115B – Linear Algebra

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This is math 115B – Linear Algebra which is the second course of the undergrad linear algebra at UCLA – continuation of 115A(H). Similar to 115AH, this class is instructed by Professor Elman, and we meet weekly on MWF from 2:00 pm to 2:50 pm. There is no official textbook used for the class. You can find the previous linear algebra notes (115AH) with other course notes through my github. Any error in this note is my responsibility and please email me if you happen to notice it.

Contents

List of Theorems

List of Definitions

§1 Lec 1: Mar 29, 2021

§1.1 Vector Spaces

Notation: if $\star : A \times B \to B$ is a map (= function) write $a \star b$ for $\star(a, b)$, e.g., $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $\mathbb{Z} =$ the integer.

Definition 1.1 (Field) — A set F is called a FIELD under

- Addition: $+ : F \times F \to F$
- Multiplication: $\cdot : F \times F \to F$

if $\forall a, b, c \in F$, we have

A1) $(a + b) + c = a + (b + c)$

A2) $\exists\ 0 \in F \ni a + 0 = a = 0 + a$

A3) A2) holds and $\exists x \in F \ni a + x = 0 = x + a$

A4) $a + b = b + a$

M1) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M2) A2) holds and $\exists 1 \neq 0 \in F$ s.t. $a \cdot 1 = a = 1 \cdot a$ (1 is unique and written 1 or 1_F)

M3) M2) holds and $\forall 0 \neq x \in F \quad \exists y \in F \ni xy = 1 = yx$ (y is seen to be unique and written x^{-1})

M4) $x \cdot y = y \cdot x$

D1) $a \cdot (b + c) = a \cdot b + a \cdot c$

D2) $(a + b) \cdot c = a \cdot c + b \cdot c$

Example 1.2

$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are fields as is

$$\mathbb{F}_2 := \{0, 1\} \text{ with } + : \text{ given by}$$

$+$	0	1
0	0	1
1	1	0

$\bullet$	0	1
0	0	0
1	0	1

Fact 1.1. Let $p > 0$ be a prime number in $\mathbb{Z}$. Then $\exists$ a field $\mathbb{F}_{p^n}$ having p^n elements write $|\mathbb{F}_{p^n}| = p^n \quad \forall n \in \mathbb{Z}^+$.

Definition 1.3 (Ring) — Let R be a set with

- $+ : R \times R \to R$
- $\cdot : R \times R \to R$

satisfying A1) – A4), M1), M2), D1), D2), then R is called a RING.
A ring is called

i) a commutative ring if it also satisfies M4).

ii) an (integral) domain if it is a commutative ring and satisfies

$$\text{M 3')}\ a \cdot b = 0 \implies a = 0 \text{ or } b = 0$$

($0 = \{0\}$ is also called a ring – the only ring with $1 = 0$)

Example 1.4 (Proof left as exercises)

1. $\mathbb{Z}$ is a domain and not a field.
2. Any field is a domain.
3. Let F be a field
$$F[t] \coloneqq \{\text{polys coeffs in } F\}$$
with usual $+, \cdot$ of polys, is a domain but not a field. So if $f \in F[t]$
$$f = a_0 + a_1 t + \ldots + a_n t^n$$
where $a_0, \ldots, a_n \in F$.
4. $\mathbb{Q} \coloneqq \left\{ \frac{a}{b} | a, b \in \mathbb{Z},\ b \neq 0 \right\} < \mathbb{C}$ ($<$ means $\subset$ and $\neq$) with usual $+, \cdot$ of fractions. (when does $\frac{a}{b} = \frac{c}{d}$?)
5. If F is a field
$$F(t) \coloneqq \left\{ \frac{f}{g} | f, g \in F[t],\ g \neq 0 \right\} \text{ (rational function)}$$
with usual $+, \cdot$ of fractions is a field.

Example 1.5 (Cont'd from above) 6. $\mathbb{Q}[\sqrt{-1}] \coloneqq \left\{\alpha + \beta\sqrt{-1} \in \mathbb{C} | \alpha, \beta \in \mathbb{Q}\right\} < \mathbb{C}$. Then $\mathbb{Q}[\sqrt{-1}]$ is a field and

$$\begin{aligned}\mathbb{Q}(\sqrt{-1}) &\coloneqq \left\{\frac{a}{b} | a, b \in \mathbb{Q}[\sqrt{-1}],\, b \neq 0\right\} \\ &= \mathbb{Q}[\sqrt{-1}] \\ &= \left\{\frac{a}{b} | a, b \in \mathbb{Z}[\sqrt{-1}],\, b \neq 0\right\}\end{aligned}$$

where $\mathbb{Z}[\sqrt{-1}] \coloneqq \left\{\alpha + \beta\sqrt{-1} \in \mathbb{C},\, \alpha, \beta \in \mathbb{Z}\right\} < \mathbb{C}$. How to show this? – rationalize ($\mathbb{Z}[\sqrt{-1}]$ is a domain not a field, $F[t] < F(t)$ if F is a field so we have to be careful).

7. F a field

$$\mathbb{M}_n F \coloneqq \{n \times n \text{ matrices entries in } F\}$$

is a ring under $+, \cdot$ of matrices.

$$1_{\mathbb{M}_n F} = I_n = n \times n \text{ identity matrix } \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$0_{\mathbb{M}_n F} = 0 = 0_n = n \times n \text{ zero matrix } \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

is <u>not</u> commutative if $n > 1$.

In the same way, if R is a ring we have

$$\mathbb{M}_n R = \{n \times n \text{ matrices entries in } R\}$$

e.g., if R is a field $\mathbb{M}_n F[t]$.

8. Let $\emptyset \neq I \subset \mathbb{R}$ be a subset, e.g., $[\alpha, \beta], \alpha < \beta \in \mathbb{R}$. Then

$$C(I) = \{f : I \to \mathbb{R} | f \text{ continuous}\}$$

is a commutative ring and not a domain where

$$\begin{aligned}(f \dot{+} g)(x) &\coloneqq f(x) \dot{+} g(x) \\ 0(x) &= 0 \\ 1(x) &= x\end{aligned}$$

for all $x \in I$.

<u>Notation</u>: Unless stated otherwise F is always a field.

Definition 1.6 (Vector Space) — Let F be a field, V a set. Then V is called a VECTOR SPACE OVER F write V is a vector space over F under

- $+ : V \times V \to V$ – Addition
- $\cdot : F \times V \to V$ – Scalar multiplication

if $\forall x, y, z \in V \quad \forall \alpha, \beta \in F$.

1. $(x + y) + z = x + (y + z)$
2. $\exists 0 \in V \ni \ x + 0 = x = 0 + x$ (0 is seen to be unique and written 0 or 0_V)
3. 2) holds and $\exists v \in V \ni \ x + v = 0 = v + x$ (v is seen to be unique and written $-x$)
4. $x + y = y + x$
5. $1_F \cdot x = x$.
6. $(\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta \cdot x)$
7. $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$
8. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

Remark 1.7. The usual properties we learned in 115A hold for V a vector space over F, e.g., $0_F V = 0_V$, general association law,...

§2 Lec 2: Mar 31, 2021

§2.1 Vector Spaces (Cont'd)

Example 2.1

The following are vector space over F

1. $F^{m\times n} \coloneqq \{m \times n \text{ matrices entries in } F\}$, usual $+$, scalar multiplication, i.e., if $A \in F^{m\times n}$, let $A_{ij} = ij^{\text{th}}$ entry of A. If $A, B \in F^{m\times n}$, then

$$(A+B)_{ij} \coloneqq A_{ij} + B_{ij}$$
$$(\alpha A)_{ij} \coloneqq \alpha A_{ij} \quad \forall \alpha \in F$$

   i.e., component-wise operations.

2. $F^n = F^{1\times n} \coloneqq \{(\alpha_1, \ldots, \alpha_n) \,|\alpha_i \in F\}$

3. Let V be a vector space over F, $\emptyset \neq S$ a set. Define

$$\mathcal{F}cn(S, V) \coloneqq \{f : S \to V | \, f \text{ a fcn}\}$$

   Then $\mathcal{F}cn(S, V)$ is a vector space over F $\forall f, g \in \mathcal{F}cn(S, V)$, $\forall \alpha \in F$. For all $x \in S$,

$$f + g : \, x \mapsto f(x) + g(x)$$
$$\alpha f : \, x \mapsto \alpha f(x)$$

   i.e.

$$(f+g)(x) = f(x) + g(x)$$
$$(\alpha f)(x) = \alpha f(x)$$

   with 0 by $0(x) = 0_V \, \forall x \in S$.

4. Let R be a ring under $+, \cdot$, F a field $\ni F \subseteq R$ with $+, \cdot$ on F induced by $+, \cdot$ on R and $0_F = 0_R$, $1_F = 1_R$, i.e.

$$\underbrace{+}_{\text{on } R}\Big|_{\underbrace{F\times F}_{\text{restrict dom}}} : F \times F \to F \text{ and } \underbrace{\cdot}_{\text{on } R}\Big|_{\underbrace{F\times F}_{\text{restrict dom}}} : F \times F \to F$$

   i.e. closed under the restriction of $+, \cdot$ on R to F and also with $0_F = 0_R$ and $1_F = 1_R$ (we call F a subring of R). Then R is a vector space over F by restriction of scalar multiplication, i.e., same $+$ on R but scalar multiplication

$$\cdot\big|_{F\times R} : F \times R \to R$$

   e.g., $\mathbb{R} \subseteq \mathbb{C}$ and $F \subseteq F[t]$.

   Note: $\mathbb{C}$ is a vector space over $\mathbb{R}$ by the above but as a vector space over $\mathbb{C}$ is different.

5. In 4) if R is also a field (so $F \subseteq R$ is a subfield). Let V be a vector space over R. Then V is also a vector space over F by restriction of scalars, e.g., $M_n\mathbb{C}$ is a vector space over $\mathbb{C}$ so is a vector space over $\mathbb{R}$ so is a vector space over $\mathbb{Q}$.

§2.2 Subspaces

Definition 2.2 (Subspace) — Let V be a vector space under $+, \cdot, \emptyset \neq W \subseteq V$ a subset. We call W a subspace of V if $\forall w_1, w_2 \in W$, $\forall \alpha \in F$,

$$\alpha w_1,\ w_1 + w_2 \in W$$

with $0_W = 0_V$ is a vector space over F under $+|_{W\times W}$ and $\cdot|_{F\times W}$ i.e., closed under the operation on V.

Theorem 2.3

Let V be a vector space over F, $\emptyset \neq W \subseteq V$ a subset. Then W is a subspace of V iff $\forall \alpha \in F$, $\forall w_1, w_2 \in W$, $\alpha w_1 + w_2 \in W$.

Example 2.4 1. Let $\emptyset \neq I \subseteq \mathbb{R}$, $C(I)$ the commutative ring of continuous function $f : I \to \mathbb{R}$. Then $C(I)$ is a vector space over $\mathbb{R}$ and a subspace of $\mathcal{F}cn(I, \mathbb{R})$.

2. $F[t]$ is a vector space over F and $n \geq 0$ in $\mathbb{Z}$.

$$F[t]_n \coloneqq \{f |\ f \in F[t],\ f = 0 \text{ or } \deg f \leq d\}$$

is a subspace of $F[t]$ (it is not a ring).

Attached is a review of theorems about vector spaces from math 115A.

§2.3 Motivation

Problem 2.1. Can you break down an object into simpler pieces? If yes can you do it uniquely?

Example 2.5

Let $n > 1$ in $\mathbb{Z}$. Then n is a product of primes unique up to order.

Example 2.6

Let V be a finite dimensional inner product space over $\mathbb{R}$ (or $\mathbb{C}$) and $T : V \to V$ a hermitian (=self adjoint) operator. Then $\exists$ an ON basis for V consisting of eigenvectors for T. In particular, T is diagonalizable. This means

$$V = E_T(\lambda_1) \perp \ldots \perp E_T(\lambda_r) \tag{*}$$

$E_T(\lambda_i) \coloneqq \{v \in V | \, Tv = \lambda_i v\} \neq 0$ eigenspace of λ_i; $\lambda_1, \ldots, \lambda_r$ the distinct eigenvalues of T. So

$$T\big|_{E_T(\lambda_i)} : E_T(\lambda_i) \to E_T(\lambda_i)$$

i.e., $E_T(\lambda_i)$ is T-invariant and

$$T\big|_{E_T(\lambda_i)} = \lambda_i 1_{E_T(\lambda_i)}$$

and (*) is unique up to order.

Goal: Generalize this to V any finite dimensional vector space over F, any F, and $T : V \to V$ linear. We have many problems to overcome in order to get a meaningful result, e.g.,

Problem 2.2.

1. V may not be an inner product space.
2. $F \neq \mathbb{R}$ or $\mathbb{C}$ is possible.
3. $F \not\subseteq$ is possible, so cannot even define an inner product.
4. V may not have any eigenvalues for $T : V \to V$.
5. If we prove an existence theorem, we may not have a uniqueness one.

We shall show: given V a finite dimensional vector space over F and $T : V \to V$ a linear operator. Then V breaks up uniquely up to order into small T-invariant subspace that we shall show are completely determined by polys in $F[t]$ arising from T.

§2.4 Direct Sums

Motivation: Generalize the concept of linear independence, Spectral Theorem Decomposition, to see how pieces are put together (if possible).

Definition 2.7 (Span) — Let V be a vector space over F, $W_i \subseteq V$, $i \in I$ – may not be finite, subspaces. Let

$$\sum_{i\in I} W_i = \sum_{i\in I} W_i \coloneqq \left\{ v \in V | \exists w_i \in W_i,\, i \in I,\ \text{almost all } w_i = 0 \ni v = \sum_{i\in I} w_i \right\}$$

when almost all zero means only finitely many $w_i \neq 0$. Warning: In a vector space/F we can only take finite linear combination of vectors. So

$$\sum_{i\in I} W_i = \text{Span}\left(\bigcup_{i\in I} W_i\right) = \left\{ \text{finite linear combos of vectors in } \bigcup_{i\in I} W_i \right\}$$

e.g., if I is finite, i.e., $|I| < \infty$, say $I = \{1, \ldots, n\}$ then

$$\sum_{i\in I} W_i = W_1 + \ldots + W_n \coloneqq \{w_1 + \ldots + w_n |\, w_i \in W_i \,\forall i \in I\}$$

cf. Linear Combinations.

Definition 2.8 (Direct Sum) — Let V be a vector space over F, $W_i \subseteq V$, $i \in I$, subspace. Let $W \subseteq V$ be a subspace. We say that W is the (internal) direct sum of the W_i, $i \in I$ write $W = \bigoplus_{i\in I} W_i$ if

$$\forall w \in W\, \exists!\, w_i \in W_i \text{ almost all } 0 \ni\ w = \sum_{i\in I} w_i$$

e.g., if $I = \{1, \ldots, n\}$, then

$$w \in W_1 \oplus \ldots \oplus W_n \text{ means } \exists!\, w_i \in W_i \ni\ w = w_1 + \ldots + w_n$$

Warning: It may not exist.

§3 Lec 3: Apr 2, 2021

§3.1 Direct Sums (Cont'd)

Definition 3.1 (Independent Subspace) — Let V be a vector space over $F, W_i \subseteq V$, $i \in I$ subspaces. We say the W_i, $i \in I$, are independent if whenever $w_i \in W_i$, $i \in I$, almost all $w_i = 0$, satisfy $\sum w_i = 0$, then $w_i = 0\, \forall i \in I$.

Theorem 3.2

Let V be a vector space over F, $W_i \subseteq V$, $i \in I$ subspaces, $W \subseteq V$ a subspace. Then the following are equivalent:

1. $W = \bigoplus_{i\in I} W_i$
2. $W = \sum_{i\in I} W_i$ and $\forall i$
$$W_i \cap \sum_{j\in I\setminus\{i\}} W_j = 0 \coloneqq \{0\}$$
3. $W = \sum_{i\in I} W_i$ and the W_i, $i \in I$, are independent.

Proof. 1) $\implies$ 2) Suppose $W = \bigoplus_{i\in I} W_i$. Certainly, $W = \sum_{i\in I} W_i$. Fix i and suppose that
$$\exists x \in W_i \cap \sum_{j\in I\setminus\{i\}} W_j$$
By definition, $\exists w_i \in W_i$, $w_j \in W_j$, $j \in I \setminus \{i\}$ almost all 0 satisfying
$$w_i = x = \sum_{j\neq i} w_j$$
So
$$0_V = 0_W = w_i - \sum_{j\neq i} w_j$$
But
$$0_W = \sum_I 0_{W_k} \qquad 0_{W_k} = 0_V \,\forall k \in I$$
By uniqueness of 1), $w_i = 0$ so $x = 0$.
2) $\implies$ 3) Let $w_i \in W_i$, $i \in I$, almost all zero satisfy
$$\sum_{i\in I} w_i = 0$$
Suppose that $w_k \neq 0$. Then
$$w_k = -\sum_{i\in I\setminus\{k\}} w_i \in W_k \cap \sum_{i\neq k} w_i = 0,$$
a contradiction. So $w_i = 0\,\forall i$
3) $\implies$ 1) Suppose $v \in \sum_{i\in I} W_i$ and $\exists w_i, w_i' \in W_i$, $i \in I$, almost all $0 \ni$
$$\sum_{i\in I} w_i = v = \sum_{i\in I} w_i'$$
Then $\sum_{i\in I}(w_i - w_i') = 0$, $w_i - w_i' \in W_i\,\forall i$. So
$$w_i - w_i' = 0, \text{ i.e., } w_i = w_i' \quad \forall i$$
and the $w_i's$ are unique. □

Warning: 2) DOES NOT SAY $W_i \cap W_j = 0$ if $i \neq j$. This is too weak. It says $W_i \cap \sum_{j\neq i} W_j = 0$.

