Math 134 – Nonlinear ODE
University of California, Los Angeles

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This is math 134 – Linear and Nonlinear System of Differential Equations taught by Professor Wink. The class lecture is prerecorded, and we have live session every Monday and Friday at 3:00 pm – 3:50 pm for Q & A. We use *Nonlinear Dynamics and Chaos* 2nd by *Steven Strogatz* as our main book for the class. Other course notes can be found through my github. Any error spotted in the notes is my responsibility, and please let me know through my email at ducvu2718@ucla.edu.

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§1 | Lec 1: Jan 4, 2021

§1.1   Intro to Dynamical Systems

There are two types of dynamical systems:

1. Discrete in time:

   - Difference equation
   - Iterated map: $a_{n+1} = f(a_n)$

2. Continuous in time: differential equation

   - Partial Differential Equation (PDE):
     e.g. heat equation
     $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$
     wave equation
     $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$
     where the derivatives w.r.t time and space.
   - Ordinary Differential Equation (ODE):
     i) Harmonic oscillator
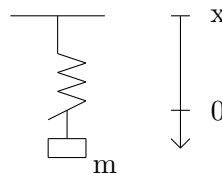
m: mass
k: spring constant

$$m\dot{x} + kx = 0$$

If $\omega^2 = \frac{k}{m}$, then
$$x(t) = x_0 \cos(\omega t) + x_1 \sin(\omega t)$$

ii) Damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0, \qquad b\text{: damping constant}$$

iii) Forced, damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = F\cos(t), \qquad F\text{: force}$$

so derivatives w.r.t time only.

Definition 1.1 (Order of ODE) — Highest occurring derivative is defined as the order of the ODE.

Remark 1.2. We can always write an ODE of nth order as a system of ODEs of 1st order.

Trick: Consider the damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

Set

$$x_1 = x$$
$$x_2 = \dot{x}$$

Then,

$$\dot{x}_1 = \dot{x} = x_2$$
$$\dot{x}_2 = \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x$$
$$= -\frac{b}{m}x_2 - \frac{k}{m}x_1$$

i.e.,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1$$

General framework: $\dot{x} = f(t, x)$

$$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

i.e.,

$$\dot{x}_1 = f_1(t_1, x_1, \ldots, x_n)$$
$$\vdots \tag{1}$$
$$\dot{x}_n = f_n(t, x_1, \ldots, x_n)$$

which is 1st order n-dimensional ODE.

> **Definition 1.3** (Linear ODE) — The ODE (1) is called <u>linear</u> if $f(t, x) = A(t) \cdot x$ for a time dependent matrix $A(t)$, otherwise we call it <u>non-linear</u>.

Example 1.4

The damped harmonic oscillator is linear.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Question 1.1. Why are linear equations special?

They satisfy the principle of superposition. If ϕ, ψ solve $\dot{x} = A(t)x$, then $y(t) = c \cdot \phi(t) + \psi(t), c \in \mathbb{R}$ also solves $\dot{x} = A(t)x$. This is valid because $\dot{y} = c\dot{\phi} + \dot{\psi} = cA\phi + A\psi = A(c\phi + \psi) = Ay$. For non-linear ODEs, the principle of superposition fails.

> **Definition 1.5** (Autonomous ODE) — The ODE (1) is called <u>autonomous</u> if f does not depend on t, i.e., $f(t, x) = f(x)$.

Example 1.6

$$m\ddot{x} + b\dot{x} + kx = F\cos(t)$$

is non-autonomous.

However, we can always consider an autonomous system instead. Set

$$x_1 = x$$
$$x_2 = \dot{x}$$
$$x_3 = t$$

Then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + F\cos(x_3)$$
$$\dot{x}_3 = 1$$

We will primarily study autonomous 1st order system in 1 or 2 variables.

Example 1.7 (Swinging Pendulum)

Consider a swinging pendulum

$$\ddot{x} + \tfrac{g}{L}\sin(x) = 0$$
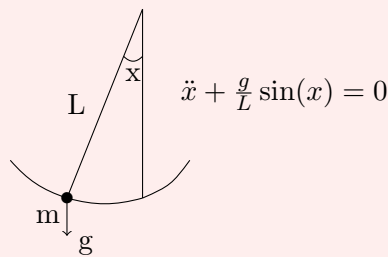
Set

$$x_1 = x$$
$$x_2 = \dot{x}$$

Then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{L}\sin(x_1)$$

1st order, non-linear autonomous ODE in 2 variables.

Question 1.2. What can we say about the behavior of a solution $x_1(t), x_2(t)$ for larger time t? How does it depend on $\frac{g}{L}$?

<u>Idea</u>: Use geometric methods, without solving $\dot{x} = f(x)$ explicitly, to make qualitative statements about the long time behavior of the solution.

§2 | Lec 2: Jan 6, 2021

§2.1   Phase Portraits

We want to study 1D autonomous dynamical systems

$$\dot{x} = f(x), \qquad f : \mathbb{R} \to \mathbb{R}$$

> **Remark 2.1.** $x(t)$ is the solution to $\dot{x} = f(x)$ with $x(0) = x_0$. Find the solution $y(t)$ with $y(t_0) = x_0$.
>   <u>Ans</u>: $y(t) = x(t - t_0)$ because $y(t_0) = x(0) = x_0$ and $\dot{y}(t) = \dot{x}(t - t_0) = f\left(x(t - t_0)\right) = f\left(y(t)\right)$.

Example 2.2

$x = \sin(x)$. Suppose $x_0 = \frac{\pi}{4}, x(t)$ solution with $x(0) = x_0$. Answer the followings

- Describe the long time behaviors of $x(t)$ as $t \to \infty$.

- How does the long time behavior depend on $x_0 \in \mathbb{R}$?

<u>Attemp 1</u>: Find explicit solution

$$\frac{dx}{dt} = \sin(x)$$

$$dt = \frac{dx}{\sin(x)}$$

$$t = -\ln\left|\frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)}\right| + c$$

We know $x(0) = x_0$, so $c = \ln\left|\frac{1+\cos(x_0)}{\sin(x_0)}\right|$. But what is $x(t) =$? This approach fails!

<u>Attempt 2</u>: Draw a phase portrait/diagram. We want to interpret the velocity $\dot{x} = f(x)$ as a vector field on the real line.
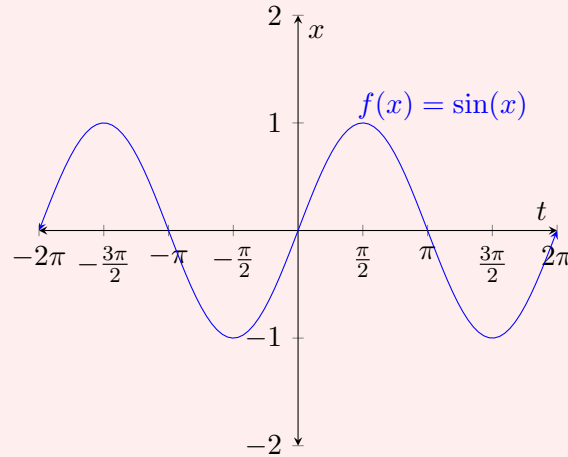
Idea:

- If $f(x_0) > 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ increase near x_0.

- If $f(x_0) < 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ decrease near x_0.

- If $f(x_0) = 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ is $x(t) = x_0$ for all $t \in \mathbb{R}$, i.e., we have a fixed point/equilibrium point.

Example 2.3

$\dot{x} = f(x) = \sin(x)$

Phase portrait:

stable fixed point   unstable fixed point
(sink, attractors)   (source,)
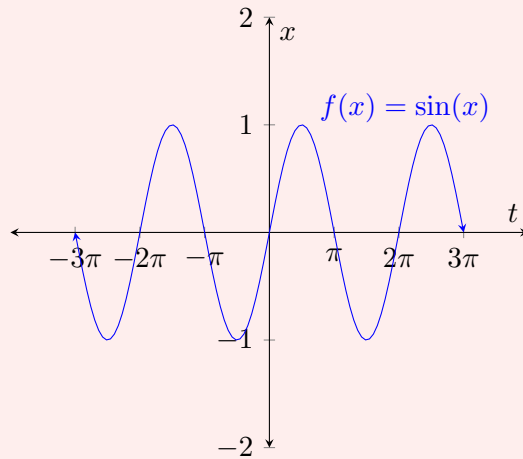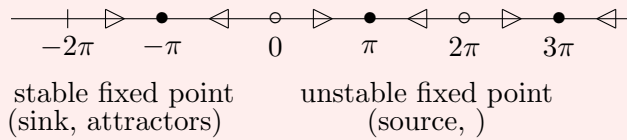
Qualitative plot of solution:

$x(t)$ is concave up
if $f(x(t))$ is increasing
$x(t)$ is concave down
if $f(x(t))$ is decreasing
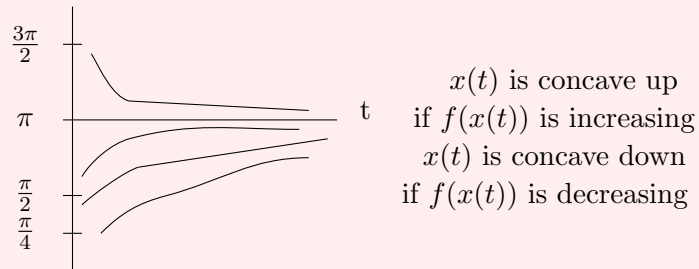
Example 2.4

$\dot{x} = x^2 - 1$. Fixed points: $f(x) = x^2 - 1 = 0 \implies x = \pm 1$
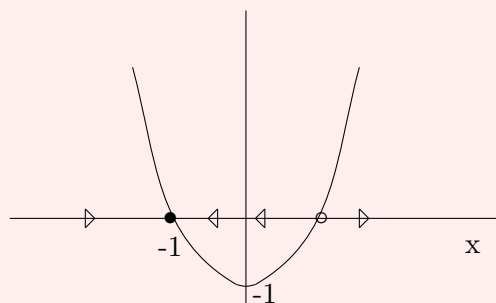
Note: If $x_0 > 1$, then solution $x(t)$ with $x(0) = x_0 > 1$ is unbounded. In fact, $x(t) \to \infty$ in finite time.

