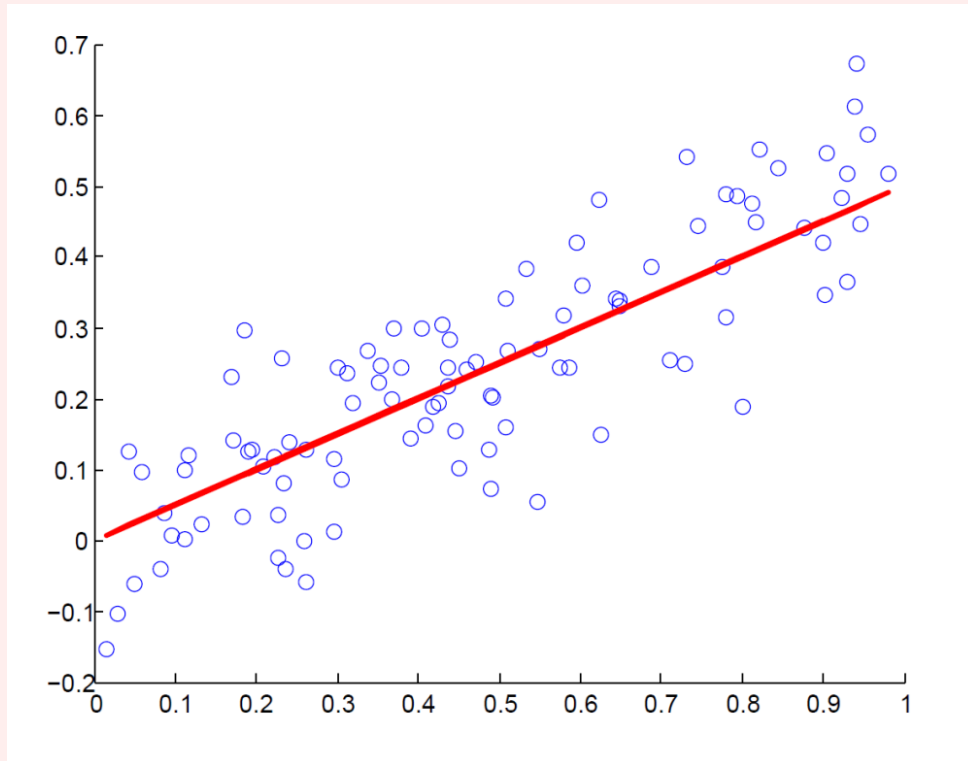
A lot of times we stack all equality constraints and/or inequality constraints into vector functions and write, e.g.,  $c(x) \geq 0$  meaning element-wise equality or inequality.

$$c(x) = \begin{bmatrix} c_1(x) \\ c_2(x) \\ c_3(x) \\ \vdots \end{bmatrix}$$



**Example 1.2** (Linear Regression)

Given a set of feature vectors  $a_i \in \mathbb{R}^n$  and outcomes  $y_i$ ,  $i = 1, \dots, N$ , find weights  $x$  that predict the outcome accurately  $x^T a_i \approx y_i$ .



We can find the optimal  $x$  by solving the least squares problem.

$$\min_x \sum_{i=1}^N (y_i - x^T a_i)^2$$

**Example 1.3**

The Netflix prize: predict how a user will rate a movie.



- Some pattern exists: users do not assign ratings completely at random – if you like Godfather I, you'll probably like Godfather II.
- We have lots and lots of data: we know how a user has rated other movies, and we know how other users have rated this (and other) movies.
- Let  $y_{ij}$  denote the rate of user  $i$  for movie  $j$ .
- The Netflix prize concerns finding a low-rank matrix  $X$  such that  $x_{ij} = y_{ij}$  for observed  $(i, j)$ , related to the following optimization problem

$$\min_X (\text{minimize}) \sum_{\text{observed}(i,j)} (x_{ij} - y_{ij})^2 \text{ subject to } \text{rank}(X) \leq r$$

or an alternative way is to

$$\min_X \text{rank}(X) \text{ s.t. } x_{ij} = y_{ij} \text{ observed}(i, j)$$

### §1.3 Classification of Optimizations

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } c_i(x) = 0, i \in \mathcal{E} \\ &\quad c_i(x) \geq 0, i \in \mathcal{I} \end{aligned}$$



- Unconstrained Optimization:  $\mathcal{E} = \mathcal{I} = \emptyset$ .
  - Many practical problems are unconstrained or the constraints can be safely discarded.
  - Unconstrained problems arise also as reformulation of constrained ones by replacing the constraints with penalization.
- Constrained Optimization: when constraints are essential for the problem.
- Linear Programming: when the objective function and all the constraints are linear function of  $x$ .
  - widely used in management, financial, and economic applications.
- Nonlinear Programming: at least some of the constraints or the objective are nonlinear functions.
  - tend to arise naturally in physical sciences, engineering, signal processing, and machine learning.
  - become more widely used in management and economic sciences as well.
- Global Optimization: aim at finding the global optimal solution, which is generally very challenging.
- Local Optimization: focuses instead on the computation and characterization of local solutions.
- Convex Optimization: the objective function is convex, the equality constraint functions are linear, and the inequality constraint functions are concave.

## §2 | Lec 2: Mar 31, 2021

### §2.1 An Overview of Linear Algebra

#### Vector Spaces:

In “linear algebra”, we denote a vector as a list of numbers,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \text{ or } \mathbb{C}^n$$

The general definition of a vector space is as follows.

- A vector space  $S$  is composed of a set of elements (called “vectors”) and members of a field  $R$  (called scalars).
  - Roughly, vectors are objects that can be added together and multiplied by numbers (namely, the scalars).
  - A field is a set of numbers for which addition and multiplication are will defined. We will typically use  $R = \mathbb{R}$  or  $R = \mathbb{C}$ .
- In a vector space, there must be two rules defined for combining vectors and scalars.
  - The first operation is *vector addition*, which associates with any two vectors  $\vec{x}, \vec{y} \in S$  the sum  $\vec{x} + \vec{y}$  which also must belong to  $S$ .
  - The second operation is *scalar multiplication*, which associates with any vector  $\vec{x} \in S$  and any scalar  $a \in R$  the scalar multiple of  $\vec{x}$  by  $a$ , denoted by  $a\vec{x}$  or  $a \cdot \vec{x}$ , which must belong to  $S$ .
- The vector addition operation must obey four rules:
  1. Commutativity:  $\vec{x} + \vec{y} = \vec{y} + \vec{x} \forall \vec{x}, \vec{y} \in S$ .
  2. Associativity:  $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z} \forall \vec{x}, \vec{y}, \vec{z} \in S$ .
  3. There is a unique “zero vector”  $\vec{0} \in S$  such that  $\vec{x} + \vec{0} = \vec{x} \forall \vec{x} \in S$ .
  4. For each  $\vec{x} \in S$ , there is a vector  $-\vec{x} \in S$  such that  $\vec{x} + (-\vec{x}) = \vec{0}$ .
- The scalar multiplication operation must also obey four rules:
  1. Distributivity:  $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$  and  $(a+b)\vec{x} = a\vec{x} + b\vec{x} \forall \vec{x}, \vec{y} \in S$  and  $a, b \in R$ .
  2. Associativity:  $(ab)\vec{x} = a(b\vec{x}) \forall \vec{x} \in S$  and  $a, b \in R$ .
  3. For the multiplicative identity of  $R$  (denoted by the scalar  $1 \in R$ ), we have  $1 \cdot \vec{x} = \vec{x} \in S$ .
  4. For the additive identity of  $R$  (denoted by the scalar  $0 \in R$ ), we have  $0 \cdot \vec{x} = \vec{0} \in S$ .

#### Linear Subspaces:

The concept of a linear subspace is useful for modeling, approximating signals, discussing the concept of bases, etc.

**Definition 2.1 (Linear Subspace)** — A nonempty subset  $T$  of a vector space  $S$  is called a subspace (or linear subspace) of  $S$  if

$$a\vec{x} + b\vec{y} \in T$$

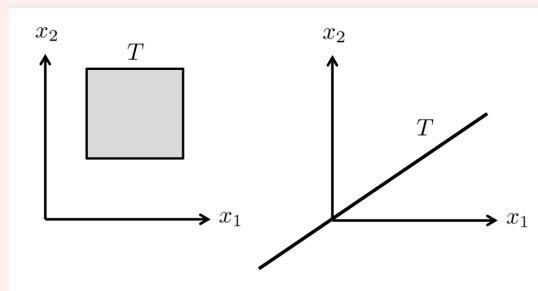
for all  $\vec{x}, \vec{y} \in T$  and all  $a, b \in R$ .

Notes:

- Any linear subspace  $T$  must contain  $\vec{0}$ .
- Any vector space  $S$  is a linear subspace (of itself).
- Any linear subspace  $T$  meets all the properties of a vector space.

### Example 2.2

Are either of these a subspace of  $S = \mathbb{R}^2$



Left: No! Right: Yes!

### Example 2.3

Which of the following are subspaces?

1.  $S = \mathbb{R}^5$  and  $T = \{\vec{x} \in \mathbb{R}^5 : x_4 = 0\}$  – Yes!
2.  $S = \mathbb{R}^5$  and  $T = \{\vec{x} \in \mathbb{R}^5 : x_4 = 1\}$  – No, add any two vectors in  $T$  and the sum will not belong to  $T$ .
3.  $S = \mathbb{R}^5$  and  $T$  is the set of vectors in  $\mathbb{R}^5$  with no more than 3 nonzero entries – No, can add certain vectors in  $T$  to get up to 5 nonzero elements.

### Linear Combinations:

Linear combinations are used to build new vectors a weighted sum of other vectors.

**Definition 2.4 (Linear Combination)** — Let  $M = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  be a collection of vectors in a vector space  $S$ . (we will stick with finite collections at the moment). A linear combination of vectors in  $M$  is a sum of the form

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n$$

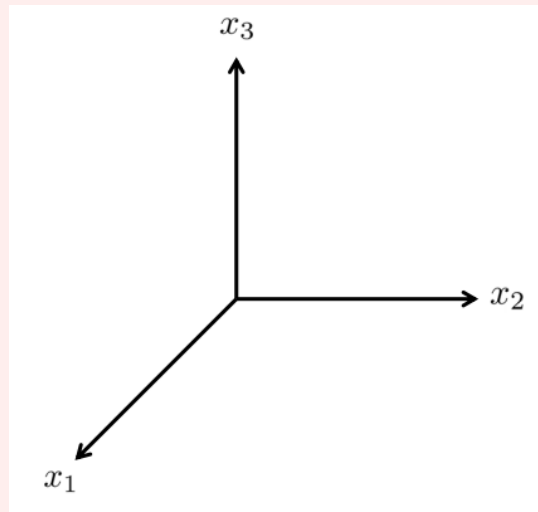
for some  $a_1, a_2, \dots, a_n \in R$ . Since  $S$  is a vector space, this sum must belong to  $S$ .

Mentally, you might find it useful to replace “linear combination” with “weighted sum”, although this is not standard terminology.

**Definition 2.5 (Span)** — Let  $M = \{\vec{v}_1, \dots, \vec{v}_n\}$  be a finite collection of vectors in a vector space  $S$ . The span of  $M$ , denoted by  $\text{span}(M)$  or  $\text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$ , is the set of all linear combinations of vectors in  $M$ .

**Example 2.6**

Consider the vectors  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ . What is  $\text{span}\{\vec{v}_1, \vec{v}_2\}$ ?



The entire  $(x_1, x_2)$ -plane.

When we are in  $\mathbb{R}^n$  for some finite  $n$ , it is common to use matrix-vector notation as shorthand for linear combinations:

- Suppose  $\vec{x} = c_1\vec{p}_1 + \dots + c_k\vec{p}_k$
- Then we may define the  $n \times k$  matrix

$$A = [\vec{p}_1 \quad \vec{p}_2 \quad \dots \quad \vec{p}_k]$$

and the  $k \times 1$  vector

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

and this allows us to write the  $n \times 1$  vector  $\vec{x}$  as  $\vec{x} = A\vec{c}$ .