Corollary 3.3

Let V be a vector space over $F, W_i \subseteq V$, $i \in I$ subspaces. Suppose $I = I_1 \cup I_2$ with $I_1 \cap I_2 = \emptyset$ and $V = \bigoplus_{i \in I} W_i$. Set

$$W_{I_1} = \bigoplus_{i \in I_1} W_i \quad \text{and} \quad W_{I_2} = \bigoplus_{j \in I_2} W_j$$

Then

$$V = W_{I_1} \oplus W_{I_2}$$

Proof. Left as exercise – Homework. □

Notation: Let V be a vector space over F, $v \in V$. Set

$$Fv \coloneqq \{\alpha v | \alpha \in F\} = \text{Span}(v)$$

if $v \neq 0$, then Fv is the line containing v, i.e., Fv is the one dimensional vector space over F with basis $\{v\}$.

Example 3.4

Let V be a vector space over F

1. If $\emptyset \neq S \subseteq V$ is a subset, then
$$\sum_{v \in S} Fv = \text{Span}(S)$$
the span of S. So
$$\text{Span } S = \{\text{all finite linear combos of vectors in } S\}$$
2. If $\emptyset \neq S$ is linearly indep. (i.e. meaning every finite nonempty subset of S is linearly indep.), then
$$\text{Span}(S) = \bigoplus_{s \in S} Fs$$
3. If S is a basis for V, then $V = \bigoplus_{s \in S} Fs$
4. If $\exists$ a finite set $S \subseteq V \ni V = \text{Span}(S)$, then $V = \sum_{s \in S} Fs$ and $\exists$ a subset $\mathscr{B} \subseteq S$ that is a basis for V, i.e., V is a finite dimensional vector space over F and $\dim V = \dim_F V = |\mathscr{B}|$ is indep. of basis $\mathscr{B}$ for V.
5. Let V be a vector space over $F, W_1, W_2 \subseteq V$ finite dimensional subspaces. Then $W_1 + W_2$, $W_1 \cap W_2$ are finite dimensional vector space over F and
$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$
So
$$W_1 + W_2 = W_1 \oplus W_2 \iff W_1 \cap W_2 = \emptyset$$
(warning: be very careful if you wish to generalize this)

Definition 3.5 (Complementary Subspace) — Let V be a finite dimensional vector space over $F, W \subseteq V$ a subspace if
$$V = W \oplus W', \quad W' \subseteq V \text{ a subspace}$$
We call W' a complementary subspace of W in V.

Example 3.6

Let $\mathscr{B}_0$ be a basis of W. Extend $\mathscr{B}_0$ to a basis $\mathscr{B}$ for V (even works if V is not finite dimensional). Then

$$W' = \bigoplus_{\mathscr{B}\setminus\mathscr{B}_0} Fv \text{ is a complement of } W \text{ in } V$$

Note: W' is not the unique complement of W in V – counter-example?

Consequences: Let V be a finite dimensional vector space over $F, W_1, \ldots, W_n \subseteq V$ subspaces, $W_i \neq 0 \, \forall i$. Then the following are equivalent

1. $V = W_1 \oplus \ldots \oplus W_n$.
2. If $\mathscr{B}_i$ is a basis (resp., ordered basis) for $W_i \, \forall i$, then $\mathscr{B} = \mathscr{B}_1 \cup \ldots \cup \mathscr{B}_n$ is a basis (resp. ordered) – with obvious order – for V.

Proof. Left as exercise (good one)! □

Notation: Let V be a vector space over F, $\mathscr{B}$ a basis for V, $x \in V$. Then, $\exists! \alpha_v \in F$, $v \in \mathscr{B}$, almost all $\alpha_v = 0$ (i.e., all but finitely many) s.t. $x = \sum_{\mathscr{B}} \alpha_v v$. Given $x \in V$,

$$x = \sum_{v \in \mathscr{B}} \alpha_v v$$

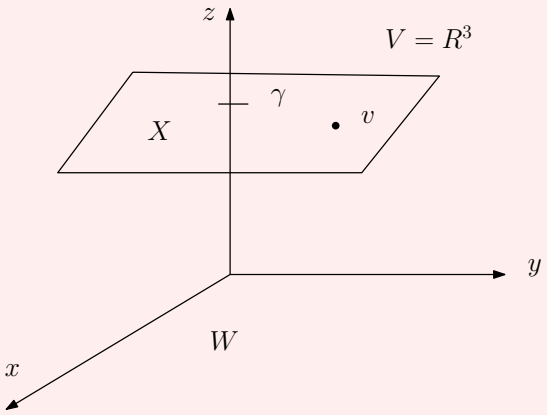
to mean α_v is the unique complement of x on v and hence $\alpha_v = 0$ for almost all $v \in \mathscr{B}$.

§3.2 Quotient Spaces

Idea: Given a surjective map $f : X \to Y$ and "nice", can we use properties of Y to obtain properties of X?

Example 3.7

Let $V = \mathbb{R}^3$, $W = X - Y$ plane. Let $X =$ plane parallel to W intersecting the z-axis at γ.



So

$$\begin{aligned} X &= \{(\alpha, \beta, \gamma) | \alpha, \beta \in \mathbb{R}\} \\ &= \{(\alpha, \beta, 0) + (0, 0, \gamma) | \alpha, \beta \in \mathbb{R}\} \\ &= W + \gamma \underbrace{e_3}_{(0,0,1)} \end{aligned}$$

<u>*Note*</u>: X is a vector space over $\mathbb{R} \iff \gamma = 0 \iff W = X$ (need 0_V). Let $v \in X$. So $v = (x, y, \gamma)$ some $x, y \in \mathbb{R}$. So

$$\begin{aligned} W + v &\coloneqq \left\{ \underbrace{(\alpha, \beta, 0)}_{\text{arbitrary}} + \underbrace{(x, y, \gamma)}_{\text{fixed}} \,|\, \alpha, \beta \in \mathbb{R} \right\} \\ &= \{(\alpha + x, \beta + y, \gamma) \,|\, \alpha, \beta \in \mathbb{R}\} \\ &= W + \gamma_{e_3} \end{aligned}$$

It follows if $v, v' \in V$, then

$$W + v = W + v' \implies v - v' \in W$$

Conversely, if $v, v' \in V$ with $X = W + v$, then

$$v' \in X \implies v' = w + v \text{ some } w \in W$$

hence

$$v' - v \in W$$

So for arbitrary $v, v' \in V$, we have the <u>conclusion</u> $W + v = W + v' \iff v - v' \in W$. We can also write $W + v$ as $v + W$.

§4 Lec 4: Apr 5, 2021

§4.1 Quotient Spaces (Cont'd)

Recall from the last example of the last lecture, we have

$$V = \bigcup_{v \in V} W + v$$

If $v, v' \in V$, then

$$0 \neq v'' \in (W + v) \cap (W + v')$$

means

$$W + v - W + v'' = W + v'$$

This means either $W + v = W + v'$ or $W + v \cap W + v' = \emptyset$, i.e., planes parallel to the xy-plane partition V into a disjoint unions of planes.

Let

$$S \coloneqq \{W + v | \, v \in V\}$$

the set of these planes. We make S into a vector space over $\mathbb{R}$ as follows: $\forall v, v' \in V$, $\forall \alpha \in \mathbb{R}$ define

$$(W + v) + (W + v') \coloneqq W + (v + v')$$
$$\alpha \cdot (W + v) \coloneqq W + \alpha v$$

We must check these two operations are well-defined and we set

$$0_S \coloneqq W$$

Then $(W + v) + W = W + v = W + (W + v)$ make S into a vector space over $\mathbb{R}$.
If $v \in V$ let $\gamma_v^1 =$ the k^{th} component of v. Define

$$S \to \{(0, 0, \gamma) | \, \gamma \in \mathbb{R}\} \to \mathbb{R}$$

by

$$W + v \mapsto (0, 0, \gamma_v) \mapsto \gamma$$

both maps are bijection and, in fact, linear isomorphism. So

$$S \cong \{(0, 0, \gamma) | \, \gamma \in \mathbb{R}\} \cong \mathbb{R}$$

Note: $\dim V = 3$, $\dim W = 2$, $\dim S = 1$ and we also have a linear transformation

$$V \to S \text{ by } (\alpha, \beta, \gamma) \mapsto W + \gamma_{e_3}$$

a surjection.
We can now generalize this.
Construction: Let V be a vector space over F, $W \subseteq V$ a subspace. Define $\equiv \mod W$ called congruent mod W on V as follows: if $x, y \in V$, then

$$x \equiv y \mod W \iff x - y \in W \iff \exists w \in W \ni x = w + y$$

Then, for all $x, y, z \in V$, $\equiv \mod W$ satisfies

1. $x \equiv x \mod W$
2. $x \equiv y \mod W \implies y \equiv x \mod W$
3. $x \equiv y \mod W$ and $y \equiv z \mod W \implies x \equiv z \mod W$

We can conclude that $\equiv \mod W$ is an equivalence relation on V.

Notation: For $x \in V$, $W \subseteq V$, let

$$\overline{x} \coloneqq \{y \in V \mid y \equiv x \mod W\}$$

We can also write $\overline{x}$ as $[x]_W$ if W is not understood. Also, $\overline{x} \subseteq V$ is a subset and not an element of V called a coset of V by W. We have

$$\begin{aligned}\overline{x} &= \{y \in V \mid y \equiv x \mod W\} \\ &= \{y \in V \mid y = w + x \text{ for some } w \in W\} \\ &= \{w + x \mid w \in W\} = W + x = x + W\end{aligned}$$

Example 4.1
$\overline{0}_V = W + 0_V = W$.

Note: $W + x$ translates every element of W by x. By 2), 3) of $\equiv \mod W$, we have check

$$y \in \overline{x} = W + x \iff x \in \overline{y} = W + y$$

and

$$x \equiv y \mod W \iff \overline{x} = \overline{y} \iff W + x = W + y$$

and check

$$\overline{x} \cap \overline{y} = \emptyset \iff (W + x) \cap (W + y) = \emptyset \iff x \not\equiv y \mod W$$

This means the $W + x$ partition V, i.e.,

$$V = \bigcup_V (W + x) \text{ with } (W + x) \cap (W + y) = \emptyset \text{ if } \overline{x} = (W + x) \neq (W + y) = \overline{y}$$

Let

$$\overline{V} \coloneqq V/W \coloneqq \{\overline{x} \mid x \in V\} = \{W + x \mid x \in V\}$$

a collection of subsets of V.

§5 Lec 5: Apr 7, 2021

§5.1 Quotient Spaces (Cont'd)

Suppose we have $W \subseteq V$ a subspace. For $x, y, z, v \in V$

$$\begin{aligned} x &\equiv y \mod W \\ z &\equiv v \mod W \end{aligned} \tag{+}$$

Then

$$(x+z)-(y+v) = \underbrace{(x-y)}_{\in W} + \underbrace{(z-v)}_{\in W} \in W$$

So

$$x+z \mod y+v \mod W$$

and if $\alpha \in F$

$$\alpha x - \alpha y = \alpha(x-y) \in W \quad \forall x, y \in V$$

So

$$\alpha x \equiv \alpha y \mod W$$

Therefore, $\overline{V} = V/W$. If $(+)$ holds, then for all $x, y, z, v \in V$ and $\alpha \in F$, we have

$$\begin{aligned} \overline{x+z} &= \overline{y+v} \in \overline{V} \\ \overline{\alpha x} &= \overline{\alpha y} \in \overline{V} \end{aligned}$$

Notice $\overline{V} = V/W$ satisfies all the axioms of a vector space with $0_{\overline{V}} = \overline{0_V} = \{y \in V \,|\, y \equiv 0 \mod W\} = W + 0_V = W$.

We call $\overline{V} = V/W$ the QUOTIENT SPACE of V by W.

We also have a map

$$- : V \to \overline{V} = V/W \text{ by } x \mapsto \overline{x} = W + x$$

which satisfies

$$\alpha v + v' \overset{-}{\mapsto} \overline{\alpha u + v'} = \alpha \overline{v} + \overline{v'}$$

for all $v, v' \in V$ and $\alpha \in F$. Then

$$\begin{aligned} \dim V &= \dim \ker^{-} \\ \dim V &= \dim W + \dim V/W \\ \dim V/W &= \dim V - \dim W \end{aligned}$$

which is called the codimension of W in V.

Proposition 5.1

Let V be a vector space over F, $W \subseteq V$ a subspace, $\overline{V} = V/W$. Let $\mathscr{B}_0$ be a basis for W and

$$\mathscr{B}_1 = \{v_i | \, i \in I, \, v_i - v_j \notin W \text{ if } i \neq j\}$$

where $\overline{v_i} \neq \overline{v_j}$ if $i \neq j$ or $w + v_i \neq w + v_j$ if $i \neq j$.

Let

$$\mathscr{C} = \{\overline{v_i} = W + v_i | \, i \in I, \, v_i \in \mathscr{B}_1\}$$

If $\mathscr{C}$ is a basis for $\overline{V} = V/W$, then $\mathscr{B}_0 \cup \mathscr{B}_1$ is a basis for V (compare with the proof of the Dimension Theorem).

Proof. Hw 2 # 3. □

§5.2 Linear Transformation

A review of linear of linear transformation can be found here.
Now, we consider

$$GL_n F \coloneqq \{A \in \mathbb{M}_n F | \, \det A \neq 0\}$$

The elements in $GL_n F$ in the ring $\mathbb{M}_n F$ are those having a multiplicative inverse. If R is a commutative ring, determinants are still as before but

$$\begin{aligned} GL_n R &\coloneqq \{A \in \mathbb{M}_n R | \, \det A \text{ is a unit in } R\} \\ &= \left\{A \in \mathbb{M}_n R | \, A^{-1} \text{ exists}\right\} \end{aligned}$$

Example 5.2

Let V be a vector space over $F, W \subseteq V$ a subspace. Recall

$$\overline{V} = V/W = \{\overline{v} = W + v | \, v \in V\}$$

a vector space over F s.t. for all $v_1, v_2 \in F$ and $\alpha \in F$

$$\begin{aligned} 0_{\overline{V}} = \overline{0_V} &= W \\ \overline{v_1} + \overline{v_2} &= \overline{v_1 + v_2} \\ \alpha \overline{v_1} &= \overline{\alpha v_1} \end{aligned}$$

Then

$$- : V \to V/W = \overline{V} \text{ by } v \mapsto \overline{v} = W + v$$

is an epimorphism with $\ker^- = W$.

Recall from 115A(H) that the most important theorem about linear transformation is Universal Property of Vector Spaces. As a result, we can deduce the following corollary

Corollary 5.3

Let V, W be vector space over F with bases $\mathscr{B}, \mathscr{C}$ respectively. Suppose there exists a bijection $f : \mathscr{B} \to \mathscr{C}$, i.e., $|\mathscr{B}| = |\mathscr{C}|$. Then $V \cong W$.

Proof. There exists a unique $T : V \to W \ni T\big|_{\mathscr{B}} = f$. T is monic by the Monomorphism Theorem (T takes linearly indep. sets to linearly indep. sets iff it's monic) and is onto as $W = \text{Span}(\mathscr{C}) = \text{Span}\,(f(\mathscr{B}))$. □

§6 Lec 6: Apr 9, 2021

§6.1 Linear Transformation (Cont'd)

Theorem 6.1

Let $T : V \to W$ be linear. Then $\exists X \subseteq V$ a subspace s.t.

$$V = \ker T \oplus X \text{ with } X \cong \text{im } T$$

Proof. Let $\mathscr{B}_0$ be a basis for $\ker T$. Extend $\mathscr{B}_0$ to a basis $\mathscr{B}$ for V by the Extension Theorem. Let $\mathscr{B}_1 = \mathscr{B} \setminus \mathscr{B}_0$, so $\mathscr{B} = \mathscr{B}_0 \vee \mathscr{B}_1$ ($\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$ and $\mathscr{B}_0 \cap \mathscr{B}_1 = \emptyset$) and let

$$X = \bigoplus_{\mathscr{B}_1} Fv$$

As $\ker T = \bigoplus_{\mathscr{B}_0} Fv$, we have

$$V = \ker T \oplus X$$

and we have to show

$$X \cong \text{im } T$$

Claim 6.1. Tv, $v \in \mathscr{B}_1$ are linearly indep.

In particular, $Tv \neq Tv'$ if $v, v' \in \mathscr{B}_1$ and $v \neq v'$. Suppose

$$\sum_{v \in \mathscr{B}} \alpha_v Tv = 0_W, \quad \alpha_v \in F \text{ almost all } \alpha_v = 0$$

Then

$$0_W = T\left(\sum_{v \in \mathscr{B}_1} \alpha_v v\right), \quad \text{i.e. } \sum_{\mathscr{B}_1} \alpha_v v \in \ker T$$

Hence

$$\sum_{\mathscr{B}_1} \alpha_v v = \sum_{\mathscr{B}_0} \beta_v v \in \ker T \text{ almost all } \beta_v \in F = 0$$

As $\sum_{\mathscr{B}_1} \alpha_v v - \sum_{\mathscr{B}_0} \beta_v v = 0$ and $\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$ is linearly indep., $\alpha_v = 0 \, \forall v$. This proves the above claim.

Let $\mathscr{C} = \{Tv | \, v \in \mathscr{B}_1\}$. By the claim

$$\mathscr{B}_1 \to \mathscr{C} \text{ by } v \mapsto Tv \text{ is } 1-1$$

and onto as $\mathscr{C}$ is linearly indep. Lastly, we must show $\mathscr{C}$ spans im T. Let $w \in \text{im } T$. Then $\exists x \in V \ni Tx = w$. Then

$$\begin{aligned} w = Tx &= T\left(\sum_{\mathscr{B}_0} \alpha_v v\right) + T\left(\sum_{\mathscr{B}_1} \alpha_v v\right) \\ &= \sum_{\mathscr{B}_0} \alpha_v Tv + \sum_{\mathscr{B}_1} \alpha_v Tv = \sum_{\mathscr{B}_1} \alpha_v Tv \end{aligned}$$

lies in span $\mathscr{C}$ as needed. □

Remark 6.2. Note that the proof is essentially the same as the proof of the Dimension Theorem.

