Math 131BH – Honors Real Analysis II

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Spring 2021

This is math 131BH – Honors Real Analysis II, and it is instructed by Professor Visan. It's the second class in the undergrad real analysis sequence at UCLA. We meet weekly on MWF from 10:00 – 10:50 am for online lectures. Similar to 131AH, there are two textbooks associated to the course, *Principles of Mathematical Analysis* by *Rudin* and *Metric Spaces* by *Copson*. You can find the previous analysis lecture notes along with the other course notes through my github. Please email me if you notice any significant mathematical errors/typos that needs to be addressed. Thank you, and I hope you find this helpful for your study!

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§1 Lec 1: Mar 29, 2021

§1.1 Compactness

Definition 1.1 (Open Cover) — Let (X, d) be a metric space and let $A \subseteq X$. An open cover of A is a family $\{G_i\}_{i\in I}$ of open sets in X such that

$$A \subseteq \bigcup_{i\in I} G_i$$

The open cover is called finite if the cardinality of I is finite. If it's not finite, the open cover is called infinite.

Definition 1.2 (Compactness & Precompactness) — Let (X, d) be a metric space and let $K \subseteq X$.

1. We say that K is a compact set if every open cover $\{G_i\}_{i\in I}$ of K admits a finite subcover, that is,

$$\exists n \geq 1 \text{ and } \exists i_1, \ldots, i_n \in I \text{ s.t. } K \subseteq \bigcup_{j=1} G_{i_j}$$

2. We say that a set $A \subseteq X$ is precompact if $\overline{A}$ is compact.

Lemma 1.3
Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$. We equip Y with the induced metric $d_1 : Y \times Y \to \mathbb{R}$, $d_1(y_1, y_2) = d(y_1, y_2)$. Let $K \subseteq Y \subseteq X$. The followings are equivalent:

1. K is compact in (X, d).
2. K is compact in (Y, d_1).

Proof. 1) $\implies$ 2) Assume K is compact in (X, d). Let $\{V_i\}_{i\in I}$ be a family of open sets in (Y, d_1) s.t.

$$K \subseteq \bigcup_{i\in I} V_i$$

For $i \in I$ fixed, V_i is open in $(Y, d_1) \implies \exists G_i \subseteq X$ open in (X, d) s.t.

$$V_i = G_i \cap Y$$

Then

$$\left.\begin{aligned} &K \subseteq \bigcup_{i\in I} V_i \subseteq \bigcup_{i\in I} G_i \\ &K \text{ compact in } (X,d) \end{aligned}\right\} \implies \exists n \geq 1 \text{ and } \exists i_1,\dots,i_n \in I \text{ s.t.}$$

$$\left.\begin{aligned} &K \subseteq \bigcup_{j=1}^n G_{i_j} \\ &K \subseteq Y \end{aligned}\right\} \implies K \subseteq \left(\bigcup_{j=1}^n G_{i_j}\right) \cap Y = \bigcup_{j=1}^n \left(G_{i_j} \cap Y\right) = \bigcup_{j=1}^n V_{i_j}$$

So K is compact in (Y, d_1).
2) $\implies$ 1) Assume K is compact in (Y, d_1). Let $\{G_i\}_{i\in I}$ be a family of open sets in (X,d) s.t.

$$\left.\begin{aligned} &K \subseteq \bigcup_{i\in I} G_i \\ &K \subseteq Y \end{aligned}\right\} \implies \left.\begin{aligned} &K \subseteq \left(\bigcup_{i\in I} G_i\right) \cap Y = \bigcup_{i\in I} \underbrace{(G_i \cap Y)}_{\text{open in } Y} \\ &K \text{ is compact in } (Y, d_1) \end{aligned}\right\} \implies$$

$$\implies \exists n \geq 1 \text{ and } \exists i_1,\dots,i_n \in I \text{ s.t. } K \subseteq \bigcup_{j=1}^n \left(G_{i_j} \cap Y\right) \subseteq \bigcup_{j=1}^n G_{i_j}.$$

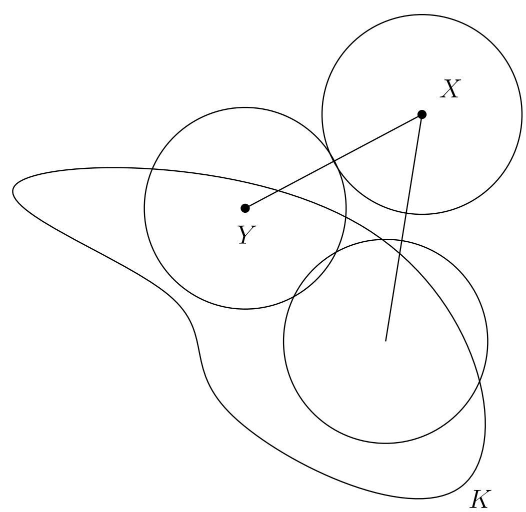
□

Proposition 1.4

Let (X,d) be a metric space and let $K \subseteq X$ be compact. Then K is closed and bounded.

Proof. Let's prove K is closed. We'll show ${}^cK$ is open.
Case 1: ${}^cK = \emptyset$. This is open.
Case 2: ${}^cK \neq \emptyset$. Let $x \in {}^cK$

For $y \in K$ let $r_y = \frac{d(x,y)}{2}$. Note $r_y > 0$ (since $x \in {}^cK$ and $y \in K$).



Note

$$\left.\begin{aligned} &K \subseteq \bigcup_{y\in K} \underbrace{B_{r_y}(y)}_{\text{open}} \\ &K \text{ is compact} \end{aligned}\right\} \implies \exists n \geq 1 \text{ and } \exists y_1,\dots,y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^n B_{r_j}(y_j)$$

where we use the shorthand $r_j = r_{y_j}$.
Let $r = \min_{1\le j\le n} r_j > 0$.
By construction, $B_r(x) \cap B_{r_j}(y_j) = \emptyset \quad \forall 1 \le j \le n$.

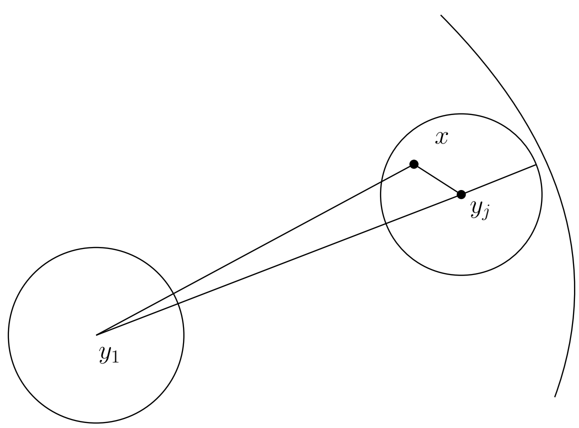
$$\begin{aligned}
&\implies B_r(x) \subseteq {}^cB_{r_j}(y_j) \quad \forall 1 \le j \le n \\
&\implies B_r(x) \subseteq \bigcap_{j=1}^n {}^cB_{r_j}(y_j) = {}^c\left(\bigcup_{j=1}^n B_{r_j}(y_j)\right) \subseteq {}^cK \\
&\implies \left.\begin{array}{l} x \in \widehat{{}^cK}^{\circ} \\ x \in {}^cK \text{ was arbitrary} \end{array}\right\} \implies {}^cK = \widehat{{}^cK}^{\circ}
\end{aligned}$$

Let's show K is bounded. Note

$$\left.\begin{array}{l} K \subseteq \bigcup_{y\in K} \underbrace{B_1(y)}_{\text{open}} \\ K \text{ compact} \end{array}\right\} \implies \exists n \ge 1 \text{ and } \exists y_1, \ldots, y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^n B_1(y_j)$$

For $2 \le j \le n$, let $r_j = d(y_1, y_j) + 1$.

Claim 1.1. $B_1(y_j) \subseteq B_{r_j}(y_1)$



Indeed, if $x \in B_1(y_j) \implies d(x, y_j) < 1$. By the triangle inequality

$$d(y_1, x) \le d(y_j, x) + d(y_1, y_j) < 1 + d(y_1, y_j) = r_j \implies x \in B_{r_j}(y_1)$$

So with $r = \max_{2\le j\le n} r_j$,

$$K \subseteq \bigcup_{j=1}^n B_1(y_j) \subseteq B_r(y_1)$$

$\square$

Proposition 1.5

Let (X, d) be a metric space and let $F \subseteq K \subseteq X$ such that F is closed in X and K is compact. Then F is compact.

Proof. Let $\{G_i\}_{i\in I}$ be a family of open sets in X s.t.

$$F \subseteq \bigcup_{i\in I} G_i$$

Then

$$\left.\begin{array}{l} K \subseteq F \cup {}^cF \subseteq \bigcup_{i\in I} G_i \cup \underbrace{{}^cF}_{\text{open in } X} \\ K \text{ compact} \end{array}\right\} \implies$$

$\implies \exists n \geq 1$ and $\exists i_1, \dots, i_n \in I$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j=1}^n G_{i_j} \cup {}^cF \\ F \subseteq K \end{array}\right\} \implies F = \left(\bigcup_{j=1}^n G_{i_j} \cup {}^cF\right) \cap F \subseteq \bigcup_{j=1}^n G_{i_j}$$

So F is compact. $\square$

Corollary 1.6

Let (X, d) be a metric space and let $F \subseteq X$ be closed and let $K \subseteq X$ be compact. Then $K \cap F$ is compact.

Proof. K is compact. So

$$\left.\begin{array}{l} K \text{ closed} \\ F \text{ closed} \end{array}\right\} \implies \left.\begin{array}{l} K \cap F \text{ is closed} \\ K \cap F \subseteq K \text{ compact} \end{array}\right\} \implies K \cap F \text{ is compact}$$

$\square$

§1.2 Sequential Compactness

Definition 1.7 (Sequential Compactness) — Let (X, d) be a metric space. A set $K \subseteq X$ is called <u>sequentially compact</u> if every sequence $\{x_n\}_{n\geq 1} \subseteq K$ admits a subsequence that converges in K.

§2 Lec 2: Mar 31, 2021

§2.1 Sequential Compactness (Cont'd)

Theorem 2.1 (Bolzano – Weierstrass)
Let (X, d) be a metric space and let $K \subseteq X$ be infinite. The following are equivalent:

1. K is sequentially compact.

2. For every infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

Proof. 1) $\implies$ 2) Let $A \subseteq K$ be infinite. As every infinite set has a countable subset we can find a sequence $\{a_n\}_{n\geq 1} \subseteq A$ such that $a_n \neq a_m \, \forall n \neq m$. As K is sequentially compact, $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

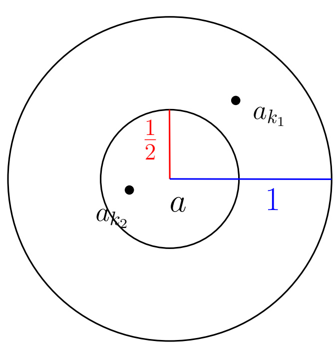
$$a_{k_n} \xrightarrow[n\to\infty]{d} a \in K$$

Claim 2.1. $a \in A' \iff \forall r > 0 \; B_r(a) \cap A \setminus \{a\} \neq \emptyset$.

Indeed, fix $r > 0$.

$$a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_r \in \mathbb{N} \text{ s.t. } d(a, a_{k_n}) < r \quad \forall n \geq n_r$$

As $a_n \neq a_m \, \forall n \neq m$, $\exists n_0 \geq n_r$ s.t. $a_{k_{n_0}} \neq a$. Then $a_{k_{n_0}} \in B_r(a) \cap A \setminus \{a\}$. We get $a \in A' \cap K$.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. We distinguish two cases:
Case 1: The sequence $\{a_n\}_{n\geq 1}$ contains a constant subsequence. That subsequence converges to an element in K.
Case 2: $\{a_n\}_{n\geq 1}$ does not contain a constant subsequence. Then $A = \{a_n : n \geq 1\}$ is infinite and $A \subseteq K$. So $A' \cap K \neq \emptyset$. Let $a \in A' \cap K$. Then $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t. $a_{k_n} \xrightarrow[n\to\infty]{d} a$.



$B_1(a) \cap A \setminus \{a\} \neq \emptyset$

$B_{\min\left\{\frac{1}{2}, d(a, a_{k_1})\right\}}(a) \cap A \setminus \{a\} \neq \emptyset$

$k_2 > k_1$

$\square$

Theorem 2.2

Let (X, d) be a metric space and let $K \subseteq X$ be compact. Then K is sequentially compact.

Proof. If K is finite, then any sequence $\{x_n\}_{n\geq 1} \subseteq K$ will have a constant subsequence.

Assume now K is infinite. We will use the Bolzano – Weierstrass theorem. It suffices to prove that for any infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

$$\left.\begin{aligned} &\text{Note } A \subseteq K \text{ then } A' \subseteq K' \\ &K \text{ compact } \implies K \text{ closed } \implies K' \subseteq K \end{aligned}\right\} \implies A' \subseteq K \implies A' \cap K = A'$$

We argue by contradiction. Assume $A' = \emptyset$. Then for $x \in K$ we have $x \notin A' \implies \exists r_x > 0$ s.t. $B_{r_x}(x) \cap A \setminus \{x\} = \emptyset$. So

$$\left.\begin{aligned} &K \subseteq \textstyle\bigcup_{x\in K} \underbrace{B_{r_x}(x)}_{\text{open}} \\ &K \text{ compact} \end{aligned}\right\} \implies \exists n \geq 1 \text{ and } \exists x_1, \dots, x_n \in K \text{ s.t.}$$

$$K \subseteq \bigcup_{j=1}^{n} B_{r_j}(x_j) \text{ where } r_j = r_{x_j}$$

In particular,

$$\left.\begin{aligned} &A = \left(\textstyle\bigcup_{j=1}^{n} B_{r_j}(x_j)\right) \cap A = \textstyle\bigcup_{j=1}^{n} \left[B_{r_j}(x_j) \cap A\right] \\ &\text{By construction, } B_{r_j}(x_j) \cap A \subseteq \{x_j\} \end{aligned}\right\} \implies \underbrace{A}_{\text{infinite}} \subseteq \underbrace{\bigcup_{j=1}^{n} \{x_j\}}_{\text{finite}}$$

– Contradiction! So $A' \neq \emptyset$. □

Proposition 2.3

Let (X, d) be a metric space and let $K \subseteq X$ be sequentially compact. Then K is closed and bounded.

Proof. Let's show K is closed $\iff K = \overline{K}$.

We know $K \subseteq \overline{K}$. We need to show $\overline{K} \subseteq K$. Let $x \in \overline{K} \implies \exists \{x_n\}_{n\geq 1} \subseteq K$ s.t. $x_n \xrightarrow[n\to\infty]{d} x$.

K sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

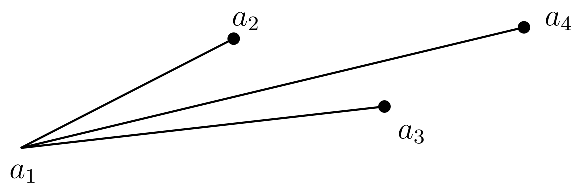
$$\left.\begin{aligned} &x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ &x_n \xrightarrow[n\to\infty]{d} x \implies x_{k_n} \xrightarrow[n\to\infty]{d} x \\ &\text{Limits of convergent sequences are unique} \end{aligned}\right\} \implies x = y \in K$$

As $x \in \overline{K}$ was arbitrary, we get $\overline{K} \subseteq K$.

Let's show K is bounded. We argue by contradiction. Assume K is not bounded. Let $a_1 \in K$.

$$\begin{aligned} K \text{ not bounded } &\implies K \not\subseteq B_1(a_1) \implies \exists a_2 \in K \text{ s.t. } d(a_1, a_2) \geq 1 \\ K \text{ not bounded } &\implies K \not\subseteq B_{1+d(a_1,a_2)}(a_1) \implies \exists a_3 \in K \text{ s.t. } d(a_1, a_3) \geq 1 + d(a_1, a_2) \end{aligned}$$

Proceeding inductively, we find a sequence $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_1, a_{n+1}) \geq 1 + d(a_1, a_n)$.



By construction,

$$|d(a_1, a_m) - d(a_1, a_n)| \geq |n - m| \quad \forall n, m \geq 1$$

By the triangle inequality,

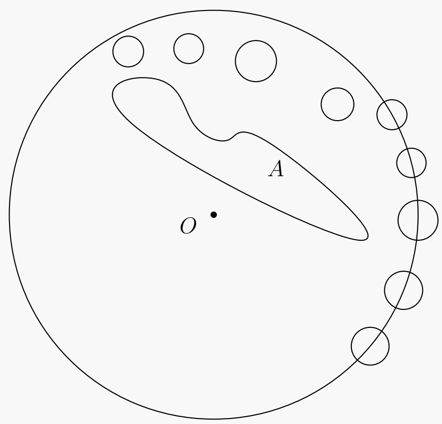
$$d(a_n, a_m) \geq |d(a_1, a_n) - d(a_1, a_m)| \geq |n - m| \quad \forall n, m \geq 1$$

This sequence cannot have a convergent (Cauchy) subsequence, thus contradiction the hypothesis that K is sequentially compact. So K is bounded. □

Definition 2.4 (Totally Bounded) — Let (X, d) be a metric space. A set $A \subseteq X$ is totally bounded if for every $\varepsilon > 0$, A can be covered by finitely many balls of radius ε.

Remark 2.5.

1. A totally bounded $\implies$ A bounded.
   Indeed, taking $\varepsilon = 1$, $\exists n \geq 1$ and $\exists x_1, \ldots, x_n \in X$ s.t.
   $$A \subseteq \bigcup_{j=1}^{n} B_1(x_j) \subseteq B_r(x_1)$$
   where $r = 1 + \max_{2 \leq j \leq n} d(x_1, x_j)$.
2. A bounded $\not\Longrightarrow$ A totally bounded.
   Consider $\mathbb{N}$ equipped with the discrete metric
   $$d(n,m) = \begin{cases} 0, \, n = m \\ 1, \, n \neq m \end{cases}$$
   Then $\mathbb{N} = B_2(1)$, but $\mathbb{N}$ cannot be covered by finitely many balls of radius $\frac{1}{2}$ since $B_{\frac{1}{2}}(n) = \{n\}$.
3. On $(\mathbb{R}^n, d_2)$, A bounded $\implies$ A totally bounded. Indeed, A bounded $\implies$ $A \subseteq B_R(0)$ for some $R > 0$. $B_R(0)$ can be covered by $10^6 \left(\frac{R}{\varepsilon}\right)^n$ many balls of radius ε.



§3 Lec 3: Apr 2, 2021

§3.1 Heine – Borel Theorem

Theorem 3.1

Let (X, d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is sequentially compact.
2. K is complete and totally bounded.

Proof. 1) $\implies$ 2) Let's show K is complete. Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence with $x_n \in K \quad \forall n \geq 1$.
K sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$\left.\begin{array}{l} x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ \{x_n\}_{n\geq 1} \text{ is Cauchy} \end{array}\right\} \implies x_n \xrightarrow[n\to\infty]{d} y \in K$$

As $\{x_n\}_{n\geq 1} \subseteq K$ was arbitrary, we get that K is complete.
Let's show K is totally bounded. Fix $\varepsilon > 0$ and $a_1 \in K$.

- If $K \subseteq B_\varepsilon(a_1)$, then K is totally bounded.
- If $K \nsubseteq B_\varepsilon(a_1)$, then $\exists a_2 \in K$ s.t. $d(a_1, a_2) \geq \varepsilon$
- If $K \subseteq B_\varepsilon(a_1) \cup B_\varepsilon(a_2)$, then K is totally bounded.
- If $K \nsubseteq B_\varepsilon(a_1) \cup B_\varepsilon(a_2)$, then $\exists a_3 \in K$ s.t. $d(a_1, a_3) \geq \varepsilon$ and $d(a_2, a_3) \geq \varepsilon$.

We distinguish two cases:
Case 1: The process terminates in finitely many steps $\implies K$ is totally bounded.
Case 2: The process does not terminate in finitely many steps. Then we find $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_n, a_m) \geq \varepsilon \quad \forall n \neq m$. This sequence does not admit a convergent subsequence, contradicting the fact that K is sequentially compact.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. K totally bounded $\implies \mathcal{J}_1$ finite and $\left\{x_j^{(1)}\right\}_{j\in\mathcal{J}_1} \subseteq X$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j\in\mathcal{J}_1} B_1(x_j^{(1)}) \\ \{a_n\}_{n\geq 1} \subseteq K \end{array}\right\} \implies \exists j_1 \in \mathcal{J}_1 \text{ s.t. } \left|\left\{n : a_n \in B_1(x_{j_1}^{(1)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(1)}\right\}_{n\geq 1}$ be the corresponding subsequence.
K totally bounded $\implies \exists \mathcal{J}_2$ finite and $\left\{x_j^{(2)}\right\}_{j\in\mathcal{J}_2} \subseteq X$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j\in\mathcal{J}_2} B_{\frac{1}{2}}(x_j^{(2)}) \\ \left\{a_n^{(1)}\right\}_{n\geq 1} \subseteq K \end{array}\right\} \implies \exists j_2 \in \mathcal{J}_2 \text{ s.t. } \left|\left\{n : a_n^{(1)} \in B_{\frac{1}{2}}(x_{j_2}^{(2)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(2)}\right\}_{n\geq 1}$ denote the corresponding subsequence.
We proceed inductively. We find that $\forall k \geq 1$

- $\left\{a_n^{(k+1)}\right\}_{n\geq 1}$ subsequence of $\left\{a_n^{(k)}\right\}_{n\geq 1}$
- $\left\{a_n^{(k)}\right\}_{n\geq 1} \subseteq B_{\frac{1}{k}}\left(x_{j_k}^{(k)}\right)$ for some $x_{j_k}^{(k)} \in X$.

We consider the subsequence $\left\{a_n^{(n)}\right\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$.

$$\begin{aligned}
\left\{a_n^{(1)}\right\}_{n\geq 1} &= \left(a_1^{(1)},\quad a_2^{(1)},\quad a_3^{(1)},\quad \ldots\right) \\
\left\{a_n^{(2)}\right\}_{n\geq 1} &= \left(\qquad a_1^{(2)},\quad a_2^{(2)},\quad a_3^{(2)},\quad \ldots\right) \\
\left\{a_n^{(3)}\right\}_{n\geq 1} &= \left(\qquad\qquad a_1^{(3)},\quad a_2^{(3)},\qquad a_3^{(3)},\quad \ldots\right)
\end{aligned}$$

For $n, m \geq k$ the $a_n^{(n)}, a_m^{(m)}$ belong to the subsequence $\left\{a_n^{(k)}\right\}_{n\geq 1}$. In particular,

$$d(a_n^{(n)}, a_m^{(m)}) \leq d(a_n^{(n)}, x_{j_k}^{(k)}) + d(a_m^{(m)}, x_{j_k}^{(k)}) < \frac{2}{k} \quad \forall n, m \geq k$$

This shows $\left\{a_n^{(n)}\right\}_{n\geq 1}$ is Cauchy and K is complete, so $a_n^{(n)} \xrightarrow[n\to\infty]{d} a \in K$. As $\{a_n\}_{n\geq 1}$ was arbitrary, we get that K is sequentially compact. □

Lemma 3.2

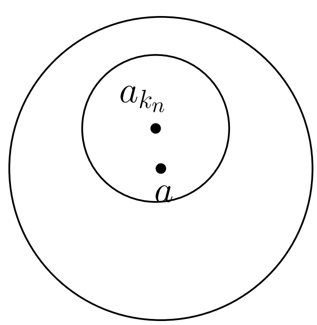
Let (X, d) be a sequentially compact metric space. Let $\{G_i\}_{i\in I}$ be an open cover of X. Then there exists $\varepsilon > 0$ such that every ball of radius ε is contained in at least one G_i.

Proof. We argue by contradiction. Then

$$\forall n \geq 1 \quad \exists a_n \in X \text{ s.t. } B_{\frac{1}{n}}(a_n) \text{ is not contained in any } G_i$$

X is sequentially compact $\implies \exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

$$\begin{gathered}
a_{k_n} \xrightarrow[n\to\infty]{d} a \in X = \bigcup_{i\in I} G_i \implies \exists i_0 \in I \text{ s.t. } a \in G_{i_0} \\
G_{i_0} \text{ open } \implies \exists r > 0 \text{ s.t. } B_r(a) \subseteq G_{i_0} \\
a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_1(r) \in \mathbb{N} \text{ s.t. } d(a_1, a_{k_n}) < \frac{r}{2} \forall n \geq n_1
\end{gathered}$$



Let $n_2(r)$ s.t. $n_2 > \frac{2}{r}$.

Claim 3.1. $\forall n \geq n_r = \max\{n_1, n_2\}$ we have $B_{\frac{1}{k_n}}(a_{k_n}) \subseteq B_r(a) \subseteq G_{i_0}$ thefore giving a contradiction!

Fix $x \in B_{\frac{1}{k_n}}(a_{k_n})$. Then

$$d(a, x) \leq d(x, a_{k_n}) + d(a_{k_n}, a) < \frac{1}{k_n} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r$$

□

Theorem 3.3

A sequentially compact metric space (X, d) is compact.

Proof. Let $\{G_i\}_{i \in I}$ be an open cover of X. Let ε be given by the previous lemma. X sequentially compact $\implies$ X totally bounded $\implies$ $\exists n \geq 1$ and

$$\left.\begin{array}{l} \exists x_1, \ldots, x_n \in X \text{ s.t. } X = \bigcup_{j=1}^n B_\varepsilon(x_j) \\ \forall 1 \leq j \leq n \quad \exists i_j \in I \text{ s.t. } B_\varepsilon(x_j) \subseteq G_{i_j} \end{array}\right\} \implies X = \bigcup_{j=1}^{n} G_{i_j}$$

□

Collecting our results so far we obtain

Theorem 3.4 (Heine – Borel)

Let (X, d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is compact,
2. K is sequentially compact,
3. K is complete and totally bounded,
4. Every infinite subset of K has an accumulation point in K.

Remark 3.5. In $\mathbb{R}^n$, K is compact $\iff$ K is closed and bounded.

Definition 3.6 (Finite Intersection Property) — An infinite family $\{F_i\}_{i \in I}$ of closed sets is said to have the finite intersection property if $\forall \mathcal{J} \subseteq I$ finite we have

$$\bigcap_{j \in \mathcal{J}} F_j \neq \emptyset$$

Theorem 3.7

A metric space (X, d) is compact if and only if every infinite family $\{F_i\}_{i\in I}$ of closed sets with the finite intersection property satisfies

$$\bigcap_{i\in I} F_i \neq \emptyset$$

Proof. " $\implies$ " We argue by contradiction. Assume $\exists \{F_i\}_{i\in I}$ closed sets with the finite intersection property s.t. $\bigcap_{i\in I} F_i = \emptyset$

$$\left.\begin{array}{l} X = {}^c\left(\bigcap_{i\in I} F_i\right) = \bigcup_{i\in I} \underbrace{{}^cF_i}_{\text{open}} \\ \\ X \text{ compact} \end{array}\right\} \implies \exists \mathcal{J} \subseteq I \text{ finite s.t. } X = \bigcup_{j\in\mathcal{J}} {}^cF_j$$

$$\implies \emptyset = {}^c\left(\bigcup_{j\in\mathcal{J}} {}^cF_j\right) = \bigcap_{j\in\mathcal{J}} F_j \text{ – Contradiction!}$$

" $\impliedby$ " We argue by contradiction. Assume $\exists \{G_i\}_{i\in I}$ open cover of X that does not admit a finite subcover.

So $\forall \mathcal{J} \subseteq I$ finite $X \neq \bigcup_{j\in\mathcal{J}} G_j \implies \emptyset \neq \bigcap_{j\in\mathcal{J}} \underbrace{{}^cG_j}_{\text{closed}}$. So $\{{}^cG_i\}_{i\in I}$ is a family of closed sets with the finite intersection property. Then

$$\bigcap_{i\in I} {}^cG_i \neq \emptyset \implies \bigcup_{i\in I} G_i \neq X$$

Contradiction! $\square$

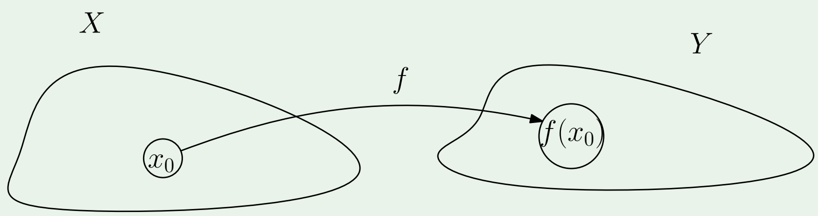
§4 Lec 4: Apr 5, 2021

§4.1 Continuity

Definition 4.1 (Continuous Function) — Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that a function $f : X \to Y$ is continuous at a point $x_0 \in X$ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } d_X(x, x_0) < \delta \text{ then } d_Y\left(f(x), f(x_0)\right) < \varepsilon$$

We say f is continuous (on X) if f is continuous at every point in X.



Remark 4.2. $f : X \to Y$ is continuous at every isolated point in X. Indeed, if $x_0 \in X$ is isolated, then $\exists \delta > 0$ s.t. $B_\delta^X(x_0) = \{x_0\}$. Then $d_X(x, x_0) < \delta \implies d_Y\left(f(x), f(x_0)\right) = 0$

Proposition 4.3

Let $(X, d_X), (Y, d_Y)$ be two metric spaces and $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous at $x_0 \in X$.
2. For any $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $x_n \xrightarrow[n\to\infty]{d_X} x_0$ we have $f(x_n) \xrightarrow[n\to\infty]{d_Y} f(x_0)$.

Proof. 1) $\implies$ 2) Let $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $x_n \xrightarrow[n\to\infty]{d_X} x_0$.
Let $\varepsilon > 0$. f continuous at $x_0 \implies \exists \delta > 0$ s.t.

$$\left.\begin{aligned} & d_X(x, x_0) < \delta \implies d_Y\left(f(x), f(x_0)\right) < \varepsilon \\ & x_n \xrightarrow[n\to\infty]{d_X} x_0 \implies \exists n_\delta \in \mathbb{N} \text{ s.t. } d_X(x_n, x_0) < \delta \, \forall n \geq n_\delta \end{aligned}\right\} \implies d_X\left(f(x_n), f(x_0)\right) < \varepsilon$$

for each $n \geq n_\delta$.
2) $\implies$ 1) We argue by contradiction. Assume

$$\exists \varepsilon_0 > 0 \text{ s.t. } \forall \delta > 0 \quad \exists x_\delta \in X \text{ s.t. } d_X(x_\delta, x_0) < \delta \text{ but } d_Y\left(f(x_\delta), f(x_0)\right) \geq \varepsilon_0$$

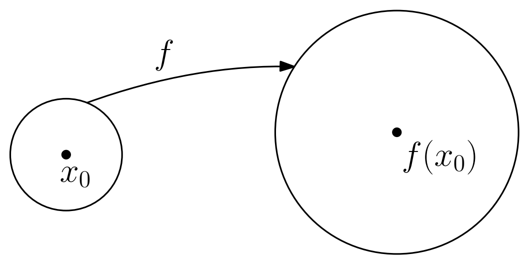
Letting $\delta = \frac{1}{n}$ we find $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $d_X(x_n, x_0) < \frac{1}{n}$ but $d_Y\left(f(x_n), f(x_0)\right) \geq \varepsilon_0$ – Contradiction! $\square$

Theorem 4.4

Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous.
2. for any G open in Y, $f^{-1}(G) = \{x \in X : f(X) \in G\}$ is open in X.
3. for any F closed in Y, $f^{-1}(F)$ is closed in X.
4. for any $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.
5. for any $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

Proof. We will show 1) $\implies$ 2) $\implies$ 3) $\implies$ 4) $\implies$ 5) $\implies$ 1).
1) $\implies$ 2) Let $G \subseteq Y$ be open.



Let $x_0 \in f^{-1}(G)$

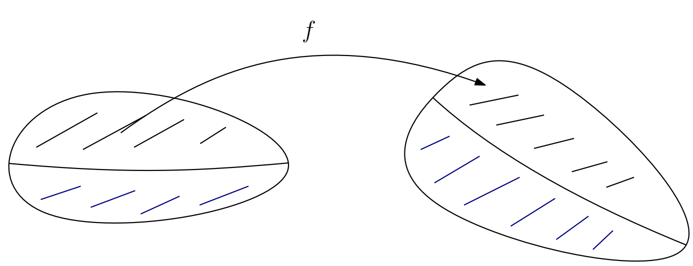
$$\implies \left.\begin{matrix} f(x_0) \in G \\ G \text{ open in } Y \end{matrix}\right\} \implies \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon^Y(f(x_0)) \subseteq G$$

f is continuous

$$\begin{aligned} &\implies \exists \delta > 0 \text{ s.t. } f\left(B_\delta^X(x_0)\right) \subseteq B_\varepsilon^Y(f(x_0)) \subseteq G \\ &\implies B_\delta^X(x_0) \subseteq f^{-1}(G) \implies x_0 \in \overset{\circ}{\widehat{f^{-1}(G)}} \end{aligned}$$

So $f^{-1}(G)$ is open in X.
2) $\implies$ 3) Let $F \subseteq Y$ be closed $\implies$ ${}^cF = Y \setminus F$ is open in Y. By assumption,

$$\left.\begin{matrix} f^{-1}\left({}^cF\right) \text{ is open in } X \\ f^{-1}\left({}^cF\right) = {}^c\left[f^{-1}(F)\right] = X \setminus f^{-1}(F) \end{matrix}\right\} \implies f^{-1}(F) \text{ is closed in } X$$



$$f^{-1}(Y \setminus F) = f^{-1}(Y) \setminus f^{-1}(F) = X \setminus f^{-1}(F)$$

3) $\implies$ 4) Let $B \subseteq Y \implies \overline{B}$ closed in Y. By assumption,

$$\left.\begin{array}{l} f^{-1}(\overline{B}) \text{ closed in } X \\ f^{-1}(\overline{B}) \supseteq f^{-1}(B) \end{array}\right\} \implies \overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$$

4) $\implies$ 5) Let $A \subseteq X$. Use the hypothesis with $B = f(A)$. We have

$$\overline{A} \subseteq \overline{f^{-1}(f(A))} \subseteq f^{-1}\left(\overline{f(A)}\right) \implies f(\overline{A}) \subseteq \overline{f(A)}$$

5) $\implies$ 1) We argue by contradiction. Assume $\exists x_0 \in X$ s.t. f is not continuous at x_0. Then $\exists \varepsilon_0 > 0$ and $\exists x_n \xrightarrow[n\to\infty]{d_X} x_0$ but $d_Y(f(x_n), f(x_0)) \geq \varepsilon_0$.
Let $A = \{x_n : n \geq 1\}$. Then $x_0 \in \overline{A}$ but $f(x_0) \notin \overline{\{f(x_n) : n \geq 1\}} = \overline{f(A)}$. On the other hand, we must have

$$\left.\begin{array}{l} f(\overline{A}) \subseteq \overline{f(A)} \\ x_0 \in \overline{A} \end{array}\right\} \implies f(x_0) \in \overline{f(A)}$$

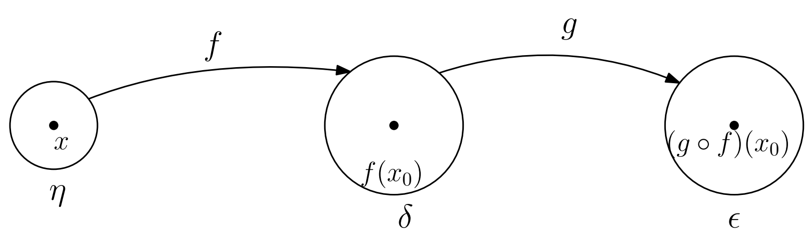
Contradiction! □

Proposition 4.5

Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces and assume $f : X \to Y$ is continuous at $x_0 \in X$ and $g : Y \to Z$ is continuous at $f(x_0) \in Y$. Then $g \circ f : X \to Z$ is continuous at x_0.