§3 | **Lec 3: Jan 8, 2021**

§3.1   **Stability Types of Fixed Points**

> **Definition 3.1** (Stability Types) — Consider the ODE $\dot{x} = f(x)$ and suppose that $f(x_*) = 0$. The fixed point x_* is called
>
> 1. Lyapunov stable if every solution $x(t)$ with $x(0) = x_0$ closed to x_* remain close to x_* for all $t \geq 0$, otherwise unstable.
>
> 2. Attracting if every solution $x(t)$ with $x(0) = x_0$ close to x_* satisfies $x(t) \to x_*$ as $t \to \infty$.
>
> 3. (asymptotically) stable if x_* is both Lyapunov stable and attracting.

Example 3.2

Let $\alpha \in \mathbb{R}, \dot{x} = \alpha x$. General solution $x(t) = x_0 e^{\alpha t}$.

- $x_* = 0$ is always an equilibrium solution.

- $x_* = 0$ is

    1. attracting if $\alpha < 0$
    2. Lyapunov stable if $\alpha \leq 0$
    3. unstable if $\alpha > 0$

Example 3.3 (RC circuit)

We have the following circuit
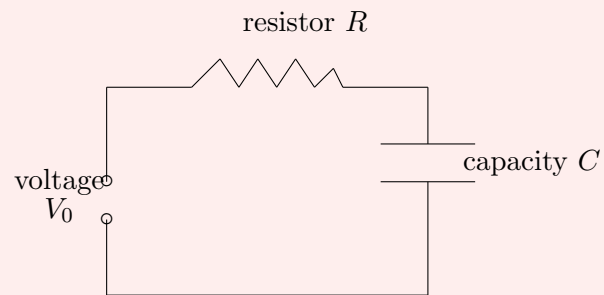
$$V_0 = RI + \frac{Q}{C}$$

$$I: \text{ current}, Q: \text{ change}$$

$$I = \dot{Q}$$

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$$

Phase portrait
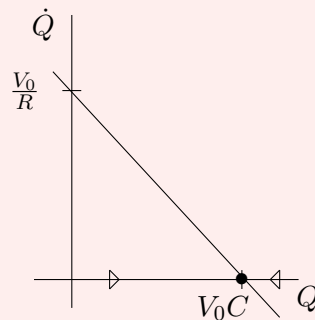
$Q_* = V_0 C$ globally stable because every $Q(t)$ approaches Q_* as $t \to \infty$.
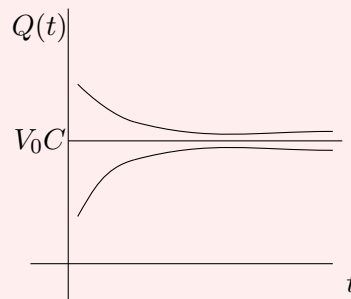
§3.2   Linear Stability Analysis

We have $\dot{x} = f(x), f(x_*) = 0$. Our task is to find an analytic criterion to decide if a fixed point x_* is stable/unstable.

<u>Picture:</u>

if $f'(x_*) > 0$, then x_* is unstable

if $f'(x_*) < 0$, then x_* is a stable fixed point
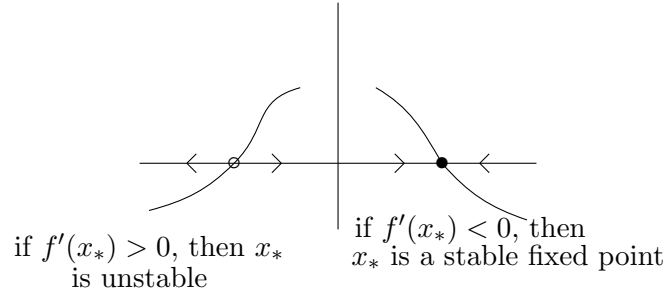
If $f'(x*) > 0$, then x_* is unstable. On the other hand, if $f'(x_*) < 0$, then x_* is a stable fixed point.

The linearization:

Consider: $\eta(t) = x(t) - x_*$ where $x(t)$ is the solution of $\dot{x} = f(x)$ with $x(0)$ close to x_*, $f(x_*) = 0$.

Note: $\dot{\eta}(t) = \dot{x}(t) = f(x(t)) = f(x(t) - x_* + x_*) = f(\eta(t) + x_*)$.

Taylor's Theorem:

$$f(x_* + \eta) = \underbrace{f(x_*)}_{=0} + f'(x_*)\eta + \underbrace{\mathcal{O}(\eta^2)}_{\text{error term and negligible if } f'(x_*) \neq 0 \text{ and } \eta \text{ is small}}$$

$\implies \quad \dot{\eta}(t) \approx f'(x_*)\eta(t)$ (as long as $\eta(t)$ is small) which is called the linearization of $\dot{x} = f(x)$ about x_*. The general solution is

$$\eta(t) = \eta_0 e^{f'(x_*) \cdot t}$$

In particular, η grows exponentially if $f'(x_*) > 0$ or decreases exponentially if $f'(x_*) < 0$.

Definition 3.4 (Characteristics Time Scale) — $\frac{1}{|f'(x_*)|}$ is called the characteristics time scale.

Example 3.5 (Logstics Equation)

$N \geq 0$ population size, $r > 0$ growth rate, $K > 0$ carrying capacity

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

Fixed points: $\dot{N} = 0 \implies N_* = 0$ or $N_* = K$.

Let $f(N) = rN\left(1 - \frac{N}{K}\right) \implies f'(N) = r - 2\frac{r}{K}N$. In particular, $f'(0) = r > 0 \implies N_* = 0$ is an unstable fixed point and $f'(K) = r - 2r = -r < 0 \implies N_* = K$ is stable.

Phase portrait:
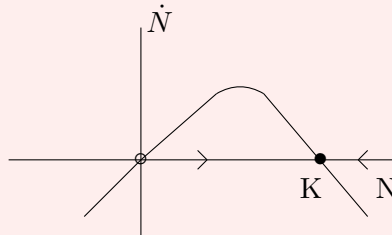
Thus, if $N(t)$ is the population with

$$N(0) = N_0 > 0 \implies N(t) \to K \text{ as } t \to \infty$$
$$N(0) = 0 \to N(t) = 0 \quad \forall t \text{ (no spontaneous outbreak)}$$

Characteristics time scale: $\frac{1}{|f'(N_*)|} = \frac{1}{r}$ for both $N_* = 0, K$.

Example 3.6

What if $f'(x_*) = 0$? Then we can't tell.

$\dot{x} = -x^3$

$x_* = 0$ stable

$\dot{x} = x^3$

$x_* = 0$ unstable

$\dot{x} = x^2$

$x_* = 0$ unstable

(semistable)

16
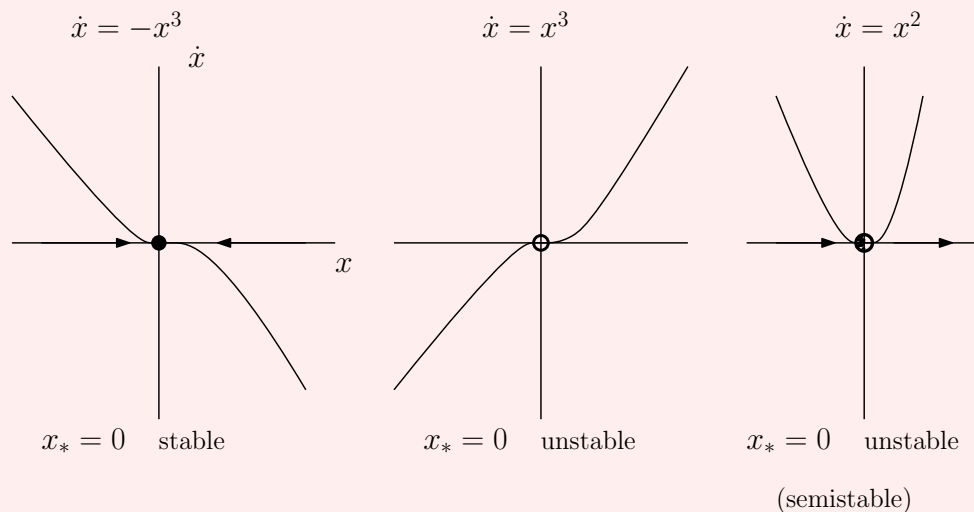
§4 | Lec 4: Jan 11, 2021

§4.1   Existence and Uniqueness

Example 4.1 (Non-uniqueness)

$\dot{x} = x^{\frac{1}{3}} \implies x_1(t) \equiv 0$ (for all t) is a solution with $x_1(0) = 0$ but $x_2(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$ is also a solution with $x_2(0) = 0$
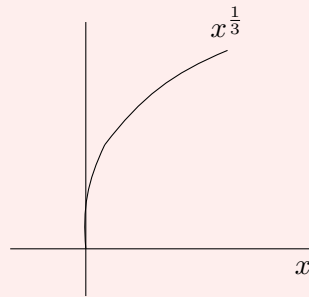
Is $x_0 = 0$ really a fixed point? No, it's unclear how it would behave (according to $x(t) = 0$ or $x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$).

Theorem 4.2 (Picard's)

Let $I = (a, b) \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ differentiable and f' continuous. Let $x_0 \in I$. Then there is $\tau > 0$ s.t. the initial value problem

$$\dot{x} = f(x), x(0) = x_0$$

has a unique solution $x : (-\tau, \tau) \to \mathbb{R}$.

Example 4.3

(The solution might not exist for all times) Consider

$$\frac{dx}{dt} = \dot{x} = 1 + x^2, \quad x(0) = 0$$

So,

$$dt = \frac{dx}{1 + x^2}$$

$$t = \int \frac{dx}{1 + x^2} = \arctan x + C$$

$$0 = 0 + C \implies C = 0$$

$$x(t) = \tan(t)$$
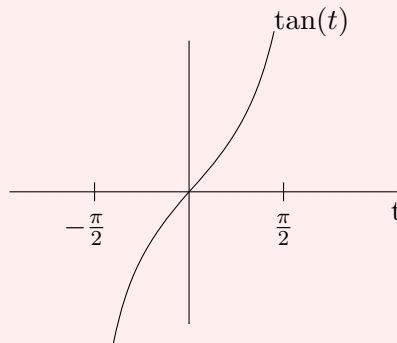
In particular,

$$x(t) \to +\infty \text{ as } t \to \frac{\pi}{2}$$

$$x(t) \to -\infty \text{ as } t \to \frac{-\pi}{2}$$

i.e., $x(t)$ reaches infinity in finite time, i.e., the solution $x(t)$ blows up in finite time.

