Math 170E – Intro to Probability

University of California, Los Angeles

Duc Vu

Fall 2020

This is math 170E taught by Professor Nguyen. The formal name of the class is **Introduction to Probability and Statistics 1: Probability**. The textbook used for the class is *Probability & Statistical Interference* 10^{th} by *Hogg, Tanis*. We meet weekly on MWF from 10:00 – 10:50 and on Tue at the same time frame for discussion with our TA, Jason Snyder. You can also find other lecture notes at my github. Let me know through my email if you notice something mathematically wrong/concerning. Thank you!

Contents

List of Definitions

§1 Lec 1: Oct 2, 2020

§1.1 Properties of Probability

Definition 1.1 (Outcome Space) — Consider the outcome of a random experiment, e.g. flipping a coin. The collection of all such outcomes, denoted by $\underbrace{S}_{\omega \text{ in other advanced prob. textbook}}$, is called the outcome space.

- A subset $A \subseteq S$ is called an event.
- If $A_1, A_2, \ldots \subseteq S$ satisfy $A_i \cap A_j = \emptyset$, $i \neq j$ then they are called "disjoint" (mutually exclusive)
- If $A_1, A_2, \ldots, A_n \subseteq$ satisfy $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = S$. Then $\{A_i\}_{i=1\ldots n}$ are called exhaustive(fully comprehensive).

Example 1.2 1. Flip two coins in order. Denote H = head, T = tail.

$$S = \{HH, HT, TH, TT\}$$
$$A = \{HH\} = \{\text{both coins are head}\}$$

$A \subseteq S$ is an event.

$$B = \{HT, TH\}$$

$B \subseteq S$ is another event.

$A \cap B = \emptyset$, they are disjoint.

2. Flip 2 coins at once.

$$S = \{HH, HT, TT\}$$
$$A = \{\text{one head, one tail}\}$$
$$A = \{HT\}\text{, is an event.}$$

Probability – A heuristic intro:
Consider an experiment and repeat n times. Let $N(A)$ = number of times A occurs. The ratio $\frac{N(A)}{n}$ is called the relative frequency of A in n repetitions of the experiment.

$$0 \leq \frac{N(A)}{n} \leq 1$$

As $n \to \infty$,

$$\frac{N(A)}{n} \to p \in [0, 1]$$

This p is called the prob. that event A occurs.

Example 1.3

(a) Flip a coin

$$S = \{H, T\}$$
$$A = \{H\}$$

What is $P(A)$?

(b) Sometimes, we can also assign prob. based on the nature of the event Pick a random point in the unit circle.

$$A = \left\{\text{chosen point} \in 1^{\text{st}}\text{quadrant}\right\}$$

$P(A) = \frac{\text{Area of first quadrant}}{\text{Area of unit circle}} = \frac{1}{4}$

(c) Pick a number randomly from $\{0, 1, \ldots, 9\}$, $B = \{2 \text{ is picked}\}$

$$P(B) = \frac{1}{10}$$

Table 1: From example 1.3 (a)

n	$N(A)$	$\frac{N(A)}{n}$
50	37	.74
500	333	.66

It is safe to assign $P(A) = 0.66$

Definition 1.4 (Probability) — Given an outcome space S, the probability of an event A $A \subseteq S$, is a number satisfying:

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $A_1, \ldots, A_n \subseteq S$ are disjoint events, i.e. $A_i \cap A_j = \emptyset, i \neq j$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) = P(A_1) + \ldots + P(A_n)$$

More generally, if $A_1, \ldots, A_n, \ldots \subseteq S$ are disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Theorem 1.5 1. Denote A' to be the complement of A in S, i.e.

$$A' \cup A = S$$
$$A' \cap A = \emptyset$$

Then

$$P(A') = 1 - P(A)$$

2. $P(\emptyset) = 0$
3. If $A \leq B$ then $P(A) \leq P(B)$
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
5. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

<u>*Note*</u>: The pattern here is add the prob. of odd event(s) and substract the prob. of even events.(for prop (4) and (5) of theorem 1.5).

Proof.

$$P(A') = 1 - P(A)$$

Since $A' \cap A = \emptyset$ (by def of A'). By property (c),

$$P(\underbrace{A' \cup A}_{S}) = P(A') + P(A)$$
$$\underbrace{P(S)}_{1\text{(by prop.(b))}} = P(A') + P(A)$$

Thus,

$$P(A') = 1 - P(A)$$

§2 Lec 2: Oct 5, 2020

Cont'd of Lec 1
(2)

$$P(\emptyset) = 1 - P(S)$$
$$= 1 - 1$$
$$= 0$$

(3)

$$P(A) \leq P(B)$$

$B \setminus A$ is the set s.t.

$$A \cup (B \setminus A) = B$$
$$A \cap (B \setminus A) = \emptyset$$
$$\text{something here}$$

implying

$$P(A) \leq P(B)$$

(4)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(5) Textbook Section 1.1. □

Definition 2.1 ("Equally Likely") — Suppose $S = \{e_1, \ldots, e_m\}$ where each e_i is a possible outcome. Denote $n(s) =$ number of outcomes $= m$. If each e_i has the same prob. of occurring, then they are called equally likely. In particular,

$$P(e_i) = \frac{1}{n(s)} = \frac{1}{m}$$

Moreover, if $A \subseteq S$ is an event s.t. $n(A) = k$. Then,

$$P(A) = \frac{n(A)}{n(s)} = \frac{k}{m}$$

Example 2.2

Draw one card from a deck of 52 cards.

$$P(\text{each card is drawn}) = \frac{1}{52}$$

$A = \{\text{a king is drawn}\}$, so $n(A) = 4$. Thus,

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}$$

§2.1 Method of Enumeration

Multiplication Principle:

Suppose an experiment E_1 has n_1 outcomes

- For each outcome from E_1, a 2nd experiment E_2 has n_2 outcomes. Then the composite E_1E_2 has $n_1 \cdot n_2$ outcomes.

Permutation of size n:

Definition 2.3 (Permutation of n objects) — Suppose there are n positions to be filled by n persons. One such arrangement is called a permutation of size n.
FACT: the total number of different such arrangements is given by "$n! = 1 \cdot 2 \cdot 3 \cdot \ldots n$"

Proof.

- $E_1 =$ fill the 1st position from n persons $\implies$ n outcomes for E_1.

- E_2 = fill the 2^{nd} pos. from $n-1$ persons left $\implies n-1$ outcomes for E_2

  $\vdots$

- E_n = fill the n^{th} pos. from 1 person left $\implies$ 1 outcome for E_n

- One arrangement $= E_1 E_2 \dots E_n$
  Thus, total number of arrangements is $n!$. □

Permutation/Combination of n objects taken k:

Definition 2.4 (Permutation/Combination of size n taken k) — Given $k \le n$ and suppose there are n objects. If k objects are taken from n with/without order, then such a selection is called permutation/combination of size n taken k.

Note: "Permutation of size n" = "permutation of size n taken n".

Fact 2.1. 1. The total number of permutation n taken k (order is important here) is denoted by ${}^nP_k$ is given by

$$ {}^nP_k = \frac{n!}{(n-k)!} $$

2. The total numbers of combination of n taken k, denoted by ${}^nC_k$ or $\binom{n}{k}$ is given by

$$ {}^nC_k = \binom{n}{k} = \frac{n!}{(n-k)!k!} $$

Proof. E_1 = fill 1^{st} pos. from n $\implies n$ for E_1

$\vdots$

E_k = fill k^{th} pos. from $n-k+1$ persons left. Thus,

$$ permk = n \cdot \ldots \cdot (n-k+1) $$

(2) Combination of n taken k :
Start with ${}_nP_k$ as follow:

- E_1 = take k from n at once, outcome $={}_n C_k = \binom{n}{k}$

- E_2 = permute k, outcomes $= k!$. Thus,

$$ {}^nP_k = \binom{n}{k} \cdot k! $$

implying

$$ \binom{n}{k} = \frac{{}^nP_k}{k!} = \frac{n!}{(n-k)!k!} $$

□

Practice 1: https://ccle.ucla.edu/pluginfile.php/3766550/mod_resource/content/1/Practice%201.pdf
1. Consider $S = \{1, \dots, 8\}$
a)

- E_1 = filling 1^{st} pos $\implies$ 8 choices.
- Same for $E_2 \implies$ 8 choices.
- Likewise, E_3 has 8 choices.