Corollary 6.3 (Dimension Theorem)
If V is a finite dimensional vector space over F, $T : V \to W$ linear then

$$\dim V = \dim \ker T + \dim \text{ im } T$$

Corollary 6.4
If V is a finite dimensional vector space over F, $W \subseteq V$ a subspace, then

$$\dim V = \dim W + \dim V/W$$

Proof. $- : V \to V/W$ by $v \mapsto \overline{v} = W + v$ is an epi. □

Important Construction: Set

$$T : V \to Z \text{ be linear}$$
$$W = \ker T$$
$$\overline{V} = V/W$$
$$- : V \to V/W \text{ by } v \mapsto \overline{v} = W + v \text{ linear}$$

$\forall x, y \in V$ we have

$$\overline{x} = \overline{y} \in \overline{V} \iff x \equiv y \mod W \iff x - y \in W \iff T(x - y) = 0_Z$$

i.e., when $W = \ker T$

$$\overline{x} = \overline{y} \iff Tx = Ty \tag{*}$$

This means

$$\overline{T} : \overline{V} \to Z \text{ defined by } W + v = \overline{v} \mapsto Tv$$

is well-defined, i.e., via function, since if $\overline{x} = \overline{y}$, then $\overline{T}(\overline{x}) := Tx = Ty =: \overline{T}(\overline{y})$. From (*),

$$\overline{x} = \overline{y} \iff \overline{T}(\overline{x}) = T(x) = T(y) =: \overline{T}(\overline{y})$$

so

$$\overline{T} : \overline{V} \to Z \text{ is also injective}$$

As $\overline{T}$ is linear, let $\alpha \in F$, $x, y \in V$, then

$$\begin{aligned}\overline{T}\left(\alpha\overline{x} + \overline{y}\right) = \overline{T}\left(\overline{\alpha x + y}\right) &= T(\alpha x + y) \\ &= \alpha Tx + Ty = \alpha\overline{T}(\overline{x}) + \overline{T}(\overline{y})\end{aligned}$$

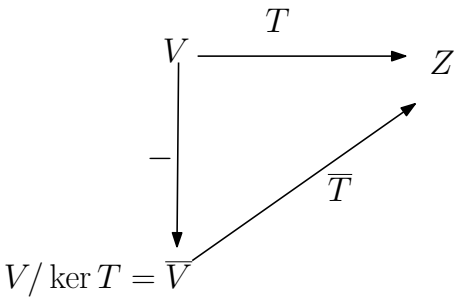
as needed. Therefore,

$$\overline{T} : \overline{V} \to Z \text{ by } \overline{x} \mapsto T(x)$$

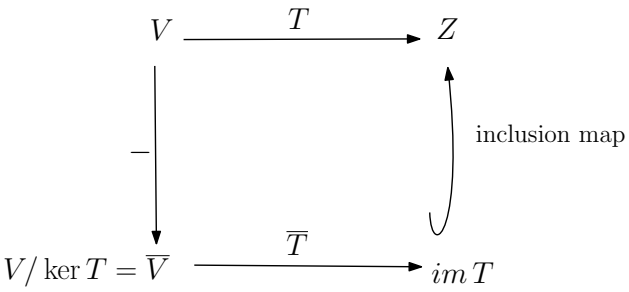
is a monomorphism, so induces an isomorphism onto im $\overline{T}$ and we recall im $\overline{T}$ = im T, so

$$\overline{V} \cong \text{im } \overline{T} = \text{im } T$$

and we have a commutative diagram



This can also be written as



Consequence: Any linear transformation $T : V \to Z$ induces an isomorphism

$$\overline{T} : V/\ker T \to \text{im } T \text{ by } \overline{v} = \ker T + v \mapsto Tv$$

This is called the First Isomorphism Theorem. We also have

$$V = \ker T \oplus X \text{ with } X \subseteq V \text{ and } X \cong \text{im } T \cong V/\ker T$$

This means that all images of linear transformations from V are determined, up to isomorphism, by V and its subspaces. It also means, if V is a finite dimensional vector space over F, we can try prove things by induction.

§6.2 Projections

Motivation: Let $m < n$ in $\mathbb{Z}^+$ and

$$\pi : \mathbb{R}^n \to \mathbb{R}^n \text{ by } (\alpha_1, \ldots, \alpha_n) \mapsto (\alpha_1, \ldots, \alpha_n, 0, \ldots, 0)$$

a linear operator onto $\bigoplus_{i=1}^{m} \Gamma e_i$ where $e_i = \left(0, \ldots, \underbrace{1}_{i^m}, \ldots, 0\right)$.

Definition 6.5 (T-invariant) — Let $T : V \to V$ be linear, $W \subseteq V$ a subspace. We say W is T-invariant if $T(W) \subseteq V$ if this is the case, then the restriction $T\big|_W$ of T can be viewed as a linear operator

$$T\big|_W : W \to W$$

Example 6.6

Let $T : V \to V$ be linear.

1. $\ker T$ and $\operatorname{im} T$ are T-invariant.
2. Let $\lambda \in F$ be an eigenvalue of T, i.e., $\exists 0 \neq v \in V \ni Tv = \lambda v$, then any subspace of the eigenspace
$$E_T(\lambda) \coloneqq \{v \in V \,|\, Tv = \lambda v\}$$
is T-invariant as $T\big|_{E_T(\lambda)} = \lambda 1_{E_T(\lambda)}$

Remark 6.7. Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Suppose that
$$V = W_1 \oplus \ldots \oplus W_n$$
with each W_i T-invariant, $i = 1, \ldots, n$ and $\mathscr{B}_i$ an ordered basis for W_i, $i = 1, \ldots, n$. Let $\mathscr{B} = \mathscr{B}_1 \cup \ldots \cup \mathscr{B}_n$ be a basis of V ordered in the obvious way.
Then the matrix representation of T in the $\mathscr{B}$ basis is
$$[T]_{\mathscr{B}} = \begin{pmatrix} \left[T\big|_{W_1}\right]_{\mathscr{B}_1} & & 0 \\ & \ddots & \\ 0 & & \left[T\big|_{W_n}\right]_{\mathscr{B}_n} \end{pmatrix}$$

Example 6.8

Suppose that $T : V \to V$ is diagonalizable, i.e., there exists a basis $\mathscr{B}$ of eigenvectors of T for V. Then, $T : V \to V$,
$$V = \bigoplus E_T(\lambda_i)$$
each $E_T(\lambda_i)$ is T-invariant.
$$T\big|_{E_T(\lambda_i)} = \lambda_i 1_{E_T(\lambda_i)}$$

Goal: Let V be a finite dimensional vector space over F, $n = \dim V$, $T : V \to V$ linear. Then $\exists W_1, \ldots, W_m \subseteq V$ all T-invariant subspaces with $m = m(T)$ with each W_i being as small as possible with $V = W_1 \oplus \ldots \oplus W_m$. This is the theory of canonical forms.
Recall: If V is a finite dimensional vector space over F, $T : V \to V$ linear, $\mathscr{B}$ an ordered basis for V, then the matrix representation $[T]_{\mathscr{B}}$ is only unique up to similarity, i.e., if $\mathscr{C}$ is an another ordered basis
$$[T]_{\mathscr{C}} = P\,[T]_{\mathscr{B}}\,P^{-1}$$
where $P = [1_V]_{\mathscr{B},\mathscr{C}} \in GL_n F$, the change of basis matrix $\mathscr{B} \to \mathscr{C}$.

Definition 6.9 (Projection) — Let V be a vector space over F, $P : V \to V$ linear. We call P a projection if $P^2 = P \circ P = P$.

Example 6.10 1. $P = 0_V$ or $1_V : V \to V$, V is a vector space over F.

2. An orthogonal projection in 115A.

3. If P is a projection, so is $1_V - P$.

If $T : V \to V$ is linear, then

$$V = \ker T \oplus X \text{ with } X \cong \text{im } T$$

Lemma 6.11

Let $P : V \to V$ be a projection. Then

$$V = \ker P \oplus \text{im } P$$

Moreover, if $v \in \text{im } P$, then

$$Pv = v$$

i.e.

$$P\big|_{\text{im } P} : \text{im } P \to \text{im } P \text{ is } 1_{\text{im } P}$$

In particular, if V is a finite dimensional vector space over $F, \mathscr{B}_1$ an ordered basis for $\ker P$, $\mathscr{B}_2$ an ordered basis for im P, then $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is an ordered basis for V and

$$[P]_{\mathscr{B}} = \begin{pmatrix} [0]_{\mathscr{B}_1} & 0 \\ 0 & [1_{\text{im } P}]_{\mathscr{B}_2} \end{pmatrix} = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

Proof. Let $v \in V$, then $v - Pv \in \ker P$, since

$$P(v - Pv) = Pv - P^2v = Pv - Pv = 0$$

Hence

$$v = (v - Pv) + Pv \in \ker P + \text{im } P$$

$\ker P \cap \text{im } P = 0$ and $P\big|_{\text{im } P} = 1_{\text{im } P}$. Let $v \in$ im P. By definition, $Pw = v$ for some $w \in V$. Therefore,

$$Pv = PPw = Pw = v$$

Hence

$$P\big|_{\text{im } P} = 1_{\text{im } P}$$

If $v \in \ker P \cap \text{im } P$, then

$$v = Pv = 0$$

□

§7 Lec 7: Apr 12, 2021

§7.1 Projection (Cont'd)

Lemma 7.1

Let V be a vector space over F, $W, X \subseteq V$ subspaces. Suppose

$$V = W \oplus X$$

Then $\exists! \, P : V \to V$ a projection satisfying

$$\begin{aligned} W &= \ker P \\ X &= \text{im } P \end{aligned} \tag{*}$$

We say such a P is the projection along W onto X.

Proof. Existence: Let $v \in V$. Then

$$\exists! w \in W, \, x \in X \ni \; v = w + x$$

Define

$$P : V \to V \text{ by } v \mapsto x$$

To show $P^2 = P$, we suppose $v \in V$ satisfies $v = w + x$, for unique $w \in W$, $x \in X$. Then

check P is linear and well defined

$$Pv = Pw + Px = Px = 1_X x = x$$

so

$$P^2 v = Px = x = Pv \qquad \forall v \in V$$

hence $P^2 = P$.
Uniqueness: Any P satisfying (*) takes a basis for W to 0 and fix a basis of X. Therefore, P is unique by the UPVS. □

Remark 7.2. Compare the above to the case that V is an inner product space over F, $W \subseteq V$ is a finite dimensional subspace and $P : V \to V$ by $v \mapsto v_W$, the orthogonal projection of P onto W.

Proposition 7.3

Let V be a vector space over F, $W, X \subseteq V$ subspaces s.t. $V = W \oplus X$, $P : V \to V$ the projection along W onto X, and $T : V \to V$ linear. Then the following are equivalent:

1. W and X are both $T-$invariant.
2. $PT = TP$.

Proof. 2) $\implies$ 1) : W is T-invariant: We have $W = \ker P$, so if $w \in W$, $Pw = 0$. Hence

$$PTw = TPw = T0 = 0$$

$Tw \in \ker P = W$ so W is T-invariant.
X is T-invariant, $X = \text{im } P$, $P|_X = 1_X$. So if $x \in X$

$$Tx = TPx = PTx \in \text{im } P = X$$

So X is T-invariant.
1) $\implies$ 2) Let $v \in V$. Then $\exists!\, w \in W$, $x \in X$ s.t.

$$v = w + x$$

As $P|_X = 1_X$ and $P|_W = 0$, so $Pv = Px$. By 1), W and X are T-invariant, so

$$\begin{aligned} PTv &= PT(w + x) = PTw + PTx \\ &= 0 + Tx = TPx = TPw + TPx = TPv \end{aligned}$$

for all $v \in V$ and $PT = TP$. □

Remark 7.4. One can easily generalize from the case

$$V = W_1 \oplus W_2$$

that we did to the case

$$V = W_1 \oplus \ldots \oplus W_n$$

by induction on n as

$$V = W_i \oplus \left(W_1 \oplus \ldots \oplus \underbrace{\hat{W}_i}_{\text{omit}} \oplus \ldots \oplus W_n \right)$$

Construction: Let

$$V = W_1 \oplus \ldots \oplus W_n$$

as above. Define

$$P_{W_i} : V \to V$$

to be the projection along $W_1 \oplus \ldots \oplus \hat{W}_i \oplus \ldots \oplus W_n$, i.e.

$$\ker P_{W_i} = W_1 \oplus \ldots \oplus \hat{W}_i \oplus \ldots \oplus W_n$$

and onto $W_i = \text{im } P_{W_i}$ as in the above Proposition. Then we have

a) Each P_{W_i} is linear (and a projection).

b) $\ker P_{W_i} = W_1 \oplus \ldots \oplus \hat{W}_i \oplus \ldots \oplus W_n$.

c) W_i is P_{W_i}-invariant and $P_{W_i}\big|_{W_i} = 1_{W_i}$. In particular, $\text{im } P_{W_i} = W_i$.

d) $P_{W_i} P_{W_j} = \delta_{ij} P_{W_i}$ where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

e) $1_V = P_{W_1} + \ldots + P_{W_n}$.

Moreover, if $T : V \to V$ is linear and each W_i is T-invariant, then

$$TP_{W_i} = P_{W_i}T, \quad i = 1, \ldots, n$$

Hence

$$\begin{aligned} T = T1_V &= T\left(P_{W_1} + \ldots P_{W_n}\right) = TP_{W_1} + \ldots + TP_{W_n} \\ &= P_{W_1}T + \ldots + P_{W_n}T \end{aligned}$$

i.e., $1_V T = T1_V$. This implies

$$T\big|_{W_i} : W_i \to W_i$$

is given by

$$T\big|_{W_i} = TP_{W_i}\big|_{W_i}$$

or T is determined by what it does to each W_i.

Remark 7.5. Compare this to the case that T is diagonalizable and the W_i are the eigenspaces.

Question 7.1. Let V be a real or complex finite dimensional inner product space, $T : V \to V$ hermitian. What can you replace $\oplus$ by? What if V is a complex finite dimensional inner product space and $T : V \to V$ is normal.

Exercise 7.1. Suppose V is a vector space over F, $P_1, \ldots, P_n : V \to V$ linear and satisfy

i) $P_i - P_j = \delta_{ij} P_i$, $i = 1, \ldots, n$

ii) $1_V = P_1 + \ldots + P_n$

iii) $W_i = \text{im } P_i$, $i = 1, \ldots, n$

Then

$$\begin{aligned} V &= W_1 \oplus \ldots \oplus W_n \\ P_i &= P_{W_i} \qquad i = 1, \ldots, n \end{aligned}$$

§7.2 Dual Spaces

Question 7.2. Let $V = \mathbb{R}^3$, $v \in V$. What is the first question that we should ask about v?

Motivation/Construction: Let V be a vector space over F, $\mathscr{B}$ a basis for V. Fix $v_0 \in \mathscr{B}$. By the UPVS, $\exists!\, f_{v_0} : V \to F$ linear satisfying

$$f_{vv_0}(v) = \begin{cases} 1 & \text{if } v_0 = v \\ 0 & \text{if } v_0 \neq v \end{cases} = \delta_{v,v_0} \quad \forall v \in \mathscr{B}$$

Example 7.6

Let $\mathscr{E}_n = \{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$ and in the above $e_1 = v_0 \dots$ Then

$$f_{e_1} : \mathbb{R}^n \to \mathbb{R} \text{ satisfies}$$

If $v = (\alpha_1, \dots, \alpha_n)$ in $\mathbb{R}^n$

$$v = \sum_{i=1}^{n} \alpha_i e_i$$

so

$$\begin{aligned} f_{e_1}(v) &= f_{e_1}\left(\sum_{i=1}^{n} \alpha_i e_i\right) \\ &= \sum_{i=1}^{n} \alpha_i f_{e_1}(e_i) = \sum_{i=1}^{n} \alpha_i \delta_{ii} = \alpha_1 \end{aligned}$$

this first coordinate of v.

Notation: If $A \subseteq B$ are sets, we write $A < B$ if $A \neq B$.
As $v_0 \neq 0$,

$$0 < \text{im } f_{v_0} \subseteq F \text{ is a subspace}$$

Notice $\dim_F F = 1$, so $\dim \text{im } f_{v_0} \leq \dim F = 1$ and

$$\dim \text{im } f_{v_0} = 1, \quad \text{i.e. im } f_0 = F$$

So $f_{v_0} : V \to F$ is a surjective linear transformation. Since this is true for all $v_0 \in \mathscr{B}$, for each $v \in \mathscr{B}$, $\exists!\, f_v : V \to F$ s.t.

$$f_v(v') = \delta_{v,v'} = \begin{cases} 1 & \text{if } v = v' \\ 0 & \text{if } v \neq v' \end{cases} \quad \forall v' \in \mathscr{B}$$

Now suppose that $x \in V$, then

$$\exists!\, \alpha_v \in F,\ v \in \mathscr{B}, \text{ almost all 0 s.t. } x = \sum_{\mathscr{B}} \alpha_v v$$

Hence

$$\begin{aligned} f_{v_0}(x) &= f_{v_0}\left(\sum_{v \in \mathscr{B}} \alpha_v v\right) = \sum_{\mathscr{B}} \alpha_v f_{v_0}(v) \\ &= \sum_{\mathscr{B}} \alpha_v \delta_{v,v_0} = \alpha_{v_0} \end{aligned}$$

Example 7.7

$\mathscr{B} = \mathscr{E}_n$ standard basis for $\mathbb{R}^n$

$$f_{e_i}(e_j) = \delta_{e_i,e_j} = \delta_{i,j} = \begin{cases} 1 & \text{if } e_i = e_j \\ 0 & \text{if } e_i \neq e_j \end{cases}$$

Then if $v = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n = V$. Then

$$f_{e_i}(v) = f_{e_i}(\alpha_1, \ldots, \alpha_n) = \alpha_i$$

So we observe in the above that if $x \in V$, then

$$x = \sum_{\mathscr{B}} f_v(x)v$$

We call f_v the coordinate function on v relative to $\mathscr{B}$.

Example 7.8

Let V be a finite dimensional inner product space over $\mathbb{R}$, $\mathscr{B} = \{v_1, \ldots, v_n\}$ an orthonormal basis. Then if $x = \sum_{\mathscr{B}} \alpha_i v_i$, then

$$\alpha_i = \langle x, v_i \rangle$$

Take

$$\begin{aligned} \langle x, v_i \rangle &= \langle \sum \alpha_j v_j, v_i \rangle = \sum \alpha_j \langle v_j, v_i \rangle \\ &= \sum \alpha_j \delta_{ij} \|v_i\|^2 = \sum \alpha_j \delta_{ij} = \alpha_i \end{aligned}$$

i.e. the linear map

$$f_{v_i} \coloneqq \langle , v_i \rangle : V \to \mathbb{R} \text{ by } x \mapsto \langle x, v_i \rangle$$

is the coordinate function on vectors relative to $\mathscr{B}$.

