

# Math 134 – Nonlinear ODE

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This is math 134 – Linear and Nonlinear System of Differential Equations taught by Professor Wink. The class lecture is prerecorded, and we have live session every Monday and Friday at 3:00 pm – 3:50 pm for Q & A. We use *Nonlinear Dynamics and Chaos* 2<sup>nd</sup> by *Steven Strogatz* as our main book for the class. Other course notes can be found through my [github](#). Any error spotted in the notes is my responsibility, and please let me know through my email at [ducvu2718@ucla.edu](mailto:ducvu2718@ucla.edu).

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# §1 | Lec 1: Jan 4, 2021

## §1.1 Intro to Dynamical Systems

There are two types of dynamical systems:

1. Discrete in time:

- Difference equation
- Iterated map:  $a_{n+1} = f(a_n)$

2. Continuous in time: differential equation

- Partial Differential Equation (PDE):  
e.g. heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

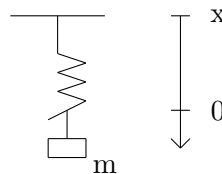
wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

where the derivatives w.r.t time and space.

- Ordinary Differential Equation (ODE):

i) Harmonic oscillator



m: mass

k: spring constant

$$m\ddot{x} + kx = 0$$

If  $\omega^2 = \frac{k}{m}$ , then

$$x(t) = x_0 \cos(\omega t) + x_1 \sin(\omega t)$$

ii) Damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0, \quad b: \text{damping constant}$$

iii) Forced, damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = F \cos(t), \quad F: \text{force}$$

so derivatives w.r.t time only.

**Definition 1.1 (Order of ODE)** — Highest occurring derivative is defined as the order of the ODE.

**Remark 1.2.** We can always write an ODE of  $n^{\text{th}}$  order as a system of ODEs of  $1^{\text{st}}$  order.

Trick: Consider the damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

Set

$$\begin{aligned}x_1 &= x \\x_2 &= \dot{x}\end{aligned}$$

Then,

$$\begin{aligned}\dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x \\ &= -\frac{b}{m}x_2 - \frac{k}{m}x_1\end{aligned}$$

i.e.,

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1\end{aligned}$$

General framework:  $\dot{x} = f(t, x)$

$$f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

i.e.,

$$\begin{aligned}\dot{x}_1 &= f_1(t, x_1, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, \dots, x_n)\end{aligned}\tag{1}$$

which is  $1^{\text{st}}$  order  $n$ -dimensional ODE.

**Definition 1.3 (Linear ODE)** — The ODE (1) is called linear if  $f(t, x) = A(t) \cdot x$  for a time dependent matrix  $A(t)$ , otherwise we call it non-linear.

#### Example 1.4

The damped harmonic oscillator is linear.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

**Question 1.1.** Why are linear equations special?

They satisfy the principle of superposition. If  $\phi, \psi$  solve  $\dot{x} = A(t)x$ , then  $y(t) = c \cdot \phi(t) + \psi(t)$ ,  $c \in \mathbb{R}$  also solves  $\dot{x} = A(t)x$ . This is valid because  $\dot{y} = c\dot{\phi} + \dot{\psi} = cA\phi + A\psi = A(c\phi + \psi) = Ay$ . For non-linear ODEs, the principle of superposition fails.

**Definition 1.5 (Autonomous ODE)** — The ODE (1) is called autonomous if  $f$  does not depend on  $t$ , i.e.,  $f(t, x) = f(x)$ .

**Example 1.6**

$$m\ddot{x} + b\dot{x} + kx = F \cos(t)$$

is non-autonomous.

However, we can always consider an autonomous system instead. Set

$$x_1 = x$$

$$x_2 = \dot{x}$$

$$x_3 = t$$

Then

$$\dot{x}_1 = x_2$$

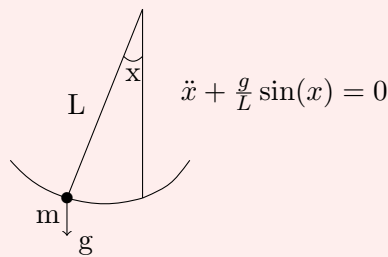
$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + F \cos(x_3)$$

$$\dot{x}_3 = 1$$

We will primarily study autonomous 1<sup>st</sup> order system in 1 or 2 variables.

**Example 1.7 (Swinging Pendulum)**

Consider a swinging pendulum



Set

$$x_1 = x$$

$$x_2 = \dot{x}$$

Then

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{L} \sin(x_1)$$

1<sup>st</sup> order, non-linear autonomous ODE in 2 variables.

**Question 1.2.** What can we say about the behavior of a solution  $x_1(t), x_2(t)$  for larger time  $t$ ? How does it depend on  $\frac{g}{L}$ ?

Idea: Use geometric methods, without solving  $\dot{x} = f(x)$  explicitly, to make qualitative statements about the long time behavior of the solution.

## §2 | Lec 2: Jan 6, 2021

### §2.1 Phase Portraits

We want to study 1D autonomous dynamical systems

$$\dot{x} = f(x), \quad f : \mathbb{R} \rightarrow \mathbb{R}$$

**Remark 2.1.**  $x(t)$  is the solution to  $\dot{x} = f(x)$  with  $x(0) = x_0$ . Find the solution  $y(t)$  with  $y(t_0) = x_0$ .

Ans:  $y(t) = x(t - t_0)$  because  $y(t_0) = x(0) = x_0$  and  $\dot{y}(t) = \dot{x}(t - t_0) = f(x(t - t_0)) = f(y(t))$ .



**Example 2.2**

$\dot{x} = \sin(x)$ . Suppose  $x_0 = \frac{\pi}{4}$ ,  $x(t)$  solution with  $x(0) = x_0$ . Answer the followings

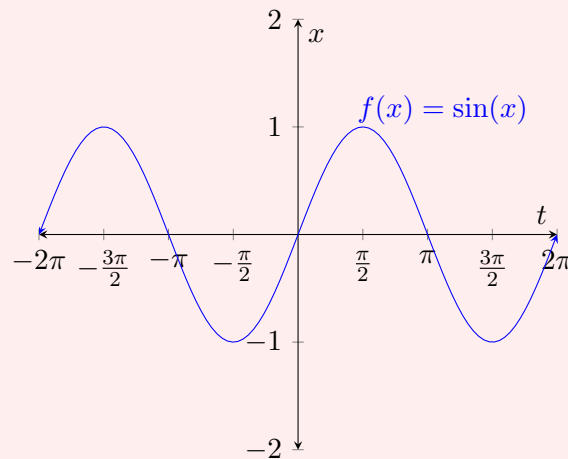
- Describe the long time behaviors of  $x(t)$  as  $t \rightarrow \infty$ .
- How does the long time behavior depend on  $x_0 \in \mathbb{R}$ ?

Attempt 1: Find explicit solution

$$\begin{aligned}\frac{dx}{dt} &= \sin(x) \\ dt &= \frac{dx}{\sin(x)} \\ t &= -\ln \left| \frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)} \right| + c\end{aligned}$$

We know  $x(0) = x_0$ , so  $c = \ln \left| \frac{1+\cos(x_0)}{\sin(x_0)} \right|$ . But what is  $x(t)$ ? This approach fails!

Attempt 2: Draw a phase portrait/diagram. We want to interpret the velocity  $\dot{x} = f(x)$  as a vector field on the real line.

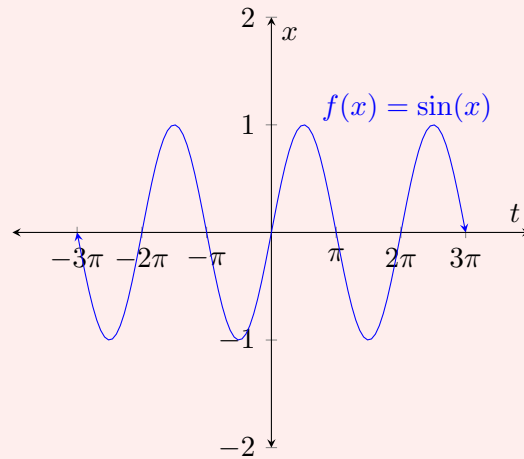


Idea:

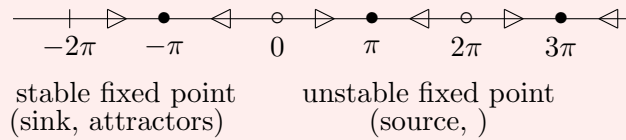
- If  $f(x_0) > 0$ , then the solution to  $\dot{x} = f(x)$ ,  $x(0) = x_0$  increase near  $x_0$ .
- If  $f(x_0) < 0$ , then the solution to  $\dot{x} = f(x)$ ,  $x(0) = x_0$  decrease near  $x_0$ .
- If  $f(x_0) = 0$ , then the solution to  $\dot{x} = f(x)$ ,  $x(0) = x_0$  is  $x(t) = x_0$  for all  $t \in \mathbb{R}$ , i.e., we have a fixed point/equilibrium point.

### Example 2.3

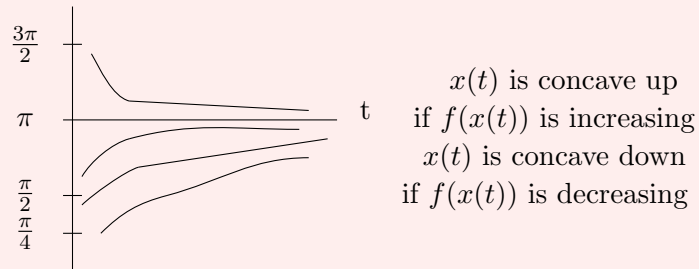
$$\dot{x} = f(x) = \sin(x)$$



Phase portrait:

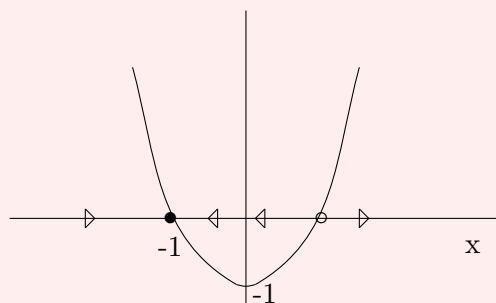


Qualitative plot of solution:



**Example 2.4**

$\dot{x} = x^2 - 1$ . Fixed points:  $f(x) = x^2 - 1 = 0 \implies x = \pm 1$



Note: If  $x_0 > 1$ , then solution  $x(t)$  with  $x(0) = x_0 > 1$  is unbounded. In fact,  $x(t) \rightarrow \infty$  in finite time.

## §3 | Lec 3: Jan 8, 2021

### §3.1 Stability Types of Fixed Points

**Definition 3.1 (Stability Types)** — Consider the ODE  $\dot{x} = f(x)$  and suppose that  $f(x_*) = 0$ . The fixed point  $x_*$  is called

1. Lyapunov stable if every solution  $x(t)$  with  $x(0) = x_0$  close to  $x_*$  remain close to  $x_*$  for all  $t \geq 0$ , otherwise unstable.
2. Attracting if every solution  $x(t)$  with  $x(0) = x_0$  close to  $x_*$  satisfies  $x(t) \rightarrow x_*$  as  $t \rightarrow \infty$ .
3. (asymptotically) stable if  $x_*$  is both Lyapunov stable and attracting.

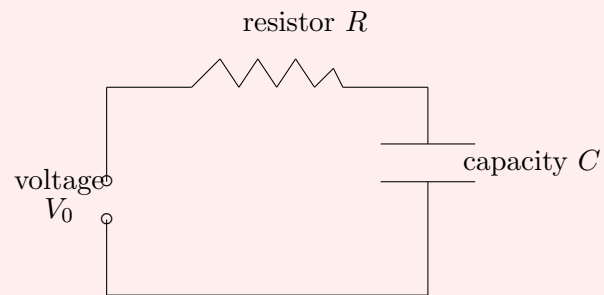
#### Example 3.2

Let  $\alpha \in \mathbb{R}, \dot{x} = \alpha x$ . General solution  $x(t) = x_0 e^{\alpha t}$ .

- $x_* = 0$  is always an equilibrium solution.
- $x_* = 0$  is
  1. attracting if  $\alpha < 0$
  2. Lyapunov stable if  $\alpha \leq 0$
  3. unstable if  $\alpha > 0$

**Example 3.3** (RC circuit)

We have the following circuit



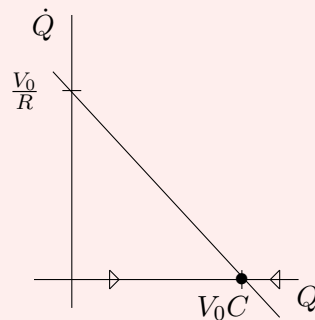
$$V_0 = RI + \frac{Q}{C}$$

$I$  : current,  $Q$  : charge

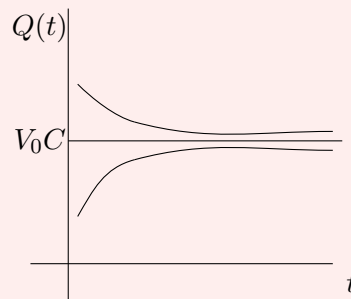
$$I = \dot{Q}$$

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$$

Phase portrait



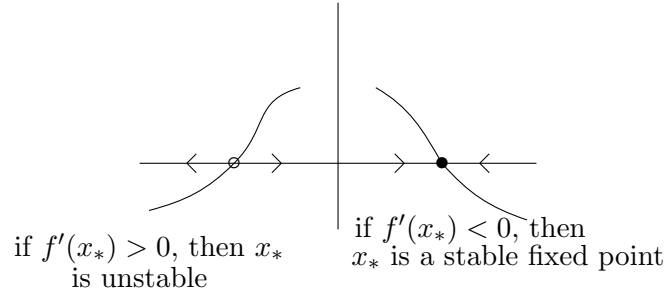
$Q_* = V_0C$  globally stable because every  $Q(t)$  approaches  $Q_*$  as  $t \rightarrow \infty$ .



### §3.2 Linear Stability Analysis

We have  $\dot{x} = f(x)$ ,  $f(x_*) = 0$ . Our task is to find an analytic criterion to decide if a fixed point  $x_*$  is stable/unstable.

Picture:



If  $f'(x_*) > 0$ , then  $x_*$  is unstable. On the other hand, if  $f'(x_*) < 0$ , then  $x_*$  is a stable fixed point.

The linearization:

Consider:  $\eta(t) = x(t) - x_*$  where  $x(t)$  is the solution of  $\dot{x} = f(x)$  with  $x(0)$  close to  $x_*$ ,  $f(x_*) = 0$ .

Note:  $\dot{\eta}(t) = \dot{x}(t) = f(x(t)) = f(x(t) - x_* + x_*) = f(\eta(t) + x_*)$ .

Taylor's Theorem:

$$f(x_* + \eta) = \underbrace{f(x_*)}_{=0} + f'(x_*)\eta + \underbrace{\mathcal{O}(\eta^2)}_{\text{error term and negligible if } f'(x_*) \neq 0 \text{ and } \eta \text{ is small}}$$

$\Rightarrow \dot{\eta}(t) \approx f'(x_*)\eta(t)$  (as long as  $\eta(t)$  is small) which is called the linearization of  $\dot{x} = f(x)$  about  $x_*$ . The general solution is

$$\eta(t) = \eta_0 e^{f'(x_*) \cdot t}$$

In particular,  $\eta$  grows exponentially if  $f'(x_*) > 0$  or decreases exponentially if  $f'(x_*) < 0$ .

**Definition 3.4** (Characteristics Time Scale) —  $\frac{1}{|f'(x_*)|}$  is called the characteristics time scale.

**Example 3.5 (Logistics Equation)**

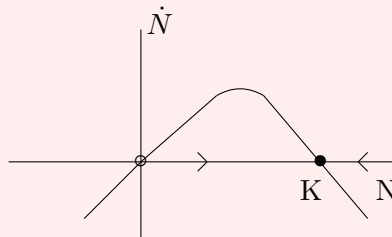
$N \geq 0$  population size,  $r > 0$  growth rate,  $K > 0$  carrying capacity

$$\dot{N} = rN \left(1 - \frac{N}{K}\right)$$

Fixed points:  $\dot{N} = 0 \implies N_* = 0$  or  $N_* = K$ .

Let  $f(N) = rN \left(1 - \frac{N}{K}\right) \implies f'(N) = r - 2\frac{r}{K}N$ . In particular,  $f'(0) = r > 0 \implies N_* = 0$  is an unstable fixed point and  $f'(K) = r - 2r = -r < 0 \implies N_* = K$  is stable.

Phase portrait:



Thus, if  $N(t)$  is the population with

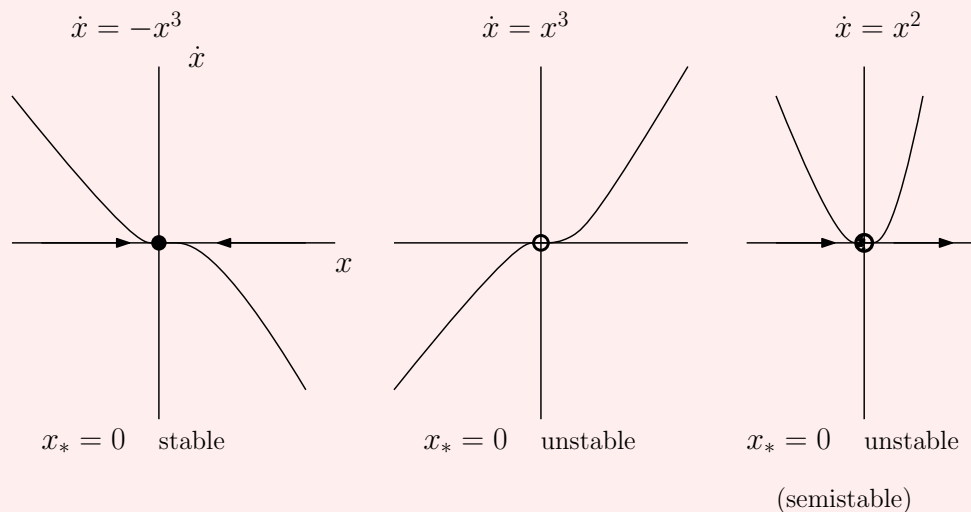
$$N(0) = N_0 > 0 \implies N(t) \rightarrow K \text{ as } t \rightarrow \infty$$

$$N(0) = 0 \rightarrow N(t) = 0 \quad \forall t \text{ (no spontaneous outbreak)}$$

Characteristics time scale:  $\frac{1}{|f'(N_*)|} = \frac{1}{r}$  for both  $N_* = 0, K$ .

**Example 3.6**

What if  $f'(x_*) = 0$ ? Then we can't tell.