Thus, the number of 3 digit numbers can be formed is 8^3
b) "3 distinct digit numbers" = "permutation of size 8 taken 3"
Thus, total such numbers is ${}_8P_3 = \frac{8!}{5!} = 8 \cdot 7 \cdot 6$
c) Considering subset where order is not taken into account
Combination of size 8 taken 3. Thus, the answer is

$$\binom{8}{3} = \frac{8!}{3!5!}$$

d) 3 digit numbers and divisible by 5

- E_1 = choose 5 for the 3^{rd} pos, so 1 choice.
- E_2 = 8 choices
- E_3 = 8 choices

Thus, the total of choices is $8 \cdot 8 = 64$.
e) 4 element subsets of S that has one even digit.

- E_1 = choose one even digit from S, so 4 choices (2,4,6,8).
- E_2 = choose 3 digits from $\{1, 3, 5, 7\}$ without order, so $\binom{4}{3}$

Thus, total = $E_1 \cdot E_2 = 4 \cdot \binom{4}{3}$.
e') What if "at least one even digit" instead of "exactly one even"?

1. Total = exactly "one even" + "two even" + "three even" + "four even"
2. Total = "4-element subset" - "4-element subset with no even digit"

§3 Lec 3: Oct 7, 2020

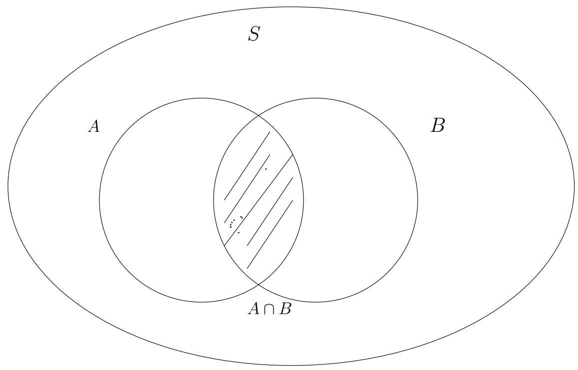
§3.1 Conditional Probability

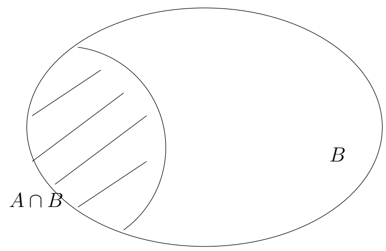
Definition 3.1 (Conditional Probability) — Let $A, B \subseteq S$ be two events. The conditional prob. of A, given that B has occurred with $P(B) > 0$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

A heuristic explanation: $A \cap B$: "the portion in B that A occurs"

$$P(A|B) = \frac{\text{"area of A in B"}}{\text{"area of B"}}$$





Example 3.2

Suppose my family has two kids. Given that there is at least a boy, what is the prob. my family has two boys?

$$S = \{\text{bb,bg,gb,gg}\}$$

Now, let $B = \{$ at least a boy$\}$. So we only look at the first three outcomes from S (B). Define $A = \{$ two boys$\}$

$$A \cap B = \{\text{bb}\}$$

Note $A = A \cap B$ since $A \subseteq B$. Thus,

$$P(A|B) = \frac{P(A \cap B}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

<u>*Note*</u>: We can also consider the alternative outcome space without order as follows

$$S = \left\{(b,b) - -\frac{1}{4}, (b,g) - -\frac{1}{2}, (g,g) - -\frac{1}{4}\right\}$$

Fact 3.1. $P(A|B)$ satisfies basic properties of probability:

- $P(A|B) \geq 0$
- $P(B|B) = 1$

  Moreover, if $B \leq C$ then
$$P(C|B) = 1$$
- If $A_1, \ldots, A_n \ldots$ are disjoint events,
$$P(\bigcup_{k=1}^{\infty} A_k|B) = \sum_{k=1}^{\infty} P(A_k|B)$$

Proof. (a) $P(A|B) = \frac{P(A\cap B)}{P(B)} \geq 0$
(b) $P(B|B) = \frac{P(B\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$
If $B \subseteq C$ then $B \cap C = B$

$$P(C|B) = \frac{P(B\cap C)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

$B \subseteq C$ means "if B occurs then C must occur".
(c) $P(\bigcup_{\infty}^{k=1} A_k|B) = \frac{P(\bigcup_{\infty}^{k=1} A_k \cap B}{P(B)}$. By distributive law,

$$\begin{aligned} &= \frac{P(\bigcup_{\infty}^{k=1}(A_k \cap B))}{P(B)} \\ &= \frac{\sum_{k=1}^{\infty} P(A_k \cap B}{P(B)} \\ &= \sum_{k=1}^{\infty} P(A_k|B) \end{aligned}$$

□

INSERT: PRACTICE 1 #3 here

Theorem 3.3
1. $P(A\cap B) = P(A|B)\cdot P(B)$ given that $P(B) > 0$
2. $P(A\cap B\cap C) = P(A)\cdot P(B|A)\cdot P(C|A\cap B)$ given $P(A), P(A\cap B) > 0$.

Proof.
1. By defn of cond. prob.

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

implying

$$P(B)P(A|B) = P(A\cap B)$$

2. $P(A\cap B\cap C) = P(C\cap(A\cap B)$. By part 1,

$$\begin{aligned} &= P(C|A\cap B)P(A\cap B)P(A\cap B) \\ &= P(C|A\cap B)P(B|A)P(A) \end{aligned}$$

□

Practice 3.1. The url: `https://ccle.ucla.edu/pluginfile.php/3776692/mod_resource/content/0/Practice%202.pdf`
INSERT: Look at the online notes

§4 Lec 4: Oct 9, 2020

Cont'd (Practice)
3)

$$A = \{\text{spade}\} \quad B = \{\text{heart}\} \quad C = \{\text{diamond}\} \quad D = \{\text{club}\}$$

$P = (A \cap B \cap C \cap D = ?$ So,

$$= P(A)P(B|A)P(C|A \cap B)P(D|A \cap B \cap C)$$

(from problem 2 in practice 2)

- $P(A) = \frac{13}{52}$
- $P(B|A) =$, now restricted to outcome space {51 cards in cluding 13 hearts} $B|A =$ { dealing a heart}. Thus,
  $$P(B|A) = \frac{13}{51}$$
- Similarly,
  $$P(C|A \cap B) = \frac{13}{50}$$
  (13 diamond from 50 cards left)
- $P(D|A \cap B \cap C) = \frac{13}{49}$ (13 clubs from 49 cards left).

Hence,

$$P(A \cap B \cap C \cap D) = \frac{13}{52}\frac{13}{51}\frac{13}{50}\frac{13}{49}$$

§4.1 Independent Events

Example 4.1

Flip a fair coin twice

$$\begin{aligned} S &= \{\text{ HH, HT , TH, TT}\} \\ A &= \left\{1^{\mathrm{st}H}\right\} \\ B &= \left\{2^{\mathrm{nd}}T\right\} \\ C &= \{\mathrm{TT}\} \end{aligned}$$

$C \subseteq B$ "2 tails" $\implies$ "2nd is T". i.e., if C occurs then B must have occurred. Thus,

$$\begin{aligned} P(B|C) &= 1 \\ P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{\frac{1}{4}}{\frac{1}{2}} \\ &= \frac{1}{2} \\ P(A) &= \frac{1}{2} \end{aligned}$$

Thus, $P(A|B) = P(A)$, i.e., B occurring does not impact the occurrence of A.

Note also that

$$\frac{P(A \cap B)}{P(B)} = P(A|B) = P(A)$$

implying

$$P(A \cap B) = P(A)P(B)$$

Definition 4.2 (Independent Events) — Given two events A, B which are called independents iff

$$P(A \cap B) = P(A)P(B)$$

Theorem 4.3

The following are equivalent

- A, B are independent
- $P(A|B) = P(B)$, provided $P(B) > 0$
- $P(B|A) = P(B)$, provided $P(A) > 0$

Proof. Left as an excercise. □

Theorem 4.4 1. If $P(A) = 0$ then A is independent with any event.

2. If A and B are independent then so are the following pairs:

$$A, B' \quad A', B \quad A', B'$$

Proof. 1. Let B an arbitrary event, we need to show $P(A \cap B) = P(A)P(B)$. Since $P(A) = 0$, $P(A)P(B) = 0$.

$$A \cap B \subseteq A$$

imply

$$0 \leq P(A \cap B) \leq P(A) = 0$$

thus $P(A \cap B) = 0$.