Definition 7.9 (Dual Space) — Let V be a vector space over F. A linear transformation $f : V \to F$ is called a linear functional. Set

$$V^* \coloneqq L(V, F) \coloneqq \{f : V \to F | f \text{ is linear}\}$$

is called the dual space of V.

Proposition 7.10

Let V, W be a vector space over F. Then

$$L(V,W) \coloneqq \{T : V \to W | T \text{ linear}\}$$

is a vector space over F. Moreover, if V, W are finite dimensional vector spaces over F

$$\dim L(V,W) = \dim V \dim W$$

In particular, if V is a finite dimensional vector space over F, then so is V^* and

$$\dim V = \dim V^*$$

so

$$V \cong V^*$$

Proof. 115A. □

Example 7.11

Let V be a vector space over F. Then the following are linear functionals

1. $0 : V \to F$
2. Let $0 \neq v_0 \in V$ then $\{v_0\}$ is a basis for Fv_0. Therefore, $\{v_0\}$ extends to a basis $\mathscr{B}$ for V. Let $fv_0 \in V^*$ be the coordinate function for V on v_0 relative to $\mathscr{B}$. Then $fv_0 \in \mathscr{B}^* \coloneqq \{fv | v \in \mathscr{B}\}$.

§8 Lec 8: Apr 14, 2021

§8.1 Dual Spaces (Cont'd)

Example 8.1 (Cont'd from Lec 7)

3. trace: $\mathbb{M}_n F \to F$ by

$$A \mapsto \sum_{i=1}^{n} A_{ii}$$

4. $\alpha < \beta \in \mathbb{R}$, then

$$I : C[\alpha, \beta] \to \mathbb{R} \text{ by } f \mapsto \int_{\alpha}^{\beta} f$$

5. Fix $\gamma \in [\alpha, \beta]$, $\alpha < \beta \in \mathbb{R}$. Then the evaluation map at γ

$$e_\gamma : C[\alpha, \beta] \to \mathbb{R} \text{ by } f \mapsto f(\gamma)$$

Lemma 8.2

Let V be a vector space over F, $\mathscr{B}$ a basis for V,

$$\mathscr{B}^* \coloneqq \{fv_0 : V \to F | \text{ coordinate function on } v_0 \text{ relative to } \mathscr{B}\}$$

so

$$fv_0(v) = \delta_{v_0, v} \qquad \forall v \in \mathscr{B}$$

the set of coordinate functions relative to $\mathscr{B}$. Then $\mathscr{B}^* \subseteq V^*$ is linearly indep.

Proof. Suppose

$$0 = 0_{V^*} = \sum_{v \in \mathscr{B}} \beta v f v, \quad \beta v \in F \text{ almost all } 0$$

We need to show $\beta v = 0 \, \forall v \in \mathscr{B}$. Evaluation at $v_0 \in \mathscr{B}$ yields

$$\begin{aligned} 0 = 0_{V^*}(v_0) &= \left(\sum_{\mathscr{B}} \beta v f v \right)(v_0) = \sum \beta v f v(v_0) \\ &= \sum_{\mathscr{B}} \beta v f_{v, v_0} = \beta v_0 \end{aligned}$$

So $\beta v = 0 \, \forall v \in \mathscr{B}$ and the lemma follows. □

Corollary 8.3

Let V be a vector space over F with basis $\mathscr{B}$. Then the linear transformation

$$D_{\mathscr{B}} : V \to V^* \text{ induced by } \mathscr{B} \to \mathscr{B}^* \text{ by } v \mapsto fv$$

is a monomorphism.
In particular, if V is a finite dimensional vector space over F, then $\mathscr{B}^*$ is a basis for V^* and

$$D_{\mathscr{B}} : V \to V^* \text{ is an isomorphism}$$

Proof. By the Monomorphism Theorem, $D_{\mathscr{B}}$ is monic in view of he lemma if V is a finite dimensional vectors space over F, then

$$\dim V = \dim V^*$$

so $V \cong V^*$ by the Isomorphism Theorem. □

Remark 8.4.

1. If $V = \mathbb{R}_f^\infty := \{(\alpha_1, \alpha_2, \ldots) \,|\alpha_i \in \mathbb{R} \text{ almost all } 0\}$, then by HW1 # 4,

$$D\mathscr{E}_\infty : V \to V^* \text{ is not an isomorphism}$$

2. $D_{\mathscr{B}} : V \to V^*$ in the corollary depends on $\mathscr{B}$. There exists no monomorphism $V \to V^*$ that does not depend on a choice of basis. However, there exists a "nice" monomorphism, i.e., defined independent of basis.

$$L : V \to (V^*)^* =: V^{**}$$

V^{**} is called the double dual of V. We now construct it.

Lemma 8.5

Let V be a vector space over F, $v \in V$. Then

$$L_v : V^* \to F \text{ by } f \mapsto L_v(f) := f(v)$$

the evaluation map at v is linear, i.e.

$$L_v \in V^{**}$$

Proof. For all $f, g \in V^*$, $\alpha \in F$

$$L_v(\alpha f + g) = (\alpha f + g)(v) = \alpha f(v) + g(v) = \alpha L_v f + L_v g$$ □

Theorem 8.6

The "natural" map

$$L : V \to V^{**} \text{ by } v \mapsto L(v) := L_v$$

is a monomorphism.

Proof. L is linear: Let $v, w \in V$, $\alpha \in F$. Then for all $f \in V^*$, as $V^{**} = (V^*)^*$

$$\begin{aligned} L(\alpha v + w)(f) &= L_{\alpha v + w}(f) = f(\alpha v + w) \\ &= \alpha f(v) + f(w) = \alpha L_v f + L_w f = (\alpha L_v + L_w)(f) \\ &= (\alpha L(v) + L(w))(f) \end{aligned}$$

So

$$L(\alpha v + w) = \alpha L(v) + L(w)$$

L is monic. Suppose $v \neq 0$. To show $L_v = L(v) \neq 0$. By example 2,

$$\exists 0 \neq f \in V^* \ni f(v) \neq 0$$

So

$$L_v f = f(v) \neq 0$$

so $L_v = L(v) \neq 0$ and L is monic. □

Corollary 8.7

If V is a finite dimensional vector space over F, then $L : V \to V^{**}$ is a natural isomorphism.

Proof. $\dim V = \dim V^* = \dim V^{**}$ and the Isomorphism Theorem. □

Identification: Let V be a finite dimensional vector space over F. Then $\forall v, w \in V$

1. $v = w \iff L_v = L_w$
2. $\forall f \in V^* \, f(v) = f(w) \iff L_v f = L_w f$

Moreover, if W is also a finite dimensional vector space over F, then if $T : V \to W$ is linear, $\exists! \, \tilde{T} : V^{**} \to W^{**}$ linear and if $\tilde{T} : V^{**} \to W^{**}$ $\exists! \, T : V \to W$ linear. In other words, V and V^{**} can be identified by

$$v \leftrightarrow L_v$$

because

$$L_v(f) = f(v) \qquad \forall v \in V \quad \forall f \in V^*$$

Construction: Let V be a finite dimensional vector space over F with basis $\mathscr{B} = \{v_1, \ldots, v_n\}$. Then

$$\mathscr{B}^* \coloneqq \{f_1, \ldots, f_n\}$$

defined by

$$f_i(v_j) = \delta_{ij} \qquad \forall i, j$$

i.e., f_i is the coordinate function on v_i relative to $\mathscr{B}$. Since

$$L_{v_i}(f_j) = f_j(v_i) = \delta_{ij} \qquad \forall i, j$$

$L_{v_i} \in V^{**}$

$$\mathscr{B}^{**} \coloneqq \{L_{v_1}, \ldots, L_{v_n}\}$$

is the dual basis of $\mathscr{B}^*$ for V^{**}. So we have if $x = \sum_{i=1}^n \alpha_i v_i \in V$, $g = \sum_{i=1}^n \beta_i f_i \in V^*$.

$$x = \sum_{i=1}^n \alpha_i v_i = \sum_{i=1}^n f_i(x) v_i$$
$$g = \sum_{i=1}^n \beta_i f_i = \sum_{i=1}^n L_{v_i}(g) f_i = \sum_{i=1}^n g(v_i) f_i$$

i.e.

$$x = \sum_{i=1}^n f_i(x) v_i \qquad \forall x \in V$$
$$g = \sum_{i=1}^n g(v_i) f_i \qquad \forall g \in V^*$$

Motivation: Let V be an inner product space over $\mathbb{R}$, $\emptyset \neq S \subseteq V$ a subset. What is $S^\perp$?
Note: $\forall v \in V$, $\langle , v\rangle : V \to \mathbb{R}$ by $x \mapsto \langle x, v\rangle$ is a linear functional. To generalize this to an arbitrary vector space over F, we define the following.

Definition 8.8 (Annihilator) — Let V be a vector space over F, $\emptyset \neq S \subseteq V$ a subset. Define the annihilator of S to be

$$\begin{aligned} S^\circ &:= \{f \in V^* |\, f(x) = 0 \,\forall x \in S\} \\ &= \{f \in V^* |\, f|_S = 0\} \subseteq V^* \end{aligned}$$

Remark 8.9. Many people write $\langle v, f\rangle$ for $f(v)$ in the above even though $f \notin v$.

§9 Lec 9: Apr 16, 2021

§9.1 Dual Spaces (Cont'd)

Lemma 9.1

Let V be a vector space over F, $\emptyset \neq S \subseteq V$ a subset. Then

1. $S^\circ \subseteq V^*$ is a subspace.
2. If V is a finite dimensional vector space over F and we identify V as V^{**} (by $v \leftrightarrow L_v$), then $S \subseteq S^{\circ\circ} \coloneqq (S^\circ)^\circ$.

Proof. 1. For all $f, g \in S^\circ$, $\alpha \in F$, we have

$$(\alpha f + g)(x) = \alpha f(x) + g(x) = 0 \qquad \forall x \in S$$

Hence $\alpha f + g \in S^\circ$ and $S^\circ \subseteq V^*$ is a subspace.

2. Let $x \in S$. Then $\forall f \in S^\circ$, we have

$$0 = f(x) = L_x f, \quad \text{so } L_x \in (S^\circ)^\circ = S^{\circ\circ}$$

□

Theorem 9.2

Let V be a finite dimensional vector space over F, $S \subseteq V$ a subspace. Then

$$\dim V = \dim S + \dim S^\circ$$

Proof. Let $\mathscr{B}_0 = \{v_1, \dots, v_k\}$ be a basis for S. Extend this to

$$\mathscr{B} = \{v_1, \dots, v_n\} \text{ a basis for } V$$
$$\mathscr{B}_0 = \{f_1, \dots, f_n\} \text{ the dual basis of } \mathscr{B}$$

Claim 9.1. $\mathscr{C} \coloneqq \{f_{k+1}, \dots, f_n\}$ is a basis for S°.

If we show this, the theorem follows. Let $f \in S^\circ$. Then

$$f = \sum_{i=1}^{n} L_{v_i}(f) f_i = \sum_{i=1}^{n} f(v_i) f_i$$
$$= \sum_{i=1}^{k} f(v_i) f_i + \sum_{i=k+1}^{n} f(v_i) f_i = \sum_{i=k+1}^{n} f(v_i) f_i$$

lies in span $\mathscr{C}$ so $\mathscr{C}$ spans. As $\mathscr{C} \subseteq \mathscr{B}^*$ which is linearly indep., so is $\mathscr{C}$. This proves the claim. □

Corollary 9.3
Let V be a finite dimensional vector space over F, $S \subseteq V$ a subspace. Then $S = S^{\circ\circ}$.

Proof. As $S \subseteq S^{\circ\circ}$, it suffices to show $\dim S = \dim S^{\circ\circ}$. By the theorem, we have

$$\dim V = \dim S + \dim S^{\circ}$$
$$\dim V^* = \dim S^{\circ} + \dim S^{\circ\circ}$$

where $\dim V = \dim V^*$. So $\dim S = \dim S^{\circ\circ}$. □

Remark 9.4. If V is an inner product space over $\mathbb{R}$, compare all this to $\emptyset \neq S \subseteq V$ a subset and $S^{\perp}, S^{\perp\perp}$.

§9.2 The Transpose

Construction: Fix $T : V \to W$ linear. For every $S : W \to X$, we have a composition

$$S \circ T : V \to X \text{ is linear}$$

So $T :\to W$ linear induces a map

$$T^{\star} : L(W, X) \to L(V, X)$$

by

$$S \mapsto S \circ T$$

Proposition 9.5
Let V, W, X be vector spaces over F, $T : V \to W$ linear. Then

$$T^{\star} : L(W, X) \to L(V, X)$$

is linear.

Proof. Let $S_1, S_2 \in L(W, X)$, $\alpha \in F$. Then

$$\begin{aligned} T^{\star}(\alpha S_1 + S_2) &= (\alpha S_1 + S_2) \circ T \\ &= \alpha S_1 \circ T + S_2 \circ T = \alpha T^{\star} S_1 + T^{\star} S_2 \end{aligned}$$

□

Corollary 9.6
Let $T : V \to W$ be linear. Then

$$T^* : W^* \to V^* \text{ by } f \mapsto f \circ T$$

is linear.

Proof. Let $X = F$ in the proposition. □

Definition 9.7 (Transpose) — Let $T : V \to W$ be linear. The linear map $T^\star : W^* \to V^*$ in the corollary is called the transpose of T and denoted by $T^\top$.

Note: The transpose "turns thing around"

$$V \xrightarrow{T} W$$
$$V^* \xleftarrow{T^\top} W^*$$

Lemma 9.8

Let $T : V \to W$ be linear. Then

$$\ker T^\top = (\text{im T})^\circ \in W^*$$

Proof. $g \in \ker T^\top \iff T^\top g = 0 \iff (T^\top g)(v) = 0 \ \forall v \in V \iff (g \circ T)(v) = 0 \ \forall v \in V \iff g(Tv) = 0 \ \forall v \in V \iff g \in (\text{im } T)^\circ$. □

Theorem 9.9

Let V, W be finite dimensional vector space over F, $T : V \to W$ linear. Then

$$\dim \text{im } T = \dim \text{im } T^\top$$

Proof. Consider:

$$\dim W^* = \dim \ker T^\top + \dim \text{im } T^\top$$
$$\dim W = \dim \text{im } T + \dim(\text{im } T)^\circ$$

Notice that $\dim W^* = \dim W$. By the lemma, $\dim \text{im } T = \dim \text{im } T^\top$. □

Computation: Let V, W be finite dimensional vector space over F.

$\mathscr{B}$, $\mathscr{B}^*$ ordered dual bases for V, V^*

$\mathscr{C}$, $\mathscr{C}^*$ ordered dual bases for W, W^*

Suppose

$$\mathscr{B} = \{v_1, \ldots, v_n\}, \quad \mathscr{B}^* = \{f_1, \ldots, f_n\}$$
$$f_i(v_j) = \delta_{ij} \quad \forall i, j$$

So

$$\mathscr{C} = \{w_1, \ldots, w_n\}, \quad \mathscr{C}^* = \{g_1, \ldots, g_n\}$$
$$g_i(w_j) = \delta_{ij} \quad \forall i, j$$

Let

$$A = [T]_{\mathscr{B},\mathscr{C}}, \qquad B = \left[T^\top\right]_{\mathscr{C}^*,\mathscr{B}^*}$$

be the matrix representation of $T, T^\top$ in the ordered bases $\mathscr{B}, \mathscr{C}$ and $\mathscr{C}^*, \mathscr{C}^*$ respectively. By definition of A and B, we have

$$Tv_k = \sum_{i=1}^{m} A_{ik} w_i \qquad k = 1, \dots, n$$

$$T^\top g_j = \sum_{i=1}^{n} B_{ij} f_i \qquad j = 1, \dots, m$$

So

$$B_{kj} = A_{jk} \qquad \forall j, k$$

So we just proved...

Theorem 9.10

Let V, W be finite dimensional vector space over F, $T : V \to W$ linear, $\mathscr{B}, \mathscr{B}^*$ ordered dual bases for V, V^* and $\mathscr{C}, \mathscr{C}^*$ ordered dual bases for W, W^*. Then

$$\left[T^\top\right]_{\mathscr{C}^*,\mathscr{B}^*} = \left([T]_{\mathscr{B},\mathscr{C}}\right)^\top$$

Definition 9.11 (Row/Column Rank) — Let $A \in F^{m\times n}$. The row (column) rank of A is the dimension of the span of the rows (columns) of A.

We know if $A \in F^{m\times n}$, we can view

$$A : F^{n\times 1} \to F^{m\times 1} \text{ by } v \mapsto A \cdot v$$

a linear transformation and the matrix representation of A is

$$A = [A]_{\mathscr{E}_{n,1},\mathscr{E}_{m,1}}$$

where $\mathscr{E}_{n,1}$, $\mathscr{E}_{m,1}$ are the standard bases for $F^{n\times 1}$ and $F^{m\times 1}$ respectively.

Corollary 9.12

Let $A \in F^{m\times n}$. Then

$$\text{row rank } A = \text{column rank } A$$

and we call this common number the rank of A.

§9.3 Polynomials

Definition 9.13 (Polynomial Division) — Let $f, g \in F[t]$, $f \neq 0$. We say that f divides $g \in F[t]$ write $f|g$ if $\exists h \in F[t]$ s.t. $g = fh$, i.e. g is multiple of f, e.g. $t+1|t^2-1$.

Lemma 9.14

If $f|g$ and $f|h$ in $F[t]$, then $f|gk + hl$ in $F[t]$ for all $k, l \in F[t]$.

Proof. By definition,

$$g = fg_1, \quad h = fh_1, \quad g_1, h_1 \in F[t]$$

So

$$gk + hl = fg_1k + fh_1l = f(g_1k + h_1l)$$

in $F[t]$. □

Remark 9.15. If $f|g \in F[t]$ and $0 \neq a \in F$, then $af|g$ and $f|ag$.