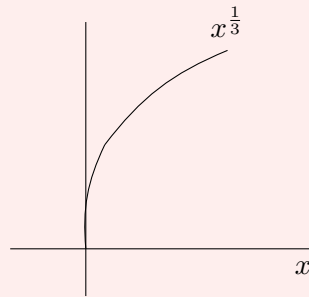


## §4 | Lec 4: Jan 11, 2021

### §4.1 Existence and Uniqueness

#### Example 4.1 (Non-uniqueness)

$\dot{x} = x^{\frac{1}{3}} \implies x_1(t) \equiv 0$  (for all  $t$ ) is a solution with  $x_1(0) = 0$  but  $x_2(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$  is also a solution with  $x_2(0) = 0$



Is  $x_0 = 0$  really a fixed point? No, it's unclear how it would behave (according to  $x(t) = 0$  or  $x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$ ).

#### Theorem 4.2 (Picard's)

Let  $I = (a, b) \subseteq \mathbb{R}$  be an open interval,  $f : I \rightarrow \mathbb{R}$  differentiable and  $f'$  continuous. Let  $x_0 \in I$ . Then there is  $\tau > 0$  s.t. the initial value problem

$$\dot{x} = f(x), x(0) = x_0$$

has a unique solution  $x : (-\tau, \tau) \rightarrow \mathbb{R}$ .



**Example 4.3**

(The solution might not exist for all times) Consider

$$\frac{dx}{dt} = \dot{x} = 1 + x^2, \quad x(0) = 0$$

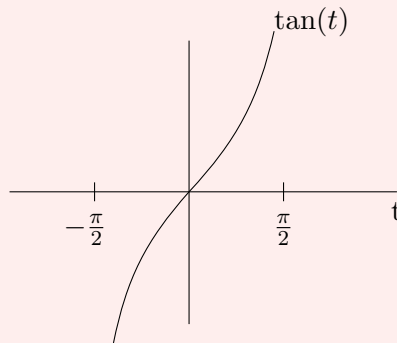
So,

$$dt = \frac{dx}{1 + x^2}$$

$$t = \int \frac{dx}{1 + x^2} = \arctan x + C$$

$$0 = 0 + C \implies C = 0$$

$$x(t) = \tan(t)$$



In particular,

$$x(t) \rightarrow +\infty \text{ as } t \rightarrow \frac{\pi}{2}$$

$$x(t) \rightarrow -\infty \text{ as } t \rightarrow \frac{-\pi}{2}$$

i.e.,  $x(t)$  reaches infinity in finite time, i.e., the solution  $x(t)$  blows up in finite time.

**Remark 4.4.** (Hw 1) If  $x_0 > 0$ , then the solution to  $\dot{x} = x^2, x(0) = x_0 > 0$  blows up in finite time. In fact, if  $\alpha > 1$ , then the solution to  $\dot{x} = x^\alpha, x(0) = x_0 > 0$  blows up in finite time.

**Theorem 4.5 (ODE Comparison)**

If  $x_1(t)$  solves  $\dot{x} = f(x)$ ,  $x_2(t)$  solves  $\dot{x} = q(x)$  and  $x_1(0) \leq x_2(0)$ ,  $f(x) < q(x)$ , then  $x_1(t) \leq x_2(t)$  for all  $t > 0$ .

In particular, if  $x_1(t) \rightarrow \infty$  in finite time, then  $x_2(t) \rightarrow \infty$  in finite time.

**Example 4.6**

The solution to  $\dot{x} = 1 + x^2 + x^3, x(0) = 0$  blows up in finite time.

Note: For  $x \geq 0$  :

$$1 + x^2 \leq 1 + x^2 + x^3$$

Recall:  $\tan(t)$  solves  $\dot{x} = 1 + x^2, x(0) = 0$ . By comparison: the solution  $x(t)$  to  $\dot{x} = 1 + x^2 + x^3, x(0) = 0$  satisfies  $x(t) \geq \tan(t)$ . Thus,  $x(t)$  blows up in finite time.

We may indeed assume that  $x(t) > 0$ . Since  $\dot{x}(0) = 1$ , it follows that  $x(t) > 0$  for  $t > 0$  small. In fact,  $\dot{x} = 1 + x^2 + x^3 > 0$  for  $x(t)$  small, i.e., whenever  $x(t)$  is close to zero, it must increase  $\implies x(t) > 0$  for  $t > 0$ .

**Example 4.7 (No Oscillating Solution in 1D)**

Let  $f \in C^1(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} | f \text{ differentiable, } f' \text{ continuous}\}$ . Suppose  $f(x_*) = 0, x(t)$  solution of  $\dot{x} = f(x)$ . If  $x(t_0) = x_*$  for some  $t_0$ . Then  $x(t) = x_*$  for all time  $t$ . Geometrically this says that a solution can never reach/cross a fixed point (unless it is a fixed point).

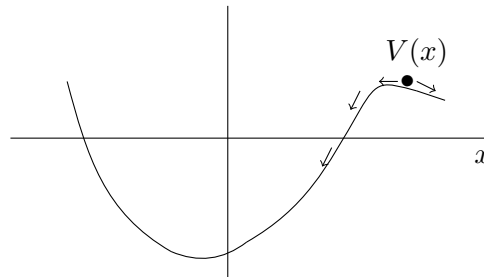
- $f(x(t)) > 0$  and  $\dot{x}(t) > 0$ , i.e.,  $x(t)$  increases.
- $f(x(t)) = 0$  and  $x(t) = \text{constant}$  for all  $t$ .
- $f(x(t)) < 0$  and  $\dot{x}(t) < 0$  i.e.,  $x(t)$  decreases.

In particular, there is no oscillating solution.

## §5 | Lec 5: Jan 13, 2021

### §5.1 Potential

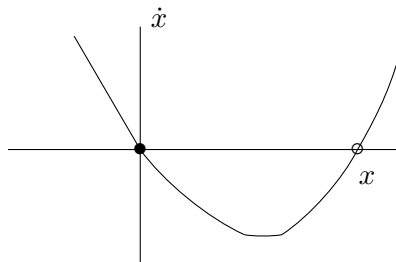
Consider the movement of a particle (with lots of friction) in a potential.



Notice:

- Particle approaches the local minimum of  $V(x)$  (minimum energy level) no fixed point.
- Local minima of  $V(x)$  are stable fixed points.
- Local maxima of  $V(x)$  are unstable fixed points.

$$\Rightarrow \dot{x} = f(x) = -\frac{dV}{dx} = -V'(x).$$



Expect  $t \rightarrow V(x(t))$  is non-increasing for a solution  $x(t)$  of  $\dot{x} = -V'(x)$ .

Indeed:

$$\begin{aligned} \frac{d}{dt}V(x(t)) &= V'(x(t)) \frac{d}{dt}x(t) \\ &= V'(x(t)) (-V'(x(t))) \\ &= -(V'(x(t)))^2 \leq 0 \end{aligned}$$

$\Rightarrow$  particle always moves towards a lower energy level.

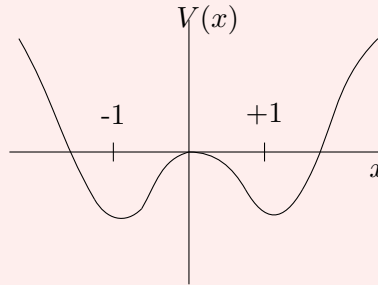
**Definition 5.1 (Potential)** — A function  $V(x)$  s.t.  $\dot{x} = f(x) = -\frac{dV}{dx}$  is called a potential.

**Example 5.2**

Graph potential for  $\dot{x} = x - x^3$ . Find/characterize equilibria (fixed points).

$$\dot{x} = f(x) = x - x^3 = -\frac{dV}{dx} \xRightarrow{f} V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$$

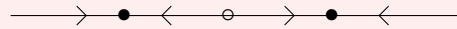
$\Rightarrow V$  is only defined up to a constant, we may choose any  $C \in \mathbb{R}$ , e.g., choose  $C = 0$ .



Local minima of  $V$  correspond to stable fixed points  $\Rightarrow 0 = -\frac{dV}{dx} = f(x) = x - x^3$ , i.e.,  $x = \pm 1$ .

Local maximum of  $V$  corresponds to an unstable fixed point at  $x = 0$ .

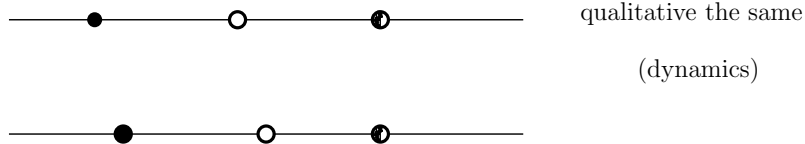
Phase portrait:



**Remark 5.3.** This system is often called bistable because it has two stable fixed points.

## §5.2 Bifurcations

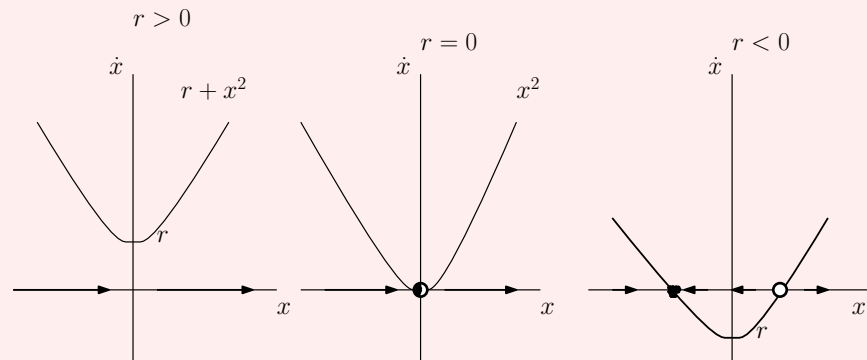
The qualitative behavior of 1D dynamical systems  $\dot{x} = f(x)$  is determined by fixed points.



If  $\dot{x} = f(r, x)$  depends on a parameter  $r$ , then the numbers of fixed points and their stability may change as  $r$  varies. This is called **bifurcation**.

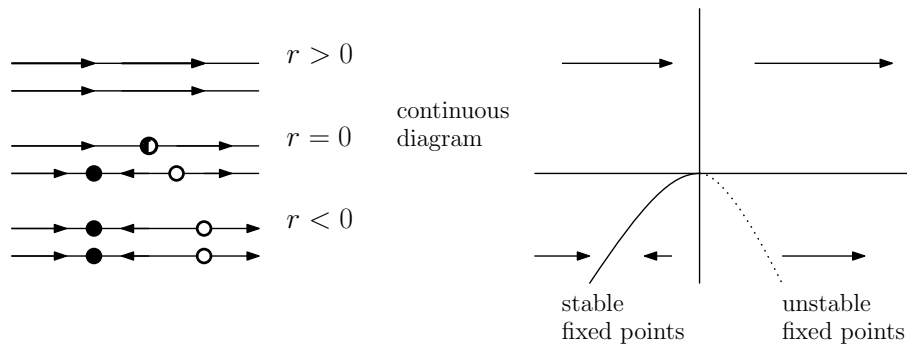
**Example 5.4** (Saddle-node, blue sky bifurcation)

$$\dot{x} = r + x^2, \quad r \in \mathbb{R}.$$

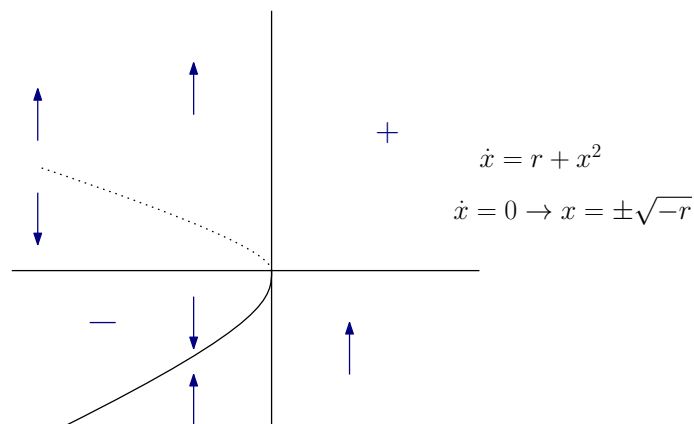


Hence, the qualitative behavior changes at  $r_* = 0$ , i.e.,  $r_* = 0$  is called a bifurcation point.

Ways to plot the dependence on the parameter:



Most common: bifurcation diagram



## §6 | Lec 6: Jan 15, 2021

### §6.1 Saddle-Node Example

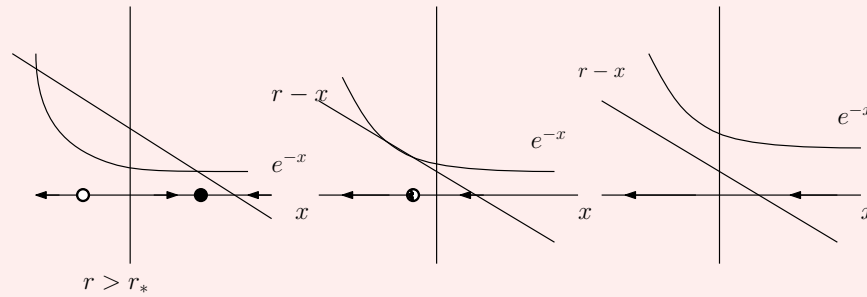
#### Example 6.1

Argue geometrically that the ODE

$$\dot{x} = r - x - e^{-x}$$

undergoes a saddle-node bifurcation. Furthermore, find the bifurcation point.

Note: Fixed points of  $\dot{x} = r - x - e^{-x}$  correspond to intersection points of the functions  $r - x, e^{-x}$  because  $r - x - e^{-x} = 0 \iff r - x = e^{-x}$ .



Indeed we have a saddle-node bifurcation.

Note: At  $r = r_*$ , the graph of  $r - x$  and  $e^{-x}$  intersect tangentially. Thus, for the bifurcation point we require:

$$\begin{aligned} 0 = \dot{x} = r - x - e^{-x} &\implies r - x = e^{-x} \\ 0 = \frac{d}{dx}(r - x - e^{-x}) &\implies \frac{d}{dx}(r - x) = \frac{d}{dx}e^{-x} \end{aligned}$$

So,

$$\begin{aligned} -1 &= -e^{-x} \\ e^{-x} &= 1 \\ x &= 0 \\ r_* &= x_* + e^{-x_*} = 0 + 1 = 1 \end{aligned}$$

Thus the bifurcation point is  $(r_*, x_*) = (1, 0)$ .

Note:

$$\begin{aligned} \dot{x} &= r - x - e^{-x} = r - x - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right) \\ &= r - 1 - \frac{1}{2}x^2 + \frac{x^3}{6} - \dots \\ &\approx (r - 1) - \frac{1}{2}x^2 \text{ for } x \text{ near } x_* = 0 \end{aligned}$$

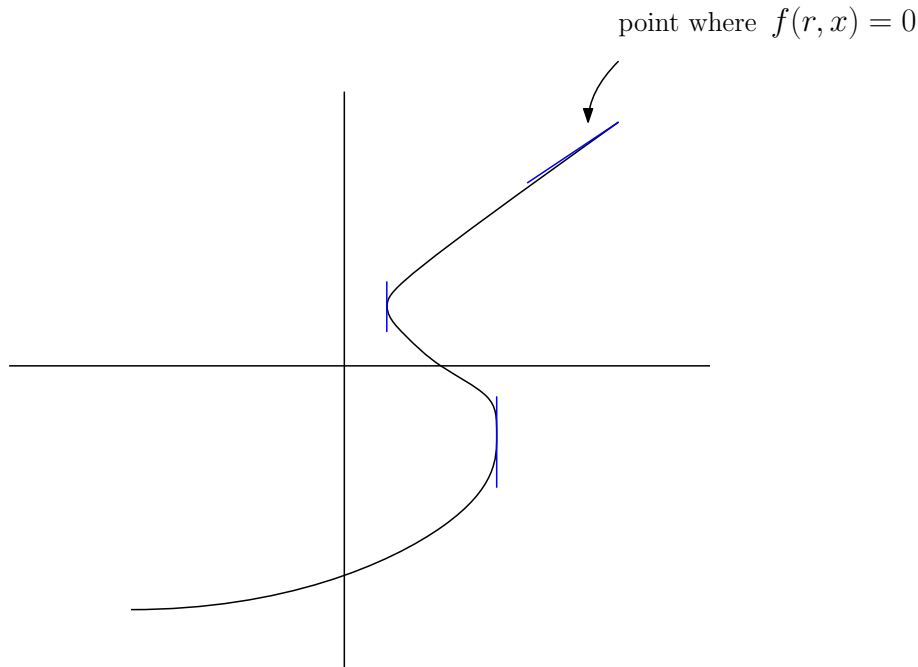
Set  $R = r - 1$ , then  $\dot{x} \approx R - \frac{1}{2}x^2$ .

Upshot: Up to appropriate rescalings/coordinate changes, every saddle-node bifurcation looks like its normal form

$$\dot{x} = r - x^2 \quad (\text{or } \dot{x} = r + x^2)$$

close to the bifurcation point  $(r_*, x_*) = (0, 0)$ .

## §6.2 Normal Forms



Recall:

- Normal vector:  $\begin{pmatrix} \partial_r f \\ \partial_x f \end{pmatrix}$
- Tangent vector:  $\begin{pmatrix} -\partial_x f \\ \partial_r f \end{pmatrix}$

Note: Bifurcation points have vertical tangent vectors, i.e.,  $\partial_x f = 0, \partial_r f \neq 0$ .

### Theorem 6.2 (Taylor's)

Suppose  $f(r_*, x_*) = 0$ .

$$\begin{aligned} f(r, x) = & f(r_*, x_*) + \underbrace{\frac{\partial f}{\partial r}(r_*, x_*)}_{p_1}(r - r_*) + \underbrace{\frac{\partial f}{\partial x}(r_*, x_*)}_{q_1}(x - x_*) \\ & + \frac{1}{2} \underbrace{\frac{\partial^2 f}{\partial r^2}(r_*, x_*)}_{p_2}(r - r_*)^2 + \underbrace{\frac{\partial^2 f}{\partial r \partial x}(r_*, x_*)}_{R}(r - r_*)(x - x_*) + \frac{1}{2} \underbrace{\frac{\partial^2 f}{\partial x^2}(r_*, x_*)}_{q_2}(x - x_*)^2 + \dots \end{aligned}$$

**Remark 6.3.** If  $q_1 \neq 0$ , then there is no bifurcation at  $(r_*, x_*)$ , linear stability (sign of  $q_1$ ) determines if  $(r_*, x_*)$  is (un)stable.