**Definition 2.7 (Linear Dependence)** — A finite set of vectors  $\vec{v}_1, \dots, \vec{v}_n$  in a vector space  $S$  is said to be linearly dependent if there exists scalars  $a_1, \dots, a_n \in R$ , not all equal to zero, such that

$$a_1\vec{v}_1 + \dots + a_n\vec{v}_n = \vec{0}$$

**Definition 2.8 (Linear Independence)** — A finite set of vectors  $\vec{v}_1, \dots, \vec{v}_n$  in a vector space  $S$  is said to be linearly independent if

$$a_1\vec{v}_1 + \dots + a_n\vec{v}_n = \vec{0}$$

only when all  $a_k = 0$ .

Every vector in the span of a linearly independent set of vectors has a unique expansion in terms of those vectors. This is formalized in the following lemma.

**Lemma 2.9**

Suppose  $\vec{v}_1, \dots, \vec{v}_n$  are linearly indep. and suppose

$$a_1\vec{v}_1 + \dots + a_n\vec{v}_n = b_1\vec{v}_1 + \dots + b_n\vec{v}_n$$

for some scalars  $a_1, \dots, a_n \in R$  and  $b_1, \dots, b_n \in R$ . Then  $a_k = b_k$  for  $k = 1, 2, \dots, n$ .

*Proof.* Note that  $\sum_{k=1}^n (a_k - b_k) \vec{v}_k = \vec{0}$ . Since  $\vec{v}_1, \dots, \vec{v}_n$  are linearly indep., it must follow that  $a_k - b_k = 0$  for all  $k = 1, 2, \dots, n$ .  $\square$

**Bases and Dimension:**

Now that we know how to combine vectors (via linear combinations), let's think about important sets of vectors we'd be interested in combining.

**Definition 2.10 (Basis)** — A finite set of vectors  $\vec{v}_1, \dots, \vec{v}_n$  in a vector space  $S$  is said to form a basis for  $S$  if the following two conditions are satisfied:

1.  $\vec{v}_1, \dots, \vec{v}_n$  are linearly independent.
2.  $\text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = S$ .

A vector space with a finite basis, as in the above definition, is said to be finite-dimensional. Any two bases for a finite-dimensional vector space contain the same number of elements. This leads to a meaningful definition of dimension.

- Definition 2.11** (Dimension) —
- For a vector space  $S$  that can be spanned using a finite set of basis vectors, the dimension of  $S$  is the number of vectors required in any basis for  $S$ .
  - For a vector space  $S$  that cannot be spanned using a finite set of basis vectors, the dimension of  $S$  is said to be infinite.

**Example 2.12** (Bases for  $S = \mathbb{R}^n$ ) • The standard, or canonical, basis for  $\mathbb{R}^n$  is given by:

$$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

- Any set of  $n$  linearly indep. vectors in  $\mathbb{R}^n$  forms a basis for  $\mathbb{R}^n$ .

## §3 | Lec 3: Apr 2, 2021

### §3.1 Lec 2 (Cont'd)

#### Normed Linear Spaces:

A norm is a function used to measure the size of vectors in a vector space.

**Definition 3.1 (Norm)** — A norm  $\|\cdot\|$  on a vector space  $S$  is a mapping  $\|\cdot\| : S \rightarrow \mathbb{R}$  with the following properties:

1.  $\|\vec{x}\| \geq 0$  for all  $\vec{x} \in S$ , and  $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$ .
2. Triangle inequality:  $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$  for all  $\vec{x}, \vec{y} \in S$ .
3.  $\|a\vec{x}\| = |a| \cdot \|\vec{x}\|$  for any  $a \in \mathbb{R}$  and  $\vec{x} \in S$ .

**Definition 3.2 (Normed Linear Space)** — A normed linear space is a vector space  $S$  together with a valid norm  $\|\cdot\| : S \rightarrow \mathbb{R}$ .

The  $l_p$  metrics for vectors in  $\mathbb{R}^n$  extend naturally to  $l_p$  norms for these same spaces:

- $l_1$  norm:  $\|\vec{x}\|_1 = \sum_{i=1}^n |x_i|$
- $l_2$  (“Euclidean”) norm:

$$\|\vec{x}\|_2 = \left( \sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}}$$

- $l_p$  norm for  $1 \leq p < \infty$ :

$$\|\vec{x}\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

- $l_\infty$  norm:

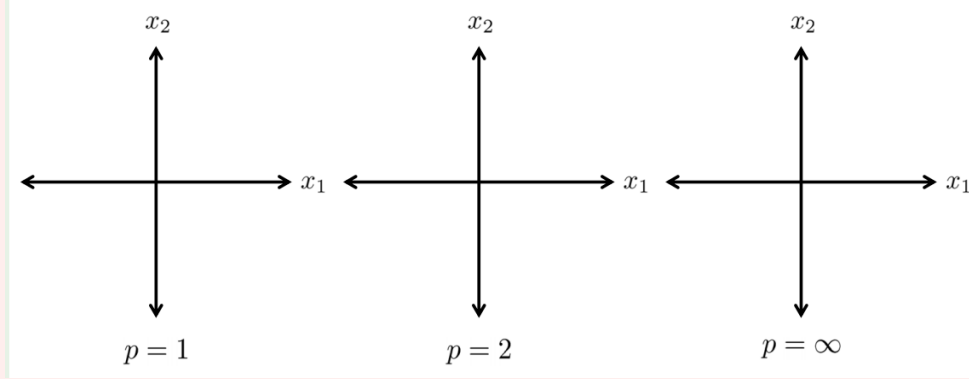
$$\|\vec{x}\|_\infty = \max_{i=1, \dots, n} |x_i|$$

- Bonus –  $l_0$  norm (not really a norm – not satisfied the definition of a norm)

$$\|\vec{x}\|_0 = \# \text{ of nonzeros in } \vec{x}$$

**Example 3.3**

The “unit ball” in a normed linear space is the set of all vectors in  $S$  having norm less than or equal to 1. Suppose  $S = \mathbb{R}^2$  and draw the  $l_p$  balls for  $p = 1, 2, \infty$ .



- $p = 1$ :  $\|\vec{x}\|_1 = |x_1| + |x_2| \leq 1$  – diamond.
- $p = 2$ :  $\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2} \leq 1$  – circle.
- $p = \infty$ :  $\|\vec{x}\|_\infty = \max\{|x_1|, |x_2|\} \leq 1$  – square.

**Inner Product Spaces:**

An inner product is a function used to compare two vectors in a vector space. The concept of an inner product will give us additional geometric structure beyond what is available in general normed linear spaces. In particular, using an inner product we can define a meaningful measure of the angle between two vectors, discuss orthonormal bases and orthogonal projections, etc.

**Definition 3.4 (Inner Product)** — An inner product  $\langle \cdot, \cdot \rangle$  on a vector space  $S$  is a mapping  $\langle \cdot, \cdot \rangle : S \times S \rightarrow R$  with the following properties:

1.  $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle^*$  for all  $\vec{x}, \vec{y} \in S$ .
2. For any  $\vec{x}, \vec{y}, \vec{z} \in S$  and any  $a, b \in R$ ,  $\langle a\vec{x} + b\vec{y}, \vec{z} \rangle = a\langle \vec{x}, \vec{z} \rangle + b\langle \vec{y}, \vec{z} \rangle$ .
3. For any  $\vec{x} \in S$ ,  $\langle \vec{x}, \vec{x} \rangle$  is real-valued and non-negative, and  $\langle \vec{x}, \vec{x} \rangle = 0$  iff  $\vec{x} = \vec{0}$ .

**Definition 3.5 (Inner Product Space)** — An inner product space is a vector space  $S$  together with a valid inner product  $\langle \cdot, \cdot \rangle : S \times S \rightarrow R$ .

**Definition 3.6 (Orthogonality)** — Two vectors  $\vec{x}$  and  $\vec{y}$  in an inner product space  $S$  are said to be orthogonal if  $\langle \vec{x}, \vec{y} \rangle = 0$ .



**Example 3.7**

When  $S = \mathbb{R}^n$ , the standard inner product between two vectors  $\vec{x}, \vec{y} \in S$  is given by

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^n x_i y_i = \vec{y}^\top \vec{x}$$

The standard inner product on  $\mathbb{R}^n$  is also known as the dot product.

$S = \mathbb{C}^n : \langle \vec{x}, \vec{y} \rangle = \vec{y}^H \vec{x} = \sum_{i=1}^n x_i y_i^*$  where  $H$  denotes conjugate transpose/hermitian.

Before we get to the connection between inner product and angles, it is worth noting that inner products can actually be used to measure the length of vectors (and thus distances between vectors as well).

In particular, any valid inner product induces a valid norm by

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$$

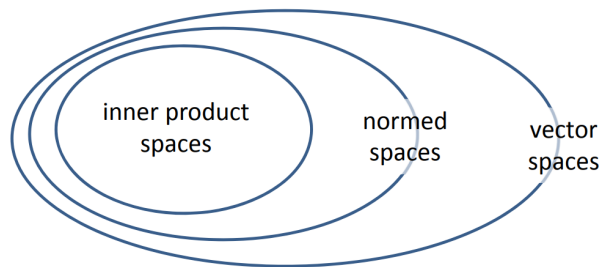
**Example 3.8**

When  $S = \mathbb{R}^n$ , the standard inner product induces the following norm:

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\sum_{i=1}^n x_i^2}$$

We can recognize this as the  $l_2$  norm.

Other  $l_p$  norm (for  $p \neq 2$ ) cannot be induced by inner products. Because every valid inner product induces a valid norm, every inner product space is also a normed linear space. But, not every normed linear space is also an inner product space:



Recall the definition of a basis in a generic, finite-dimensional vector space. In inner product spaces, a particularly useful class of bases are orthogonal bases.

**Definition 3.9 (Orthogonal Basis)** — A finite sets of non-zero vectors  $\vec{v}_1, \dots, \vec{v}_n$  in an inner product space  $S$  is said to form an orthogonal basis for  $S$  if the following two conditions are satisfied:

1.  $\langle \vec{v}_k, \vec{v}_l \rangle = 0$  for all  $k \neq l$  (note that this implies the vectors are linearly indep.)
2.  $\text{span} \{ \vec{v}_1, \dots, \vec{v}_n \} = S$

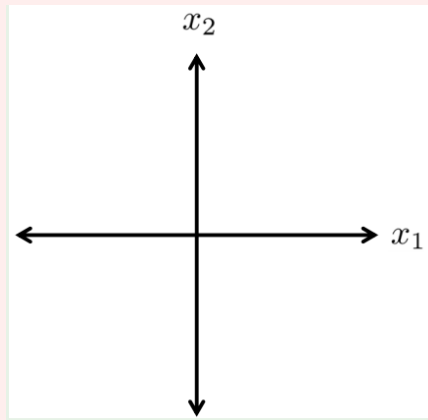
**Definition 3.10 (Orthonormal Basis)** — An orthogonal basis is called orthonormal basis or orthobasis if every basis vector  $\vec{v}_k$  has unit norm (i.e.,  $\|\vec{v}_k\| = 1$ ) according to the induced norm in the inner product space.

**Example 3.11**

Using the standard inner product,

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

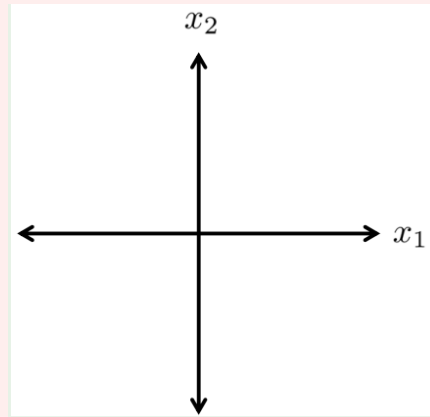
form an orthobasis for  $\mathbb{R}^2$ .



**Example 3.12** (Rotation of 45 degree from the last example)

Another possible orthobasis in  $\mathbb{R}^2$  is given by

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

**§3.2 Linear Operators**

Operators are transformations that map vectors in some vector space to vectors in some other (possibly different) vector space.