Definition 9.16 (Polynomial Degree and Leading Coefficient) — Let

$$0 \neq f = at^n + a_{n-1}t^{n-1} + \ldots + a_1t + a_0 \in F[t]$$

with $a, a_0, \ldots, a_{n-1} \in F$ and $a \neq 0$. We call n the degree of f write $\deg f = n$ and a the leading coefficient of F write lead $f = a$. If $a = 1$, we say f is monic.

We can define the degree of $0 \in F[t]$ to be the symbol $-\infty$ or just do not define it at all.

Remark 9.17. Let $f, g \in F[t] \setminus \{0\}$. Then

$$\text{lead}(fg) = \text{lead}(f) \cdot \text{lead}(g) \neq 0 \in F$$

So

$$\deg(fg) = \deg f + \deg g$$

§10 Lec 10: Apr 19, 2021

§10.1 Polynomials (Cont'd)

Division Algorithm: Let $0 \neq f \in F[t]$, $g \in F[t]$. Then

$$\exists!\ q,\ r \in F[t]$$

satisfying

$$g = fq + r \quad \text{with} \quad r = 0 \quad \text{or} \quad \deg r < \deg f$$

Definition 10.1 (Greatest Common Divisor) — Let $f, g \in F[t] \setminus \{0\}$. We say d in $F[t]$ is a gcd (greatest common divisor) of f, g if

i) d is monic.

ii) $d|f$ and $d|g$ in $F[t]$.

iii) if $e|f$ and $e|g$ in $F[t]$, then $e|d$ in $F[t]$.

Remark 10.2. If a gcd of f, g exists, then it is unique.

Remark 10.3. If $d = 1$ is a gcd of $f, g \in F[t]$, we say that f, g are relatively bear.

Remark 10.4. Compare the above with analogous in $\mathbb{Z}$.

Theorem 10.5

Let $f, g \in F[t] \setminus \{0\}$. Then a gcd of f, g exists and is unique write gcd(f,g) for the gcd of f, g. Moreover, we have an equation

$$d = fk + gl \in F[t] \text{ for some } k, l \in F[t] \qquad (\star)$$

Proof. The existence and $(\star)$ follow from the Euclidean Algorithm. Let $f, g \in F[t] \setminus \{0\}$. Then iteration of the Division Algorithm produces equations in $F[t]$, if $f + g \in F[t]$,

$$\begin{aligned}
g &= q_1 f + r_1 && \deg r_1 < \deg f \\
f &= q_2 r_1 + r_2 && \deg r_2 < \deg r_1 \\
&\vdots \\
r_{n-3} &= q_{n-1} r_{n-2} + r_{n-1} && \deg r_{n-1} < \deg r_{n-2} \\
r_{n-2} &= q_n r_{n-1} + r_n && \deg r_{n-1} < \deg r_n \\
r_{n-1} &= q_{n+1} + r_n
\end{aligned}$$

where r_n is the remainder of least degree ($r_n \neq 0$).
This must stop in $\leq \deg f$ steps. Plugging from the bottom up and using the lemma shows

$$r_n = fk + gl \in F[t]$$

and if $e|r_1 \to e|r_2 \to \ldots \to e|r_n$ then $(\text{lead } r_n)^{-1} r_n$ is the gcd of f and g in $F[t]$ if $a = \text{lead } f$

$$a^{-1} r_n = a^{-1} fk + a^{-1} gl$$ □

Definition 10.6 (Irreducible Polynomial) — $f \in F[t] \setminus F$ is called irreducible if there does not exist $g, h \in F[t] \ni f = gh$ with $\deg g, \deg h < \deg f$. Equivalently, if

$$f = gh \in F[t], \quad \text{then } 0 \neq g \in F \text{ or } 0 \neq h \in F$$

Example 10.7

If $f \in F[t]$, $\deg f = 1$, then f is irreducible.

Remark 10.8. If $f, g \in F[t] \setminus F$ with f irreducible, then either f and g are relatively prime or $f|g$ since only $a, af, 0 \neq a \in F$ can divide f.

Lemma 10.9 (Euclid)

Let $f \in F[t]$ be irreducible and $f|gh$ in $F[t]$. Then $f|g$ or $f|h$.

Proof. Suppose $f \times g$ where $\times$ means does not divide. Then f and g are relatively prime. By the Euclidean Algorithm, there exists an equation

$$1 = fk + gl \in F[t]$$

Hence

$$h = fhk + ghl \in F[t]$$

As $f|fhk$ and $f|ghl$ in $F[t]$, $f|h$ by the lemma. □

Remark 10.10. In $\mathbb{Z}$ the analog of an irreducible element is called a prime element.

Remark 10.11. Euclid's lemma is the key idea. The "correct" generalization of "prime" is the conclusion of Euclid's lemma. This generalization is profound as, in general, there is difference between the two conditions "irreducible" and "prime", although not for $\mathbb{Z}$ or $F[t]$.

We know that any positive integer is a product of positive primes unique up to order n. If we allow $n < 0$ such is unique up to ± 1.

Theorem 10.12 (Fundamental Theorem of Arithmetic (Polynomial Case))
Let $g \in F[t] \setminus F$. Then there exists uniquely $a \in F$, $r \in \mathbb{Z}^+$, $p_1, \dots, p_r \in F[t]$ distinct monic irreducible polynomial, $e_1, \dots, e_r \in \mathbb{Z}^+$ s.t. we have a factorization

$$g = ap_1^{e_1} \dots p_r^{e_r}$$

unique up to order.

Proof. (Sketch) Existence: We induct on $n = \deg g \geq 1$. If g is irreducible, $a, (\text{lead } g)^{-1}g$, $a = \text{lead } g$ work. If g is reducible,

$$g = fh \in F[t], \qquad 1 < \deg f, \quad \deg h < \deg g$$

By induction, f, h have factorization hence we're done as $g = fh$.
Uniqueness: We induct on $n = \deg g \geq 1$. If

$$ap_1^{e_1} \dots p_r^{e_r} = g = bq_1^{f_1} \dots q_s^{f_s}$$

with p_i, q_i monic irreducible, $a, b \in F$, $e_i, f_j \in \mathbb{Z}^+$ for all i, j, $\deg q_1 \geq 1$, so $\deg q_1 \times a$. By Euclid's lemma

$$q_i | p_j \text{ for some } j$$

Changing notation, we may assume that $j = 1$. As p_1 is irreducible $p_1 = q_1$ and by $(M3')$

$$g_0 \coloneqq ap_1^{e_1 - 1}p_2^{e_2} \dots p_r^{e_r} = bq_1^{f_1 - 1}q_2^{f_2} \dots q_s^{f_s}$$

As $\deg g_0 < \deg g$, induction yields

$$r = s,\ e_1 - 1 = f_1 - 1,\ e_i = f_i,\ i > 1,\ a = b = \text{lead } g_0,\ p_i = q_i\ \forall i,\ e_i = f_i\ \forall i \qquad \square$$

Remark 10.13. Applying the Euclidean Algorithm is relatively fast to compute, (for $f|g$ takes $\leq \deg f$ steps to get a gcd). Factoring into the irreducible is not.

§11 Lec 11: Apr 21, 2021

§11.1 Minimal Polynomials

We use the following theorem from 115A, Matrix Theory Theorem.

Remark 11.1. Let $T : V \to V$ be linear. If $f = a_n t^n + \ldots + a_1 t + a_0 \in F[t]$, we can plug T in for t to get

$$f(T) = a_n T^n + \ldots + a_1 T + a_0 1_V \in L(V, V)$$

More precisely

$$e_T : F[t] \to L(V, V) \text{ by } t \mapsto T$$

i.e. $f = \sum a_i t^i \mapsto f(T) = \sum a_i T^i$ is a ring homomorphism. Since we have

$$T^n = T \underbrace{\circ \ldots \circ}_{n} T, \qquad n \geq 0$$

Can we use the remark if V is a finite dimensional vector space over F?

Lemma 11.2

Let V be a finite dimensional vector space over F, $f, g, h \in F[t]$, $\mathscr{B}$ an ordered basis for V, $T : V \to V$ linear. Then

1. $[g(T)]_{\mathscr{B}} = g\left([T]_{\mathscr{B}}\right)$
2. If $f = gh \in F[t]$, then
$$f(T) = g(T)h(T)$$

Proof.

- By MTT, if $g = \sum_{i=0}^{n} a_i t^i \in F[t]$, then
$$[g(T)]_{\mathscr{B}} = \left[\sum_{i=0}^{n} a_i T^i\right]_{\mathscr{B}} = \sum_{i=0}^{n} a_i \left[T^i\right]_{\mathscr{B}}$$
$$= \sum a_i \left[T\right]^i_{\mathscr{B}} = g\left([T]_{\mathscr{B}}\right)$$
- Left as exercise.

□

Lemma 11.3

Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Then $\exists q \in F[t] \setminus \{0\} \ni q(T) = 0$ and if $a = \text{lead } q$, then $q_0 \coloneqq a^{-1} q$ is moinc and satisfies $q_0(T) = 0$

$$q \in \ker e_T \coloneqq \{f \in F[t] | \, f(T) = 0\}$$

Proof. Let $n = \dim V$. By MTT

$$\dim L(V, V) = \dim \mathbb{M}_n F = n^2 < \infty$$

So

$$1_V, T, T^2, \ldots, T^{n^2} \in L(V,V)$$

are linearly dependent. So $\exists a_0, \ldots, a_{n^2} \in F$ not all 0 s.t.

$$\sum_{i=0}^{n^2} a_i T^i = 0$$

Then $q = \sum_{i=0}^{n^2} a_i t^i$ works. □

Theorem 11.4

Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Then $\exists! 0 \neq q_T \in F[t]$ monic called the minimal polynomial of T having the following properties:

1. $q_T(T) = 0$
2. If $g \in F[t]$ satisfies $g(T) = 0$, then $q_T | g \in F[t]$. In particular, if $0 \neq g \in F[t]$ satisfies $g(T) = 0$, then $\deg g \geq \deg q_T$ and if $\deg g = \deg q_T$, then $g = (\text{lead } g) q_T$

Proof. By the lemma, $\exists 0 \neq q \in F[t]$ monic s.t. $q(T) = 0$. Among all such q, choose one with $\deg q$ minimal.

Claim 11.1. q works.

Let $g \neq 0$ in $F[t]$ satisfy $g(T) = 0$. To show $q|g \in F[t]$. Write $g = qh + r$ in $F[t]$ with $r = 0$ or $\deg r < \deg q$. Then

$$0 = g(T) = q(T)hh(T) + r(T) = r(T)$$

If $r \neq 0$, then $r_0 = (\text{lead } r)^{-1} r$ is a monic poly satisfying $r_0(T) = 0$, $\deg r_0 < \deg q$, contradicting the minimality of $\deg q$. So $r_0 = 0$ and $q|g \in F[t]$. If q' also satisfies 1) and 2), then

$$q|q' \text{ and } q'|q \in F[t] \text{ both monic so } q = q'$$

The last statement follows as if

$$h, g \in F[t], \quad g|h,\ h \neq 0, \text{ then } \deg h \geq \deg q$$ □

Corollary 11.5

Let V be a finite dimensional vector space over F, $\mathscr{B}$ an ordered basis for V_1 and $T : V \to V$ linear. Then

$$q_T = q_{[T]_{\mathscr{B}}}$$

In particular, if $A, B \in \mathbb{M}_n F$ are similar write $A \sim B$. Then

$$q_A = q_B$$

Proof. $q_T = q_{[T]_{\mathscr{B}}}$ by MTT and the first lemma. □

Note:By the theorem, if V is a finite dimensional vector space over F $g \in F[t]$ $g \neq 0$, and $\deg g < \deg q_T$, then $q(T) \neq 0$.
Goal: Let V be a finite dimensional vector space over F, $\mathscr{B}$ an ordered basis of V, $T : V \to V$ linear. Call

$$tI - [T]_{\mathscr{B}} \text{ the characteristics matrix of } T \text{ relative to } \mathscr{B}$$

Recall the characteristics polynomial f_T of T is defined to be

$$f_T \coloneqq f_{[T]_{\mathscr{B}}} = \det\left(tI - [T]_{\mathscr{B}}\right) \in F[t]$$

We want to show f_T satisfies the

Theorem 11.6 (Cayley-Hamilton)
If V is a finite dimensional vector space over F, $T : V \to V$ linear, then

$$q_T | f_T, \quad \text{hence } f_T(T) = 0$$

In particular, $\deg q_T \leq \deg f_T$.

Remark 11.7.
1. There exists a determinant proof of this – essentially Cramer's rule.
2. A priori we only know $\deg q_T \leq n^2$, where $n = \dim V$.
3. f_T is independent of $\mathscr{B}$ depends on properties of $\det : \mathbb{M}_n F[t] \to F[t]$

$$\begin{aligned} \det(tI - A) &= \det\left(P(tI - A)P^{-1}\right) \\ &= \det\left(tI - PAP^{-1}\right) \end{aligned}$$

for each $P \in GL_n F$

Proposition 11.8
Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Then q_T and f_T have the same roots in F, the eigenvalues of T.

Proof. Let λ be a root of q_T. To show λ is an eigenvalue of T, i.e., a root of f_T. As λ is a root of q_T, using the Division Algorithm that

$$q_T = (t - \lambda)h \in F[t]$$

So

$$0 = q_T(T) = (T - \lambda 1_V)h(T)$$

As

$$0 \leq \deg h < \deg q_T, \quad \text{we have } h(T) \neq 0$$

Since $h(T) \neq 0 \exists 0 \neq v \in V$ s.t.

$$w = h(T)v \neq 0$$

Then

$$0 = q_T(T)v = (T - \lambda 1_V)\, h(T)v = (T - \lambda 1_V)\, w$$

So $0 \neq w \in E_T(\lambda)$ and λ is an eigenvalue of T.
Conversely, suppose λ is a root of f_T so an eigenvalue of T. Let $0 \neq v \in E_T(\lambda)$. Then $t - \lambda \in F[t]$ satisfies $(T - \lambda)w = 0$ for all $w \in Fv$, i.e. it is the minimal poly of $T|_{Fv} : Fv \to Fv$. But $q_T(T) = 0$ on V so $t - \lambda | q_T$ by the definition that $t - \lambda$ is the minimal poly of $T|_{Fv}$. □

§11.2 Algebraic Aside

Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Te minimality poly q_T of T is algebraically more interesting than f_T. Recall we have a ring homomorphism

$$e_T : F[t] \to L(V, V)$$

given by

$$\sum a_i t^i \mapsto \sum a_i T^i$$

so e_T is not only a linear transformation but a ring homomorphism, i.e., it also follows that

$$(fg)(T) = f(T)g(T) \qquad \forall f, g \in F[t]$$

We know that

$$\dim_F F[t] = \infty$$

which has $\{1, t, \ldots, t^n, \ldots\}$ is a basis for $F[t]$ and

$$\dim_F L(V, V) = (\dim V)^2 < \infty$$

by MTT. So

$$0 < \ker e_T \coloneqq \{f \in F[t] | e_T f = f(T) = 0\}$$

is a vector space over F and a subspace of $F[t]$. This induces a linear transformation

$$\overline{e_T} : V/\ker e_T \to \text{im } e_T = F[T]$$

which is an isomorphism. If $\overline{V} = V/\ker T$, we have

$$\begin{aligned} \overline{e_T}\left(\overline{\sum a_i t^i}\right) = \overline{e_T\left(\sum a_i t^i\right)} &= \sum \overline{a_i}\overline{T}^i \\ &= \sum a_i \overline{T}^i = \sum a_i T^i \end{aligned}$$

Check that $\bar{e}_T$ is also a ring isomorphism onto im e_T. By definition, if $f(T) = 0$, $f \in F[t]$, then

$$q_T | f \in F[t]$$

It follows that

$$\ker e_T = \{q_t g | g \in F[t]\} \subseteq F[t]$$

called an ideal in the ring $F[t]$.
The first isomorphism of rings gives rise to $\ker e_T$ whit quotient isomorphic to $F[t] \subseteq L(V, V)$. So we are at a higher level of algebra. Then this allows us to view $F[t]$ as acting on V, i.e. there exists a map

$$F[t] \times V \to V \qquad (*)$$

by

$$f \cdot v \coloneqq f(T)v$$
$$q_T(T) = 0$$

This turns V into what is called an $F[t]$-module, i.e., V via (*) satisfies the axioms of a vector space over F but the scalars $F[t]$ are now a ring rather than only a field.

§12 | Lec 12: Apr 23, 2021

§12.1 Triangularizability

Proposition 12.1

Let V be a finite dimensional vector space over F, $T : V \to V$ linear, $W \subseteq V$ a T-invariant subspace. Then T induces a linear transformation

$$\overline{T} : V/W \to V/W \text{ by } \overline{T}(\overline{v}) := \overline{T(v)}$$

where $\overline{v} = W + v$, $\overline{V} = V/W$ and

$$q_{\overline{T}} | q_T \in F[t]$$

Proof. By the hw, we need only to prove that

$$q_{\overline{T}} | q_T \in F[t]$$

But also by the hw,

$$q_T(\overline{T}) = \overline{q_T(T)}$$

As $q_T(T) = 0$,

$$0 = \overline{q_T(T)} = q_T(\overline{T})$$

so

$$q_{\overline{T}} | q_T$$

by the defining property of $q_{\overline{T}}$. □

Definition 12.2 (Triangularizability) — Let V be a finite dimensional vector space over F, $T : V \to V$ linear. We say T is triangularizable if $\exists$ an ordered basis $\mathscr{B}$ for V s.t. $A = [T]_{\mathscr{B}}$ satisfies $A_{ij} = 0 \ \forall i < j$, i.e.

$$A = \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \text{ is lower triangular} \qquad (*)$$

Note: If $\mathscr{B} = \{v_1, \ldots, v_n\}$ in (*) and $\mathscr{C} = \{v_n, v_{n-1}, \ldots, v_1\}$, then

$$[T]_{\mathscr{C}} = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \text{ is upper triangular}$$

Hence, by Change of Basis Theorem,

$$[T]_{\mathscr{B}} \sim [T]_{\mathscr{C}}$$

Remark 12.3. Suppose V is a finite dimensional vector space over F, $\dim V = n$, $T : V \to V$ linear, $\mathscr{B}$ an ordered basis for V, $A = [T]_{\mathscr{B}}$ is triangular (upper or lower). Then

$$f_T = (t - A_{11}) \dots (t - A_{nn}) \in F[t]$$

and $A_{11}, \dots, A_{nn}$ are all the eigenvalues of T (not necessarily distinct) and hence roots of q_T.