Remark 4.4. (Hw 1) If $x_0 > 0$, then the solution to $\dot{x} = x^2, x(0) = x_0 > 0$ blows up in finite time. In fact, if $\alpha > 1$, then the solution to $\dot{x} = x^\alpha$, $x(0) = x_0 > 0$ blows up in finite time.

Theorem 4.5 (ODE Comparison)

If $x_1(t)$ solves $\dot{x} = f(x)$, $x_2(t)$ solves $\dot{x} = q(x)$ and $x_1(0) \leq x_2(0)$, $f(x) < q(x)$, then $x_1(t) \leq x_2(t)$ for all $t > 0$.

In particular, if $x_1(t) \to \infty$ in finite time, then $x_2(t) \to \infty$ in finite time.

Example 4.6

The solution to $\dot{x} = 1 + x^2 + x^3, x(0) = 0$ blows up in finite time.
Note: For $x \geq 0$:

$$1 + x^2 \leq 1 + x^2 + x^3$$

Recall: $\tan(t)$ solves $\dot{x} = 1 + x^2, x(0) = 0$. By comparison: the solution $x(t)$ to $\dot{x} = 1 + x^2 + x^3, x(0) = 0$ satisfies $x(t) \geq \tan(t)$. Thus, $x(t)$ blows up in finite time. We may indeed assume that $x(t) > 0$. Since $\dot{x}(0) = 1$, it follows that $x(t) > 0$ for $t > 0$ small. In fact, $\dot{x} = 1 + x^2 + x^3 > 0$ for $x(t)$ small, i.e., whenever $x(t)$ is close to zero, it must increase $\implies x(t) > 0$ for $t > 0$.

Example 4.7 (No Oscillating Solution in 1D)

Let $f \in C^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} | f \text{ differentiable}, f' \text{ continuous}\}$. Suppose $f(x_*) = 0, x(t)$ solution of $\dot{x} = f(x)$. If $x(t_0) = x_*$ for some t_0. Then $x(t) = x_*$ for all time t. Geometrically this says that a solution can never reach/cross a fixed point (unless it is a fixed point).

- $f(x(t)) > 0$ and $\dot{x}(t) > 0$, i.e., $x(t)$ increases.

- $f(x(t)) = 0$ and $x(t) = $ constant for all t.

- $f(x(t)) < 0$ and $\dot{x}(t) < 0$ i.e., $x(t)$ decreases.

In particular, there is no oscillating solution.

§5 | Lec 5: Jan 13, 2021

§5.1   Potential

Consider the movement of a particle (with lots of friction) in a potential.
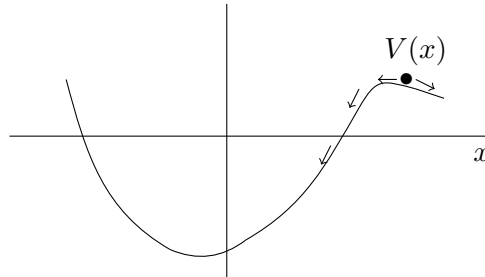
Notice:

- Particle approaches the local minimum of $V(x)$ (minimum energy level) no fixed point.

- Local minima of $V(x)$ are stable fixed points.

- Local maxima of $V(x)$ are unstable fixed points.

$\implies \dot{x} = f(x) = -\frac{dV}{dx} = -V'(x)$.
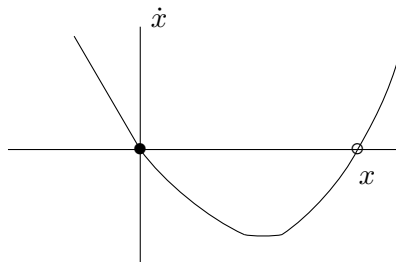
Expect $t \to V(x(t))$ is non-increasing for a solution $x(t)$ of $\dot{x} = -V'(x)$.
Indeed:

$$
\begin{aligned}
\frac{d}{dt}V(x(t)) &= V'(x(t))\frac{d}{dt}x(t) \\
&= V'(x(t))\left(-V'(x(t))\right) \\
&= -\left(V'(x(t))\right)^2 \leq 0
\end{aligned}
$$

$\implies$ particle always moves towards a lower energy level.

> **Definition 5.1** (Potential) — A function $V(x)$ s.t. $\dot{x} = f(x) = -\frac{dV}{dx}$ is called a potential.

Example 5.2

Graph potential for $\dot{x} = x - x^3$. Find/characterize equilibria (fixed points).

$$\dot{x} = f(x) = x - x^3 = -\frac{dV}{dx} \overset{\int}{\Longrightarrow} V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$$

$\Longrightarrow V$ is only defined up to a constant, we may choose any $C \in \mathbb{R}$, e.g., choose $C = 0$.
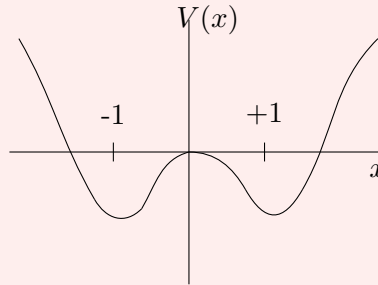
Local minima of V correspond to stable fixed points $\Longrightarrow 0 = -\frac{dV}{dx} = f(x) = x - x^3$, i.e., $x = \pm 1$.

Local maximum of V corresponds to an unstable fixed point at $x = 0$.

Phase portrait:
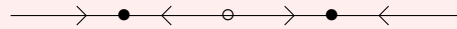
Remark 5.3. This system is often called <u>bistable</u> because it has two stable fixed points.

§5.2   Bifurcations

The qualitative behavior of 1D dynamical systems $\dot{x} = f(x)$ is determined by fixed points.

qualitative the same

(dynamics)
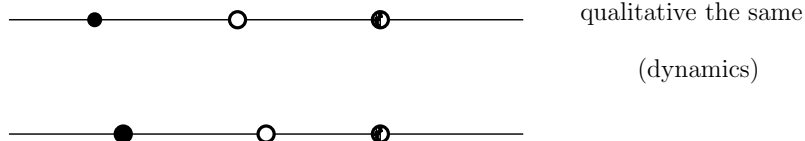
If $\dot{x} = f(r, x)$ depends on a parameter r, then the numbers of fixed points and their stability may change as r varies. This is called bifurcation.

Example 5.4 (Saddle-node, blue sky bifurcation)
$\dot{x} = r + x^2, \quad r \in \mathbb{R}.$
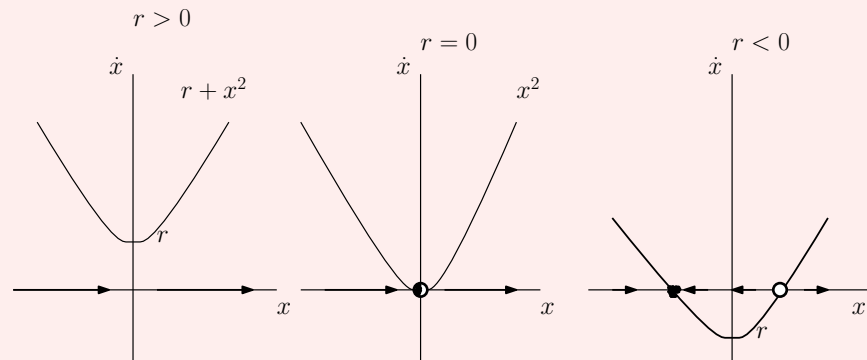
Hence, the qualitative behavior changes at $r_* = 0$, i.e., $r_* = 0$ is called a bifurcation point.

Ways to plot the dependence on the parameter:
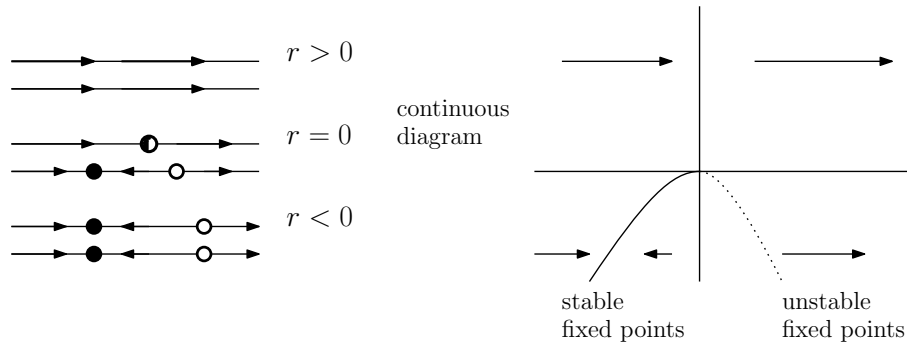
Most common: bifurcation diagram

$\dot{x} = r + x^2$

$\dot{x} = 0 \rightarrow x = \pm\sqrt{-r}$
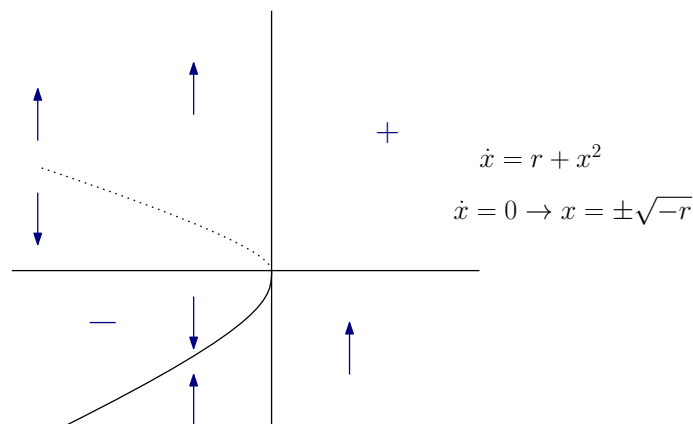
§6 | Lec 6: Jan 15, 2021

§6.1   Saddle-Node Example

Example 6.1

Argue geometrically that the ODE

$$\dot{x} = r - x - e^{-x}$$

undergoes a saddle-node bifurcation. Furthermore, find the bifurcation point.