2. Textbook(section 1.5)

$\square$

Practice 4.1. Practice 2 – Problem 4:
Let's consider C and D first

$$\begin{aligned} D &= \{ \text{ sum of two rolls } = 12\} \\ &= \{(6,6)\} \end{aligned}$$

Thus, $D \subsetneq C = \{\text{first roll is } 6\}$. Hence, C and D are dependent.
A v.s. B

$$\begin{aligned} P(A) &= \frac{5}{6} \\ B &= \{ \text{ sum is even}\} \\ &= \{ \text{ first and second roll are even}\} \cup \{\text{first and second roll are odd}\} \\ P(B) &= P(\text{first even})P(\text{second even}) + P(\text{first odd})P(\text{second odd}) \\ &= \frac{3}{6}\frac{3}{6} + \frac{3}{6}\frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

Now, consider $A \cap B = \left\{1^{\text{st}} \neq 3,\ \text{sum is even}\right\}$. So,

$$\begin{aligned} A \cap B &= \left\{1^{\text{st}} \neq 3, 1^{\text{st}} \text{ odd}, 2^{\text{nd}} \text{ odd}\right\} \cup \left\{1^{\text{st}} \neq 3, 1^{\text{st}} \text{ even}, 2^{\text{nd}} \text{ even}\right\} \\ P(A \cap B) &= P(1^{\text{st}} \neq, 1^{\text{st}} \text{ odd})P(2^{\text{nd}} \text{ odd}) + P(1^{\text{st}} \neq 3, 1^{\text{st}} \text{ even})P(2^{\text{nd}} \text{ even}) \\ &= \frac{2}{6}\frac{3}{6} + \frac{3}{6}\frac{3}{6} \\ &= \frac{5}{12} \end{aligned}$$

Since $P(A \cap B) = \frac{5}{12} = \frac{5}{6}\frac{1}{2} = P(A)P(B)$, A and B are independent.

§5 Lec 5: Oct 12, 2020

§5.1 Independent Events (cont'd)

Definition 5.1 (Mutually Independent Events) — A, B, C are called "mutually independent" if followings hold:

- pairwise independent

$$P(A \cap B) = P(A)P(B) \quad P(B \cap C) = P(B)P(C) \quad P(A \cap C) = P(A)P(C)$$

- "triple" wise independent, i.e.,

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Note:analogous defn holds for $A_1, \ldots, A_n, \ldots$ in which any pairs, triple, quadruple and so on must satisfy the similarly multiplication rules. Usually, the term "mutually" is dropped but it is understood that "independence" means "mutually independence".

> **Remark 5.2.** In general, pairwise independence does not imply triple-wise independence.

Practice 5.1. 2 – Problem 5:

$$A = \{1, 2\}, \quad B = \{1, 3\}, \quad C = \{1, 4\}$$

$P(A) = \frac{2}{4} = P(B) = P(C)$

$$A \cap B = \{1\} = B \cap C = A \cap C$$
$$P(A \cap B) = P(B \cap C) = P(C \cap A) = \frac{1}{4}$$

Thus,

$$P(A \cap B) = \frac{1}{4} = P(A)P(B)$$

Same for B, C and A, C – so pairwise independent.
Triple:

$$A \cap B \cap C = \{1\}$$

$P(A \cap B \cap C) = \frac{1}{4}$; on the other hand, $P(A)P(B)P(C) = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$. They are not equal! Therefore, A, B, C are not mutually independent.

§5.2 Bayes's Theorem

Definition 5.3 (Partition of Outcome Space) — The events $B_1, \ldots, B_n$ (n may be finite or ∞) are called a partition of the outcome space S if followings hold

- disjoint: $B_i \cap B_k = \emptyset, i \neq k$
- exhausted: $\bigcup_n^{i=1} B_i = S$

then,

$$P(B_1) + \ldots + P(B_n) = P(S) = 1$$

Theorem 5.4 (Law of total Probability)
Suppose $B_1, \dots, B_n$ is a partition of S with $P(B_i) > 0$ for $i = 1, \dots, n$. If A is an event in S, then

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

where $P(B_i)$ is called the prior probability.

Proof. (sketch)

$$\begin{aligned} P(A) &= P(\bigcup_{n}^{i=1}(A \cap B_i)) \\ &= \sum_{i=1}^{n} P(A \cap B_i) \\ &= \sum_{i=1}^{n} P(A|B_i)P(B_i) \end{aligned}$$

□

Practice 5.2. 3 – problem 1:

$$\begin{aligned} P(I) &= .35 \\ P(II) &= .25 \\ P(III) &= .4 \end{aligned}$$

$A = \{$ a spring is defective$\}$, $\quad P(A) =$? We know

$$\begin{aligned} P(A|I) &= .02 \\ P(A|II) &= .01 \\ P(A|III) &= .03 \end{aligned}$$

By law of total prob:

$$\begin{aligned} P(A) &= P(A|I)P(I) + P(A|II)P(II) + P(A|III)P(III) \\ &= 0.0215 \end{aligned}$$

Theorem 5.5 (Bayes's Theorem)
Suppose $\{B_i\}_{i=1,\dots,n}$ is a partition of S with $P(B_i) > 0$. If A with $P(A) > 0$, then for all $i = 1, \dots, n$

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{k=1}^{n} P(A|B_k)P(B_k)}$$

where $P(B_i|A)$ is called posterior probability.

Proof.

$$\begin{aligned} P(B_i|A) &= \frac{P(B_i \cap A)}{P(A)} \\ &= \frac{P(A \cap B_i)}{P(A)} \\ &= \frac{P(A|B_i)P(B_i)}{P(A)} \\ &= \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + \ldots + P(A|B_n)P(B_n)} \end{aligned}$$

□

Practice 5.3. 3 – problem 2: $A = \{$ person has disease $\}, \quad P(A) = .005.$

$$\begin{aligned} + &= \{\text{test } +\} \\ - &= \{\text{ test } -\} \\ P(+|A) &= .99 \\ P(\underbrace{+|A'}_{\text{false positive}}) &= .03 \\ P(A|+) &= ? \end{aligned}$$

By Bayes's Theorem:

$$\begin{aligned} P(A|+) &= \frac{P(+|A)P(A)}{P(+|A)P(A) + P(+|A')P(A')} \\ &= \frac{(.99)(.005)}{(.99)(.005) + (.03)(.995)} \end{aligned}$$

$\{A, A'\}$ is a partition of S.

§6 Lec 6: Oct 14, 2020

Practice 6.1. 3 – Problem 3: Trial: know at least 1 girl

$$P(\text{GG}|\text{at least a girl}) = \frac{1}{3}$$

However, the above approach is not correct.
Intuition: The moment the girl opens the door, the first child's gender is determined – which makes the other kid's gender is now independent of the girl. Thus, $P(\text{other kid is girl}) = \frac{1}{2}$.
Correct approach:

$$\begin{aligned} A &= \{\text{ a girl opens the door}\} \\ P(GG|A) &= ? \end{aligned}$$

- $P(A|GG) = 1$
- $P(A|BB) = 0$
- $P(A|GB) = \frac{1}{2}$

- $P(A|BG) = \frac{1}{2}$

By Bayes' Theorem

$$P(GG|A) = \frac{P(A|GG)P(GG)}{P(A|GG)P(GG) + P(A|BB)P(BB) + P(A|BG)P(BG) + P(A|GB)P(GB)} = \frac{1}{2}$$

§6.1 Random Variables with Discrete Type

Example 6.1

Flip a coin

$$S = \{\text{H, T}\}$$

Define

$$X : S \to \mathbb{R}$$

$$\triangle \mapsto X(s) \in \mathbb{R}$$

s.t. $X(H) = 0, \quad X(T) = 1$

$$\text{H} \xrightarrow{X} 0$$

$$\text{T} \longrightarrow 1$$

The function X is called a random variable (RV). Since S is discrete space, X is called a RV of discrete-type.

Definition 6.2 (Random Variable) — Given an outcome space S, a function X that assigns $X(s) = x \in \mathbb{R}$ for each $s \in S$ is called a random variable.
The space(range) of X is the collection of real numbers, denoted by S_x,

$$S_x = \{x \in \mathbb{R} : \quad \exists s \in S, X(s) = x\}$$

S_x is also called the "support" of X.
When the outcome space S is discrete, then X is called a discrete random variable.

Example above:

$$S_x = \{0, 1\}$$

Note: the space of X is denoted by S in the textbook. Here we will use S_x.

Remark 6.3. Under the above definition, for $x \in S_x$,

$$P(X = x) = P(\{s \in S : X(s) = x\})$$

Example 6.4

Roll a fair dice

$$\begin{aligned} S &= \{1, 2, \ldots, 6\} \\ X &: S \to \mathbb{R} \\ s &\mapsto X(s) = x \\ S_x &= \{1, 2, \ldots, 6\}\,(= S) \end{aligned}$$

For each $k \in S_x$,

$$P(X = k) = P(\{k\}) = \frac{1}{6}$$

Also,

$$\sum_{k \in S_x} P(X = k) = \sum_{k=1}^{6} \frac{1}{6} = 1$$

Definition 6.5 (Probability Mass Function) — The probability mass function (pmf) $f(x)$ of a discrete random variable X is a function satisfying the followings:

- $f(x) > 0, \quad x \in S_x$.
- $\sum_{x \in S_x} f(x) = 1$.
- If $A \subseteq S_x$,

$$P(X \in A) = \sum_{x \in A} f(x)$$

Note: if $x \notin S_x$, then we assign $f(x) = 0 (P(X = x) = 0)$.