**Definition 3.13** (Linear Operator) — Suppose  $X$  and  $Y$  are vector spaces. We say the operator  $A : X \rightarrow Y$  is a linear operator if

$$A(\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2) = \alpha_1 A\vec{x}_1 + \alpha_2 A\vec{x}_2$$

for all  $\alpha_1, \alpha_2 \in R$  and  $\vec{x}_1, \vec{x}_2 \in X$ .

**Fact 3.1.** If  $X = \mathbb{R}^n$  and  $Y = \mathbb{R}^m$ , any linear operator from  $X$  to  $Y$  can be represented as multiplication by an  $m \times n$  matrix.

Therefore, a particularly interesting class of linear operators for us will simply be matrices.

**§3.3 Operator Norms**

Roughly speaking, operator norms help us talk about the “gain” of a system.

**Definition 3.14 (Operator Norm)** — Let  $X$  and  $Y$  be normed linear spaces with corresponding norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  and suppose  $A : X \rightarrow Y$  is linear operator. The operator norm  $\|A\|$  is defined as

$$\|A\| := \sup_{\vec{0} \neq \vec{x} \in X} \frac{\|A\vec{x}\|_Y}{\|\vec{x}\|_X}$$

This is equivalent to

$$\|A\| := \sup_{\vec{x} \in X, \|\vec{x}\|_X = 1} \|A\vec{x}\|_Y$$

## §4 | Lec 4: Apr 5, 2021

### §4.1 Operator Norms (Cont'd)

Any operator norm as defined in the last lecture will satisfy the following properties

1.  $\|A\| \geq 0$  with equality iff  $A = 0$
2.  $\|\alpha A\| = |\alpha| \|A\|$  for all  $\alpha \in \mathbb{R}$
3.  $\|A + B\| \leq \|A\| + \|B\|$  for all linear operators  $A$  and  $B$  between the vector spaces  $X$  and  $Y$ .
  - Examining these first three properties, we see that  $\|A\|$  is a valid norm on the vector space of linear operators!
4.  $\|A\vec{x}\|_Y \leq \|A\| \|\vec{x}\|_X$  for all  $\vec{x} \in X$ 
  - Therefore, the operator norm helps us bound how much an operator can “amplify” a signal.
5.  $\|AB\| \leq \|A\| \|B\|$
6. If  $X = Y$  and if  $\|A\| < 1$ , then we can write

$$\sum_{i=0}^{\infty} A^i = (I - A)^{-1}$$

just as for a scalar  $a \in \mathbb{R}$  with  $|a| < 1$ , we can write  $\sum_{i=0}^{\infty} a_i = \frac{1}{1-a}$

Let's restrict our attention to the special case where  $X = \mathbb{R}^n$  and  $Y = \mathbb{R}^m$ . For these choices of  $X$  and  $Y$ , recall that any linear operator  $A : X \rightarrow Y$  can be represented as multiplication by an  $m \times n$  matrix. In such a case, the operator norm  $\|A\|$  is also called a matrix norm.

When  $X$  and  $Y$  are both equipped with the  $l_p$  norm for  $p \in [1, \infty]$ , we can write

$$\|A\|_p := \sup_{\vec{x} \in X, \|\vec{x}\|_p=1} \|A\vec{x}\|_p$$

We can relate  $\|A\|_p$  to certain properties of the matrix  $A$ :

- In the case  $p = \infty$ , letting  $\vec{y} = A\vec{x}$ , we have

$$\|A\|_{\infty} := \sup_{\|\vec{x}\|_{\infty}=1} \underbrace{\|A\vec{x}\|_{\infty}}_{\vec{y}} = \sup_{\|\vec{x}\|_{\infty}=1} \left\| \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \right\|_{\infty}$$

Note that  $|y_i| = \left| \sum_{j=1}^n a_{ij} x_j \right|$ . Overall  $\vec{x}$  with  $\|\vec{x}\|_{\infty} = 1$ , the largest  $|y_i|$  for a given  $i = 1, 2, \dots, m$  is achieved by taking  $x_j = \text{sign}(a_{ij})$  for  $j = 1, 2, \dots, n$ , and for this choice of  $i$  and  $\vec{x}$ , we will have

$$|y_i| = \sum_{j=1}^n |a_{ij}| = \text{absolute sum of row } i \text{ of } A$$

Thus,

$$\|A\|_\infty = \max_{i=1,2,\dots,m} \sum_{j=1}^n |a_{ij}| = \text{maximum absolute row sum of } A$$

- Similarly, in the case  $p = 1$ , we have

$$\|A\|_1 = \max_{j=1,2,\dots,n} \sum_{i=1}^m |a_{ij}| = \text{maximum absolute column sum of } A$$

- When  $p = 2$ ,  $\|A\|_2$  is also referred to as the spectral norm of  $A$ . We can understand  $\|A\|_2$  geometrically: the operator  $A$  maps the  $l_2$  unit ball in  $\mathbb{R}^n$  to an ellipsoid in  $\mathbb{R}^m$ .

The length of the major axis of the ellipsoid is equal to  $\|A\|_2$ . We can also write

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^\top A)} = \sigma_{\max}(A)$$

where  $\lambda_{\max}$  denotes the largest eigenvalue of a matrix and  $\sigma_{\max}$  denotes the largest singular value of a matrix. If the matrix  $A$  happens to be symmetric (i.e. if  $A = A^\top$ ) then we can also write

$$\|A\|_2 = \max_i |\lambda_i(A)|$$

There is also a special type of “matrix norm” that does not actually follow the definition of an operator norm,

$$\|A\| := \sup_{\vec{x} \in X, \|\vec{x}\|_X=1} \|A\vec{x}\|_Y$$

In particular, the Frobenius form of a matrix  $A$  is defined to be

$$\|A\|_F := \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2} = \sqrt{\text{trace}(A^H A)}$$

Note that Frobenius form  $\|A\|_F$  is not an operator norm, because  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .

$$\|A\|_F = \sqrt{2} > 1 = \sup_{\|x\|_X=1} \|Ax\|_Y = \|x\|_X = 1$$

$$\|A\|_F = \sqrt{1+1} = \sqrt{2}$$

$$\|A\|_2 = \sup_{\|x\|_2=1} \|Ax\|_2 = 1$$

How about  $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ ?

$$\|A\|_F = \sqrt{1+1+4+0} = \sqrt{6}$$

## §4.2 Inverse Operator

**Definition 4.1 (Invertibility)** — A linear operator  $A : X \rightarrow Y$  between two vector spaces  $X$  and  $Y$  is said to be invertible if there exists an operator  $A^{-1} : Y \rightarrow X$  s.t.

- $AA^{-1} = I$ , i.e.,  $AA^{-1}\vec{y} = \vec{y}$  for all  $\vec{y} \in Y$  and
- $A^{-1}A = I$ , i.e.,  $A^{-1}A\vec{x} = \vec{x}$  for all  $\vec{x} \in X$ .

In such a case,  $A^{-1}$  is referred to as the inverse of  $A$ .

### Lemma 4.2

If  $A$  is an invertible linear operator,  $A^{-1}$  is itself a linear operator.

Invertibility is a topic of interest when we want to find an exact solution to a linear equation. Let us again restrict our attention to the special case where  $X = \mathbb{R}^n$  and  $Y = \mathbb{R}^m$ . For these choices of  $X$  and  $Y$ , recall that any linear operator  $A : X \rightarrow Y$  can be represented as multiplication by an  $m \times n$  matrix.

**Fact 4.1.** An  $m \times n$  matrix  $A$  cannot be invertible unless it is square.

Not all square matrices are invertible. An invertible matrix is also known as the nonsingular matrix.

### Proposition 4.3

If  $A$  is a square matrix, the following statements are all equivalent

- $A$  is invertible
- $A$  is nonsingular
- $\det(A) \neq 0$
- $A\vec{x} = \vec{0} \iff \vec{x} = \vec{0}$
- The rows of  $A$  are linearly indep.
- The columns of  $A$  are linearly indep.
- $\dim(\mathcal{N}(A)) = 0$ , i.e.,  $\mathcal{N} = \{\vec{0}\}$
- $\dim(\mathcal{R}(A)) = n$ .
- $A$  is full rank
- All eigenvalues of  $A$  are nonzero
- The matrix  $A^\top A$  is positive definite.
- $A^\top$  is invertible.

### §4.3 Adjoint Operators

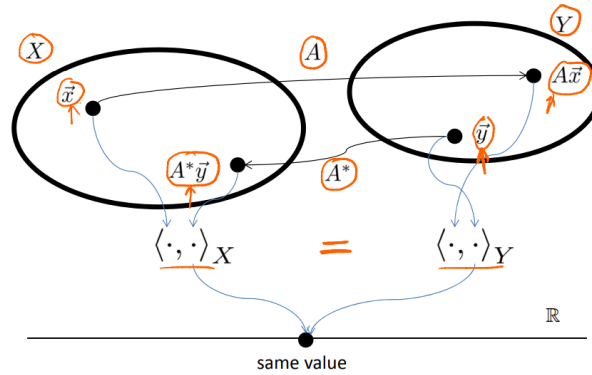
**Definition 4.4 (Adjoint of an Operator)** — Let  $A : X \rightarrow Y$  be a bounded ( $\|A\| < \infty$ ) linear operator between two inner product spaces  $X$  and  $Y$ . The adjoint of  $A$ , denoted  $A^* : Y \rightarrow X$  is the unique operator such that

$$\langle A\vec{x}, \vec{y} \rangle_Y = \langle \vec{x}, A^*\vec{y} \rangle_X$$

for all  $\vec{x} \in X$  and  $\vec{y} \in Y$ .

**Definition 4.5 (Self-Adjoint Operator)** — An operator  $A : X \rightarrow X$  is said to be self-adjoint if  $A = A^*$ .

An illustration:



#### Lemma 4.6

If  $A$  is bounded linear operator with adjoint  $A^*$  then  $A^*$  is itself a bounded linear operator, and

$$\|A^*\| = \|A\|$$

#### Lemma 4.7

If  $A$  is bounded linear operator with adjoint  $A^*$  then  $(A^*)^* = A$

#### Lemma 4.8

If  $A$  is an invertible bounded linear operator with adjoint  $A^*$  and bounded inverse  $A^{-1}$ , then

$$(A^{-1})^* = (A^*)^{-1}$$



In the special case where  $X = \mathbb{R}^n$  and  $Y = \mathbb{R}^m$ , recall that any linear operator  $A : X \rightarrow Y$  can be represented as multiplication by an  $m \times n$  matrix.

**Question 4.1.** What is the adjoint of a matrix?

Using the standard inner product on  $X$  and  $Y$ , the adjoint of  $A$  is the unique operator s.t.

$$\vec{y}^H A \vec{x} = \langle A \vec{x}, \vec{y} \rangle = \langle \vec{x}, A^* \vec{y} \rangle = \vec{y}^H (A^*)^H \vec{x}$$

for all  $\vec{x}$  and  $\vec{y}$ . This requires that  $A = (A^*)^H$  which is satisfied by taking

$$A^* = A^H$$

Therefore, the adjoint of a matrix is simply its conjugate transpose (not its inverse). Self-adjoint matrices satisfy  $A = A^H$ . These are also known as symmetric (if real), conjugate symmetric or Hermitian (if complex).

#### Theorem 4.9

Let  $A$  be a real-valued  $m \times n$  matrix. For a fixed  $\vec{y} \in \mathbb{R}^m$ , the vector  $\vec{x} \in \mathbb{R}^n$  is a minimizer of  $\|\vec{y} - A\vec{x}\|_2 \iff$

$$A^\top A \vec{x} = A^\top \vec{y}$$

If  $A^\top A$  is invertible, then the unique minimizer of  $\|\vec{y} - A\vec{x}\|_2$  is given by

$$\vec{x} = (A^\top A)^{-1} A^\top \vec{y}$$

The same theorem holds if  $A, \vec{x}$  and  $\vec{y}$  are all complex-valued.