<u>Note</u>: Fixed points of $\dot{x} = r - x - e^{-x}$ correspond to intersection points of the functions $r - x, e^{-x}$ because $r - x - e^{-x} = 0 \iff r - x = e^{-x}$.

$r > r_*$
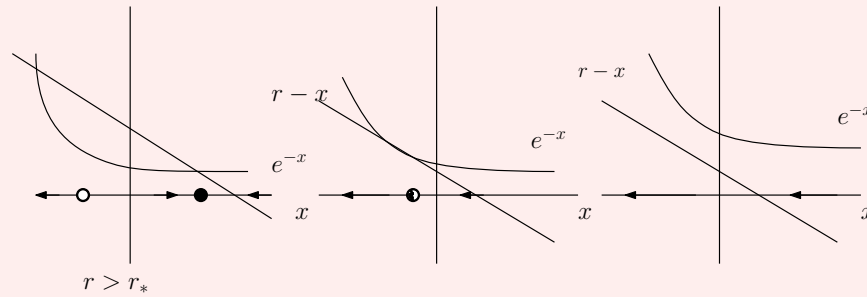
Indeed we have a saddle-node bifurcation.

<u>Note</u>: At $r = r_*$, the graph of $r - x$ and e^{-x} intersect tangentially. Thus, for the bifurcation point we require:

$$0 = \dot{x} = r - x - e^{-x} \implies r - x = e^{-x}$$

$$0 = \frac{d}{dx}(r - x - e^{-x}) \implies \frac{d}{dx}(r - x) = \frac{d}{dx}e^{-x}$$

So,

$$-1 = -e^{-x}$$
$$e^{-x} = 1$$
$$x = 0$$
$$r_* = x_* + e^{-x_*} = 0 + 1 = 1$$

Thus the bifurcation point is $(r_*, x_*) = (1, 0)$.

<u>Note</u>:

$$\dot{x} = r - x - e^{-x} = r - x - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right)$$

$$= r - 1 - \frac{1}{2}x^2 + \frac{x^3}{6} - \dots$$

$$\approx (r - 1) - \frac{1}{2}x^2 \text{ for x near } x_* = 0$$

Set $R = r - 1$, then $\dot{x} \approx R - \frac{1}{2}x^2$.

Upshot: Up to appropriate rescalings/coordinate changes, every saddle-node bifurcation looks like its normal form

$$\dot{x} = r - x^2 \qquad (\text{or } \dot{x} = r + x^2)$$

close to the bifurcation point $(r_*, x_*) = (0, 0)$.

§6.2   Normal Forms

point where $f(r, x) = 0$
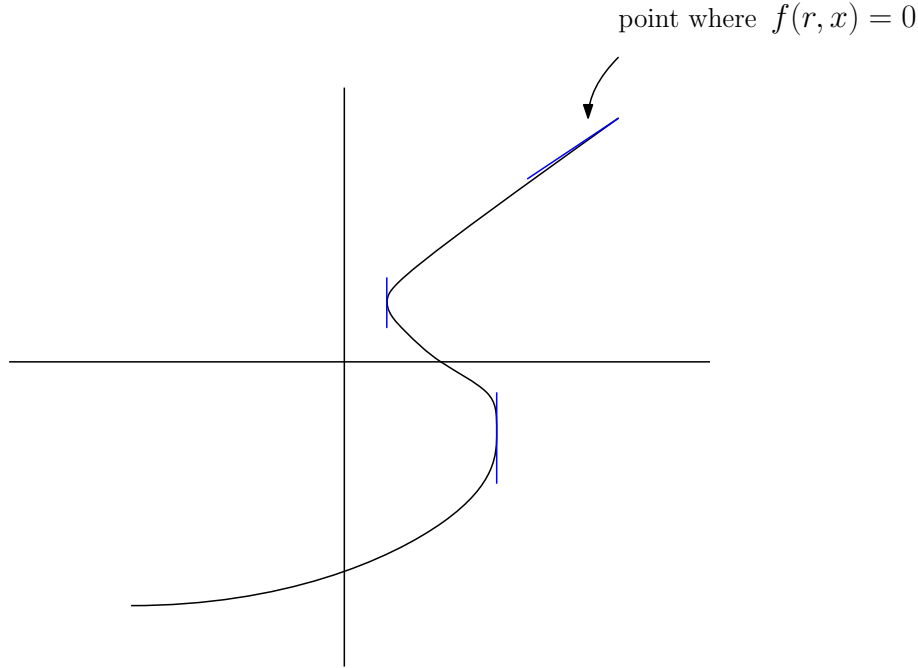
Recall:

- Normal vector: $\begin{pmatrix} \partial_r f \\ \partial_x f \end{pmatrix}$

- Tangent vector: $\begin{pmatrix} -\partial_x f \\ \partial_r f \end{pmatrix}$

<u>Note</u>: Bifurcation points have vertical tangent vectors, i.e., $\partial_x f = 0, \partial_r f \neq 0$.

Theorem 6.2 (Taylor's)

Suppose $f(r_*, x_*) = 0$.

$$f(r, x) = f(r_*, x_*) + \underbrace{\frac{\partial f}{\partial r}(r_*, x_*)(r - r_*)}_{p_1} + \underbrace{\frac{\partial f}{\partial x}(r_*, x_*)(x - x_*)}_{q_1}$$

$$+ \underbrace{\frac{1}{2}\frac{\partial^2 f}{\partial r^2}(r_*, x_*)(r - r_*)^2}_{p_2} + \underbrace{\frac{\partial^2 f}{\partial r \partial x}(r_*, x_*)(r - r_*)(x - x_*)}_{R} + \underbrace{\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(r_*, x_*)(x - x_*)^2}_{q_2} + \ldots$$

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Remark 6.3. If $q_1 \neq 0$, then there is no bifurcation at (r_*, x_*), linear stability (sign of q_1) determines if (r_*, x_*) is (un)stable.

Theorem 6.4

Suppose that $f(r_*, x_*) = 0, q_1 = 0, p_1 \neq 0, q_2 \neq 0$, then $\dot{x} = f(r, x)$ undergoes a saddle node bifurcation at (r_*, x_*) and

$$\dot{x} = \frac{\partial f}{\partial r}(r^*, x^*)(r - r^*) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

for $|r - r_*| < \epsilon^2, \quad |x - x_*| < \epsilon$.

Remark 6.5.   i) Note that the constant $(r - r_*)(x - x_*)$ is $\mathcal{O}(\epsilon^3)$

ii) With a coordinate change $(t, x, r) \mapsto (s, y, R)$ we can arrange that ODE looks like

$$\frac{d}{ds}y = R + y^2$$

near $(0, 0) = (R(r_*), y(x_*))$

Example 6.6

$\dot{x} = e^r - x - e^{-x}$ undergoes a saddle-node bifurcation near $(r_*, x_*) = (0,0)$. Apply the theorem 6.4,

$$f(r, x) = e^r - x - e^{-x}$$
$$f(0,0) = 1 - 0 - 1 = 0$$
$$\frac{\partial f}{\partial x}(r, x) = -1 + e^{-x} \implies \frac{\partial f}{\partial x}(0,0) = 0$$
$$\frac{\partial f}{\partial r}(r, x) = e^r \implies \frac{\partial f}{\partial r}(0,0) = 1 \neq 0$$
$$\frac{\partial^2 f}{\partial x^2}(r, x) = -e^{-x} \implies \frac{\partial^2 f}{\partial x^2}(0,0) = -1 \neq 0$$

Therefore, by theorem 6.4, $(r_*, x_*) = (0,0)$ is a bifurcation point of a saddle-node bifurcation.

Normal form near $(r_*, x_*) = (0,0)$:

$$\dot{x} = e^r - x - e^{-x}$$
$$= 1 + r + \frac{r^2}{2} + \mathcal{O}(r^3) - x - \left(1 - x + \frac{x^2}{2} + \mathcal{O}(x^3)\right)$$
$$= r + \underbrace{\frac{r^2}{2}}_{\mathcal{O}(\epsilon^4)} - \frac{x^2}{2} + \mathcal{O}(r^3) + \mathcal{O}(x^3)$$
$$= r \underbrace{- \frac{x^2}{2}}_{\mathcal{O}(\epsilon^2)} + \mathcal{O}(\epsilon^3) \text{ if } |r - r_*| = |r| < \epsilon^2$$

$$\text{if } |x - x_*| = |x| < \epsilon$$

Set $y = \frac{x}{2}$, then
$$\dot{y} = \frac{1}{2}\dot{x} = \frac{r}{2} - \frac{x^2}{4} + \mathcal{O}(\epsilon^3) = \frac{r}{2} - y^2 + \mathcal{O}(\epsilon^3)$$

Set $s = -t$, then
$$\frac{d}{ds}y = -\frac{d}{dt}y = -\frac{r}{2} + y^2 + \mathcal{O}(\epsilon^3)$$

Set $R = -\frac{r}{2}$, then
$$\underbrace{\frac{d}{ds}y = R + y^2}_{\text{normal form of a saddle-node bifurcation}} + \mathcal{O}(\epsilon^3)$$

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§7 | Lec 7: Jan 20, 2021

§7.1   Classification of Bifurcations

Let's rewrite $\dot{x}$ in theorem 6.4 as

$$\dot{x} = p(r - r_*) + \frac{c}{2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

if $|r - r_*| < \epsilon^2, |x - x_*| < \epsilon$. After a coordinate change $(t, x, r) \mapsto (s, y, R)$ such that

$$s = t$$
$$y = \frac{c}{2}(x - x_*)$$
$$R = p\frac{c}{2}(r - r_*)$$

the ODE is represented by the normal form.

$$\frac{d}{ds}y = \dot{y} = R + y^2 + \mathcal{O}(\epsilon^3)$$

for $|R| < \epsilon^2, |y| < \epsilon$.