Definition 12.4 (Splits) — We say $g \in F[t] \setminus F$ splits in $F[t]$ if g is a product of linear polys in $F[t]$, i.e.,

$$g = (\text{lead } g)(t - \alpha_1) \dots (t - \alpha_n) \in F[t]$$

Example 12.5

If V is a finite dimensional vector space over F, $T : V \to V$ linear and T is triangularizable, then f_T splits in $F[t]$.

Note: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{M}_2\mathbb{R}$ is not triangularizable as it has no eigenvalues.

Theorem 12.6

Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Then T is triangularizable if and only if q_T splits in $F[t]$.

Proof. " $\implies$ " We induct on $n = \dim V$.
$n = 1$: It's obvious.
$n > 1$: We proceed by induction: let λ be a root of q_T in F (q_T splits in $F[t]$). Then λ is a root of q_T hence an eigenvalue of T. Let $0 \neq v_n \in E_T(\lambda)$, so $W = Fv_n$ is T-invariant. By the Proposition, T induces a linear map

$$\overline{T} : V/W \to V/W \text{ by } \overline{v} \mapsto \overline{T(v)}$$

and

$$q_{\overline{T}} | q_T \in F[t]$$

We also know that

$$W = \ker(- : V \to V/W) \text{ by } v \mapsto \overline{v}$$

and

$$\dim V/W = \dim V - \dim W = n - 1$$

as $- : v \to \overline{v}$ is epic. Since q_T splits in $F[t]$ and $q_{\overline{T}} | q_T$ in $F[t]$, $q_{\overline{T}}$ also splits in $F[t]$ by Fundamental Theorem of Algebra. Thus, by induction,

$$\exists v_1, \dots, v_{n-i} \in V \ni \mathscr{C} = \{\overline{v_1}, \dots, \overline{v}_{n-1}\}$$

is an ordered basis for $\overline{V} = V/W$ with $A = [\overline{T}]_{\mathscr{C}}$ is lower triangular, i.e., $A_{ij} = 0$ if $i < j \leq n - 1$. Thus

$$\overline{T}\overline{v}_j = \sum_{i=j}^{n-1} A_{ij}\overline{v}_i, \qquad 1 \leq j \leq n - 1$$

hence

$$0 = \overline{T}\overline{v}_j - \sum_{i=j}^{n-1} A_{ij}\overline{v}_i = \overline{Tv_j - \sum_{i=j}^{n-1} A_{ij}v_i}$$

$1 \le j \le n-1$ in $\overline{V} = V/W$. Therefore,

$$Tv_j - \sum_{i=j}^{n-1} A_{ij}v_i \in \ker^- = W = Fv_n$$

by definition as $W = \ker^- : V \to V/W$.
In particular, $\exists A_{nj} \in F$, $1 \le j \le n-1$ satisfying

$$Tv_j - \sum_{i=j}^{n-1} A_{ij}v_i = A_{nj}v_n$$

So

$$Tv_j = \sum_{i=j}^{n} A_{ij}v_n \qquad 1 \le j \le n-1$$

By choice, $A_{ij} = 0$, $i < j \le n-1$ and

$$Tv_n = \lambda v_n$$

By hw 2 # 3, $\mathscr{B} = \{v_1, \dots, v_n\}$ is an ordered basis for V and

$$[T]_{\mathscr{B}} = \begin{pmatrix} [\overline{T}]_{\mathscr{C}} & 0 \\ & \vdots \\ & 0 \\ A_{n1} \dots A_{n,n-1} & \lambda \end{pmatrix}$$

which is lower triangular, as needed. " $\implies$ " Let $\mathscr{B} = \{v_1, \dots, v_n\}$ be an ordered basis for V. $A = [T]_{\mathscr{B}}$ is lower triangular. Then

$$f_T = \prod_{i=1}^{n} (t - A_{ii}) \text{ splits in } F[t]$$

$A_{11}, \dots, A_{nn}$ are the (not necessarily distinct) eigenvalues of T and hence roots of q_T.
Let $\lambda_i = A_{ii}$, $i = 1, \dots, n$. We have

$$Tv_j = \sum_{i=1}^{n} A_{ij}v_i = \lambda_j v_j + \sum_{i=j+1}^{n} A_{ij}v_i, \qquad 1 \le j \le n-1$$
$$Tv_n = \lambda_n v_n$$

So

$$(T - \lambda_j 1_V)v_j = \sum_{i=j+1}^{n} A_{ij}v_i \in \text{Span}\,(v_{j+1}, \dots, v_n) \quad \forall 1 \le j \le n-1 \qquad (*)$$

Now

$$(T - \lambda_n 1_V)v_n = 0$$

So

$$(T - \lambda_n 1_V)v_{n-1} \in \text{Span}(v_n) \text{ by } (*)$$

This implies

$$(T - \lambda_n 1_V)(T - \lambda_{n-1} 1_V)v_{n-1} = 0$$

By induction, we may assume that

$$(T - \lambda_n 1_V) \dots (T - \lambda_j 1_V)v_j = 0$$

So by (*),

$$(T - \lambda_n 1_V) \dots (T - \lambda_j 1_V)(T - \lambda_{j-1} 1_V)v_{j-1} = 0$$

Therefore,

$$f_T(T)v_i = (T - \lambda_n 1_V) \dots (T - \lambda_i 1_V)v_i = 0$$

for $i = 1, \dots, n$. As $\mathscr{B}$ is a basis for V, $f_T(T) = 0$. Thus $q_T | f_T \in F[t]$. In particular, q_T splits in $F[t]$. □

Corollary 12.7

Let V be a finite dimensional vector space over F, $T : V \to V$ a triangularizable linear operator. Then

$$q_T | f_T \in F[t]$$

In particular,

$$f_T(T) = 0$$

Definition 12.8 (Algebraically Closed) — A field F is called algebraically closed if every $f \in F[t] \setminus F$ splits in $F[t]$. Equivalently, $f \in F[t] \setminus F$ has a root in F.

Corollary 12.9 (Cayley-Hamilton – Special Case)

Let F be algebraically closed, V a finite dimensional vector space over F, $T : V \to V$ linear. Then

1. T is triangularizable.
2. $q_T | f_T$
3. $f_T(T) = 0$

Theorem 12.10 (Fundamental Theorem of Algebra)

(FTA) $\mathbb{C}$ is algebraically closed.

Proof. It's assumed (proven in 132 – Complex Analysis or 110C – Algebra). □

§13 Lec 13: Apr 26, 2021

§13.1 Triangularizability (Cont'd)

Remark 13.1. Let V be a finite dimensional vector space over F, $T : V \to V$ linear, $\mathscr{B}$ an ordered basis for V, $A = [T]_{\mathscr{B}}$. So $q_A = q_T$ and $f_A = f_T$.

Let $n = \dim V$. Given a field F, $\exists \tilde{F}$ an algebraically closed field satisfying $F \subseteq \tilde{F}$ is a subfield. Then

$$A \in \mathbb{M}_n F \subseteq \mathbb{M}_n \tilde{F}$$

So by the corollary,

$$f_A(A)v = 0 \qquad \forall v \in \tilde{F}^{n\times 1}$$

where we view $A : \tilde{F}^{n\times 1} \to \tilde{F}^{n\times 1}$ linear. Then

$$f_A(A)v = 0 \qquad \forall v \in F^{n\times 1} \subseteq \tilde{F}^{n\times 1}$$

viewing

$$A : F^{n\times 1} \to F^{n\times 1} \text{ linear}$$

Thus,

$$f_A(A) = 0$$

Hence $f_T(T) = 0$ and $q_T = q_A | f_A = f_T$. So $q_T | f_T$ in $F[t]$. Thus, if we knew such an $\tilde{F}$ exists in general, we would have proven the Cayley-Hamilton Theorem in general, i.e., if V is a finite dimensional vector space over F and $T : V \to V$ linear, then

$$q_T | f_T \in F[t]$$
$$f_T(T) = 0$$

This is, in fact, true (and proven in Math 110C). Of course, assuming FTA, this proves Cayley-Hamilton for all fields $F \subseteq \mathbb{C}$.

Remark 13.2. The symmetric matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2 \mathbb{F}_2 \text{ and } \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \in \mathbb{M}_2 \mathbb{F}_5$$

are both triangularizable, but not diagonalizable.

§13.2 Primary Decomposition

Algebraic Motivation: Let $f \in F[t] \setminus F$ be monic. Write

$$f = p_1^{e_1} \dots p_r^{e_r},\ p_1, \dots, p_r \text{ distinct monic}$$

irreducible polys in $F[t]$, $e_i > 0\, \forall i$. Set

$$q = \frac{f}{p_i^{e_i}} = p_1^{e_1} \dots p_i^{e_i} \dots p_r^{e_r}$$

Then p_i, q_i are relatively prime so there exists an equation

$$1 = p_i^{e_i} k_i + q_i g_i \in F[t], \quad i = 1, \dots, n \tag{*}$$

if we plug a linear operator $T : V \to V$ into (*), we get

$$1_V = p_i^{e_i}(T) k_1(T) + q_i(T) g_i(T) \qquad \forall i$$

Linear Algebra Motivation: Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Suppose

$$V = W_1 \oplus W_2, \quad W_1, W_2 \subseteq V \text{ subspaces}$$

with W_1, W_2 both T-invariant.
Let $\mathscr{B}_i$ be an ordered basis for W_i, $i = 1, 2$ and $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ an ordered basis for V. Then

$$[T]_{\mathscr{B}} = \begin{pmatrix} [T|_{W_1}]_{\mathscr{B}_1} & 0 \\ 0 & [T|_{W_2}]_{\mathscr{B}_2} \end{pmatrix}$$

Let $P_{W_i} : V \to V$ be the projection onto W_i along W_j, $j \neq i$. Then we know

$$\begin{aligned} 1_V &= P_{W_1} + P_{W_2} \\ P_{W_i} P_{W_j} &= \delta_{ij} P_{W_j} \\ P_{W_i} T &= T P_{W_i}, \quad i = 1, 2 \\ T &= T P_{W_1} + T P_{W_2} = T|_{W_1} + T|_{W_2} \end{aligned}$$

By hw 4 # 6

$$q_T = \operatorname{lcm}\left(q_T|_{W_1},\, q_T|_{W_2}\right)$$

This easily extends to more blocks.

Lemma 13.3

Let $f \in F[t]$, $T : V \to V$ linear. Then $\ker f(T)$ is T-invariant.

Proof. If $v \in \ker f(T)$, to show $Tv \in \ker f(T)$. But

$$f(T)\, Tv = T f(T) v = 0$$

so this is immediate. □

Lemma 13.4

Let $g, h \in F[t] \setminus F$ be relatively prime. Set $f = gh \in F[t]$. Suppose $T : V \to V$ is linear and $f(T) = 0$. Then

$$\ker g(T) \text{ and } \ker h(T) \text{ are } T\text{-invariant}$$

subspaces of V and

$$V = \ker g(T) \oplus \ker h(T) \tag{+}$$

Proof. By the lemma we just proved, we need only show (+). Since g, h are relatively prime, there exists equation

$$1 = gk + hl \in F[t]$$

Hence

$$1_V = g(T)k(T) = h(T)l(T)$$

as linear operators on V i.e. $\forall v \in V$

$$v = g(T)k(T)v + h(T)l(T)v \tag{*}$$

Since $f(T) = 0$ we have

$$0 = f(T)k(T)v = h(T)g(T)k(T)v$$

Therefore,

$$g(T)k(T)v \in \ker h(T)$$

and

$$0 = f(T)l(T)v = g(T)h(T)l(T)v$$

so

$$h(T)l(T)v \in \ker g(T)$$

It follows by (*), $\forall v \in V$

$$v = g(T)k(T)v + h(T)l(T)v \in \ker h(T) + \ker g(T)$$

where

$$V = \ker g(T) + \ker h(T)$$

By (*), if $v \in \ker g(T) \cap \ker h(T)$, then

$$v = g(T)k(T)v + h(T)l(T)v = 0$$

Hence

$$V = \ker g(T) \oplus \ker h(T)$$

as needed. □

§14 | Lec 14: Apr 28, 2021

§14.1 Primary Decomposition (Cont'd)

Proposition 14.1

Let V be a finite dimensional vector space over F, $T : V \to V$ linear, $g, h \in F[t] \setminus F$ monic and relatively prime. Suppose that

$$q_T = gh \in F[t]$$

Then $\ker g(T)$ and $\ker h(T)$ are T-invariant.

$$V = \ker g(T) \oplus \ker h(T)$$

and

$$g = q_T\big|_{\ker g(T)} \text{ and } h = q_T\big|_{\ker h(T)}$$

Proof. By the last lemma in last lecture, we need only prove the last statement. By definition, we have

$$g(T)\big|_{\ker g(T)} = 0 \text{ and } h(T)\big|_{\ker h(T)} = 0$$

So by definition,

$$q_T\big|_{\ker q(T)} | g \text{ and } q_T\big|_{\ker h(T)} | h \in F[t]$$

As g and h are relatively prime, by the FTA, so are

$$q_T\big|_{\ker g(T)} \text{ and } q_T\big|_{\ker h(T)}$$

Therefore, we have

$$\begin{aligned} f &\coloneqq \operatorname{lcm}\left(q_T\big|_{\ker g(T)}, q_T\big|_{\ker h(T)}\right) \\ &= q_T\big|_{\ker q(T) q_T}\big|_{\ker h(T)} \end{aligned}$$

Since

$$\begin{aligned} V &= \ker g(T) \oplus \ker h(T) \\ f(T)v &= 0 \quad \forall v \in V \end{aligned}$$

Hence

$$q_T | f \in F[t]$$

By (+) and FTA

$$f | gh = q_T$$

As both f and q_T are monic,

$$f = q_T$$

Applying FTA again, we conclude that

$$g = q_T\big|_{\ker g(T)} \text{ and } h = q_T\big|_{\ker h(T)}$$ □

We now generalize the proposition to an important result that decomposes a finite dimensional vector space over F relative to a linear operator $T : V \to V$.

Theorem 14.2 (Primary Decomposition)

Let V be a finite dimensional vector space over F, $T : V \to V$ linear, and $q_T = p_1^{e_1} \dots p_r^{e_r}$, with $p_1, \dots, p_r$ distinct monic irreducible polys in $F[t]$, $e_1, \dots, e_r \in \mathbb{Z}^+$. Then there exists a direct sum decomposition of V into subspaces $W_1, \dots, W_r$

$$V = W_1 \oplus \dots \oplus W_r \tag{*}$$

satisfying all of the following:

i) Each W_i is T-invariant, $i = 1, \dots, r$

ii) $q_T|_{W_i} = p_i^{e_i}$, $i = 1, \dots, r$

iii) $q_T = \prod_{i=1}^r p_i^{e_i} = \prod_{i=1}^r q_{T|_{W_i}}$

iv) If $\mathscr{B}_i$ is an ordered basis for W_i, $i = 1, \dots, r$, $\mathscr{B} = \mathscr{B}_1 \cup \dots \cup \mathscr{B}_r$ is an ordered basis for V with

$$[T]_{\mathscr{B}} = \begin{pmatrix} [T|_{W_1}]_{\mathscr{B}_1} & & 0 \\ & \ddots & \\ 0 & & [T|_{W_r}] \end{pmatrix}$$

Moreover, any direct sum decomposition (*) of V satisfying $i), ii), iii)$ is uniquely determined by T and the $p_1, \dots, p_r$ up to order. If in addition, this is the case, then

$$W_i = \ker p_i^{e_i}(T) \qquad i = 1, \dots, r$$

Proof. We induct on r.

- $r = 1$ is immediate
- $r > 1$ By TFA, $p_1^{e_1}$ and $g = p_2^{e_2} \dots p_r^{e_r}$ are relatively prime, so by the Proposition

$$V = W_1 \oplus V_1$$

where

$$\begin{aligned} W_1 &= \ker p_1^{e_1}(T) \text{ and } W_1 \text{ is } T\text{-invariant} \\ V_1 &= \ker g(T) \text{ and } V_1 \text{ is } T\text{-invariant} \\ q_{T|_{W_1}} &= p_1^{e_1} q_{T|_{V_1}} && = p_2^{e_2} \dots p_r^{e_r} \end{aligned}$$

Let

$$T_1 = T|_{V_1} : V_1 \to V_1$$

By induction on r, we may assume all of the following:

$$\begin{aligned} V_1 &= W_2 \oplus \dots \oplus W_r \\ W_i &= \ker p_i^{e_i}(T_1) \text{ and is } T_1\text{-invariant} \\ q_{T_1|_{W_i}} &= p_i^{e_i} \text{ for } i = 2, \dots, r \end{aligned}$$

Note:

$$\ker p_i^{e_i}(T_1) \cap \sum_{\substack{j=2 \\ j\neq i}}^{r} \ker p_j(T_1) = 0 \quad \forall i > 0$$

Claim 14.1. Let $2 \leq i \leq r$. Then

$$\ker p_i^{e_i}(T) = \ker p_i^{e_i}(T_1)$$

Let $v \in \ker p_i^{e_i}(T)$, $i > 1$. So

$$p_i^{e_i}(T)v = 0$$

Hence

$$0 = \prod_{j=2}^{r} p_j^{e_j}(T)v = g(T)v,$$

i.e.,

$$v \in \ker g(T) = V_1$$

So

$$Tv = T|_{V_1}v = T_1 v$$

and

$$0 = p_i^{e_i}(T)v = p_i^{e_i}(T_1)v$$

as needed.