## §5 | Lec 5: Apr 7, 2021

### §5.1 Fundamental Subspaces of Linear Operators

**Definition 5.1 (Range)** — Let  $A : X \rightarrow Y$  be a linear operator between two vector spaces  $X$  and  $Y$ . The range or range space of  $A$ , denoted by  $\mathcal{R}(A)$ , is defined to be

$$\mathcal{R}(A) := \{\vec{y} \in Y : A\vec{x} = \vec{y} \text{ for some } \vec{x} \in X\}$$

The range space is a linear subspace of  $Y$ .

**Definition 5.2 (Nullspace)** — Let  $A : X \rightarrow Y$  a linear operator between two vector spaces  $X$  and  $Y$ . The nullspace of  $A$ , denoted by  $\mathcal{N}(A)$ , is defined to be

$$\mathcal{N}(A) := \{\vec{x} \in X : A\vec{x} = \vec{0}\}$$

The nullspace is a linear subspace of  $X$ .

Again, we consider the case where  $X = \mathbb{R}^n$  and  $Y = \mathbb{R}^m$  (or where  $X = \mathbb{C}^n$  and  $Y = \mathbb{C}^m$ ) and  $A : X \rightarrow Y$  can be represented as multiplication by an  $m \times n$  matrix.

When  $A$  is a matrix,  $\mathcal{R}(A)$  is just the span of the columns of  $A$ :

$$\mathcal{R}(A) = \text{colspan}(A)$$

For any  $m \times n$  matrix  $A$ ,

$$\text{rank}(A) \leq \min\{m, n\}$$

We say that  $A$  is full rank if  $\text{rank}(A) = \min\{m, n\}$ , otherwise we call it rank deficient.

We can relate the rank of  $A$  to the dimensions of the four fundamental subspaces of  $A$ :

- $\dim(\mathcal{R}(A)) = \dim(\text{colspan}(A)) = \text{rank}(A)$
- $\dim(\mathcal{N}(A)) = n - \text{rank}(A)$
- $\dim(\mathcal{R}(A^*)) = \dim(\text{rowspan}(A)) = \text{rank}(A)$
- $\dim(\mathcal{N}(A^*)) = m - \text{rank}(A)$
- $\dim(\mathcal{R}(A)) + \dim(\mathcal{N}(A)) = n = \# \text{ columns of } A$ .

For two matrices  $A$  and  $B$ , we have

- $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$
- $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$

## §5.2 Projection Operators

**Definition 5.3 (Projection Operator)** — A linear operator  $P : X \rightarrow X$  from a vector space  $X$  into itself is called a projection or a projection operator if

$$P^2 = P$$

i.e.,  $P(P(\vec{x})) = P(\vec{x})$  for all  $\vec{x} \in X$ .

For  $P^2 = P$ ,  $P$  is called idempotent operator.

**Definition 5.4 (Orthogonal Projection Operator)** — A projection operator  $P$  in an inner product space  $X$  is called an orthogonal projection or an orthogonal projection operator if

$$\mathcal{R}(P) \perp \mathcal{N}(P)$$

i.e., if  $\langle \vec{x}, \vec{y} \rangle = 0$  for all  $\vec{x} \in \mathcal{R}(P)$  and  $\vec{y} \in \mathcal{N}(P)$ .

We notice

$$\vec{x} = \underbrace{P\vec{x}}_{\in \mathcal{R}(P)} + \underbrace{(I - P)\vec{x}}_{\in \mathcal{N}(P)}$$

If  $P$  is an orthogonal projection operator,

$$\langle P\vec{x}, (I - P)\vec{x} \rangle = 0$$

### Lemma 5.5

A bounded linear operator  $P : X \rightarrow X$  on an inner product space  $X$  is an orthogonal projection iff

1.  $P^2 = P$  and
2.  $P = P^*$

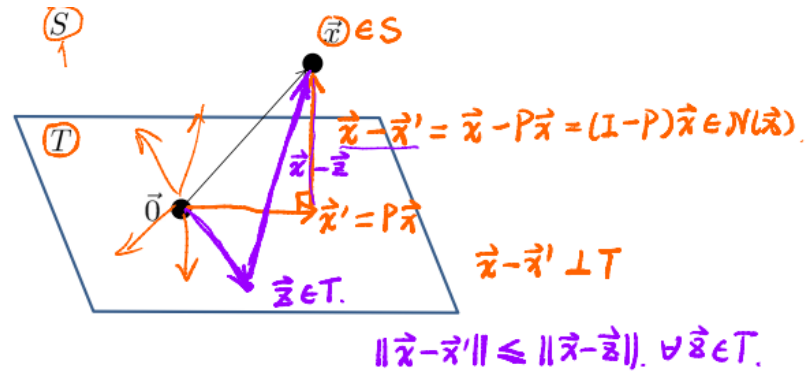
### Theorem 5.6

Suppose  $S$  is an inner product space and suppose  $T$  is a linear subspace of  $S$ . For a given vector  $\vec{x} \in S$ , there is a unique vector  $\vec{x}' \in T$  such that  $\|\vec{x} - \vec{x}'\| \leq \|\vec{x} - \vec{z}\|$  for all  $\vec{z} \in T$ . Furthermore, this minimizer has the property that

$$\vec{x} - \vec{x}' \perp T$$

i.e.  $\langle \vec{x} - \vec{x}', \vec{y} \rangle = 0$  for all  $\vec{y} \in T$ .

The minimizing vector  $\vec{x}'$  is referred to as the orthogonal projection of  $\vec{x}$  onto  $T$ . In other words,  $\vec{x}' = P\vec{x}$ , where  $P$  is an orthogonal projection operator with  $\mathcal{R}(P) = T$ .



For an  $n \times n$  matrix  $P$ ,

- $P$  is a projection if  $P^2 = P$
- $P$  is orthogonal projection if  $P^2 = P$  and  $P^\top = P$

### Example 5.7

Consider the operator  $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined as

$$P \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

We can express  $P$  as the  $2 \times 2$  matrix  $P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$  Consider:

- Does  $P^2 = P$ ? Yes!
- Does  $P^\top = P$ ? No!
- What is  $\mathcal{R}(P)$ ? the line  $x_2 = x_1$
- What is  $\mathcal{N}(P)$ ?  $x_2$ -axis
- Is  $\mathcal{R}(P) \perp \mathcal{N}(P)$ ? No!

**Example 5.8**

Consider the operator  $P : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined as

$$P \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Consider:

- Does  $P^2 = P$ ? Yes!
- Does  $P^\top = P$ ? Yes!
- What is  $\mathcal{R}(P)$ ? the plane  $x_1 - x_2$
- What is  $\mathcal{N}(P)$ ?  $x_3$ -axis
- Is  $\mathcal{R}(P) \perp \mathcal{N}(P)$ ? Yes!

Consider a set of  $m$  linearly indep. vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$  or  $\mathbb{C}^n$ . We can construct an orthogonal projection matrix  $P$  onto the subspace  $T = \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$  as follows:

1. Construct an  $n \times m$  matrix

$$A = [\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_m]$$

Note that  $\text{colspan}(A) = T$ .

2. Let

$$P = A(A^\top A)^{-1} A^\top = AA^\dagger$$

**Example 5.9**

Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The orthogonal projection can be constructed as follows

$$P = AA^\dagger = A(A^\top A)^{-1} A^\top = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## §6 | Lec 6: Apr 9, 2021

### §6.1 Motivating Examples

Consider a  $2 \times 2$  matrix

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

For what nonzero vectors  $\vec{x} \in S$  (eigenvectors) and scalar  $\lambda \in \mathbb{C}$  (eigenvalue), do we have that  $A\vec{x} = \lambda\vec{x}$ ?

- We know that  $\vec{x}$  and  $\lambda$  must satisfy  $(A - \lambda I)\vec{x} = \vec{0}$ .
- Thus,  $\vec{x}$  must be in  $\mathcal{N}(A - \lambda I)$ .
- Thus,  $A - \lambda I$  must have a nontrivial nullspace.
- Thus,  $A - \lambda I$  must be singular.
- We can solve for  $\lambda$  s.t.  $\det(A - \lambda I) = 0$ :

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1 - \lambda & \frac{1}{2} \\ \frac{1}{2} & 1 - \lambda \end{bmatrix}\right) = (1 - \lambda)^2 - \frac{1}{4} = 0$$

which equals 0 for  $\lambda = 1.5$  or  $\lambda = 0.5$

- Now we know the eigenvalues. What are the corresponding eigenvectors?
- For  $\lambda = 1.5$ , we need  $A\vec{x} = 1.5\vec{x}$ .

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.5x_1 \\ 1.5x_2 \end{bmatrix}$$

which requires  $x_1 = x_2$ . To have unit norm, we can choose

$$\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

but notice that any rescaling of this  $\vec{x}$  is also an eigenvector.

- For  $\lambda = 0.5$ , we need  $A\vec{x} = 0.5\vec{x}$ .

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5x_1 \\ 0.5x_2 \end{bmatrix}$$

This requires  $x_1 = -x_2$ . To have unit norm, we can choose

$$\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

but notice that any rescaling of this  $\vec{x}$  is also an eigenvector.

- We say that the eigenvectors of  $A$  are

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

with the understanding that any rescaling of either one is also an eigenvector.

## §6.2 Eigenvalues and Eigenvectors

Big picture:

- first find the eigenvalues
  - Suppose  $A$  is an  $n \times n$  matrix.
  - We want to know: for what values of  $\lambda \in \mathbb{C}$  does there exist a non-zero  $\vec{x} \in \mathbb{C}^n$  s.t.

$$A\vec{x} = \lambda\vec{x}?$$

- We know that for such a  $(\lambda, \vec{x})$  pair to exist, we must have

$$(A - \lambda I)\vec{x} = \vec{0}$$

and therefore,  $\vec{x} \in \mathcal{N}(A - \lambda I)$ .

- For such a non-zero  $\vec{x}$  to exist in the nullspace of  $A - \lambda I$ , we need

$$\dim(\mathcal{N}(A - \lambda I)) > 0$$

Since

$$\dim(\mathcal{R}(A - \lambda I)) + \dim(\mathcal{N}(A - \lambda I)) = n = \# \text{ columns of } A - \lambda I$$

which means we require

$$\dim(\mathcal{R}(A - \lambda I)) < n$$

so  $A - \lambda I$  cannot be full rank. We also know this requires

$$\det(A - \lambda I) = 0$$

- Hence, by finding all  $\lambda$  s.t.  $\det(A - \lambda I) = 0$ , we get the eigenvalues of  $A$ .
- the find the eigenvectors
  - Suppose  $\lambda$  is an eigenvalue of  $A$ , which has size  $n \times n$ .
  - Then every  $\vec{x} \in \mathcal{N}(A - \lambda I)$  is considered an eigenvector of  $A$ , corresponding to the eigenvalue  $\lambda$ .
  - Because  $\mathcal{N}(A - \lambda I)$  is a linear subspace of  $\mathbb{C}^n$ , we conventionally just specify enough vectors to span this subspace, i.e., a basis for  $\mathcal{N}(A - \lambda I)$ .

Let's work through an example.

**Example 6.1**

Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Solving  $\det(A - \lambda I) = 0$ , we want

$$\det(A - \lambda I) = (2 - \lambda) \left( \left( \frac{3}{2} - \lambda \right)^2 - \frac{1}{4} \right) = 0$$

which is satisfied if  $\lambda = 2, 1$ .

- For  $\lambda = 1$  (with multiplicity one)

- We want to find all  $\vec{x} \in \mathbb{C}^3$  s.t.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- This requires that  $x_1 = 0$  and that  $\frac{x_2}{2} + \frac{x_3}{2} = 0$ . We have two linear equations, which imply that  $\mathcal{N}(A - \lambda I)$  is a line in  $\mathbb{C}^3$ .

- So we can pick  $\vec{v}_1 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$  as our first eigenvector.

- For  $\lambda = 2$  (with multiplicity two)

- We want to find  $\vec{x} \in \mathbb{C}^3$  s.t.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- This requires that  $-\frac{x_2}{2} + \frac{x_3}{2} = 0 \implies x_2 = x_3$ . We have one linear equation, which implies that  $\mathcal{N}(A - \lambda I)$  is a plane in  $\mathbb{C}^3$ .