**Theorem 6.4**

Suppose that  $f(r_*, x_*) = 0, q_1 = 0, p_1 \neq 0, q_2 \neq 0$ , then  $\dot{x} = f(r, x)$  undergoes a saddle node bifurcation at  $(r_*, x_*)$  and

$$\dot{x} = \frac{\partial f}{\partial r}(r^*, x^*)(r - r^*) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

for  $|r - r_*| < \epsilon^2, \quad |x - x_*| < \epsilon.$

**Remark 6.5.** i) Note that the constant  $(r - r_*)(x - x_*)$  is  $\mathcal{O}(\epsilon^3)$

ii) With a coordinate change  $(t, x, r) \mapsto (s, y, R)$  we can arrange that ODE looks like

$$\frac{d}{ds}y = R + y^2$$

near  $(0, 0) = (R(r_*), y(x_*))$



**Example 6.6**

$\dot{x} = e^r - x - e^{-x}$  undergoes a saddle-node bifurcation near  $(r_*, x_*) = (0, 0)$ . Apply the theorem 6.4,

$$\begin{aligned} f(r, x) &= e^r - x - e^{-x} \\ f(0, 0) &= 1 - 0 - 1 = 0 \\ \frac{\partial f}{\partial x}(r, x) &= -1 + e^{-x} \implies \frac{\partial f}{\partial x}(0, 0) = 0 \\ \frac{\partial f}{\partial r}(r, x) &= e^r \implies \frac{\partial f}{\partial r}(0, 0) = 1 \neq 0 \\ \frac{\partial^2 f}{\partial x^2}(r, x) &= -e^{-x} \implies \frac{\partial^2 f}{\partial x^2}(0, 0) = -1 \neq 0 \end{aligned}$$

Therefore, by theorem 6.4,  $(r_*, x_*) = (0, 0)$  is a bifurcation point of a saddle-node bifurcation.

Normal form near  $(r_*, x_*) = (0, 0)$  :

$$\begin{aligned} \dot{x} &= e^r - x - e^{-x} \\ &= 1 + r + \frac{r^2}{2} + \mathcal{O}(r^3) - x - \left(1 - x + \frac{x^2}{2} + \mathcal{O}(x^3)\right) \\ &= r + \underbrace{\frac{r^2}{2}}_{\mathcal{O}(\epsilon^4)} - \frac{x^2}{2} + \mathcal{O}(r^3) + \mathcal{O}(x^3) \\ &= \underbrace{r - \frac{x^2}{2}}_{\mathcal{O}(\epsilon^2)} + \mathcal{O}(\epsilon^3) \text{ if } |r - r_*| = |r| < \epsilon^2 \\ &\quad \text{if } |x - x_*| = |x| < \epsilon \end{aligned}$$

Set  $y = \frac{x}{2}$ , then

$$\dot{y} = \frac{1}{2}\dot{x} = \frac{r}{2} - \frac{x^2}{4} + \mathcal{O}(\epsilon^3) = \frac{r}{2} - y^2 + \mathcal{O}(\epsilon^3)$$

Set  $s = -t$ , then

$$\frac{d}{ds}y = -\frac{d}{dt}y = -\frac{r}{2} + y^2 + \mathcal{O}(\epsilon^3)$$

Set  $R = -\frac{r}{2}$ , then

$$\underbrace{\frac{d}{ds}y = R + y^2}_{\text{normal form of a saddle-node bifurcation}} + \mathcal{O}(\epsilon^3)$$

## §7 | Lec 7: Jan 20, 2021

### §7.1 Classification of Bifurcations

Let's rewrite  $\dot{x}$  in theorem 6.4 as

$$\dot{x} = p(r - r_*) + \frac{c}{2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

if  $|r - r_*| < \epsilon^2, |x - x_*| < \epsilon$ . After a coordinate change  $(t, x, r) \mapsto (s, y, R)$  such that

$$\begin{aligned} s &= t \\ y &= \frac{c}{2}(x - x_*) \\ R &= p\frac{c}{2}(r - r_*) \end{aligned}$$

the ODE is represented by the normal form.

$$\frac{d}{ds}y = \dot{y} = R + y^2 + \mathcal{O}(\epsilon^3)$$

for  $|R| < \epsilon^2, |y| < \epsilon$ .

If  $f(x_*, r_*) = 0$ , and also  $\frac{\partial f}{\partial x}(x_*, r_*) = 0 = \frac{\partial f}{\partial r}(x_*, r_*)$ , then the second derivatives determines the bifurcation type.

$$\text{Hessian Hess}f = \begin{pmatrix} \frac{\partial^2 f}{\partial r^2} & \frac{\partial^2 f}{\partial r \partial x} \\ \frac{\partial^2 f}{\partial r \partial x} & \frac{\partial^2 f}{\partial x^2} \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Second test: if  $AC - B^2 > 0$ ,  $(r_*, x_*)$  is a local maximum/minimum. In particular,  $(r_*, x_*)$  is an isolated fixed point. (irrelevant case)

Practically relevant case: If  $AC - B^2 < 0$  :  $(r_*, x_*)$  is a saddle. If also  $C \neq 0$ : transcritical bifurcation.

$$\dot{y} = Ry - y^2 + \mathcal{O}(\epsilon^2)$$

for  $|R| < \epsilon, |y| < \epsilon$  (after an appropriate coordinate change)

$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

If also  $C = 0$  : Pitchfork bifurcation

- Supercritical Pitchfork bifurcation:

$$y' = Ry - y^3 + \mathcal{O}(\epsilon^3)$$

- Subcritical Pitchfork bifurcation

$$y' = Ry + y^3 + \mathcal{O}(\epsilon^3)$$

for  $|R| < \epsilon^2, |y| < \epsilon$

Again,

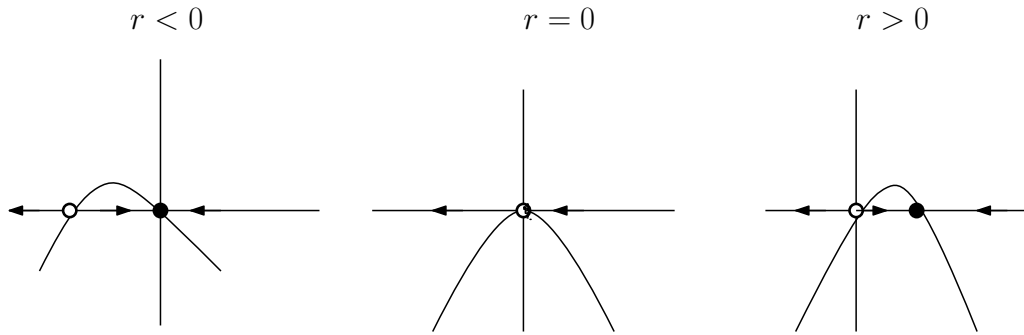
$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

## §7.2 Transcritical Bifurcation

Normal form:

$$\dot{x} = rx - x^2 = x(r - x)$$

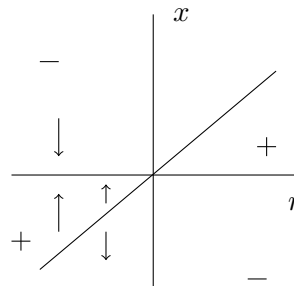
In particular,  $x_* = 0$  is always a fixed point but it changes stability.



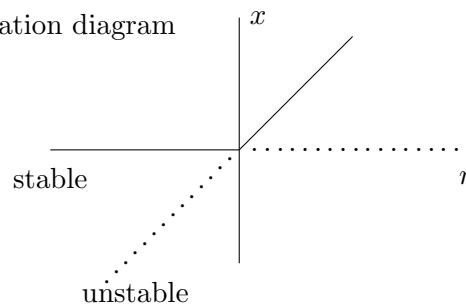
Bifurcation diagram:  $\dot{x} = x(r - x) = rx - x^2 = f(x)$ . Fixed points:

$$x_* = 0, \quad x_* = r \quad r \in \mathbb{R}$$

intermediate step:  
draw fixed points  
(without stability)



bifurcation diagram



## §8 | Lec 8: Jan 22, 2021

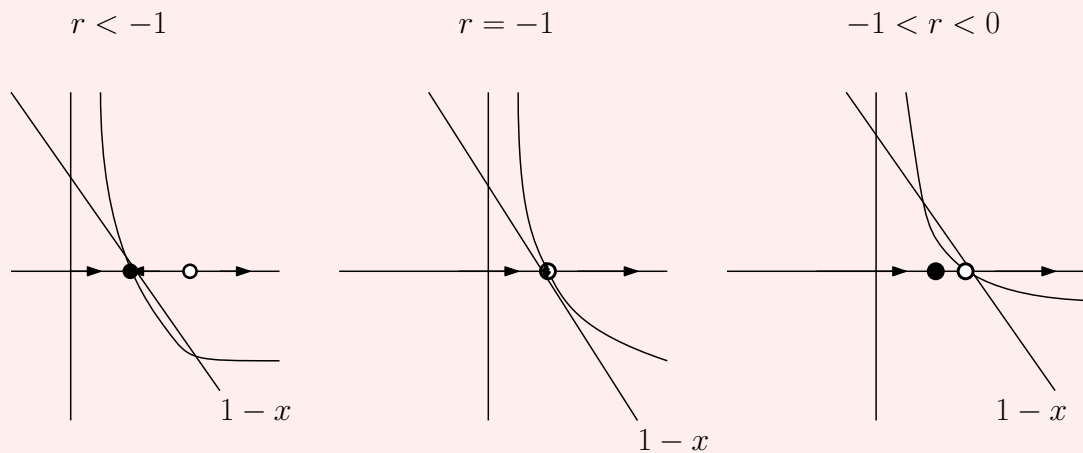
### §8.1 Example of Transcritical Bifurcation

#### Example 8.1

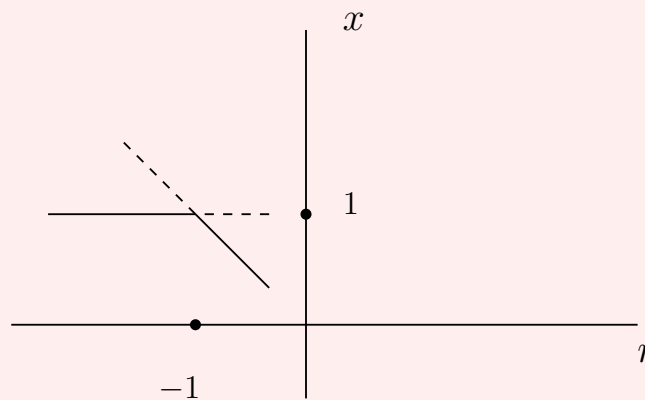
$\dot{x} = r \ln(x) + x - 1$  has a transcritical bifurcation at  $(r_*, x_*) = (-1, 1)$ .

Geometric approach:

$$\dot{x} = 0 \iff r \ln(x) = 1 - x$$



Bifurcation near  $(r_*, x_*) = (-1, 1)$



Normal form:  $\dot{x} = r \ln(x) + x - 1$ .

**Remark 8.2.**  $\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \quad |x| < 1$

So,

$$\begin{aligned}
 \dot{x} &= r \ln(x) + x - 1 \\
 &= r(x - 1 - \frac{1}{2}(x - 1)^2 + \mathcal{O}((x - 1)^3)) + x - 1 \\
 &= (r + 1)(x - 1) - \frac{1}{2}((r + 1) - 1)(x - 1)^2 + \mathcal{O}(r(x - 1)^3) \\
 &= (r + 1)(x - 1) + \frac{1}{2}(x - 1)^2 + \mathcal{O}(\epsilon^3)
 \end{aligned}$$

if  $|r - (-1)| < \epsilon$  and  $|x - 1| < \epsilon$ .

Now, set  $R = r + 1, y = c \cdot (x - 1)$ . Then,

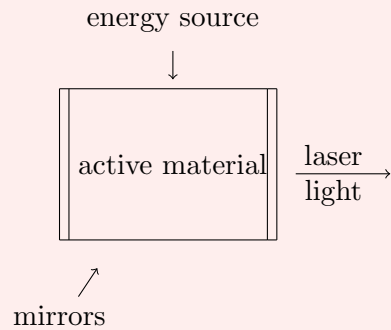
$$\begin{aligned}
 \dot{y} &= c\dot{x} \\
 &= (r + 1)c(x - 1) + \frac{1}{2}c(x - 1)^2 + \mathcal{O}(\epsilon^3) \\
 &= Ry + \frac{1}{2c}(c(x - 1))^2 + \mathcal{O}(\epsilon^3) \\
 &= Ry + \underbrace{\frac{1}{2c}}_{=1} y^2 = Ry + y^2
 \end{aligned}$$

for  $c = \frac{1}{2}$ .

## §8.2 Application of Transcritical Bifurcations

**Example 8.3 (Laser Threshold)**

Consider



Simple model:

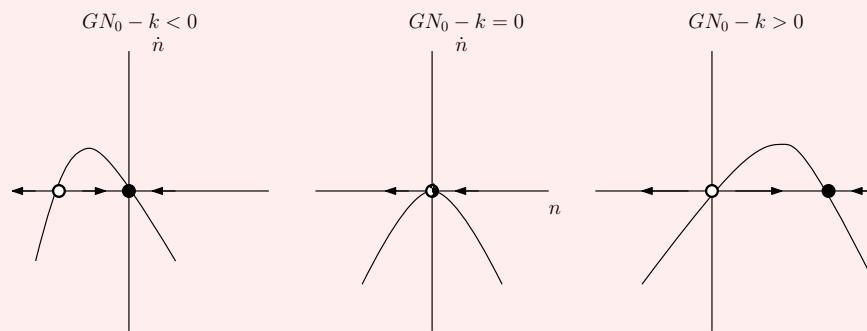
$$n = n(t) = \# \text{ photons in the laser}$$

Then

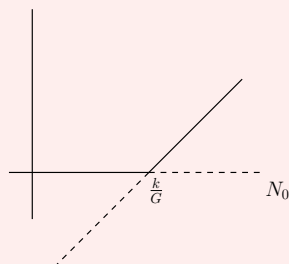
$$\begin{aligned}
 \dot{n} &= G \cdot \underbrace{N}_{\# \text{ excited atoms}} \cdot n - kn \\
 &= N_0 - \alpha \cdot n \\
 &= G(N_0 - \alpha n)n - kn \\
 &= (GN_0 - k)n - \alpha Gn^2
 \end{aligned}$$

where  $G, k, \alpha > 0$ . Fixed points:

$$\dot{n} = 0 \iff n = 0 \text{ or } n = \frac{GN_0 - k}{\alpha G}$$



Bifurcation diagram



transcritical bifurcation at

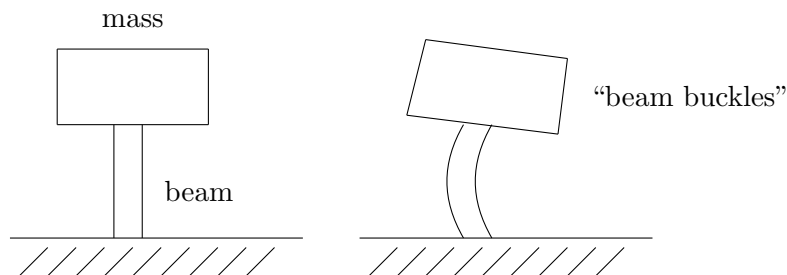
$$(N, n) = \left(\frac{k}{G}, 0\right)$$

 $\frac{k}{G} = \text{laser threshold}$

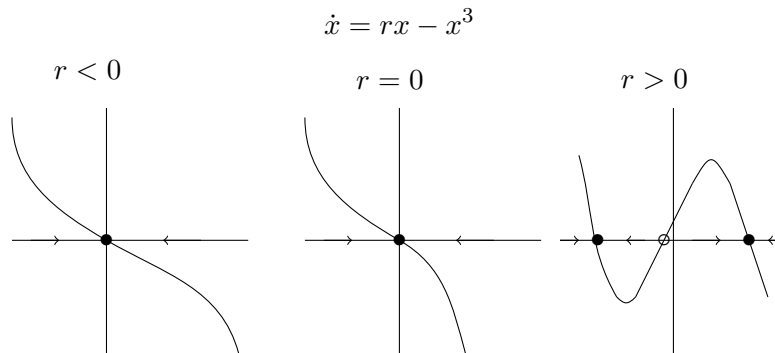
## §9 | Lec 9: Jan 25, 2021

### §9.1 Supercritical Pitchfork Bifurcation

Fixed points appear/disappear in symmetric pairs



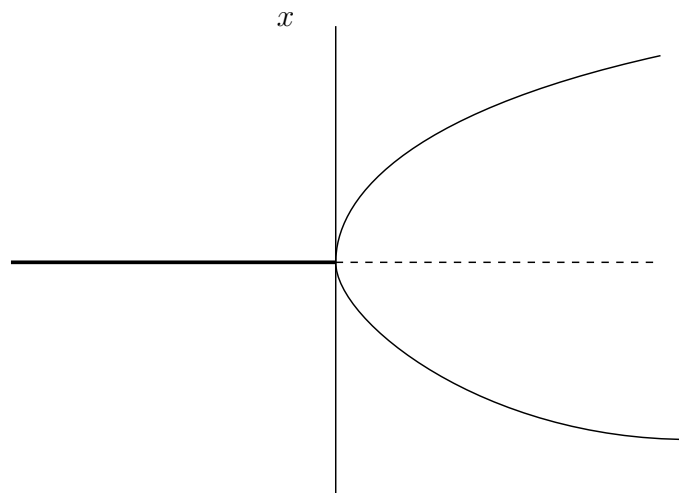
Supercritical Pitchfork Bifurcation:



**Remark 9.1.** Decay towards  $x_* = 0$  is not exponential in time for  $r = 0$ .

Bifurcation diagram:

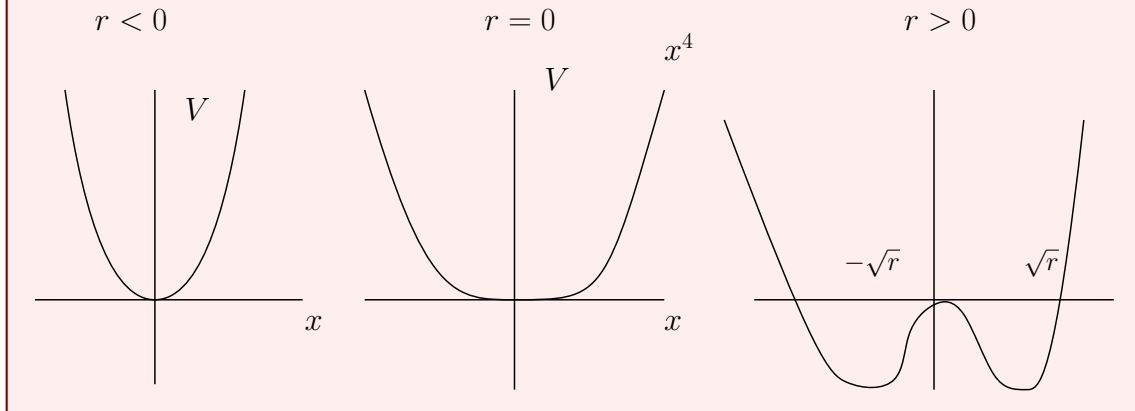
$$\begin{aligned} \dot{x} &= rx - x^3 = 0 \\ \Rightarrow x &= 0, \quad x = \pm\sqrt{r}, \quad r > 0 \end{aligned}$$