If $f(x_*, r_*) = 0$, and also $\frac{\partial f}{\partial x}(x_*, r_*) = 0 = \frac{\partial f}{\partial r}(x_*, r_*)$, then the second derivatives determines the bifurcation type.

$$\text{Hessian}\quad \text{Hessf} = \begin{pmatrix} \frac{\partial^2 f}{\partial r^2} & \frac{\partial^2 f}{\partial r \partial x} \\ \frac{\partial^2 f}{\partial r \partial x} & \frac{\partial^2 f}{\partial x^2} \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Second test: if $AC - B^2 > 0$, (r_*, x_*) is a local maximum/minimum. In particular, (r_*, x_*) is an isolated fixed point. (irrelevant case)

Practically relevant case: If $AC - B^2 < 0$: (r_*, x_*) is a saddle. If also $C \neq 0$: transcritical bifurcation.

$$\dot{y} = Ry - y^2 + \mathcal{O}(\epsilon^2)$$

for $|R| < \epsilon, |y| < \epsilon$ (after an appropriate coordinate change)

$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

If also $C = 0$: Pitchfork bifurcation

- Supercritical Pitchfork bifurcation:

$$y' = Ry - y^3 + \mathcal{O}(\epsilon^3)$$

- Subcritical Pitchfork bifurcation

$$y' = Ry + y^3 + \mathcal{O}(\epsilon^3)$$

  for $|R| < \epsilon^2, |y| < \epsilon$

Again,

$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

§7.2   Transcritical Bifurcation

Normal form:
$$\dot{x} = rx - x^2 = x(r - x)$$

In particular, $x_* = 0$ is always a fixed point but it changes stability.
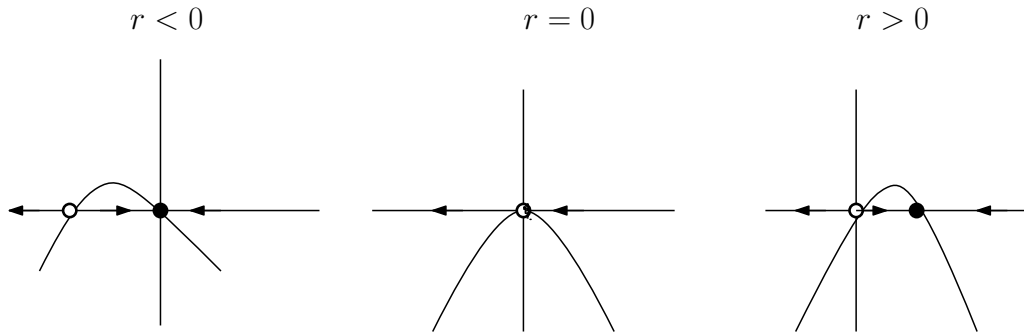
Bifurcation diagram: $\dot{x} = x(r - x) = rx - x^2 = f(x)$. Fixed points:

$$x_* = 0, \quad x_* = r \quad r \in \mathbb{R}$$

intermediate step:
draw fixed points
(without stability)
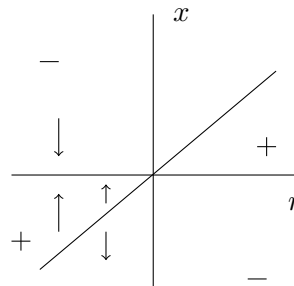
bifurcation diagram
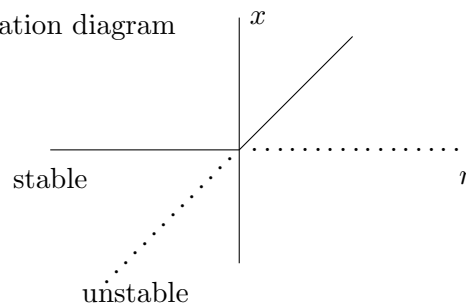
§8 | Lec 8: Jan 22, 2021

§8.1   Example of Transcritical Bifurcation

Example 8.1

$\dot{x} = r\ln(x) + x - 1$ has a transcritical bifurcation at $(r_*, x_*) = (-1, 1)$.
Geometric approach:

$$\dot{x} = 0 \iff r\ln(x) = 1 - x$$

$r < -1$ \qquad\qquad\qquad $r = -1$ \qquad\qquad\qquad $-1 < r < 0$
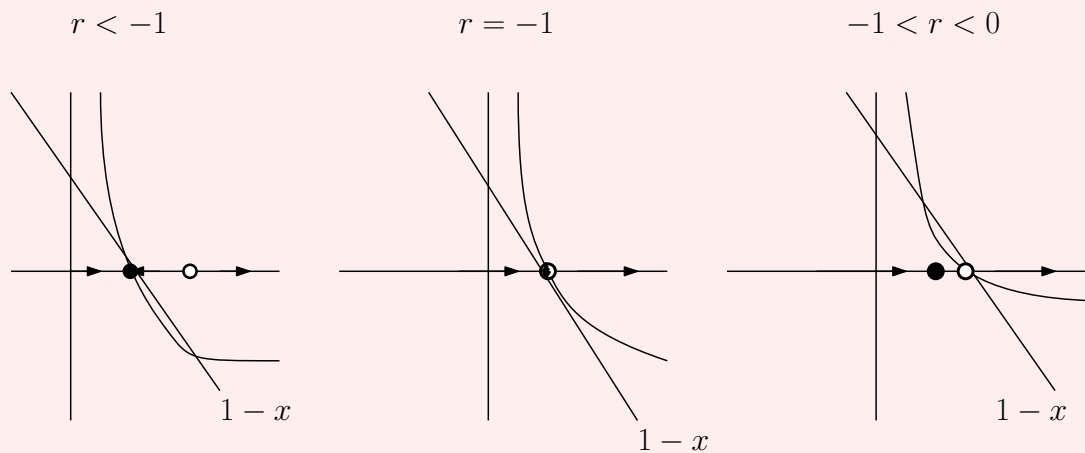
Bifurcation near $(r_*, x_*) = (-1, 1)$
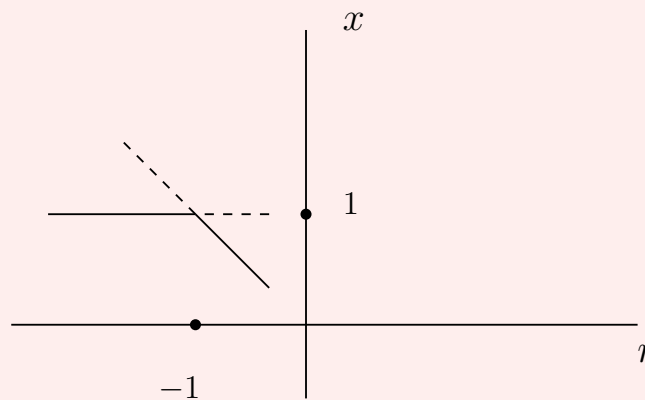
Normal form: $\dot{x} = r\ln(x) + x - 1$.

Remark 8.2. $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \quad |x| < 1$

So,

$$\dot{x} = r\ln(x) + x - 1$$
$$= r\left(x - 1 - \frac{1}{2}(x-1)^2 + \mathcal{O}((x-1)^3)\right) + x - 1$$
$$= (r+1)(x-1) - \frac{1}{2}\left((r+1) - 1\right)(x-1)^2 + \mathcal{O}\left(r(x-1)^3\right)$$
$$= (r+1)(x-1) + \frac{1}{2}(x-1)^2 + \mathcal{O}(\epsilon^3)$$

if $|r - (-1)| < \epsilon$ and $|x - 1 < \epsilon|$.
Now, set $R = r + 1, y = c \cdot (x - 1)$. Then,

$$\dot{y} = c\dot{x}$$
$$= (r+1)c(x-1) + \frac{1}{2}c(x-1)^2 + \mathcal{O}(\epsilon^3)$$
$$= Ry + \frac{1}{2c}\left(c(x-1)\right)^2 + \mathcal{O}(\epsilon^3)$$
$$= Ry + \underbrace{\frac{1}{2c}}_{=1} y^2 = Ry + y^2$$

for $c = \frac{1}{2}$.

§8.2   Application of Transcritical Bifurcations

Example 8.3 (Laser Threshold)

Consider

energy source

$\downarrow$

active material $\quad \dfrac{\text{laser}}{\text{light}} \longrightarrow$

$\nearrow$

mirrors
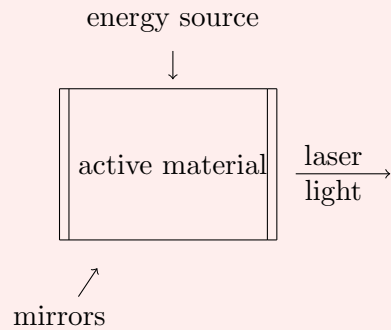
Simple model:

$$n = n(t) = \# \text{ photons in the laser}$$

Then

$$\dot{n} = G \cdot \underbrace{N}_{\# \text{ excited atoms}} \cdot n - kn$$

$$= N_0 - \alpha \cdot n$$

$$= G(N_0 - \alpha n)n - kn$$

$$= (GN_0 - k)n - \alpha Gn^2$$

where $G, k, \alpha > 0$. Fixed points:

$$\dot{n} = 0 \iff n = 0 \text{ or } n = \frac{GN_0 - k}{\alpha G}$$

$GN_0 - k < 0$ $GN_0 - k = 0$ $GN_0 - k > 0$

$\dot{n}$ $\dot{n}$

n
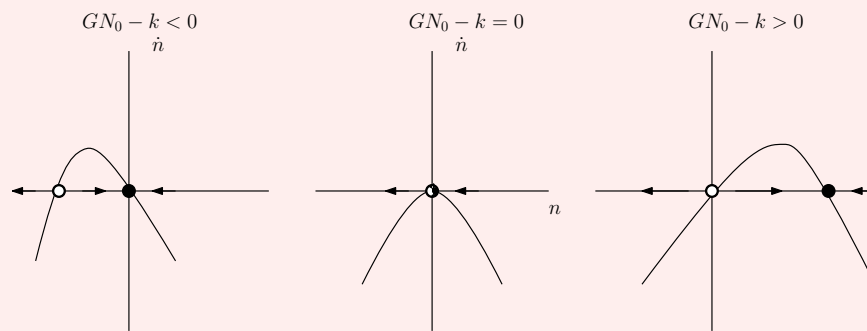
Bifurcation diagram
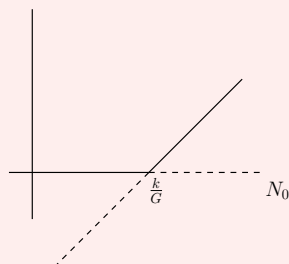
transcritical bifurcation at