Let $v \in \ker p_i^{e_i}(T_1), i > 1$. By definition, $v \in V_1$, so

$$\begin{aligned} 0 = p_i^{e_i}(T_1)v &= p_i^{e_i}\left(T|_{V_1}\right)v \\ &= p_i^{e_i}(T)|_{V_1}v = p_i^{e_i}(T)v \end{aligned}$$

This proves the claim.

The existence of (*), $i), ii), iii)$ nad $W_i = \ker p_i^{e_i}(T)$, $i = 1, \ldots, r$, now follow. Moreover, $i)$ and (*) yield $iv)$.

Uniqueness: Suppose that

$$V = W_1 \oplus \ldots \oplus W_r$$

satisfies i), ii), iii). If we show

$$W_i = \ker p_i^{e_i}(T), \qquad i = 1, \ldots, r$$

the result will follow. It suffices to do the case $i = 1$. Let

$$\begin{aligned} V_1 &= W_2 \oplus \ldots \oplus W_r \\ V &= W_1 \oplus V_1 \end{aligned}$$

As each W_i is T-invariant and V_1 is T-invariant. As before

$$p_1^{e_1} \text{ and } g = p_2^{e_2} \ldots p_r^{e_r}$$

and relatively prime by FTA. So by hw 4 # 6

$$q_T = \operatorname{lcm}\left(q_{T|_{V_1}}, q_{T|_{V_1}}\right)$$

It follows that

$$q_{T|_{V_1}} = p_2^{e_2} \dots p_r^{e_r} = g$$

Moreover, we have an equation

$$1 = p_1^{e_1} k + gl \in F[t]$$

So

$$1_V = p_1^{e_1}(T)k(T) + g(T)l(T) \tag{+}$$

Claim 14.2. $W_1 = \ker p_1^{e_1}(T)$ and hence we are done.

Since

$$q_{T|_{W_1}} = p_1^{e_1}$$

We have

$$p_1^{e_1}(T)v = 0 \qquad \forall v \in W_1$$

Hence

$$W_1 \subseteq \ker p_1^{e_1}(T)$$

To finish, we must know

$$\ker p_1^{e_1}(T) \subseteq W_1$$

Let

$$v \in \ker p_1^{e_1}(T) \subseteq V = W_1 \oplus V_1$$

So $\exists! w_1 \in W_1$, $v_1 \in V_1$ s.t.

$$v = w_1 + v_1$$

Since $W_1 \subseteq \ker p_1^{e_1}(T)$,

$$p_1^{e_1}(T)W_1 = 0$$

By assumption, $p_1^{e_1}(T)v = 0$, so

$$p_1^{e_1}(T)v_1 = 0$$

As $V_1 = W_2 \oplus \dots \oplus W_r$

$$p_i^{e_i} = q_{T|_{W_i}}, \qquad i = 2, \dots, r \text{ by (ii)}$$

We have

$$p_2^{e_2}(T) \dots p_r^{e_r}(T)v_1 = 0$$

Hence by (+)

$$v_1 = 1_V v_1 = p_1^{e_1}(T)k(T)v_1 + p_2^{e_2}(T) \dots p_r^{e_r}(T)l(T)v_1 = 0$$

Therefore,

$$v = w_1 + v_1 = w_1 \in W_1$$

and it follows that $\ker p_1^{e_1}(T) \subseteq W_1$ as needed. □

Recall: Let V be a finite dimensional vector space over F, $T : V \to V$ linear is called diagonalizable if there exists an ordered basis $\mathscr{B}$ for V consisting of eigenvectors of T. By hw 2 # 2, this is equivalent to

$$V = \bigoplus_{\lambda} E_T(\lambda)$$

§15 | Lec 15: Apr 30, 2021

§15.1 Primary Decomposition (Cont'd)

Recall: Let V be a finite dimensional vector space over F, $T : V \to V$ linear is called diagonalizable if there exists an ordered basis $\mathscr{B}$ for V consisting of eigenvectors of T. By hw 2 # 2, this is equivalent to

$$V = \bigoplus_{\lambda} E_T(\lambda)$$

Theorem 15.1

Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Then T is diagonalizable iff q_T splits in $F[t]$ and has no repeated roots in F. If this is the case, then

$$q_T = \prod_{i=1}^{r}(t - \lambda_i), \quad \lambda_1, \ldots, \lambda_r \text{ the distinct roots of } q_T$$

Proof. " $\Longleftarrow$ " $q_T = \prod_{i=1}^{r}(t - \lambda_i)$, $\lambda_1, \ldots, \lambda_r$ the distinct roots of q_T. Let $V_i = \ker(T - \lambda_i 1_V) = E_T(\lambda_i)$, $i = 1, \ldots, r$. Then by the Primary Decomposition Theorem,

$$V = V_1 \oplus \ldots \oplus V_r$$

SO T is diagonalizable.
" $\Longrightarrow$ " Let $\mathscr{B} = \{v_1, \ldots, v_n\}$ be an ordered basis for V consisting of eigenvectors of T with λ_i the eigenvalue of v_i and ordered s.t.

$$\lambda_1, \ldots, \lambda_r \text{ are the distinct eigenvalues of } T$$

For each j, $1 \leq j \leq n$, we have

$$(T - \lambda_i 1_V)\, v_j = Tv_j - \lambda_i v_j = (\lambda_j - \lambda_i)\, v_j, \quad j = 1, \ldots, n$$

So

$$\prod_{i=1}^{r}(T - \lambda_i 1_V)\, v_j = 0 \quad \text{for } j = 1, \ldots, n$$

i.e.,

$$\prod_{i=1}^{r}(T - \lambda_i 1_V) \text{ vanishes on a basis for } V$$

hence vanishes on all of V. It follows that

$$q_T \Big| \prod_{i=1}^{r}(t - \lambda_i) \in F[t]$$

In particular, q_T splits in $F[t]$ and has no multiple roots in F by FTA. As every eigenvalue of T is a root of f_T, we have

$$t - \lambda_i | q_T, \qquad i = 1, \ldots, r$$

using f_T and q_T have the same roots. Therefore,

$$q_T = \prod_{i=1}^{r} (t - \lambda_i) \in F[t]$$ □

§15.2 Jordan Blocks

Definition 15.2 (Jordan Block Matrix) — $J \in \mathbb{M}_n F$ is called a Jordan block matrix of eigenvalue λ of size n if

$$J = J_n(\lambda) \coloneqq \begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & 1 & & \\ & & \ddots & \lambda \\ 0 & & & 1 \end{pmatrix} \in \mathbb{M}_n F$$

Note: $f_{J_n}(\lambda) = \det(tI - J_n(\lambda)) = (t - \lambda)^n \in F[t]$, so splits with just one root of multiplicity.

Definition 15.3 (Nilpotent) — $T : V \to V$ linear is called nilpotent if $q_T = t^m$, some m, i.e., $\exists M \in \mathbb{Z}^+ \ni T^M = 0$.

Example 15.4

$J = J_n(0)$ is nilpotent and has $q_J = t^m$ for some m. In fact, $q_J = t^n$ – why?
In fact, let $A \in \mathbb{M}_n F$, $A : F^{n\times 1} \to F^{n\times 1}$ linear with $A \sim N$ with

$$N = J_n(\lambda_- \lambda I_n = J_n(0)$$

Then as N is nilpotent and

$$A = PNP^{-1}, \quad \text{some } P \in GL_n F,$$

we have

$$A^n = \left(PNP^{-1}\right)^n = PNP^{-1}PNP^{-1}\dots PNP^{-1} = PN^nP^{-1} = 0$$

So A is nilpotent. Now N is nilpotent.
If $\mathscr{S} = \{e_1, \dots, e_n\}$ is the standard basis for $F^{n\times 1}$

$$\begin{aligned} Ne_i &= e_{i+1}, \quad i \le n-1 \\ Ne_n &= 0 \\ N^2e_i &= N - Ne_i = e_{i+2}, \quad i \le n-2 \\ &\vdots \end{aligned}$$

In any case, we have

$$\left.\begin{aligned} \dim \operatorname{im} N^r &= n - r \\ \dim \ker N^r &= r \end{aligned}\right\} \text{if } r \le n$$

$$\left.\begin{aligned} \dim \operatorname{im} N^r &= 0 \\ \dim \operatorname{im} \ker N^r &= n \end{aligned}\right\} \text{if } r > n$$

Lemma 15.5

Let $J = J_n(\lambda) \in \mathbb{M}_n F$. Then

1. λ is the only eigenvalue of J.
2. $\dim E_J(\lambda) = 1$
3. $t_J = q_J = (t-\lambda)^n$
4. $f_J(J) = 0$

Proof. Let

$$N = J - \lambda I \in \mathbb{M}_n F$$

the characteristics matrix of J

$$N^{n-1} = \begin{pmatrix} 0 & & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{M}_n F$$

is not the zero matrix, but

$$N^n = 0$$

So

$$q_T | (t-\lambda)^n \text{ and } q_J \not| (t-\lambda)^{n-1}$$

It follows that $q_J = (t-\lambda)^n = f_J$. This shows 3) and 4). By the computation,

$$\dim \ker N = 1$$

and

$$\ker N = E_T(\lambda)$$

This gives 2) as $f_T = (t-\lambda)^n$, 1) is clear. □

Remark 15.6. $J_n(\lambda)$ has only a line as an eigenspace, so among triangulariazable operator away from being diagonalizable when $n \geq 1$.

Proposition 15.7

Let $A \in \mathbb{M}_n F$ be triangularizable. Suppose $f_A = (t-\lambda)^n$ for some $\lambda \in F$. Then A is diagonalizable iff $q_A = (t-\lambda)$ iff $A = \lambda I$.

Proof. If $q_A = t - \lambda$, then $A = \lambda I$ as

$$F^{n \times 1} = \ker (A - \lambda I)$$

The converse is immediate. □

Computation: Let V be a finite dimensional vector space over F, $\dim V = n$, $T : V \to V$ linear. Suppose there exists $\mathscr{B} = \{v_1, \dots, v_n\}$ an ordered basis for V satisfying

$$[T]_{\mathscr{B}} = J_n(\lambda)$$

Then by definition

$$\begin{aligned} Tv_1 &= \lambda v_1 + v_2 \quad \text{i.e. } (T - \lambda 1_V)v_1 = v_2 \\ Tv_2 &= \lambda v_2 + v_3 \quad \text{i.e. } (T - \lambda 1_V)v_2 = v_3 \\ &\vdots \\ Tv_{n-1} &= \lambda v_{n-1} + v_n \quad \text{i.e. } (T - \lambda 1_V)\, v_{n-1} = v_n \\ Tv_n &= \lambda v_n \end{aligned} \tag{+}$$

So

$$E_\lambda(\lambda) = Fv_n$$

$v_1, \ldots, v_{n-1}$ are not eigenvectors, but do satisfy

$$\begin{aligned} (T - \lambda 1_V)v_i &= v_{i+1} \qquad i = 1, \ldots, n-1 \\ (T - \lambda 1_V)^{n-i} v_i &= v_n \qquad \text{, an eigenvector} \end{aligned}$$

So we can compute $v_1, \ldots, v_{n-1}$ from the eigenvalue v_n.

§16 | Lec 16: May 3, 2021

§16.1 Jordan Blocks (Cont'd)

Definition 16.1 (Sequence of Generalized Eigenvectors) — Let $T : V \to V$ be linear, $0 \neq v_n \in E_T(\lambda)$. We say $v_1, \dots, v_n$ is an (ordered) sequence of generalized eigenvectors of eigenvalue λ of length n if $(+)$ above holds, i.e.,

$$\begin{aligned}(T - \lambda 1_V)v_i &= v_{i+1}, \qquad i = 1, \dots, n-1\\ (T - \lambda 1_V)v_n &= 0\end{aligned}$$

We let

$$\begin{aligned}g_n(\lambda) = g_n(v_n, \lambda) &\coloneqq \{v_1, \dots, v_n\}\\ &= \left\{v_1, (T - \lambda 1_V)^{n-1} v_1\right\}\end{aligned}$$

be an ordered sequence of generalized eigenvectors for T of length n relative to λ.

Note: We should really write

$$g_n(v_n, \lambda, v_1, \dots, v_{n-1})$$

Lemma 16.2

Let V be a vector space over F, $T : V \to V$ linear, $0 \neq v_n \in E_T(\lambda)$, $v_1, \dots, v_n$ an ordered sequence of generalized eigenvectors of T of length n, $g_n(\lambda) = \{v_1, \dots, v_n\}$. Then

1. $g_n(\lambda)$ is linearly independent.
2. If V is a finite dimensional vector space over F, $\dim V = n$, then
   - i) $g_n(\lambda)$ is an ordered basis for V
   - ii) $[T]_{g_n(\lambda)} = J_n(\lambda)$

Proof. 1. We have seen that $(*)$ implies

$$\begin{aligned}(T - \lambda 1_V)^{n-i} v_i &= v_n \qquad i < n\\ (T - \lambda 1 - V)v_n &= 0\end{aligned}$$

So

$$(T - \lambda 1_V)^k v_i = 0 \qquad \forall k > n - i$$

Suppose

$$\alpha_1 v_1 + \ldots + \alpha_n v_n = 0, \qquad \alpha_i \in F \text{ not all } 0$$

Choose the least k s.t. $\alpha_k \neq 0$. Then

$$0 = (T - \lambda 1_V)^{n-k} \left(\alpha_k v_k + \ldots + \alpha_n v_n\right) = \alpha_k v_n$$

As $v_n \neq 0$, $\alpha_k = 0$, a contradiction.
So 1) follows and 1) $\to$ 2). □

Definition 16.3 (Jordan Canonical Form) — $A \in \mathbb{M}_n F$ is called a matrix in Jordan canonical form (JCF) if A has the block form

$$A = \begin{pmatrix} J_{r_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{r_m}(\lambda_m) \end{pmatrix}$$

$\lambda_1, \ldots, \lambda_m$ not necessarily distinct.

Definition 16.4 (Jordan Basis) — Let V be a finite dimensional vector space over F, $T : V \to V$ linear. An ordered basis $\mathscr{B}$ for V is called a Jordan basis (if it exists) for V relative to T if $\mathscr{B}$ is the union

$$g_{r_1}(v_{1,r_1}, \lambda_1) \cup \ldots \cup g_{r_m}(v_{m,r_m}, \lambda_m) \qquad (\star)$$

where $g_{r_j}(v_{j,r_j}, \lambda_j)$ is an ordered sequence of generalized eigenvectors of T relative to λ_j ending at eigenvector v_{j,r_j}. The $\lambda_1, \ldots, \lambda_m$ need not be distinct.

Proposition 16.5

Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Then V has a Jordan basis relative to $T \iff T$ has a matrix representation in Jordan canonical form (JCF).

Proof. Let $w_i = g_{r_i}(v_{i,r_i}, \lambda_i)$ in $(\star)$. The only thing to show is: W_i is T-invariant, but this follows from our computation. □

Conclusion: Let $T : V \to V$ be linear with V having a Jordan basis relative to T. Gathering all the Jordan blocks with the same eigenvalues together and ordering these into increasing size, we can write such a Jordan basis as follows:

$$\lambda_1, \ldots, \lambda_m \text{ the distinct eigenvalues of } T$$

$$\begin{aligned} \mathscr{B} = {} & g_{r_{11}}(v_{11}, \lambda_1) \cup \ldots \cup g_{r_1,n_1}(v_{1,n_1}, \lambda_1) \\ & \cup g_{r_{21}}(v_{21}, \lambda_2) \cup \ldots \cup g_{r_{2,n_2}}(v_{2,n_2}, \lambda_2) \\ & \vdots \\ & \cup g_{r_m,1}(v_{m,1}, \lambda_m) \cup \ldots \cup g_{r_m,n_m}(v_{m,r_m}, \lambda_m) \end{aligned}$$

with

$$r_{i1} \leq r_{i2} \leq \ldots \leq r_i n_i, \qquad 1 \leq i \leq m$$

e.g.

$$[T]_{\mathscr{B}} = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & 1 & 0 & & & \\ & 1 & 1 & & & \\ & & 0 & 2 & 0 & 0 \\ & & & 1 & 2 & 0 \\ & & & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} J_1(1) & & & \\ & J_1(1) & & \\ & & J_2(1) & \\ & & & J_3(2) \end{pmatrix}$$

Let

$$W_{ij} = \text{Span } g_{r_i,j}(v_{ij}, \lambda_i) \qquad \forall i, j$$

These are all T-invariant. We have

$$f_T = \prod_{i,j} (t - \lambda_i)^{r_{ij}}$$

and

$$\begin{aligned} q_T &= \prod_i \text{lcm}\left((t - \lambda_i)^{r_{ij}} | j = 1, \ldots, n_i\right) \\ &= \prod_i (t - \lambda_i)^{r_{in_i}} \end{aligned}$$

So

$$q_T | f_T \text{ and } f_T(T) = 0$$

Also

$$q_{T|W_{ij}} = f_{T|W_{ij}} = (t - \lambda_i)^{r_{ij}}$$

for all $1 \leq j \leq n_j$, $1 \leq i \leq m$. There are called the elementary divisors of T

$$V = W_{11} \oplus \ldots \oplus W_{1,n_1} \oplus \ldots \oplus W_{m1} \oplus \ldots \oplus W_{mn_m}$$

Now let P_{ij} be the projection onto W_{ij} along

$$W_{11} \oplus \ldots \oplus \underbrace{\widehat{W_{ij}}}_{\text{omit}} \oplus \ldots \oplus W_{m,n_m}$$

Then

$$P_{ij} P_{kl} = \delta_{ik} \delta_{jl} P_{jl} = \begin{cases} P_{jl} \text{ if } i = k \text{ and } j = l \\ 0 \text{ otherwise} \end{cases}$$

$$1_V = P_{11} + \ldots + P_{mn_m}$$

$$TP_{ij} = P_{ij} T$$

$$T = TP_{11} + \ldots + TP_{mn_m} = T\big|_{W_{11}} + \ldots + T\big|_{W_{mn_m}}$$

Abusing notation

$$\lambda_1, \ldots, \lambda_m \text{ are the distinct eigenvalues of } T$$

Let

$$W_i = W_{i1} \oplus \ldots \oplus W_{in_i} \quad i = 1, \ldots, m$$

As $r_{i1} \leq \ldots \leq r_{in_i}$,

$$(T - \lambda_i 1_V)^{r_{in_i}}\big|_{W_{ij}} = 0, \qquad 1 \leq j \leq n_i$$
$$(T - \lambda_i 1_V)^{r_i n_i - 1}\big|_{W_{ij}} \neq 0$$

showing

$$q_T | W_i = (t - \lambda_i)^{r_{in_i}}$$

So

$$V = W_1 \oplus \ldots \oplus W_m$$

is the unique primary decomposition of V relative to T.
Note: The Jordan canonical form of T above is completely determined by the elementary divisors of T.