**Example 9.2**

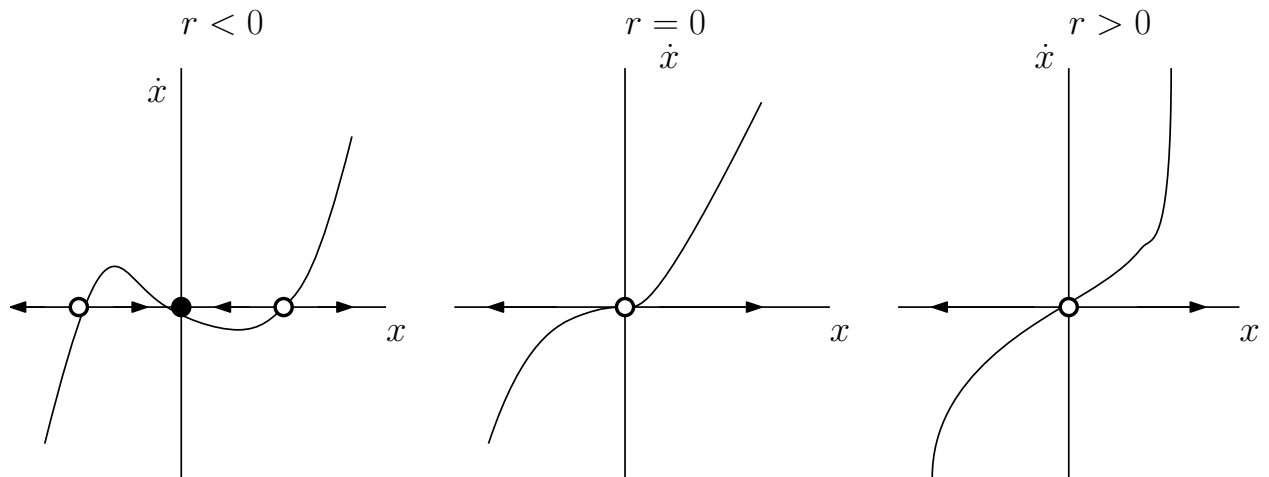
Potential for  $\dot{x} = rx - x^3 = -\frac{dV}{dx}$

$$\Rightarrow V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4 + \underbrace{C}_{=0}$$



**§9.2 Subcritical Pitchfork Bifurcation**

$$\dot{x} = rx + x^3$$

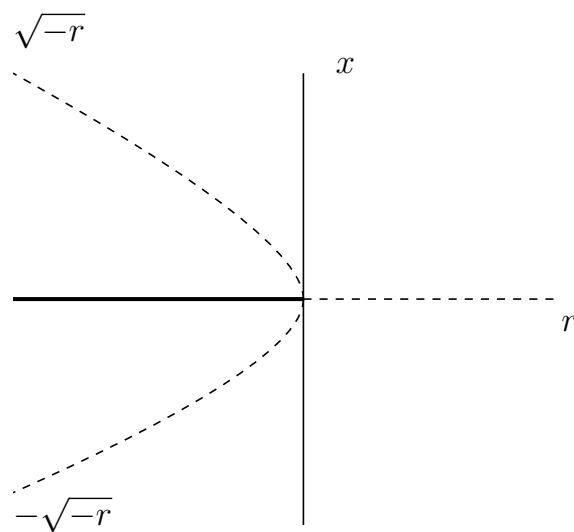


Fixed points:

$$\begin{aligned} \dot{x} &= rx + x^3 = 0 \\ \Rightarrow x &= 0, \quad x = \pm\sqrt{-r}, \quad r < 0 \end{aligned}$$

Bifurcation Diagram:





**Remark 9.3.** If  $r > 0, x_0 > 0$ , then the solution  $x(t)$  with  $x(0) = x_0 > 0$  blows up in finite time (cf. homework). Interpretation:  $+x^3$  is destabilizing.

Physically more realistic scenario:

$$\dot{x} = rx + x^3 - x^5$$

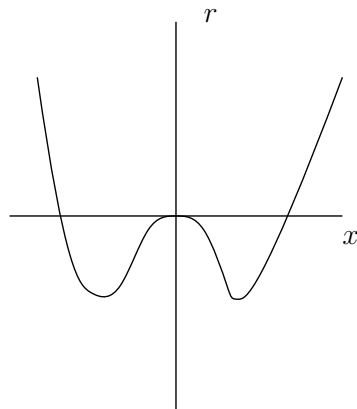
where  $x^5$  is the stabilizing higher order term.

Fixed points:

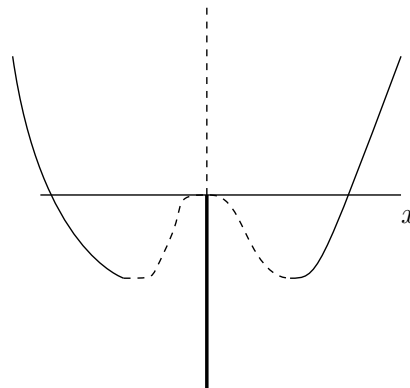
$$\dot{x} = 0 \iff x = 0, \quad r = -x^2 + x^4$$

Bifurcation diagram:

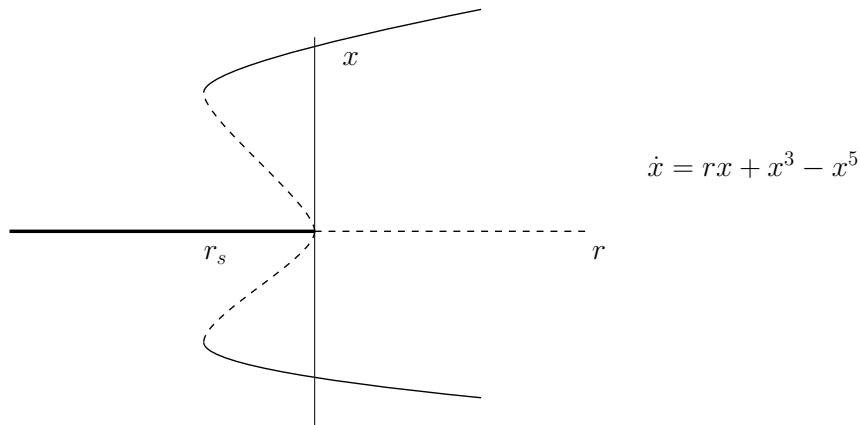
1. Intermediate step



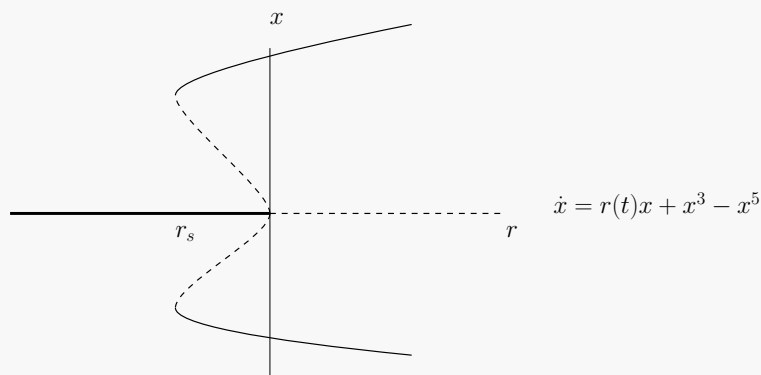
2. Stability Types



3. Change axes: bifurcation diagram



**Remark 9.4.** i) Subcritical pitchfork bifurcation at  $(r_*, x_*) = (0, 0)$  and saddle node bifurcation at  $(r_s, x_*) = (-\frac{1}{4}, \pm\sqrt{2})$ .



ii) jump at  $r_* = 0$  : A small perturbation of a stable fixed point at  $(0, r)$  with  $r < 0$  jumps to the stable large amplitude branch as  $r$  becomes positive, but does not jump back until  $r < r_s$ .

This non-reversibility is called hysteresis.

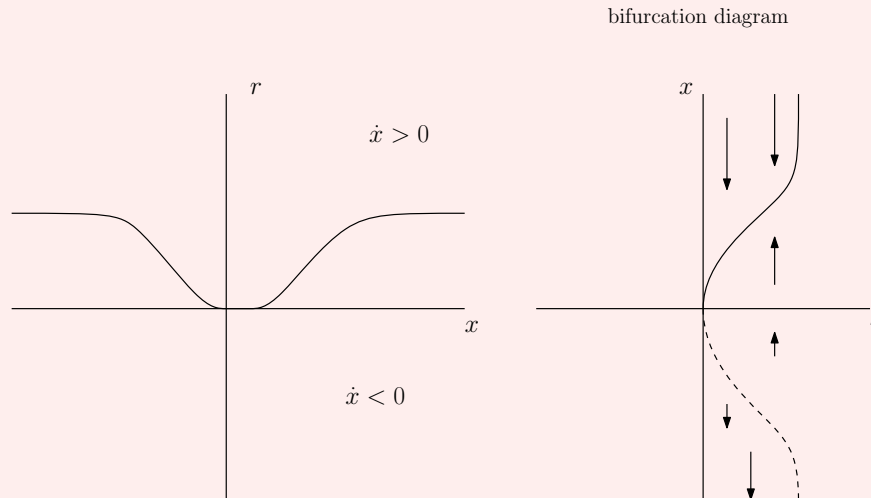
## §10 | Lec 10: Jan 27, 2021

### §10.1 Bifurcation at Infinity

#### Example 10.1

$$\dot{x} = r - \frac{x^2}{1+x^2}$$

$$\text{Fixed points: } \dot{x} = 0 \iff r = \frac{x^2}{1+x^2}$$



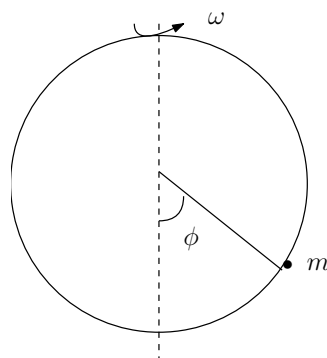
Note:

- At  $(r_*, x_*)$  we have a saddle node bifurcation.
- If  $r \in (0, 1)$  we have two fixed points.
- For  $r \geq 1$  we have no fixed points.

Thus, we have a bifurcation at (spatial) infinity.

### §10.2 Dimensional Analysis and Scaling

Over-damped bead over a hoop:



forces: gravitation:  $-mg\vec{e}_2$

centrifugal:  $mr \sin \phi \omega^2 \vec{e}_x$

damping:  $-b\dot{\phi} \vec{e}_\phi$

Physics:  $mr\ddot{\phi} = -b\dot{\phi} - mg \sin \phi + mr\omega^2 \sin \phi \cos \phi$

Experiment: Provided  $\omega$  large enough, bead slides slowly towards a fixed angle, after an initial acceleration phase.

**Question 10.1.** When we can neglect second order term  $\ddot{\phi}$ ?

**Problem 10.1.** We're working with different dimensions, e.g.

$$[m] = kg$$

$$[b] = \frac{kg \cdot m}{s}$$

What is small – what quantity is actually small so we can neglect the second order term?

Idea: Non-dimensionalize

- small means  $\ll 1$
- reduce the numbers of parameters
- no general algorithm

Quantity  $\omega$  large, time scale  $T$ .

Set  $\tau = \frac{t}{T} \implies d\tau = \frac{1}{T} dt$ , where  $T$  is the characteristics time scale.

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{d\phi}{d\tau} \frac{d\tau}{dt} = \frac{1}{T} \frac{d\phi}{d\tau}$$

$$\text{Similarly, } \ddot{\phi} = \frac{1}{T^2} \frac{d^2\phi}{d\tau^2}$$

$$mr\ddot{\phi} = -b\dot{\phi} - mg \sin \phi + mr\omega^2 \sin \phi \cos \phi \quad (1)$$

So

$$\implies \frac{mr}{T^2} \frac{d^2\phi}{d\tau^2} = -\frac{b}{T} \frac{d\phi}{d\tau} - mg \sin \phi + mr\omega^2 \sin \phi \cos \phi \quad (\text{unit force})$$

$$\implies \frac{r}{gT^2} \frac{d^2\phi}{d\tau^2} = -\frac{b}{mgT} \frac{d\phi}{d\tau} - \sin \phi + \frac{r\omega^2}{g} \sin \phi \cos \phi \quad (\text{dimensionless})$$

Thus 1<sup>st</sup> order term  $\frac{d\phi}{d\tau}$  dominates  $\frac{d^2\phi}{d\tau^2}$  if  $\frac{r}{gT^2} \ll 1$  and  $\frac{b}{mgT} \approx \mathcal{O}(1)$ , i.e.,  $\frac{b}{mgT} = 1$  and  $\epsilon = \frac{r}{gT^2}$

$$\implies T = \frac{b}{mg}$$

$$\implies \epsilon = \frac{rgm^2}{b^2} \ll 1$$

Set  $\gamma = \frac{r\omega^2}{g}$ . Then the non-dimensionalize equation becomes

$$\epsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin \phi + \gamma \sin \phi \cos \phi$$

Overdamped limit:  $\epsilon \rightarrow 0$

$$\frac{d\phi}{d\tau} = -\sin \phi + \gamma \sin \phi \cos \phi$$

$$= \sin \phi (\gamma \cos \phi - 1)$$

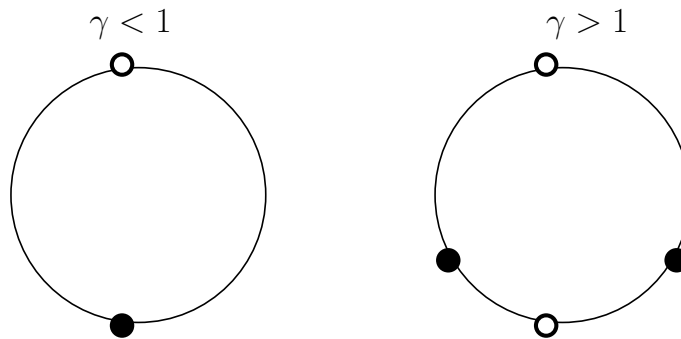
Dynamics:  $\frac{d\phi}{d\tau} = 0$  (fixed points)

$$\implies \sin \phi = 0 \iff \phi = 0, \pi \text{ (bottom/top of hoop)}$$

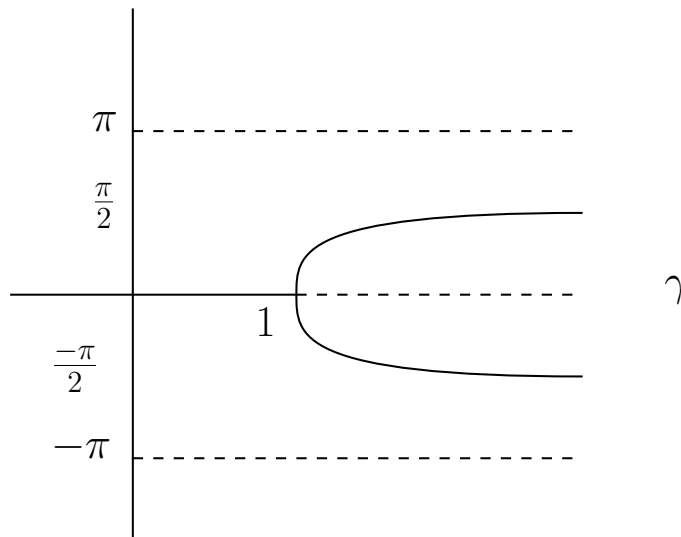
or

$$\cos \phi = \frac{1}{\gamma} \in (0, 1] \implies \gamma \geq 1$$

Fixed points:



Bifurcation Diagram:



In particular, we have a supercritical pitchfork bifurcation at  $\gamma = 1$ .

# §11 | Lec 11: Jan 29, 2021

## §11.1 Imperfect Bifurcation and Catastrophes

$$\dot{x} = h + rx - x^3$$

- If  $h = 0$  : symmetry, if  $x(t)$  is a solution then  $-x(t)$  is also a solution (supercritical pitchfork bifurcation).
- If  $h \neq 0$  : imperfect parameter, breaks symmetry.

Aim: Study qualitative behavior of ODE as parameters vary.