Example 6.6 (above)

the pmf of X is given by $f(k) = \frac{1}{6}$ for $k = 1, \ldots, 6$

$$\begin{aligned} A = \{1, 2, 3\} &= \text{“}X < 4\text{”} \\ A &\subseteq S_x \\ P(X \in A) = \sum_{k \in A} f(k) &= \sum_{k=1}^{3} \frac{1}{6} = \frac{1}{2} \end{aligned}$$

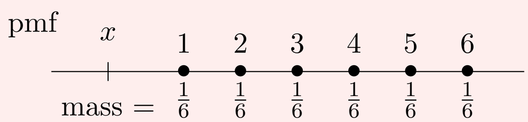
Definition 6.7 (Cumulative Distribution Function) — Cumulative distribution function (cdf) $F(x)$ of a RV x is a function given by

$$F(x) = P(X \leq x), \quad -\infty < x < \infty$$

<u>*Note*</u>: $F(x)$ is usually called distribution function, "cumulative" is dropped.

Example 6.8

Rolling a fair dice



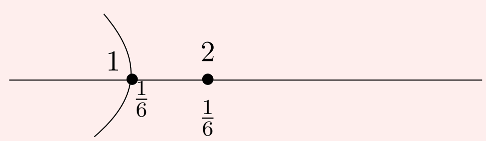
$$\begin{aligned} \text{cdf } F(x) &= P(X \leq x) \\ &= \text{total mass cumulated starting from the left up to x} \end{aligned}$$

$x < 1$,

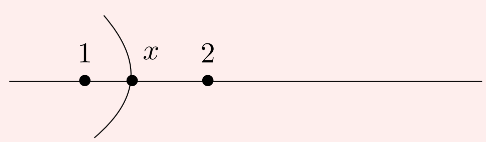
$$\begin{aligned} F(x) &= P(X \leq x) \\ &= 0 \text{ (no mass up to } x < 1) \end{aligned}$$

$x = 1$,

$$F(1) = P(X \leq 1)$$

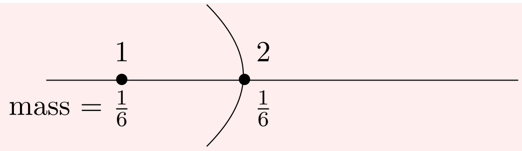


$F(1) = \frac{1}{6}$ (mass up to and including location 1).
$1 < x < 2$



$$\begin{aligned} F(x) &= P(X \leq 1) \\ &= P(X = 1) \\ &= \frac{1}{6} \end{aligned}$$

$x = 2$



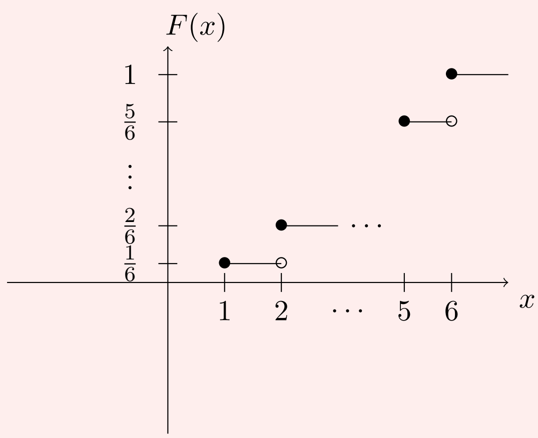
$$\begin{aligned} F(2) &= P(X \leq 2) \\ &= P(X = 1) + P(X = 2) \\ &= \frac{2}{6} \end{aligned}$$

Likewise, $2 < x < 3$

$$F(x) = \frac{2}{6}$$

$$\vdots x = 6, \quad F(X) = P(X \leq 6) = 1$$

$x > 6, F(x) = 1$

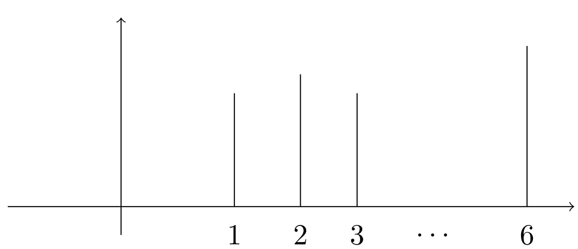


§7 Lec 7: Oct 16, 2020

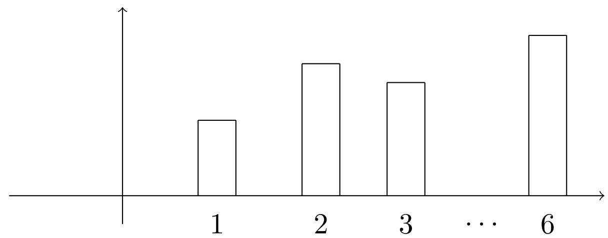
§7.1 Lec 6 (Cont'd)

In order to graph the prob. mass function:

- Line graph



- Histogram



Practice 7.1. 4 – Problem 1:

$$X = \text{ max of two rolls}$$
$$S_X = \{1, 2, \ldots, 6\}$$

For $k \in S_X$. Determine $f(k) = P(X = k) =?$

- 1^{st} approach:

1^{st} roll \ 2^{nd} roll	1	2	3	4	5	6
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
2	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
3	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
$\vdots$						
6	$(6,1)$	$(6,2)$	$\cdots$			

1^{st} roll \ 2^{nd} roll	1	2	3	$\cdots$	6
1	1	2	3	$\cdots$	6
2	2	2	3	$\cdots$	6
3	3	3	3	$\cdots$	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
6	6	6	6	$\cdots$	6

$$f(1) = P(X = 1) = \frac{1}{36}$$
$$f(2) = P(X = 2) = \frac{3}{36}$$
$$f(3) = P(X = 3) = \frac{5}{36}$$
$$\vdots$$
$$f(6) = P(X = 6) = \frac{11}{36}$$

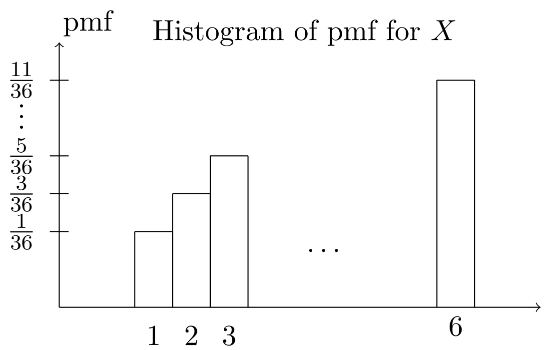
- 2^{nd} approach: for $k = 1, \ldots, 6$ (disjoint sub-events)

$$\begin{aligned}\{X = k\} = \{\max\ = k\} &= \left\{1^{\text{st}}\text{roll} = k, 2^{\text{nd}}\ < k\right\} \\ &\cup \left\{1^{\text{st}}\text{roll}\ < k, 2^{\text{nd}} = k\right\} \\ &\cup \left\{1^{\text{st}}\ \text{roll} = 2^{\text{nd}} = k\right\}\end{aligned}$$

Thus,

$$\begin{aligned}P(X = k) &= P(1^{\text{st}}\ \text{roll}\ = k)P(2^{\text{nd}} < k) + P(1^{\text{st}} < k)P(2^{\text{nd}} = k) + P(1^{\text{st}} = k)P(2^{\text{nd}} = k) \\ &= \frac{1}{6}\frac{k-1}{6} + \frac{k-1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} \\ &= \frac{2k-1}{36}\end{aligned}$$

Note: $\sum_{k=1}^{6} \frac{2k-1}{36} = 1$.



Similarly, we can calculate Y = min of 2 rolls.

Remark 7.1. Suppose $X = \max\{U, V\}$ where U, V are 2 discrete random variables. Then pmf of X can be calculated as follows:

$$\begin{aligned}f(k) &= P(X = k) \\ &= P(U = k, V < k) + P(U < k, V = k) + P(U = k, V = k)\end{aligned}$$

and we can often use indep. on each of the above events. On the other hand, for $Y = \min\{U, V\}$ then

$$P(Y = k) = P(U = k, V > k) + P(U > k, V = k) + P(U = k, V = k)$$

and use indep. on the above events.