$(N, n) = (\frac{k}{G}, 0)$

$\frac{k}{G}$ N_0

$\frac{k}{G} = $ laser threshold

§9 | Lec 9: Jan 25, 2021

§9.1   Supercritical Pitchfork Bifurcation

Fixed points appear/disappear in symmetric pairs

Supercritical Pitchfork Bifurcation:

$$\dot{x} = rx - x^3$$

$r < 0$ $r = 0$ $r > 0$
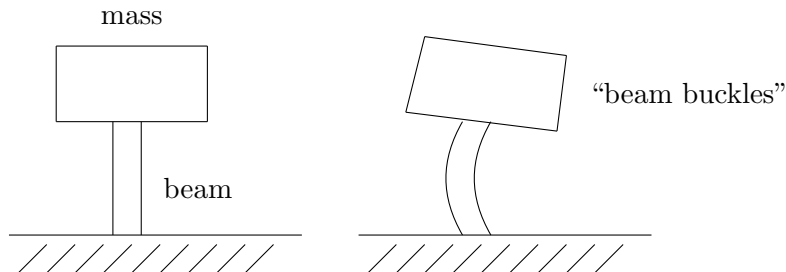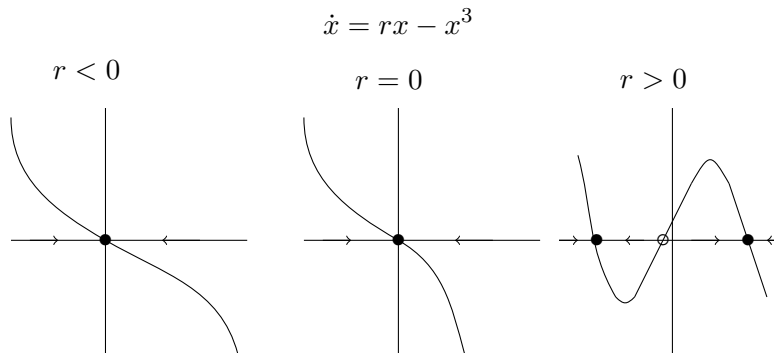
Remark 9.1. Decay towards $x_* = 0$ is not exponential in time for $r = 0$.

Bifurcation diagram:
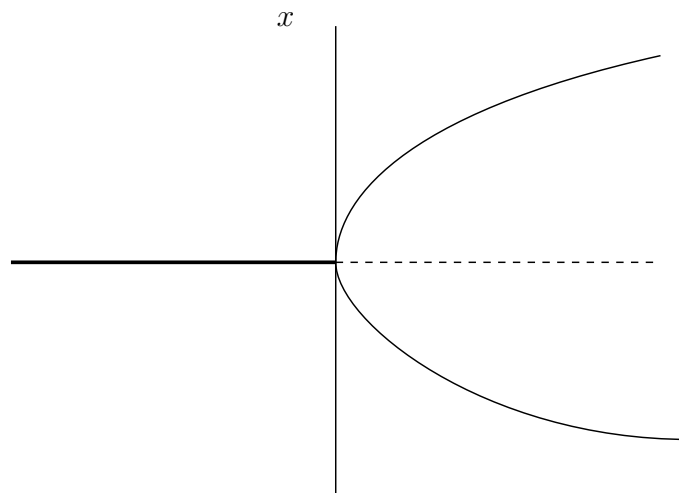
$$\dot{x} = rx - x^3 = 0$$
$$\implies x = 0, \qquad x = \pm\sqrt{r}, \quad r > 0$$

Example 9.2

Potential for $\dot{x} = rx - x^3 = -\frac{dV}{dx}$

$$\implies V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4 + \underbrace{C}_{=0}$$

$r < 0$　　　　　　　　　$r = 0$　　　　　　　　　$r > 0$
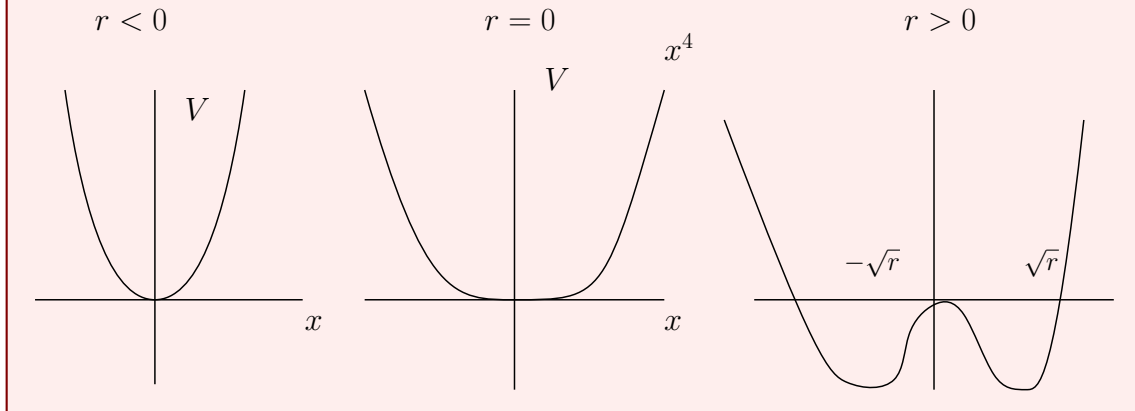
§9.2　　Subcritical Pitchfork Bifurcation

$$\dot{x} = rx + x^3$$

$r < 0$　　　　　　　　　$r = 0$　　　　　　　　　$r > 0$
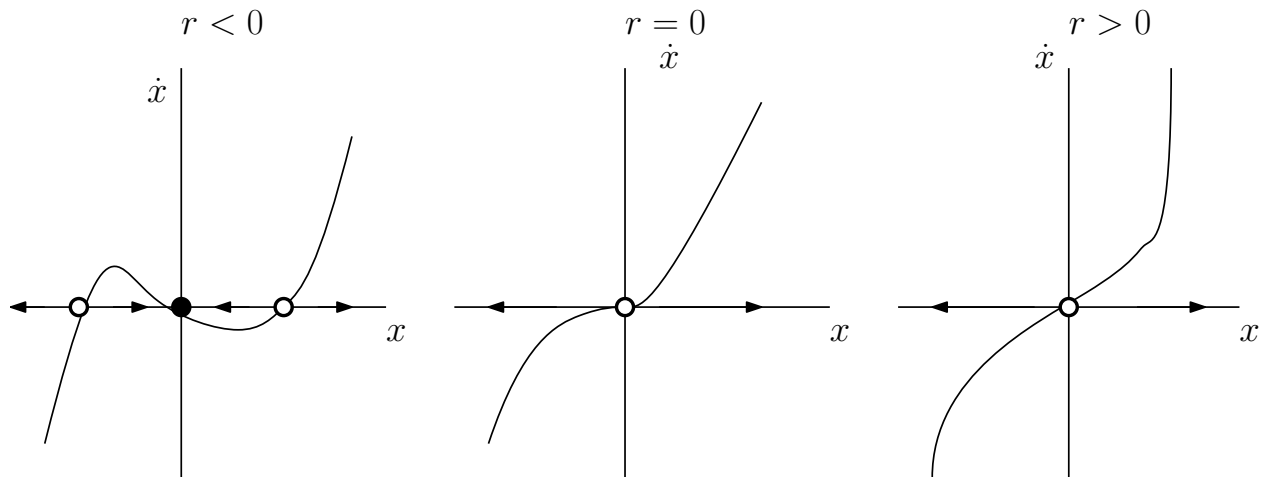
Fixed points:

$$\dot{x} = rx + x^3 = 0$$
$$\implies x = 0, \qquad x = \pm\sqrt{-r}, \quad r < 0$$

Bifurcation Diagram:
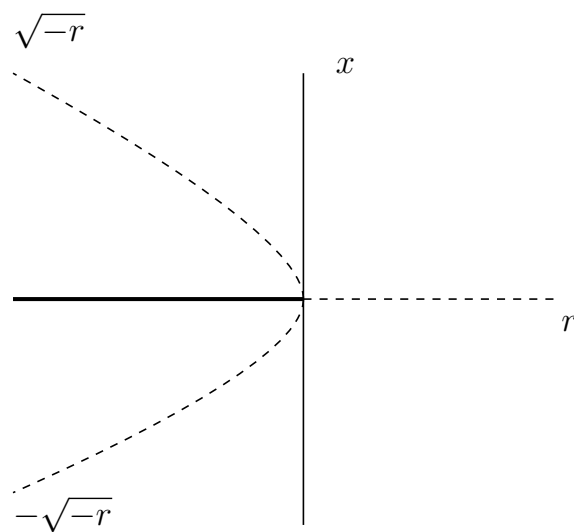
Remark 9.3. If $r > 0, x_0 > 0$, then the solution $x(t)$ with $x(0) = x_0 > 0$ blows up in finite time (cf. homework). Interpretation: $+x^3$ is destabilizing.