§16.2 Jordan Canonical Form

Theorem 16.6

Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Suppose that q_T splits in $F[t]$. Then there exists a Jordan basis $\mathscr{B}$ for V relative to T. Moreover, $[T]_{\mathscr{B}}$ is unique up to the order of the Jordan blocks. In addition, all such matrix representations are similar.

Proof. Reduction 1: We may assume that

$$q_T = (t - \lambda)^r$$

Suppose that

$$q_T = (t - \lambda_1)^{r_1} \ldots (t - \lambda_m)^{r_m} \in F[t]$$

$\lambda_1, \ldots, \lambda_m$ distinct. Set

$$W_i = \ker (T - \lambda_i 1_V)^{r_i}, \quad i = 1, \ldots, m$$

By the Primary Decomposition Theorem,

$$V = W_1 \oplus \ldots \oplus W_m$$

W_i is T-invariant, $i = 1, \ldots, n$

$$q_{T|_{W_i}} = (t - \lambda_i)^{r_i}, \qquad i = 1, \ldots, m$$

So we need only find a Jordan basis for each W_i. □

§17 | Lec 17: May 5, 2021

§17.1 Jordan Canonical Form (Cont'd)

Proof. (Cont'd from Lec 16) Reduction 2: We may assume that $q_T = t^r$, i.e., $\lambda = 0$. Suppose that we have proven the case for $\lambda = 0$. Let $S = T - \lambda 1_V$, T as in Reduction 1. Then

$$S^r = (T - \lambda 1_V)^r = 0 \text{ and } S^{r-1} = (T - \lambda 1_V)^{r-1} \neq 0$$

Therefore,

$$q_S = t^r$$

if $\mathscr{B}$ is a Jordan basis for V relative to S, then

$$[S]_{\mathscr{B}} = [T]_{\mathscr{B}} - \lambda I$$

is a JCF with diagonal entries 0. Hence

$$[T]_{\mathscr{B}} = [S]_{\mathscr{B}} + \lambda I$$

is a JCF with diagonal entries λ and $\mathscr{B}$ is also a Jordan basis for V relative to T. Reduction 2 now follows easily. We turn to
Existence: We have reduced to the case

$$q_T = t^r, \quad \text{i.e.,} \quad T^r = 0, \quad T^{r-1} \neq 0$$

In particular, T is nilpotent. We induct on $\dim V$.

- $\dim V = 1$ is immediate.
- $\dim V > 1$: T is singular, so $0 < \ker T$, as $\lambda = 0$ is an eigenvalue. Since V is a finite dimensional vector space over F, by the Dimension Theorem, T is not onto, i.e.,

  $$\text{im } T < V$$

  As im T is T-invariant, we can (and do) view

  $$T\big|_{\text{im } T} : \text{ im } T \to \text{ im } T \text{ linear}$$

  As $T^r = 0$, certainly $(T|_{\text{im } T})^r = 0$, so

  $$T\big|_{\text{im } T} \text{ is also nilpotent}$$

  and

  $$q_T\big|_{\text{im } T} | q_T \in F[t]$$

  since

  $$q_T\left(T\big|_{\text{im } T}\right) = 0 = q_T(T)$$

  So $q_{T|_{\text{im } T}}$ splits in $F[t]$ and

  $$q_{T\big|_{\text{im } T}} = t^s, \quad \text{for some } s \leq r$$

by FTA. By induction on $\dim V$, $\text{im } T$ has a Jordan basis relative to $T|_{\text{im } T}$. So

$$\text{im } T = W_1 \oplus \ldots \oplus W_m, \text{ some } m$$

with each W_i being $T|_{\text{im } T}-$ (hence $T-$) invariant and W_i has a basis of an ordered sequence of generalized eigenvectors for $T|_{W_i}$, hence for $T|_{\text{im } T}$ and T,

$$g_{r_i}(0) = \left\{w_i, Tw_i, \ldots, T^{r_i-1}w_i\right\}, \quad r_i \geq 1$$

Thus we have

$$\begin{aligned} T^{r_i}w_i &= 0, \qquad i = 1, \ldots, m \\ q_{T\big|_{W_i}} &= t^{r_i}, \qquad i = 1, \ldots, m \end{aligned}$$

Since $w_i \in W_i \subseteq \text{ im } T$,

$$\exists v_i \in V \ni Tv_i = w_i, \quad i = 1, \ldots, m$$

So we also have

$$T^{r_i+1}v_i = T^{r_i}Tv_i = T^{r_i}w_i = 0$$

and

$$T^{r_i}v_i = T^{r_i-1}Tv_i = T^{r_i-1}w_i \neq 0$$

Therefore, $v_i, Tv_i, \ldots, T^{r_i}v_i$ is an ordered sequence of generalized eigenvalues for T in V, and, in particular, linearly independent. For each $i = 1, \ldots, m$, let

$$V_i = \text{Span}\left\{v_i, Tv_i, \ldots, T^{r_i}v_i\right\}$$

So

$$\begin{aligned} V_i &= \left\{\sum_{j=0}^{r_i} \alpha_j T^j v_i | \alpha_j \in F\right\} \\ &= \{f(T)v_i | f \in F[t],\ f = 0 \text{ or } \deg f \leq r_i\} \\ &= F[T]_{V_i} \end{aligned}$$

Since each V_i is spanned by an ordered sequence of generalized eigenvectors for T, each V_i is T-invariant, $i = 1, \ldots, m$.

<u>*Note*</u>: If $f \in F[t]$ and $f(T)w_i = 0$, then $f(T) = 0$ in W_i and similarly if $f \in F[t]$ and $f(T)v_i = 0$, then $f(T) = 0$ on V_i as $f(T)w_i = 0$ implies

$$0 = T^j f(T)w_i = f(T)T^j w_i = 0 \quad \forall i$$

Set

$$V' = V_1 + \ldots + V_m$$

Each V_i is T-invariant, so V' is T-invariant.

Claim 17.1. $V' = V_1 \oplus \ldots \oplus V_m$

In particular,

$$\mathscr{B}_0 = \{v_1, Tv_1, \ldots, T^{r_i}v_1, \ldots, v_m, Tv_m, \ldots, T^{r_m}v_m\}$$

is a basis for V'.

□

§18 | Lec 18: May 7, 2021

§18.1 Jordan Canonical Form (Cont'd)

Proof. (Cont'd) Suppose $u_i \in V_i$, $i = 1, \ldots, m$ satisfies

$$u_1 + \ldots + u_m = 0 \tag{1}$$

To show $u_i = 0$, $i = 1, \ldots, m$. As $u_i \in V_i$, $\exists f_i \in F[t] \ni$

$$u_i = f_i(T)v_i$$

where we let $f_i = 0$ if $u_i = 0$. So (1) becomes

$$f_i(T)v_1 + \ldots + f_m(T)v_m = 0 \tag{2}$$

Since $Tf(T) = f(T)T \ \forall f \in F[t]$ and

$$w_i = Tv_i \qquad i = 1, \ldots, m$$

taking T of (2) yields

$$f_1(T)w_1 + \ldots + f_m(T)w_m = 0$$

As the T-invariant W_i satisfying

$$W_1 + \ldots + W_m = W_1 \oplus \ldots \oplus W_m \tag{*}$$

We have

$$f_i(T)w_i = 0, \qquad i = 1, \ldots, m$$

Hence

$$f_i(T) = 0 \text{ on } W_i, \qquad i = 1, \ldots, m$$

Thus

$$t^{r_i} = q_{T\big|_{W_i}} \big| f_i \in F[t], \quad i = 1, \ldots, m$$

In particular, since $r_i \geq 1 \ \forall i$, we can write

$$f_i = tg_i \in F[t], \qquad i = 1, \ldots, m$$
$$\deg g_i < \deg f_i, \qquad i = 1, \ldots, m \text{ if } f_i \neq 0$$

Since

$$f_i(T) = Tg_i(T) = g_i(T)T$$

and

$$w_i = Tv_i, \qquad i = 1, \ldots, m$$

(2) now becomes

$$g_1(T)w_1 + \ldots + g_m(T)w_m = 0 \tag{3}$$

Since each W_i is T-invariant, by (*)

$$g_i(T)w_i = 0, \quad \text{hence } g_i(T) = 0 \text{ on } W_i$$

for $i = 1, \ldots, m$ by the definition of W_i. Therefore, for each i, $i = 1, \ldots, m$

$$t^{r_i} = q_{T\big|_{W_i}} \big| g_i \in F[t]$$

In particular, we can write

$$g_i = t^{r_i} h_i \in F[t], \qquad i = 1, \ldots, m$$

So

$$f_i = t^{r_i+1} h_i \in F[t], \qquad i = 1, \ldots, m$$

Thus we have

$$u_i = f_i(T)v_i = h_i(T)T^{r_i+1}v_i = 0, \quad i = 1, \ldots, m$$

This establishes claim 1. As

$$w_i = Tv_i \in W_i, \qquad i = 1, \ldots, m$$

We have

$$\begin{aligned} TV' &= TV_1 \oplus \ldots \oplus TV_m \\ &= W_1 \oplus \ldots \oplus W_m = TV \end{aligned} \tag{$\star$}$$

since each W_i, V_i is T-invariant and

$$TV_i = W_i, \qquad i = 1, \ldots, m$$

Therefore,

$$T\big|_{V'} = T\big|_{V_1} + \ldots + T\big|_{V_m}$$

Claim 18.1. $V = \ker T + V'$

Let $v \in V$. Since

$$TV' = TV$$

by $(\star)$, we have $\forall v \in V$

$$\exists v' \in V' \ni Tv' = Tv,$$

so

$$v - v' \in \ker T$$

and

$$v = v' + w \text{ some } w \in \ker T$$

i.e.

$$v \in V' + \ker T$$

as needed.
Now by construction, we have a Jordan basis $\mathscr{B}_0$ for the T-invariant subspace V' relative to $T|_{V'}$. Let

$$\mathscr{C} = \{u_1, \ldots, u_k\} \text{ be a basis for } \ker T = E_T(0)$$

Modifying the Toss In Theorem, we get a basis for V as follows. If $u_1 \notin \text{Span } \mathscr{B}_0$, let $\mathscr{B}_1 = \mathscr{B}_0 \cup \{u_1\}$. Otherwise, let $\mathscr{B}_1 = \mathscr{B}_0$. If $u_2 \notin \text{Span } \mathscr{B}_1$, let $\mathscr{B}_2 = \mathscr{B}_1 \cup \{u_2\}$. Otherwise,

let $\mathscr{B}_2 = \mathscr{B}_1$. In either case, $\mathscr{B}_2$ is a linearly independent set. Continuing in this way, since $\mathscr{B}_0 \cup \mathscr{C}$ spans V, we get a spanning set of V

$$\mathscr{B} = \mathscr{B}_0 \cup \{u_{j_1}, \ldots, u_{j_r}\} \subseteq V$$

with

$$T_{u_{j_i}} = 0$$

for some u_{j_i} constructed above, $1 \leq i \leq s$.
Using claim 1, we have

$$\begin{aligned} V &= V' \oplus \text{Span } \{u_{j_1}, \ldots, u_{j_s}\} \\ &= V_1 \oplus \ldots \oplus V_m \oplus Fu_{j_1} \oplus \ldots \oplus Fu_{j_s} \end{aligned}$$

and $[T]_{\mathscr{B}}$ is in Jordan canonical form. This proves existence.
Note: Fu_{j_i} are the $g_1(u_{j_i}, 0)$ and the u_{j_i} are eigenvectors that cannot be extended to $g_i(v_i, 0)$ of longer length.
Uniqueness: By reduction 1) and 2), we have

$$q_T = t^r, \quad T^r = 0, \quad T^{r-1} \neq 0$$

Let $\mathscr{C}$ be an ordered basis for V. Then by MTT

$$m_j = \dim \text{im } T^j = \text{rank } \left[T^j\right]_{\mathscr{C}} = \text{rank } [T]^j_{\mathscr{C}} \tag{*}$$

Let $\mathscr{B}$ be any Jordan basis for V relative to T, say

$$[T]_{\mathscr{B}} = \begin{pmatrix} J_{r_1}(0) & & 0 \\ & \ddots & \\ 0 & & J_{r_m}(0) \end{pmatrix}$$

the corresponding Jordan canonical form. Prior computation showed for each i, $1 \leq i \leq m$,

$$\begin{cases} \text{rank } J^j_{r_i}(0) = r_i - j \\ \dim \ker J^j_{r_i}(0) = j \end{cases} \quad \text{if } j < r_i$$

and

$$\begin{cases} \text{rank } J^j_{r_i}(0) = 0 \\ \dim \ker J^j_{r_i}(0) = r_i \end{cases} \quad \text{if } j \geq r_i$$

Clearly, for each i,

$$[T]^j_{\mathscr{B}} = \begin{pmatrix} J^j_{r_1}(0) & & \\ & \ddots & \\ & & J^j_{r_m}(0) \end{pmatrix}$$

as $[T]_{\mathscr{B}}$ is in block form. So by (*),

$$m_j = \text{rank } [T]^j_{\mathscr{B}} = \sum_{i=1}^{m} \text{rank } J^j_{r_i}(0)$$

It follows that we have

$$\begin{aligned} m_{j-1} - m_j &= \text{rank } [T]_{\mathscr{B}}^{j-1} - \text{rank } [T]_{\mathscr{B}}^{j} \\ &= \#\text{ of } l \times l \text{ Jordan blocks } J_l(0) \text{ in } (+) \text{ with } l \geq j \end{aligned}$$

We also have, in the same way,

$$\begin{aligned} m_j - m_{j+1} &= \text{rank } [T]_{\mathscr{B}}^{j} - \text{rank } [T]_{\mathscr{B}}^{j+1} \\ &= \#\text{ of } l \times l \text{ Jordan blocks } J_l(0) \text{ in } (+) \text{ with } l \geq j+1 \end{aligned}$$

Consequently, there are precisely

$$(m_{j-1} - m_j) - (m_j - m_{j+1}) = m_{j-1} - 2m_j + m_{j+1}$$

which equals the number of $l \times l$ Jordan blocks $J_l(0)$ in $(+)$ with $l = j$. This number is independent of $\mathscr{B}$ as it is

$$\text{rank } T^{j-1} - 2 \text{ rank } T^j + \text{rank } T^{j+1}$$

Thus, $[T]_{\mathscr{B}}$ is unique up to order of the Jordan blocks. This proves uniqueness.
If $\mathscr{B}'$ is another Jordan basis, then

$$[T]_{\mathscr{B}'} \sim [T]_{\mathscr{B}}$$

by the Change of Basis Theorem. This finishes the proof **(phewww... such a long proof!)** □

Corollary 18.1

Let $A \in \mathbb{M}_n F$. If $q_A \in F[t]$ splits in $F[t]$, then A is similar to a matrix in JCF unique up to the order of the Jordan blocks.

Corollary 18.2

Let F be an algebraically closed field, e.g., $F = \mathbb{C}$. Then every $A \in \mathbb{M}_n F$ is similar to a matrix in JCF unique up to the order of the Jordan blocks and for every V, a finite dimensional vector space over F, and $T : V \to V$ linear, V has a Jordan basis relative to T. Moreover, the Jordan blocks of $[T]_{\mathscr{B}}$ are completely determined by the elementary divisors (minimal polys) that correspond to the Jordan blocks.

Theorem 18.3

Let F be an algebraically closed field, e.g., $F = \mathbb{C}$, $A, B \in \mathbb{M}_n F$. Then, the following are equivalent

1. $A \sim B$
2. A and B have the same JCF (up to block order)
3. A and B have the same elementary divisors counted with multiplicities.

Corollary 18.4
Let F be an algebraically closed field. Then $A \sim A^\top$.

Proof. For any $B \in \mathbb{M}_n F$, $q_B = q_{B^\top}$. □

§18.2 Companion Matrix

Definition 18.5 (Companion Matrix) — Let $g = t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0 \in F[t]$, $n \geq 1$. The matrix

$$C(g) \coloneqq \begin{pmatrix} 0 & 0 & \ldots & 0 & — & a_0 \\ 1 & 0 & & 0 & — & a_1 \\ 0 & 1 & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ & & & 0 & — & a_{n-2} \\ 0 & 0 & \ldots & 1 & — & a_{n-1} \end{pmatrix}$$

is called the companion matrix of g.

Example 18.6
$C\left(t^n\right) = J_n(0)$.

Note: If $f, g \in F[t]$ are monic, then

$$f = g \iff C(f) = C(g)$$

Lemma 18.7
Let $g \in F[t] \setminus F$ be moinc. Then

$$f_{C(g)} = g$$

Proof. Let $g = t^n + a_{n-1}t^{n-1} + \ldots + a_0 \in F[t] \setminus F$. We induct on n, using properties about determinants.

- $n = 1$ is immediate
- $n > 1$ Expanding on the determinant

$$f_{C(g)} = \det\left(tI - C(g)\right) = \det \begin{pmatrix} t & 0 & \ldots & 0 & a_0 \\ -1 & t & & \vdots & \\ 0 & -1 & & \vdots & \\ \vdots & 0 & & \vdots & \\ 0 & \ldots & \ldots & -1 & t + a_{n-1} \end{pmatrix}$$

along the top row and induction yields

$$t\left(t^{n-1}+a_{n-1}t^{n-2}+\ldots+a_1\right)+(-1)^{n-1}a_0(-1)^{n-1}=g$$ □

Lemma 18.8

Let $g \in F[t] \setminus F$ be monic. Then

$$q_{C(g)} = f_{C(g)} = g$$

In particular,

$$f_{C(g)}\left(C(g)\right) = 0$$