§7.2 Expectation & Special Math Expectations

Definition 7.2 (Mathematical Expectation) — Suppose X is a discrete random variable with S_X, pmf $f(x)$. Let $u(x)$ be a function, then if the sum $\sum_{x\in S_X} u(x)f(x)$ exists (finite) then the sum is mathematical expectation (expected value) of $u(X)$ and is denoted by

$$E[u(X)] \coloneqq \sum_{x\in S_X} u(x)f(x)$$

Practice 7.2. 5 – Problem 1: $S_X = \{1,\dots,6\}$. For $x \in S_X$, $u(x) = x - 3.5$

$$\begin{aligned} \text{average income} &= E[u(x)] \\ &= \sum_{x\in S_X} u(x)f(x) \\ &= \sum_{k=1}^{6} (k-3.5)\cdot\frac{1}{6} \\ &= 0 \end{aligned}$$

"After one game, on average, I do not gain/lose any money."

Theorem 7.3

When it exists, the expectation E satisfies:

- If c is a constant, then
$$E[c] = c$$
- If c is a constant and $u(X)$ is a function, then
$$E[c\cdot u(X)] = cE[u(X)]$$
- If c_1, c_2 are constants and $u_1(X), u_2(X)$ are functions.
$$E[c_1u_1(X) + c_2u_2(X)] = c_1E[u_1(X)] + c_2E[u_2(X)]$$

Remark 7.4. Part (c) can be generalized for 2 discrete random variables X, Y.

$$E[c_1u_1(X) + c_2u_2(Y)] = c_1E[u_1(X)] + c_2E[u_2(Y)]$$

Proof. Textbook. □

Definition 7.5 (Mean, Variance, & Standard Deviation) — For a random variable X,

- the mean (of X) is denoted by
$$u := E[x]$$
- the variance (of X) is denoted by
$$\sigma^2 := E[(x-\mu)^2]$$
- the standard deviation
$$\sigma := \sqrt{\sigma^2}$$

Example 7.6
Suppose X has pmf

x	-2	0	1
$f(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

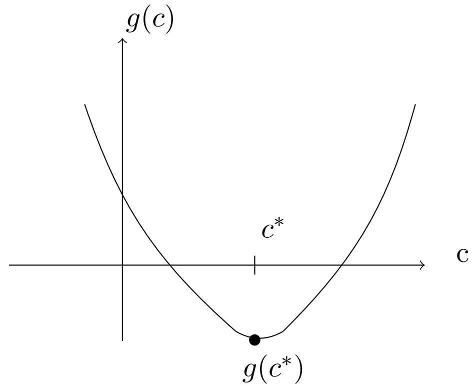
$$\begin{aligned}
\text{mean} = \mu &= E[x] \\
&= \sum_{x\in S_X} x \cdot f(x) \\
&= (-2)\frac{1}{2} + 0\frac{1}{3}1\frac{1}{6} \\
&= -\frac{5}{6} \\
\text{variance} = \sigma^2 &= E[(x-\mu)^2] \\
&= \sum_{x\in S_X} (x-\mu)^2 f(x) \\
&= (-2-(-\frac{5}{6})^2\frac{1}{2} + (0-(-\frac{5}{6}))^2\frac{1}{3} + \dots
\end{aligned}$$

σ^2 intepretation:

For a constant $c \in \mathbb{R}$, define $g(c) := E[(x-c)^2]$. Note that

$$\begin{aligned}
g(c) &= E[(X-c)^2] \\
&= E[X^2 - 2cX + c^2] \\
&= E[X^2] + E[-2cX] + E[c^2] \\
&= E[X^2] - 2cE[X] + c^2 \\
&= c^2 - 2cE[X] \cdot + E[X^2] \\
&= c^2 - 2\mu \cdot c + E[X^2]
\end{aligned}$$

"u and $E[X^2]$ are constant with respect to c".



$g(c^*) = \min g(c)$ where c^* satisfies

$$g'(c^*) = 0$$
$$g'(x) = 2c - 2\mu$$

Thus

$$g'(c^*) = 0 = 2c^* - 2\mu$$

i.e., $c^* = \mu$. Hence,

$$\sigma^2 = E[(x-\mu)^2] = g(\mu)$$

minimizes $g(c) = E[(x-c)^2]$, i.e.,

$$\sigma^2 = \underbrace{\min}_{c\in\mathbb{R}} E[(x-c)^2] = E[(x-\mu)^2]$$

"σ^2 measures fluctuation of X around its mean μ."

§8 Lec 8: Oct 19, 2020

§8.1 Info about 1^{st} midterm

1^{st} Midterm 11/2, Monday, 10am PT. Due: 10am PT – Tuesday 11/3.
2^{nd} Midterm, after Thanksgiving.

§8.2 Lec 7 (Cont'd)

Review geometric series: for $|q| < 1$,

$$\sum_{k=0}^{\infty} q^k = 1 + q + q^2 + \ldots = \frac{1}{1-q}$$

Differentiating both sides,

$$\sum_{k=1}^{\infty} kq^{k-1} = 1 + 2q + 3q^2 + \ldots = \frac{1}{(1-q)^2}$$

Practice 8.1. 5 – Problem 2:
$S_X = \{1, 2, \ldots\}$. The pmf $f(f) = P(X = k) = P(1^{\text{st}}$ k-1 shots are missed and k shot succcessful.
a) $E[X] =$?

$$\begin{aligned}
A_k &= \left\{k^{\text{th}} \text{ shot is successful}\right\} \\
P(A_k) &= p \\
P(A_k') &= 1 - p = q = P\left(\left\{k^{\text{th}} \text{ shot is missed}\right\}\right)
\end{aligned}$$

$$\begin{aligned}
P(X = k) &= P\left(\underbrace{A_1' \cap A_2' \cap \ldots \cap A_{k-1}'}_{\text{miss} 1^{\text{st}}\ k-1 \text{ shots}} \cap \underbrace{A_k}_{\text{make at} k^{\text{th}} \text{ shots}}\right) \\
&\overset{\text{independence}}{=} P(A_1')P(A_2') \ldots P(A_{k-1}')P(A_k) \\
&= q \cdot q \ldots q \cdot p \\
&= q^{k-1} \cdot p
\end{aligned}$$

for each $k = 1, 2, 3, \ldots$. Note that pmf $f(k) = P(X = k)$ indeed satisfies:

$$\begin{aligned}
\sum_{k=1}^{\infty} f(k) &= \sum_{k=1}^{\infty} q^{k-1} \cdot p \\
&= p\left(1 + q + q^2 + \ldots\right) \\
&= p \cdot \frac{1}{1-q} \\
&= p \cdot \frac{1}{p} \\
&= 1
\end{aligned}$$

Now,

$$\begin{aligned}
\mu = E[x] &= \sum_{x \in S_X} x f(x) \\
&= \sum_{k=1}^{\infty} k \cdot f(k) \\
&= \sum_{k=1}^{\infty} k \cdot q^{k-1} \cdot p \\
&= p \sum_{k=1}^{\infty} k \cdot q^{k-1} \\
&= p \cdot (1 + 2q + 3q^2 + \ldots) \\
&= p \cdot \frac{1}{(1-q)^2} \\
&= p \cdot \frac{1}{p^2} \\
&= \frac{1}{p}
\end{aligned}$$

Definition 8.1 (Moment Generating Function) — Given a discrete RV X and δ_X and pmf $f(x)$, if $\exists$ a positive constant h s.t. for all $t \in (-h, h)$, the following expectation function

$$E[e^{tX}] = \sum_{x \in S_X} e^{tx} f(x)$$

exists then $E[e^{tx}]$ is called the mgf of X and is denoted by $M_X(t)$.

Note: $(-h, h)$ needs not be a symmetric interval. But it has to contain the origin 0.

Example 8.2

Suppose X has the following pmf,

x	-2	0	1
$f(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$$\begin{aligned} E[e^{tX}] = M_X(t) &= \sum_{x \in S_X} e^{tx} f(x) \\ &= \frac{1}{2}e^{-2t} + \frac{1}{3} + \frac{1}{6}e^t \end{aligned}$$

which is finite for all $t \in \mathbb{R}$.

Theorem 8.3

MGF determines RV X, i.e., if X and Y are 2 RV s.t.

$$M_X(t) = M_Y(t)$$

then

$$S_X = S_y$$

and

$$\underbrace{f_X(x)}_{\text{pmf of X}} = \underbrace{f_Y(x)}_{\text{pmf of Y}} \quad \text{for } x \in S_X (= S_Y)$$

Example 8.4 (above)

Suppose Y has mgf

$$M_Y(t) = \frac{1}{2}e^{-2t} + \frac{1}{3} + \frac{1}{6}e^t$$

then

$$S_Y = \{-2, 0, 1\}$$

and $f_Y(-2) = \frac{1}{2}$, $f_Y(0) = \frac{1}{3}$, $f_Y(1) = \frac{1}{6}$. So that X and Y have same space and same pmf.