Physically more realistic scenario:

$$\dot{x} = rx + x^3 - x^5$$

where x^5 is the stabilizing higher order term.
Fixed points:

$$\dot{x} = 0 \iff x = 0, \quad r = -x^2 + x^4$$

Bifurcation diagram:
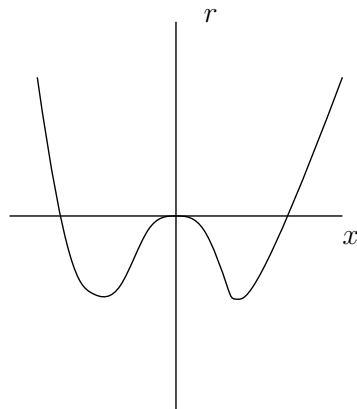
1. Intermediate step   2. Stability Types
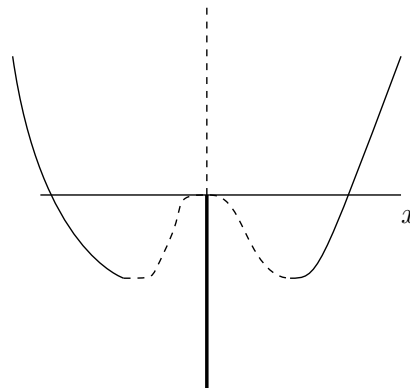
3. Change axes: bifurcation diagram

$$\dot{x} = rx + x^3 - x^5$$
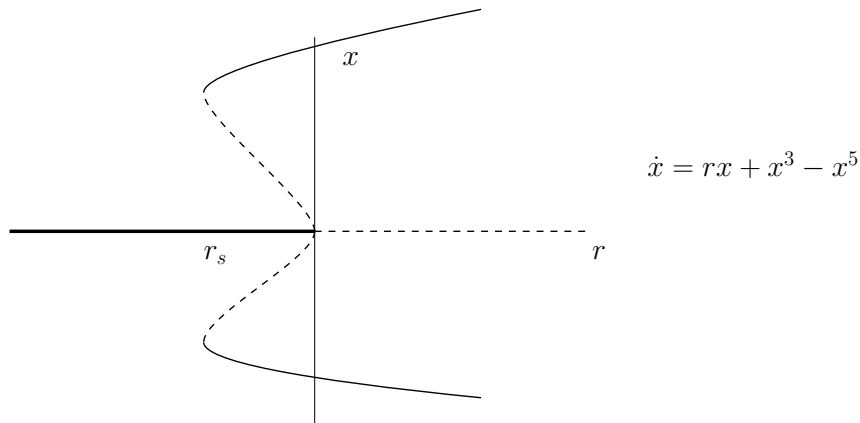
Remark 9.4.   i) Subcritical pitchfork bifurcation at $(r_*, x_*) = (0,0)$ and saddle node bifurcation at $(r_s, x_*) = \left(-\frac{1}{4}, \pm\sqrt{2}\right)$.

$$\dot{x} = r(t)x + x^3 - x^5$$
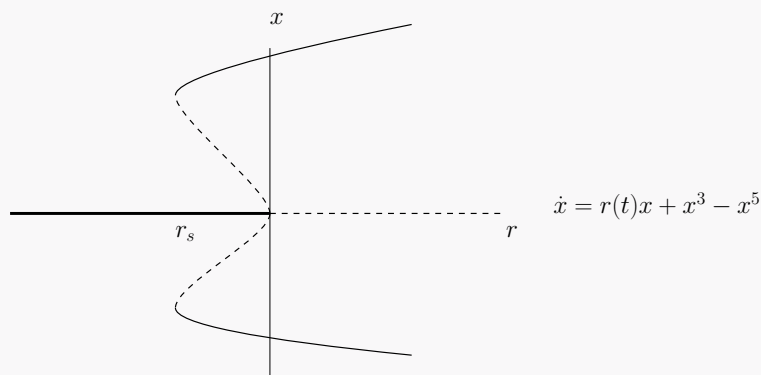
ii) jump at $r_* = 0$: A small perturbation of a stable fixed point at $(0, r)$ with $r < 0$ jumps to the stable large amplitude branch as r becomes positive, but does not jump back until $r < r_s$.

This non-reversibility is called hysteresis.

§10 | Lec 10: Jan 27, 2021

§10.1   Bifurcation at Infinity

> **Example 10.1**
>
> $\dot{x} = r - \frac{x^2}{1+x^2}$
>
> Fixed points: $\dot{x} = 0 \iff r = \frac{x^2}{1+x^2}$
>
> bifurcation diagram
>
>
>
> $\dot{x} > 0$
>
> $\dot{x} < 0$
>
> <u>Note</u>:
>
> - At (r_*, x_*) we have a saddle node bifurcation.
>
> - If $r \in (0,1)$ we have two fixed points.
>
> - For $r \geq 1$ we have no fixed points.
>
> Thus, we have a <u>bifurcation at</u> (spatial) <u>infinity</u>.
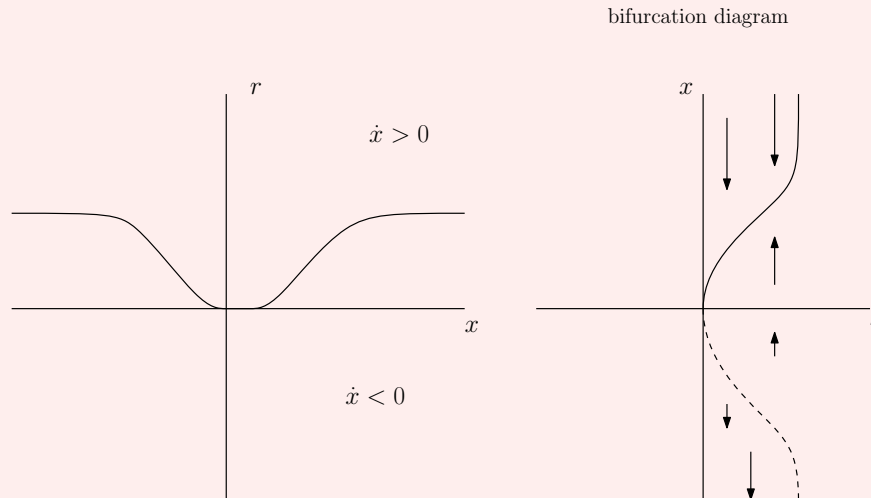
§10.2   Dimensional Analysis and Scaling

Over-damped bead over a hoop:
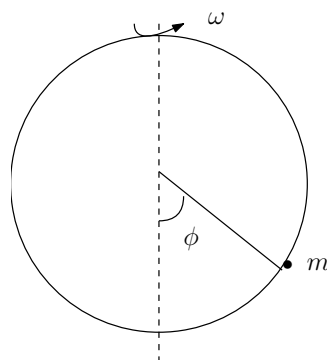
forces: gravitation: $-mg\vec{e_2}$

centrifugal: $mr\sin\phi\omega^2\vec{e_x}$

damping: $-b\phi\vec{e_\phi}$

Physics: $mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$

Experiment: Provided ω large enough, bead slides slowly towards a fixed angle, after an initial acceleration phase.

Question 10.1. When we can neglect second order term $\ddot{\phi}$?

Problem 10.1. We're working with different dimensions, e.g.

$$[m] = kg$$
$$[b] = \frac{kg \cdot m}{s}$$

What is small – what quantity is actually small so we can neglect the second order term?

Idea: Non-dimensionalize

- small means $\ll 1$

- reduce the numbers of parameters

- no general algorithm

Quantity ω large, time scale T.

Set $\tau = \frac{t}{T} \implies d\tau = \frac{1}{T}dt$, where T is the characteristics time scale.

$\dot{\phi} = \frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d\phi}{d\tau}$

Similarly, $\ddot{\phi} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}$

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi \tag{1}$$

So

$$\implies \frac{mr}{T^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{T}\frac{d\phi}{d\tau} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi \qquad \text{(unit force)}$$

$$\implies \frac{r}{gT^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{mgT}\frac{d\phi}{d\tau} - \sin\phi + \frac{r\omega^2}{g}\sin\phi\cos\phi \qquad \text{(dimensionless)}$$

Thus 1$^{\text{st}}$ order term $\frac{d\phi}{d\tau}$ dominates $\frac{d^2\phi}{d\tau^2}$ if $\frac{r}{gT^2} \ll 1$ and $\frac{b}{mgT} \approx \mathcal{O}(1)$, i.e., $\frac{b}{mgT} = 1$ and $\epsilon = \frac{r}{gT^2}$

$$\implies T = \frac{b}{mg}$$

$$\implies \epsilon = \frac{rgm^2}{b^2} \ll 1$$

Set $\gamma = \frac{r\omega^2}{g}$. Then the non-dimensionalize equation becomes

$$\epsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma\sin\phi\cos\phi$$

Overdamped limit: $\epsilon \to 0$

$$\frac{d\phi}{d\tau} = -\sin\phi + \gamma\sin\phi\cos\phi$$
$$= \sin\phi(\gamma\cos\phi - 1)$$

<u>Dynamics</u>: $\frac{d\phi}{d\tau} = 0$ (fixed points)

$$\implies \sin\phi = 0 \iff \phi = 0, \pi \text{ (bottom/top of hoop)}$$

or

$$\cos\phi = \frac{1}{\gamma} \in (0, 1] \implies \gamma \geq 1$$

Fixed points:
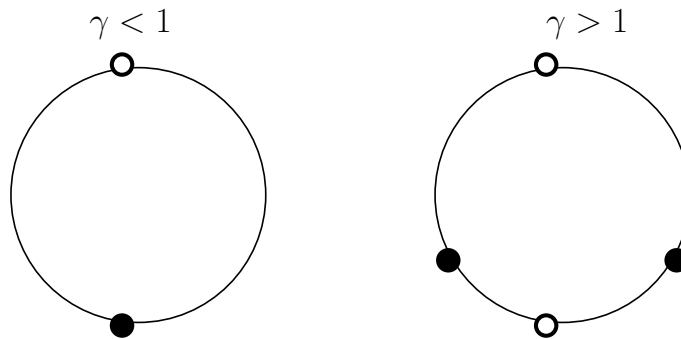
Bifurcation Diagram:
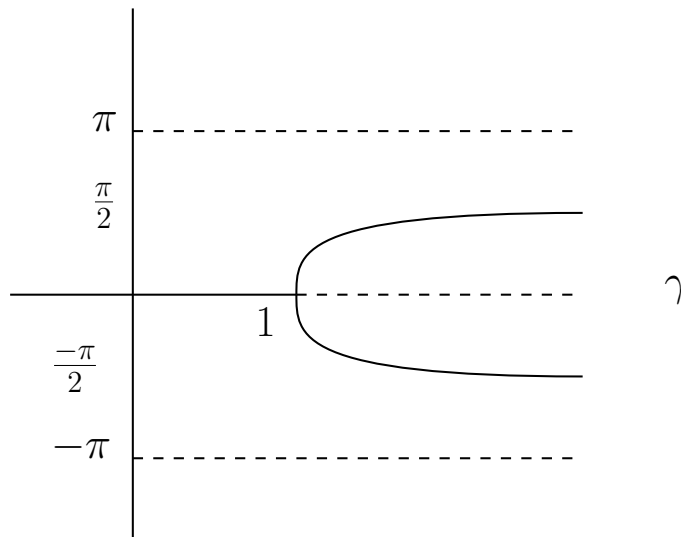
In particular, we have a supercritical pitchfork bifurcation at $\gamma = 1$.