Proof. Fix $\varepsilon > 0$.

$$\begin{aligned} g \text{ continuous at } f(x_0) &\implies \exists \delta > 0 \text{ s.t. } d_Y(y, f(x_0)) < \delta \implies d_Z(g(y), g(f(x_0))) < \varepsilon \\ f \text{ continuous at } x_0 &\implies \exists \eta > 0 \text{ s.t. } d_X(x, x_0) < \eta \implies d_Y(f(x), f(x_0)) < \delta \end{aligned}$$



So if $d_X(x, x_0) < \eta$ then $d_Z(g(f(x)), g(f(x_0))) < \varepsilon$. □

Exercise 4.1. Let (X, d) be a metric space and let $f, g : X \to \mathbb{R}$ be continuous at $x_0 \in X$. Then $f \pm g, f \cdot g$ are continuous at x_0. If $g(x_0) \neq 0$ then $\frac{f}{g} : X \to \mathbb{R}$ is continuous at x_0.

Exercise 4.2. Let (X, d) be a metric space and let $f_1, \ldots, f_n : X \to \mathbb{R}$. Then $f = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ is continuous at $x_0 \in X$ if and only if $f_1, \ldots, f_n$ are continuous at x_0.

<u>Hint</u>: $|f_i(x) - f_i(x_0)| \leq d_2(f(x), f(x_0)) = \sqrt{\sum_{j=1}^n |f_j(x) - f_j(x_0)|^2}$.

§4.2 Continuity and Compactness

Theorem 4.6

Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be continuous. If K is compact in X, then $f(K)$ is compact in Y.

Proof. Method 1: Let $\{G_i\}_{i\in I}$ be a family of open sets in Y s.t.

$$f(K) \subseteq \bigcup_{i\in I} G_i \implies K \subseteq f^{-1}\left(\bigcup_{i\in I} G_i\right) = \bigcup_{i\in I} \underbrace{f^{-1}(G_i)}_{\text{open in } X}$$

K compact $\implies \exists n \geq 1$ and $\exists i, \ldots, i_n \in I$ s.t.

$$K \subseteq \bigcup_{j=1}^{n} f^{-1}\left(G_{i_j}\right) = f^{-1}\left(\bigcup_{j=1}^{n} G_{i_j}\right) \implies f(K) \subseteq \bigcup_{j=1}^{n} G_{i_j}$$

Method 2: Let's show $f(K)$ is sequentially compact. Let $\{y_n\}_{n\geq 1} \subseteq f(K)$.

$$y_n \in f(K) \implies \exists x_n = f^{-1}(y_n) \in K$$

As K is sequentially compact, $\exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$\left.\begin{array}{l} x_{k_n} \xrightarrow[n\to\infty]{d_X} x_0 \in K \\ f \text{ is continuous} \end{array}\right\} \implies \underbrace{f\left(x_{k_n}\right)}_{=y_{k_n}} \xrightarrow[n\to\infty]{d_Y} f(x_0) \in f(K)$$

□

§5 Lec 5: Apr 7, 2021

§5.1 Continuity and Compactness (Cont'd)

Corollary 5.1

Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}^n$ be continuous. Then $f(X)$ is closed and bounded.

Corollary 5.2

Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then there exists $x_1, x_2 \in X$ s.t.

$$f(x_1) = \inf\{f(x) : x \in X\} \text{ and } f(x_2) = \sup\{f(x) : x \in X\}$$

Proof. $f(x)$ is closed and bounded.

$$\begin{aligned} \text{Boundedness} &\implies \inf f(x) \text{ and } \sup f(x) \text{ are well defined} \\ \text{Closedness} &\implies \inf f(x), \sup f(x) \in \overline{f(x)} = f(x) \end{aligned}$$

□

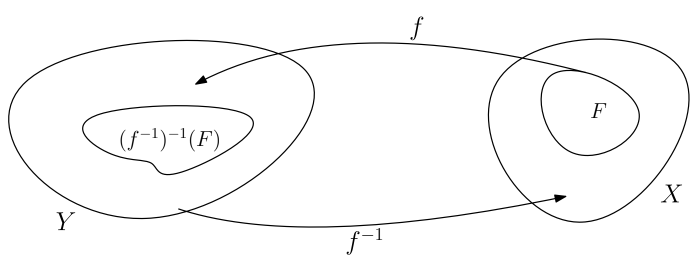
Proposition 5.3

Let $(X, d_X), (Y, d_Y)$ be metric spaces s.t. X is compact. Let $f : X \to Y$ be bijective and continuous. Then $f^{-1} : Y \to X$ is continuous.

Proof. If suffices to show that for every closed set $F \subseteq X$, we have

$$\left(f^{-1}\right)^{-1}(F) = \left\{y \in Y : f^{-1}(y) \in F\right\}$$

is closed in Y.



But $\left(f^{-1}\right)^{-1}(F) = f(F)$.

$$\left.\begin{aligned} &F \text{ closed in } X \text{ compact} \implies F \text{compact} \\ &f : X \to Y \text{ is continuous} \end{aligned}\right\} \implies f(F) \text{ is compact and closed}$$

□

Definition 5.4 (Uniform Continuity) — Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say that a function $f : X \to Y$ is uniformly continuous if

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) \text{ s.t. } d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Compare this with $g : X \to Y$ is continuous if

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon, x) \text{ s.t. } d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Remark 5.5.
1. Continuity is defined pointwise. Uniform continuity is a property of a function on a set.
2. Uniform continuity $\implies$ continuity.
3. There are continuous functions that are not uniformly continuous.

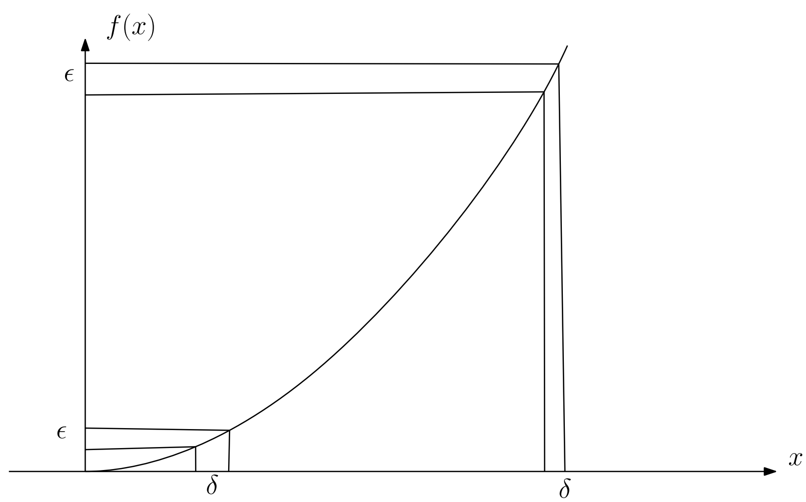
For example, consider

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

Let $x_n = n + \frac{1}{n}$, $y_n = n$

$$|x_n - y_n| = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} > 2$$



Theorem 5.6

Let $(X, d_X), (Y, d_Y)$ be metric spaces with X compact. Let $f : X \to Y$ continuous. Then f is uniformly continuous.

Proof. We argue by contradiction. Assume f is not uniformly continuous $\implies \exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0 \, \exists x_\delta, y_\delta \in X$ s.t. $d_X(x_\delta, y_\delta) < \delta$ but $d_Y\left(f(x_\delta), f(y_\delta)\right) \geq \varepsilon_0$.

Let $\delta = \frac{1}{n}$ to get $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1} \subseteq X$ s.t. $d_X(x_n, y_n) < \frac{1}{n}$ but $d_Y\left(f(x_n), f(y_n)\right) \geq \varepsilon_0$
X compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$x_{k_n} \xrightarrow[n\to\infty]{d_X} x_0 \in X$$

By the triangle inequality,

$$d\left(y_{k_n}, x_0\right) \leq \underbrace{d\left(x_{k_n}, y_{k_n}\right)}_{<\frac{1}{k_n} \leq \frac{1}{n} \xrightarrow[n\to\infty]{} 0} + \underbrace{d\left(x_{k_n}, x_0\right)}_{\xrightarrow[n\to\infty]{} 0} \xrightarrow[n\to\infty]{} 0 \implies y_{k_n} \xrightarrow[n\to\infty]{d_X} x_0$$

$$f \text{ continuous} \implies \begin{cases} f\left(x_{k_n}\right) \xrightarrow[n\to\infty]{d_Y} f(x_0) \\ f\left(y_{k_n}\right) \xrightarrow[n\to\infty]{d_Y} f(x_0) \end{cases}$$

But

$$\varepsilon_0 \leq d_Y\left(f(x_{k_n}), f(y_{k_n})\right) \leq \underbrace{d_Y\left(f(x_{k_n}), f(x_0)\right)}_{\to 0} + \underbrace{d_Y\left(f(x_0), f(y_{k_n})\right)}_{\to 0} \xrightarrow[n\to\infty]{} 0$$

Contradiction! $\square$

§5.2 Continuity and Connectedness

Theorem 5.7

Let $(X, d_X), (Y, d_Y)$ be metric spaces s.t. X is connected. Let $f : X \to Y$ be continuous. Then $f(X)$ is connected.

Proof. Method 1: Abusing notation we write $f : X \to f(x)$. It suffices to show that if $\emptyset \neq B \subseteq f(x)$ is both open and closed in $f(x)$ then $B = f(x)$.

As f is continuous, $f^{-1}(B) \neq \emptyset$ is both open and closed in X. But X is connected which implies $f^{-1}(B) = X$ and $f(x) = B$.
Method 2: Assume that $f(x)$ is not connected. Then $\exists \emptyset \neq B_1 \subseteq Y$, $\exists \emptyset \neq B_2 \subseteq Y$ s.t. $f(x) \subseteq B_1 \cup B_2$ and

$$\overline{B_1} \cap B_2 = \emptyset = B_1 \cap \overline{B_2}$$

let

$$A_1 = f^{-1}(B_1) \neq \emptyset$$
$$A_2 = f^{-1}(B_2) \neq \emptyset$$

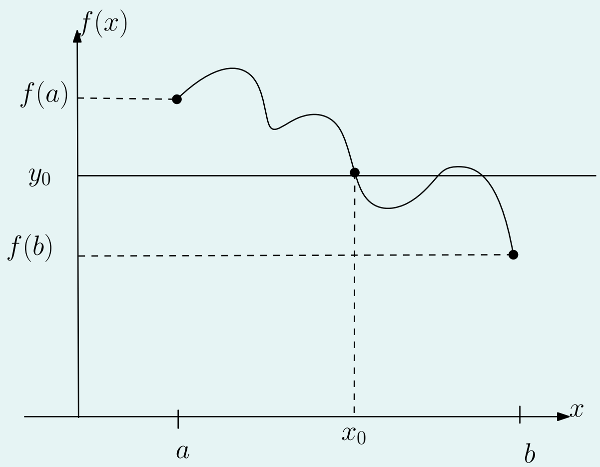
Have

$$\begin{aligned} f(X) \subseteq B_1 \cup B_2 \implies X \subseteq f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) = A_1 \cup A_2 \\ \overline{A_1} \cap A_2 = \overline{f^{-1}(B_1)} \cap f^{-1}(B_2) \subseteq f^{-1}(\overline{B_1}) \cap f^{-1}(B_2) = f^{-1}(\overline{B_1} \cap B_2) \\ = f^{-1}(\emptyset) = \emptyset \end{aligned}$$

Similarly, $\overline{A_2} \cap A_1 = \emptyset$. exercise
This contradicts that X is connected. $\square$

Corollary 5.8 (Darboux's Property)
Let (X, d_X) be a metric space and let $f : X \to \mathbb{R}$ be continuous. If $A \subseteq X$ is connected then $f(A)$ is an interval in $\mathbb{R}$.

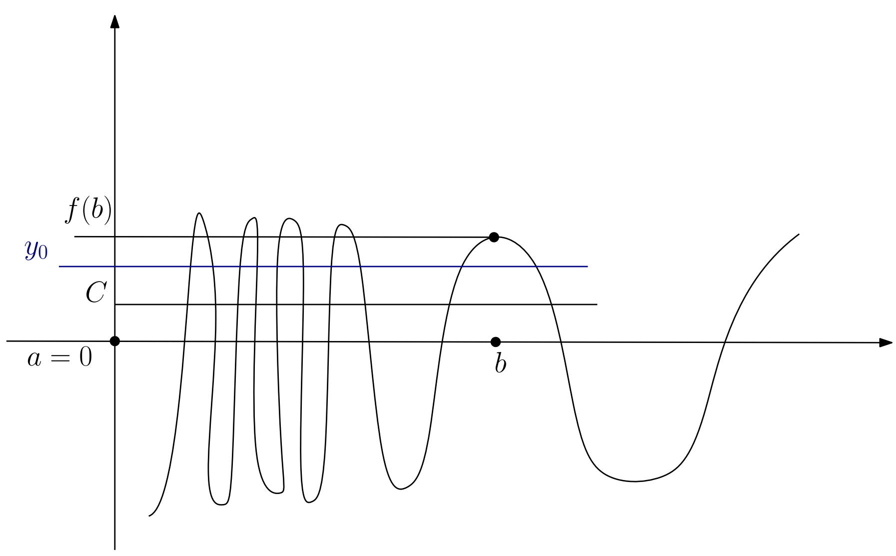
In particular, if $X = \mathbb{R}$, and $a, b \in \mathbb{R}$ s.t. $a < b$ and y_0 lies between $f(a)$ and $f(b)$, then $\exists x_0 \in (a, b)$ s.t. $f(x_0) = y_0$.



Remark 5.9. There are function that have the Darboux property, but are not continuous.

For example, consider

$$f : [0, \infty) \to \mathbb{R}, \quad f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ c, & x = 0 \end{cases} \quad \text{where } c \in [-1, 1]$$



Notice f is continuous on $(0, \infty)$ implies f has the Darboux property on $(0, \infty)$. f has the Darboux property on $[0, \infty)$, but is not continuous at $x = 0$.

§6 Lec 6: Apr 9, 2021

§6.1 Continuity and Connectedness (Cont'd)

Proposition 6.1

Let (X, d_X) and (Y, d_Y) be two connected metric spaces. Then $(X \times Y, d)$ where

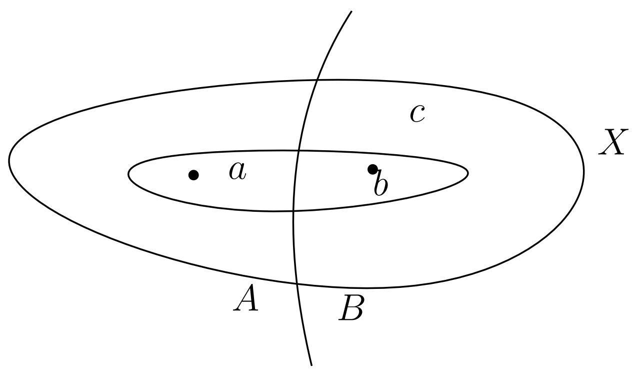
$$d\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

is a connected metric space.

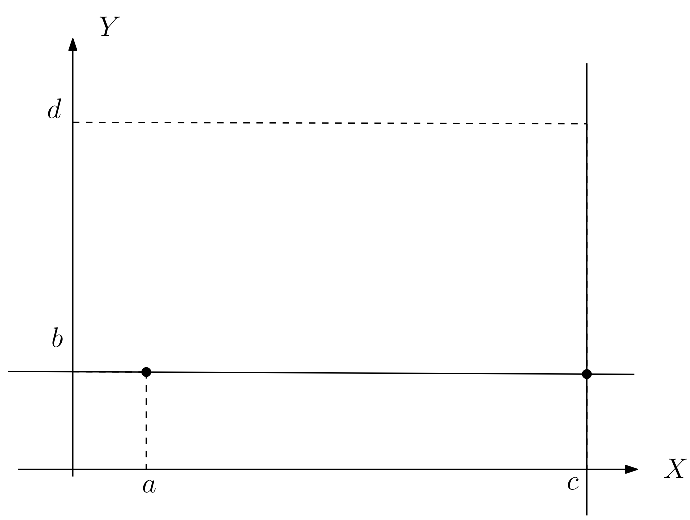
Remark 6.2. One could replace the distance d by

$$d_1\left((x_1, y_1), (x_2, y_2)\right) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$
$$d_\infty\left((x_1, y_1), (x_2, y_2)\right) = \max\left\{d_X(x_1, x_2), d_Y(y_1, y_2)\right\}$$

Proof. We will use the fact that a metric space is connected if and only if any two points are contained in a connected subset of the metric space.



So to show $X \times Y$ is connected if suffices to show that if $(a, b), (c, d) \in X \times Y$, then there exists $C \subseteq X \times Y$ connected s.t. $(a, b), (c, d) \in C$.



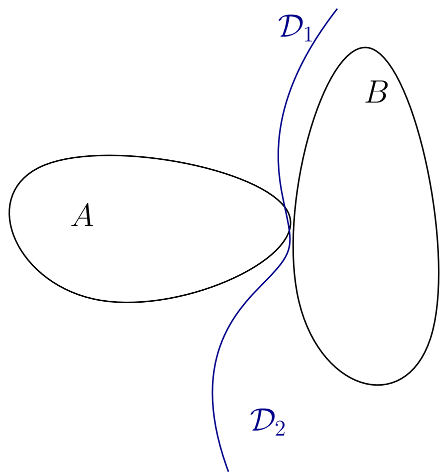
Let $f : X \to X \times Y$ where $f(x) = (x, b)$

Claim 6.1. f is continuous.

Take $\delta = \varepsilon$ in the definition of continuity. As X is connected, $f(X) = X \times \{b\}$ is connected.

Similarly, $g : Y \to X \times Y$, $g(y) = (c, y)$ is continuous and since Y is connected, $g(Y) = \{c\} \times Y$ is connected.

Finally, $f(x) \cap g(y) \ni (c, b)$ and so $f(x)$, $g(y)$ are not separated. As the union of two connected not separated sets is connected we get $f(x) \cup g(y)$ is connected.



Note $(a, b), (c, d) \in f(x) \cup g(y)$. □

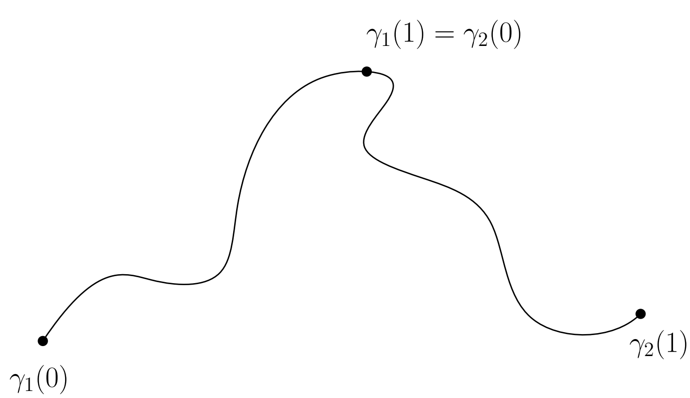
Definition 6.3 (Path) — Let (X, d) be a metric space. A path is a continuous function $\gamma : [0, 1] \to X$. $\gamma(0)$ is called the origin of the path and $\gamma(1)$ is called the end of the path.

As $[0, 1]$ is compact and connected and γ is continuous, $\gamma([0, 1])$ is compact and connected.

Given $\gamma : [0, 1] \to X$ a path, we define

$$\gamma^- : [0, 1] \to X, \qquad \gamma^-(t) = \gamma(1 - t) \text{ is a path}$$

Given $\gamma_1, \gamma_2 : [0, 1] \to X$ paths s.t. $\gamma_1(1) = \gamma_2(0)$.



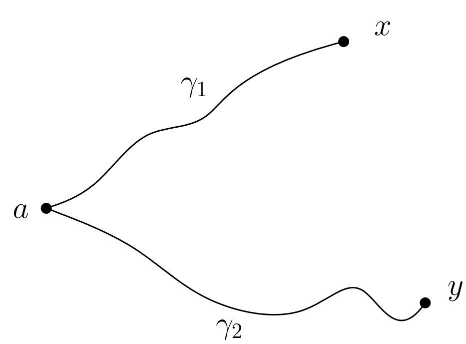
We define
$$\gamma_1 \vee \gamma_2 : [0,1] \to X$$
via
$$\gamma_1 \vee \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Proposition 6.4

Let (X,d) be a metric space and let $A \subseteq X$. Then 1) $\iff$ 2) $\implies$ 3) where

1. $\exists a \in A$ s.t. $\forall x \in A \, \exists \gamma_x : [0,1] \to A$ path s.t.
$$\gamma_x(0) = a \text{ and } \gamma_x(1) = x$$
2. $\forall x, y \in A \, \exists \gamma_{x,y} : [0,1] \to A$ path s.t.
$$\gamma_{x,y}(0) = x \text{ and } \gamma_{x,y}(1) = y$$
3. A is connected.

Proof. 1) $\implies$ 2) Let $x, y \in A$. By hypothesis, $\exists \gamma_x, \gamma_y : [0,1] \to A$ paths s.t.
$$\gamma_x(0) = \gamma_y(0) = a, \quad \gamma_x(1) = x, \quad \gamma_y(1) = y$$

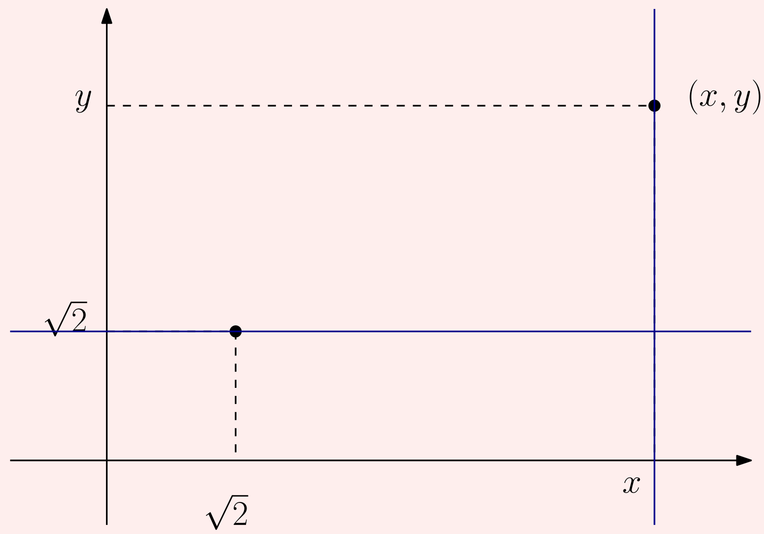


Then $\gamma_x^- \vee \gamma_y : [0,1] \to A$ is the desired path.
2) $\implies$ 1)Choose $a \in A$ arbitrary.
1) $\implies$ 3) Given $x \in A$, let $A_x = \gamma_x([0,1])$ connected. Note
$$a \in \bigcap_{x \in A} A_x \implies \text{ no two sets } A_x,\, A_y \text{ are separated}$$
Then $A = \bigcup_{x \in A} A_x$ is connected. $\square$

Definition 6.5 (Path Connected) — If either 1) or 2) holds in the Proposition 6.4, we say that A is path connected. Note A is path connected implies A is connected.

Example 6.6

$\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected.



We will show that any $(x, y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ can be joined via path in $\mathbb{R}^2 \setminus \mathbb{Q}^2$ to $\left(\sqrt{2}, \sqrt{2}\right)$.

$$(x, y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2 \implies x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q}$$

Say $x \notin \mathbb{Q}$. Then $\{x\} \times \mathbb{R} \subseteq \mathbb{R}^2 \setminus \mathbb{Q}^2$. Note also that $\mathbb{R} \times \left\{\sqrt{2}\right\} \subseteq \mathbb{R}^2 \setminus \mathbb{Q}^2$. Let $\gamma : [0, 1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2$, $\gamma = \gamma_1 \vee \gamma_2$ where

$$\gamma_1 : [0, 1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2,\ \gamma_1(t) = \left(\sqrt{2} + t(x - \sqrt{2}), \sqrt{2}\right) \text{ path}$$
$$\gamma_2 : [0, 1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2,\ \gamma_2(t) = \left(x, \sqrt{2} + t(y - \sqrt{2})\right) \text{ path}$$

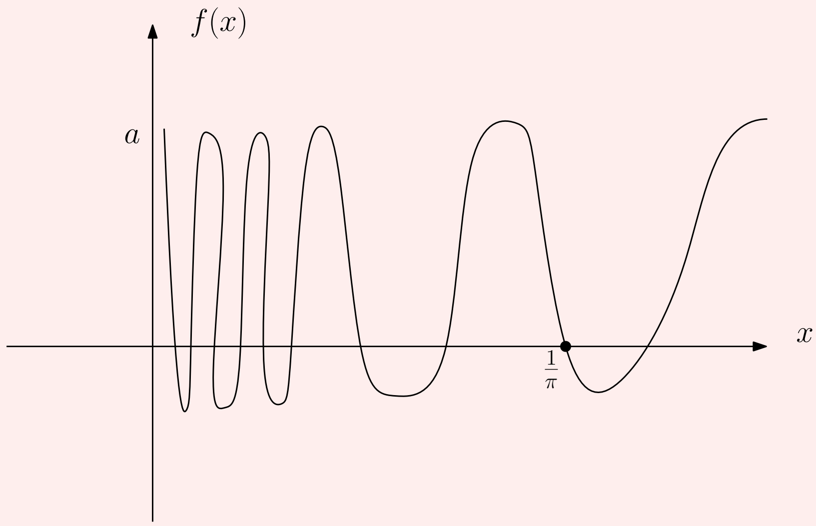
Example 6.7

A connected set which is not path connected. Let $f : [0, \infty) \to \mathbb{R}$ s.t.

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ a, & x = 0 \end{cases}$$

where $a \in [-1, 1]$ fixed.

Then $\Gamma_f = \{(x, f(x)) : x \in [0, \infty)\}$ is connected, but not path connected.



Let's show Γ_f is connected. The function $g : [0, \infty) \to \mathbb{R}^2$, $g(x) = (x, f(x))$ is continuous on $(0, \infty) \implies g((0, \infty))$ is connected.

Also, $g(\{0\}) = \{(0, a)\}$ is connected. We will show that $(0, a) \in \overline{g((0, \infty))}$ and so $\{(0, a)\}, g((0, \infty))$ are not separated. Then

$$\Gamma_f = g([0, \infty)) = g(\{0\}) \cup g((0, \infty)) \text{ is connected}$$

To see $(0, a) \in \overline{g(0, \infty)}$ we need to find $x_n \to 0$ s.t.

$$\sin\left(\frac{1}{x_n}\right) = a$$

Take $x_n = \frac{1}{\arcsin a + 2n\pi}$ where $\arcsin a \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Example 6.8 (Cont'd from above)

Now let's show Γ_f is not path connected. Assume towards a contradiction that there exists $\gamma : [0,1] \to \Gamma_f$ a path s.t.

$$\gamma(0) = (0,a), \qquad \gamma(1) = \left(\frac{1}{\Pi}, 0\right)$$

Note $\Pi_1 \circ \gamma : [0,1] \to \mathbb{R}$ is continuous

$$\left(\Pi_1 \circ \gamma\right)(0) = 0, \quad \left(\Pi_1 \circ \gamma\right)(1) = \frac{1}{\pi}$$

Let $b \in [-1,1] \setminus \{a\}$. By the Darboux property, $\exists t_n \in \left(0, \frac{1}{\pi}\right)$ s.t.

$$\left(\Pi_1 \circ \gamma\right)(t_n) = \frac{1}{\arcsin b + 2n\pi} \text{ where } \arcsin b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

As $[0,1]$ is compact, $\exists t_{k_n} \underset{n\to\infty}{\longrightarrow} t_\infty \in [0,1]$.

$$\left.\begin{aligned} &\gamma \text{ continuous } \implies \gamma\left(t_{k_n}\right) \underset{n\to\infty}{\longrightarrow} \gamma(t_\infty) \\ &\gamma\left(t_{k_n}\right) = \left(\tfrac{1}{\arcsin b + 2k_n\pi}, b\right) \underset{n\to\infty}{\longrightarrow} (0,b) \end{aligned}\right\} \implies \gamma(t_\infty) = (0,b) \notin \Gamma_f$$

§7 Lec 7: Apr 12, 2021

§7.1 Continuity and Connectedness (Cont'd)

Example 7.1

Two connected sets $A, B \subseteq [-1,1] \times [-1,1]$ s.t. $(-1,-1),(1,1) \in A$, $(-1,1),(1,-1) \in B$, $A \cap B = \emptyset$. Let $f : [-1,1] \to [-1,1]$,

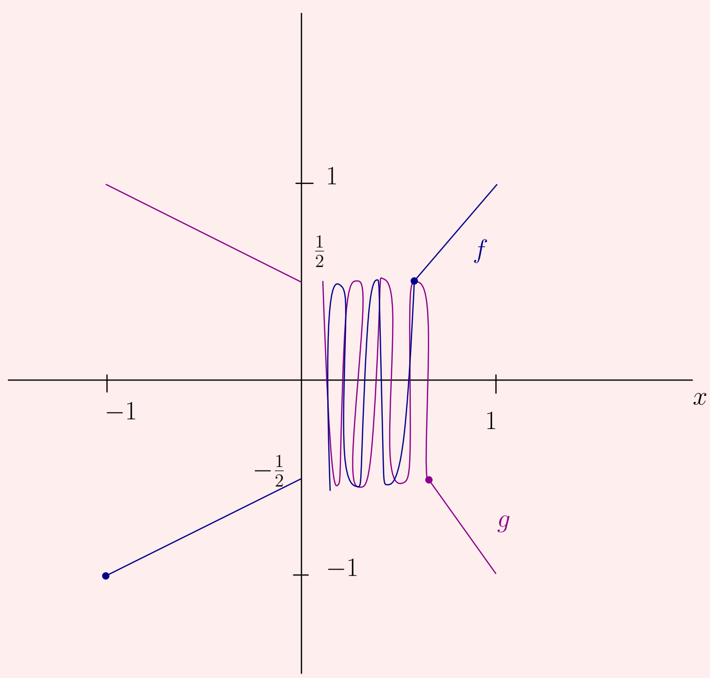
$$f(x) = \begin{cases} \frac{x-1}{2}, & -1 \le x \le 0 \\ x - \frac{1}{2}\sin\frac{\pi}{x}, & 0 < x \le \frac{1}{2} \\ x, & \frac{1}{2} \le x \le 1 \end{cases}$$

Let $g : [-1,1] \to [-1,1]$,

$$g(x) = \begin{cases} \frac{1-x}{2}, & -1 \le x \le 0 \\ -x - \frac{1}{2}\sin\frac{\pi}{x}, & 0 < x \le \frac{1}{2} \\ -x, & \frac{1}{2} \le x \le 1 \end{cases}$$

Let

$$A = \Gamma_f = \{(x_1 f(x)) : x \in [-1,1]\}$$
$$B = \Gamma_g = \{(x_1 g(x)) : x \in [-1,1]\}$$



Example 7.2 (Cont'd from above)

Let's prove $A \cap B = \emptyset$. If

$$\begin{aligned} -1 \le x \le 0, &\quad f(x) = g(x) \iff \frac{x-1}{2} = \frac{1-x}{2} \iff x = 1 \\ 0 < x \le \frac{1}{2}, &\quad f(x) = g(x) \iff x = 0 \\ \frac{1}{2} \le x \le 1, &\quad f(x) = g(x) \iff x = 0 \end{aligned}$$

Also

$$\begin{aligned} f(-1) = -1 &\implies (-1,-1) \in A \\ f(1) = 1 &\implies (1,1) \in A \\ g(-1) = 1 &\implies (-1,1) \in B \\ g(1) = -1 &\implies (1,-1) \in B \end{aligned}$$

Let's show that A is connected. A similar argument can be used to prove that B is connected.

We write $A = A_1 \cup A_2$ where $A_1 = \{(x, f(x)) : -1 \le x \le 0\}$ and $A_2 = \{(x, f(x)) : 0 < x \le 1\}$. Note that $h : [-1,1] \to \mathbb{R}^2$ where $h(x) = (x, f(x))$ is continuous on $[-1,0]$ and $(0,1]$.

Since $[-1,0]$ and $(0,1]$ are connected sets, we get that $h([-1,0]) = A_1$ and $h((0,1]) = A_2$ are connected.

To show that $A = A_1 \cup A_2$ is connected, it suffices to show that A_1 and A_2 are not separated. We will show $\left(0, -\frac{1}{2}\right) \in A_1 \cap \overline{A_2}$. It's clear that $f(0) = -\frac{1}{2} \implies \left(0, -\frac{1}{2}\right) \in A_1$. To show that $\left(0, -\frac{1}{2}\right) \in \overline{A_2}$ we need to find a decreasing sequence $x_n \to 0$ s.t.

$$f(x_n) = x_n - \frac{1}{2} \sin \frac{\pi}{x_n} \xrightarrow[n\to\infty]{} -\frac{1}{2}$$

We take x_n s.t. $\sin \frac{\pi}{x_n} = 1 \iff \frac{\pi}{x_n} = \frac{\pi}{2} + 2n\pi \iff x_n = \frac{2}{4n+1} \to 0$. Notice that

$$f(x_n) = \frac{2}{4n+1} - \frac{1}{2} \xrightarrow[n\to\infty]{} -\frac{1}{2}$$

§7.2 Convergent Sequences of Functions

Definition 7.3 (Pointwise Convergence) — Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\ge1}$ converges pointwise if for all $x \in X$ the sequence $\{f_n(x)\}_{n\ge1}$ converges in Y. The limit $\lim_{n\to\infty} f_n(x) = f(x)$ defines a function $f : X \to Y$.