- So we can pick two linearly indep. vectors from this nullspace, e.g.,

$$\vec{v}_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Thus, for this particular matrix  $A$ , we have two distinct eigenvalues but three linearly indep. eigenvectors.

In the case where the eigenvalues of the matrix are distinct, we have an important result:



**Theorem 6.2**

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

*Proof.* Left for readers to figure out on your own :) (just kidding, I guess I am just lazy).  $\square$

Eigenvectors corresponding to repeated eigenvalues could be linearly indep.

**Example 6.3**

Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \lambda_1 = \lambda_2 = 1, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

in which case  $A - \lambda I$  has a two-dimensional nullspace.

**Example 6.4**

Consider:

$$A = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}, \lambda_1 = \lambda_2 = 4, \vec{v}_1 = \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

in which case  $A - \lambda I$  has a one-dimensional nullspace.

## §7 | Lec 7: Apr 12, 2021

### §7.1 Diagonalization

If an  $n \times n$  matrix  $A$  happens to have  $n$  linearly indep. eigenvectors, then it can be written (or “diagonalized”) as

$$A = T\Lambda T^{-1}$$

where

- $T$  is an  $n \times n$  invertible matrix
- $\Lambda$  is an  $n \times n$  diagonal matrix

Construction:

- Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of  $A$  (not necessarily distinct)
- Let  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  be the corresponding eigenvectors (not necessarily linearly indep.)
- For all  $i = 1, 2, \dots, n$ , we know that  $A\vec{v}_i = \lambda_i\vec{v}_i$ .
- We can stack these  $n$  equations in the form of a matrix equation:

$$A \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \dots & \lambda_n\vec{v}_n \end{bmatrix}$$

that is,

$$\underbrace{A \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix}}_T = \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix}}_T \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_\Lambda$$

- Because the  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  are linearly indep., then  $T$  must be invertible. Then

$$A = T\Lambda T^{-1}$$

#### Lemma 7.1 (Eigenvalues of a Hermitian Matrix)

If  $A = A^H$ , then all eigenvalues of  $A$  are real-valued (even if  $A$  has complex entries).

*Proof.* Let  $\lambda$  be an eigenvalue of  $A$  and let  $\vec{x}$  be an eigenvector corresponding to  $\lambda$ . Then

$$\langle A\vec{x}, \vec{x} \rangle = \langle \lambda\vec{x}, \vec{x} \rangle = \lambda \langle \vec{x}, \vec{x} \rangle$$

But also,

$$\langle \vec{x}, A\vec{x} \rangle = \langle \vec{x}, \lambda\vec{x} \rangle = \lambda^* \langle \vec{x}, \vec{x} \rangle$$

Since  $A = A^H$ , then  $\langle A\vec{x}, \vec{x} \rangle$  must equal  $\langle \vec{x}, A\vec{x} \rangle$ , and so this implies that  $\lambda = \lambda^*$ . Thus,  $\lambda$  is real.  $\square$

This lemma does not mean that all the eigenvalues must be distinct (only that they must be real). So what can we say about the eigenvectors? Will they be linearly indep.?

**Lemma 7.2** (Eigenvectors of a Hermitian Matrix)

If  $A = A^H$ , then there exists a set of  $n$  orthonormal eigenvectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  s.t.

$$A\vec{v}_i = \lambda_i \vec{v}_i$$

for all  $i = 1, 2, \dots, n$ .

This result holds even if there are repeated eigenvalues, but it uses the assumption that  $A = A^H$ .

Let  $A$  be an  $n \times n$  matrix and suppose  $A = A^H$ . Then choosing an orthonormal set of eigenvectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  and letting  $T = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$  as before, we have

$$A = T\Lambda T^{-1}$$

However, since the  $\{\vec{v}_i\}$  are orthonormal, then  $T$  is unitary. Therefore,  $T^{-1} = T^H$  and so

$$A = T\Lambda T^H$$

Note: If  $A$  is real, it is possible to choose  $T$  real and have  $A = T\Lambda T^\top$ .

**Example 7.3**

Let

$$A = A^H = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Then  $\lambda_1 = 2$  and  $\lambda_2 = 1$ , both of which are real since  $A = A^H$ . We can derive

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus,  $A = T\Lambda T^H$ , where

$$T = [\vec{v}_1 \ \vec{v}_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

## §7.2 Positive Definite Matrices

Let  $A$  be an  $n \times n$ , Hermitian, symmetric matrix. Recall that we say  $A$  is positive definite if

$$\vec{x}^H A \vec{x} > 0, \quad A \succ 0$$

holds for all non-zero  $\vec{x} \in \mathbb{R}^n$  (or  $\mathbb{C}^n$ ). Similarly, we say that  $A$  is positive semi-definite if

$$\vec{x}^H A \vec{x} \geq 0, \quad A \succeq 0$$

holds for all non-zero  $\vec{x} \in \mathbb{R}^n$  (or  $\mathbb{C}^n$ ). Such matrices are called symmetric, positive (semi-) definite.

If  $A = A^H$ , we already know the eigenvalues of  $A$  are real. Furthermore, if  $A$  is positive definite, then all eigenvalues of  $A$  are positive.

*Proof.* Let  $\vec{v}$  be an eigenvector of  $A$  and let  $\lambda$  be the corresponding eigenvalue. Assume  $\vec{v} \neq \vec{0}$ . Then, because  $A$  is positive definite,

$$\vec{v}^H \underbrace{A\vec{v}}_{\lambda\vec{v}} > 0$$

Substituting,

$$\vec{v}^H(\lambda\vec{v}) > 0 \implies \lambda\vec{v}^H\vec{v} > 0 \implies \lambda > 0$$

because  $\|\vec{v}\| > 0$ . Similarly, if  $A$  is positive semi-definite, then all eigenvalues of  $A$  are non-negative.  $\square$

Positive definite matrices can be used to define variations on the standard  $l_2$  inner product. In particular, suppose  $A$  is a symmetric, positive definite matrix. Then

$$\langle \vec{x}, \vec{y} \rangle_A := \vec{y}^H A \vec{x}$$

defines a valid inner product on  $\mathbb{C}^n$ . Consequently,

$$\|\vec{x}\|_A = \sqrt{\langle \vec{x}, \vec{x} \rangle_A} = \sqrt{\vec{x}^H A \vec{x}}$$

defines a valid induced norm on  $\mathbb{C}^n$ .

Consider the optimization problems

$$\max_{\vec{x} \in \mathbb{C}^n} \frac{\|\vec{x}\|_A^2}{\|\vec{x}\|_2^2} = \max_{\substack{\vec{x} \in \mathbb{C}^n \\ \|\vec{x}\|_2=1}} \vec{x}^H A \vec{x}$$

and

$$\min_{\vec{x} \in \mathbb{C}^n} \frac{\|\vec{x}\|_A^2}{\|\vec{x}\|_2^2} = \min_{\substack{\vec{x} \in \mathbb{C}^n \\ \|\vec{x}\|_2=1}} \vec{x}^H A \vec{x}$$

The maximum value of the first problem is given by  $\lambda_{\max}(A)$  and occurs when  $\vec{x}$  equals the corresponding eigenvector of  $A$ . Similarly, the minimum value of the second problem is given by  $\lambda_{\min}(A)$  and occurs when  $\vec{x}$  equals the corresponding eigenvector of  $A$ .

*Proof.* Refer to the [lecture note](#).  $\square$

Recall the 2-norm (spectral norm) of a matrix  $A$ :

$$\|A\|_2 = \sup_{\substack{\vec{x} \in \mathbb{C}^n \\ \|\vec{x}\|_2=1}} \|A\vec{x}\|_2$$

Note that

$$\|A\vec{x}\|_2 = \sqrt{(A\vec{x})^H A\vec{x}} = \sqrt{\vec{x}^H A^H A \vec{x}} = \|\vec{x}\|_{A^H A}$$

For any matrix  $A$ , it turns out that  $A^H A$  is positive semi-definite. Thus it follows that

$$\|A\|_2 = \sup_{\|\vec{x}\|_2=1} \|\vec{x}\|_{A^H A} = \sqrt{\lambda_{\max}(A^H A)}$$

Now, consider the special case where  $A = A^H$ . In this case,

$$\|\vec{x}\|_{A^H A} = \|\vec{x}\|_{A^2}$$

Also, since  $A = T\Lambda T^H$  with  $T$  unitary, then  $A^2 = T\Lambda^2 T^H$ , and so  $(\lambda_i(A))^2 = \lambda_i(A^2)$ . Thus, when  $A = A^H$  we have

$$\|A\|_2 = \sup_{\|\vec{x}\|_2=1} \|\vec{x}\|_{A^2} = \sqrt{\lambda_{\max}(A^2)} = \max_i |\lambda_i(A)|$$

Similarly, it follows that

$$\|A^{-1}\|_2 = \frac{1}{\min_i |\lambda_i(A)|}$$

Notice

$$\begin{aligned} A = T\Lambda T^H &\implies A^{-1} = T\Lambda^{-1}T^H \implies \lambda_i(A^{-1}) = \frac{1}{\lambda_i(A)} \\ \|A^{-1}\|_2 &= \max_i |\lambda_i(A^{-1})| = \max_i \frac{1}{|\lambda_i(A)|} = \frac{1}{\min_i |\lambda_i(A)|} \end{aligned}$$

## §8 | Lec 8: Apr 14, 2021

### §8.1 Some Properties of Eigenvalues

Let  $A$  be an  $n \times n$  matrix. Then,

- $\det(A) = \prod_{i=1}^n \lambda_i(A)$
- $\text{trace}(A) = \sum_{i=1}^n \lambda_i(A)$

Rank:

- If  $A$  is not full rank, at least one of its eigenvalues must equal 0.
- If  $A = A^H$ , we have seen that we can write  $\underbrace{A = U\Lambda U^H}_{T\Lambda T^H}$  for some unitary  $U$ .
  - $\text{rank}(A) = \# \text{ nonzero eigenvalues of } A$ .
  - Writing  $U = [\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n]$ , we have

$$\begin{aligned} A &= U\Lambda U^H \\ &= \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^H \\ \vdots \\ u_n^H \end{bmatrix} \\ &= \sum_{i=1}^n \lambda_i \vec{u}_i \vec{u}_i^H \end{aligned}$$

If  $A$  is not full rank, some of the terms in this summation equal 0.

### §8.2 Singular Value Decomposition

We've seen that a square Hermitian matrix  $A$  can be factored as

$$A = U\Lambda U^H$$

where  $U$  is orthonormal and  $\Lambda$  is diagonal.

The SVD allows us to generalize this type of factorization to any matrix, even those that are not square. Let  $A$  be an  $m \times n$  matrix, with real or complex entries. Then  $A$  can be factored as

$$A = U\Sigma V^H \leftarrow \text{SVD}$$

where

- $U$  is  $m \times m$  and orthonormal.
- $\Sigma$  is  $m \times n$  and diagonal.
- $V$  is  $n \times n$  and orthonormal.

Let  $p = \min(m, n)$ . Then  $m \times n$  diagonal matrix  $\Sigma$  has the form

$$\begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_p \end{bmatrix} \quad \text{if } m \leq n$$

or

$$\begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_p \end{bmatrix} \quad \text{if } m \geq n$$

Terminology: The elements  $\sigma_1, \sigma_2, \dots, \sigma_p$  are known as the singular values of  $A$ . For any matrix  $A$ , the singular values are always real and non-negative. Thus, it is customary to order them as follows:

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p \geq 0$$

The columns of  $U$  are known as the left singular vectors. The columns of  $V$  are known as the right singular vectors.