Practice 8.2. 5 – Problem 2b: X has geometric distribution with parameter $p \in [0, 1]$ denoted by $X \sim \text{Geom}(P)$.
with pmf $f(k) = q^{k-1}p$ for $k = 1, 2, \ldots, \quad q = 1 - p$. MGF of X is given by

$$\begin{aligned} M_X(t) &= \sum_{k=1}^{\infty} e^{tk} f(k) \\ &= \sum_{k=1}^{\infty} e^{tk} q^{k-1} p \\ &= p(e^t + e^{t2}q + e^{t3}q^2 + \ldots) \\ &= p \cdot e^t \left(1 + (e^t q) + (e^t q)^2 + (e^t q)^3 + \ldots\right) \\ &= pe^t \frac{1}{1 - e^t q} \end{aligned}$$

which is finite for t,

$$\begin{aligned} 0 < e^t \cdot q < 1 \\ e^t < \frac{1}{q} \\ t < \ln\left(\frac{1}{q}\right) \end{aligned}$$

Thus,

$$M_X(t) = \frac{pe^t}{1 - qe^t}, \quad \text{with } t < \ln\left(\frac{1}{q}\right)$$

Definition 8.5 (n^{th} Moment) — For each n positive integer, if $E[X^n] = \sum_{x \in S_X} x^n f(x)$ exists then $E[X^n]$ is called the n^{th} moment of X.

Remark 8.6. Properties of MGF $M_X(t)$

- $t = 0$, $M_X(0) = E[e^{0 \cdot X}] = E[1] = 1$.
- Derivatives of $M_X(t)$ is given by

$$\begin{aligned} \frac{d}{dt}[M_X(t)] &= \frac{d}{dt}\left[E[e^{tX}]\right] \\ &= E\left[\frac{d}{dt}e^{tX}\right] \quad \text{assume } \frac{d}{dt} \text{ and E are interchangeable} \\ M'_X(t) &= E\left[Xe^{tX}\right] \end{aligned}$$

  Thus,

$$M'_X(t)\Big|_{t=0} = E[Xe^{0\cdot X}] = E[X], \text{ first moment of X}$$

- Similarly, 2^{nd} derivative of $M_X(t)$ given by

$$M''_X(t) = E\left[X^2e^{tX}\right]$$

$$M''_X(t)\Big|_{t=0} = E[x^2], \text{ second moment of X}$$

- More generally, the $n^{\text{th}}-$ derivative of M_X satisfies

$$M_X^(n)(t)\Big|_{t=0} = E[x^n]$$

  hence the name “mgf”.

Example 8.7

$X \sim \text{Geom}(p)$.

$$\begin{aligned} M_X(t) &= \frac{pe^t}{1-qe^t}, \quad q = 1-p \\ M'_X(t) &= \frac{pe^t}{(1-qe^t)^2} \\ M'_X(0) &= \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} = E[x] \end{aligned}$$

§9 Lec 9: Oct 21, 2020

§9.1 Binomial Distribution

Definition 9.1 (Bernoulli Trial) — Bernoulli trial is a random experiment such that the outcomes can be classified in one of two mutually exclusive and exhaustive ways.

Example 9.2 1. Flipping a coin $S = \{H, T\}$.

2. A sequence of Bernoulli trials occurs when the experiment is performed several times and the prob. of success is the same in every trial and the trials are independent.

3. A player shooting the throws in basket ball

   - Making the shots has prob. $p \in (0, 1)$.
   - Missing.

   Each throw is a Bernoulli trial. A sequence of throw is a sequence of Bernoulli trial.

Definition 9.3 (Bernoulli Random Variable) — Let X be the random variable associated with a Bernoulli trial. Then X is called a Bernoulli R.V with the pmf

$$P(X = 1(\text{success})) = p$$
$$P(X = 0(\text{failure})) = 1 - p$$

which can also be rewritten as:

$$f(x) = p^x (1-p)^{1-x}, \quad x \in \{0, 1\}$$

<u>*Note*</u>: A formula of variance

$$\begin{aligned}
\sigma^2 &= E[(X - \mu)^2] \\
&= E[X^2 - 2\mu X + \mu^2] \\
&= E[X^2] - 2\mu E[X] + \mu^2 \\
&= E[X^2] - 2\mu^2 + \mu^2 \\
&= E[X^2] - \mu^2 \\
&= E[X^2] - (E[X])^2 \\
&= M_X''(0) - \left(M_X'(0)\right)^2
\end{aligned}$$

Practice 9.1. 6 – Problem 1: Let $X \sim$ Bernoulli R.V with p

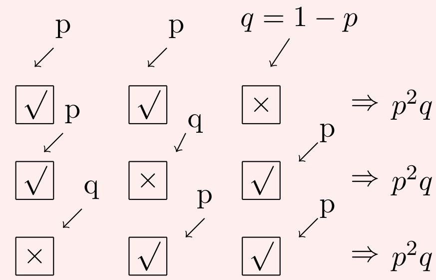
$$\begin{aligned}
\mu = E[X] &= 1 \cdot P(X = 1) + 0 \cdot P(X = 0) \\
&= p \\
E[X^2] &= 1^2 \cdot P(X = 1) + 0^2 \cdot P(X = 0) \\
&= p
\end{aligned}$$

Thus,

$$\begin{aligned}
\sigma^2 &= E[X^2] - (E[X])^2 \\
&= p - p^2 \\
&= p(1 - p) \\
&= pq
\end{aligned}$$

Example 9.4

Suppose the player shoots three times. Let X be the number of times of making the shot. $P(X = 2) = ?$



In total

$$P(X = 2) = 3p^2q = \binom{3}{2}p^2q$$

Definition 9.5 (Binomial Distribution) — Given a Bernoulli trial, let X be the number of successes in n Bernoulli trials. Then X is called the binomial distribution and is denoted by

$$X \sim B(n, p) \quad \text{or} \quad X \sim \text{ Binom}(n, p)$$

The pmf of X is given by

$$\begin{aligned} f(k) &= P(X = k), \quad k \in S_X = \{0, \dots, n\} \\ &= \binom{n}{k}p^k(1-p)^{n-k} \end{aligned}$$

Explanation:

- choose k trials for success:

$$\# \text{ ways } = \binom{n}{k}$$

- for each choice, prob of success $= \underbrace{p \cdot p \dots p}_{k \text{ times}}$ and failures $= \underbrace{(1-p)\dots(1-p)}_{n-k}$.

$$\implies \quad \binom{n}{k}p^k(1-p)^{n-k}$$

Note:the pmf of $B(n, p)$ satisfies

$$\begin{aligned} \sum_{k=0}^{n} f(k) &= \sum_{k=0}^{n} \binom{n}{k}p^k(1-p)^{n-k} \\ &= (p + 1 - p)^n \quad \text{by Binomial Expansion Formula} \\ &= 1 \end{aligned}$$

Practice 9.2. 6 – Problem 2: mgf of $B(n,p)$:

$$\begin{aligned}
E[e^{tX}] &= \sum_{k=0}^{n} e^{tk} P(X=k) \\
&= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum \binom{n}{k} e^{tk} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(pe^t\right)^k (1-p)^{n-k} \\
&= (pe^t + 1 - p)^n \quad \text{by Binomial Expansion}
\end{aligned}$$

Note that $n = 1$, $B(1,p)$ is simply a Bernoulli trial mgf if Bernoulli trial is given by

$$\left(pe^t + 1 - p\right)^1 = pe^t + 1 - p$$

Now, we can calculate the mean

$$\begin{aligned}
\mu = E[X] &= \sum_{x \in S_X} x f(x) \\
&= \underbrace{\sum k \binom{n}{k} p^k (1-p)^{n-k}}_{\text{time consuming but doable}}
\end{aligned}$$

MGF approach:

$$\begin{aligned}
\mu = E[X] &= M_X'(t)\Big|_{t=0} \\
M_X(t) &= (pe^t + 1 - p)^n) \\
M_X'(0) &= np
\end{aligned}$$

Variance:

$$\begin{aligned}
\sigma^2 &= E[X^2] - (E[X])^2 \\
E[X^2] &= M_X''(0) \\
M_X''(0) &= n(n-1)p^2 + np
\end{aligned}$$

Thus,

$$\begin{aligned}
\sigma^2 &= E[X^2] - (E[X])^2 \\
&= n(n-1)p^2 + np - (np)^2 \\
&= np(1-p)
\end{aligned}$$

"Recalling variance of Bernoulli trial is $p(1-p)$."