Remark 7.4. $\{f_n\}_{n\ge1}$ converges pointwise to f if

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists n(\varepsilon, x) \in \mathbb{N} \text{ s.t. } d_Y(f_n(x), f(x)) < \varepsilon \quad \forall n \ge n(\varepsilon, x)$$

Note that for $\varepsilon > 0$ fixed, $n(\varepsilon, \cdot) : X \to \mathbb{N}$ can be bounded or unbounded. If it is bounded, we get the following

Definition 7.5 (Uniform Convergence) — Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\geq 1}$ converges uniformly to a function $f : X \to Y$ if

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d_Y\left(f(x), f_n(x)\right) < \varepsilon \quad \forall n \geq n_\varepsilon \, \forall x \in X$$

We denote $f_n \xrightarrow[n\to\infty]{u} f$.

Remark 7.6. Let $(X, d_X), (Y, d_Y)$ be metric spaces, $B(X, Y) = \{f : X \to Y;\, f \text{ is bounded}\}$, $d : B(X, Y) \times B(X, Y) \to \mathbb{R}$ via

$$d(f, g) = \sup_{x\in X} d_Y\left(f(x), g(x)\right)$$

Exercise 7.1. Show that $(B(X, Y), d)$ is a metric space.

Note that $f_n \xrightarrow[n\to\infty]{u} f \iff M_n = d\left(f_n, f\right) \xrightarrow[n\to\infty]{} 0$.
" $\Longleftarrow$ " $\forall \varepsilon > 0\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $M_n < \varepsilon\ \forall n \geq n_\varepsilon$

$$\begin{aligned}\implies d(f_n, f) = \sup_{x\in X} d_Y\left(f_n(x), f(x)\right) < \varepsilon \quad \forall n \geq n_\varepsilon \\ \implies d_Y\left(f_n(x), f(x)\right) < \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall x \in X\end{aligned}$$

" $\implies$ "

$$\begin{aligned} f_n \xrightarrow[n\to\infty]{u} f \implies \forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d_Y\left(f_n(x), f(x)\right) < \frac{\varepsilon}{2} \quad \forall n \geq n_\varepsilon \, \forall x \in X \\ \implies \underbrace{\sup_{x\in X} d_Y\left(f_n(x), f(x)\right)}_{d(f_n, f) = M_n} \leq \frac{\varepsilon}{2} < \varepsilon \quad \forall n \geq n_\varepsilon \end{aligned}$$

Remark 7.7.
1. Uniform convergence $\implies$ pointwise convergence
2. Pointwise convergence $\not\Longrightarrow$ uniform convergence

$f_n : [0, 1] \to \mathbb{R},\ f_n(x) = x^n$

$$\{f_n\}_{n\geq 1} \text{ converges pointwise : } \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Let

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Note $f_n \underset{n\to\infty}{\overset{u}{\not\longrightarrow}} f$ since

$$d(f_n, f) = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} |x^n| = 1 \underset{n\to\infty}{\not\longrightarrow} 0$$

Theorem 7.8 (Weierstrass)
Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f_n : X \to Y$ be a sequence of functions that converges uniformly to a function $f : X \to Y$. If $\forall n \geq 1$, f_n is continuous at $x_0 \in X$ then f is continuous at x_0.

Corollary 7.9
A uniform limit of continuous functions is a continuous function.

Proof. (of theorem) Fix $\varepsilon > 0$.

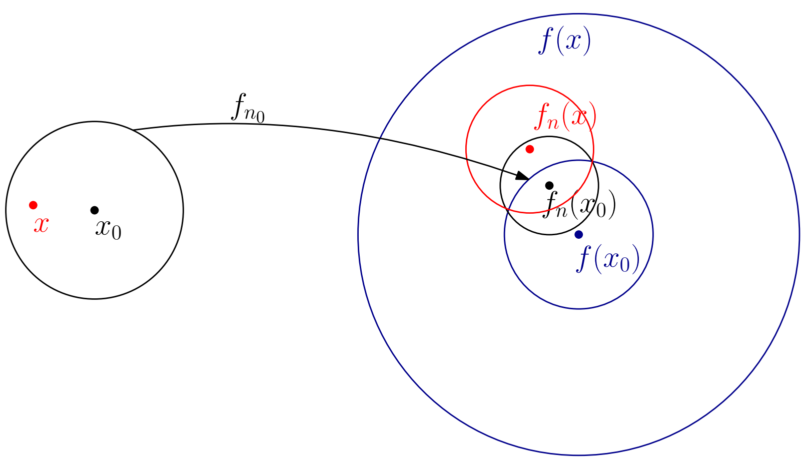
$$f_n \xrightarrow[n\to\infty]{u} f \implies \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d_Y\left(f_n(x), f(x)\right) < \frac{\varepsilon}{3} \quad \forall n \geq n_\varepsilon \, \forall x \in X$$

Fix $n_0 \geq n_\varepsilon$. f_{n_0} is continuous at x_0

$$\implies \exists \delta > 0 \text{ s.t. if } d_X(x_0, x) < \delta$$

then

$$d_Y\left(f_{n_0}(x_0), f_{n_0}(x)\right) < \frac{\varepsilon}{3}$$



Then for $x \in B_\delta(x_0)$ we have

$$\begin{aligned} d_Y\left(f(x), f(x_0)\right) &\leq d_Y\left(f(x), f_{n_0}(x)\right) + d\left(f_{n_0}(x), f_{n_0}(x_0)\right) + d\left(f_{n_0}(x_0), f(x_0)\right) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

By definition, f is continuous at x_0. $\square$

§8 Lec 8: Apr 14, 2021

§8.1 Convergent Sequences of Functions (Cont'd)

Theorem 8.1 (Dini)
Let (X, d) be a compact metric space and let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions that converges pointwise to a continuous function $f : X \to \mathbb{R}$. Assume that $\{f_n\}_{n\geq 1}$ is monotone in the sense that either $\{f_n(x)\}_{n\geq 1}$ is increasing for all $x \in X$ or $\{f_n(x)\}_{n\geq 1}$ is decreasing for all $x \in X$. Then,

$$f_n \xrightarrow[n\to\infty]{u} f \text{ i.e. } d(f_n, f) = \sup_{x\in X} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$$

Proof. Assume that $\{f_n\}_{n\geq 1}$ is increasing. Then $\{f - f_n\}_{n\geq 1}$ is decreasing and for all $x \in X$ we have

$$\lim_{n\to\infty} [f(x) - f_n(x)] = \inf_{n\to\infty} [f(x) - f_n(x)] = 0$$

Then $\forall \varepsilon > 0 \quad \exists n(\varepsilon, x) \in \mathbb{N}$ s.t. $\forall n \geq n(\varepsilon, x)$ we have

$$0 \leq f(x) - f_n(x) \leq f(x) - f_{n_{\varepsilon,x}}(x) < \varepsilon$$

As $f - f_{n_{\varepsilon,x}}$ is continuous at x, $\exists \delta(\varepsilon, x) > 0$ s.t.

$$d(x, y) < \delta_{\varepsilon,x} \implies \left|\left[f(x) - f_{n_{\varepsilon,x}}(x)\right] - \left[f(y) - f_{n_{\varepsilon,x}}(y)\right]\right| < \varepsilon$$

By the triangle inequality, we get

$$\begin{aligned} 0 \leq f(y) - f_{n_{\varepsilon,x}}(y) &\leq \left|\left[f(x) - f_{n_{\varepsilon,x}}(x)\right] - \left[f(y) - f_{n_{\varepsilon,x}}(y)\right]\right| + f(x) - f_{n_{\varepsilon,x}}(x) \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

whenever $y \in B_{\delta_{\varepsilon,x}}(x)$. In particular,

$$0 \leq f(y) - f_n(y) \leq f(y) - f_{n_{\varepsilon,x}}(y) < 2\varepsilon \quad \forall n \geq n_{\varepsilon,x}, \forall y \in B_{\delta_{\varepsilon,x}}(x) \tag{*}$$

Note

$$\left.\begin{array}{l} X = \bigcup_{x\in X} B_{\delta_{\varepsilon,x}}(x) \\ X \text{ compact} \end{array}\right\} \implies \exists \mathcal{J} \subseteq \mathbb{N} \text{ finite and } \exists \{x_j\}_{j\in\mathcal{J}} \in X$$

s.t. $X = \bigcup_{j\in\mathcal{J}} B_{\delta_j}(x_j)$ and where $\delta_j = \delta(\varepsilon, x_j)$.
Let $n_\varepsilon = \max_{j\in\mathcal{J}} n(\varepsilon, x_j)$. Fix $n \geq n_\varepsilon$ and $x \in X$. As $x \in X = \bigcup_{j\in\mathcal{J}} B_{\delta_j}(x_j) \implies j \in \mathcal{J}$ s.t. $x \in B_{\delta_j}(x_j)$. By (*), we have

$$0 \leq f(x) - f_n(x) < 2\varepsilon$$

As $x \in X$ was arbitrary we get

$$d(f, f_n) \leq 2\varepsilon \qquad \forall n \geq n_\varepsilon$$

□

Remark 8.2. The compactness of X is necessary in Dini's theorem.

Example 8.3

$f_n : (0,1) \to \mathbb{R}$, $f_n(x) = x^n$ continuous

$$f_{n+1}(x) \le f_n(x) \quad \forall n \ge 1 \quad \forall x \in (0,1)$$
$$f_n(x) \underset{n\to\infty}{\longrightarrow} 0 \quad \forall x \in (0,1)$$

Let $f : (0,1) \to \mathbb{R}$, $f(x) = 0 \quad \forall x \in (0,1)$. It's continuous. But

$$d(f_n, f) = \sup_{x\in(0,1)} |x^n| = 1 \underset{n\to\infty}{\not\longrightarrow} 0 \implies f_n \underset{n\to\infty}{\overset{u}{\not\longrightarrow}} f$$

Note that $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$ continuous, $\{f_n\}_{n\ge1}$ is decreasing and converge pointwise to $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} \quad \text{which is not continuous}$$

This also shows that the continuity of the limit function is necessary in Dini's theorem.

Remark 8.4. Monotonicity is necessary in Dini's theorem.

Example 8.5

$f_n : [0,1] \to \mathbb{R}$ is continuous. $\{f_n\}_{n\ge1}$ converges pointwise to $f : [0,1] \to \mathbb{R}$, $f(x) = 0 \, \forall x \in [0,1]$ figure here f is continuous. But

$$d(f_n, f) = \sup_{x\in[0,1]} |f_n(x)| = 1 \underset{n\to\infty}{\not\longrightarrow} 0 \implies f_n \underset{n\to\infty}{\overset{u}{\not\longrightarrow}} f$$

Note that $\{f_n\}_{n\ge1}$ is not monotone!

§8.2 Space of Functions

Fix $a, b \in \mathbb{R}$, $a < b$. We define

$$C([a,b]) = \{f : [a,b] \to \mathbb{R};\ f \text{ is continuous}\}$$

We equip $C([a,b])$ with the metric $d : C([a,b]) \times C([a,b]) \to \mathbb{R}$, given by

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

Then $(C([a,b]), d)$ is a metric space.
Completeness: Let $\{f_n\}_{n\ge1} \subseteq C([a,b])$ be Cauchy. So $\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N}$ s.t. $d(f_n, f_m) < \varepsilon$ $\forall n, m \ge n_\varepsilon$

$$\implies |f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \ge n_\varepsilon \quad \forall x \in [a,b]$$

So $\{f_n(x)\}_{n\geq 1}$ is Cauchy $\forall x \in [a,b]$. As $\mathbb{R}$ is complete,

$$\forall x \in [a,b] \quad f_n(x) \xrightarrow[n\to\infty]{} f(x) \in \mathbb{R}$$

This defines a function $f : [a,b] \to \mathbb{R}$. Recall that for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ s.t.

$$\begin{aligned} |f_n(x) - f(x)| \leq \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall x \in [a,b] \\ \implies d\left(f_n, f\right) \leq \varepsilon \quad \forall n \geq n_\varepsilon \end{aligned}$$

So $f_n \xrightarrow[n\to\infty]{u} f$. By Weierstrass, $f \in C\left([a,b]\right)$. Thus $\left(C\left([a,b]\right), d\right)$ is a complete metric space.
Compactness: Note that $\left(C\left([a,b]\right), d\right)$ is not bounded and so not compact.

Example 8.6

$f_n : [a,b] \to \mathbb{R}$, $f_n(x) = n$ for all $x \in [a,b]$.

Connectedness: $\left(C\left([a,b]\right), d\right)$ is path connected and so connected.

Let $f, g \in C\left([a,b]\right)$. Define $\gamma : [0,1] \to C\left([a,b]\right)$ via $\gamma(t) = f + t(g-f)$. Note $\forall t \in [0,1]$, $\gamma(t) \in C\left([a,b]\right)$ and

$$\gamma(0) = f, \quad \gamma(1) = g$$

To see that γ is a path we compute

$$\begin{aligned} d\left(\gamma(t), \gamma(s)\right) &= \sup_{x\in[a,b]} |\gamma(t;x) - \gamma(s;x)| \\ &= \sup_{x\in[a,b]} |t-s|\,|g(x) - f(x)| \\ &= |t-s| \underbrace{d(g,f)}_{\in\mathbb{R}} \xrightarrow[|t-s|\to 0]{} 0 \end{aligned}$$

So γ is a continuous function and so a path.

§9 Lec 9: Apr 16, 2021

§9.1 Arzela–Ascoli Theorem

For $a, b \in \mathbb{R}$ with $a < b$, we define

$$C([a, b]) = \{f : [a, b] \to \mathbb{R};\ f \text{ continuous}\}$$

We equip $C([a, b])$ with the uniform metric

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

We showed that $(C([a, b]), d)$ is a complete, connected metric space, but it's not compact.

Definition 9.1 (Equicontinuity) — We say that a set $\mathcal{F} \subseteq C([a, b])$ is equicontinuous if

$$\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \text{ s.t. } |f(x) - f(y)| < \varepsilon \quad \forall x, y \in [a, b] \text{ with } |x - y| < \delta(\varepsilon)$$

and for all $f \in \mathcal{F}$.

Note: For a fixed function $f \in \mathcal{F} \subseteq C([a, b])$, we have that f is uniformly continuous (since f is continuous on $[a, b]$ compact) which means for all $\varepsilon > 0$, there exists $\delta(\varepsilon, f) > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \quad \forall x, y \in [a, b] \text{ with } |x - y| < \delta(\varepsilon, f)$$

Note that for an equicontinuous family $\mathcal{F}$, δ_ε can be chosen uniformly for $f \in \mathcal{F}$.

Definition 9.2 (Uniformly Bounded) — We say that a set $\mathcal{F} \subseteq C([a, b])$ is uniformly bounded if $\exists M > 0$ s.t. $|f(x)| \leq M$ $\forall x \in [a, b]$ $\forall f \in \mathcal{F}$.

Note: For a fixed $f \in \mathcal{F} \subseteq C[a, b]$ we have that $f([a, b])$ is bounded (since f continuous and $[a, b]$ compact which implies $f([a, b])$ is compact and so bounded). So $\exists M_f > 0$ s.t. $|f(x)| \leq M_f$ $\forall x \in [a, b]$. For a uniformly bounded family $\mathcal{F}$, we can choose the bound M uniformly for $f \in \mathcal{F}$.

Theorem 9.3 (Arzela-Ascoli)
Let $\mathcal{F} \subseteq C([a, b])$. The following are equivalent:

1. $\mathcal{F}$ is uniformly bounded and equicontinuous.
2. Every sequence in $\mathcal{F}$ admits a convergent subsequence.

Caution: We cannot guarantee that the limit of the convergent subsequence belongs to $\mathcal{F}$, unless $\mathcal{F}$ is closed in $C([a, b])$. If $\mathcal{F}$ is closed in $C([a, b])$, then the theorem becomes

$$\mathcal{F} \text{ is compact } \iff \mathcal{F} \text{ is uniformly bounded and equicontinuous}$$

Proof. 2) $\implies$ 1)

Claim 9.1. $\mathcal{F}$ is totally bounded.

Fix $\varepsilon > 0$. Let $f_1 \in \mathcal{F}$.

$$\begin{aligned}
&\text{If } \mathcal{F} \subseteq B_\varepsilon(f_1) \text{ then } \mathcal{F} \text{ is totally bounded} \\
&\text{If } \mathcal{F} \nsubseteq B_\varepsilon(f_1) \text{ then } \exists f_2 \in \mathcal{F} \text{ s.t. } d\left(f_1, f_2\right) \geq \varepsilon \\
&\text{If } \mathcal{F} \subseteq B_\varepsilon(f_1) \cup B_\varepsilon(f_2) \text{ then } \mathcal{F} \text{ is totally bounded} \\
&\text{If } \mathcal{F} \nsubseteq B_\varepsilon(f_1) \cup B_\varepsilon(f_2) \text{ then } \exists f_3 \in \mathcal{F} \text{ s.t. } \begin{cases} d\left(f_1, f_3\right) \geq \varepsilon \\ d\left(f_2, f_3\right) \geq \varepsilon \end{cases}
\end{aligned}$$

If the process terminates in finitely many steps, then $\mathcal{F}$ is totally bounded. Otherwise, we find $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$ s.t. $d\left(f_n, f_m\right) \geq \varepsilon \, \forall n \neq m$. This sequence does not admit a convergent subsequence, leading a contradiction.

Let's show that $\mathcal{F}$ is uniformly bounded. As $\mathcal{F}$ is totally bounded, $\exists n \geq 1$ and $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t.

$$\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_1(f_j) \subseteq B_r(f_1)$$

where $r = 1 + \max_{2\leq j\leq n} d\left(f_1, f_j\right)$. In particular, for all $f \in \mathcal{F}$,

$$d\left(f, f_1\right) < r$$

f_1 is continuous on compact $[a, b] \implies \exists M_{f_1} > 0$ s.t.

$$|f_1(x)| \leq M_{f_1} \quad \forall x \in [a, b]$$

So for $f \in \mathcal{F}$

$$|f(x)| \leq |f(x) - f_1(x)| + |f_1(x)| \leq d(f, f_1) + M_{f_1} < r + M_{f_1} \quad \forall x \in [a, b]$$

So $\mathcal{F}$ is uniformly bounded.
Let's show that $\mathcal{F}$ is equicontinuous. Let $\varepsilon > 0$. As $\mathcal{F}$ is totally bounded, $\exists n \geq 1$ and $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t.

$$\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_{\frac{\varepsilon}{3}}(f_j)$$

For each $1 \leq j \leq n$, f_j is uniformly continuous on $[a, b]$. So $\exists \delta_j(\varepsilon) > 0$ s.t.

$$|f_j(x) - f_j(y)| < \frac{\varepsilon}{3} \quad \forall x, y \in [a, b] \text{ with } |x - y| < \delta_j(\varepsilon)$$

Let $\delta_\varepsilon = \min_{1\leq j\leq n} \delta_j(\varepsilon) > 0$.
Fix $f \in \mathcal{F} \implies \exists 1 \leq j \leq n$ s.t. $f \in B_{\frac{\varepsilon}{3}}(f_j)$. Then for $x, y \in [a, b]$ with $|x - y| < \delta_\varepsilon$ we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\
&\leq 2d(f, f_j) + |f_j(x) - f_j(y)| \\
&\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

This shows $\mathcal{F}$ is equicontinuous.
1) $\implies$ 2) Let $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$. As $\mathcal{F}$ is uniformly bounded,

$$\exists M > 0 \text{ s.t. } |f(x)| \leq M \quad \forall x \in [a,b]\ \forall f \in \mathcal{F}$$

In particular, $|f_n(x)| \leq M\ \forall x \in [a,b]\ \forall n \geq 1$.
Let $\{r_n\}_{n\geq 1}$ denote an enumeration of the rationals in $[a,b]$. As $\{f_n(r_1)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded by M, $\exists \left\{f_n^{(1)}\right\}_{n\geq 1}$ subsequence of $\{f_n\}_{n\geq 1}$ s.t. $\left\{f_n^{(1)}(r_1)\right\}_{n\geq 1}$ converges. $\left\{f_n^{(1)}(r_2)\right\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded by $M \implies \exists \left\{f_n^{(2)}\right\}_{n\geq 1}$ subsequence of $\left\{f_n^{(1)}\right\}_{n\geq 1}$ s.t. $\left\{f_n^{(2)}(r_2)\right\}_{n\geq 1}$ converges.
Proceeding inductively we find $\forall k \geq 1$ $\left\{f_n^{(k+1)}\right\}_{n\geq 1}$ is a subsequence of $\left\{f_n^{(k)}\right\}_{n\geq 1}$ and $\left\{f_n^{(k)}(r_k)\right\}_{n\geq 1}$ converges.
We consider $\left\{f_n^{(n)}\right\}_{n\geq 1}$ subsequence of $\{f_n\}_{n\geq 1}$.
For $n, m \geq k$, $f_n^{(n)}$, $f_m^{(m)}$ are elements in $\left\{f_n^{(k)}\right\}_{n\geq 1}$. So $\left\{f_n^{(n)}\right\}_{n\geq 1}$ converges at r_k.
<u>Caution</u>: The convergence is not uniform in k.
Fix $\varepsilon > 0$. As $\mathcal{F}$ is equicontinuous, $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{3} \quad \forall x, y \in [a,b]\ |x-y| < \delta, \forall f \in \mathcal{F}$$

In particular,

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3} \quad \forall x, y \in [a,b]\ |x-y| < \delta, \forall n \geq 1 \tag{*}$$

Let $r_1, \ldots, r_N \in \mathbb{Q} \cap [a,b]$ s.t. $a = r_0 < r_1 < \ldots < r_N < r_{N+1} = b$ and

$$|r_{j+1} - r_j| < \delta \qquad 0 \leq j \leq N$$

Note $N \sim \frac{|a-b|}{\delta}$. For each $1 \leq j \leq N$, $\exists n_j(\varepsilon) \in \mathbb{N}$ s.t.

$$\left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| < \frac{\varepsilon}{3} \qquad \forall n, m \geq n_j(\varepsilon)$$

Let $n_\varepsilon = \max_{1\leq j\leq N} n_j(\varepsilon)$. Note

$$\left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| < \frac{\varepsilon}{3} \quad \forall n, m \geq n_\varepsilon \quad \forall 1 \leq j \leq N \tag{**}$$

Let $x \in [a,b] \implies \exists 1 \leq j \leq N$ s.t. $|x - r_j| < \delta$. Then

$$\begin{aligned}\left|f_n^{(n)}(x) - f_m^{(m)}(x)\right| &\leq \left|f_n^{(n)}(x) - f_n^{(n)}(r_j)\right| + \left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| + \left|f_m^{(m)}(r_j) - f_m^{(m)}(x)\right| \\ &\text{By (*) and (**)} < 2 \cdot \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \forall n, m \geq n_\varepsilon\end{aligned}$$

So $\left\{f_n^{(n)}\right\}_{n\geq 1}$ is uniformly Cauchy and so uniformly convergent. □

> **Remark 9.4.** One can replace $[a,b]$ by any other compact metric space (X,d).

§10 Lec 10: Apr 19, 2021

§10.1 Arzela-Ascoli Theorem (Cont'd)

Remark 10.1. The compactness of the set on which the functions are defined is necessary in Arzela-Ascoli.

Example 10.2

$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R};\ |f(x) - f(y)| \le |x-y|\ \forall x, y \in \mathbb{R}$ and $\sup_{x\in\mathbb{R}} |f(x)| \le 1\}$. Note $\mathcal{F}$ is equicontinuous and uniformly bounded. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$

Claim 10.1. $f \in \mathcal{F}$.

Indeed,

$$\sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+x^2} = 1$$

Moreover, for $x, y \in \mathbb{R}$

$$\begin{aligned} |f(x) - f(y)| = \left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| &= \frac{\left|x^2 - y^2\right|}{(1+x^2)(1+y^2)} \\ &= |x-y| \cdot \frac{|x+y|}{(1+x^2)(1+y^2)} \\ &\le |x-y| \, (\underbrace{\frac{|x|}{1+x^2}}_{\le \frac{1}{2}} + \underbrace{\frac{|y|}{1+y^2}}_{\le \frac{1}{2}}) \\ &\le |x-y| \end{aligned}$$

So $f \in \mathcal{F}$.
For $n \ge 1$, let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = f(x-n)$. Note $f_n \in \mathcal{F}$ since $\sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+(x-n)^2} = 1$.

$$\begin{aligned} |f_n(x) - f_n(y)| = |f(x-n) - f(y-n)| &\le |(x-n) - (y-n)| \\ &= |x-y| \end{aligned}$$

Note that $\{f_n\}_{n\ge1}$ converge pointwise to $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ since $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+(x-n)^2} = 0$. However, $\{f_n\}_{n\ge1}$ does not admit a subsequence that converges uniformly since $\forall n \ge 1$

$$d(f_n, f) = \sup_{x\in\mathbb{R}} |f_n(x)| = 1 \overset{n\to\infty}{\not\to} 0$$

Remark 10.3. Uniform boundedness is necessary in Arzela-Ascoli.

Example 10.4

$\mathcal{F} = \{f : \underbrace{[0,1]}_{\text{compact}} \to \mathbb{R};\, f \text{ is continuous and } \underbrace{\sup_{x\in[0,1]} |f(x)| \le 1}_{\text{uniformly bounded}}\}$.

Claim 10.2. $\mathcal{F}$ is not equicontinuous.

For $n \ge 1$, let $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \sin(nx)$. Note $f_n \in \mathcal{F}$. Let $x_n = \frac{3\pi}{2n}$, $y_n = \frac{\pi}{2n}$. Then $|x_n - y_n| = \frac{\pi}{n} \underset{n\to\infty}{\longrightarrow} 0$ but

$$|f_n(x_n) - f_n(y_n)| = 2$$

So $\{f_n\}_{n\ge1}$ is not equicontinuous $\implies \mathcal{F}$ is not equicontinuous.

Claim 10.3. $\{f_n\}_{n\ge1}$ does not admit a convergent subsequence.

Assume, towards a contradiction, that there exists a subsequence $\{f_{k_n}\}_{n\ge1}$ of $\{f_n\}_{n\ge1}$ that converges uniformly to $f : [0,1] \to \mathbb{R}$. By Weierstrass,

$$\left.\begin{array}{l} f \in C([0,1]) \\ \left.\begin{array}{l} f_{k_n}(0) = 0 \quad \forall n \ge 1 \\ f_{k_n}(0) \underset{n\to\infty}{\longrightarrow} f(0) \end{array}\right\} \implies f(0) = 0 \end{array}\right\} \implies \forall \varepsilon > 0\, \exists \delta > 0 \text{ s.t. } |f(x)| < \varepsilon \,\forall 0 < x < \delta$$

$f_{k_n} \underset{n\to\infty}{\overset{u}{\longrightarrow}} f \implies \exists n_\varepsilon \in \mathbb{N}$ s.t. $d(f_{k_n}, f) < \varepsilon \ \forall n \ge n_\varepsilon$. In particular, for $0 < x < \delta$ and $n \ge n_\varepsilon$ we have

$$|f_{k_n}(x)| \le |f_{k_n}(x) - f(x)| + |f(x)| < d(f_{k_n}, f) + \varepsilon < 2\varepsilon$$

Choosing $\varepsilon \le \frac{1}{2}$ and N large so that $N \ge n_{\varepsilon=\frac{1}{2}}$ and $\frac{\pi}{2N} < \delta_{\varepsilon=\frac{1}{2}}$ we find

$$1 = \left|f_{k_N}\left(\frac{\pi}{2N}\right)\right| < 2\varepsilon \le 1 \qquad \text{Contradiction!}$$

§10.2 The oscillation of a Real Function

Definition 10.5 (Oscillation of a Function) — Let (X,d) be a metric space and let $f : X \to \mathbb{R}$ be a function. For $\emptyset \ne A \subseteq X$, the oscillation of f on A is

$$\omega(f, A) = \sup_{x\in A} f(x) - \inf_{x\in A} f(x) = \sup_{x,y\in A} [f(x) - f(y)] \ge 0$$

Note that if $A \subseteq B$ then

$$\omega(f, A) \le \omega(f, B)$$

For $x_0 \in X$, the oscillation of f at x_0 is given by

$$\omega(f, x_0) = \inf_{\delta>0} \omega(f, B_\delta(x_0))$$

Proposition 10.6

Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then f is continuous at a point $x_0 \in X$ if and only if $\omega(f, x_0) = 0$.

Proof. " $\implies$ " Fix $\varepsilon > 0$. As f is continuous at x_0, $\exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \frac{\varepsilon}{4}$ $\forall x \in B_\delta(x_0)$.

$$\begin{aligned}
&\implies |f(x) - f(y)| \le |f(x) - f(x_0)| + |f(x_0) - f(y)| < \frac{\varepsilon}{2} \quad \forall x, y \in B_\delta(x_0) \\
&\implies \omega(f, B_\delta(x_0)) = \sup_{x,y \in B_\delta(x_0)} [f(x) - f(y)] \le \frac{\varepsilon}{2} < \varepsilon \\
&\implies \omega(f, x_0) \le \omega(f, B_\delta(x_0)) < \varepsilon
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, $\omega(f, x_0) = 0$.
" $\impliedby$ " Fix $\varepsilon > 0$. Then $\omega(f, x_0) = 0 < \varepsilon$ implies $\exists \delta > 0$ s.t. $\omega(f, B_\delta(x_0)) < \varepsilon$

$$\begin{aligned}
&\implies |f(x) - f(y)| < \varepsilon \qquad \forall x, y \in B_\delta(x_0) \\
&\implies |f(x) - f(x_0)| < \varepsilon \qquad \forall x \in B_\delta(x_0)
\end{aligned}$$

So f is continuous at x_0. $\square$

Lemma 10.7

Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then for any $\alpha > 0$,

$$\{x \in X : \omega(f, x) < \alpha\} \text{ is open in } X$$

Proof. Fix $\alpha > 0$ and let $A = \{x \in X : \omega(f, x) < \alpha\}$. Fix $x_0 \in A \implies \omega(f, x_0) = \inf_{\delta > 0} \omega(f, B_\delta(x_0)) < \alpha$.

$$\implies \exists \delta > 0 \text{ s.t. } \omega(f, B_\delta(x_0)) < \alpha$$

Claim 10.4. $B_\delta(x_0) \subseteq A$ (which implies $x_0 \in \mathring{A}$ and so $A = \mathring{A}$).

Let $x \in B_\delta(x_0)$. Then $r = \delta - d(x, x_0) > 0$ and $B_r(x) \subseteq B_\delta(x_0)$

$$\begin{aligned}
&\implies \omega(f, B_r(x)) \le \omega(f, B_\delta(x_0)) < \alpha \\
&\implies \omega(f, x) \le \omega(f, B_r(x)) < \alpha \implies x \in A
\end{aligned}$$

$\square$

Remark 10.8. Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then

$$\begin{aligned}
\{x \in X : f \text{ is continuous at } x\} &= \{x \in X : \omega(f, x) = 0\} \\
&= \bigcap_{n \ge 1} \underbrace{\left\{x \in X : \omega(f, x) < \frac{1}{n}\right\}}_{=G_n}
\end{aligned}$$

By the lemma, $G_n = \mathring{G}_n$ $\forall n \ge 1$. Also, $G_{n+1} \subseteq G_n$ $\forall n \ge 1$. This observation allows us to prove that there are no functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every rational point and discontinuous at every irrational point.

§11 Lec 11: Apr 21, 2021

§11.1 Oscillation of a Function (Cont'd)

Recall from last lecture that there are no functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every rational point and discontinuous at every irrational point.