The SVD and eigenvalue decomposition are closely related. For any  $m \times n$  matrix  $A$  with SVD given by  $A = U\Sigma V^H$ , consider the matrix

$$A^H A = (U\Sigma V^H)^H (U\Sigma V^H) = V\Sigma^H \underbrace{U^H U}_{=I} \Sigma V^H = V \underbrace{\Sigma^H \Sigma}_{=\Sigma^2} V^H = V\Lambda V^H$$

Let  $\Lambda = \Sigma^H \Sigma = \Sigma^2$ . Then  $\Lambda$  is an  $n \times n$  diagonal matrix with the following entries along the main diagonal

$$\lambda_i = \begin{cases} \sigma_i^2, & i \leq p \\ 0, & i > p \end{cases}$$

Thus, we see that the singular values of  $A$  are the square root of the eigenvalues of  $A^H A$ , and the right singular vectors of  $A$  are the eigenvectors of  $A^H A$ . Similar statements can be made for  $AA^H$ , since

$$AA^H = (U\Sigma V^H) (U\Sigma V^H)^H = U\Sigma V^H V\Sigma^H U^H = U\Sigma \Sigma^H U^H = U\Sigma^2 U^H = U\Lambda U^H$$

Thus, we see that the singular values of  $A$  are the square roots of the eigenvalues of  $AA^H$ , and the left singular vectors of  $A$  are the eigenvectors of  $AA^H$ .

An  $m \times n$  matrix  $A$  can have at most  $p = \min(m, n)$  nonzero singular values. Suppose that a matrix  $A$  has fewer than  $p$  nonzero singular values. In other words, suppose for some  $r < p$  that

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

but that

$$\sigma_{r+1} = \sigma_{r+2} = \dots = \sigma_p = 0$$

That is,  $A$  has only  $r$  nonzero singular values. Then, we can write

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

where  $\Sigma_1$  is  $r \times r$  and diagonal (with non-zeros on the diagonal, and thus it is invertible), and  $\Sigma_2$  is  $(m-r) \times (n-r)$  and all zeros.

We can similarly partition

$$U = [U_1 \ U_2] \text{ and } V^\top = \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}$$

and note that

$$U_1^\top U_1 = I, \quad V_1^\top V_1 = I, \quad U_1^\top U_2 = 0, \quad V_1^\top V_2 = 0$$

This allows us to write the “compact” or “reduced” form of the SVD

$$A = [U_1 \ U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix} = U_1 \Sigma_1 V_1^H$$

where  $U_1$  is  $m \times r$  with orthonormal columns,  $\Sigma_1$  is diagonal with positive real entries along the diagonal, and  $V_1$  is  $n \times r$  with orthonormal columns. Equivalently, we can write

$$A = \sum_{i=1}^r \sigma_i \vec{u}_i \vec{v}_i^\top$$

which decomposes  $A$  as a sum of  $r$  rank-1 matrices.

The SVD reveals the rank of the matrix  $A$  as

$$\text{rank}(A) = r = \# \text{ of nonzero singular values of } A$$

The SVD also reveals the four fundamental subspaces of  $A$

- $\mathcal{R}(A) = \text{colspan}(U_1)$
- $\mathcal{N}(A) = \text{colspan}(V_2)$
- $\mathcal{R}(A^\top) = \text{colspan}(V_1)$
- $\mathcal{N}(A^\top) = \text{colspan}(U_2)$

For example, suppose  $\vec{x} \in \text{colspan}(V_2)$ . Then  $V_1^\top \vec{x} = \vec{0}$  because the columns of  $V$  are orthonormal (and so  $V_1^\top V_2 = 0$ ). So  $A\vec{x} = \vec{0}$ .

### §8.3 Gradient, Hessian, Jacobian, and Chain Rule

- Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a real-valued function of  $n$  variables that is continuously differentiable. The gradient of  $f$  at  $x$ , denoted by,  $\nabla f(x)$ , is

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \in \mathbb{R}^n$$

Here  $\partial f / \partial x_i$  represents the partial derivative of  $f$  with respect to  $x_i$ .

- A gradient with respect to only a subset of the unknowns can be expressed by means of subscript on the symbol  $\nabla$ . For example,  $\nabla_x f(x, z)$  denotes the gradient with respect to  $x$  while holding  $z$  constant.



- If  $f$  is twice continuously differentiable, the matrix of second-order partial derivatives of  $f$  is known as the Hessian, and is defined as

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- The Hessian is a symmetric matrix since  $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$
- When  $f$  is a vector-valued function, that is,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , we define  $\nabla f(x)$  to be the  $n \times m$  matrix whose  $i^{\text{th}}$  columns is  $\nabla f_i(x)$ , that is

$$\nabla f(x) = \begin{bmatrix} \left| \begin{array}{c} \nabla f_1(x) \\ \vdots \end{array} \right| & \left| \begin{array}{c} \nabla f_2(x) \\ \vdots \end{array} \right| & \cdots & \left| \begin{array}{c} \nabla f_m(x) \\ \vdots \end{array} \right| \end{bmatrix}$$

- The rows of the gradient are indexed by variable components, while the columns by function components, and the  $(i, j)^{\text{th}}$  entry is  $[\nabla f(x)]_{ij} = \frac{\partial f_j(x)}{\partial x_i}$ .
- Often it's easier to work with the transpose of this matrix called Jacobian and usually denoted by  $Df(x)$ ,  $J_f(x)$  or  $\frac{\partial(f_1, \dots, f_m)}{\partial(x_1, \dots, x_n)}$

$$Df(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} = (\nabla f(x))^{\top} \in \mathbb{R}^{m \times n}$$

- The rows of Jacobian are indexed by function components, while the columns by variable components, and the  $(i, j)^{\text{th}}$  entry is  $[Df(x)]_{ij} = \frac{\partial f_i(x)}{\partial x_j}$ .

**Question 8.1.** Is  $\nabla^2 f(x) = \nabla(\nabla f(x))$  or  $= D(\nabla f(x))$

Ans: Both! For  $f : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\begin{aligned} \nabla f(x) &= \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \triangleq g(x) \quad g : \mathbb{R}^n \rightarrow \mathbb{R}^n \\ \nabla(\nabla f(x)) &= \nabla g(x) = \begin{bmatrix} \left| \begin{array}{c} \nabla g_1(x) \\ \vdots \end{array} \right| & \left| \begin{array}{c} \nabla g_2(x) \\ \vdots \end{array} \right| & \cdots & \left| \begin{array}{c} \nabla g_n(x) \\ \vdots \end{array} \right| \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} = \nabla^2 f(x) \\ D(\nabla f(x)) &= Dg(x) = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} = \nabla^2 f(x) \end{aligned}$$

## §9 | Lec 9: Apr 16, 2021

### §9.1 Lec 8 (Cont'd)

Chain Rule: Let  $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $f : \mathbb{R}^m \rightarrow \mathbb{R}^p$ , and their composition  $h = f \circ g : \mathbb{R}^n \rightarrow \mathbb{R}^p$  be defined via  $h(x) = f(g(x))$  for  $x \in \mathbb{R}^n$ . Then

$$\begin{aligned}\nabla h(x) &= [\dots \quad \nabla h_i(x) \quad \dots] = [\dots \quad \sum_{j=1}^m \frac{\partial f_i}{\partial g_j} \nabla g_j(x) \quad \dots] \\ &= [\dots \quad \nabla g(x) \nabla f_i(g) \quad \dots] \\ &= \nabla g(x) \nabla f(g(x))\end{aligned}$$

Or simply

$$\nabla(f \circ g) = \nabla g \nabla f(g) \implies (\nabla(f \circ g))^\top = (\nabla g \nabla f(g))^\top = (\nabla f(g))^\top (\nabla g)^\top$$

Or in terms of Jacobian,

$$J_{f \circ g}(x) = J_f(g(x)) J_g(x), \quad D(f \circ g) = Df Dg$$

#### Example 9.1

Calculate the gradient and Hessian for  $f(x) = x_1^3 + 3x_1x_2^2$

- Gradient:

$$\nabla f(x) = \begin{bmatrix} 3x_1^2 + 3x_2^2 \\ 6x_1x_2 \end{bmatrix}$$

- Hessian:

$$\nabla^2 f(x) = \begin{bmatrix} 6x_1 & 6x_2 \\ 6x_2 & 6x_1 \end{bmatrix}$$

**Example 9.2**

Calculate the gradient and Jacobian for  $f(x) = Ax + b$  for  $A \in \mathbb{R}^{m \times n}$ .

Notice  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , denote

$$A = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}, f(x) = Ax + b = \begin{bmatrix} a_1^\top x + b_1 \\ \vdots \\ a_m^\top x + b_m \end{bmatrix}$$

- Gradient:

$$g(x) \triangleq a^\top x = \sum_{i=1}^n \alpha_i \gamma_i = a_1 \gamma_1 + a_2 \gamma_2 + \dots$$

$$\nabla g(x) = \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a$$

$$\nabla f(x) = [\nabla f_1(x) \quad \dots \quad \nabla f_m(x)] = [a_1 \quad \dots \quad a_m] = A^\top$$

- Jacobian:

$$Df(x) = [\nabla f(x)]^\top = A$$

# §10 | Lec 10: Apr 19, 2021

## §10.1 Lec 9 (Cont'd)

### Example 10.1

How about  $f(A) = Ax + b$ ?

Hint:

1. Vectorize  $A$
2. Connect it to Ex 2.

Have:

$$\begin{aligned}
 A &= \begin{bmatrix} | & & | \\ a_1 & \dots & a_n \\ | & & | \end{bmatrix}, \text{vec}(A) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \\
 f(\text{vec}(A)) &= f(A) = Ax + b = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + b \\
 &= \sum_{i=1}^n x_i a_i + b = \sum_{i=1}^n (x_i I) a_i + b \\
 &= \begin{bmatrix} x_1 I & \dots & x_n I \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + b
 \end{aligned}$$

So

$$\begin{aligned}
 Df(A) &= \begin{bmatrix} x_1 I & \dots & x_n I \end{bmatrix} \\
 &= x^\top \otimes I
 \end{aligned}$$

where  $\otimes$  denotes the Kronecker product and

$$\underbrace{A}_{m \times n} \otimes \underbrace{B}_{p \times q} = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} = \underbrace{C}_{mp \times nq}$$

So

$$\nabla f(A) = \begin{bmatrix} x_1 I \\ \vdots \\ x_n I \end{bmatrix} = x \otimes I$$

**Example 10.2**

Calculate the gradient and Hessian for  $f(x) = \frac{1}{2}\|Ax - b\|_2^2$

Hint: Chain rule

$$\underbrace{g(x) \triangleq Ax - b}_{g: \mathbb{R}^n \rightarrow \mathbb{R}^n}, \quad \underbrace{h(g) = \frac{1}{2}\|g\|_2^2}_{h: \mathbb{R}^m \rightarrow \mathbb{R}} \implies f(x) = h(g(x))$$

$$\nabla f(x) = \nabla g(x) \nabla h(g(x)) = A^\top g(x) = A^\top (Ax - b)$$

$$\nabla^2 f(x) = D(\nabla f(x)) = D\left(A^\top (Ax - b)\right) = D\left(A^\top Ax\right) = A^\top A$$

**§10.2 Taylor's Theorem**

The foundational result for many algorithms in smooth nonlinear optimization is Taylor's theorem. Taylor's theorem shows how smooth functions can be approximated locally by low-order (linear or quadratic) functions. The next iterate of many iterative algorithms can be obtained by minimizing a local approximation of the objective function around the previous iterate. Therefore, the convergence property of these algorithms on the accuracy of this approximation.

Given a continuously differentiable function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  and  $x, p \in \mathbb{R}^n$ , we have

$$\text{mean-value version: } f(x+p) = f(x) + \underbrace{\nabla f(x+tp)^\top}_{\text{slope in scalar case}} p \text{ for some } t \in (0, 1)$$

$$\text{integral version: } f(x+p) = f(x) + \int_0^1 \nabla f(x+tp)^\top p \, dt$$

If  $f$  is twice continuously differentiable then

$$\nabla f(x+p) = \nabla f(x) + \int_0^1 \nabla^2 f(x+tp) p \, dt$$

$$f(x+p) = f(x) + \nabla f(x)^\top p + \frac{1}{2} p^\top \nabla^2 f(x+tp) p \text{ for some } t \in (0, 1) \quad (\star)$$

$\star$  is also known as **Taylor's Expansion Theorem**. We can use it to compute gradient and Hessian.