Strategy: keep  $h$  fixed and vary  $r$

- $h = 0$  : supercritical pitchfork bifurcation

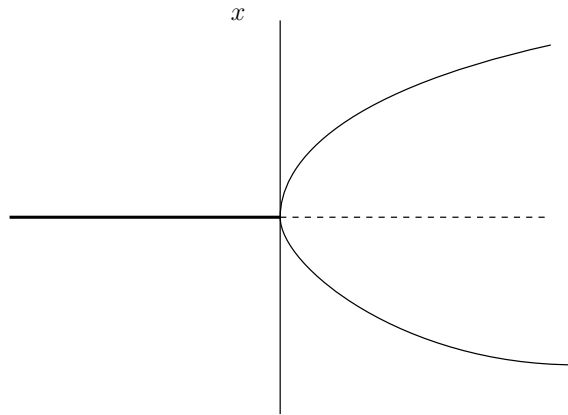


Figure 1: Bifurcation Diagram

- $h > 0$  : fixed points:  $\dot{x} = 0 \iff x^3 = h + rx$

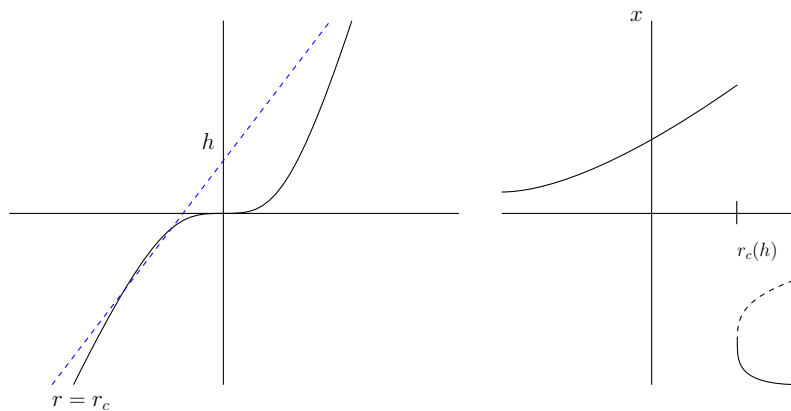


Figure 2: Bifurcation Diagram

- $h < 0$  : Fixed points:  $x^3 = h + rx$

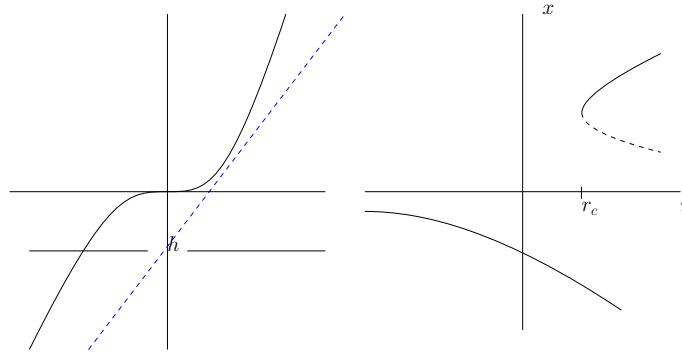


Figure 3: Bifurcation Diagram

Note: We have saddle node bifurcation at  $r_c = r(h)$

### Bifurcation Curves

$$\left\{ (h, r) \mid (h, r, x) \text{ solves } f = 0, \frac{\partial f}{\partial x} = 0 \right\}$$

in our example  $\dot{x} = h + rx - x^3$

$$0 = \frac{\partial f}{\partial x} = r - 3x^2 \implies x = \pm \sqrt{\frac{r}{3}}$$

$$0 = f = h + rx - x^3 \implies h = x^3 - rx$$

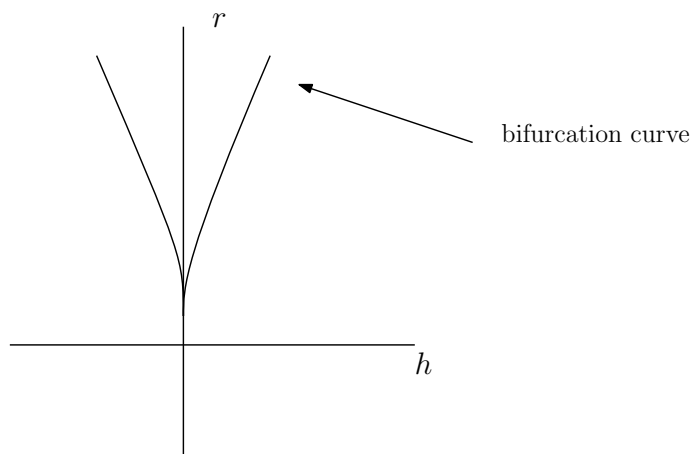
$$\implies h = x^3 - rx = \pm \frac{2\sqrt{3}}{9} r^{\frac{3}{2}}$$

$$h = h_c(r) = \pm \frac{2\sqrt{3}}{9} r^{\frac{3}{2}}$$

$$\implies r = r_c(h) = \left( \frac{9}{2\sqrt{3}} |h| \right)^{\frac{2}{3}}$$

Stability Diagram:

Plot the bifurcation curves in the parameters space  $(= (h, r) \text{ plane})$ .



Note: qualitative behavior of ode changes as  $(h, r)$  cross bifurcation curve.

In example:

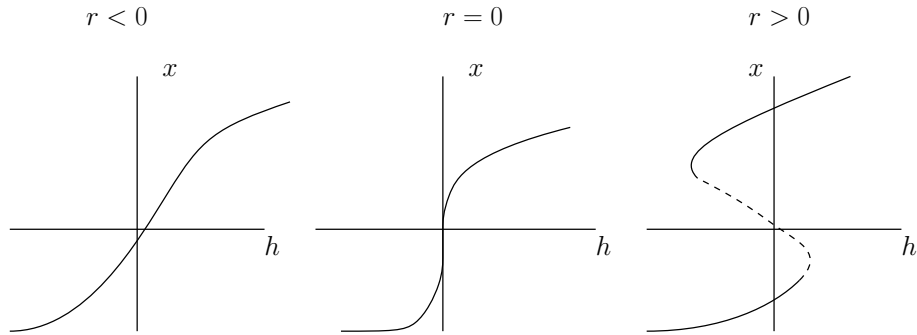
- “below” bifurcation curve: ODE has one (stable) fixed point.
- “on” bifurcation curve: two fixed points.
- “above” bifurcation curve: three fixed points.

**Remark 11.1.** • Saddle-node bifurcation occurs along bifurcation curve for  $(h, r) \neq (0, 0)$

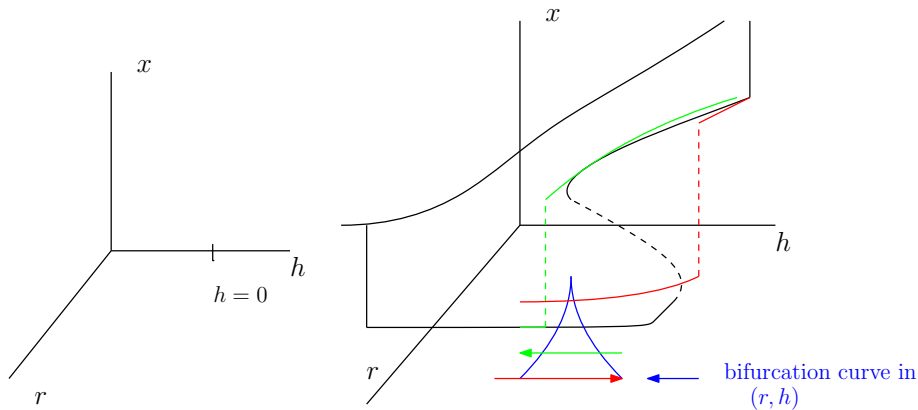
- At  $(h, r) = (0, 0)$ , the branches  $r_c(h) = \left(\frac{9}{2\sqrt{3}}|h|\right)^{\frac{2}{3}}$  for  $h > 0$  and  $h < 0$  meet tangentially, and we have a cusp point at  $(h, r) = (0, 0)$ . This is an example of a codimension 2 bifurcation (i.e., we need two parameters to model this type of bifurcation).

Bifurcation diagrams for fixed  $r \in \mathbb{R}$ .

$$\dot{x} = h + rx - x^3 = 0 \iff h = x^3 - rx$$



3D plot( $h, r$ , fixed points  $x$ )

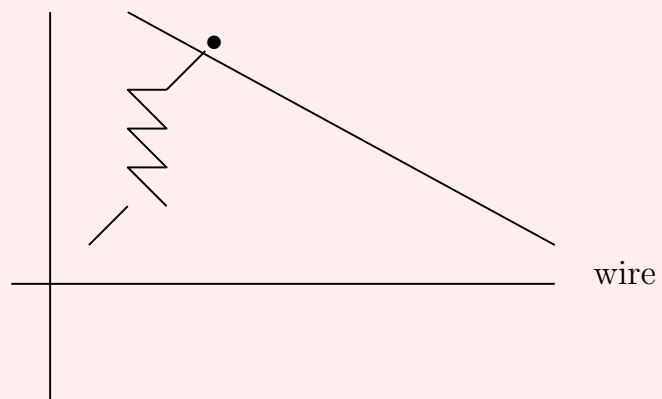


Picture/surface of cusp catastrophe solutions close to “upper” stable fixed points drop to “lower” stable fixed points as  $(r, h)$  vary (and vice versa).



**Example 11.2** (practical)

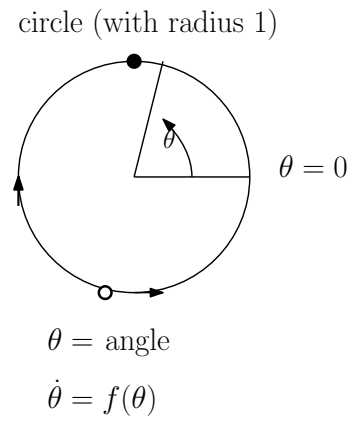
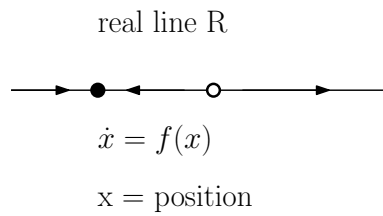
Details in the book, page 74



## §12 | Midterm 1: Feb 1, 2021

# §13 | Lec 12: Feb 3, 2021

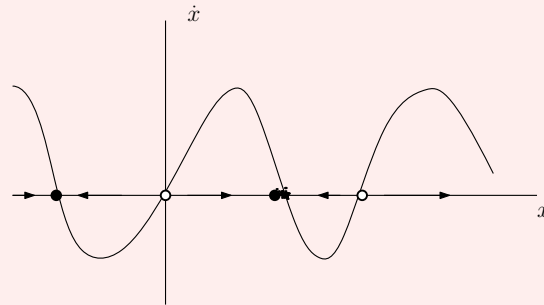
## §13.1 Flows on the Circle



**Example 13.1** i)  $\dot{x} = \sin(x)$ . Fixed points:  $\dot{x} = 0$

$$\iff x = \dots, -\pi, 0, \pi, 2\pi, \dots$$

i.e.,  $x = k\pi, k \in \mathbb{Z}$ .



$$\dot{\theta} = \sin \theta$$

$$\dot{\theta} = 0$$

$$\iff \theta = 0 \text{ or } \theta = \pi$$

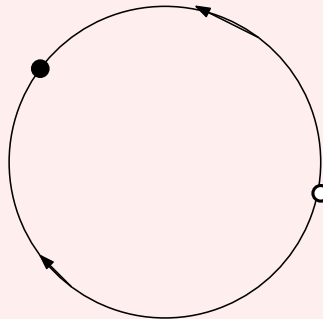
$$\underbrace{\theta = 2\pi}_{\text{same position on circle}}$$

same position on circle

i.e.,  $\theta$  is defined up to multiples of  $2\pi$ .

Note: If  $f(\theta) > 0$  : flow is counterclockwise, and if  $f(\theta) < 0$  : flow is clockwise.

$$\dot{\theta} = \sin(\theta)$$



ii)  $\dot{x} = x$  where  $f(x) = x$  is not periodic.

Thus  $\dot{\theta} = \theta$  does not work, because  $\theta = 0, \theta = 2\pi$  describe the same position on the circle but  $f(\theta) = \theta$  yields different values at  $\theta = 0, 2\pi$ , i.e.  $f(\theta)$  is not a vector field on the circle.

Correspondence:

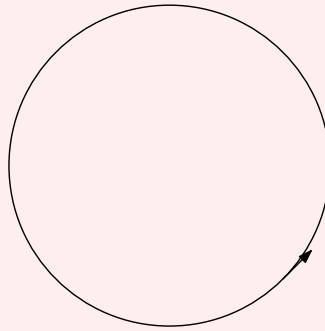
$f(x)$  is  $2\pi$ -periodic, i.e.  $f(x + 2\pi) = f(x)$ , and  $f$  is continuously differentiable  
 $\iff f(\theta)$  defines a vector field on the circle.

**Example 13.2** iii)  $\dot{x} = c > 0$

$$x(t) = ct + x_0$$



$\dot{\theta} = \omega > 0$  – uniform oscillator



$$\theta(t) = \omega t + \theta_0$$

Period T:

$$\theta(T) = \theta(0) + 2\pi$$

$$\omega T + \theta_0 = \theta_0 + 2\pi$$

$$T = \frac{2\pi}{\omega}$$

In particular, periodic solutions are possible.

**Example 13.3**

Two runners are on a circular track, running in the same direction, with constant speed:

- Runner 1: period  $T_1 = \frac{2\pi}{\omega_1}$ , angle  $\theta_1$
- Runner 2: period  $T_2 = \frac{2\pi}{\omega_2}$ , angle  $\theta_2$

Runner 1, 2 start at the same position. Suppose  $T_1 < T_2$ , i.e. Runner 1 is faster than runner 2.

**Question 13.1.** How long does it take runner 1 to lap runner 2?

Ans:  $T_{\text{lap}}$  = time when phase difference

$$\begin{aligned}\phi &= \theta_1 - \theta_2 \text{ is } 2\pi \\ \dot{\phi} &= \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2, \phi(0) = 0 \\ \implies \phi(t) &= (\omega_1 - \omega_2)t \\ \implies T_{\text{lap}} &= \frac{2\pi}{\omega_1 - \omega_2} = \frac{1}{\frac{1}{T_1} - \frac{1}{T_2}} = \left( \frac{1}{T_1} - \frac{1}{T_2} \right)^{-1}\end{aligned}$$

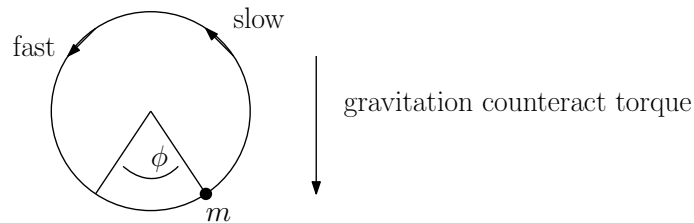
i.e. Runner 1,2 are in phase after  $T_{\text{lap}}$  again. This is called beat phenomenon.

# §14 | Lec 13: Feb 5, 2021

## §14.1 Non-uniform Oscillator

$$\dot{\theta} = \omega - a \sin \theta, \quad \omega > 0, a > 0$$

Practical example: overdamped limit of pendulum driven by constant torque.



$$\dot{\phi} = \omega - a \sin \phi$$

Consider:  $\dot{\theta} = \omega - a \sin \theta$

For  $0 < a < \omega$ :

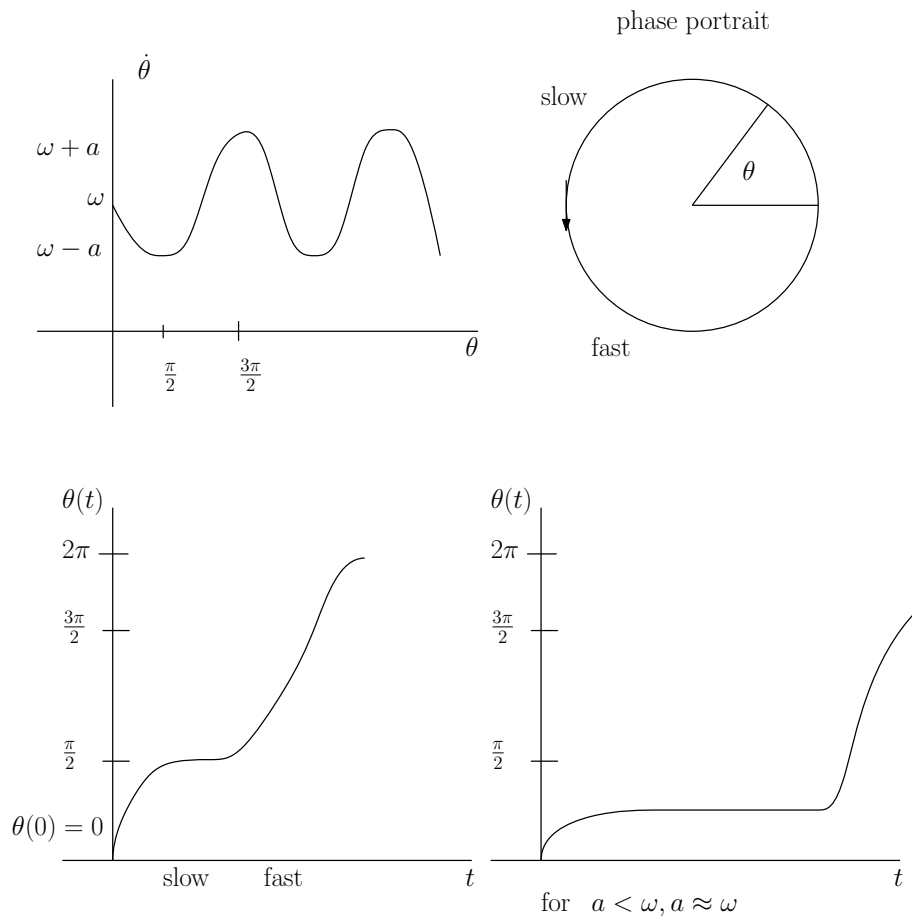
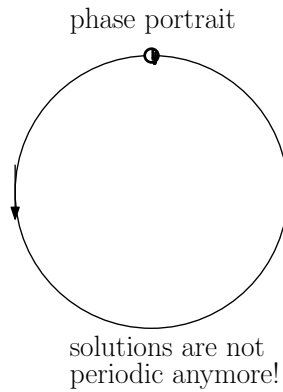
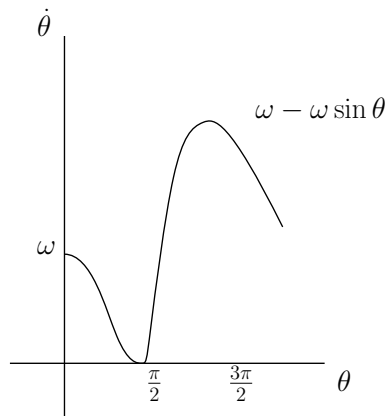
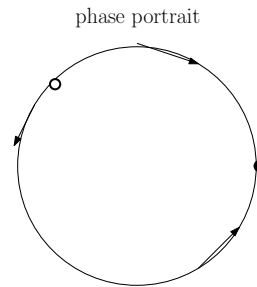
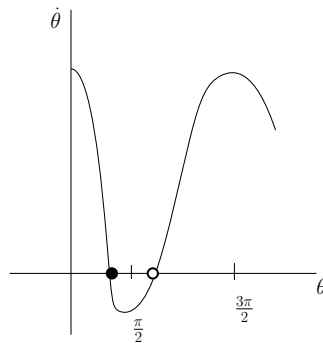


Figure 4: bottle neck remnants or "ghost" of a saddle-node bifurcation

For  $a = \omega$

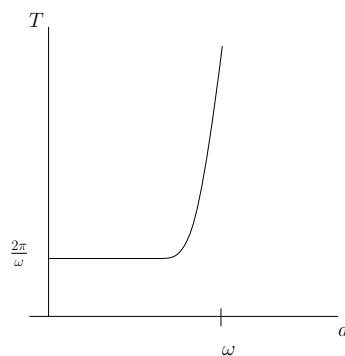


For  $w < \omega$  :



Oscillation period for  $a < \omega$ :

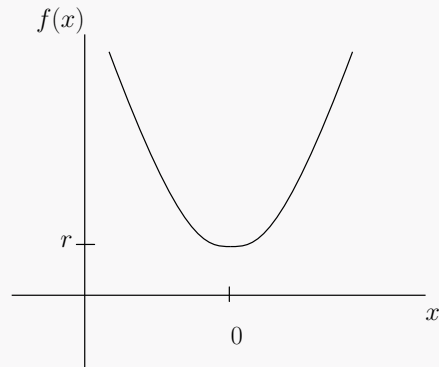
$$\begin{aligned}
 T &= \int dt = \int_0^{2\pi} \frac{dt}{d\theta} d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a \sin \theta} \\
 &= \dots = \frac{2\pi}{\sqrt{\omega^2 - a^2}} = \frac{2\pi}{\sqrt{\omega + a}} \cdot \frac{1}{\sqrt{\omega - a}} \\
 &\approx \frac{2\pi}{\sqrt{2\omega}} \cdot \underbrace{\frac{1}{\sqrt{\omega - a}}}_{\text{blow up as } a \rightarrow \omega}
 \end{aligned}$$





**Remark 14.1.** Bottlenecks/this scaling law are a general feature of saddle-node bifurcations:

$$\text{Normal form: } \frac{dx}{dt} = \dot{x} = r + x^2$$



$$\begin{aligned} T_{\text{bottleneck}} &\approx \int dt \\ &= \int_{-\infty}^{\infty} \frac{dt}{dx} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{r + x^2} dx \\ T_{\text{bottleneck}} &= \frac{\pi}{\sqrt{r}} \end{aligned}$$

blows up like  $\sim r^{-\frac{1}{2}} = \frac{1}{\sqrt{r}}$  as  $r \rightarrow 0$  and  $r > 0$ .