Proof. (Sketch) Assume, towards a contradiction, that $f : \mathbb{R} \to \mathbb{R}$ is such a function. Then

$$\mathbb{Q} = \{x \in \mathbb{R} : f \text{ is continuous at } x\} = \bigcap_{n \geq 1} G_n \text{ with } G_n \text{ open in } \mathbb{R}$$

Note $\forall n \geq 1$, $Q \subseteq G_n$

$$\begin{aligned} &\implies \mathbb{R} = \overline{\mathbb{Q}} \subseteq \overline{G_n} \subseteq \mathbb{R} \\ &\implies \overline{G_n} = \mathbb{R} \text{ i.e. } G_n \text{ is dense in } \mathbb{R} \end{aligned}$$

Let $\{q_n\}_{n\geq 1}$ be an enumeration of $\mathbb{Q}$. For each $n \geq 1$, let $H_n = \mathbb{R}\backslash\{q_n\} = (-\infty, q_n) \cup (q_n, \infty)$. Note H_n is open and dense ($\overline{H_n} = \mathbb{R}$) in $\mathbb{R}$. Also

$$\bigcap_{n \geq 1} H_n = \mathbb{R} \setminus \mathbb{Q}$$

So

$$\bigcap_{n \geq 1} G_n \cap \bigcap_{n \geq 1} H_n = \mathbb{Q} \cap \mathbb{R} \setminus \mathbb{Q} = \emptyset$$

This contradicts the following property of $\mathbb{R}$:

Exercise 11.1. If $\{A_n\}_{n\geq 1}$ is a countable collection of open and dense subsets of $\mathbb{R}$, then

$$\overline{\bigcap_{n \geq 1} A_n} = \mathbb{R}$$

Apply this exercise with $\{A_n : n \geq 1\} = \{G_n : n \geq 1\} \cup \{H_n : n \geq 1\}$. □

§11.2 Weierstrass Approximation Theorem

Theorem 11.1 (Weierstrass Approximation)
Fix $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ with $\deg P_n \leq n \; \forall n \geq 1$ s.t.

$$P_n \xrightarrow[n\to\infty]{u} f \quad \text{on} \quad [a, b]$$

Proof. First, we reduce to the case when $[a, b]$ is $[0, 1]$. Let $\phi : [0, 1] \to [a, b]$, $\phi(t) = a + t(b - a)$. Note ϕ is a continuous, bijective function with the inverse

$$\phi^{-1} : [a, b] \to [0, 1]\,, \quad \phi^{-1}(x) = \frac{x - a}{b - a} \text{ continuous}$$

As $f : [a, b] \to \mathbb{R}$ is continuous, $f \circ \phi : [0, 1] \to \mathbb{R}$ is continuous.
If $\{P_n\}_{n\geq 1}$ is a sequence of polynomials with $\deg P_n \leq n$ s.t.

$$P_n \xrightarrow[n\to\infty]{u} f \circ \phi \text{ on } [0,1]$$

then $P_n \circ \phi^{-1} \xrightarrow[n\to\infty]{u} f$ on $[a, b]$. Indeed,

$$\sup_{x\in[a,b]} \left| \left(P_n \circ \phi^{-1}\right)(x) - f(x)\right| \underset{x=\phi(t)}{=} \underbrace{\sup_{t\in[0,1]} |P_n(t) - (f \circ \phi)(t)|}_{\xrightarrow[n\to\infty]{} 0}$$

Therefore, we may assume $f : [0, 1] \to \mathbb{R}$ is continuous. Define the Bernstein polynomials via

$$P_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \qquad \deg P_n \leq n$$

Note that if f is a constant, say $f(x) = c\ \forall x \in [0, 1]$ then

$$P_n(x) = c\sum_{k=0}^{n}\binom{n}{k} x^k(1-x)^{n-k} = c\,(x+1-x)^n = c \quad \forall x \in [0,1]\ \forall n \geq 1$$

We want to show $P_n \xrightarrow[n\to\infty]{u} f$ on $[0, 1]$. Fix $x \in [0, 1]$. Consider

$$\begin{aligned}
|f(x) - P_n(x)| &= \left| f(x)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}\right| \\
&= \left|\sum_{k=0}^{n}\left[f(x) - f\left(\frac{k}{n}\right)\right]\binom{n}{k}x^k(1-x)^{n-k}\right| \\
&\leq \sum_{k=0}^{n}\left|f(x) - f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}
\end{aligned}$$

To estimate the sum we use the following

- when $\frac{k}{n}$ is close to x, we use the continuity of f.
- when $\frac{k}{n}$ is far from x, we use the fact that $x \overset{g}{\mapsto} x^k(1-x)^{n-k}$ has a local maximum at $x = \frac{k}{n}$.

$$\begin{aligned}
g'(x) &= kx^{k-1}(1-x)^{n-k} - (n-k)x^k(1-x)^{n-k-1} \\
&= x^{k-1}(1-x)^{n-k-1}\left\{k(1-x) - (n-k)x\right\} \\
&= x^{k-1}(1-x)^{n-k-1}\left\{k - nx\right\} \\
&= \begin{cases} > 0 & \text{if } x < \frac{k}{n} \\ = 0 & \text{if } x = \frac{k}{n} \\ < 0 & \text{if } x > \frac{k}{n} \end{cases}
\end{aligned}$$

$f : [0,1] \to \mathbb{R}$ is continuous $\implies$ f is uniformly continuous. Fix $\varepsilon > 0$. Then $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad x, y \in [0,1], \quad |x-y| < \delta$$

$f : [0,1] \to \mathbb{R}$ is continuous $\implies$ f is bounded. Let $M > 0$ be s.t.

$$|f(x)| \le M \qquad \forall x \in [0,1]$$

We estimate

$$\begin{aligned}
|f(x) - P_n(x)| &\le \sum_{\substack{0\le k\le n \\ \left|x-\frac{k}{n}\right|<\delta}} \underbrace{\left|f(x) - f\left(\frac{k}{n}\right)\right|}_{<\varepsilon} \binom{n}{k} x^k (1-x)^{n-k} \\
&+ \sum_{\substack{0\le k\le n \\ \left|x-\frac{k}{n}\right|\ge\delta}} \underbrace{\left|f(x) - f\left(\frac{k}{n}\right)\right|}_{\le 2M} \binom{n}{k} x^k (1-x)^{n-k} \\
&\le \varepsilon \sum_{0\le k\le n} \binom{n}{k} x^k (1-x)^{n-k} + 2M \sum_{0\le k\le n} \frac{\left(x-\frac{k}{n}\right)^2}{\delta^2} \binom{n}{k} x^k (1-x)^{n-k} \\
&\le \varepsilon + \frac{2M}{n^2\delta^2} \sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k}
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} &= n^2x^2 \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}}_{=1} \\
&- 2nx \sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} + \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} &= x \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} \\
&= nx \underbrace{\sum_{l=0}^{n-1} \frac{(n-1)!}{l!(n-1-l)!} x^l (1-x)^{n-1-l}}_{=(x+1-x)^{n-1}} \\
&= nx
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} &= nx \sum_{k=1}^{n} \frac{k(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= nx \sum_{k=1}^{n} \frac{(k-1+1)(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= n(n-1)x^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} x^{k-2}(1-x)^{n-k} \\
&+ nx \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= n(n-1)x^2 + nx
\end{aligned}$$

So

$$\begin{aligned}
\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} &= n^2x^2 - 2n^2x^2 + n(n-1)x^2 + nx \\
&= nx(1-x)
\end{aligned}$$

We get

$$\begin{aligned}
|f(x) - P_n(x)| &\le \varepsilon + \frac{2M}{n^2\delta^2} \cdot nx(1-x) \\
&\le \varepsilon + \frac{2M}{n\delta^2} \sup_{x\in[0,1]} x(1-x) \\
&\le \varepsilon + \frac{M}{2\delta^2 n} < 2\varepsilon
\end{aligned}$$

provided $n > \frac{M}{2\delta^2\varepsilon}$. So $P_n \xrightarrow[n\to\infty]{u} f$ on $[0,1]$. $\square$

§12 Lec 12: Apr 23, 2021

§12.1 Weierstrass Approximation Theorem (Cont'd)

Corollary 12.1
Let $M > 0$. Then there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ s.t.

$$\begin{cases} \deg P_n \leq n & \forall n \geq 1 \\ P_n(0) = 0 & \forall n \geq 1 \\ P_n \xrightarrow[n\to\infty]{u} |x| \text{ on } [-M, M] \end{cases}$$

Proof. Let $f : [-M, M] \to \mathbb{R}$, $f(x) = |x|$. Then f is continuous and $[-M, M]$ compact. By Weierstrass Approximation, $\exists \{Q_n\}_{n\geq 1}$ sequence of polynomials s.t.

$$\begin{cases} \deg Q_n \leq n & \forall n \geq 1 \\ Q_n \xrightarrow[n\to\infty]{u} f \text{ on } [-M, M] \end{cases}$$

Note $Q_n \xrightarrow[n\to\infty]{u} f \implies Q_n(0) \xrightarrow[n\to\infty]{} f(0) = 0$.
Let $P_n(x) = Q_n(x) - Q_n(0)$. Then

$$\begin{cases} \deg P_n \leq n & \forall n \geq 1 \\ P_n(0) = 0 & \forall n \geq 1 \end{cases}$$

For $x \in [-M, M]$,

$$\begin{aligned} |P_n(x) - f(x)| &\leq |Q_n(x) - f(x)| + |Q_n(0)| \leq d(Q_n, f) + |Q_n(0)| \\ \implies d(P_n, f) &\leq d(Q_n, f) + |Q_n(0)| \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

□

§12.2 Stone-Weierstrass Theorem

Definition 12.2 (Algebra) — Let (X, d) be a metric space and let

$$\mathcal{A} \subseteq \{f : X \to \mathbb{R}(\text{or } \mathbb{C});\ f \text{ is a function}\}$$

We say that $\mathcal{A}$ is an algebra if

1. $f + g \in \mathcal{A} \quad \forall f, g \in \mathcal{A}$.
2. $fg \in \mathcal{A} \quad \forall f, g \in \mathcal{A}$
3. $\lambda f \in \mathcal{A} \quad \forall f \in \mathcal{A}\ \forall \lambda \in \mathbb{R}(\text{or } \mathbb{C})$

We say that the algebra $\mathcal{A}$ separates points if whenever $x, y \in X$ with $x \neq y$ then $\exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$.
We say that the algebra $\mathcal{A}$ vanishes at no point in X if $\forall x \in X\ \exists f \in \mathcal{A}$ s.t. $f(x) \neq 0$.

Lemma 12.3

Let (X, d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra. Then its closure $\overline{\mathcal{A}}$ with respect to the uniform topology is also an algebra.

Proof. Let $f, g \in \mathcal{A}$. Then

$$\begin{cases} \exists f_n \in \mathcal{A} \text{ s.t. } f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ \exists g_n \in \mathcal{A} \text{ s.t. } g_n \xrightarrow[n\to\infty]{u} g \text{ on } X \end{cases}$$

$$\left.\begin{aligned} &d\left(f_n + g_n, f + g\right) \le d(f_n, f) + d(g_n, g) \xrightarrow[n\to\infty]{} 0 \\ &f_n + g_n \in \mathcal{A} \text{ (because } \mathcal{A} \text{ is an algebra)} \end{aligned}\right\} \implies f + g \in \overline{\mathcal{A}}$$

Similarly, for $\lambda \in \mathbb{R}$,

$$\left.\begin{aligned} &d\left(\lambda f_n, \lambda f\right) \le |\lambda| d\left(f_n, f\right) \xrightarrow[n\to\infty]{} 0 \\ &\lambda f_n \in \mathcal{A} \text{ (because } \mathcal{A} \text{ is an algebra)} \end{aligned}\right\} \implies \lambda f \in \overline{\mathcal{A}}$$

Then

$$\begin{aligned} d\left(f_n g_n, fg\right) &= \sup_{x\in X} |f_n(x) g_n(x) - f(x) g(x)| \\ &\le \sup_{x\in X} \left[|f_n(x) - f(x)|\,|g_n(x)| + |f(x)|\,|g_n(x) - g(x)|\right] \\ &\le d(f_n, f) \sup_{x\in X} |g_n(x)| + d(g_n, g) \sup_{x\in X} |f(x)| \end{aligned}$$

By Weierstrass,

$$\left.\begin{aligned} &f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ &f_n \in C(X) \end{aligned}\right\} \implies \left.\begin{aligned} &f \in C(X) \\ &X \text{ compact} \end{aligned}\right\} \implies \exists M > 0 \text{ s.t. } \sup_{x\in X} |f(x)| \le M$$

Similarly, $g \in C(X) \implies \exists M_2 > 0$ s.t. $\sup_{x\in X} |g(x)| \le M_2$

$$d(g_n, 0) \le d(g_n, g) + d(g, 0) \le 1 + M_2 \qquad \forall n \ge n_1$$

Let $M_3 = \max\left\{1 + M_2, \underbrace{d(g_1, 0)}_{<\infty}, \ldots, \underbrace{d(g_{n_1}, 0)}_{<\infty}\right\}$. So $d(g_n, 0) \le M_3 \,\forall n \ge 1$. Thus

$$\left.\begin{aligned} &d\left(f_n g_n, fg\right) \le d(f_n, f) \cdot M_3 + d(g_n, g) \cdot M_1 \xrightarrow[n\to\infty]{} 0 \\ &f_n g_n \in \mathcal{A} \text{ (since } \mathcal{A} \text{ is an algebra)} \end{aligned}\right\} \implies f \cdot g \in \overline{\mathcal{A}}$$

□

Lemma 12.4

Let (X, d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes at no point in X. Then

$$\forall \alpha, \beta \in \mathbb{R} \quad \forall x_1, x_2 \in X \text{ s.t. } x_1 \ne x_2 \quad \exists f \in \mathcal{A} \text{ s.t. } \begin{cases} f(x_1) = \alpha \\ f(x_2) = \beta \end{cases}$$

Proof. Fix $\alpha, \beta \in \mathbb{R}$. Fix $x_1, x_2 \in X$ s.t. $x_1 \neq x_2$. We would like

$$f(x) = \alpha \cdot \frac{u(x)}{u(x_1)} + \beta \cdot \frac{v(x)}{v(x_1)}$$

for $u, v \in \mathcal{A}$ s.t.

$$\begin{aligned} u(x_1) \neq 0 \quad &\text{and} \quad u(x_2) = 0 \\ v(x_1) = 0 \quad &\text{and} \quad v(x_2) \neq 0 \end{aligned}$$

Then $f \in \mathcal{A}$ (because $\mathcal{A}$ is an algebra) is the desired function.

As $\mathcal{A}$ separates points, $\exists g \in \mathcal{A}$ s.t. $g(x_1) \neq g(x_2)$.

As $\mathcal{A}$ vanishes at no point in X,

$$\begin{cases} \exists h \in \mathcal{A} \text{ s.t } h(x_1) \neq 0 \\ \exists k \in \mathcal{A} \text{ s.t. } k(x_2) \neq 0 \end{cases}$$

Then, we define

$$\begin{aligned} u(x) &= [g(x) - g(x_2)] \cdot h(x) \in \mathcal{A} \\ v(x) &= [g(x) - g(x_1)] \cdot k(x) \in \mathcal{A} \end{aligned}$$

□

Theorem 12.5 (Stone-Weierstrass)

Let (X, d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes no point in X. Then $\mathcal{A}$ is dense in $C(X)$, i.e., $\overline{\mathcal{A}} = C(X) = \{f : X \to \mathbb{R};\ f \text{ continuous}\}$.

Proof. Want to show $\forall f \in C(X)\ \forall \varepsilon > 0\ \exists g \in \mathcal{A}$ s.t. $d(f, g) < \varepsilon$.

Step 1: If $f \in \overline{\mathcal{A}}$ then $|f| \in \overline{\mathcal{A}}$. Let $f \in \overline{\mathcal{A}} \implies \exists f_n \in \mathcal{A}$ s.t.

$$\left.\begin{aligned} &f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ &f_n \in C(X) \end{aligned}\right\} \implies f \in C(X)$$

As X is compact, $\exists M > 0$ s.t. $|f(x)| \leq M\ \forall x \in X$. By the previous Corollary 12.1, $\exists \{P_n\}_{n\geq 1}$ sequence of polynomials with $\deg P_n \leq n\ \forall n \geq 1$ s.t.

$$\begin{cases} P_n \xrightarrow[n\to\infty]{u} |x| \text{on } [-M, M] \\ P_n(0) = 0 \end{cases} \implies P_n(f) \xrightarrow[n\to\infty]{u} |f| \text{ on } X$$

If $P_n(x) = \sum_{k=1}^n c_k x^k$ then $P_n(f) = \sum_{k=1}^n c_k f^k \in \mathcal{A}$ which implies $|f| \in \overline{\mathcal{A}}$.

Step 2: If $f, g \in \overline{\mathcal{A}}$ then $\max\{f, g\}$, $\min\{f, g\} \in \overline{\mathcal{A}}$.

$$\begin{aligned} \max\{f, g\} &= \frac{f+g}{2} + \frac{|f-g|}{2} \in \overline{\mathcal{A}} \\ \min\{f, g\} &= \frac{f+g}{2} - \frac{|f-g|}{2} \in \overline{\mathcal{A}} \end{aligned}$$

Step 3: $\forall f \in C(X)$, $\forall x \in X$, $\forall \varepsilon > 0$, $\exists g \in \overline{\mathcal{A}}$ s.t.

$$g(x) = f(x) \quad \text{and} \quad g(y) > f(y) - \varepsilon \quad \forall y \in X$$

Continue in the next lecture. □

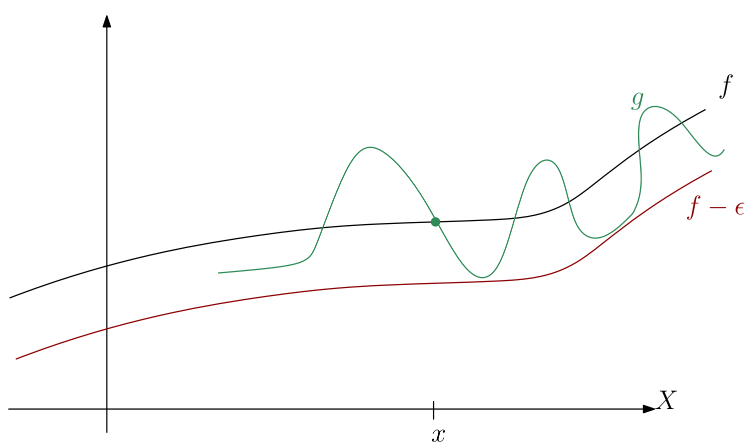
§13 Lec 13: Apr 26, 2021

§13.1 Stone-Weierstrass Theorem (Cont'd)

We continue with the proof of Stone-Weierstrass from lecture 12. Recall that we are at step 3 so far.

Proof. **Step 3**: For any $f \in C(X)$, $x \in X$, $\varepsilon > 0$, there exists $g \in \overline{\mathcal{A}}$ s.t.

$$\begin{cases} g(x) = f(x) \\ g(y) > f(y) - \varepsilon \qquad \forall y \in X \end{cases}$$



For any $y \in X$, there exists $h_y \in \overline{\mathcal{A}}$ s.t.

$$h_y(x) = f(x)$$
$$h_y(y) = f(y)$$

As $h_y \in \overline{\mathcal{A}}$, h_y is continuous. Thus, $h_y - f$ is continuous at y. So $\exists \delta_y > 0$ s.t. $|h_y(z) - f(z)| < \varepsilon$, $\forall z \in B_{\delta_y}(y)$. In particular,

$$h_y(z) > f(z) - \varepsilon \qquad \forall z \in B_{\delta_y}(y)$$

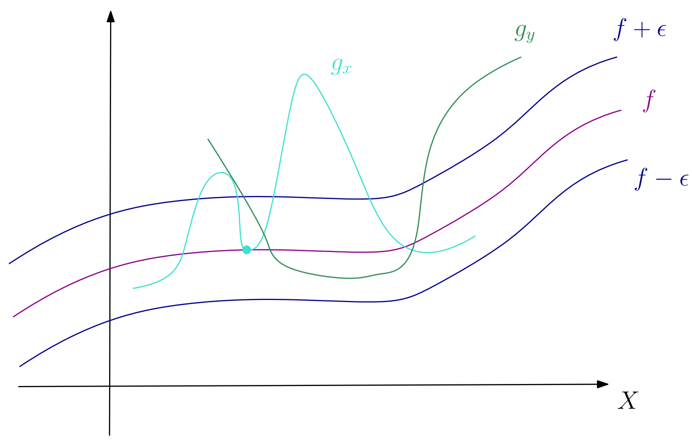
Note that

$$\left.\begin{array}{l} X = \bigcup_{y \in X} B_{\delta_y}(y) \\ X \text{ compact} \end{array}\right\} \implies \exists N \geq 1 \text{ and } \exists y_1, \ldots, y_N \in X$$

s.t. $X = \bigcup_{n=1}^N B_{\delta_n}(y_n)$ where $\delta_n = \delta_{y_n}$.
Take $g = \max\{h_{y_1}, \ldots, h_{y_N}\}$ (by step 2). By construction, $g(x) = f(x)$. Also if $y \in X$, $\exists 1 \leq n \leq N$ s.t. $y \in B_{\delta_n}(y_n)$. So

$$g(y) \geq h_{y_n}(y) > f(y) - \varepsilon$$

Step 4: For all $f \in C(X)$ and $\varepsilon > 0$, $\exists g \in \overline{\mathcal{A}}$ s.t. $d(f, g) < \varepsilon$. Fix $f \in C(X)$, $\varepsilon > 0$



For $x \in X$, let $g_x \in \overline{\mathcal{A}}$ be the function given by step 3. In particular, $g_x(x) = f(x)$,

$$g_x(y) > f(y) - \varepsilon \qquad \forall y \in X$$

As $g_x \in \overline{\mathcal{A}}$, the function $g_x - f$ is continuous at x. So $\exists \delta_x > 0$ s.t. $|g_x(y) - f(y)| < \varepsilon$, $\forall y \in B_{\delta_x}(x)$. In particular,

$$g_x(y) < f(y) + \varepsilon \qquad \forall y \in B_{\delta_x}(x)$$

Note

$$\left.\begin{matrix} X = \bigcup_{x\in X} B_{\delta_x}(x) \\ X \text{ compact} \end{matrix}\right\} \implies \exists N \geq 1 \text{ and } \exists x_1, \ldots, x_N \in X \text{ s.t.}$$

$X = \bigcup_{n=1}^N B_{\delta_n}(x_n)$ where $\delta_n = \delta_{x_n}$.
Take $g = \min\{g_{x_1}, \ldots, g_{x_N}\} \in \overline{\mathcal{A}}$ (by step 2).
For $y \in X$, $\exists 1 \leq n \leq N$ s.t. $y \in B_{\delta_n}(x_n)$ and so

$$g(y) \leq g_{x_n}(y) < f(y) + \varepsilon$$

Moreover, as $g_{x_n}(y) > f(y) - \varepsilon$, $\forall y \in X$, $\forall 1 \leq n \leq N$, we have

$$g(y) > f(y) - \varepsilon \qquad \forall y \in X$$

This shows $C(X) \subseteq \overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}} \subseteq C(X)$. □

§13.2 Differentiation

Definition 13.1 (Limit) — Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $\emptyset \neq A \subseteq X$, let $f : A \to Y$. For $x_0 \in A'$ and $y_0 \in Y$ we write

$$f \xrightarrow[x\to x_0]{} y_0 \qquad \text{or} \qquad \lim_{x\to x_0} f(x) = y_0$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(f(x), y_0) < \varepsilon$ whenever $0 < d_X(x, x_0) < \delta$.
Equivalently, $\lim_{x\to x_0} f(x) = y_0$ if

$$\lim_{n\to\infty} f(x_n) = y_0 \text{ for every sequence } \{x_n\}_{n\geq 1} \subseteq A \setminus \{x_0\} \text{ s.t. } x_n \xrightarrow[n\to\infty]{d_X} x_0$$

Note also that if $x_0 \in A' \cap A$ then f is continuous at $x_0 \iff \lim_{x\to x_0} f(x) = f(x_0)$.

Exercise 13.1. Let (X, d) be a metric space, $\emptyset \neq A \subseteq X$, $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions. Assume that at a point $a \in A'$ we have

$$\lim_{x\to x_0} f(x) = \alpha \quad \text{and} \quad \lim_{x\to x_0} g(x) = \beta$$

Then

1. $\lim_{x\to x_0} (\lambda f(x)) = \lambda\alpha$, $\lambda \in \mathbb{R}$
2. $\lim_{x\to x_0} (f(x) + g(x)) = \alpha + \beta$
3. $\lim_{x\to x_0} (f(x)g(x)) = \alpha \cdot \beta$
4. If $\beta \neq 0$ then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\alpha}{\beta}$

Definition 13.2 (Differentiability) — Let I be an open interval and let $f : I \to \mathbb{R}$ be a function. We say that f is differentiable at $a \in I$ if

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} \text{ exists and is finite}$$

in which case we denote it $f'(a)$.

Example 13.3

Fix $n \geq 1$ and let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$. For $a \in \mathbb{R}$ and $x \neq a$

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} &= \frac{x^n - a^n}{x - a} \\ &= x^{n-1} + x^{n-2}a + \ldots + a^{n-1} \underset{x\to a}{\longrightarrow} na^{n-1} \end{aligned}$$

So f is differentiable at a and $f'(a) = na^{n-1}$.

Theorem 13.4

Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable at $a \in I$. Then f is continuous at a.

Proof. For $x \in I \setminus \{a\}$, we write

$$f(x) = \underbrace{\frac{f(x) - f(a)}{x - a}}_{\underset{x\to a}{\longrightarrow} f'(a)} \cdot \underbrace{(x - a)}_{\underset{x\to a}{\longrightarrow} 0} + \underbrace{f(a)}_{\underset{x\to a}{\longrightarrow} f(a)} \underset{x\to a}{\longrightarrow} f(a)$$

□

Theorem 13.5

Let I be an open interval and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be two functions differentiable at $a \in I$. Then

1. $\forall \lambda \in \mathbb{R}$, λf is differentiable at a and
$$(\lambda f)'(a) = \lambda f'(a)$$
2. $f + g$ is differentiable at a and
$$(f+g)'(a) = f'(a) + g'(a)$$
3. $f \cdot g$ is differentiable at a and
$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$
4. $\frac{f}{g}$ is differentiable at a if $g(a) \neq 0$ and
$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$$

Proof. For $x \neq a$

1. Consider
$$\frac{\lambda f(x) - \lambda f(a)}{x-a} = \lambda \cdot \frac{f(x)-f(a)}{x-a} \xrightarrow[x \to a]{} \lambda f'(a)$$
2. Consider
$$\frac{(f(x)+g(x)) - (f(a)+g(a))}{x-a} = \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a} \xrightarrow[x \to a]{} f'(a) + g'(a)$$
3. Consider
$$\underbrace{\frac{f(x)-f(a)}{x-a}}_{\xrightarrow[x\to a]{} f'(a)} \cdot \underbrace{g(x)}_{\xrightarrow[x\to a]{} g(a)} + \underbrace{f(a)}_{\xrightarrow[x\to a]{} f(a)} \cdot \underbrace{\frac{g(x)-g(a)}{x-a}}_{\xrightarrow[x\to a]{} g'(a)} \xrightarrow[x \to a]{} f'(a)g(a) + f(a)g'(a)$$
4. Consider
$$\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} \underbrace{\frac{f(x)-f(a)}{x-a}}_{\xrightarrow[x\to a]{} f'(a)} \cdot \underbrace{\frac{1}{g(x)}}_{\xrightarrow[x\to a]{} \frac{1}{g(a)}} + f(a) \cdot \underbrace{\frac{g(a)-g(x)}{x-a}}_{\xrightarrow[x\to a]{} -g'(a)} \cdot \underbrace{\frac{1}{g(x)}}_{\xrightarrow[x\to a]{} \frac{1}{g(a)}} \cdot \frac{1}{g(a)}$$
$$\xrightarrow[x\to a]{} \frac{f'(a)}{g(a)} - \frac{g'(a)}{g^2(a)} f(a)$$

□

§14 Lec 14: Apr 28, 2021

§14.1 Chain Rule

Theorem 14.1 (Chain Rule)
Let I and J be two open intervals and let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions. Assume that f is differentiable at $a \in I$ and that g is differentiable at $f(a) \in J$. Then $g \circ f$ is well defined on a neighborhood of a, $g \circ f$ is differentiable at a, and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Proof. Consider:

$$\left.\begin{matrix} f(a) \in J \\ J \text{ is open} \end{matrix}\right\} \implies \exists \varepsilon > 0 \text{ s.t. } (f(a) - \varepsilon, f(a) + \varepsilon) \subseteq J$$

f is differentiable at $a \implies f$ is continuous at $a \implies \exists \delta > 0$ s.t. $f((a-\delta, a+\delta) \cap I) \subseteq (f(a) - \varepsilon, f(a) + \varepsilon)$. As $a \in I$ and I is open, shrinking δ if necessary, me may assume that $(a - \delta, a + \delta) \subseteq I$.

Then $g \circ f$ is well-defined on $(a - \delta, a + \delta)$.

$$\underbrace{(a-\delta, a+\delta)}_{\subseteq I} \xrightarrow{f} \underbrace{(f(a) - \varepsilon, f(a) + \varepsilon)}_{\subseteq J} \xrightarrow{g} \mathbb{R}$$

<u>Caution</u>: The following argument does not work

$$\frac{g(f(x)) - g(f(a))}{x - a} = \underbrace{\frac{g(f(x)) - g(f(a))}{f(x) - f(a)}}_{\xrightarrow{x \to a} g'(f(a))} \cdot \underbrace{\frac{f(x) - f(a)}{x - a}}_{\xrightarrow{x \to a} f'(a)}$$

because f is continuous at $a \implies f(x) \xrightarrow{x \to a} f(a)$

Instead, we argue as follows: Define $h : J \to \mathbb{R}$,

$$h(y) = \begin{cases} \frac{g(y) - g(f(a))}{y - f(a)}, & \text{if } y \in J \setminus \{f(a)\} \\ g'(f(a)), & \text{if } y = f(a) \end{cases}$$

As g is differentiable at $f(a)$, h is continuous at $f(a)$. Moreover, we can write

$$g(y) - g(f(a)) = h(y) \cdot (y - f(a)) \qquad \forall y \in J$$

For $x \in (a - \delta, a + \delta) \implies f(x) \in J$. So for $x \in (a - \delta, a + \delta) \setminus \{a\}$,

$$\frac{g(f(x)) - g(f(a))}{x - a} = \underbrace{h(f(x))}_{\xrightarrow{x \to a} h(f(a))} \cdot \underbrace{\frac{f(x) - f(a)}{x - a}}_{\xrightarrow{x \to a} f'(a)}$$

So $\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = h(f(a)) f'(a) = g'(f(a)) \cdot f'(a)$. ∎

Lemma 14.2

Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. If f is increasing then $f'(x) \geq 0\, \forall x \in (a, b)$ or decreasing then $f'(x) \leq 0\, \forall x \in (a, b)$.

Proof. Assume f is increasing (if f is decreasing, replace f by $-f$ in what follows). Fix $x \in (a, b)$ and let $\{x_n\}_{n\geq 1}$ be an increasing from (a, b) with $\lim_{n\to\infty} x_n = x$.
Then $f'(x) = \lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n-x} \geq 0$ where $f(x_n) - f(x) \leq 0$ and $x_n - x < 0$. □

Theorem 14.3

Let $f : (a, b) \to \mathbb{R}$ be a function. Assume that $x_0 \in (a, b)$ is a point of local maximum/minimum for f. Assume also that f is differentiable at x_0. Then $f'(x_0) = 0$.

Proof. Assume that x_0 is a point of local maximum for f (if x_0 is a point of local minimum, replace f by $-f$ in what follows).

Then $\exists \delta > 0$ s.t. $f(x) \leq f(x_0) \quad \forall x \in (x_0 - \delta, x_0 + \delta) \cap (a, b)$. For $x_n \in (x_0 - \delta, x_0) \cap (a, b)$ s.t. $x_n \underset{n\to\infty}{\longrightarrow} x_0$, we have

$$f'(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0) \leq 0}{x_n - x_0 < 0} \geq 0$$

On the other hand, for $y_n \in (x_0, x_0 + \delta) \cap (a, b)$ s.t. $y_n \underset{n\to\infty}{\longrightarrow} x_0$, we have

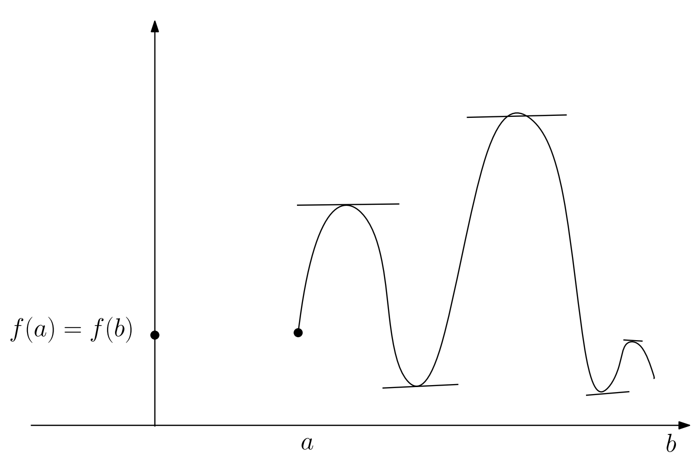
$$f'(x_0) = \lim_{n\to\infty} \frac{f(y_n) - f(x_0) \leq 0}{y_n - x_0 > 0} \leq 0$$

Thus, we get $f'(x_0) = 0$. □

§14.2 Mean Value Theorem

Theorem 14.4 (Rolle)

Let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on the $[a, b]$, differentiable on (a, b), and s.t. $f(a) = f(b)$. Then there exists (at least one) $x \in (a, b)$ s.t. $f'(x) = 0$.



Proof. Consider:

$$\left.\begin{aligned} &f : [a,b] \to \mathbb{R} \text{ continuous} \\ &[a,b] \ \text{ compact} \end{aligned}\right\} \implies \exists x_0, y_0 \in [a,b]$$

s.t.

$$f(x_0) = \sup_{x\in[a,b]} f(x) \quad \text{ and } \quad f(y_0) = \inf_{x\in[a,b]} f(x)$$

So $f(y_0) \le f(x) \le f(x_0) \quad \forall x \in [a,b]$.
Case 1: We have

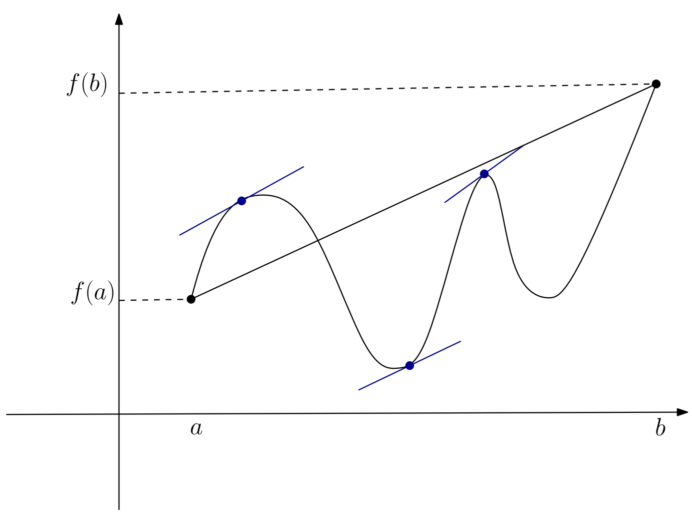
$$\left.\begin{aligned} &\{x_0, y_0\} \subseteq \{a,b\} \\ &f(a) = f(b) \end{aligned}\right\} \implies f(x_0) = f(y_0) \implies f \text{ constant} \implies f'(x) = 0 \, \forall x \in (a,b)$$

Case 2: $\{x_0, y_0\} \nsubseteq \{a,b\} \implies x_0 \notin \{a,b\}$ or $y_0 \notin \{a,b\}$. Say $x_0 \notin \{a,b\} \implies x_0 \in (a,b)$. By Theorem 14.3, we get $f'(x_0) = 0$. ☐

Theorem 14.5 (Mean Value)
Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists (at least one) $y \in (a,b)$ s.t.

$$f'(y) = \frac{f(b) - f(a)}{b - a}$$

Remark 14.6. The Mean Value Theorem implies Rolle's Theorem. We will see from the proof that Rolle's Theorem implies the Mean Value Theorem, so the two are equivalent.