**Example 10.3**

Compute  $\nabla f(x)$  and  $\nabla^2 f(x)$  with Taylor's Expansion Theorem where  $f(x) = \frac{1}{2}\|Ax - b\|_2^2$  for  $A \in \mathbb{R}^{m \times n}$ .

Have

$$\begin{aligned}
 f(x) &= \frac{1}{2}(Ax - b)^\top (Ax - b) \\
 &= \frac{1}{2}x^\top A^\top Ax - \frac{1}{2}b^\top Ax - \frac{1}{2}x^\top A^\top b + \frac{1}{2}b^\top b \\
 &= \underbrace{\frac{1}{2}x^\top A^\top Ax}_{g(x)} - \underbrace{b^\top Ax}_{\nabla(b^\top Ax)} + \frac{1}{2}b^\top b \\
 g(x+p) &= \frac{1}{2}(x+p)^\top A^\top A(x+p) \\
 &= \underbrace{\frac{1}{2}x^\top A^\top Ax}_{g(x)} + \underbrace{\frac{1}{2}p^\top A^\top Ax + \frac{1}{2}x^\top A^\top Ap}_{=x^\top A^\top Ap} + \frac{1}{2}p^\top \underbrace{A^\top A}_{\nabla^2 g(x)} p
 \end{aligned}$$

Then

$$\begin{aligned}
 \nabla f(x) &= A^\top Ax - A^\top b = A^\top (Ax - b) \\
 \nabla^2 f(x) &= A^\top A
 \end{aligned}$$

# §11 | Lec 11: Apr 21, 2021

## §11.1 Taylor's Theorem (Cont'd)

A brief intro to norm: a norm  $\xi$  is a function from a vector space to the non-negative real numbers, that satisfies the following three properties for any  $x, y \in \mathbb{R}^n$ .

- Non-negative:  $\xi(x) \geq 0$  and  $\xi(x) = 0 \iff x = 0$ .
- Absolutely homogeneous:  $\xi(ax) = |a|\xi(x)$  for any  $a \in \mathbb{R}$ .
- Triangle inequality:  $\xi(x + y) \leq \xi(x) + \xi(y)$

Common examples: the  $l_p$  norm ( $p \geq 1$ ) in  $\mathbb{R}^n$  is  $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ .  
When  $p = 2$ , we have

$$\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$$

which for convenience is often written as  $\|x\|$ . It defines the Euclidean distance of a vector and is by far the most commonly used norm in  $\mathbb{R}^n$ .

**Fact 11.1.**  $\|x\| = \sqrt{x^\top x} = \sqrt{\langle x, x \rangle}$

Cauchy-Schwarz Inequality:

$$|\langle x, y \rangle| \leq \|x\|_2 \cdot \|y\|_2 \quad (x, y \in \mathbb{R}^n)$$

A crucial quantity in optimization is the Lipschitz constant  $L$  for the gradient of  $f$ , which is defined to satisfy

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \leftarrow l_2\text{-norm}$$

for all  $x, y$  in the domain of  $f$ .

Given  $f$  with  $\nabla f$  uniformly Lipschitz continuous with constant  $L$ , we have for any  $x, y$  in the domain of  $f$  that

$$f(y) \leq \underbrace{f(x) + \nabla f(x)^\top (y - x) + \frac{L}{2}\|y - x\|_2^2}_{\triangleq Q(y)}$$

where “ $\leq$ ” is sometimes known as the quadratic upper bound.

*Proof.* Denote  $p = y - x$ . Using the integral-version of Taylor's theorem,

$$\begin{aligned} f(y) &= f(x + p) = f(x) + \int_0^1 \nabla f(x + tp)^\top p \, dt \\ &= f(x) + \int_0^1 (\nabla f(x + tp) - \nabla f(x) + \nabla f(x))^\top p \, dt \\ &= f(x) + \nabla f(x)^\top (y - x) + \int_0^1 (\nabla f(x + tp) - \nabla f(x))^\top p \, dt \end{aligned}$$

$$\begin{aligned} \left| (\nabla f(x + t(y - x)) - \nabla f(x))^\top (y - x) \right| &\leq \|\nabla f(x + t(y - x)) - \nabla f(x)\|_2 \cdot \|y - x\|_2 \\ &\leq Lt\|y - x\|_2^2 \end{aligned}$$

Finally,

$$\begin{aligned} f(y) &\leq f(x) + \nabla f(x)^\top (y - x) + \int_0^1 Lt \|y - x\|_2^2 dt \\ &= f(x) + \nabla f(x)^\top (y - x) + \frac{1}{2} \|y - x\|_2^2 \end{aligned}$$

□

This means that  $f$  can be upper bounded by a quadratic function whose value at  $x$  is equal to  $f(x)$ . If  $f$  is twice continuously differentiable, then  $\nabla f$  is Lipschitz continuous with Lipschitz constant  $L \iff$  for  $x$  we have all the eigenvalues of  $\nabla^2 f(x)$  are between  $-L$  and  $L$ .

$$-L \leq \lambda_{\min}(\nabla^2 f(x)) \leq \lambda_{\max}(\nabla^2 f(x)) \leq L$$

## §11.2 Solution and Optimality Conditions

Let  $f : \mathcal{D} \rightarrow \mathbb{R}$  where  $\mathcal{D} \subseteq \mathbb{R}^n$ . We define different notions of minimizers.

- $x^* \in \mathcal{D}$  is a local minimizer of  $f$  if there is a neighborhood  $\mathcal{N}$  of  $x^*$  s.t.  $f(x) \geq f(x^*)$  for all  $x \in \mathcal{N} \cap \mathcal{D}$ .
- $x^* \in \mathcal{D}$  is a global minimizer of  $f$  if  $f(x) \geq f(x^*)$  for all  $x \in \mathcal{D}$ .
- $x^* \in \mathcal{D}$  is a strict local minimizer if it is a local minimizer and in addition  $f(x) > f(x^*)$  for all  $x \in \mathcal{N}$  with  $x \neq x^*$ .

**Definition 11.1 (Critical/Stationary Point)** — A point  $x^*$  with  $\nabla f(x^*) = 0$  is called a critical point or a stationary point.

### Theorem 11.2 (Necessary Conditions for Smooth Unconstrained Optimization)

There are two necessary conditions:

1. Suppose that  $f$  is continuously differentiable. Then if  $x^*$  is a local minimizer of the unconstrained optimization  $\min_x f(x)$ , then  $\nabla f(x^*) = 0$ .
2. Suppose that  $f$  is twice continuously differentiable. Then if  $x^*$  is a local minimizer of the unconstrained optimization  $\min_x f(x)$ , then  $\nabla f(x^*) = 0$  and  $\nabla^2 f(x^*)$  is positive semi-definite, i.e.  $\nabla^2 f(x^*) \succeq 0$

(1) is called the first-order necessary condition and (2) is called the second-order necessary condition.

A matrix  $A$  is positive semi-definite if

1.  $A$  is symmetric.
2.  $\text{eigs}(A)$  are non-negative.
3. For all nonzero  $x \in \mathbb{R}^n$ ,  $x^\top A x \geq 0$ .



## §12 | Lec 12: Apr 23, 2021

### §12.1 Solution and Optimality Conditions (Cont'd)

*Proof.* 1. Assume  $x^*$  is a local min but  $\nabla f(x^*) \neq 0$ . Consider  $x(\alpha) = x^* - \alpha \nabla f(x^*)$ ,  $\alpha > 0$

$$f(x(\alpha)) = f(x^* - \alpha \nabla f(x^*)), \quad \alpha > 0$$

$$\begin{aligned} \text{Taylor's (mean-value)} &= f(x^*) + \nabla f(x^* - \alpha t \nabla f(x^*))^\top (-\alpha \nabla f(x^*)), \text{ for some } t \in (0, 1) \\ &= f(x^*) - \alpha \nabla f(x^* - \alpha t \nabla f(x^*))^\top \nabla f(x^*) \end{aligned}$$

Have

$$\lim_{\alpha \rightarrow 0} \nabla f(x^* - \alpha t \nabla f(x^*))^\top \nabla f(x^*) = \|\nabla f(x^*)\|_2^2 > 0$$

By definition of limit, there exists  $\alpha_1$  s.t. for all  $\alpha \in [0, \alpha_1)$ , we have

$$\begin{aligned} \nabla f(x^* - \alpha t \nabla f(x^*))^\top \nabla f(x^*) &> \frac{1}{2} \|\nabla f(x^*)\|_2^2 \\ \implies f(x(\alpha)) &< f(x^*) - \underbrace{\frac{1}{2} \alpha \|\nabla f(x^*)\|_2^2}_{>0} < f(x^*), \quad \forall \alpha \in [0, \alpha_1) \end{aligned}$$

2. By (1), we have  $\nabla f(x^*) = 0$ . Suppose  $\nabla^2 f(x^*) \not\geq 0$ . Then  $\exists v \neq 0$ , s.t.  $v^\top \nabla^2 f(x^*) v = \lambda < 0$ . Consider  $x(\alpha) = x^* + \alpha v$

$$f(x(\alpha)) = f(x^* + \alpha v)$$

$$\begin{aligned} \text{Taylor's Thm} &= f(x^*) + \nabla f(x^*)^\top (\alpha v) + \frac{1}{2} \alpha^2 v^\top \nabla^2 f(x^* + \alpha t v) v, \text{ for some } t \in (0, 1) \\ &= f(x^*) + \frac{1}{2} \alpha^2 v^\top \nabla^2 f(x^* + \alpha t v) v, \text{ for some } t \in (0, 1) \end{aligned}$$

Notice

$$\lim_{\alpha \rightarrow 0} v^\top \nabla^2 f(x^* + \alpha t v) v = v^\top \nabla^2 f(x^*) v = \lambda < 0$$

By definition of limit, there exists  $\alpha_1$  s.t. for all  $\alpha \in [0, \alpha_1)$ , we have

$$v^\top \nabla^2 f(x^* + \alpha t v) v < \frac{\lambda}{2} \implies f(x(\alpha)) < f(x^*), \quad \forall \alpha \in [0, \alpha_1) \quad \square$$

#### Theorem 12.1 (Sufficient Condition for Smooth Unconstrained Optimization)

Suppose that  $f$  is twice continuously differentiable and for some  $x^*$ , we have  $\nabla f(x^*) = 0$  and  $\nabla^2 f(x^*)$  is positive definite. Then  $x^*$  is a strict local minimizer of the unconstrained optimization  $\min_x f(x)$ .

*Proof.* Have

$$\begin{aligned}\nabla^2 f(x^*) \succ 0 &\implies \nabla^2 f(x^* + tp) \succ 0 \text{ for any } \|p\| \leq r, t \in (0, 1) \\ f(x^* + p) &= f(x^*) + \nabla f(x^*)^\top p + \frac{1}{2} p^\top \nabla^2 f(x^* + tp) p, \text{ for some } t \in (0, 1) \\ &= f(x^*) + \frac{1}{2} p^\top \underbrace{\nabla^2 f(x^* + tp)}_{>0} p, \text{ for some } t \in (0, 1) \\ &> f(x^*), \text{ for any } \|p\| \leq r\end{aligned}$$

Choose  $\mathcal{N} = \{x^* + p : \|p\| \leq r\}$ . □

**Definition 12.2 (Local Maximizer)** — If  $\nabla f(x^*) = 0$  and all the eigenvalues of  $\nabla^2 f(x^*)$  are negative,  $x^*$  is a local maximizer.

**Definition 12.3 (Saddle Point)** — If  $\nabla f(x^*) = 0$  and  $\nabla^2 f(x^*)$  has both positive and negative eigenvalues,  $x^*$  is a saddle points.

## §12.2 Midterm

- Date: Next Friday, Apr 30, 2021 at 9:00 am
- Deadline: May 1, 2021 at 9:00 am
- Materials covered: Lec 1, Lec 2.1 – Lec 2.4 on CCLE
- Review:
  - Notes/Examples on slides
  - Questions in HW1 & HW2
- Open note/book.