§10 Lec 10: Oct 23, 2020

§10.1 Practice 6 Problem 3

Practice 10.1. 6 – Problem 3: $p = 0.95$
a) Let X be the number of days without an accident in next 7 days. Then $X \sim B(n = 7, p = 0.95)$.

$$\begin{aligned} P(X = 7) &= \binom{7}{7} .95^7 (1 - .95)^{7-7} \\ &= .95^7 \end{aligned}$$

b) Y = number of days in October without accident. $Y \sim B(n = 31, p = .95)$.

$$P(Y = 29) = \binom{31}{29} .95^{29} (.05)^2$$

c)

$$\begin{aligned} A &= \{\text{today, no accident}\} \\ B &= \{\text{no accident from day 2 to day 5}\} \\ C &= \{\text{at least one day with accident between day 6 to day 10}\} \\ C' &= \{\text{no accident between day 6 and day 10}\} \end{aligned}$$

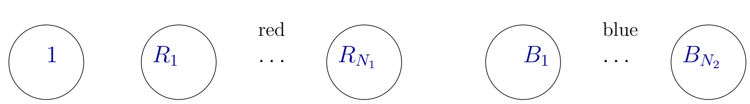
$P(B \cap C|A) =?$ Note that A, B, C are mutually independent. Thus,

$$\begin{aligned} P(B \cap C|A) &= P(B \cap C) \\ &= \underbrace{P(B)}_{(n=4,p=0.95)} \underbrace{P(C)}_{(n=5,p=.95)} \\ &= \binom{4}{4} (.95)^4 (.05)^0 \left[1 - P(C')\right] \left[1 - \binom{5}{5} (.95)^5 (.05)^0\right] \\ &= (.95)^4 \left[1 - (.95)^5\right] \end{aligned}$$

> **Remark 10.1.** It might be helpful to consider compliment when dealing with "at least" event.

§10.2 Hypergeometric Distribution

Practice 10.2. 7 – Problem 1: draw n = x reds + $(n - x)$ blues



Denote $X = \#$ red balls from n drawn.

$$S_X = \begin{cases} x \in \mathbb{N} : 0 \leq x \leq n, \\ \qquad 0 \leq x \leq N_1, \\ \qquad 0 \leq n - x \leq N_2 \end{cases}$$

For $x \in S_X$, $P(X = x) =?$
Ways to drawn n balls from $N_1 + N_2$: $\binom{N_1+N_2}{n}$

- $E_1 =$ pick x reds from N_1 which is $\binom{N_1}{x}$
- $E_2 =$ pick $n - x$ blues from $N_2 \implies \binom{N_2}{n-x}$
- $E_1E_2 =$ number of ways to pick n balls from $N_1 + N_2$ and pick exactly x red balls. $\implies \binom{N_1}{x}\binom{N_2}{n-x}$. Thus,

$$P(X = x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N_1+N_2}{n}}$$

Note that X is denoted as $X \sim HG(N_1, N_2, n)$. The pmf indeed satisfies

$$\sum_{x \in S_X} f(x) = \sum_{x \in S_X} \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N_1+N_2}{x}} = 1$$

Fact 10.1. Let $X \sim HG(N_1, N_2, n)$ then

$$\mu = E[X] = n\frac{N_1}{N_1 + N_2}$$

Proof. See textbook 2.5. □

§11 Lec 11: Oct 26, 2020

§11.1 Negative Binomial Distribution

Definition 11.1 (Negative Binomial Distribution) — Considering the experiment of performing Bernoulli trials until r successes occur (r is a fixed pos. integer). $X =$ number needed to observe the r^{th} success. Then X is called a negative binomial distribution.

1^{st} success $\quad 2^{nd}$ success $\quad r^{th}$ success

$$\underbrace{\boxed{\times}\ \boxed{\times}\ \boxed{\times}\ \boxed{\times}\ \boxed{\times}\ \cdots\ \boxed{\times}\ \boxed{\surd}}_{\text{X trials in total}}$$

X is denoted as $X \sim NB(r, p)$

> **Remark 11.2.** When $r = 1$, $X = \#$ needed to observe the first success ($\sim$ Geom (p))

Fact 11.1. The pmf of $X \sim NB(r, p)$ is given by
for $k \geq r$,

$$f(k) = \binom{k-1}{r-1} p^r \left(1-p\right)^{k-r}$$

where p is the probability of success (from Bernoulli trial). The space $S_X = \{r, r+1, \ldots\}$.

Proof. Given $k \geq r$, $P(X = k) = ?$

$$\overbrace{\square \quad \square \quad \cdots \quad \square \quad \square}^{\text{k-1 trials and r-1 successes}} \quad \underset{\substack{\uparrow \\ k^{th}}}{\overset{\substack{r^{th} \\ \downarrow}}{\boxed{\checkmark}}}$$

$$\begin{aligned} P(X = k) &= P(\text{in the first k-1 trials, there are exactly r-1 successes)} \\ &\quad \text{and the } k^{\text{th}} \text{ trial is successful} \\ &= P(r-1 \text{ successes from k-1 trials}) \cdot P(k^{\text{th}} \text{ trial is successful}) \\ &= \binom{k-1}{r-1} p^{r-1} (1-p)^{(k-1)-(r-1)} \cdot p \\ &= \binom{k-1}{r-1} p^r (1-p)^{k-r} \end{aligned}$$

$\square$

<u>*Note*</u>: The pmf of $NB(r, p)$ satisfies

$$\sum_{k=r}^{\infty} f(k) = \sum_{k=r}^{\infty} \binom{k-1}{r-1} p^r (1-p)^{k-r} = 1$$

We need Taylor expansion for the above formula, for $|w| < 1$,

$$\frac{1}{(1-w)^r} = \sum_{k=1}^{\infty} \binom{k-1}{r-1} w^{k-r}$$

So,

$$\begin{aligned} \sum_{k=r}^{\infty} f(k) &= p^r \sum_{k=r}^{\infty} \binom{k-1}{r-1} (1-p)^{k-r} \\ &= p^r \frac{1}{(1-(1-p))^r} \\ &= 1 \end{aligned}$$

Fact 11.2. $X \sim NB(r,p)$ then

$$M_X(t) = \left[\frac{pe^t}{1-(1-p)e^t}\right]^r$$

Mean:

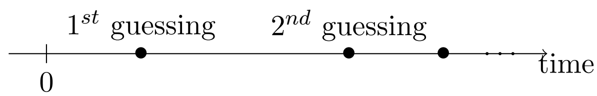
$$\mu = E[X] = \frac{r}{p}$$

Variance:

$$\sigma^2 = \mathrm{Var(X)} = \frac{r(1-p)}{p^2}$$

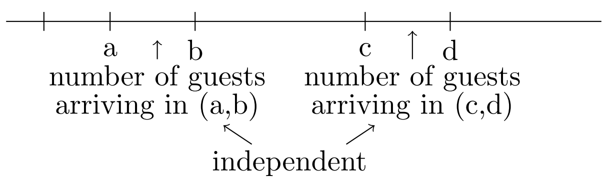
§11.2 Poisson Distribution

Motivation: Considering the arrivals(of guests at a bank or a restaurant,etc) in a continuous time interval

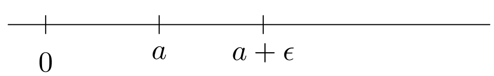


We assume the followings:

- The number of arrivals in non-overlapping intervals are mutually independent.



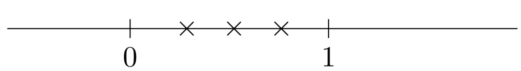
- There exists a fixed $\lambda > 0$ s.t. for all $\epsilon > 0$ efficiently small $P(\text{exactly one arrival in } [a, a+\epsilon]) = \lambda\epsilon$ and $P(\text{at least two arrivals in } [a, a+\epsilon)) = 0$



Note that we also have

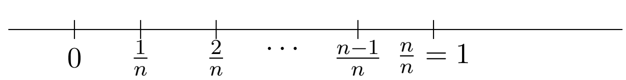
$$P(\text{no arrival in } [a, a+\epsilon)) = 1 - \lambda\epsilon$$

Question 11.1. $X = \#$ arrivals in one hour



$P(X = k) =?$

Approach: for n large



By the second assumption,

$$P(\text{one arrival in one subinterval}) = \lambda \cdot \frac{1}{n} = \frac{\lambda}{n}$$

By the first assumption, subintervals arrivals are independent. Thus,

$$P(X = k) \cong P(\text{k subintervals have one arrival each, among n subintervals})$$

" a subinterval having one arrival is a success with prob. $\frac{\lambda}{n}$ " where

$$P(X = k) \cong \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

Practice 11.1. 8 – Problem 1: For $k \geq 0$

$$S_n \sim B(n, \frac{\lambda}{n})$$
$$\lim_{n\to\infty} P(S_n = k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k (1 - \frac{\lambda}{n})^{n-k}$$

Everything converges to 1 except $\frac{\alpha^k}{\lambda^k}$ and

$$\lim_{n\to\infty} (1 - \frac{\alpha}{n})^n = \lim_{y\to\infty} \left[(1 - \frac{1}{y})^y\right]^\lambda$$

Notice that

$$\lim_{y\to\infty} [(1 - \frac{1}{y})^y]^\lambda = (e^{-1})^\lambda = e^{-\lambda}$$

Hence,

$$\lim_{n\to\infty} (S_n = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Definition 11.3 (Poisson Distribution) — Let X be a r.v. taking values in $\{0, 1, 2 \ldots\}$ with pmf $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$ for a fixed $\lambda > 0$. Thus X is called a Poisson distribution, $X \sim \text{Pois}(\lambda)$

§12 Dis 1: Oct 6, 2020

§12.1 Set Theory

Definition 12.1 (Set) — A set is a collection of items.