Proof. We define $l : [a,b] \to \mathbb{R}$ where

$$l(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

Note that l is continuous on $[a, b]$, differentiable on (a, b), and

$$l'(x) = \frac{f(b) - f(a)}{b - a} \qquad \forall x \in (a, b)$$

Let $g : [a, b] \to \mathbb{R}$, $g(x) = f(x) - l(x)$. Then g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = 0 = g(b)$. Then Rolle's implies that $\exists y \in (a, b)$ s.t.

$$g'(y) = 0 \implies f'(y) - l'(y) = 0 \implies f'(y) = \frac{f(b) - f(a)}{b - a}$$

□

Corollary 14.7

If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = 0 \, \forall x \in (a, b)$, then f is a constant.

Proof. Assume f is not a constant. Then $\exists a < x_1 < x_2 < b$ s.t.

$$f(x_1) \neq f(x_2)$$

Then f is continuous on $[x_1, x_2]$, differentiable on (x_1, x_2). By Mean Value, $\exists y \in (x_1, x_2)$ s.t.

$$f'(y) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \neq 0$$

Contradiction! □

Corollary 14.8

If $f, g : (a, b) \to \mathbb{R}$ are differentiable s.t. $f'(x) = g'(x) \, \forall x \in (a, b)$, then $\exists c \in \mathbb{R}$ s.t.

$$f(x) = g(x) + c \qquad \forall x \in (a, b)$$

§15 Lec 15: Apr 30, 2021

§15.1 Mean Value Theorem (Cont'd)

Theorem 15.1

Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists (at least one) $c \in (a, b)$ s.t.

$$f'(c)\,[g(b) - g(a)] = g'(c)\,[f(b) - f(a)]$$

Remark 15.2. Taking $g(x) = x$ we recover the Mean Value theorem. In fact, the two results are equivalent, as can be seen from the proof.

Proof. We define $h : [a, b] \to \mathbb{R}$

$$h(x) = f(x)\,[g(b) - g(a)] - g(x)\,[f(b) - f(a)]$$

Note that h is continuous on $[a, b]$ and differentiable on (a, b). Moreover,

$$\left.\begin{aligned} h(a) &= f(a)\,[g(b) - g(a)] - g(a)\,[f(b) - f(a)] = f(a)g(b) - g(a)f(b) \\ h(b) &= f(b)\,[g(b) - g(a)] - g(b)\,[f(b) - f(a)] = -f(b)g(a) + g(b)f(a) \end{aligned}\right\} \implies h(a) = h(b)$$

By Rolle's theorem, $\exists c \in (a, b)$ s.t $h'(c) = 0$. □

Corollary 15.3

Let $f : (a, b) \to \mathbb{R}$ be differentiable.

1. If $f'(x) > 0$ $\forall x \in (a, b)$ then f is strictly increasing.
2. If $f'(x) \geq 0$ $\forall x \in (a, b)$ then f is increasing.
3. If $f'(x) < 0$ $\forall x \in (a, b)$ then f is strictly decreasing.
4. If $f'(x) \leq 0$ $\forall x \in (a, b)$ then f is decreasing.

Proof. We only present the details for (1).
Fix $a < x_1 < x_2 < b$. f is differentiable on $(a, b) \implies f$ is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Mean Value theorem, $\exists c \in (x_1, x_2)$ s.t.

$$0 < f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \implies f(x_1) < f(x_2)$$

As $a < x_1 < x_2 < b$ were arbitrary, f is strictly increasing. □

Example 15.4

The derivative of a differentiable function need not be continuous

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f is continuous on $\mathbb{R} \setminus \{0\}$. To see that it's continuous at 0,

$$|f(x) - f(0)| = \left| x^2 \sin \frac{1}{x} \right| \leq x^2 \underset{x \to 0}{\longrightarrow} 0 \tag{*}$$

f is differentiable on $\mathbb{R} \setminus \{0\}$. To see that it's differentiable at 0, we compute

$$x \neq 0 : \quad \frac{f(x) - f(0)}{x - 0} = x \sin \frac{1}{x} \underset{x \to 0}{\longrightarrow} 0 \quad \text{(as in (*))}$$

So $f'(0) = 0$. Thus,

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot \frac{-1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f' is continuous on $\mathbb{R} \setminus \{0\}$ (not continuous at 0). While $\lim_{x \to 0} 2x \sin \frac{1}{x} = 0$, for each $\lambda \in [-1, 1]$, there exists $x_n(\lambda) \underset{n \to \infty}{\longrightarrow} 0$ s.t. $\cos \frac{1}{x_n(\lambda)} = \lambda$. Nevertheless, the derivative of a differentiable function has the Darboux property.

Theorem 15.5 (Intermediate Value for Derivatives)

Let $f : (a, b) \to \mathbb{R}$ be differentiable. Then f' has the Darboux property, that is, if $a < x_1 < x_2 < b$ and λ lies between $f'(x_1)$ and $f'(x_2)$, then there exists $c \in (x_1, x_2)$ s.t.

$$f'(c) = \lambda$$

Proof. Let $g : (a, b) \to \mathbb{R}$, $g(x) = f(x) - \lambda x$. g is differentiable on $(a, b) \implies g$ is continuous on (a, b). Fix $a < x_1 < x_2 < b$ and assume without loss of generality

$$f'(x_1) < \lambda < f'(x_2)$$

Then

$$g'(x_1) = f'(x_1) - \lambda < 0$$
$$g'(x_2) = f'(x_2) - \lambda > 0$$

g is continuous on $[x_1, x_2]$

$$\implies \exists c \in [x_1, x_2] \text{ s.t. } g(c) = \inf_{x \in [x_1, x_2]} g(x)$$

If we can prove that $c \in (x_1, x_2)$ then $g'(c) = 0$. To see that $c \neq x_1$ we argue as follows:

$$0 > g'(x_1) = \lim_{x \to x_1} \frac{g(x) - g(x_1)}{x - x_1} \implies \exists \delta_1 > 0$$

s.t. if $0 < |x - x_1| < \delta_1$ then
$$\frac{g(x) - g(x_1)}{x - x_1} < 0$$
In particular, for $x \in (x_1, x_1 + \delta_1)$ we have
$$\frac{g(x) - g(x_1)}{\underbrace{x - x_1}_{>0}} < 0 \implies g(x) < g(x_1)$$
$$\implies g \text{ cannot attain its minimum at } x_1$$

Similarly,
$$0 < g'(x_2) = \lim_{x \to x_2} \frac{g(x) - g(x_2)}{x - x_2} \implies \exists \delta_2 > 0$$
s.t. if $0 < |x - x_2| < \delta_2$ then
$$\frac{g(x) - g(x_2)}{x - x_2} > 0$$
In particular, if $x \in (x_2 - \delta_2, x_2)$ then
$$\frac{g(x) - g(x_2)}{\underbrace{x - x_2}_{<0}} \implies g(x) < g(x_2)$$
$$\implies g \text{ cannot attain its minimum at } x_2$$
□

§15.2 Derivative of Inverse Functions

Theorem 15.6

Let I be an open interval and let $f : I \to \mathbb{R}$ be continuous and injective. Then $f(I) = J$ is an interval and $f : I \to J$ is bijective. If f is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$ then $f^{-1} : J \to I$ is differentiable at $y_0 = f(x_0)$ and
$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{-1}(y_0)\right)}$$

Proof. The proof uses the following two exercises:

Exercise 15.1. Let I be an interval and let $f : I \to \mathbb{R}$ be continuous and injective. Then f is strictly monotone.

Exercise 15.2. Let I be an interval and let $f : I \to \mathbb{R}$ be strictly increasing and so that $f(I)$ is an interval. Then f is continuous.

Using exercise 1, we find that f is strictly monotone. Assume f is strictly increasing $\implies f^{-1}$ is strictly increasing.
Using exercise 2 with $g = f^{-1} : J \to I$, we find that f^{-1} is continuous.

Claim 15.1. J is an open interval.

Assume, towards a contradiction, that $\inf J \in J = f(I) \implies \exists a \in I$ s.t. $f(a) = \inf J$.

$$\left.\begin{array}{l} I \text{ open } \implies \exists \delta > 0 \text{ s.t. } (a-\delta, a+\delta) \subseteq I \\ f \text{ is strictly increasing} \end{array}\right\} \implies J = f(I) \ni f\left(a - \frac{\delta}{2}\right) < f(a) = \inf J$$

Contradiction!
Similarly, one can show that $\sup J \notin J$

$$\left.\begin{array}{l} f \text{ is diff at } x_0 \implies f'(x_0) = \lim \frac{f(x)-f(x_0)}{x-x_0} \\ f'(x_0) \neq 0 \text{ and } f(x) \neq f(x_0) \quad \forall x \neq x_0 \end{array}\right\} \implies$$

$$\implies \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

$$\implies \forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } 0 < |x - x_0| < \delta \implies \left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

f^{-1} is continuous at $y_0 \implies \exists \eta > 0$ s.t. $0 < |y - y_0| < \eta$ implies

$$0 < \left| f^{-1}(y) - f^{-1}(y_0) \right| < \delta$$

So for $0 < |y - y_0| < \eta$ we get

$$\left| \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

which implies

$$\left(f^{-1}\right)'(y_0) = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)} \qquad \square$$

§16 Lec 16: May 3, 2021

§16.1 L'Hopital Rule

Definition 16.1 (Existence of Limit) — Let $-\infty \le a < b \le \infty$ and let $f : (a,b) \to \mathbb{R}$ be a function. For $c \in (a,b) \cup \{a\}$ we write

$$\lim_{x\to c^+} f(x) = L \in \mathbb{R} \cup \{\pm\infty\}$$

if for every sequence $\{x_n\}_{n\ge 1} \subseteq (c,b)$ s.t. $\lim_{n\to\infty} x_n = c$ we have

$$\lim_{n\to\infty} f(x_n) = L$$

For $c \in (a,b) \cup \{b\}$ we write

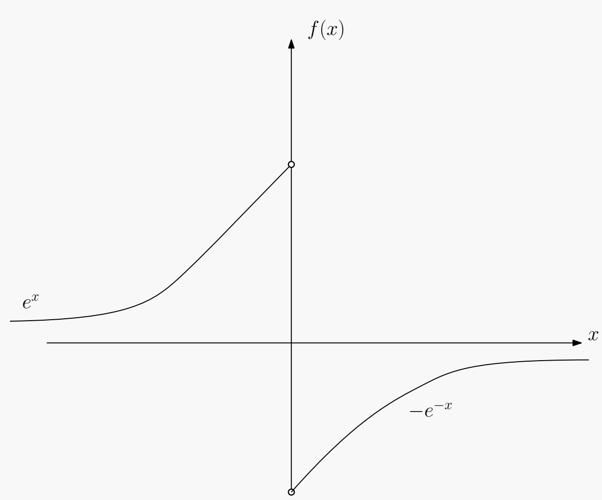
$$\lim_{x\to c^-} f(x) = M \in \mathbb{R} \cup \{\pm\infty\}$$

if for every sequence $\{x_n\}_{n\ge 1} \subseteq (a,c)$ s.t. $\lim_{n\to\infty} x_n = c$ we have

$$\lim_{n\to\infty} f(x_n) = M$$

Remark 16.2. In general, if $c \in (a,b)$ we have

$$f(c) \ne \lim_{x\to c^-} f(x) \ne \lim_{x\to c^+} f(x) \ne f(c)$$



Theorem 16.3 (L'Hopital)
Let $-\infty \leq a < b \leq \infty$ and let $f, g : (a,b) \to \mathbb{R}$ be differentiable. Assume that $g'(x) \neq 0$ $\forall x \in (a,b)$ and that

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{\pm\infty\}$$

Assume also that either

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0 \tag{1}$$

or

$$\lim_{x \to a^+} |g(x)| = \infty \tag{2}$$

Then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

Remark 16.4. $\lim_{x \to a^+}$ in the theorem can be replaced by $\lim_{x \to b^-}$ or by $\lim_{x \to c}$ for some $c \in (a,b)$.

Proof. We'll present the details for $L \in \mathbb{R}$. We'll prove

Claim 16.1. $\forall \varepsilon > 0$ $\exists \delta_1(\varepsilon) > 0$ s.t.

$$\frac{f(x)}{g(x)} < L + \varepsilon \qquad \forall x \in (a, a + \delta_1)$$

Claim 16.2. $\forall \varepsilon > 0$ $\exists \delta_2(\varepsilon) > 0$ s.t.

$$L - \varepsilon < \frac{f(x)}{g(x)} \qquad \forall x \in (a, a + \delta_2)$$

Then taking $\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$ we get

$$\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon \qquad \forall x \in (a, a + \delta)$$

$\implies \lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.
Note: If $L = -\infty$ then it suffices to prove Claim 1 with $L + \varepsilon$ replaced by $M < 0$.
If $L = \infty$ then it suffices to prove Claim 2 with $L - \varepsilon$ replaced by $M > 0$.
By assumption, $g'(x) \neq 0$ $\forall x \in (a,b)$. As g is differentiable on (a,b), g' has the Darboux property. So either $g'(x) < 0$ $\forall x \in (a,b)$ or $g'(x) > 0$ $\forall x \in (a,b)$.
Assume $g'(x) < 0$ $\forall x \in (a,b) \implies g$ strictly decreasing on (a,b). In case 1,

$$\lim_{x \to a^+} g(x) = 0$$

As g is strictly decreasing, we get

$$g(x) < 0 \qquad \forall x \in (a,b)$$

In case 2,

$$\lim_{x \to a^+} |g(x)| = \infty$$

As g is strictly decreasing, we get

$$\lim_{x\to a^+} g(x) = \infty$$

and so $\exists c \in (a,b)$ s.t. $g(x) > 0\ \forall x \in (a,c)$ (**). In particular, in both cases $g(x) \neq 0$ $\forall x \in (a,c)$. We prove claim 1:

Fix $\varepsilon > 0$. As $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, $\exists \delta_1(\varepsilon) > 0$ s.t.

$$\frac{f'(x)}{g'(x)} < L + \frac{\varepsilon}{2} \qquad \forall x \in (a, a+\delta_1)$$

Fix $a < x < y < \min(a+\delta_1, c)$. By (an equivalent formulation of) Mean Value theorem, $\exists z \in (x,y)$ s.t.

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(z)}{g'(z)} < L + \frac{\varepsilon}{2} \tag{*}$$

In case 1, take the limit $x \to a^+$ in (*) to get

$$\frac{f(y)}{g(y)} \le L + \frac{\varepsilon}{2} < L + \varepsilon \qquad \forall a < y < \min(a+\delta_1, c)$$

In case 2, we write

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(y)}{g(x)-g(y)} \cdot \frac{g(x)-g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

By (**) we have $g(x) > g(y) > 0 \implies \frac{g(x)-g(y)}{g(x)} > 0$. So

$$\begin{aligned} \frac{f(x)}{g(x)} &< \left(L + \frac{\varepsilon}{2}\right) \frac{g(x)-g(y)}{g(x)} + \frac{f(y)}{g(x)} \\ &= \left(L + \frac{\varepsilon}{2}\right)\left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)} \\ &= L + \frac{\varepsilon}{2} + \frac{f(y) - \left(L + \frac{\varepsilon}{2}\right) g(y)}{g(x)} \end{aligned}$$

For y fixed, $\lim_{x\to a^+} \frac{f(y)-\left(L+\frac{\varepsilon}{2}\right)g(y)}{g(x)} = 0$

$$\implies \exists \tilde{\delta_1}(\varepsilon) > 0 \text{ s.t. } \left| \frac{f(y) - \left(L + \frac{\varepsilon}{2}\right) g(y)}{g(x)} \right| < \frac{\varepsilon}{2} \qquad \forall x \in \left(a, a + \tilde{\delta_1}\right)$$

In particular,

$$\frac{f(x)}{g(x)} < L + \varepsilon \qquad \forall a < x < \min\left\{a + \delta_1, a + \tilde{\delta_1}, c\right\}$$

Exercise 16.1. Prove claim 2. □

§16.2 Taylor's Theorem

Definition 16.5 (Taylor Expansion) — Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable of any order. For $x_0 \in I$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is called the Taylor expansion of f about x_0. For $n \geq 1$, we define the remainder

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Theorem 16.6 (Taylor)

Let $n \geq 1$ and assume $f : (a,b) \to \mathbb{R}$ is n times differentiable. Let $x_0 \in (a,b)$. Then for any $x \in (a,b) \setminus \{x_0\}$ there exists y between x and x_0 s.t.

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-x_0)^n$$

In particular,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n)}(y)}{n!}(x-x_0)^n$$

Proof. Fix $x \in (a,b) \setminus \{x_0\}$. Define $M \in \mathbb{R}$ to be the unique solution to the equation

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + M \cdot \frac{(x-x_0)^n}{n!}$$

We want to show that there exists y between x and x_0 s.t.

$$M = f^{(n)}(y)$$

Let $g : (a,b) \to \mathbb{R}$

$$g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(t-x_0)^k - M \cdot \frac{(t-x_0)^n}{n!}$$

Note g is n times differentiable. For $1 \leq l \leq n-1$,

$$g^{(l)}(t) = f^{(l)}(t) - \sum_{k \geq l}^{n-1} \frac{f^{(k)}(x_0)}{(k-l)!}(t-x_0)^{k-l} - M\frac{(t-x_0)^{n-l}}{(n-l)!}$$
$$g^{(n)}(t) = f^{(n)}(t) - M$$

In particular, if $0 \leq l \leq n-1$,

$$g^{(l)}(x_0) = f^{(l)}(x_0) - f^{(l)}(x_0) = 0$$

Also $g(x) = 0$ by contradiction.
g is continuous on $[x, x_0]$, differentiable on (x, x_0) and

$$g(x) = g(x_0) = 0 \implies \exists x_1 \in (x, x_0) \text{ s.t. } g'(x_1) = 0$$

By Rolle's theorem,

$$\begin{aligned} \exists x_2 \in (x_1, x_0) \quad & \text{s.t.} \quad g''(x_2) = 0 \\ & \vdots \\ \exists x_n \in (x_{n-1}, x_0) \quad & \text{s.t.} \quad g^{(n)}(x_n) = 0 \end{aligned}$$

Set $y = x_n$. □

§17 Lec 17: May 5, 2021

§17.1 Taylor's Theorem (Cont'd)

Corollary 17.1

Fix $a > 0$ and let $f : (-a, a) \to \mathbb{R}$ be a function differentiable of any order. Assume that all derivatives of f are uniformly bounded on $(-a, a)$, that is,

$$\exists M > 0 \text{ s.t. } \left|f^{(n)}(x)\right| \le M \quad \forall x \in (-a, a), \quad \forall n \ge 1$$

Then

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k \xrightarrow[n\to\infty]{u} 0 \text{ on } (-a, a)$$

Proof. Fix $x \in (-a, a) \setminus \{0\}$. By Taylor, there exists y between x and 0 s.t.

$$\begin{aligned} R_n(x) &= \frac{f^{(n)}(y)}{n!} x^n \\ \implies |R_n(x)| &\le M \frac{|x|^n}{n!} \le M \frac{a^n}{n!} \\ \implies \sup_{x \in (-a,a)} |R_n(x)| &\le M \cdot \frac{a^n}{n!} \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

□

Example 17.2

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$

$$f^{(n)}(x) = \begin{cases} -\sin x, & n = 1 + 4k \\ -\cos x, & n = 2 + 4k \\ \sin x, & n = 3 + 4k \\ \cos x, & n = 4k \end{cases} \qquad \text{for } k \ge 0$$

So $\left|f^{(n)}(x)\right| \le 1 \ \forall x \in \mathbb{R} \ \forall n \ge 0$. We get

$$f(x) = u - \lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \quad \text{on } (-a, a) \text{ for any } a > 0$$

Let $n = 2l$

$$\begin{aligned} &\implies f^{(n)}(0) = \begin{cases} -1, & \text{if } l \text{ odd} \\ 1, & \text{if } l \text{ even} \end{cases} = (-1)^l \\ &\implies f(X) = \sum_{n\ge 0} \frac{f^{(n)}(0)}{n!} x^n = \sum_{l \ge 0} \frac{(-1)^l}{(2l)!} x^{2l} \end{aligned}$$

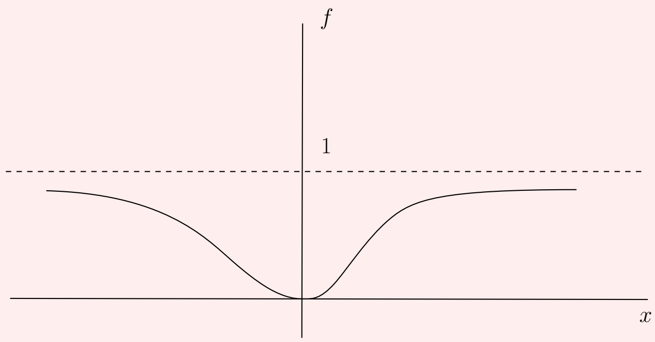
A similar argument gives

$$\sin x = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Example 17.3

$f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$



Note f is differentiable of any order on $\mathbb{R}$. Clearly, this holds on $\mathbb{R} \setminus \{0\}$. In fact, for $x \in \mathbb{R} \setminus \{0\}$,

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$$

where

$$P_n\left(\frac{1}{x}\right) = \left(\frac{2}{x^3}\right)^n + \dots$$

To see that f is differentiable at 0 we compute

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim_{t\to\infty} \frac{t}{e^{t^2}} = \lim_{t\to\infty} \frac{1}{2te^{t^2}} = 0$$

Similarly,

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{t\to -\infty} \frac{t}{e^{t^2}} = 0$$

Proceeding inductively, we can prove that f is differentiable of any order at 0 and

$$f^{(n)}(0) = 0$$

We consider

$$\lim_{x\to 0^+} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0^+} \frac{P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}}{x} \lim_{t\to\infty} \frac{tP_n(t)}{e^{t^2}} = 0$$

and

$$\lim_{x\to 0^-} \frac{f^{(n)}(x)}{x} = 0$$

Example 17.4 (Cont'd from above)

Thus,

$$\sum_{n\geq 0} \frac{f^{(n)}(0)}{n!} x^n \equiv 0$$

At leading order as $x \to 0$,

$$f^{(n)}(x) \sim 2^n \cdot \left(\frac{1}{x^2}\right)^{\frac{3n}{2}} e^{-\frac{1}{x^2}} \sim 2^n e^{-\frac{1}{x^2} + \frac{3n}{2} \ln \frac{1}{x^2}}$$

The function $g : (0, \infty) \to \mathbb{R}$, $g(t) = -t + \frac{3n}{2} \ln t$ achieves its maximum at

$$g'(t) = 0 \iff -1 + \frac{3n}{2t} = 0 \iff t = \frac{3n}{2}$$

So $f^{(n)}\left(\sqrt{\frac{2}{3n}}\right) \sim 2^n e^{-\frac{3n}{2} + \frac{3n}{2} \ln \frac{3n}{2}} \sim 2^n e^{\frac{3n}{2} \ln\left(\frac{3n}{2e}\right)} \sim 2^n \left(\frac{3n}{2e}\right)^{\frac{3n}{2}} \xrightarrow[n\to\infty]{} \infty$.

Theorem 17.5

Assume that $f_n : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Assume also that

1. $\{f_n'\}_{n\geq 1}$ converges uniformly on (a, b)
2. $\{f_n\}_{n\geq 1}$ converges at some x_0 in $[a, b]$

Then $\{f_n\}_{n\geq 1}$ converges uniformly on $[a, b]$ to some function f. Moreover, f is differentiable on (a, b) and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad \forall x \in (a, b)$$

Remark 17.6. We can restate the conclusion as follows:

$$\lim_{y\to x} \lim_{n\to\infty} \frac{f_n(y) - f_n(x)}{y - x} = \lim_{y\to x} \frac{f(y) - f(x)}{y - x} = f'(x) = \lim_{n\to\infty} \lim_{y\to x} \frac{f_n(y) - f_n(x)}{y - x}$$

Proof. Let's prove that $\{f_n\}_{n\geq 1}$ converges uniformly on $[a, b]$. Fix $\varepsilon > 0$.
$\{f_n'\}_{n\geq 1}$ converges uniformly on (a, b) which implies $\{f_n'\}_{n\geq 1}$ is uniformly Cauchy on (a, b) which also implies $\exists n_1(\varepsilon) \in \mathbb{N}$ s.t.

$$\left|f_n'(x) - f_m'(x)\right| < \varepsilon \quad \forall n, m \geq n_1(\varepsilon) \quad \forall x \in (a, b)$$

Also, we know that $\{f_n(x_0)\}_{n\geq 1}$ converges which means $\{f_n(x_0)\}$ is Cauchy which implies $\exists n_2(\varepsilon) \in \mathbb{N}$ s.t.

$$|f_n(x_0) - f_m(x_0)| < \varepsilon \quad \forall n, m \geq n_2(\varepsilon)$$

For $x \in [a, b] \setminus \{x_0\}$,

$$|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]|$$

By the Mean Value theorem, there exists y between x and x_0 s.t.

$$|\,[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]\,| = \left|f_n'(y) - f_m'(y)\right| |x - x_0| < \varepsilon(b - a)$$

So for $n, m \geq n(\varepsilon) = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$ we get

$$\begin{aligned} &|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + \varepsilon(b - a) \leq \varepsilon(1 + b - a) \\ \implies\ & sup_{x\in[a,b]}\,|f_n(x) - f_m(x)| \leq \varepsilon(1 + b - a) \qquad \forall n, m \geq n(\varepsilon) \end{aligned}$$

So $\{f_n\}_{n\geq 1}$ are uniformly Cauchy on $[a, b]$ and so converge to a function $f = \lim_{n\to\infty} f_n$. It remains to show that f is differentiable on (a, b) and

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

which we will prove in the next lecture. □

§18 Lec 18: May 7, 2021

§18.1 Taylor's Theorem (Cont'd)

Proof. (Cont'd from lecture 17) Fix $x \in (a,b)$. We want to show that f is differentiable at x and
$$f'(x) = \lim_{n\to\infty} f_n'(x)$$
We define
$$g : [a,b] \setminus \{x\} \to \mathbb{R}, \quad g(y) = \frac{f(y)-f(x)}{y-x}$$
$$g_n : [a,b] \setminus \{x\} \to \mathbb{R}, \quad g_n(y) = \frac{f_n(y)-f_n(x)}{y-x}$$
Since $f_n \xrightarrow[n\to\infty]{u} f$ we have
$$\lim_{n\to\infty} g_n(y) = g(y)$$
Since f_n is differentiable at x,
$$\lim_{y\to x} g_n(y) = f_n'(x)$$
Let $L(x) = \lim_{n\to\infty} f_n'(x)$. We want to show that
$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } |g(y) - L(x)| < \varepsilon \text{ whenever } 0 < |y-x| < \delta \ y \in [a,b]$$
Fix $\varepsilon > 0$. By the triangle inequality,
$$|g(y) - L(x)| \le |g(y) - g_n(y)| + \left|g_n(y) - f_n'(x)\right| + \left|f_n'(x) - L(x)\right|$$
We have $\{f_n'\}_{n\ge 1}$ converges uniformly on $(a,b) \implies \{f_n'\}_{n\ge1}$ is uniformly Cauchy on (a,b) $\implies \exists n_1(\varepsilon) \in \mathbb{N}$ s.t.
$$\left|f_n'(z) - f_m'(z)\right| < \varepsilon \qquad \forall n,m \ge n_1(\varepsilon) \quad \forall z \in (a,b) \tag{1}$$
Letting $m \to \infty$ we get
$$\left|f_n'(z) - L(z)\right| \le \varepsilon \qquad \forall n \ge n_1(\varepsilon) \quad \forall z \in (a,b)$$
For $y \in [a,b] \setminus \{x\}$, by the Mean Value theorem, we can find a point z between x and y so that
$$\begin{aligned} |g_n(y) - g_m(y)| &= \left| \frac{f_n(y)-f_n(x)}{y-x} - \frac{f_m(y)-f_m(x)}{y-x} \right| \\ &= \frac{|[f_n(y)-f_m(y)] - [f_n(x)-f_m(x)]|}{|y-x|} \\ &= \left|f_n'(z) - f_m'(z)\right| \overset{(1)}{<} \varepsilon \quad \forall n,m \ge n_1(\varepsilon) \end{aligned}$$
Letting $m \to \infty$ we find
$$|g_n(y) - g(y)| \le \varepsilon \qquad \forall n \ge n_1(\varepsilon) \quad \forall y \in [a,b] \setminus \{x\} \tag{3}$$

Fix $n \geq n_1(\varepsilon)$. As f_n is differentiable at x we find $\delta = \delta(\varepsilon, n) > 0$ s.t.

$$\left|g_n(y) - f_n'(x)\right| < \varepsilon \qquad \forall 0 < |y - x| < \delta \quad y \in [a, b] \tag{4}$$

Thus for this $n \geq n_1(\varepsilon)$ and $0 < |y - x| < \delta$ we have

$$\begin{aligned} |g(y) - L(x)| &\leq |g(y) - g_n(y)| + \left|g_n(y) - f_n'(x)\right| + \left|f_n'(x) - L(x)\right| \\ \text{by (2), (3), (4)} &\leq 3\varepsilon \end{aligned}$$

□

Example 18.1

$f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{1+nx^2}$, f_n is differentiable and

$$f_n'(x) = \frac{1}{1+nx^2} - \frac{x \cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

Now

$$f_n \xrightarrow[n\to\infty]{u} f \equiv 0$$

$$f_n'(x) \xrightarrow[n\to\infty]{} \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Note that f_n' do not converge uniformly since their limit is not continuous.

$$\lim_{n\to\infty} \lim_{y\to 0} \frac{f_n(y) - f_n(0)}{y - 0} = \lim_{n\to\infty} f_n'(0) = 1$$

but

$$\lim_{y\to 0} \lim_{n\to\infty} \frac{f_n(y) - f_n(0)}{y - 0} = \lim_{y\to 0} 0 = 0$$

§18.2 Darboux Integral

Definition 18.2 (Partition) — Let $f : [a, b] \to \mathbb{R}$ be a <u>bounded</u> function. If $S \subseteq [a, b]$ we denote

$$M(f; S) = \sup_{x\in S} f(x) \quad \text{and} \quad m(f; S) = \inf_{x\in S} f(x)$$

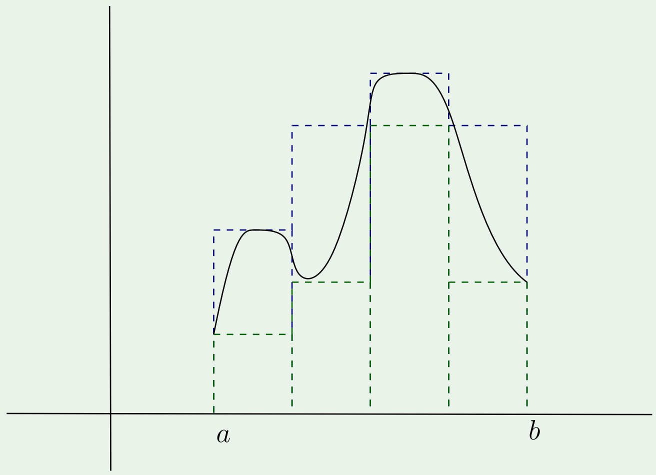
A <u>partition</u> of $[a, b]$ is a finite ordered set $P \subseteq [a, b]$. We write

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

for some $n \geq 1$.

Definition 18.3 (Darboux Sum) — The upper Darboux sum of f with respect to P is

$$U(f;P) = \sum_{k=1}^{n} M\left(f;[t_{k-1},t_k]\right)\left(t_k - t_{k-1}\right)$$



The lower Darboux sum of f with respect to P is

$$L(f;P) = \sum_{k=1}^{n} m\left(f;[t_{k-1},t_k]\right)\left(t_k - t_{k-1}\right)$$

Note that

$$m\left(f;[a,b]\right)(b-a) \le L(f;P) \le U(f;P) \le M\left(f;[a,b]\right)(b-a)$$

So

$$\{L(f;P) :\ P \text{ partition of } [a,b]\} \text{ is bounded above}$$
$$\{U(f;P) :\ P \text{ partition of } [a,b]\} \text{ is bounded below}$$

Definition 18.4 (Darboux Integral) — The upper Darboux integral of f on $[a,b]$ is

$$U(f) = \inf\left\{U(f;P) :\ P \text{ partition of } [a,b]\right\}$$

The lower Darboux integral of f on $[a,b]$ is

$$L(f) = \sup\left\{L(f;P) :\ P \text{ partition of } [a,b]\right\}$$

We say that f is Darboux integrable on $[a,b]$ if $U(f) = L(f)$. In this case we write

$$\int_a^b f(x)\,dx = U(f) = L(f)$$

Example 18.5

Let $f : [0, M] \to \mathbb{R}$, $f(x) = x^3$. Then f is Darboux integrable.
Let $P = \{0 = t_0 < \ldots < t_n = M\}$ be a partition of $[0, M]$ and

$$\begin{aligned} U(f; P) &= \sum_{k=1}^{n} M\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} t_k^3 (t_k - t_{k-1}) \end{aligned}$$

Similarly,

$$L(f; P) = \sum_{k=1}^{n} m\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1}) = \sum_{k=1}^{n} t_{k-1}^3 (t_k - t_{k-1})$$

Take $t_k = \frac{kM}{n}$ $0 \le k \le n$. Then

$$U(f; P) = \sum_{k=1}^{n} \left(\frac{kM}{n}\right)^3 \cdot \frac{M}{n} = \frac{M^4}{n^4} \sum_{k=1}^{n} k^3 = \frac{M^4}{n^4}\left[\frac{n(n+1)^2}{2}\right] \xrightarrow[n\to\infty]{} \frac{M^4}{4}$$

$$L(f; P) = \sum_{k=1}^{n} \left(\frac{(k-1)M}{n}\right)^3 \cdot \frac{M}{n} = \frac{M^4}{n^4} \sum_{k=0}^{n-1} k^3 = \frac{M^4}{n^4}\left[\frac{n(n-1)^2}{2}\right] \xrightarrow[n\to\infty]{} \frac{M^4}{4}$$

So, $U(f) \le \frac{M^4}{4}$ and $L(f) \ge \frac{M^4}{4}$ and we will show that $L(f) \le U(f)$ which imply $U(f) = L(f) = \frac{M^4}{4}$. So f is Darboux integrable and $\int_0^M f(x)\,dx = \frac{M^4}{4}$.

Example 18.6

Given

$$f : [0, 1] \to \mathbb{R}, \quad f(x) = \begin{cases} 1, & x \in [0, 1] \cap \mathbb{Q} \\ 0, & x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

f is not Darboux integrable. For any partition P, $U(f; P) = 1$ and $L(f; P) = 0$ which implies $U(f) = 1$ and $L(f) = 0$.