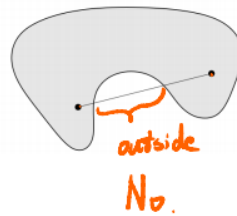
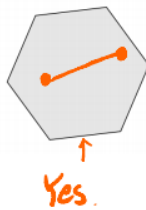
## §12.3 Convexity

Convex functions take a central role in optimization – they are the class of functions that are guaranteed to find global minimizer within a reasonable amount of time.

**Definition 12.4 (Convex Set)** — A set  $\omega \subseteq \mathbb{R}^n$  is convex if for any  $x, y \in \omega$  and any  $\alpha \in [0, 1]$ , one has  $(1 - \alpha)x + \alpha y \in \omega$ .

Geometrically, for all pairs of points in a convex set, the line segment between them is also contained in the set.

Which of the following sets are convex?



### Example 12.5

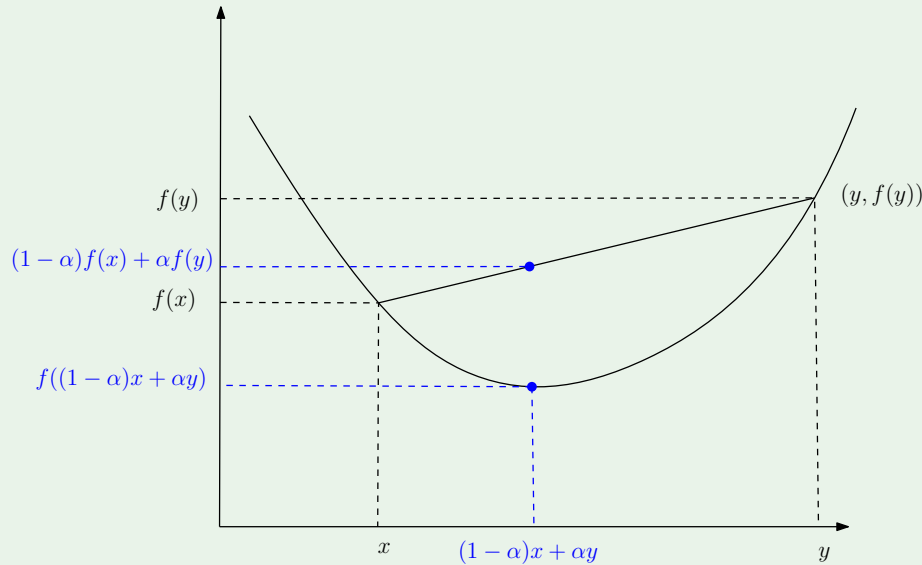
Examples of convex sets:

- Hyperplane:  $\{x : a^\top x = b\}$  with  $a \neq 0 \in \mathbb{R}^n$  and  $b \in \mathbb{R}$
- Halfspace:  $\{x : a^\top x \leq b\}$  with  $a \neq 0 \in \mathbb{R}^n$  and  $b \in \mathbb{R}$
- Norm ball:  $\{x : \|x - x_c\| \leq r\}$
- Non-negative Orthant:  $\mathbb{R}_+^n = \{x : x \geq 0\}$
- Positive semi-definite cone:  $S_+^n = \{X : X \text{ is symmetric, } X \succeq 0\}$

**Definition 12.6 (Convex Function)** — A function  $f : \omega \rightarrow \mathbb{R}$  where  $\omega \subseteq \mathbb{R}^n$  is a convex function if its domain  $\omega$  is a convex set and for all  $x, y \in \omega$  and all  $\alpha \in [0, 1]$ , one has

$$f((1 - \alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y)$$

Geometrically, for convex function, the line segment connecting  $(x, f(x))$  and  $(y, f(y))$  lie above the graph of the function  $f$ .

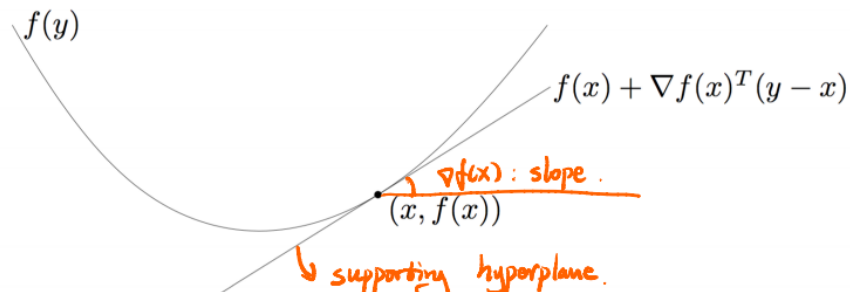


A function  $f$  is concave if  $-f$  is convex.

A continuously differentiable function  $f$  is convex iff its domain is convex and for all  $x, y \in \text{dom}(f)$

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

A twice continuously differentiable function  $f$  is convex iff its domain is convex and its Hessian is positive semi-definite, that is,  $\nabla^2 f(x) \succeq 0$  for all  $x \in \text{dom}(f)$ .



## §13 | Lec 13: Apr 26, 2021

### §13.1 Strongly Convex Functions

When  $f : \omega \rightarrow \mathbb{R}$  is continuously differentiable, we call  $f$  is strongly convex with modulus of convexity  $m$  if  $\omega$  is convex and there exists  $m > 0$  s.t.

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \frac{m}{2} \|y - x\|_2^2$$

Just as the Lipschitz constant of the gradient provides an upper bound on the eigenvalues of the Hessian, the strongly convex parameter provides a lower bound for twice continuously differentiable functions.

More precisely, suppose  $f$  is twice continuously differentiable, then  $f$  has modulus of convexity  $m \iff \lambda_{\min}(\nabla^2 f(x)) \geq m > 0$  for all  $x$ .

**Exercise 13.1.** Answer whether or when the following functions are (strongly) convex. Hint:

1. Check domain of  $f$  is convex.
  2. Compute  $\nabla^2 f(x)$
  3. If  $\lambda_{\min}(\nabla^2 f(x)) \geq m > 0 \implies$  strongly convex.
  4.  $\lambda_{\min}(\nabla^2 f(x)) \geq 0 \implies$  convex.
- $x^2, x \in \mathbb{R}$ 
    1. Domain is convex
    2.  $\nabla f(x) = 2x \implies \nabla^2 f(x) = 2 \implies$  strongly convex
  - $x^4, x \in \mathbb{R}$ 
    1. Domain is convex
    2.  $\nabla f(x) = 4x^3, \nabla^2 f(x) = 12x^2 \geq 0 \implies$  convex
  - $e^{ax}$  on  $\mathbb{R}$ 
    1. Domain is convex
    2.  $\nabla f(x) = ae^{ax}, \nabla^2 f(x) = a^2 e^{ax} \geq 0 \implies$  convex
  - $-\log(x)$  on  $\underbrace{\mathbb{R}_{++}}_{x>0}$ 
    1. Domain is convex
    2.  $\nabla f(x) = -\frac{1}{x}, \nabla^2 f(x) = \frac{1}{x^2} \geq 0 \implies$  convex
  - $f(x) = \|x\|_2^2, x \in \mathbb{R}^n$ 
    1. Domain is convex
    2.  $\nabla f(x) = 2x, \nabla^2 f(x) = 2I \implies$  strongly convex

- $f(x) = a^\top x + b, x \in \mathbb{R}^n$ 
  1. Domain is convex
  2.  $\nabla f(x) = a, \nabla^2 f(x) = 0 \implies$  convex and concave
- $f(x) = \frac{1}{2}x^\top Px + q^\top x + r, x \in \mathbb{R}^n$  and  $P = P^\top$ 
  1. Domain is convex
  2.  $\nabla f(x) = Px + q, \nabla^2 f(x) = P$ . If  $P \succeq 0 \implies$  convex. If  $P \succ 0 \implies$  strongly convex. If  $P \preceq 0 \implies$  concave. If  $P \prec 0 \implies$  strongly concave

**Theorem 13.1**

Suppose  $f$  is continuously differentiable and convex. Then if  $\nabla f(x^*) = 0$ , then  $x^*$  is a global minimizer of  $\min_x f(x)$ . When, in addition,  $f$  is strongly convex, then  $x^*$  is the unique global minimizer.

*Proof.* Take  $x = x^*$ ,

$$f(y) \geq f(x^*) + \underbrace{\nabla f(x^*)^\top}_{=0} (y - x^*) = f(x^*), \quad \forall y$$

for convex  $f$ , and

$$f(y) \geq f(x^*) + \underbrace{\nabla f(x^*)^\top}_{=0} (y - x^*) + \frac{m}{2} \|y - x^*\|_2^2 = f(x^*) + \underbrace{\frac{m}{2} \|y - x^*\|_2^2}_{>0 \text{ if } y \neq x^*} > f(x^*)$$

for strongly convex  $f$  and  $y \neq x^*$ . □

## §14 | Lec 14: Apr 28, 2021

### §14.1 Examples of Finding Global Minimizers

#### Example 14.1

$$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = (x - 3)^2$$

1. Is  $f(x)$  strongly convex?
  2. Find the global minimizers of  $f(x)$ . Is it unique?
- $\text{dom}(f) = \mathbb{R}$  is a convex set
  - $\nabla f(x) = 2(x - 3)$ ,  $\nabla^2 f(x) = 2 \implies f(x)$  is a strongly convex function.  
 $\nabla f(x) = 0 \implies x = 3$  is the unique global minimizer of  $f(x)$ .

#### Example 14.2

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, f(x) = \|x - x^*\|_2^2, x^* \in \mathbb{R}^n \text{ is given.}$$

1. Is  $f(x)$  (strongly) convex?
  2. Find the global minimizers of  $f(x)$ . Is it unique?
- $\text{dom}(f) = \mathbb{R}^n$  is a convex set
  - Consider:

$$\begin{aligned} f(x + d) &= \|x + d - x^*\|_2^2 = \|(x - x^*) + d\|_2^2 \\ &= \langle (x - x^*) + d, (x - x^*) + d \rangle \\ &= \langle x - x^*, x - x^* \rangle + \langle x - x^*, d \rangle + \langle d, x - x^* \rangle + \langle d, d \rangle \\ &= \underbrace{\|x - x^*\|_2^2}_{f(x)} + \underbrace{\langle 2(x - x^*), d \rangle}_{\nabla f(x)} + \frac{1}{2} d^\top \underbrace{\left( \begin{matrix} 2I \end{matrix} \right)}_{\nabla^2 f(x)} d \end{aligned}$$

$\implies f(x)$  is a strongly convex function, so  $\nabla f(x) = 2(x - x^*) = 0 \implies x = x^*$  is the unique global minimizer.

**Example 14.3**

$f : \mathbb{R}^d \rightarrow \mathbb{R}$ . Given  $x_1, \dots, x_n \in \mathbb{R}^d$ . Find the global minimizer of

$$f(x) = \sum_{k=1}^n \|x - x_k\|_2^2$$

Hint: Show  $f(x)$  is a convex function, so any critical point is a global minimizer.

- $\text{dom}(f = \mathbb{R}^d)$  is a convex set.
- Consider

$$\begin{aligned} f(x+d) &= \sum_{k=1}^n \|x+d-x_k\|_2^2 = \sum_{k=1}^n \|(x-x_k)+d\|_2^2 \\ &= \sum_{k=1}^n \left( \|x-x_k\|_2^2 + \langle 2(x-x_k), d \rangle + \frac{1}{2}d^\top (2I)d \right) \\ &= \underbrace{\sum_{k=1}^n \|x-x_k\|_2^2}_{f(x)} + \underbrace{\langle 2 \sum_{k=1}^n (x-x_k), d \rangle}_{\nabla f(x)} + \frac{1}{2}d^\top \underbrace{(2nI)}_{\nabla^2 f(x)} d \end{aligned}$$

$\implies f(x)$  is a strongly convex function.

$$\nabla f(x) = 2nx - 2 \sum_{k=1}^n x_k = 0 \implies x = \frac{1}{n} \sum_{k=1}^n x_k$$

is the unique global minimizer of  $f(x)$ .



§15 | Midterm: Apr 30, 2021 – :D



Figure 1: Better study now!