Example 12.2

$$\begin{gathered}T=\{1,2,3,\text{red},\text{blue}\}\\ S=\{1,3,\text{red}\}\\ R=\{1,2,4\}\\ S\subseteq T\\ S'=S^c=\{2,\text{blue}\}\\ R\not\subseteq T\\ 3\quad\underbrace{\in}_{\text{is an element of}}\quad T\\ \{3\}\subseteq T\end{gathered}$$

Example 12.3

$$\begin{aligned}A&=\{1,3,7\} & A\cup B&=\{1,2,3,4,7\}\\ B&=\{2,3,4\} & A\cap B&=\{3\}\\ A\setminus B&=\{1,7\} & B\setminus A&=\{2,4\}\end{aligned}$$

De Morgan Laws:

$$\begin{aligned}(A\cup B)'&=A'\cap B'\\ (A_1\cup A_2\cup\ldots\cup A_n)&=A_1'\cap A_2'\cap\ldots\cap A_n'\\ (A\cap B)'&=A'\cup B'\end{aligned}$$

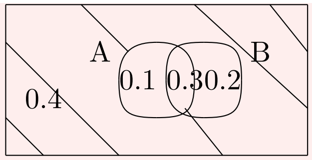
If have a <u>sample space</u> S, and subset of S are called <u>events</u>. A <u>probability function</u> is a function $\mathbb{P}$ that assigns a real number each event with three rules:

1. $P(A)\geq 0$
2. $P(S)=1$
3. $A_1,A_2,\ldots,A_n$ with $A_i\cap A_j=\emptyset=\{\}$, then $P(A_1\cup\ldots\cup A_n)=P(A_1)+\ldots+P(A_n)$

Example 12.4

1.1-6 (from the book): $P(A)=0.4,\quad P(B)=0.5,\quad P(A\cap B)=0.3$
Find

- $P(A\cup B)=.1+.3+.2=.6$
- $P(A\cap B)'=.1$
- $P(A'\cap B)=.2$



Note: (P, S) : probability space on all subsets of S

Example 12.5

1.2-5: How many four letter codes can be made from the letters in IOWA if

- Letters may not be repeat: $4! = 24$ ways.
- Letters may repeat: $4^4 = 256$ ways.

§13 Dis 2: Oct 13, 2020

1.4.16: An urn has 5 balls. One is marked "win" and the other are marked "lose". You and another player each take balls out one at a time until somebody picks win. You pick first.
W/o replacement: $P(\text{winning}) = \frac{1}{5} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot 1 = \frac{3}{5}$
With replacement:

$$\begin{aligned} P(\text{winning}) &= \frac{1}{5} + \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} + \frac{4}{5}\frac{4}{5}\frac{4}{5}\frac{4}{5}\frac{1}{5} + \dots \\ &= \frac{\frac{1}{5}}{1 - \frac{16}{25}} = \frac{5}{9} \end{aligned}$$

§13.1 Conditional Probabilities

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

or $P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$

Example 13.1

1.3.7: An urn has 4 balls. 2 are red and 2 are blue. We pull out 2 balls. We are told that at least one is red. What's the probability that they're both red?

$P(\text{both red}|\text{at least one red}) = \frac{P(\text{both red and at least one red})}{P(\text{at least one red})} = \frac{P(\text{both red})}{P(\text{at least red})} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}$.

§13.2 Bayes's Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Example 13.2

1.5-8: Four types of tablets: B_1, B_2, B_3, B_4 with % of sales $0.4, 0.3, 0.2, 0.1$ and % tablet needs repair $0.1, 0.05, 0.03, 0.02$, respectively. What is the probability that a tablet needing repair is B_i?

$$\begin{aligned} P(B_1|\text{need repair}) &= \frac{P(\text{need repair}|B_1)\cdot P(B_1)}{P(\text{need repair})} \\ &= \frac{(0.1)(0.4)}{(0.40)(0.10)+(0.30)(0.05)+(0.20)(0.03)+(0.10)(0.02)} \\ &\approx 63.5\% \\ P(B_2|\text{need repair}) &= \frac{(0.30)(0.05)}{0.063} \approx 23.8\% \\ P(B_3|\text{need repair}) &\approx 9.5\% \\ P(B_4|\text{need repair}) &\approx 3.2\% \end{aligned}$$

§14 Dis 3: Oct 20,2020

§14.1 Recap of Terminology/Functions

We have a situation with a set of possible outcomes

- This set is called the sample space denoted S or Ω.
- Elements of S are called outcomes.
- Subsets of S are called events.
- A probability function is a function where

$$P : \{\text{ subset of } S\} \to [0,1]$$

Example 14.1

$$\begin{aligned} S &= \{HH, HT, TH, TT\} \\ A &= \{HH, HT\} \\ B &= \{HH\} \\ P(A) &= 0.5 \\ P(B) &= P(\{HH\}) = 0.25 \end{aligned}$$

A random variable, denoted X, is a function

$$X : \underbrace{S}_{\text{sample space}} \to \underbrace{S_X}_{\text{the space support}} \subseteq \mathbb{R}$$

"$X = a$" $\leftrightarrow \{w \in S \text{ s.t. } X(w) = a\}$.

Example 14.2

Define $X(w)$ to be the number of tails in the outcome w.

$$\begin{aligned} X(HH) &= 0 \\ X(HT) &= 1 \\ X(TH) &= 1 \\ X(TT) &= 2 \\ (X = 1) &= \{HT, TH\} \\ (X = 2) &= \{TT\} \\ (X = 0) &= \{HH\} \\ (X = 3) &= \emptyset \end{aligned}$$

The probability mass function or pmf of a r.v. X is a function $f_x : S_X \to [0, 1]$ defined by

$$\begin{aligned} f(x) &= P(X = x) \\ f(a) &= P(X = a) \end{aligned}$$

Example 14.3

$$f_x(a) = \begin{cases} 0.25 & a = 0 \\ 0.5 & a = 1 \\ 0.25 & a = 2 \end{cases}$$

Also,

$$\begin{aligned} P(X = 1) &= P(\{HT, TH\}) = 0.5 \\ P(X < 2) &= P(\{HH, HT, TH\}) = 0.75 \end{aligned}$$

The cumulative distribution function or cdf of a r.v. X is a function $F_x : S_x \to [0, 1]$ defined by

$$F(a) = P(X \leq a)$$

Example 14.4

$$F_x(a) = \begin{cases} 0.25 & a = 0 \\ 0.75 & a = 1 \\ 1 & a = 2 \end{cases}$$

The expectation or mean of X is

$$E[x] = \sum_{a \in S_x} af(a)$$
$$E[g(x)] = \sum_{a \in S_x} g(a)f(a)$$

Example 14.5 (above)

$$\begin{aligned} E[x] &= (0)0.25 + (1)0.5 + (2)0.25 \\ &= 1 \\ E[x^2] &= (0^2)0.25 + (1^2)0.5 + (2^2)0.5 \\ &= 2.5 \neq E[x]^2 \end{aligned}$$

The moment of generating function or mgf of X is

$$M_x(t) = E[e^{tX}] = \sum_{a \in S_x} e^{ta} f(a)$$

Example 14.6

$M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t} = \sum_{a \in \{1,2,3\}} e^{at} f(a)$. Find mean, variance, pmf.
$S_x = \{1, 2, 3\}$. The pmf is

$$f_x(a) = \begin{cases} \frac{2}{5} & a = 1 \\ \frac{1}{5} & a = 2 \\ \frac{2}{5} & a = 3 \end{cases}$$

The mean is

$$E[x] = (1)\frac{2}{5} + (2)\frac{1}{5} + (3)\frac{2}{5} = 2$$

Variance is

$$\begin{aligned} \sigma^2 = \text{Var}(X) &= E[x^2] - E[x]^2 \\ &= \left((1^2)\frac{2}{5} + (2^2)\frac{1}{5}(3^2)\frac{2}{5}\right) - 2^2 \\ &= \frac{4}{5} \end{aligned}$$