§19 Lec 19: May 10, 2021

§19.1 Darboux Integral (Cont'd)

Recall: If $f : [a, b] \to \mathbb{R}$ bounded

$$P = \{a = t_0 < \ldots < t_n = b\} \text{ partition of } [a, b]$$

then

$$U(f; P) = \sum_{k=1}^{n} M\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1})$$
$$L(f; P) = \sum_{k=1}^{n} m\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1})$$

are the upper and lower Darboux sum associated with P, respectively f is Darboux integrable if $U(f) = L(f)$ where

$$U(f) = \inf_P U(f; P) \quad \text{and} \quad L(f) = \sup_P L(f; P)$$

Proposition 19.1

Let $f : [a, b] \to \mathbb{R}$ be two bounded and let P and Q be partitions of $[a, b]$ s.t. $P \subseteq Q$. Then

$$L(f; p) \le L(f; Q) \le U(f; Q) \le U(f; P)$$

Proof. We will prove the third inequality. The first inequality follows from a similar argument. Arguing by induction, it suffices to prove the claim when the partition Q contains exactly one extra point compared to the partition P. Let

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$
$$Q = \{a = t_0 < \ldots < t_{l-1} < s < t_l < \ldots < t_n = b\}$$

for some $1 \le l \le n$.

$$U(f; Q) = \sum_{k=1}^{l-1} M\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1}) + M\left(f; [t_{l-1}, s]\right)(s - t_{l-1}) + M\left(f; [s, t_l]\right)(t_l - s)$$
$$+ \sum_{k=l+1}^{n} M\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1})$$

Clearly,

$$M\left(f; [t_{l-1}, s]\right) \le M\left(f; [t_{l-1}, t_l]\right)$$
$$M\left(f; [s, t_l]\right) \le M\left(f; [t_{l-1}, t_l]\right)$$

So

$$U(f; Q) \le \sum_{k=1}^{n} M\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1}) = U(f; P) \qquad \square$$

Corollary 19.2

Let $f : [a, b] \to \mathbb{R}$ be bounded and let P, Q be two partitions of $[a, b]$. Then

$$L(f; P) \le U(f; Q)$$

Consequently,

$$L(f) \le U(f)$$

Proof. Consider the partition $P \cup Q$. We have

$$\begin{aligned} L(f; P) \le L(f; P \cup Q) \le U(f; P \cup Q) \le U(f; Q) \\ \implies L(f) = \sup_P L(f; P) \le U(f; Q) \\ \implies L(f) \le \inf_Q U(f; Q) = U(f) \end{aligned}$$

□

Theorem 19.3

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Darboux integrable if and only if

$$\forall \varepsilon > 0 \quad \exists P \text{ partitions of } [a, b] \quad \ni \quad U(f; P) - L(f; P) < \varepsilon$$

Proof. " $\impliedby$ " Fix $\varepsilon > 0$. Then there exists P partition of $[a, b]$ s.t. $U(f; P) - L(f; P) < \varepsilon$

$$\begin{aligned} &\implies U(f) \le U(f; P) < L(f; P) + \varepsilon \le L(f) + \varepsilon \\ &\implies \left.\begin{matrix} U(f) < L(f) + \varepsilon \\ \varepsilon > 0 \text{ was arbitrary} \end{matrix}\right\} \implies \left.\begin{matrix} U(f) \le L(f) \\ L(f) \le U(f) \end{matrix}\right\} \implies U(f) = L(f) \\ &\implies f \text{ is Darboux integrable} \end{aligned}$$

" $\implies$ " Fix $\varepsilon > 0$, f is Darboux integrable implies

$$U(f) = L(f)$$

Then

$$\begin{aligned} U(f) = \inf_P U(f; P) &\implies \exists P_1 \text{ partition of } [a, b] \text{ s.t. } U(f; P_1) < U(f) + \frac{\varepsilon}{2} \\ L(f) = \sup_P L(f; P) &\implies \exists P_2 \text{ partition of } [a, b] \text{ s.t. } L(f; P_2) > L(f) - \frac{\varepsilon}{2} \end{aligned}$$

Consider the partition $P_1 \cup P_2$. Then

$$L(f; P_2) \le L(f; P_1 \cup P_2) \le U(f; P_1 \cup P_2) \le U(f; P_1)$$

So

$$U(f; P_1 \cup P_2) - L(f; P_1 \cup P_2) < U(f) + \frac{\varepsilon}{2} - \left(L(f) - \frac{\varepsilon}{2}\right) = \varepsilon$$

□

Definition 19.4 (Mesh) — Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a, b]$. The mesh of P is given by

$$\text{mesh}(P) = \max_{1 \le k \le n} (t_k - t_{k-1})$$

Theorem 19.5

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Darboux integrable if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. if } P \text{ is a partition of } [a,b] \text{ with } \text{mesh}(P) < \delta$$

then

$$U(f; P) - L(f; P) < \varepsilon$$

Proof. " $\Longleftarrow$ " By the previous theorem, it suffices to show that $\forall \delta > 0\ \exists P$ partition of $[a, b]$ with $\text{mesh}(P) < \delta$. For $\delta > 0$, let $P = \{a = t_0 < \ldots < t_n = b\}$ where

$$t_k = a + k \cdot \frac{\delta}{2} \quad \text{for} \quad 0 \le k \le \lfloor \frac{2(b-a)}{\delta} \rfloor = n - 1$$

and $t_n = b$. Clearly,

$$\text{mesh}(P) = \frac{\delta}{2} < \delta$$

" $\Longrightarrow$ " Fix $\varepsilon > 0$. By the previous theorem, as f is Darboux integrable, there exists a partition $P_0 = \{a = s_0 < \ldots < s_m = b\}$ of $[a, b]$ s.t.

$$U(f; P_0) - L(f; P_0) < \frac{\varepsilon}{2}$$

Let $0 < \delta < \text{mesh}(P_0)$ to be chosen later and let $P = \{a = t_0 < \ldots < t_n = b\}$ be a partition of $[a, b]$ with $\text{mesh}(P) < \delta$

$$\begin{aligned} U(f; P) - L(f; P) &\le U(f; P) - U(f; P_0) + U(f; P_0) - L(f; P_0) + L(f; P_0) - L(f; P) \\ &\le \frac{\varepsilon}{2} + U(f; P) - U(f; P_0) + L(f; P_0) - L(f; P) \end{aligned}$$

Consider the partition $P \cup P_0$. Then

$$U(f; P) - U(f; P_0) \le U(f; P) - U(f; P \cup P_0)$$

As $\text{mesh}(P) < \delta < \text{mesh}(P_0)$, there must be at most one point from P_0 in each $[t_{k-1}, t_k]$. Only subintervals $[t_{k-1}, t_k]$ with an $s_j \in P_0 \cap [t_{k-1}, t_k]$ contribute to $U(f; P) - U(f; P_0 \cup P)$. There are only m many such intervals. The contribution of one such interval to $U(f; P) - U(f; P_0 \cup P)$ is

$$M(f; [t_{k-1}, t_k])(t_k - t_{k-1}) - M(f; [t_{k-1}, s_j])(s_j - t_{k-1}) - M(f; [s_j, t_k])(t_k - s_j)$$

As f is bounded, $\exists M > 0$ s.t. $|f(x)| \le M\ \forall x \in [a, b]$. Note

$$\begin{aligned} &M(f; [t_{k-1}, t_k]) \le M \\ &M(f; [t_{k-1}, s_j]) \ge -M; \qquad M(f; [s_j, t_k]) \ge -M \end{aligned}$$

So

$$M\left(f;\left[t_{k-1},t_k\right]\right)\left(t_k-t_{k-1}\right)-M\left(f;\left[t_{k-1},s_j\right]\right)\left(s_j-t_{k-1}\right)-M\left(f;\left[s_j,t_k\right]\right)\left(t_k-s_j\right)$$

which is smaller than or equal to

$$M\left(t_k-t_{k-1}\right)-(-M)\left[\left(s_j-t_{k-1}\right)+\left(t_k-s_j\right)\right]=2M\left(t_k-t_{k-1}\right)<2M\cdot\operatorname{mesh}(P)$$

Thus

$$U(f;P)-U(f;P_0)<m\cdot 2M\cdot\operatorname{mesh}(P)$$

Similarly,

$$L(f;P_0)-L(f;P)<m\cdot 2M\cdot\operatorname{mesh}(P)$$

which requires

$$4Mm\cdot\operatorname{mesh}(P)<\frac{\varepsilon}{2}\iff\operatorname{mesh}(P)<\frac{\varepsilon}{8Mm}$$

Thus, $\delta<\min\left\{\frac{\varepsilon}{8Mm},\operatorname{mesh}(P_0)\right\}$. □

§20 Lec 20: May 12, 2021

§20.1 Riemann Integral

Definition 20.1 (Riemann Sum) — Let $f : [a, b] \to \mathbb{R}$ be a function and let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a, b]$. A Riemann sum of f associated to P is a sum of the form

$$S = \sum_{k=1}^{n} f(x_k)\,(t_k - t_{k-1}) \qquad \text{where } x_k \in [t_{k-1}, t_k] \quad \forall 1 \le k \le n$$

Note: If S is a Riemann sum associated with a partition P of $[a, b]$ then

$$L(f; P) \le S \le U(f; P)$$

Definition 20.2 (Riemann Integrable) — We say that f is Riemann integrable if $\exists r \in \mathbb{R}$ s.t. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$|S - r| < \varepsilon$$

for any Riemann sum S of f associated with a partition P with $\operatorname{mesh}(P) < \delta$. Then r is called the Riemann integral of f and we write

$$r = \mathcal{R}\int_a^b f(x)\,dx$$

Lemma 20.3

If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then f is bounded.

Proof. Let $r = \mathcal{R}\int_a^b f(x)\,dx$. Taking $\varepsilon = 1$ we find $\delta > 0$ s.t. $|S - r| < 1$ for any Riemann sum S of f associated to a partition P with $\operatorname{mesh}(P) < \delta$.

Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ with $\operatorname{mesh}(P) < \delta$. Fix $1 \le k \le n$. Fix $x_l \in [t_{l-1}, t_l]$ for $1 \le l \le n$, $l \ne k$. For $x \in [t_{k-1}, t_k]$ we have

$$\left| \sum_{l \ne k} f(x_l)\,(t_l - t_{l-1}) + f(x)\,(t_k - t_{k-1}) - r \right| < 1$$

$$\left.\begin{array}{l} \frac{r-1-\sum_{l\ne k} f(x_l)(t_l - t_{l-1})}{t_k - t_{k-1}} < f(x) < \frac{1+r-\sum_{l\ne k} f(x_l)(t_l - t_{l-1})}{t_k - t_{k-1}} \\ x \in [t_{k-1}, t_k] \text{ is arbitrary} \end{array}\right\} \implies$$

$$\implies \left.\begin{array}{l} f \text{ is bounded on } [t_{k-1}, t_k] \\ 1 \le k \le n \text{ is arbitrary} \end{array}\right\} \implies f \text{ is bounded on } [a, b]$$

□

Theorem 20.4

Let $f : [a,b] \to \mathbb{R}$. The following are equivalent

1. f is Riemann integrable.

2. f is bounded and Darboux integrable.

If either conditions holds, then the integrals agree.

Proof. 2) $\implies$ 1) Fix $\varepsilon > 0$.

f is Darboux integrable $\implies$ $\exists \delta > 0$ s.t. $U(f;P) - L(f;P) < \varepsilon$ for any partition P with $\text{mesh}(P) < \delta$. Let P be a partition of $[a,b]$ with $\text{mesh}(P < \delta)$. If S is a Riemann sum of f associated to P, then

$$\left.\begin{aligned} S \le U(f;P) < L(f;P) + \varepsilon \le L(f) + \varepsilon = \textstyle\int_a^b f(x)\,dx + \varepsilon \\ S \ge L(f;P) > U(f;P) - \varepsilon \ge U(f) - \varepsilon = \textstyle\int_a^b f(x)\,dx - \varepsilon \end{aligned}\right\} \implies \left| s - \int_a^b f(x) dx \right| < \varepsilon$$

By definition, f is Riemann integrable and $\mathcal{R}\int_a^b f(x)dx = \int_a^b f(x)dx$.

1) $\implies$ 2) By the previous lemma, f is bounded. Fix $\varepsilon > 0$. Let $r = \mathcal{R}\int_a^b f(x)dx$. Then $\exists \delta > 0$ s.t.

$$|S - r| < \frac{\varepsilon}{2}$$

for any Riemann sum of f associated with a partition of P with $\text{mesh}(P) < \delta$. Fix $P = \{a = t_0 < t_1 < \ldots . < t_n = b\}$ be a partition with $(\text{mesh}(P) < \delta$. There exist $x_k, y_k \in [t_{k-1}, t_k]$ s.t.

$$f(x_k) > M\left(f; [t_{k-1}, t_k]\right) - \frac{\varepsilon}{2(b-a)}$$
$$f(y_k) < m\left(f; [t_{k-1}, t_k]\right) + \frac{\varepsilon}{2(b-a)}$$

Then

$$\begin{aligned} S_1 = \sum_{k=1}^n f(x_k)\left(t_k - t_{k-1}\right) &> U(f;P) - \frac{\varepsilon}{2(b-a)} \sum_{k=1}^n (t_k - t_{k-1}) \\ &= U(f;P) - \frac{\varepsilon}{2} \\ S_2 = \sum_{k=1}^n f(y_k)\left(t_k - t_{k-1}\right) &< L(f;P) + \frac{\varepsilon}{2(b-a)} \sum_{k=1}^n (t_k - t_{k-1}) \\ &= L(f;P) + \frac{\varepsilon}{2} \end{aligned}$$

However, $|S_1 - r| < \frac{\varepsilon}{2}$ and $|S_2 - r| < \frac{\varepsilon}{2}$. So

$$\left.\begin{aligned} U(f;P) - \tfrac{\varepsilon}{2} < S_1 < r + \tfrac{\varepsilon}{2} \implies U(f) \le U(f;P) < r + \varepsilon \\ r - \tfrac{\varepsilon}{2} < S_2 < L(f;P) + \tfrac{\varepsilon}{2} \implies r - \varepsilon < L(f;P) \le L(f) \end{aligned}\right\} \implies$$

$$\implies \left.\begin{aligned} &r - \varepsilon < L(f) \le U(f) < r + \varepsilon \\ &\varepsilon > 0 \text{ arbitrary} \end{aligned}\right\} \implies f \text{ is Darboux integrable and } \int_a^b f(x)\,dx = r$$

$\square$

Theorem 20.5

Let $f : [a, b] \to \mathbb{R}$ be monotonic. Then f is integrable.

Proof. Assume f is increasing. Then

$$f(a) \le f(x) \le f(b) \qquad \forall x \in [a, b]$$

So f is bounded.
Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ with $\operatorname{mesh}(P) < \delta$ for δ to be chosen later. Then

$$\begin{aligned} U(f;P) - L(f;P) &= \sum_{k=1}^{n} \left[M\left(f;[t_{k-1},t_k]\right) - m\left(f;[t_{k-1},t_k]\right)\right](t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} \left[f(t_k) - f(t_{k-1})\right](t_k - t_{k-1}) \\ &\le \operatorname{mesh}(P) \sum_{k=1}^{n} \left[f(t_k) - f(t_{k-1})\right] \\ &< \delta \cdot \left[f(b) - f(a)\right] \end{aligned}$$

Taking $\delta < \frac{\varepsilon}{f(b)-f(a)+1}$ we see that f is Darboux integrable. □

Theorem 20.6

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is integrable.

Proof. We have

$$\left.\begin{array}{l} f : [a,b] \to \mathbb{R} \text{ continuous} \\ [a,b] \text{ compact} \end{array}\right\} \implies f \text{ is bounded}$$

Fix $\varepsilon > 0$. As f is continuous on $[a, b]$ compact, f is uniformly continuous. So $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad \forall x, y \in [a, b] \text{ with } |x - y| < \delta$$

Let $P = \{a = t_0 < \ldots < t_n = b\}$ with $\operatorname{mesh}(P) < \delta$.

$$U(f;P) - L(f;P) = \sum_{k=1}^{n} \left[M\left(f;[t_{k-1},t_k]\right) - m\left(f;[t_{k-1},t_k]\right)\right](t_k - t_{k-1})$$

f continuous on $[t_{k-1}, t_k]$ compact implies $\exists x_k, y_k \in [t_{k-1}, t_k]$ s.t.

$$\begin{aligned} f(x_k) &= M\left(f;[t_{k-1},t_k]\right) \\ f(y_k) &= m\left(f;[t_{k-1},t_k]\right) \end{aligned}$$

So

$$\begin{aligned} U(f;P) - L(f;P) &= \sum_{k=1}^{n} \left[f(x_k) - f(y_k)\right](t_k - t_{k-1}) \\ &< \sum_{k=1}^{n} \frac{\varepsilon}{b-a}(t_k - t_{k-1}) = \varepsilon \end{aligned}$$

Then f is Darboux integrable. □

Theorem 20.7

Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable.

1. For any $\alpha \in \mathbb{R}$, αf is Riemann integrable and
$$\int_a^b (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx$$
2. $f + g$ is Riemann integrable and
$$\int_a^b (f+g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

Proof. 1. If $\alpha = 0$ this is clear. Assume $\alpha > 0$. For any $S \subseteq [a, b]$
$$\begin{aligned} M(\alpha f; S) &= \alpha M(f; S) \\ m(\alpha f; S) &= \alpha m(f; S) \end{aligned}$$

For by partition P of $[a, b]$,
$$\begin{aligned} U(\alpha f; P) = \alpha U(f; P) \implies U(\alpha f) &= \sup_P U(\alpha f; P) \\ &= \sup_P [\alpha \cdot U(f; P)] \\ &= \alpha \sup_P U(f; P) = \alpha U(f) \end{aligned}$$

Similarly,
$$\begin{aligned} L(\alpha f) &= \alpha L(f) \\ L(f) &= U(f) \end{aligned}$$

$\implies \alpha f$ is Darboux integrable and $\int_a^b (\alpha f)(x) dx = \alpha \int_a^b f(x) dx$. $\square$

§21 Lec 21: May 14, 2021

§21.1 Riemann Integral (Cont'd)

Recall from last lecture, we have the following theorem,

Theorem 21.1

Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable.

1. For any $\alpha \in \mathbb{R}$, αf is Riemann integrable and
$$\int_a^b (\alpha f)(x)\,dx = \alpha \int_a^b f(x)\,dx$$
2. $f+g$ is Riemann integrable and
$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Proof. 1. Last time we proved the result for $\alpha \ge 0$. Assume $\alpha < 0$. For $S \subseteq [a,b]$, we have
$$M(\alpha f; S) = \alpha m(f;S) \quad \text{and} \quad m(\alpha f; S) = \alpha M(f;S)$$
If P is a partition of $[a,b]$,
$$U(\alpha f; P) = \alpha L(f;P) \quad \text{and} \quad L(\alpha f; P) = \alpha U(f;P)$$
Thus,
$$\left.\begin{array}{l} U(\alpha f) = \inf_P U(\alpha f; P) = \inf_P \alpha L(f;P) = \alpha \sup_P L(f;P) = \alpha L(f) \\ L(\alpha f) = \ldots = \alpha U(f) \\ f \text{ is Riemann integrable} \implies f \text{ bounded and } L(f) = U(f) = \int_a^b f(x)\,dx \end{array}\right\} \implies$$
$$\implies \alpha f \text{ is bounded and } L(\alpha f) = U(\alpha f) = \alpha \int_a^b f(x)\,dx$$
$$\implies \alpha f \text{ is Riemann integrable and } \int_a^b (\alpha f)(x)\,dx = \alpha \int_a^b f(x)\,dx$$

2. As f, g are Riemann integrable, $f+g$ is bounded and f, g are Darboux integrable. Fix $\varepsilon > 0$. Then, f is Darboux integrable implies $\exists P_1$ partition of $[a,b]$ s.t.
$$U(f;P_1) - L(f;P_1) < \frac{\varepsilon}{2}$$
g is Darboux integrable implies $\exists P_2$ partition of $[a,b]$ s.t.
$$U(g;P_2) - L(g;P_2) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$. Then, we have

$$U(f;P) - L(f;P) < \frac{\varepsilon}{2} \quad \text{and} \quad U(g;P) - L(g;P) < \frac{\varepsilon}{2}$$

For $S \subseteq [a,b]$,

$$\begin{aligned} M(f+g;S) &\le M(f;S) + M(g;S) \\ m(f+g;S) &\ge m(f;S) + m(g;S) \end{aligned}$$

So

$$\left.\begin{aligned} U(f+g;P) &\le U(f;P) + U(g;P) \\ L(f+g;P) &\ge L(f;P) + L(g;P) \end{aligned}\right\} \implies$$

$$\implies U(f+g;P) - L(f+g;P) \le U(f;P) - L(f;P) + U(g;P) - L(g;P) < \varepsilon$$

$$\implies \left.\begin{aligned} &f+g \text{ is Darboux integrable} \\ &f+g \text{ is bounded} \end{aligned}\right\} \implies f+g \text{ is Riemann integrable}$$

Moreover,

$$\begin{aligned} U(f+g) \le U(f+g;P) &\le U(f;P) + U(g;P) \\ &< L(f;P) + L(g;P) + \varepsilon \\ &\le L(f) + L(g) + \varepsilon = \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon \end{aligned}$$

Similarly,

$$\begin{aligned} L(f+g) \ge L(f+g;P) &\ge L(f;P) + L(g;P) \\ &> U(f;P) + U(g;P) - \varepsilon \\ &\ge U(f) + U(g) - \varepsilon = \int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon \end{aligned}$$

Let $\varepsilon \to 0$, we get

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad \square$$

Theorem 21.2

Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable. Assume $f(x) \le g(x)\ \forall x \in [a,b]$. Then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Proof. By the previous theorem, $h : [a,b] \to \mathbb{R}$, $h = g - f$ is Riemann integrable. Moreover, since $h \ge 0$, we have

$$\int_a^b h(x)\,dx = L(h) = \sup_P L(h;P) \ge 0$$

which implies

$$0 \le \int_a^b h(x)\,dx = \int_a^b (g-f)(x)\,dx = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \qquad \square$$

Theorem 21.3

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then $|f|$ is Riemann integrable and

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

Proof. Let f is Riemann integrable. Then, f is bounded and Darboux integrable. So $|f|$ is bounded. For $S \subseteq [a, b]$ we have

$$\begin{aligned} M\left(|f|; S\right) - m\left(|f|; S\right) &= \sup_{x \in S} |f(x)| - \inf_{y \in S} |f(y)| \\ &= \sup_{x \in S} |f(x)| + \sup_{y \in S} -|f(y)| \\ &= \sup_{x, y \in S} \left\{ |f(x)| - |f(y)| \right\} \\ &\leq \sup_{x, y \in S} |f(x) - f(y)| \\ &= \sup_{x, y \in S} \left\{ f(x) - f(y) \right\} \\ &= \sup_{x \in S} f(x) - \inf_{y \in S} f(y) \\ &= M(f; S) - m(f; S) \end{aligned}$$

So for any partition P of $[a, b]$ we have

$$U(|f|; P) - L(|f|; P) \leq U(f; P) - L(f; P)$$

f Darboux integrable $\implies \forall \varepsilon > 0\ \exists P$ partition of $[a, b]$ s.t.

$$\begin{aligned} &U(f; P) - L(f; P) < \varepsilon \\ \implies &\forall \varepsilon > 0\, \exists P \text{ partition of } [a, b] \text{ s.t. } U(|f|; P) - L(|f|; P) < \varepsilon \\ \implies &\left.\begin{array}{l} |f| \text{ is Darboux integrable} \\ |f| \text{ is bounded} \end{array}\right\} \implies |f| \text{ is Riemann integrable} \end{aligned}$$

We have

$$-|f(x)| \leq f(x) \leq |f(x)| \qquad \forall x \in [a, b]$$

By the previous theorem,

$$-\int_a^b |f(x)|\, dx = \int_a^b -|f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx$$

which implies

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

□

Theorem 21.4

Let $f : [a, b] \to \mathbb{R}$ be a function and let $a < c < b$. Assume f is Riemann integrable on $[a, c]$ and on $[c, b]$. Then f is Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Proof. f is Riemann integrable on $[a, c]$ and on $[c, b]$

$$\begin{aligned} &\implies f \text{ bounded on } [a,c] \text{ and on } [c,b] \\ &\implies f \text{ bounded on } [a,b] \end{aligned}$$

Fix $\varepsilon > 0$. As f is Riemann integrable on $[a, c]$, f is Darboux integrable on $[a, c]$

$$\implies \exists P_1 \text{ partition of } [a,c] \text{ s.t. } U_a^c(f; P_1) - L_a^c(f; P_1) < \frac{\varepsilon}{2}$$

Similarly, as f is Riemann integrable on $[c, b]$ $\implies$ f Darboux integrable on $[c, b]$

$$\implies \exists P_2 \text{ partition of } [c,b] \text{ s.t. } U_c^b(f; P_2) - L_c^b(f; P_2) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$ partition on $[a, b]$ and

$$\begin{aligned} U(f;P) &= U_a^c(f;P_1) + U_c^b(f;P_2) \\ L(f;P) &= L_a^c(f;P_1) + L_c^b(f;P_2) \end{aligned}$$

So

$$U(f;P) - L(f;P) < \frac{\varepsilon}{2}$$

Therefore, as f is Darboux integrable and bounded on $[a, b]$, f is Riemann integrable on $[a, b]$. Moreover,

$$\begin{aligned} U(f) \le U(f;P) = U_a^c(f;P_1) + U_c^b(f;P_2) &< L_a^c(f;P_1) + L_c^b(f;P_2) + \varepsilon \\ &\le \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \varepsilon \end{aligned}$$

Similarly,

$$L(f) \ge \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \varepsilon$$

Since $\varepsilon > 0$ is arbitrary,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

□

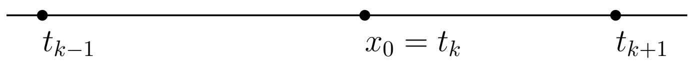
Lemma 21.5

Let $f, g : [a, b] \to \mathbb{R}$ be functions s.t. f is Riemann integrable and $g(x) = f(x)$ except at finitely many points in $[a, b]$. Then g is Riemann integrable and

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$$

Proof. Arguing by induction, we may assume that there exists exactly one point $x_0 \in [a, b]$ s.t. $f(x_0) \neq g(x_0)$. Let $B > 0$ s.t. $|f(x)| \leq B$ and $|g(x)| \leq B$ $\forall x \in [a, b]$. Let $P = \{a = t_0 < \ldots < t_n = b\}$. We consider

$$\begin{gathered} U(f; P) - U(g; P) \\ L(f; P) - L(g; P) \end{gathered}$$



The largest contribution occurs when $x_0 = t_k$ for some $1 \leq k \leq n - 1$.

$$\begin{aligned} |M(f; [t_{k-1}, t_k]) - M(g; [t_{k-1}, t_k])| &\leq [B - (-B)](t_k - t_{k-1}) \\ &\leq 2B \text{ mesh}(P) \\ \implies |U(f; P) - U(g; P)| &\leq 4B \text{ mesh}(P) \end{aligned}$$

Similarly,

$$\begin{aligned} |m(f; [t_{k-1}, t_k]) - m(g; [t_{k-1}, t_k])| &\leq 2B \text{ mesh}(P) \\ \implies |L(f; P) - L(g; P)| &\leq 4B \text{ mesh}(P) \end{aligned}$$

Thus,

$$\begin{aligned} U(g; P) - L(g; P) &\leq U(f; P) - L(f; P) + |U(f; P) - U(g; P)| \\ &\quad + |L(f; P) - L(g; P)| \\ &\leq U(f; P) - L(f; P) + 8B \text{ mesh}(P) \end{aligned}$$

f Darboux integrable $\implies \forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$U(f; P) - L(f; P) < \frac{\varepsilon}{2} \qquad \forall P \text{ partition with mesh}(P) < \delta$$

Choose δ even smaller if necessary so that

$$8B\delta < \frac{\varepsilon}{2} \iff \delta < \frac{\varepsilon}{16B}$$

Then $U(g; P) - L(g; P) < \varepsilon$ for all P partition with mesh$(P) < \delta$.

$$\left.\begin{aligned} &g \text{ is Darboux integrable} \\ &g \text{ bounded} \end{aligned}\right\} \implies g \text{ is Riemann integrable}$$

Exercise 21.1. Show $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$. $\square$

§22 Lec 22: May 17, 2021

§22.1 Riemann Integral (Cont'd)

Definition 22.1 (Piecewise Monotone) — We say that a function $f : [a, b] \to \mathbb{R}$ is piecewise monotone if there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ s.t. f is monotone on (t_{k-1}, t_k) for each $1 \leq k \leq n$.

Definition 22.2 (Piecewise Continuous) — We say that $f : [a, b] \to \mathbb{R}$ is piecewise continuous if there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ s.t. f is uniformly continuous on (t_{k-1}, t_k) for each $1 \leq k \leq n$.

Theorem 22.3

Let $f : [a, b] \to \mathbb{R}$ be a function that satisfies

1. f is bounded and piecewise monotone.

   or

2. f is piecewise continuous.

Then f is Riemann integrable.

Proof. Let $P = \{a = t_0 < \ldots < t_n = b\}$ be a partition of $[a, b]$ s.t. 1) f is monotone **or** 2) f is uniformly continuous on (t_{k-1}, t_k) $\forall 1 \leq k \leq n$.

If f is monotone on (t_{k-1}, t_k), then f can be extended to a monotone function on f_k on $[t_{k-1}, t_k]$. For example, if f is increasing on (t_{k-1}, t_k) we define

$$f_k(t) = \begin{cases} \inf_{t \in (t_{k-1}, t_k)} f(t), & t = t_{k-1} \\ f(t), & t \in (t_{k-1}, t_k) \\ \sup_{t \in (t_{k-1}, t_k)} f(t), & t = t_k \end{cases}$$

As f_k is monotone on $[t_{k-1}, t_k]$, f_k is Riemann integrable on $[t_{k-1}, t_k]$. As f differs from f_k at most two points, f is Riemann integrable on $[t_{k-1}, t_k]$ and

$$\int_{t_{k-1}}^{t_k} f(t)\, dt = \int_{t_{k-1}}^{t_k} f_k(t)\, dt$$

If f is uniformly continuous on (t_{k-1}, t_k), then f admits a continuous extension f_k to $[t_{k-1}, t_k]$. Then f_k is Riemann integrable on $[t_{k-1}, t_k]$ and so f is Riemann integrable on $[t_{k-1}, t_k]$ and

$$\int_{t_{k-1}}^{t_k} f(t)\, dt = \int_{t_{k-1}}^{t_k} f_k(t)\, dt$$

By the last theorem from last lecture, we conclude that f is Riemann integrable on $[a, b]$ and

$$\int_a^b f(t)\,dt = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f(t)\,dt \qquad \square$$

Theorem 22.4 (Intermediate Value Property for Integrals)
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a, b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Proof. f is continuous on $[a, b]$ compact which implies there exist $x_0, y_0 \in [a, b]$ s.t.

$$\begin{cases} f(x_0) = \inf_{x\in[a,b]} f(x) \\ f(y_0) = \sup_{x\in[a,b]} f(x) \end{cases}$$

So

$$(b-a)f(x_0) = \int_a^b f(x_0)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(y_0)\,dx = (b-a)f(y_0)$$

$$\left.\begin{array}{r} \implies f(x_0) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(y_0) \\ f \text{ is continuous} \implies f \text{ has the Darboux property} \end{array}\right\} \implies$$

$\implies \exists c$ between x_0 and y_0 s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$. $\square$

§22.2 Fundamental Theorem of Calculus

Definition 22.5 (Riemann Integrable – "Extension") — We say that a function $f : (a, b) \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if every extension of f to $[a, b]$ is Riemann integrable. In this case, $\int_a^b f(t)dt$ does not depend on the values of the extension at a and at b.

Theorem 22.6 (Fundamental Theorem of Calculus Part II)
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If f' is Riemann integrable on $[a, b]$ then

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Proof. Fix $\varepsilon > 0$. As f' is Riemann integrable on $[a, b]$, $\exists P = \{a = t_0 < \ldots < t_n = b\}$ s.t.

$$U(f'; P) - L(f'; P) < \varepsilon$$

where f is continuous on $[t_{k-1}, t_k]$ and differentiable on (t_{k-1}, t_k). So, by the Mean Value theorem, $\exists x_k \in (t_{k-1}, t_k)$ s.t.

$$f'(x_k) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}}$$

In particular,

$$\sum_{k=1}^{n} f'(x_k)(t_k - t_{k-1}) = \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] = f(b) - f(a)$$

is a Riemann sum of f' associated to the partition P. Moreover,

$$\left.\begin{aligned} L(f';P) \le f(b) - f(a) \le U(f';P) < L(f';P) + \varepsilon \\ L(f';P) \le \textstyle\int_a^b f'(x)\,dx \le U(f';P) < L(f';P) + \varepsilon \end{aligned}\right\} \implies$$

$$\left.\begin{aligned} &\implies \left|\textstyle\int_a^b f'(x)\,dx - [f(b) - f(a)]\right| < 2\varepsilon \\ &\varepsilon > 0 \text{ was arbitrary} \end{aligned}\right\} \implies \int_a^b f'(x)\,dx = f(b) - f(a)$$

□

Theorem 22.7 (Integration by Parts)

Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If f' and g' are Riemann integrable on $[a,b]$, then

$$\int_a^b f(x)g'(x)\,dx + \int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a)$$

Proof. By Exc 1 from Hw 8, the product of two Riemann integrable functions is Riemann integrable. In particular, $f'g$ and fg' are Riemann integrable. Let $h : [a,b] \to \mathbb{R}$, $h(x) = f(x)g(x)$. We have h is continuous on $[a,b]$, differentiable on (a,b) and

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

h' is Riemann integrable on $[a,b]$. By Fundamental Theorem of Calculus Part II,

$$\int_a^b h'(x)\,dx = h(b) - h(a)$$

$$\implies \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a)$$

□

Theorem 22.8 (Fundamental Theorem of Calculus Part I)

Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. For $x \in [a,b]$, we define

$$F(x) = \int_a^x f(t)\,dt$$

Then F is continuous on $[a,b]$. Moreover, if f is continuous at a point $x_0 \in (a,b)$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0)$$

Proof. For $a \le x < y \le b$,

$$\begin{aligned} F(y) - F(x) &= \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_a^x f(t)\,dt + \int_x^y f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_x^y f(t)\,dt \end{aligned}$$

f is Riemann integrable $\implies$ f is bounded $\implies$ $\exists M > 0$ s.t.

$$|f(x)| \le M \qquad \forall x \in [a,b]$$

So

$$|F(y) - F(x)| \le \int_x^y |f(t)|\,dt \le M\,|y-x|$$

This shows F is uniformly continuous on $[a,b]$. For each $\varepsilon > 0$ if $|y-x| < \frac{\varepsilon}{M}$ then

$$|F(y) - F(x)| < \varepsilon$$

Assume f is continuous at $x_0 \in (a,b)$. For $x \in [a,b] \setminus \{x_0\}$,

$$\begin{aligned} \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) &= \frac{1}{x - x_0} \int_{x_0}^x f(t)\,dt - f(x_0) \\ &= \frac{1}{x - x_0} \int_{x_0}^x f(t)\,dt - \frac{1}{x - x_0} \int_{x_0}^x f(x_0)\,dt \\ &= \frac{1}{x - x_0} \int_{x_0}^x \left[f(t) - f(x_0)\right]\,dt \end{aligned}$$

Fix $\varepsilon > 0$. As f is continuous at x_0, $\exists \delta > 0$ s.t.

$$|f(x) - f(x_0)| < \varepsilon \qquad \forall\, |x - x_0| < \delta \quad x \in [a,b]$$

So for $x \in [a,b]$ with $0 < |x - x_0| < \delta$,

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &\le \frac{1}{|x - x_0|} \int_{x_0}^x |f(t) - f(x_0)|\,dt \\ &< \frac{1}{|x - x_0|} \int_{x_0}^x \varepsilon\,dt = \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, F is differentiable at x_0 and $F'(x_0) = f(x_0)$. $\square$

§23 Lec 23: May 19, 2021

§23.1 Change of Variables

Theorem 23.1 (Change of Variables)
Let J be an open interval in $\mathbb{R}$ and let $u : J \to \mathbb{R}$ be differentiable with u' continuous on J. Let I be an open interval in $\mathbb{R}$ s.t. $u(J) \subseteq I$ and let $f : I \to \mathbb{R}$ be continuous. Then $f \circ u : J \to \mathbb{R}$ is continuous and for any $a, b \in J$ with $a < b$ we have

$$\int_a^b f(u(x)) \cdot u'(x)\, dx = \int_{u(a)}^{u(b)} f(y)\, dy$$

Proof. As $f \circ u$ and u' are continuous on $[a, b]$, the function $x \mapsto (f \circ u)(x) \cdot u'(x)$ is continuous on $[a, b]$ and so it's Riemann integrable on $[a, b]$.