**Example 14.2**

Draw all qualitatively different phase portraits of

$$\dot{\theta} = \omega - a \sin \theta \quad (\text{where } \omega > 0 \text{ fixed})$$

Bifurcation points:  $\dot{\theta} = f(\theta) = 0$ ,  $\frac{\partial f}{\partial \theta} = 0$ . Thus,  $0 = -a \cos \theta \implies a = 0$  or  $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ .

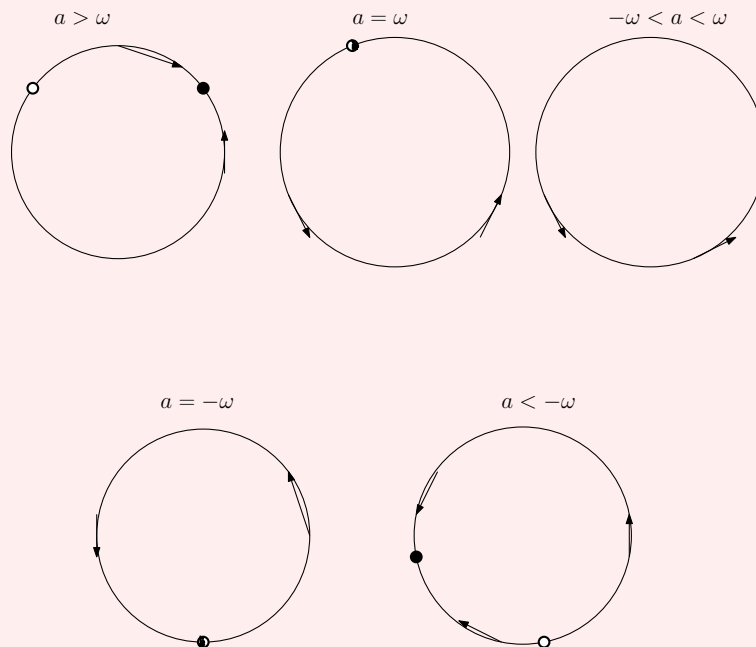
If  $a = 0$ :  $\dot{\theta} = \omega > 0$  (no bifurcation)

If  $\theta = \frac{\pi}{2}$ :  $0 = \dot{\theta} = \omega - a \implies a = \omega$

If  $\theta = \frac{3\pi}{2}$ :  $0 = \dot{\theta} = \omega + a \implies a = -\omega$

Bifurcation points  $(a_*, \theta_*) = (\omega, \frac{\pi}{2}), (-\omega, \frac{3\pi}{2})$ .

$$\dot{\theta} = \omega - a \sin \theta$$

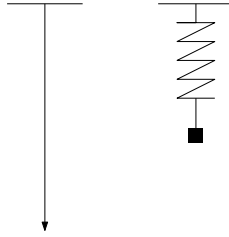
**§14.2 2D Dynamical Systems**

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

Introduction & Linear Systems:

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ i.e. } \dot{x} = Ax$$

Harmonic Oscillator:  $m\ddot{x} + kx = 0$



$$\ddot{x} + \omega^2 x = 0, \omega^2 = \frac{k}{m}$$

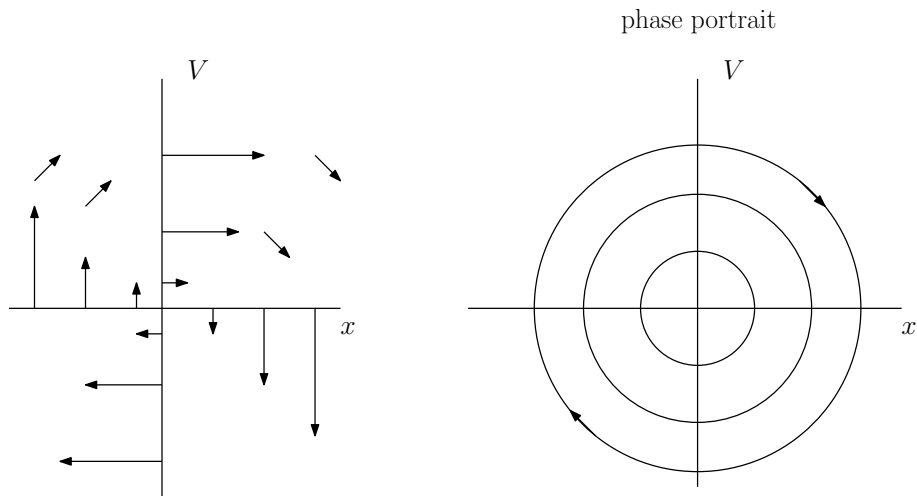
where  $k$  : spring constant and  $m$  : mass,  $x$  : position,  $v$  : velocity.

$$\dot{x} = v$$

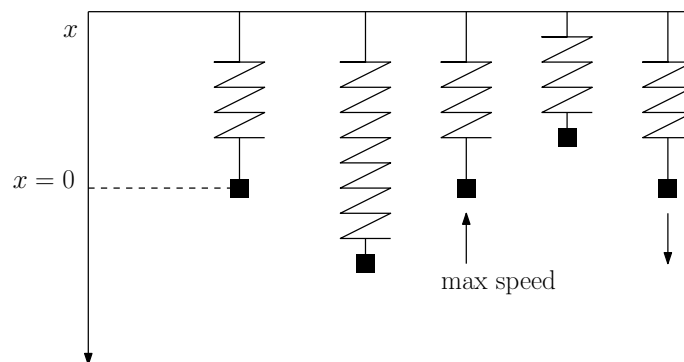
$$\dot{v} = \ddot{x} = -\omega^2 x$$

$$\frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2 x \end{pmatrix}$$

Note: the last matrix defines vector field on phase plane.



Harmonic oscillator:



**Remark 14.3.** Have:

$$\begin{aligned}\frac{d}{dt}(\omega^2 x^2 + v^2) &= 2\omega^2 x\dot{x} + 2v\dot{v} \\ &= 2\omega^2 xv - 2\omega^2 vx = 0 \\ &\implies \omega^2 x^2 + v^2 = \text{const}\end{aligned}$$

$\implies$  trajectories  $\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$  describe as closed orbits i.e. correspond to periodic solutions.

## §15 | Dis 1: Jan 7, 2021

### §15.1 Fixed points and Stability

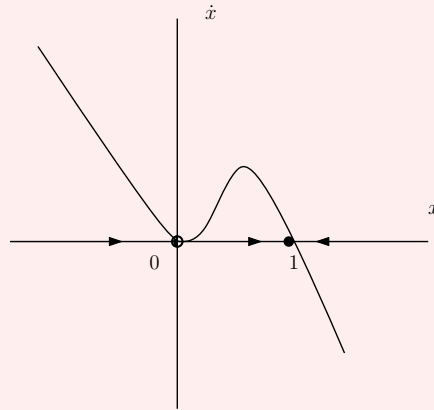
$$\dot{x} = f(x)$$

**Example 15.1**

$$\dot{x} = -x^3 + x^2$$

a) Sketch the vector field, classify the fixed points.

“vector fields” = x-axis with arrows



so:

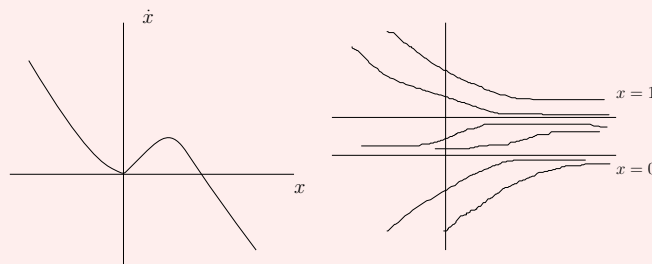
- $\dot{x} > 0 \implies x(t)$  increasing
- $\dot{x} < 0 \implies x(t)$  decreasing

“Fixed point”  $\iff x_*$  s.t.  $f(x_*) = 0 \iff x_*$  s.t. the constant function  $x(t) = x_*$  is a solution.

We have 2 fixed points:

- $x_* = 0$  is semi-stable.
- $x_* = 1$  is stable.

b) Sketch various solutions of  $x(t)$ .

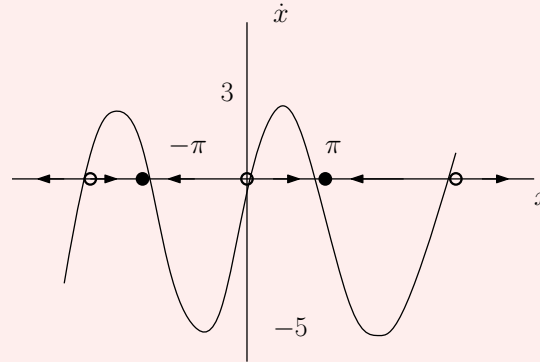


- $\dot{x} = 0$  for  $x = 0, 1 \implies x(t) = 0, 1$  are solutions.
- $\dot{x} > 0$  for  $x < 0 \implies x(t)$  increasing.
- $\dot{x} > 0$  for  $0 < x < 1 \implies x(t)$  increasing.
- $\dot{x} < 0$  for  $x > 1 \implies x(t)$  decreasing.

**Example 15.2**

$$\dot{x} = -1 + 4 \sin x$$

a) Sketch vector field, classify fixed points.

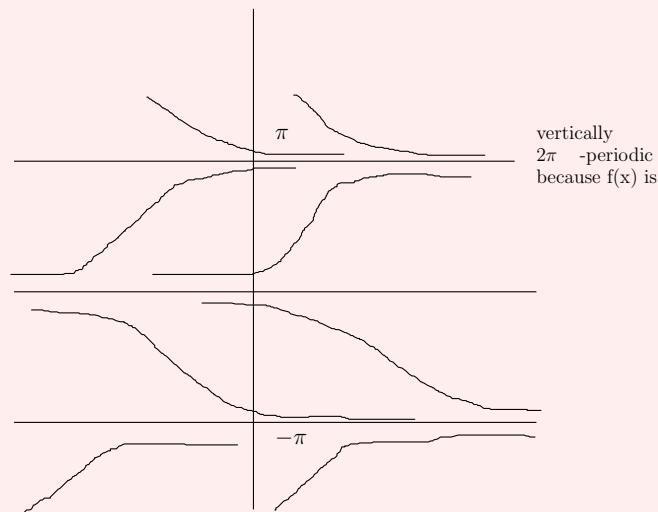


Fixed points:

$$\sin(x_*) = \frac{1}{4}$$

- $x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$  for  $n = 0, \pm 1, \dots$  are unstable.
- $x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$  for  $n = 0, \pm 1, \dots$  are stable.

b) Sketch various solutions  $x(t)$ .

**§15.2 First Order Autonomous System**

$\vec{x} = \vec{f}(\vec{x})$  – first order and autonomous.

**Example 15.3**

A unit mass with displacement  $x(t)$  attached to a spring with spring constant 6 obeys:

$$\ddot{x} = -6x - b(t)\dot{x}$$

where  $b(t) \geq 0$  is the friction coefficient.

a) Show that this can be expressed as a first order autonomous system

$$\begin{aligned} x_1 &= x, & x_2 &= \dot{x} \\ \dot{x}_2 &= \ddot{x} = -6x_1 - b(t)x_2 \\ \vec{x} &:= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \vec{\dot{x}} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -6x_1 - b(x_3)x_2 \end{pmatrix} \end{aligned}$$

where  $x_3 = t \implies \dot{x}_3 = 1$ .

b) In the case  $b(t) = 5$ , find the explicit solution for  $x(0) = x_0, \dot{x}(0) = v_0$ .

$$\ddot{x} = -6x - 5\dot{x} \implies \ddot{x} + 5\dot{x} + 6x = 0$$

Try  $x(t) = e^{kt}$  :

$$\begin{aligned} 0 &= \ddot{x} + 5\dot{x} + 6x = k^2 e^{kt} + 5k e^{kt} + 6e^{kt} \\ &= e^{kt}(k^2 + 5k + 6) \implies k = -3, -2 \end{aligned}$$

Now,  $x(t) = c_1 e^{-3t} + c_2 e^{-2t}, c_1, c_2 \in \mathbb{R}$ . Using the initial conditions, we obtain

$$x(t) = (-2x_0 - v_0)e^{-3t} + (3x_0 + v_0)e^{-2t}$$



# §16 | Dis 2: Jan 14, 2021

## §16.1 Linearization and Potentials

### Example 16.1

$$\dot{x} = -x^3 + x^2$$

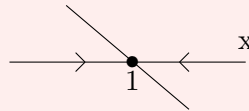
- a) Use linear stability analysis to classify the fixed points. If it fails, use a graphical argument.

Idea: For  $x$  near a fixed point  $x_*$ ,  $\dot{x} = f(x) \approx f(x_*) (= 0) + f'(x_*)(x - x_*) = \dots$

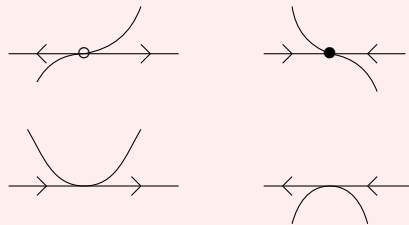
$$f(x) = -x^3 + x^2, \quad f'(x) = -3x^2 + 2x$$

$$0 = f(x_*) = -x_*^2(x_* - 1) \implies x_* = 0, 1$$

- $x_* = 1 : f'(1) = -1 < 0 \implies$  stable



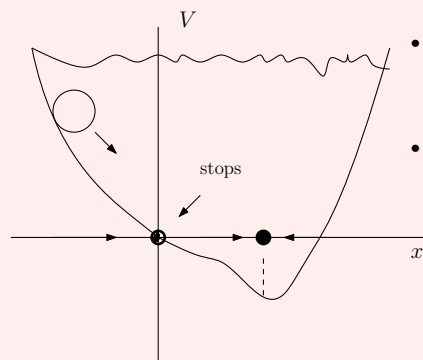
- $x_* = 0 : f'(0) = 0 : \text{inconclusive}$



- b) Find and plot a potential function.

“Potential function”  $\iff V(x)$  s.t.  $\dot{x} = -\frac{dV}{dx}$

$$\dot{x} = -x^3 + x^2 = -V'(x) \implies V(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 + C \text{ (choose 0)}$$



- fixed points
  - $\iff f(x_*) = 0$
  - $\iff V'(x_*) = 0$
  - $\iff$  critical point

- Trajectories move toward decreasing  $V$ , like a ball rolling down the graph of  $V$

**Example 16.2**

$$\dot{x} = 4 \sin x - 1.$$

a) Use linear stability analysis to classify the fixed points.

$$f(x) = 4 \sin x - 1, \quad f'(x) = 4 \cos x$$

Last time: Fixed points are

$$\bullet x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \dots$$

$$f'(x_*) = 4 \cos\left(\sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) > 0$$

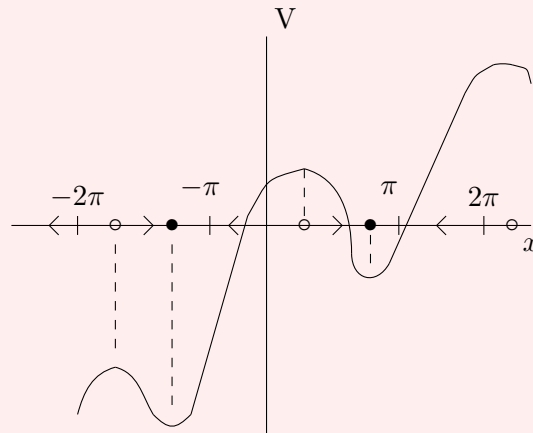
$\implies$  unstable

$$\bullet x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \dots$$

$$f'(x_*) = 4 \cos\left(\pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) < 0$$

$\implies$  Stable.

b) Plot potential  $-1 + 4 \sin x = -V'(x) \implies V(x) = x + 4 \cos x$



## §16.2 Existence of Solutions

**Example 16.3** a) Let  $a > 0$  be a constant. Show that the solution of

$$\begin{cases} \dot{x} = ax^2 \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

$$\begin{aligned} \frac{dx}{dt} = ax^2 &\implies \int \frac{dx}{x^2} = \int a \, dt \\ &\implies -\frac{1}{x} = at + c \\ &\implies x(t) = \frac{1}{c - at} \quad \forall c \in \mathbb{R} \\ x(0) > 0 &\implies c > 0 \implies \lim_{t \rightarrow T} x(t) = +\infty \end{aligned}$$

for some  $T > 0$ . In fact,  $c = \frac{1}{x_0}$  and so  $T = \frac{c}{a} = \frac{1}{ax_0}$ .

b) Let  $0 < \epsilon < 1$  be a constant. Show that the solution of

$$\begin{cases} \dot{x} = x^2 (1 + \epsilon \sin x) \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

Idea:  $\dot{x} \geq ax^2$  for some  $a > 0$ , so our solution grows faster than a function which blows up, and must blow up too.

$$\begin{aligned} |\sin x| \leq 1 &\implies 1 + \epsilon \sin x \geq 1 - \epsilon \\ \implies \dot{x} = x^2 (1 + \epsilon \sin x) &\geq \underbrace{1 - \epsilon}_{>0} x^2 \end{aligned}$$

Let  $x(t)$  be the solution to

$$\begin{cases} \dot{x} = x^2 (1 + \epsilon \sin x) \\ x(0) = x_0 \end{cases}$$

Let  $y(t)$  be the solution to

$$\begin{cases} \dot{x} = (1 - \epsilon)x^2 \\ x(0) = x_0 \end{cases}$$

By part a),  $y(t)$  blows up at some time  $T > 0$ . Since  $x(0) = y(0)$  and  $\dot{x} \geq \dot{y}$ , then  $x(t) \geq y(t)$  for all  $t \geq 0$  (ODE Comparison Lec 4). Therefore,  $x(t)$  must blow up in finite time. In fact, blow up time must be  $\leq T = \frac{1}{(1-\epsilon)x_0}$ .