Fix $c \in I$ and consider $F(x) = \int_c^x f(t)dt$. By Fundamental Theorem of Calculus Part I, F is differentiable on I (because f is continuous on I) and $F'(x) = f(x)\ \forall x \in I$. Consider $x \mapsto (F \circ u)(x)$ is differentiable on J and

$$(F \circ u)'(x) = f(u(x)) \cdot u'(x) \qquad \forall x \in J$$

By the Fundamental Theorem of Calculus Part II,

$$\int_a^b (F \circ u)'(x)\, dx = (F \circ u)(b) - (F \circ u)(a)$$

which implies

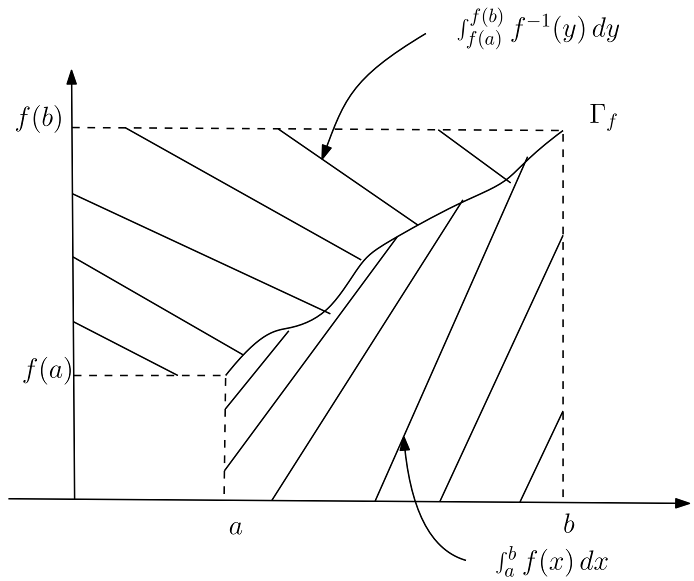
$$\implies \int_a^b f(u(x)) \cdot u'(x)\, dx = \int_c^{u(b)} f(y)\, dy - \int_c^{u(a)} f(y)\, dy = \int_{u(a)}^{u(b)} f(y)\, dy \qquad \square$$

Exercise 23.1. Let I be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be injective and differentiable with f' continuous on I. Then $J = f(I)$ is an open interval and $f^{-1} : J \to I$ is differentiable.

Then for any $a, b \in I$ with $a < b$ we have

$$\int_a^b f(x)\, dx + \int_{f(a)}^{f(b)} f^{-1}(y)\, dy = bf(b) - af(a)$$

Proof. Consider:



$$\Gamma_f = \{(x, f(x)) : a \le x \le b\} = \left\{\left(f^{-1}(y), y\right) : y \text{ between } f(a) \text{ and } f(b)\right\}$$

We perform a change of variables:

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,dy = \int_a^b f^{-1}\left(f(x)\right) f'(x)\,dx$$

where $y = f(x)$ and $dy = f'dx$

$$\begin{aligned}\int_a^b f^{-1}\left(f(x)\right) f'(x)\,dx &= \int_a^b x f'(x)\,dx \\ &= x f(x)\Big|_{x=a}^{x=b} - \int_a^b f(x)\,dx \\ &= b f(b) - a f(a) - \int_a^b f(x)\,dx\end{aligned}$$ □

Theorem 23.2

Let $f_n : [a, b] \to \mathbb{R}$ be Riemann integrable s.t. $f_n \xrightarrow[n\to\infty]{u} f$ on $[a, b]$. Then f is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx$$

Proof. For $n \ge 1$ let $d_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$. As $f_n \xrightarrow[n\to\infty]{u} f$ on $[a, b]$ we have $d_n \xrightarrow[n\to\infty]{} 0$. In particular, $f_n(x) - d_n \le f(x) \le f_n(x) + d_n$ for all $x \in [a, b]$ (and thus f is bounded). For any partition P of $[a, b]$, we have

$$\begin{cases} U(f_n; P) - d_n(b - a) \le U(f; P) \le U(f_n; P) + d_n(b - a) \\ L(f_n; P) - d_n(b - a) \le L(f; P) \le L(f_n; P) + d_n(b - a) \end{cases}$$

So

$$U(f;P) - L(f;P) \le U(f_n;P) - L(f_n;P) + 2d_n(b-a)$$

Fix $\varepsilon > 0$. As $d_n \underset{n\to\infty}{\longrightarrow} 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t.

$$d_n < \frac{\varepsilon}{4(b-a)} \qquad \forall n \ge n_\varepsilon$$

Then for each $n \ge n_\varepsilon$ (fixed) there exists a partition $P = P(\varepsilon, n)$ of $[a,b]$ s.t.

$$U(f_n;P) - L(f_n;P) < \frac{\varepsilon}{2}$$

For $n \ge n_\varepsilon$ and $P = P(\varepsilon, n)$ as above we get

$$U(f;P) - L(f;P) < \varepsilon$$

As $\varepsilon > 0$ is arbitrary, this shows that f is Riemann integrable (since it's Darboux integrable and bounded). Moreover,

$$\begin{aligned}\int_a^b f(x)\,dx \le U(f;P) &\le U(f_n;P) + d_n(b-a)\\ &< L(f_n;P) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}\\ &\le \int_a^b f_n(x)\,dx + \frac{3\varepsilon}{4}\end{aligned}$$

Similarly,

$$\begin{aligned}\int_a^b f(x)\,dx \ge L(f;P) &\ge L(f_n;P) - d_n(b-a)\\ &> U(f_n;P) - \frac{\varepsilon}{2} - \frac{\varepsilon}{4}\\ &\ge \int_a^b f_n(x)\,dx - \frac{3\varepsilon}{4}\end{aligned}$$

Thus,

$$\implies \left|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx\right| < \frac{3\varepsilon}{4} \qquad \forall n \ge n_\varepsilon$$

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

□

§23.2 Lebesgue Criterion

Definition 23.3 (Zero Outer Measure) — A set $A \subseteq \mathbb{R}$ is said to have zero outer measure if for every $\varepsilon > 0$ there exists a countable collection of open intervals $\{(a_n, b_n)\}_{n\ge1}$ s.t.

$$\begin{cases} A \subseteq \bigcup_{n\ge1}(a_n,b_n)\\ \sum_{n\ge1}(b_n - a_n) < \varepsilon\end{cases}$$

Remark 23.4.
1. If $A \subseteq \mathbb{R}$ has zero outer measure and $B \subseteq A$, then B has zero outer measure.
2. If $\{A_n\}_{n\geq 1}$ is a sequence of zero outer measure sets, then $\bigcup_{n\geq 1} A_n$ has zero outer measure.
3. If A is a set that is at most countable, then A has zero outer measure.

Proof. 2. Fix $\varepsilon > 0$. For each $n \geq 1$, let $\left\{\left(a_m^{(n)}, b_m^{(n)}\right)\right\}_{m\geq 1}$ be open intervals s.t.

$$\begin{cases} A_n \subseteq \bigcup_{m\geq 1}\left(a_m^{(n)}, b_m^{(n)}\right) \\ \sum_{n\geq 1}\left(b_m^{(n)} - a_m^{(n)}\right) < \frac{\varepsilon}{2^n} \end{cases}$$

Then $\left\{\left(a_m^{(n)}, b_m^{(n)}\right)\right\}_{m,n\geq 1}$ is a countable collection of open intervals s.t.

$$\begin{cases} \bigcup_{n\geq 1} A_n \subseteq \bigcup_{n,m\geq 1}\left(a_m^{(n)}, b_m^{(n)}\right) \\ \sum_{n\geq 1}\sum_{m\geq 1}\left(b_m^{(n)} - a_m^{(n)}\right) < \sum_{n\geq 1}\frac{\varepsilon}{2^n} = \varepsilon \end{cases}$$

□

Theorem 23.5 (Lebesgue Criterion)
Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if the set

$$\mathscr{D}_f = \{x \in [a,b] : f \text{ is not continuous at } x\}$$

has zero outer measure.

Proof. We have

$$\mathscr{D}_f = \{x \in [a,b] : \omega(f,x) = 0\}$$

where

$$\begin{aligned} \omega(f,x) &= \inf_{\delta>0} \omega\left(f, B_\delta(x)\right) \\ &= \inf_{\delta>0}\left[\sup_{y\in B_\delta(x)} f(y) - \inf_{y\in B_\delta(x)} f(y)\right] \\ &= \inf_{\delta>0}\left[M\left(f; B_\delta(x)\right) - m\left(f; B_\delta(x)\right)\right] \end{aligned}$$

Then

$$\begin{aligned} \mathscr{D}_f &= \{x \in [a,b] : \omega(f,x) > 0\} \\ &= \bigcup_{n\geq 1} \underbrace{\left\{x \in [a,b] : \omega(f,x) \geq \frac{1}{n}\right\}}_{:=F_n} \end{aligned}$$

Key Observation: If $P = \{a = t_0 < \ldots < t_n = b\}$ then

$$\begin{aligned} U(f;P) - L(f;P) &= \sum_{k=1}^{n} \left[M\left(f;[t_{k-1},t_k]\right) - m\left(f;[t_{k-1},t_k]\right)\right]\left(t_k - t_{k-1}\right) \\ &= \sum_{k=1}^{n} \omega\left(f;[t_{k-1},t_k]\right)\left(t_k - t_{k-1}\right) \end{aligned}$$

We will continue with this proof in the next lecture. □

§24 Lec 24: May 21, 2021

§24.1 Lebesgue Criterion (Cont'd)

Proof. (Cont'd) " $\implies$ " Assume that f is Riemann integrable. We denote

$$\begin{aligned}\mathscr{D}_f &= \{x \in [a,b] : \omega(f,x) > 0\} \\ &= \bigcup_{n\geq 1} \left\{x \in [a,b] : \omega(f,x) \geq \frac{1}{n}\right\}\end{aligned}$$

For $n \geq 1$, let $F_n = \left\{x \in [a,b] : \omega(f,x) \geq \frac{1}{n}\right\}$. To show that $\mathscr{D}_f$ has zero outer measure, it suffices to prove that F_n has zero outer measure for all $n \geq 1$.

Fix $N \geq 1$ and $\varepsilon > 0$. As f is Riemann integrable, there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ s.t.

$$U(f;P) - L(f;P) < \frac{\varepsilon}{N}$$

Let $I = \{1 \leq k \leq n : F_N \cap (t_{k-1}, t_k) \neq \emptyset\}$. Then

$$F_N \subseteq \bigcup_{k\in I} (t_{k-1}, t_k) \cup P$$

As P is finite, it has zero outer measure. Thus, it suffices to show that

$$\sum_{k\in I} (t_k - t_{k-1}) < \varepsilon$$

Then,

$$\begin{aligned}\frac{\varepsilon}{N} > U(f;P) - L(f;P) &= \sum_{k=1}^{n} \left[M\left(f;[t_{k-1},t_k]\right) - m\left(f;[t_{k-1},t_k]\right)\right](t_k - t_{k-1}) \\ &\geq \sum_{k\in I} \omega\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1}) \\ &\geq \frac{1}{N}\sum_{k\in I}(t_k - t_{k-1})\end{aligned}$$

which implies

$$\sum_{k\in I}(t_k - t_{k-1}) < \varepsilon$$

" $\impliedby$ " Assume that $\mathscr{D}_f$ has zero outer measure.

$$f \text{ bounded } \implies \exists M > 0 \text{ s.t. } |f(x)| \leq M \qquad \forall x \in [a,b]$$

Fix $\varepsilon > 0$ and let $\alpha > 0$ to be chosen later. Consider

$$\left.\begin{aligned}&F_\alpha = \{x \in [a,b] : \omega(f,x) \geq \alpha\} \subseteq \mathscr{D}_f \\ &\mathscr{D}_f \text{ has zero outer measure}\end{aligned}\right\} \implies F_\alpha \text{ has zero outer measure}$$

$$\implies \exists \{(a_n,b_n)\}_{n\geq 1} \text{ s.t. } \begin{cases} F_\alpha \subseteq \bigcup_{n\geq 1}(a_n,b_n) \\ \sum_{n\geq 1}(b_n - a_n) < \varepsilon \end{cases}$$

Let $A = [a,b] \setminus F_\alpha$. For any $x \in A$, $\omega(f,x) < \alpha \implies \exists (c_x, d_x)$ neighborhood of x s.t.

$$\omega(f;[c_x,d_x]) < \alpha$$

So

$$\left.\begin{array}{l} [a,b] = F_\alpha \cup A \subseteq \bigcup_{n\geq 1}(a_n,b_n) \cup \bigcup_{x\in A}(c_x,d_x) \\ [a,b] \text{ is compact} \end{array}\right\}$$

which implies there exists $n_0 \in \mathbb{N}$ and $J \subseteq A$ finite s.t.

$$[a,b] \subseteq \bigcup_{k=1}^{n_0}(a_k,b_k) \cup \bigcup_{x\in J}(c_x,d_x)$$

Let P be a partition of $[a,b]$ formed by the points

$$\left(\{a,b\} \cup \bigcup_{k=1}^{n_0} \{a_x,b_x\} \cup \bigcup_{x\in J} \{c_x,d_x\}\right) \cap [a,b]$$

Say $P = \{a = t_0 < \ldots < t_n = b\}$. For any $1 \le l \le n$, we have

$$[t_{l-1},t_l] \subseteq [a_k,b_k] \text{ for some } 1 \le k \le n_0$$

or

$$[t_{l-1},t_l] \subseteq [c_x,d_x] \text{ for some } x \in J$$

Let

$$\begin{aligned} I_1 &= \{1 \le l \le n : [t_{l-1},t_l] \subseteq [a_k,b_k] \text{ for some } 1 \le k \le n_0\} \\ I_2 &= \{1,\ldots,n\} \setminus I_1 \end{aligned}$$

Note that

$$\sum_{l\in I_1}(t_l - t_{l-1}) \le \sum_{k=1}^{n_0}(b_k - a_k) < \varepsilon$$
$$l \in I_2,\ \ \omega(f;[t_{l-1},t_l]) \le \omega\left(f;[c_x,d_x]\right) < \alpha$$

Then,

$$\begin{aligned} U(f;P) - L(f;P) &= \sum_{l=1}^{n}\left[M\left(f;[t_{l-1},t_l]\right) - m\left(f;[t_{l-1},t_l]\right)\right](t_l - t_{l-1}) \\ &= \sum_{l\in I_1}\left[M\left(f;[t_{l-1},t_l]\right) - m\left(f;[t_{l-1},t_l]\right)\right](t_l - t_{l-1}) \\ &+ \sum_{l\in I_2}\omega\left(f;[t_{l-1},t_l]\right)(t_l - t_{l-1}) \end{aligned}$$

Notice that

$$\sum_{l\in I_1}\left[M\left(f;[t_{l-1},t_l]\right) - m\left(f;[t_{l-1},t_l]\right)\right](t_l - t_{l-1}) \le 2M\sum_{l\in I_1}(t_l - t_{l-1}) < 2M\varepsilon$$

So

$$\begin{aligned}\sum_{l\in I_2}\omega\left(f;[t_{l-1},t_l]\right)(t_l-t_{l-1}) &< \alpha\sum_{l\in I_2}(t_l-t_{l-1})\\ &\le \alpha\sum_{l=1}^{n}(t_l-t_{l-1})\\ &= \alpha(b-a)\end{aligned}$$

Choose $\alpha < \frac{\varepsilon}{b-a}$ to get

$$U(f;P)-L(f;P) < 2M\varepsilon+\varepsilon$$

As ε is arbitrary, this shows that f is Darboux integrable, and thus Riemann integrable. □

§24.2 Improper Riemann Integrals

Definition 24.1 (Locally Riemann Integrable) — Let $-\infty < a < b \le \infty$. We say that $f:[a,b)\to\mathbb{R}$ is locally Riemann integrable if f is integrable on $[a,c]$ for any $c\in(a,b)$.

Definition 24.2 (Improper Riemann Integral) — Let $-\infty < a < b \le \infty$ and $f:[a,b)\to\mathbb{R}$ is locally Riemann integrable. In addition,

$$\lim_{c\to b}\int_a^c f(x)\,dx \text{ exists in } \mathbb{R}$$

We denote it $\int_a^b f(x)dx$ and we call it the improper Riemann integral of f. In this case we say that the improper Riemann integral of f converges. If

$$\lim_{c\to b}\int_a^c f(x)\,dx = \pm\infty$$

then we write $\int_a^b f(x)dx=\pm\infty$ and we say that the improper Riemann integral of f diverges to $\pm\infty$.

Remark 24.3. One can make a similar definition if $-\infty\le a<b<\infty$ and $f:(a,b]\to\mathbb{R}$ or if $-\infty\le a<b\le\infty$ and $f:(a,b)\to\mathbb{R}$.

Theorem 24.4

Let $-\infty < a < b < \infty$ and let $f:[a,b)\to\mathbb{R}$ be locally Riemann integrable and bounded. Then the improper Riemann integral $\int_a^b f(x)dx$ converges. Moreover, any extension $\tilde{f}:[a,b]\to\mathbb{R}$ of f to $[a,b]$ is Riemann integrable and

$$\int_a^b \tilde{f}(x)\,dx = \int_a^b f(x)\,dx$$

Proof. Let $\tilde{f} : [a, b] \to \mathbb{R}$ be an extension of f to $[a, b]$. As f is bounded, $\exists M > 0$ s.t.

$$\left|\tilde{f}(x)\right| \leq M \qquad \forall x \in [a, b]$$

For $c \in (a, b)$,

$$U_a^b(\tilde{f}) = U_a^c(\tilde{f}) + U_c^b(\tilde{f}) = \int_a^c f(x)\,dx + U_c^b(\tilde{f}) \tag{*}$$

$$L_a^b(\tilde{f}) = L_a^c(\tilde{f}) + L_c^b(\tilde{f}) = \int_a^c f(x)\,dx + L_c^b(\tilde{f})$$

$$\left.\begin{aligned} &\implies U_a^b(\tilde{f}) - L_a^b(\tilde{f}) = U_c^b(\tilde{f}) - L_c^b(\tilde{f}) \\ &U_c^b(\tilde{f}) \leq M(b-c) \\ &\left|L_c^b(\tilde{f})\right| \leq M(b-c) \end{aligned}\right\} \implies U_a^b(\tilde{f}) - L_a^b(\tilde{f}) \leq \underbrace{2M(b-c)}_{\xrightarrow{c\to b} 0}$$

This shows that $\tilde{f}$ is Riemann integrable. Moreover, by (*),

$$\int_a^b \tilde{f}(x)\,dx = \lim_{c\to b} \int_a^c f(x)\,dx$$

Thus, the improper Riemann integral of f converges and

$$\int_a^b f(x)\,dx = \int_a^b \tilde{f}(x)\,dx$$

□

§25 Lec 25: May 24, 2021

§25.1 Improper Riemann Integrals (Cont'd)

Proposition 25.1

Let $-\infty < a < b \leq \infty$ and let $f, g : [a, b) \to \mathbb{R}$ be locally Riemann integrable s.t. the improper Riemann integrals of f and g converge. Then

1. For any $\alpha \in \mathbb{R}$, the improper Riemann integral of αf converges and
$$\int_a^b (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx$$

2. The improper Riemann integral of $f + g$ converges and
$$\int_a^b (f+g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

Proof. 1. Consider:
$$\begin{aligned}\mathbb{R} \ni \alpha \int_a^b f(x)\, dx = \alpha \lim_{c \to b} \int_a^c f(x)\, dx &= \lim_{c \to b} \alpha \int_a^c f(x)\, dx \\ (f \text{ is locally Riemann integrable}) &= \lim_{c \to b} \int_a^c (\alpha f)(x)\, dx\end{aligned}$$

So the improper Riemann integral of αf converges and
$$\int_a^b (\alpha f)(x)\, dx = \lim_{c \to b} \int_a^c (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx$$

2. Consider:
$$\begin{aligned}\mathbb{R} \ni \int_a^b f(x)\, dx + \int_a^b g(x)\, dx &= \lim_{c \to b} \int_a^c f(x)\, dx + \lim_{c \to b} \int_a^c g(x)\, dx \\ &= \lim_{c \to b} \left[\int_a^c f(x)\, dx + \int_a^c g(x)\, dx \right] \\ &= \lim_{c \to b} \int_a^c \left[f(x) + g(x) \right]\, dx\end{aligned}$$

So the improper Riemann integral of $f + g$ converges and
$$\int_a^b (f+g)(x)\, dx = \lim_{c \to b} \int_a^c (f+g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$ □

Remark 25.2. If $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable functions, then

- $|f|$ is Riemann integrable.
- $f \cdot g$ is Riemann integrable.

However, if $f, g : [a, b)$ are locally integrable functions s.t. the improper Riemann integrals of f and g converge, then

- the improper Riemann integral of $|f|$ need not converge.
- the improper Riemann integral of $f \cdot g$ need not converge.

Example 25.3

Let $f, g : (0, 1] \to \mathbb{R}$, $f(x) = g(x) = \frac{1}{\sqrt{x}}$. The improper Riemann integral of f converges

$$\int_c^1 f(x)\, dx = \int_c^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\Big|_{x=c}^{x=1} = 2 - 2\sqrt{c} \underset{c\to 0}{\longrightarrow} 2$$

The improper Riemann integral of $f \cdot g$ does not converge

$$\int_c^1 f(x)g(x)\, dx = \int_c^1 \frac{1}{x}\, dx = \ln x\Big|_{x=c}^{x=1} = -\ln c \underset{c\to 0}{\longrightarrow} \infty$$

More generally, we can take $f, g : (0, 1] \to \mathbb{R}$

$$f(x) = \frac{1}{x^\alpha}, \quad g(x) = \frac{1}{x^\beta} \quad \text{with} \quad 0 < \alpha, \beta < 1 \quad \text{and} \quad \alpha + \beta \geq 1$$

Lemma 25.4 (Cauchy Criterion)

Let $-\infty < a < b \leq \infty$. Let $f : [a, b) \to \mathbb{R}$ be locally integrable. Then the improper Riemann integral of f converges if and only if

$$\forall \varepsilon > 0 \quad \exists c_\varepsilon \in (a, b) \text{ s.t. } \left|\int_{c_1}^{c_2} f(x)\, dx\right| < \varepsilon \quad \forall c_\varepsilon < c_1 < c_2 < b$$

Proof. " $\implies$ " Assume that the improper Riemann integral of f converges. Let

$$\alpha = \int_a^b f(x)\, dx \in \mathbb{R}$$

We have

$$\alpha = \lim_{c\to b} \int_a^c f(x)\, dx$$

Then $\forall \varepsilon > 0\ \exists c_\varepsilon \in (a, b)$ s.t.

$$\left|\alpha - \int_a^c f(x)\, dx\right| < \frac{\varepsilon}{2} \qquad \forall c_\varepsilon < c < b$$

For $c_\varepsilon < c_1 < c_2 < b$ we have

$$\begin{aligned}\left|\int_{c_1}^{c_2} f(x)\,dx\right| &= \left|\int_a^{c_2} f(x)\,dx - \int_a^{c_1} f(x)\,dx\right| \\ &\le \left|\int_a^{c_2} f(x)\,dx - \alpha\right| + \left|\alpha - \int_a^{c_1} f(x)\,dx\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

" $\impliedby$ " Fix $\varepsilon > 0$ and let $c_\varepsilon \in (a,b)$ s.t.

$$\left|\int_{c_1}^{c_2} f(x)\,dx\right| < \varepsilon \qquad \forall c_\varepsilon < c_1 < c_2 < b$$

Let $\{c_n\}_{n\ge1} \subseteq (a,b)$ s.t. $c_n \underset{n\to\infty}{\longrightarrow} b$. Then $\exists n_\varepsilon \in \mathbb{N}$ s.t. $c_\varepsilon < c_n < b$ for all $n \ge n_\varepsilon$. In particular,

$$\begin{aligned}\left|\int_a^{c_m} f(x)\,dx - \int_a^{c_n} f(x)\,dx\right| &= \left|\int_{c_n}^{c_m} f(x)\,dx\right| < \varepsilon \qquad n,m \ge n_\varepsilon \\ \implies \left\{\int_a^{c_n} f(x)\,dx\right\}_{n\ge1} &\subseteq \mathbb{R} \text{ is Cauchy and so convergent}\end{aligned}$$

Let $\alpha = \lim_{n\to\infty} \int_a^{c_n} f(x)\,dx$. To prove that the Riemann integral of f converges, we need to show that α does not depend on $\{c_n\}_{n\ge1}$. Let $\{d_n\}_{n\ge1} \subseteq (a,b)$ s.t. $\lim_{n\to\infty} d_n = b$. Consider

$$x_n = \begin{cases} c_k & \text{if } n = 2k \\ d_k & \text{if } n = 2k-1 \end{cases} \qquad \text{for } k \ge 1$$

Then $x_n \underset{n\to\infty}{\longrightarrow} b$. From the same argument used for the sequence $\{c_n\}_{n\ge1}$, we conclude that $\left\{\int_a^{x_n} f(x)\,dx\right\}_{n\ge1}$ is Cauchy and so convergent. So

$$\begin{aligned}\lim_{n\to\infty}\int_a^{x_{2n}} f(x)\,dx &= \lim_{n\to\infty}\int_a^{x_{2n-1}} f(x)\,dx \\ \alpha = \lim_{n\to\infty}\int_a^{c_n} f(x)\,dx &= \lim_{n\to\infty}\int_a^{d_n} f(x)\,dx\end{aligned}$$ □

Theorem 25.5 (Abel Criterion)

Let $-\infty < a < b \le \infty$ and let $f, g : [a,b) \to \mathbb{R}$ be locally integrable. Assume that g is decreasing and $\lim_{x\to b} g(x) = 0$. Assume also that there exists $M > 0$ s.t.

$$\left|\int_a^c f(x)\,dx\right| \le M \qquad \forall a < c < b$$

Then the improper Riemann integral of $f \cdot g$ converges.

Remark 25.6. Compare this with the series version

$$\left.\begin{array}{l}\{a_n\}_{n\geq 1} \text{ is decreasing with } \lim_{n\to\infty} a_n = 0 \\ \exists M > 0 \text{ s.t. } \left|\sum_{k=1}^n b_k\right| \leq M \quad \forall n \geq 1\end{array}\right\} \implies \sum_{n\geq 1} a_n b_n \text{ converges}$$

Proof. We'll use the Cauchy Criterion. Fix $\varepsilon > 0$.

$$\lim_{x\to b} g(x) = 0 \implies \exists c_\varepsilon \in (a,b) \text{ s.t. } |g(x)| < \varepsilon \quad \forall c_\varepsilon < x < b$$

Fix $c_\varepsilon < c_1 < c_2 < b$ and consider $\int_{c_1}^{c_2} f(x)g(x)dx$. Using exercise #6 in HW8, we can find $x_0 \in [c_1, c_2]$ s.t.

$$\begin{aligned}\int_{c_1}^{c_2} f(x)g(x)\,dx &= g(c_1)\int_{c_1}^{x_0} f(x)\,dx + g(c_2)\int_{x_0}^{c_2} f(x)\,dx \\ &= g(c_1)\left[\int_a^{x_0} f(x)\,dx - \int_a^{c_1} f(x)\,dx\right] \\ &+ g(c_2)\left[\int_a^{c_2} f(x)\,dx - \int_a^{x_0} f(x)\,dx\right]\end{aligned}$$

which implies

$$\begin{aligned}\left|\int_{c_1}^{c_2} f(x)g(x)\,dx\right| &\leq g(c_1)\left[\left|\int_a^{x_0} f(x)\,dx\right| + \left|\int_a^{c_1} f(x)\,dx\right|\right] \\ &+ g(c_2)\left[\left|\int_a^{c_2} f(x)\,dx\right| + \left|\int_a^{x_0} f(x)\,dx\right|\right] \\ &< 4M\varepsilon\end{aligned}$$

As $c_\varepsilon < c_1, c_2 < b$ are arbitrary and $\varepsilon > 0$ is arbitrary, we conclude that the improper Riemann integral of fg converges. $\square$

§26 Lec 26: May 26, 2021

§26.1 Improper Riemann Integrals (Cont'd)

Exercise 26.1. Show that the improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\,dx \quad \text{converges}$$

but the improper Riemann integral

$$\int_0^\infty \left|\frac{\sin x}{x}\right|\,dx \quad \text{does not converge}$$

Proof. To show that $\int_0^\infty \frac{\sin x}{x} dx$ converges, we have to prove that

$$\lim_{M\to\infty} \int_0^M \frac{\sin x}{x}\,dx \quad \text{exists in } \mathbb{R}$$

Note that

$$x \mapsto \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is continuous on on $[0,\infty)$ and so it is Riemann integrable on $[0,M]$ for each $M > 0$. For $M > 1$, we write

$$\int_0^M \frac{\sin x}{x}\,dx = \underbrace{\int_0^1 \frac{\sin x}{x}\,dx}_{\in\mathbb{R}} + \int_1^M \frac{\sin x}{x}\,dx$$

Note that $f, g : [1,\infty) \to \mathbb{R}$, $f(x) = \sin x$ and $g(x) = \frac{1}{x}$ are continuous and so Riemann integrable on $[1, M]$ $\forall M > 1$. Also,

- g is decreasing and $\lim_{x\to\infty} g(x) = 0$
- In addition,

$$\left|\int_1^M \sin x\,dx\right| = |\cos 1 - \cos M| \le 2 \qquad \forall M > 1$$

So by the Abel Criterion, the improper Riemann integral $\int_1^\infty \frac{\sin(x)}{x} dx$ converges. Moreover,

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{M\to\infty}\int_0^M \frac{\sin x}{x}\,dx &= \int_0^1 \frac{\sin x}{x}\,dx + \lim_{M\to\infty}\int_1^M \frac{\sin x}{x}\,dx \\ &= \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx\end{aligned}$$

Let's show that the improper Riemann integral $\int_0^\infty \frac{|\sin x|}{x} dx$ diverges to ∞. We'll use that

$$|\sin x| \ge \frac{1}{2} \quad \text{on} \quad \left[k\pi + \frac{\pi}{6}, k\pi + \frac{5\pi}{6}\right]$$

for all $k \geq 0$. So

$$\begin{aligned}
\int_0^\infty \frac{|\sin x|}{x}\, dx &\geq \sum_{k\geq 0} \int_{k\pi+\frac{\pi}{6}}^{k\pi+\frac{5\pi}{6}} \frac{|\sin x|}{x}\, dx \\
&\geq \sum_{k\geq 0} \frac{1}{2}\cdot \frac{1}{k\pi+\frac{5\pi}{6}} \cdot \left[\left(k\pi+\frac{5\pi}{6}\right)-\left(k\pi+\frac{\pi}{6}\right)\right] \\
&\geq \sum_{k\geq 0} \frac{1}{2}\cdot\frac{1}{(k+1)\pi}\cdot\frac{2\pi}{3} = \frac{1}{3}\sum_{k\geq 0}\frac{1}{k+1} = \infty
\end{aligned}$$

□

Proposition 26.1

Let $-\infty < a < b \leq \infty$ and let $f : [a, b) \to \mathbb{R}$ be locally Riemann integrable s.t. the improper Riemann integral of $|f|$ converges. Then the improper Riemann integral of f converges and

$$\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx$$

Proof. As the improper Riemann integral of $|f|$ converges, by the Cauchy Criterion we have

$$\forall \varepsilon > 0 \quad \exists c_\varepsilon \in (a, b) \text{ s.t. } \int_{c_1}^{c_2} |f(x)|\, dx < \varepsilon \quad \forall c_\varepsilon < c_1 < c_2 < b$$

As f is locally integrable, f is integrable on $[c_1, c_2]$ and

$$\left|\int_{c_1}^{c_2} f(x)\, dx\right| \leq \int_{c_1}^{c_2} |f(x)|\, dx < \varepsilon \qquad \forall c_\varepsilon < c_1 < c_2 < b$$

By the Cauchy Criterion, the improper Riemann integral of f converges. Moreover,

$$\begin{aligned}
\left|\int_a^b f(x)\, dx\right| = \left|\lim_{c\to b}\int_a^c f(x)\, dx\right| &= \lim_{c\to b}\left|\int_a^c f(x)\, dx\right| \\
(f \text{ is locally integrable}) &\leq \lim_{c\to b}\int_a^c |f(x)|\, dx \\
&= \int_a^b |f(x)|\, dx
\end{aligned}$$

□

Definition 26.2 (Absolute Convergence – Integral) — Let $-\infty < a < b \leq \infty$ and $f : [a, b) \to \mathbb{R}$ be locally integrable. We say that the improper Riemann integral of f converges absolutely if the improper Riemann integral of $|f|$ converges.