## §17 | Dis 3: Jan 21, 2021

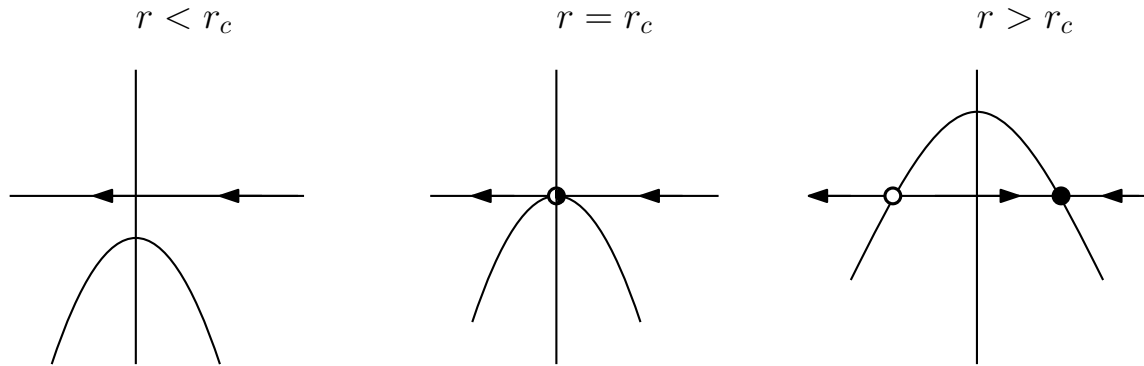
### §17.1 Bifurcations

$\dot{x} = f(x, r)$ ,  $r$  parameter

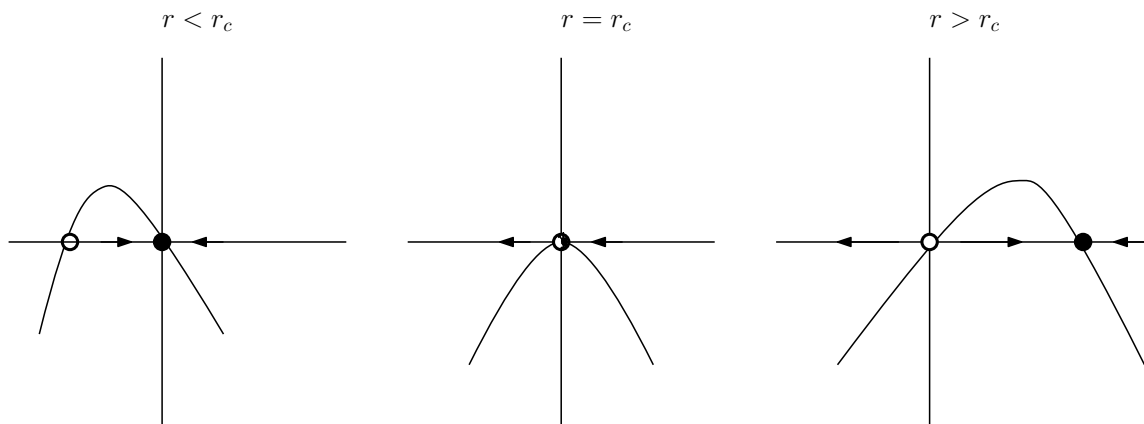
“Bifurcation”  $\iff$  change in the number or stability of fixed points.

There are two types:

- Saddle-node:  $0 \rightarrow 1 \rightarrow 2$  fixed points



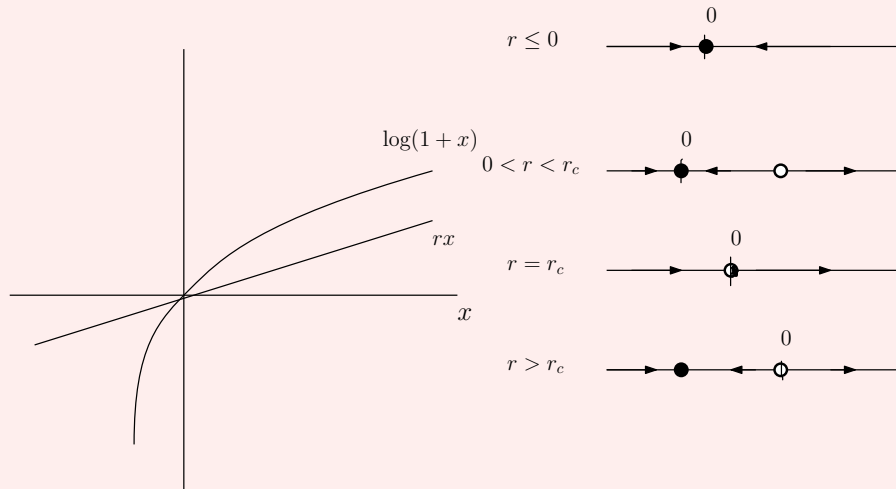
- Transcritical:  $2 \rightarrow 1 \rightarrow 2$  fixed points



**Example 17.1**

$$\dot{x} = rx - \log(1+x)$$

- (a) Sketch all qualitatively different vector fields, sketch bifurcation diagram, find and classify bifurcations.

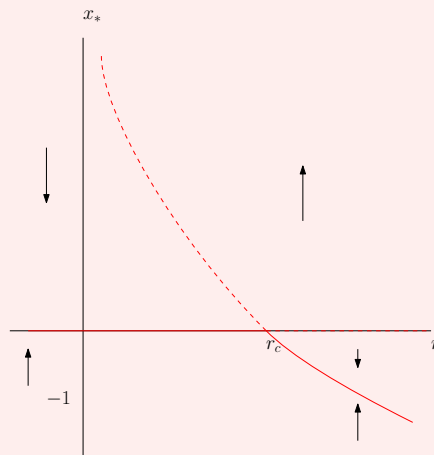


Transcritical bifurcation at  $r = r_c$

“Bifurcation diagram”  $\iff$  Plot of the fixed points  $x_*$  as a function of  $r$ .

$$0 = rx_* - \log(1+x_*)$$

$$r = \frac{1}{x_*} \log(1+x_*) \quad \text{or} \quad x_* = 0$$



- Vertical slices of constant  $r$  are vector fields
- Whole regions have same arrow direction

Bifurcation point:  $(r_c, x_c) = (1, 0)$

**Example 17.2** (Cont'd of example 17.1)

From the above example,

- (b) Show that there is a transcritical bifurcation at  $(r_c, x_c) = (1, 0)$  using normal forms. Taylor expand about  $r = 1, x = 0$

$$\begin{aligned}\dot{x} &= rx - \log(1+x) \\ &= (r-1)x + x - \left(x - \frac{1}{2}x^2 + \mathcal{O}(x^3)\right) \\ &= (r-1)x + \frac{1}{2}x^2 + \mathcal{O}(x^3)\end{aligned}$$

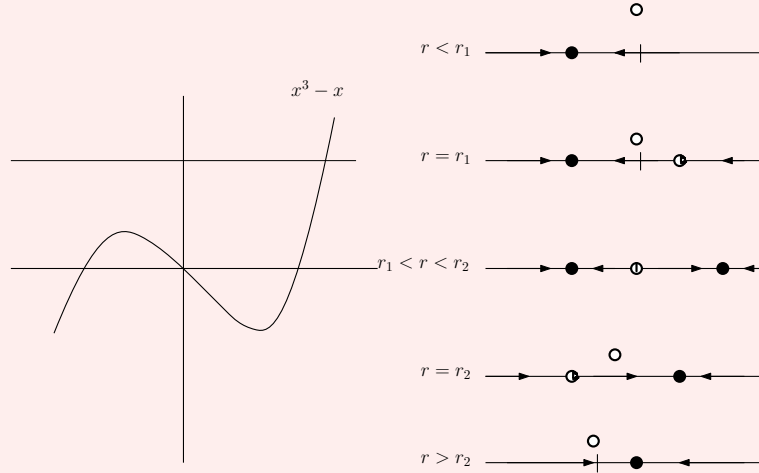
$$\dot{x} = (r-1)x + \frac{1}{2}x^2 + \mathcal{O}(\epsilon^3) \text{ for } |x| < \epsilon, |r-1| < \epsilon$$

This is normal form for transcritical bifurcation.

**Example 17.3**

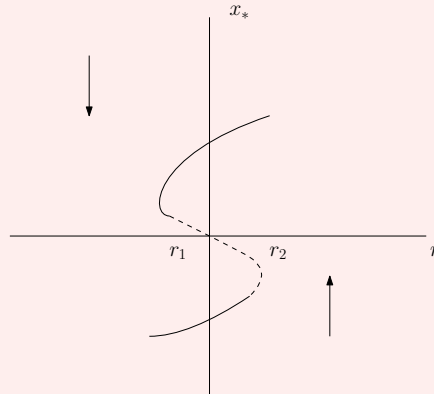
$$\dot{x} = r + x - x^3$$

(a) Sketch all vector field, sketch bifurcation diagram, find and classify bifurcations.



2 saddle node bifurcations at  $r = r_1, r_2$

$$0 = r + x_* - x_*^3 \implies r = x_*^3 - x_*$$



Bifurcation point  $(x_c, r_c)$  satisfies:

- Fixed point:  $0 = f(x_c, r_c) = r_c + x_c - x_c^3$
- $0 = \frac{\partial f}{\partial x}(x_c, r_c) = 1 - 3x_c^2$

$$0 = 1 - 3x_c^2 \implies x_c = \pm \frac{1}{\sqrt{3}}$$

$$0 = r_c + x_c - x_c^3 \implies r_1 = -\frac{2}{3\sqrt{3}}, r_2 = \pm \frac{2}{3\sqrt{3}}$$

$$(r_c, x_c) = \left( \frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), \left( -\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

**Example 17.4** (Cont'd of example 17.3) (b) Show that there is a saddle-node bifurcation at  $(r_c, x_c) = \left(\frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$  using normal forms.

Taylor expand about  $r = \frac{2}{3\sqrt{3}}, x = -\frac{1}{\sqrt{3}}$ .

$$\dot{x} = r + x - x^3, \quad r = \frac{2}{3\sqrt{3}} + \left(r - \frac{2}{3\sqrt{3}}\right)$$

$$x - x^3 = -\frac{2}{3\sqrt{3}} + 0\left(x + \frac{1}{\sqrt{3}}\right) + \frac{1}{2} \cdot \frac{6}{\sqrt{3}}\left(x + \frac{1}{\sqrt{3}}\right)^2 + \mathcal{O}\left(\left(x + \frac{1}{\sqrt{3}}\right)^3\right)$$

Plug these in, get

$$\dot{x} = \left(r - \frac{2}{3\sqrt{3}}\right) + \sqrt{3}\left(x + \frac{1}{\sqrt{3}}\right)^2 + \mathcal{O}(\epsilon^3)$$

for  $\left|r - \frac{2}{3\sqrt{3}}\right| < \epsilon^2, \left|x + \frac{1}{\sqrt{3}}\right| < \epsilon$ . This is normal form for a saddle-node bifurcation ( $\dot{y} = R + y^2$ ).



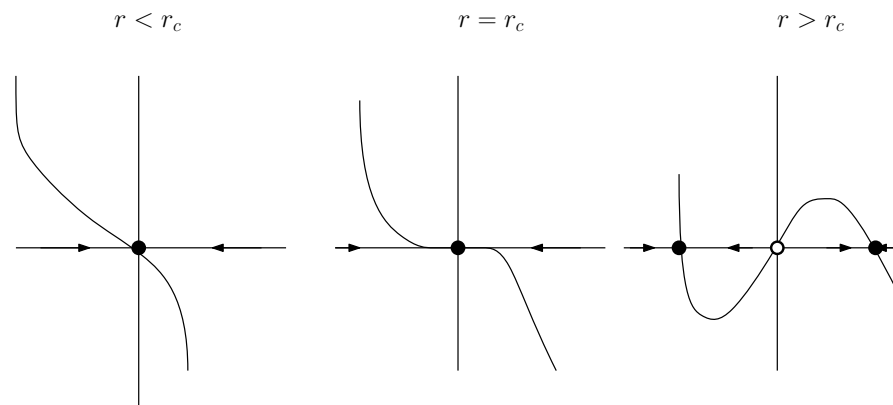
# §18 | Dis 4: Jan 28, 2021

## §18.1 Review of Bifurcations

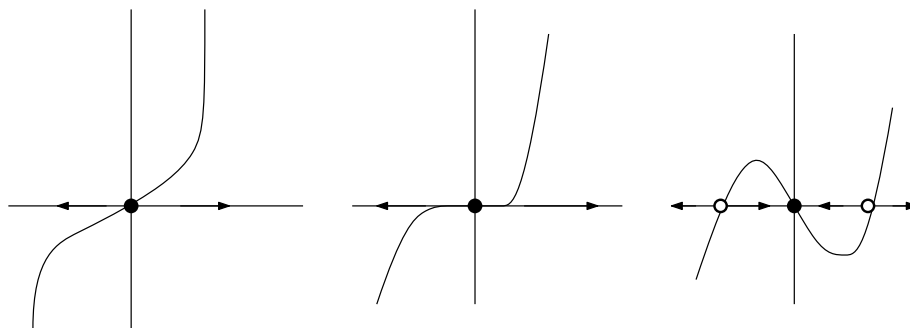
We've seen 3 types of bifurcations so far

- saddle-node:  $0 \rightarrow 1 \rightarrow 2$  fixed points
- transcritical:  $2 \rightarrow 1 \rightarrow 2$  swap stability
- sub-/supercritical pitchfork:  $1 \rightarrow 1 \rightarrow 3$

supercritical



subcritical



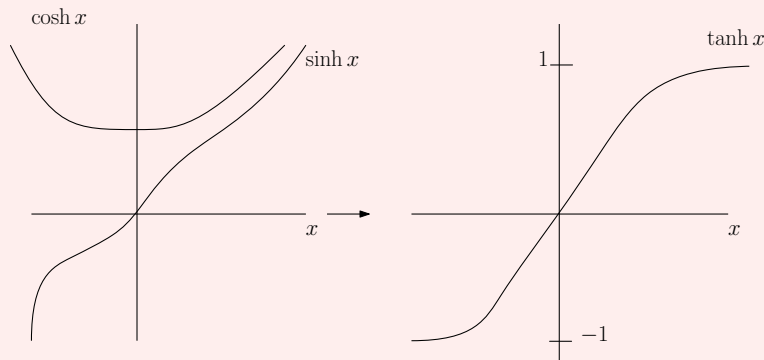
**Example 18.1**

$\dot{x} = r \tanh x - x$ . Sketch all qualitatively different phase portraits, sketch bifurcation diagram, find any classify bifurcations. Recall

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$



$$\dot{x} = r \tanh x - x$$

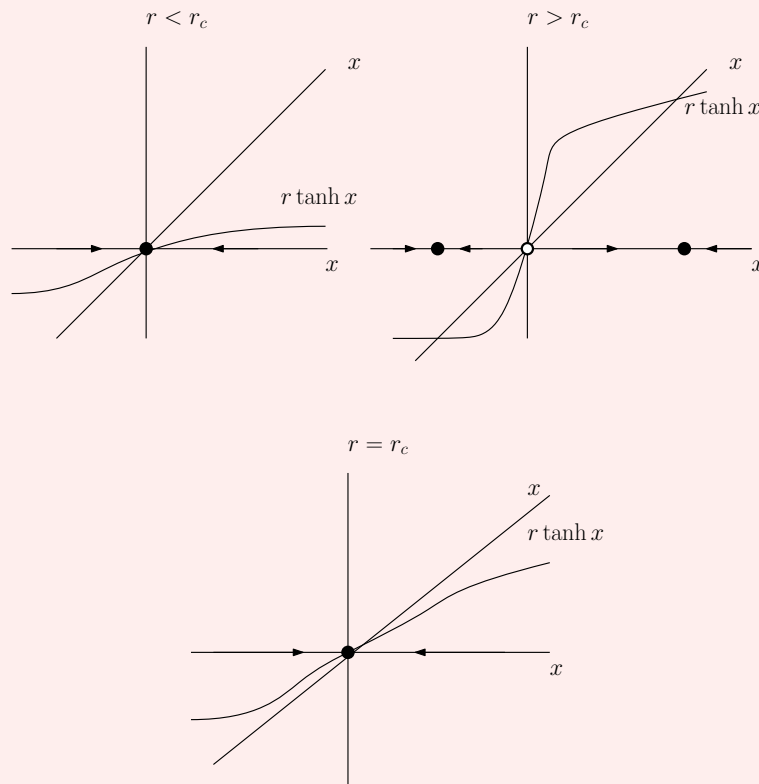
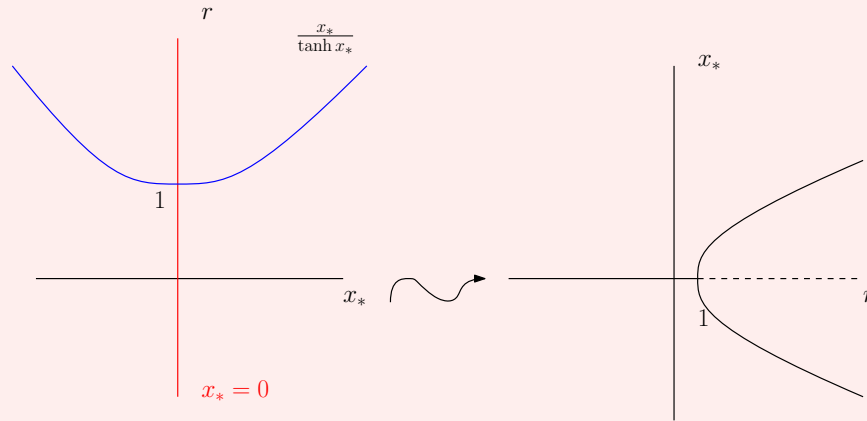


Figure 5: Supercritical pitchfork at  $r = r_c$

**Example 18.2** (Cont'd from above)

$$0 = r \tanh x_* - x_* \implies r = \frac{x_*}{\tanh x_*}, \quad x_* = 0$$

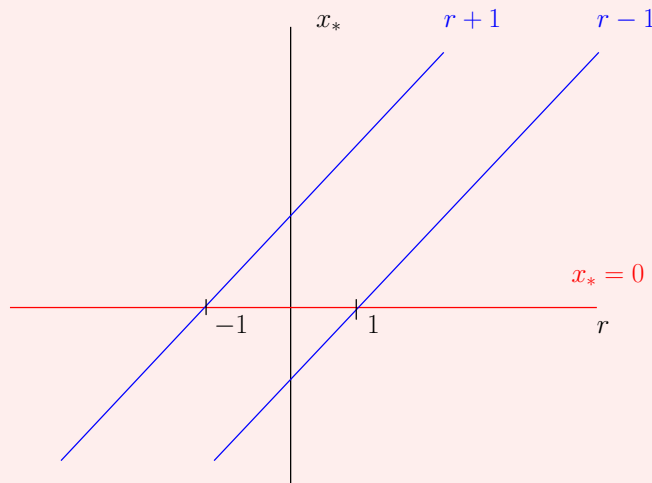


Near  $x = 0$ ,  $\tanh x = \frac{x + \dots}{1 + \dots} = x + \dots \implies \frac{x}{\tanh x} = 1 + \dots$   
 Bifurcation point:  $(r_c, x_c) = (1, 0)$ .