Remark 26.3.

1. If the improper Riemann integral of f converges absolutely, then it converges.
2. The improper Riemann integral of f converges absolutely if and only if

$$\lim_{c\to b}\int_a^c |f(x)|\,dx \in \mathbb{R} \iff \exists M>0 \text{ s.t. } \int_a^c |f(x)|\,dx \le M \quad \forall c\in[a,b)$$

3. If $f,g:[a,b)\to\mathbb{R}$ are locally integrable s.t. $|f(x)|\le|g(x)|\ \forall x\in[a,b)$ and the improper Riemann integral of g converges absolutely, then the improper Riemann integral of f converges absolutely.
4. If $f,g:[a,b)\to\mathbb{R}$ are locally integrable and their improper Riemann integrals converge absolutely, then the improper Riemann integral of $f+g$ converges absolutely.
5. If $f,g:[a,b)\to\mathbb{R}$ are locally integrable s.t. f is bounded and the improper Riemann integral of g converges absolutely, then the improper Riemann integral of $f\cdot g$ converges absolutely.

§26.2 Continuous 1-Periodic Functions

Definition 26.4 (Convolution) — Let $f,g:\mathbb{R}\to\mathbb{C}$ be continuous functions with period 1, that is,

$$f(x+1)=f(x) \quad\text{and}\quad g(x+1)=g(x) \qquad x\in\mathbb{R}$$

Their convolution $f*g:\mathbb{R}\to\mathbb{C}$ is defined via

$$(f*g)(x)=\int_0^1 f(y)g(x-y)\,dy$$

<u>**Claim 1**</u>:

$$(f*g)(x)=\int_a^{a+1} f(y)g(x-y)\,dy \qquad \forall a\in\mathbb{R},\ \forall x\in\mathbb{R}$$

This is obviously true if $a=k\in\mathbb{Z}$. For $y=k+z$,

$$\begin{aligned}\int_k^{k+1} f(y)g(x-y)\,dy &= \int_0^1 f(k+z)g(x-z-k)\,dz\\ (f\&g\text{ periodic}) &= \int_0^1 f(z)g(x-z)\,dz=(f*g)(x)\end{aligned}$$

Next, decomposing $a = \underbrace{[a]}_{\in\mathbb{Z}} + \underbrace{\{a\}}_{\in[0,1)}$ we see that it suffices to prove the claim for $a \in (0,1)$.

$$\begin{aligned}\int_a^{a+1} f(y)g(x-y)\,dy &= \int_a^1 f(y)g(x-y)\,dy + \int_1^{1+a} f(y)g(x-y)\,dy \\ &= \int_a^1 f(y)g(x-y)\,dy + \int_0^a f(z+1)g(x-z-1)\,dz \\ &= \int_a^1 f(y)g(x-y)\,dy + \int_0^a f(z)g(x-z)\,dz \\ &= \int_0^1 f(y)g(x-y)\,dy = (f*g)(x)\end{aligned}$$

Claim 2: $f*g$ is 1-periodic.

$$(f*g)(x+1) = \int_0^1 f(y)g(x+1-y)\,dy = \int_0^1 f(y)g(x-y)\,dy = (f*g)(x)$$

Claim 3: $f*g$ is continuous

$$\begin{aligned}|(f*g)(x_1)-(f*g)(x_2)| &= \left|\int_0^1 f(y)\left[g(x_1-y)-g(x_2-y)\right]\,dy\right| \\ &\le \int_0^1 |f(y)|\,|g(x_1-y)-g(x_2-y)|\,dy\end{aligned}$$

g continuous on $[0,2]$ compact $\implies$ g is uniformly continuous on $[0,2]$, and since g is 1-periodic, we conclude that g is uniformly continuous on $\mathbb{R}$. So $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$|g(x)-g(y)| < \varepsilon \qquad \forall\, |x-y| < \delta$$

f is continuous on $[0,1]$ compact $\implies M > 0$ s.t.

$$|f(x)| \le M \qquad \forall x \in [0,1]$$

So

$$|(f*g)(x_1)-(f*g)(x_2)| \le \int_0^1 M\cdot\varepsilon\,dy = M\cdot\varepsilon \quad \forall\, |x_1-x_2| < \delta$$

Claim 4: $f*g = g*f$. For $z = x-y$,

$$\begin{aligned}(g*f)(x) = \int_0^1 g(y)f(x-y)\,dy &= -\int_x^{x-1} g(x-z)f(z)\,dz \\ &= \int_{x-1}^x f(y)g(x-y)\,dy \\ &= \int_0^1 f(y)g(x-y)\,dy \\ &= (f*g)(x)\end{aligned}$$

Claim 5: For all $\alpha \in \mathbb{C}$,

$$(\alpha f)*g = f*(\alpha g) = \alpha(f*g)$$

Claim 6: If f, g, h are continuous, 1-periodic functions,

$$\begin{cases} f*(g+h) = f*g+f*h \\ (f*g)*h = f*(g*h)\end{cases}$$

Left as exercise!

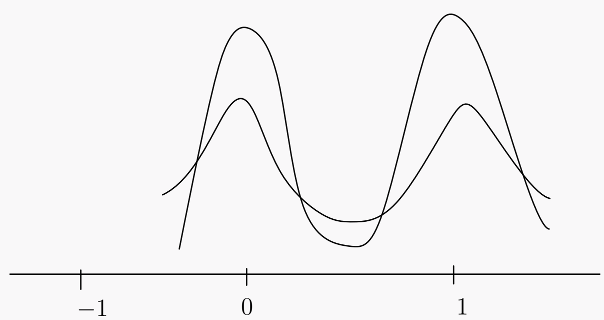
§27 Lec 27: May 28, 2021

§27.1 Continuous 1-Periodic Functions (Cont'd)

Definition 27.1 (Approximation to the Identity) — A sequence of continuous, 1-periodic functions $K_n : \mathbb{R} \to \mathbb{C}$ is called an approximation to the identity if it satisfies the following:

1. $\int_0^1 K_n(x)\,dx = 1 \;\forall n \geq 1$
2. $\exists M > 0$ s.t. $\int_0^1 |K_n(x)|\,dx \leq M \;\forall n \geq 1$
3. $\forall \delta > 0$, $\int_\delta^{1-\delta} |K_n(x)|\,dx \xrightarrow[n\to\infty]{} 0$.

Remark 27.2. While 1) says that K_n assigns mass 1 to each period, 3) says that this mass is concentrating at the integers as $n \to \infty$.



Theorem 27.3

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous, 1-periodic function and let $\{K_n\}_{n\geq 1}$ be an approximation to the identity. Then

$$K_n * f \xrightarrow[n\to\infty]{u} f \text{ on } \mathbb{R}$$

Proof. Fix $x \in \mathbb{R}$.

$$\begin{aligned}
(K_n * f)(x) - f(x) &= \int_0^1 K_n(y) f(x-y)\,dy - f(x)\int_0^1 K_n(y)\,dy \\
&= \int_0^1 K_n(y)\left[f(x-y) - f(x)\right]\,dy \\
\implies |(K_n * f)(x) - f(x)| &\leq \int_0^1 |K_n(y)|\,|f(x-y) - f(x)|\,dy
\end{aligned}$$

f is continuous and 1-periodic $\implies$ f is uniformly continuous.

Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ for all $|x - y| < \delta$

$$\begin{aligned}
\int_0^\delta |K_n(y)| \underbrace{|f(x-y) - f(x)|}_{<\varepsilon} \, dy &< \varepsilon \int_0^\delta |K_n(y)| \, dy \\
&\leq \varepsilon \int_0^1 |K_n(y)| \, dy \leq \varepsilon M \\
\int_{1-\delta}^1 |K_n(y)| \, |f(x-y) - f(x)| \, dy &\overset{y=1+z}{=} \int_{-\delta}^0 |K_n(1+z)| \, |f(x-z-1) - f(x)| \, dz \\
&= \int_{-\delta}^0 |K_n(z)| \underbrace{|f(x-z) - f(x)|}_{<\varepsilon} \, dz \\
&< \varepsilon \int_{-1}^0 |K_n(z)| \, dz \leq \varepsilon M \\
\int_\delta^{1-\delta} |K_n(y)| \, |f(x-y) - f(x)| \, dy &\leq \int_\delta^{1-\delta} |K_n(y)| \left[|f(x-y)| + |f(x)| \right] dy \\
&\leq 2 \sup_{x \in [0,1]} |f(x)| \int_\delta^{1-\delta} |K_n(y)| \, dy
\end{aligned}$$

As $\int_\delta^{1-\delta} |K_n(y)| \, dy \underset{n\to\infty}{\longrightarrow} 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t.

$$\int_\delta^{1-\delta} |K_n(y)| \, dy < \frac{\varepsilon}{2\|f\|_\infty + 1}$$

So collecting our estimates, we get

$$|(K_n * f)(x) - f(x)| \leq 2\varepsilon M + \varepsilon \qquad \forall x \in \mathbb{R}, \ \forall n \geq n_\varepsilon$$

As $\varepsilon > 0$ is arbitrary, we get $K_n * f \xrightarrow[n\to\infty]{u} f$. $\square$

§27.2 Fourier Series

Definition 27.4 (Orthonormal Family) — For $n \in \mathbb{Z}$, let $e_n(x) = e^{2\pi i n x} = \cos(2\pi n x) + i \sin(2\pi n x)$. Note $e_n : \mathbb{R} \to \mathbb{C}$ is continuous, 1-periodic.

$$\int_0^1 e_n(x) \, dx = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

So

$$\int_0^1 e_n(x) \overline{e_m(x)} \, dx = \int_0^1 e_{n-m}(x) \, dx = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

$\implies \{e_n\}_{n\geq 1}$ form an orthonormal family.

Definition 27.5 (Trigonometric Polynomial) — A trigonometric polynomial takes the form

$$\sum_{|n|\le N} c_n e_n(x)$$

where $c_n \in \mathbb{C}$ for all $|n| \le N$.

Definition 27.6 (Fourier Series) — Given a continuous, 1-periodic function $f : \mathbb{R} \to \mathbb{C}$, we define its n^{th} Fourier coefficient via

$$\hat{f}(n) = \int_0^1 f(x)\overline{e_n(x)}\, dx = \int_0^1 f(x) e^{-2\pi i n x}\, dx$$

The Fourier series of f is given by $\sum_{n\in\mathbb{Z}} \hat{f}(n) e_n(x)$.

Question 27.1. Can we recover f from its Fourier series?

If $f \in C^2$, then

$$\sum_{n\in\mathbb{Z}} \hat{f}(n) e_n(x) \xrightarrow[n\to\infty]{u} f(x)$$

In 1966, Carleson proved that the Fourier series of an integrable function converges pointwise to f outside a set of measure zero.
For $N \ge 0$, let

$$\begin{aligned} S_N(f)(x) = \sum_{|n|\le N} \hat{f}(n) e_n(x) &= \sum_{|n|\le N} \int_0^1 f(y)\overline{e_n(y)}\, dy \cdot e_n(x) \\ &= \sum_{|n|\le N} \int_0^1 f(y) e_n(x-y)\, dy \\ &= \int_0^1 f(y) \left(\sum_{|n|\le N} e_n \right)(x-y)\, dy \\ &= \left[f * \left(\sum_{|n|\le N} e_n \right) \right](x) \end{aligned}$$

For $N \ge 0$, let $D_N = \sum_{|n|\le N} e_n$ denote the Dirichlet Kernel. Note that

$$\int_0^1 D_N(x)\, dx = \sum_{|n|\le N} \int_0^1 e_n(x)\, dx = 1 \qquad \forall N \ge 0$$

$\{D_N\}_{N\ge 0}$ do not form an approximation to the identity since

$$\int_0^1 |D_N(x)|\, dx \xrightarrow[N\to\infty]{} \infty$$

We have

$$\begin{aligned} D_N &= \sum_{|n|\le N} e_n \\ (e_1-1)D_N &= \sum_{n=-N+1}^{N+1} e_n - \sum_{n=-N}^{N} e_n = e_{N+1} - e_{-N} \\ \implies D_N &= \frac{e_{N+1}-e_{-N}}{e_1-1} \end{aligned} \tag{1}$$

In addition,

$$\begin{aligned} D_N(x) = \frac{e^{2\pi i(N+1)x} - e^{-2\pi i Nx}}{e^{2\pi ix}-1} &= \frac{e^{\pi ix}\left(e^{2\pi i\left(N+\frac{1}{2}\right)x} - e^{-2\pi i\left(N+\frac{1}{2}\right)x}\right)}{e^{\pi ix}\left(e^{\pi ix}-e^{-\pi ix}\right)} \\ &= \frac{\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)}{\sin(\pi x)} \end{aligned}$$

Also,

$$\begin{aligned} \int_0^1 |D_N(x)|\,dx &\ge \int_0^1 \frac{\left|\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)\right|}{\pi x}\,dx \\ &\underset{y=2\pi\left(N+\frac{1}{2}\right)x}{=} \int_0^{2\pi\left(N+\frac{1}{2}\right)} \frac{|\sin(y)|}{\pi\cdot\frac{y}{2\pi\left(N+\frac{1}{2}\right)}}\cdot\frac{dy}{2\pi\left(N+\frac{1}{2}\right)} \\ &= \frac{1}{\pi}\int_0^{2\pi\left(N+\frac{1}{2}\right)} \frac{|\sin(y)|}{y}\,dy \xrightarrow[N\to\infty]{} \infty \end{aligned}$$

The average of the Dirichlet kernels do form an approximation to the identity. For $N \ge 1$, let $F_N = \frac{D_0+\ldots+D_{N_1}}{N}$ denote the Fejer Kernels. Note that

$$\int_0^1 F_N(x)\,dx = \frac{1}{N}\sum_{k=0}^{N-1}\int_0^1 D_k(x)\,dx = 1 \qquad N \ge 1$$

We will show that $F_N \ge 0$ and so

- $\int_0^1 |F_N(x)|\,dx = \int_0^1 F_N(x)dx = 1\ \forall N \ge 1$
- $\forall \delta > 0$, $\int_\delta^{1-\delta} |F_N(x)|\,dx \xrightarrow[N\to\infty]{} 0$

Consequently, we obtain the following

Theorem 27.7

If $f : \mathbb{R} \to \mathbb{C}$ is a continuous, 1-periodic function, then

$$F_N * f \xrightarrow[N\to\infty]{u} f \text{ on } \mathbb{R}$$

if and only if

$$\sigma(f) = \frac{1}{N}\sum_{k=0}^{N-1} S_N(f) \xrightarrow[N\to\infty]{u} f \text{ on } \mathbb{R}$$

Corollary 27.8

If $f : \mathbb{R} \to \mathbb{C}$ is a continuous, 1-periodic function, with $\hat{f}(n) = 0 \ \forall n \in \mathbb{Z}$, then $f \equiv 0$.

Corollary 27.9

Every continuous, 1-periodic function can be approximated uniformly by trigonometric polynomials.

§28 Lec 28: Jun 2, 2021

§28.1 Fourier Series (Cont'd)

Recall that for $n \in \mathbb{Z}$ we define the character $e_n : \mathbb{R} \to \mathbb{C}$

$$e_n(x) = e^{2\pi inx}$$

For a continuous, 1-periodic function $f : \mathbb{R} \to \mathbb{C}$, we define its n^{th} Fourier coefficient via

$$\hat{f}(n) = \int_0^1 f(x)\overline{e_n(x)}\,dx = \int_0^1 f(x)e^{-2\pi inx}\,dx \qquad \forall n \in \mathbb{Z}$$

and the partial Fourier series

$$[S_N(f)](x) = \sum_{|n|\le N} \hat{f}(n)e_n(x) \qquad \forall N \ge 0$$

We observed $S_N(f) = f * D_N$ where D_N denotes the Dirichlet kernel

$$D_N = \sum_{|n|\le N} e_n \qquad \forall N \ge 0$$

Using

$$D_N = \frac{e_{N+1} - e_{-N}}{e_1 - 1} \tag{1}$$

We obtained the explicit formula

$$D_N(x) = \frac{\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)}{\sin(\pi x)}$$

and computed

$$\int_0^1 |D_N(x)|\,dx \xrightarrow[N\to\infty]{} \infty$$

In particular, $\{D_N\}_{N\ge 1}$ do not form an approximation to the identity. Instead, we define the Fejer Kernel

$$F_N = \frac{D_0 + \ldots + D_{N-1}}{N} \qquad \forall N \ge 1$$

So

$$\sigma(f) = f * F_N = \frac{1}{N}\sum_{n=0}^{N-1} f * D_n = \frac{1}{N}\sum_{n=0}^{N-1} S_n(f)$$

Claim 28.1. $\{F_N\}_{N\ge 1}$ form an approximation to the identity and thus $\sigma(f) \xrightarrow[n\to\infty]{u} f$ for any continuous, 1-periodic $f : \mathbb{R} \to \mathbb{C}$.

Proof. First, we have

$$\int_0^1 e_n(x)\,dx = \int_0^1 \cos(2\pi nx)\,dx + i\int_0^1 \sin(2\pi ni)\,dx = \begin{cases} 1, & n = 0 \\ 0, & n \ne 0 \end{cases}$$

we get

$$\int_0^1 D_N(x)\,dx = \sum_{|n|\le N}\int_0^1 e_n(x)\,dx = 1 \qquad \forall N \ge 0$$

and so

$$\int_0^1 F_N(x)\,dx = \frac{1}{N}\sum_{n=0}^{N-1}\int_0^1 D_n(x)\,dx = 1 \qquad \forall N \ge 1$$

Net, we compute an explicit formula for F_N

$$\begin{aligned} NF_N &= D_0 + \ldots + D_{N-1} \\ &\overset{(1)}{=} \frac{e_1 - e_0}{e_1 - 1} + \frac{e_2 - e_{-1}}{e_1 - 1} + \ldots + \frac{e_N - e_{-N+1}}{e_1 - 1} \\ &= \frac{(e_1 + e_2 + \ldots + e_N) - (e_0 + e_{-1} + \ldots + e_{-N+1})}{e_1 - 1} \\ &= \frac{(e_1 - 1)(e_1 + e_2 + \ldots + e_N) - (e_1 - 1)(e_0 + e_{-1} + \ldots + e_{-N+1})}{(e_1 - 1)^2} \end{aligned}$$

Notice that

$$\begin{aligned} (e_1 - 1)(e_1 + \ldots + e_N) &= e_2 + \ldots + e_{N+1} - e_1 - \ldots - e_N = e_{N+1} - e_1 \\ (e_1 - 1)(e_0 + \ldots + e_{-N+1}) &= e_1 + \ldots + e_{-N+2} - e_0 - \ldots - e_{-N+1} = e_1 - e_{-N+1} \end{aligned}$$

So

$$\begin{aligned} NF_N(x) &= \frac{e_{N+1}(x) + e_{-N+1}(x) - 2e_1(x)}{\left(e^{2\pi i x} - 1\right)^2} \\ &= \frac{e_1(x)\left(e^{2\pi i N x} + e^{-2\pi i N x} - 2\right)}{e_1(x)\left(e^{\pi i x} - e^{-\pi i x}\right)^2} \\ &= \frac{2\left(\cos(2\pi N x) - 1\right)}{\left[2i\sin(\pi x)\right]^2} \\ &= \left[\frac{\sin(\pi N x)}{\sin(\pi x)}\right]^2 \end{aligned}$$

which implies

$$F_N(x) = \frac{1}{N}\left[\frac{\sin(\pi N x)}{\sin(\pi x)}\right]^2 \ge 0 \qquad \forall N \ge 1$$

Thus,

$$\int_0^1 |F_N(x)|\,dx = \int_0^1 F_N(x)\,dx = 1 \qquad \forall N \ge 1$$

Lastly, we have to verify that $\forall 0 < \delta < 1$

$$\int_\delta^{1-\delta} |F_N(x)|\,dx \underset{N\to\infty}{\longrightarrow} 0$$

Fix $\delta > 0$. Then

$$\delta \le x \le 1 - \delta \implies \pi\delta \le \pi x \le \pi - \pi\delta$$

$\implies \exists c_\delta > 0$ s.t.

$$|\sin(\pi x)|^2 \geq c_\delta \qquad \forall x \in [\delta, 1-\delta]$$

So

$$\begin{aligned}\int_\delta^{1-\delta} |F_N(x)|\, dx &= \frac{1}{N}\int_\delta^{1-\delta} \left|\frac{\sin(\pi N x)}{\sin(\pi x)}\right|^2 dx \\ &\leq \frac{1}{N}\int_\delta^{1-\delta} \frac{1}{c_\delta}\, dx \\ &= \frac{1}{N}\frac{1-2\delta}{c_\delta} \xrightarrow[N\to\infty]{} 0\end{aligned}$$

This proves that $\{F_N\}_{N\geq 1}$ form an approximation to the identity. $\square$

§28.2 Topology Addendum

Lemma 28.1

Let (X,d) be a metric space. A set $A \subseteq X$ is dense in X if and only if $A \cap W \neq \emptyset$ for every non-empty open set $W \subseteq X$.

Proof. " $\implies$ " Let $A \subseteq X$ be such that $\overline{A} = X$. Assume, towards a contradiction that $\exists \emptyset \neq W = \mathring{W} \subseteq X$ s.t.

$$\begin{aligned} A \cap W = \emptyset &\implies W \subseteq {}^cA \\ &\implies W = \mathring{W} \subseteq \widehat{{}^cA}^{\circ} = {}^c(\overline{A}) = {}^cX = \emptyset\end{aligned}$$

which is a contradiction as $W \neq \emptyset$.
" $\impliedby$ " Assume, towards a contradiction, that

$$\overline{A} \neq X \implies \left.\begin{aligned} &{}^c(\overline{A}) \neq \emptyset \\ &{}^c(\overline{A}) = \widehat{{}^cA}^{\circ}\end{aligned}\right\} \implies \widehat{{}^cA}^{\circ} \neq \emptyset$$

which implies

$$\exists x \in {}^cA \text{ and } \exists r > 0 \text{ s.t. } B_r(x) \subseteq {}^cA$$

So $\underbrace{B_r(x)}_{\neq\emptyset \text{ open}} \cap A \neq \emptyset$ – contradiction! $\square$

Theorem 28.2

Let (X,d) be a complete metric space. Then X has the property of Baire, that is, for every sequence $\{A_n\}_{n\geq 1}$ of open dense sets we have

$$\overline{\bigcap_{n\geq 1} A_n} = X$$

Proof. Using the lemma, it suffices to show

$$\bigcap_{n\geq 1} A_n \cap W \neq \emptyset \qquad \forall \emptyset \neq W = \mathring{W} \subseteq X$$

Fix $\emptyset \neq W = \mathring{W} \subseteq X$.

$$\overline{A_1} = x \implies A_1 \cap W \neq \emptyset \implies \exists x_1 \in \underbrace{A_1 \cap W}_{\text{open}} \implies \exists 0 < r_1 < 1 \text{ s.t.}$$

$$K_{r_1}(x_1) = \{y \in X : d(y, x_1) \leq r_1\} \subseteq A_1 \cap W$$

$$\overline{A_2} = X \implies A_2 \cap B_{r_1}(x_1) \neq \emptyset \implies \exists x_2 \in \underbrace{A_2 \cap B_{r_1}(x_1)}_{\text{open}} \implies \exists 0 < r_2 < \frac{1}{2} \text{ s.t.}$$

$$K_{r_2}(x_2) \subseteq A_1 \cap B_{r_1}(x_1)$$

Proceeding inductively, we find a sequence $\{x_n\}_{n\geq 1} \subseteq X$ and $\{r_n\}_{n\geq 1}$ s.t.

$$\begin{cases} 0 < r_n < \frac{1}{n} & \forall n \geq 1 \\ K_{r_{n+1}}(x_{n+1}) \subseteq A_{n+1} \cap B_{r_n}(x_n) \subseteq K_{r_n}(x_n) & \forall n \geq 1 \end{cases}$$

Note that $\{K_{r_n}(x_n)\}_{n\geq 1}$ is a sequence of nested closed sets whose diameters decrease to zero. As (X, d) is complete, we find

$$\bigcap_{n\geq 1} K_{r_n}(x_n) = \{x\}$$

for some $x \in X$. In addition,

$$\{x\} = \bigcap_{n\geq 1} K_{r_n}(x_n) \subseteq A_1 \cap W \cap \bigcap_{n\geq 2} A_n \cap B_{r_{n-1}}(x_{n-1}) \subseteq \left(\bigcap_{n\geq 1} A_n\right) \cap W$$

which implies $\left(\bigcap_{n\geq 1} A_n\right) \cap W \neq \emptyset$. ☐

Lemma 28.3

Let (X, d) be a metric space. Then the following are equivalent:

1. For every $\{A_n\}_{n\geq 1}$ of open dense sets we have $\overline{\bigcap_{n\geq 1} A_n} = X$.
2. For every $\{F_n\}_{n\geq 1}$ of closed sets with empty interiors, we have

$$\overbrace{\bigcup_{n\geq 1} F_n}^{\circ} = \emptyset$$

Proof. Left as exercise. ☐

§29 Lec 29: Jun 4, 2021

§29.1 Topology Addendum (Cont'd)

Lemma 29.1

Let (X, d) be a metric space that has the Baire property. If $\emptyset \neq W = \mathring{W} \subseteq X$, then W has the Baire property.

Proof. Fix $\emptyset \neq W = \mathring{W} \subseteq X$. Let $\{D_n\}_{n\geq 1}$ be open dense sets in W.
D_n open in $W \implies \exists G_n$ open in X s.t. $D_n = G_n \cap W$ open in X as G_n and W are open.
D_n dense in $W \implies \overline{D_n} \cap W = W \implies W \subseteq \overline{D_n} \implies \overline{W} \subseteq \overline{D_n}$.
Define $A_n = D_n \cup {}^c(\overline{W})$ open in X.

$$\overline{A_n} = \overline{D_n \cup {}^c(\overline{W})} = \overline{D_n} \cup \overline{{}^c(\overline{W})} = \overline{D_n} \cup {}^c\left(\mathring{\overline{W}}\right) \supseteq \overline{W} \cup {}^c(\overline{W}) = X$$

Thus $\{A_n\}_n$ are dense open sets in X and as X has the Baire property,

$$\overline{\bigcap_{n\geq 1} A_n} = X$$

Then,

$$X = \overline{\bigcap_{n\geq 1} A_n} = \overline{\bigcap_{n\geq 1} \left[D_n \cup {}^c(\overline{W})\right]} = \overline{\left(\bigcap_{n\geq 1} D_n\right) \cup {}^c(\overline{W})} = \overline{\bigcap_{n\geq 1} D_n} \cup {}^c\left(\mathring{\overline{W}}\right)$$

which implies

$$\left.\begin{aligned} W &= \left[\overline{\bigcap_{n\geq 1} D_n} \cup {}^c\left(\mathring{\overline{W}}\right)\right] \cap W \\ &= \left[\overline{\textstyle\bigcap_{n\geq 1} D_n} \cap W\right] \cup \left[{}^c\left(\mathring{\overline{W}}\right) \cap W\right] \\ \mathring{\overline{W}} \supseteq \mathring{W} = W &\implies {}^c\left(\mathring{\overline{W}}\right) \subseteq {}^cW \implies {}^c\left(\mathring{\overline{W}}\right) \cap W = \emptyset \end{aligned}\right\}$$

$\implies \overline{\bigcap_{n\geq 1} D_n} \cap W = W$ i.e. $\bigcap_{n\geq 1} D_n$ is dense in W. □

Theorem 29.2

Let (X, d) be a metric space with the Baire property. Let $f_n : X \to \mathbb{R}$ be continuous function that converges pointwise to a function $f : X \to \mathbb{R}$. Then the set

$$C = \{x \in X : \ f \text{ is continuous at } x\} \text{ is dense in } X$$

Proof. We can observe that it suffices to prove the theorem under the additional hypothesis

$$|f_n(x)| \leq 1 \qquad \forall x \in X \quad \forall n \geq 1$$

Indeed, if $\{f_n\}_{n\geq 1}$ is as in the theorem, then we consider

$\phi : \mathbb{R} \to (-1,1)$, $\quad \phi(x) = \dfrac{x}{1+|x|}$ continuous, bijective, with the inverse $\phi^{-1}(y) = \dfrac{y}{1-|y|}$

So $\phi \circ f_n : X \to (-1,1)$ is continuous and $|\phi \circ f_n(x)| \leq 1$ for all $n \geq 1$ and $x \in X$. Also, $f_n \underset{n\to\infty}{\longrightarrow} f$ pointwise $\implies$ $\phi \circ f_n \underset{n\to\infty}{\longrightarrow} \phi \circ f$ pointwise. If the theorem holds with the additional uniform boundedness hypothesis, we get

$$\left.\begin{matrix} \{x \in X :\ \phi \circ f \text{ is continuous at } x\} \\ \{x \in X :\ f \text{ is continuous at } x\} \end{matrix}\right\} \text{ is dense in } X$$

So without the loss of generality, we assume

$$|f_n(x)| \leq 1 \qquad \forall n \geq 1 \quad \forall x \in X \tag{1}$$

Then,

$$\begin{aligned} C &= \{x \in X :\ f \text{ is continuous at } x\} \\ &= \{x \in X :\ \omega(f,x) = 0\} \\ &= \bigcap_{n\geq 1} \underbrace{\left\{x \in X :\ \omega(f,x) < \frac{1}{n}\right\}}_{=:G_n \text{ open in } X} = \bigcap_{n\geq 1} G_n \end{aligned}$$

As X has the Baire property, to prove $\overline{C} = X$ it suffices to show $\overline{G_n} = X\ \forall n \geq 1$. Fix $N \geq 1$. We will show that $G_N = \left\{x \in X : \omega(f,x) < \frac{1}{N}\right\}$ is dense in X. By a lemma from last lecture, it suffices to show

$$G_N \cap W \neq \emptyset \qquad \forall \emptyset \neq W = \mathring{W} \subseteq X$$

Fix $\emptyset \neq W = \mathring{W} \subseteq X$. For $n \geq 1$ and $x \in X$, we define

$$u_n(x) = \inf_{m\geq n} f_m(x) \quad \text{and} \quad v_n(x) = \sup_{m \geq n} f_m(x)$$

Then $\{u_n(x)\}_{n\geq 1}$ is increasing and $\{v_n(x)\}_{n\geq 1}$ is decreasing. As $\lim_{n\to\infty} f_n(x) = f(x)$, we have

$$\lim_{n\to\infty} u_n(x) = f(x) = \lim_{n\to\infty} v_n(x) \tag{2}$$

For $n \geq 1$, let

$$\begin{aligned} F_n &= \left\{x \in X :\ v_n(x) - u_n(x) \leq \frac{1}{4N}\right\} \\ &= \left\{x \in X :\ \sup_{m\geq n} f_m(x) - \inf_{l\geq n} f_l(x) < \frac{1}{4N}\right\} \\ &= \left\{x \in X :\ \sup_{m,l\geq n} [f_m(x) - f_l(x)] \leq \frac{1}{4N}\right\} \\ &= \bigcap_{m,l\geq n} \left\{x \in X :\ f_m(x) - f_l(x) \leq \frac{1}{4N}\right\} \\ &\overset{(1)}{=} \bigcap_{m,l\geq n} (f_m - f_l)^{-1}\left(\left[-2, \frac{1}{4N}\right]\right) \end{aligned}$$

$f_m - f_l$ is continuous $\forall m, l \geq n$ and $\left[-2, \frac{1}{4N}\right]$ is closed, so

$$(f_m - f_l)^{-1}\left(\left[-2, \frac{1}{4N}\right]\right) \text{ is closed} \qquad \forall m, l \geq n$$

So F_n is closed in X for all $n \geq 1$. Also,

$$X = \bigcup_{n\geq 1} F_n \qquad \text{by (2)}$$

So

$$\left.\begin{array}{l} W = \left(\bigcup_{n\geq 1} F_n\right) \cap W = \bigcup_{n\geq 1}(F_n \cap W) \\ W = \mathring{W} \neq \emptyset \\ W \text{ has the Baire property} \end{array}\right\} \implies \exists n_1 \in \mathbb{N} \text{ s.t. } \overset{\circ}{\widehat{F_{n_1} \cap W}} \neq \emptyset$$

Let $x_0 \in \overset{\circ}{\widehat{F_{n_1} \cap W}}$ and let $\delta > 0$ s.t. $B_\delta(x_0) \subseteq F_{n_1} \cap W$. As f_{n_1} is continuous at x_0, shrinking δ if necessary, we may assume

$$\omega(f_{n_1}, B_\delta(x_0)) < \frac{1}{4N}$$

We compute

$$\begin{aligned} \omega(f, x_0) \leq \omega(f, B_\delta(x_0)) &= \sup_{x\in B_\delta(x_0)} f(x) - \inf_{y\in B_\delta(x_0)} f(y) \\ &= \sup_{x,y\in B_\delta(x_0)} [f(x) - f(y)] \\ &\leq \sup_{x,y\in B_\delta(x_0)} [v_{n_1}(x) - u_{n_1}(y)] \\ &= \sup_{x,y\in B_\delta(x_0)} [v_{n_1}(x) - u_{n_1}(x) + v_{n_1}(y) - u_{n_1}(y) + u_{n_1}(x) - v_{n_1}(y)] \\ (B_\delta(x_0) \subseteq F_{n_1}) &\leq \frac{1}{4N} + \frac{1}{4N} + \sup_{x,y\in B_\delta(x_0)} [u_{n_1}(x) - v_{n_1}(y)] \\ &\leq \frac{1}{2N} + \sup_{x,y\in B_\delta(x_0)} [f_{n_1}(x) - f_{n_1}(y)] \\ &= \frac{1}{2N} + \omega(f_{n_1}; B_\delta(x_0)) \\ &\leq \frac{1}{2N} + \frac{1}{4N} < \frac{1}{N} \end{aligned}$$

This proves $x_0 \in G_n \cap W \implies G_N \cap W \neq \emptyset$. As $\emptyset \neq W = \mathring{W} \subseteq X$ was arbitrary, we conclude G_N is dense in X. $\square$