**Example 18.3**

Same for  $\dot{x} = x - x(x - r)^2$

$$0 = x_* (1 - (x_* - r)^2) \implies x_* = 0, \quad x_* = r \pm 1$$



2 transcritical bifurcations at  $r_c = \pm 1, x_c = 0$ .

**Example 18.4** (Cont'd of example 18.3)

$$\dot{x} = x(1 - (x - r)^2) = -x(x - (r + 1))(x - (r - 1))$$

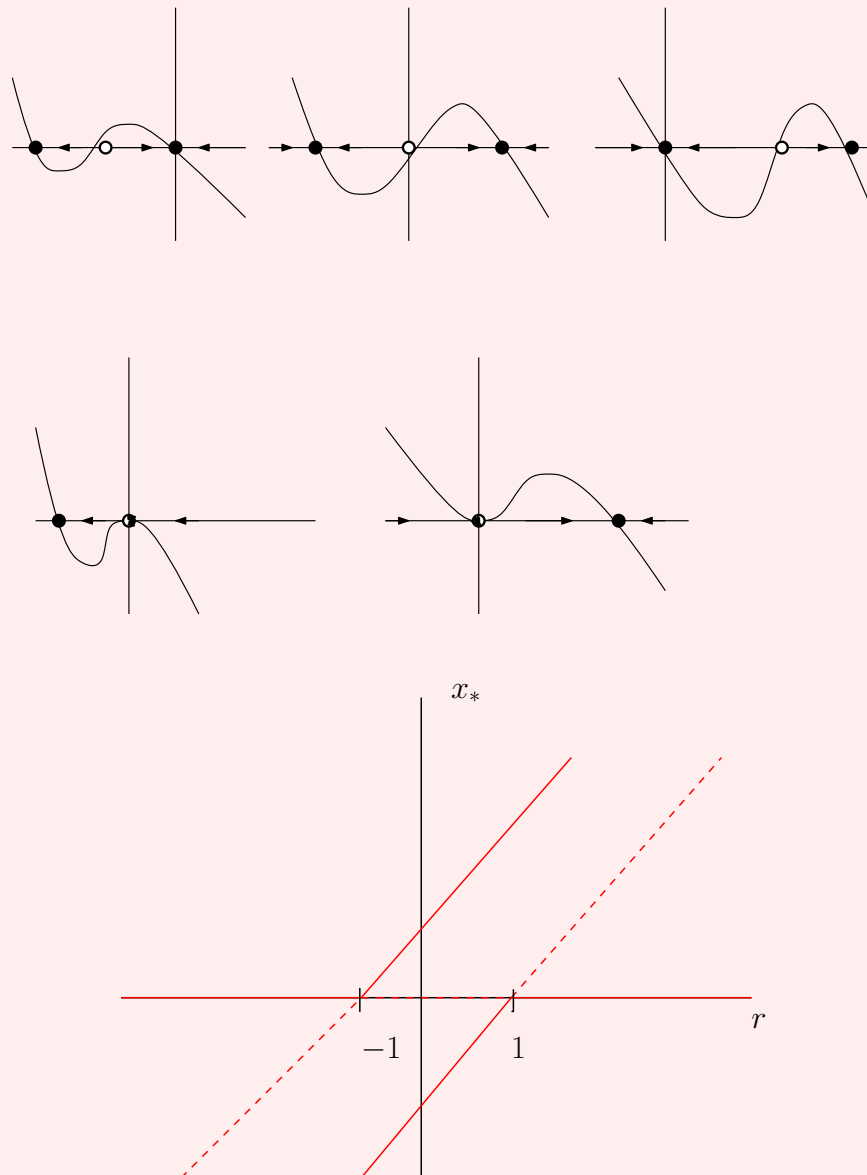
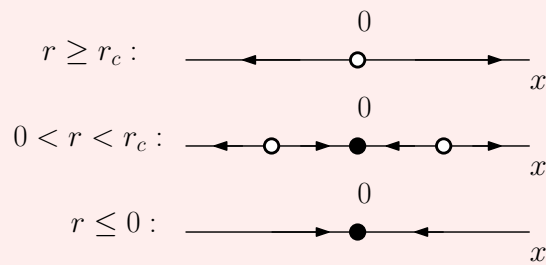
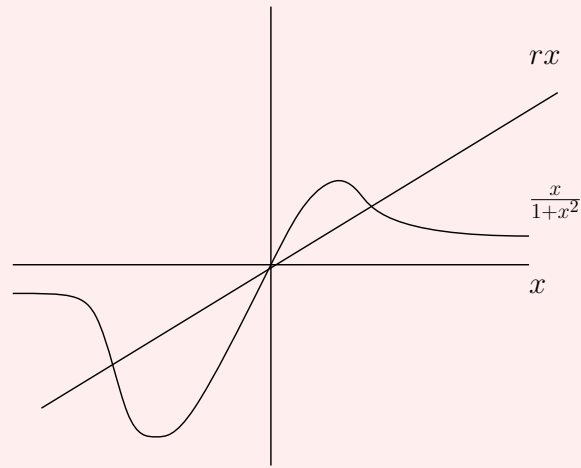


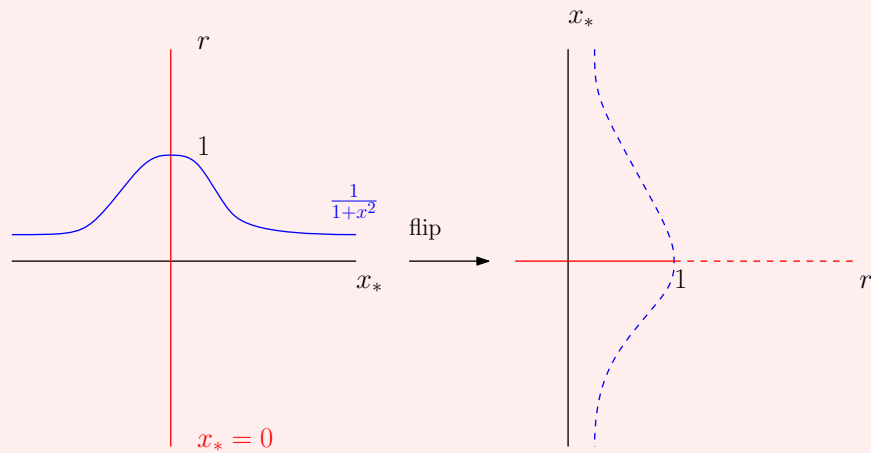
Figure 6: Bifurcation Diagram

**Example 18.5**

Same for  $\dot{x} = rx - \frac{x}{1+x^2}$



$$0 = x_* \left( r - \frac{1}{1+x_*^2} \right) \Rightarrow x_* = 0, \quad r = \frac{1}{1+x_*^2}$$

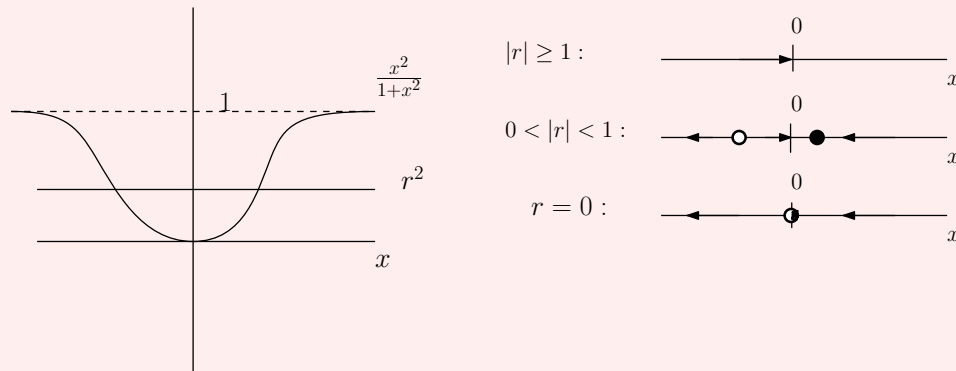


Subcritical pitchfork at  $(r_c, x_c) = (1, 0)$ .

## §18.2 Other Bifurcations

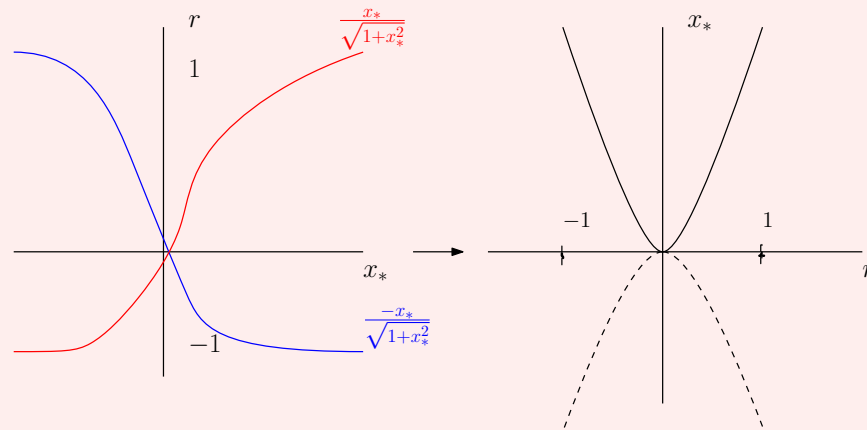
### Example 18.6

$\dot{x} = r^2 - \frac{x^2}{1+x^2}$ . Sketch all qualitatively different phase portraits, sketch bifurcation diagrams, find bifurcation point.



Bifurcation point  $(r_c, x_c) = (0, 0)$  and still satisfies  $f(r_c, x_c) = 0$ ,  $\frac{\partial f}{\partial x}(r_c, x_c) = 0$ . Bifurcation diagram:

$$0 = r^2 - \frac{x_*^2}{1+x_*^2} \implies r = \pm \sqrt{\frac{x_*^2}{1+x_*^2}} = \pm \frac{x_*}{\sqrt{1+x_*^2}}$$



This is not one of our 3 types of bifurcations

- Graphically, it's not transcritical because both fixed points are moving.
- Analytically, we can check that the Taylor expansion at  $(r, x) = (0, 0)$  :

$$f(r, x) = r^2 - x^2 + \mathcal{O}(x^3)$$

doesn't match one of the 3 normal forms we know.

## §19 | Dis 5: Feb 4, 2021

### §19.1 Bifurcations (Cont'd)

#### Example 19.1

$\dot{N} = rN - aN(N - b)^2$  population model,  $r > 0, a > 0, b \in \mathbb{R}$  parameters. Show that this can be rewritten in the dimensionless form  $\frac{dx}{d\tau} = x - x(x - c)^2$ . What are  $x, \tau, c$  in terms of the original quantities?

“dimensionless”  $\iff$  change of variables s.t. new variables have no units

$N$  has units pop (population) and  $\dot{N} = \frac{dN}{dt}$  has units pop/time.

$$\underbrace{\dot{N}}_{\frac{\text{pop}}{\text{time}}} = r \underbrace{N}_{\text{pop}} - a \underbrace{N^3}_{\text{pop}^3} + 2ab \underbrace{N^2}_{\text{pop}^2} - ab^2 \underbrace{N}_{\text{pop}}$$

$$\implies r \sim \frac{1}{\text{time}}, a \sim \frac{1}{\text{pop}^2 \cdot \text{time}}, b \sim \text{pop}$$

Rescale:  $N = nx, t = T\tau$  where  $n, T$  are constant has units and  $\tau$  is new variable with no units.

$$\frac{dN}{dt} = \underbrace{\frac{dN}{dx}}_n \frac{dx}{dt} = n \frac{dx}{d\tau} \underbrace{\frac{d\tau}{dt}}_{\frac{1}{T}} = \frac{n}{T} \frac{dx}{d\tau}$$

$$\frac{n}{T} \frac{dx}{d\tau} = rnx - anx(nx - b)^2$$

$$\frac{dx}{d\tau} = rTx - aTx(nx - b)^2$$

$$= \underbrace{rTx}_{=1} - \underbrace{aTn^2}_{=1} x \left( x - \underbrace{\frac{b}{n}}_{=c} \right)^2$$

$$\implies T = \frac{1}{r}, n = \frac{1}{\sqrt{aT}} = \sqrt{\frac{T}{a}}.$$

Units:  $T \sim \frac{1}{\frac{1}{\text{time}}} = \text{time}, n \sim \sqrt{\frac{\frac{1}{\text{time}}}{\text{pop}^2 \cdot \text{time}}} = \text{pop}$  where  $\tau$  no units and  $x$  no units.

$\implies \frac{dx}{d\tau} = x - x(x - c)^2$  with  $x = \sqrt{\frac{a}{r}}N, \tau = rt, c = b\sqrt{\frac{a}{r}}$  which is defined since  $a, r > 0$ .

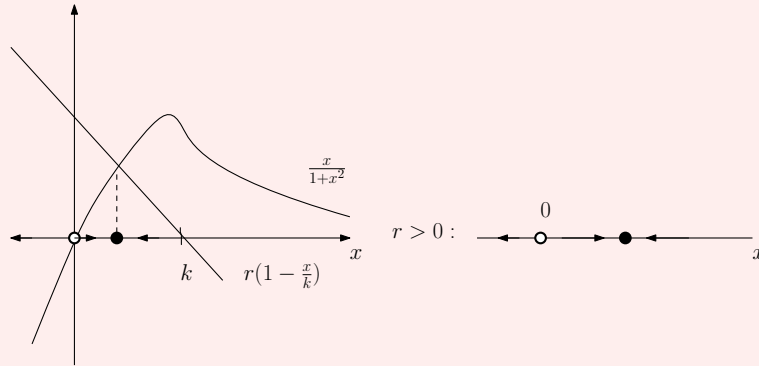
**Example 19.2**

Population model  $\dot{x} = rx \left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$  where  $r, k > 0$ .

- (a) Sketch the 2 qualitatively different bifurcation diagrams  $(r, x_*)$  for  $k \leq k_*$  and  $k > k_*$ .

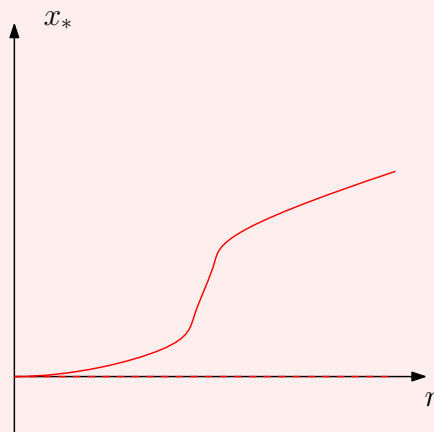
$$\dot{x} = x \left[ r \left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2} \right]$$

$0 < k \leq k_*$  :

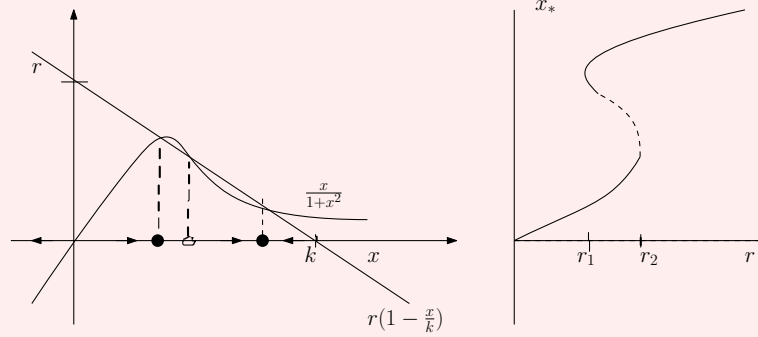


$$0 = x_* \left[ r \left(1 - \frac{x_*}{k}\right) - \frac{x_*}{1+x_*^2} \right]$$

$$\Rightarrow x_* = 0 \text{ or } 0 = r \left(1 - \frac{x_*}{k}\right) - \frac{x_*}{1+x_*^2} \text{ (with 1 solution)}$$





**Example 19.3** (Cont'd of example 19.2) $k > k_*$  :

$x_* = 0$  or  $0 = r \left(1 - \frac{x_*}{k}\right) - \frac{x_*}{1+x_*^2}$  (with 1-3) solutions.

- (b) Sketch regions in  $(k, r)$  plane with qualitatively different vector fields, classify bifurcations on boundaries.

Fixed points:

$$0 = x \left[ r \left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2} \right]$$

$$\Rightarrow x = 0 \text{ (bifurcation)}, \quad r \left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2} \text{ (includes 2 saddle-node bif)}$$
(1)

Tangent:

$$0 = \frac{\partial}{\partial x} \left\{ x \left[ r \left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2} \right] \right\}$$

$$= \underbrace{r \left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2}}_{=0 \text{ by (1)}} \underbrace{x}_{\neq 0} \left[ -\frac{r}{k} - \frac{1}{1+x^2} + \frac{2x^2}{(1+x^2)^2} \right]$$

$$\Rightarrow -\frac{r}{k} = \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$
(2)

$$\Rightarrow \frac{x}{1+x^2} \stackrel{(1)}{=} r + \left(-\frac{r}{k}\right)x \stackrel{(2)}{=} r + \frac{x-x^3}{1+x^2}$$

$$\Rightarrow r = \frac{2x^3}{(1+x^2)^2} \stackrel{(2)}{\Rightarrow} k = \frac{2x^3}{x^2-1}$$

for  $x > 1$  (so  $r, k > 0$ ).

**Example 19.4** (Cont'd of example 19.2)

So