§11 | Lec 11: Jan 29, 2021

§11.1   Imperfect Bifurcation and Catastrophes

$\dot{x} = h + rx - x^3$

- If $h = 0$: symmetry, if $x(t)$ is a solution then $-x(t)$ is also a solution (supercritical pitchfork bifurcation).

- If $h \neq 0$: imperfect parameter, breaks symmetry.

Aim: Study qualitative behavior of ODE as parameters vary.
Strategy: keep h fixed and vary r

- $h = 0$: supercritical pitchfork bifurcation
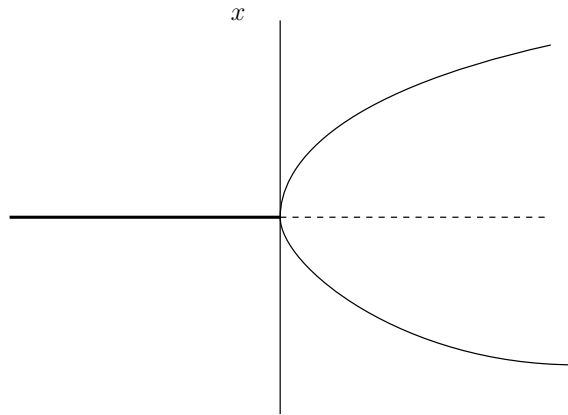
Figure 1: Bifurcation Diagram

- $h > 0$: fixed points: $\dot{x} = 0 \iff x^3 = h + rx$
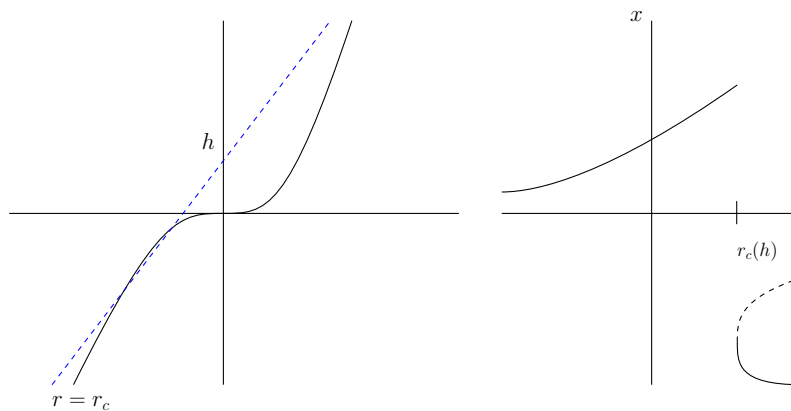
Figure 2: Bifurcation Diagram

- $h < 0$: Fixed points: $x^3 = h + rx$
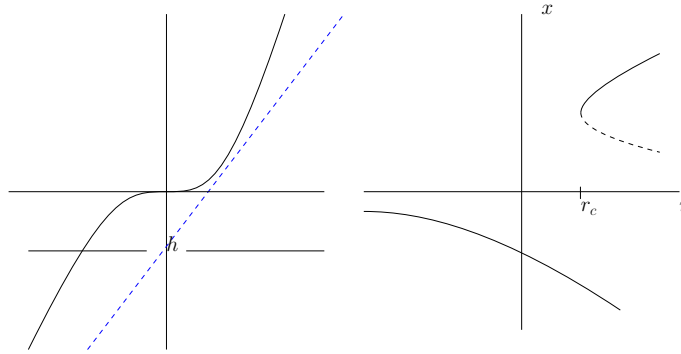
Figure 3: Bifurcation Diagram

Note:We have saddle node bifurcation at $r_c = r(h)$

Bifurcation Curves

$$\left\{ (h,r)|(h,r,x) \text{ solves } f = 0, \frac{\partial f}{\partial x} = 0 \right\}$$

in our example $\dot{x} = h + rx - x^3$

$$0 = \frac{\partial f}{\partial x} = r - 3x^2 \implies x = \pm\sqrt{\frac{r}{3}}$$

$$0 = f = h + rx - x^3 \implies h = x^3 - rx$$

$$\implies h = x^3 - rx = \pm\frac{2\sqrt{3}}{9}r^3$$

$$h = h_c(r) = \pm\frac{2\sqrt{3}}{9}r^{\frac{3}{2}}$$

$$\implies r = r_c(h) = \left(\frac{9}{2\sqrt{3}}|h| \right)^{\frac{2}{3}}$$

Stability Diagram:
Plot the bifurcation curves in the parameters space ($= (h,r)$ plane).
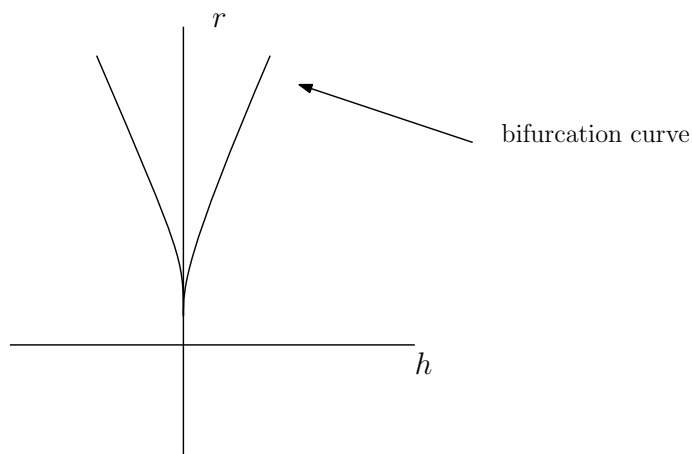
bifurcation curve

Note: qualitative behavior of ode changes as (h, r) cross bifurcation curve.
In example:

- "below" bifurcation curve: ODE has one (stable) fixed point.

- "on" bifurcation curve: two fixed points.

- "above" bifurcation curve: three fixed points.

Remark 11.1.   • Saddle-node bifurcation occurs along bifurcation curve for $(h, r) \neq (0, 0)$

- At $(h, r) = (0, 0)$, the branches $r_c(h) = \left(\frac{9}{2\sqrt{3}} |h| \right)^{\frac{2}{3}}$ for $h > 0$ and $h < 0$ meet tangentially, and we have a cusp point at $(h, r) = (0, 0)$. This is an example of a codimension 2 bifurcation (i.e., we need two parameters to model this type of bifurcation).

Bifurcation diagrams for fixed $r \in \mathbb{R}$.

$$\dot{x} = h + rx - x^3 = 0 \iff h = x^3 - rx$$
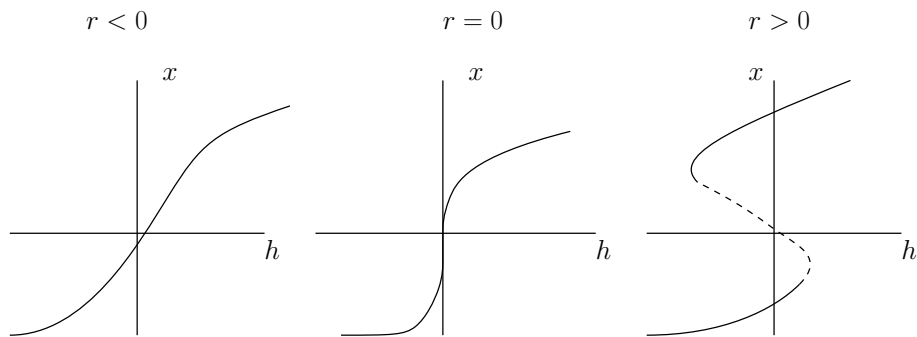
3D plot(h, r, fixed points x)
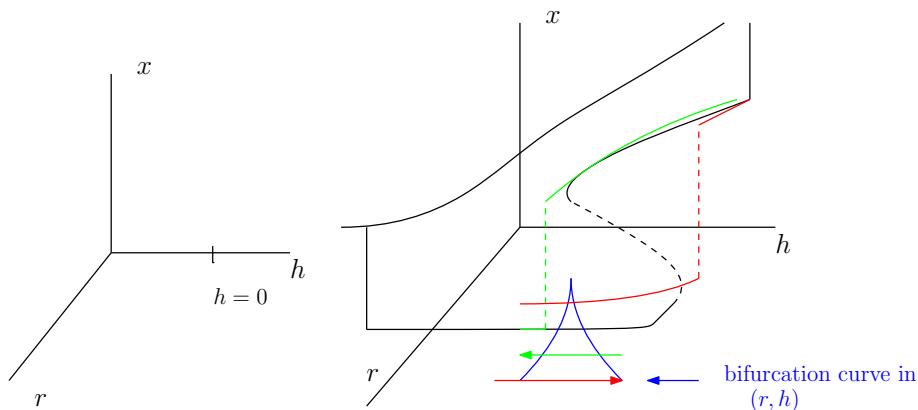
Picture/surface of cusp catastrophe solutions close to "upper" stable fixed points drop to "lower" stable fixed points as (r, h) vary (and vice versa).

Example 11.2 (practical)

Details in the book, page 74

wire
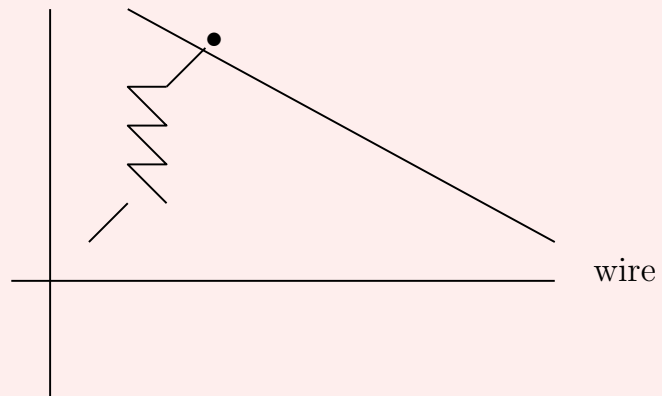
§12 | Midterm 1: Feb 1, 2021

§13 | Lec 12: Feb 3, 2021

§13.1   Flows on the Circle

real line R

$\dot{x} = f(x)$

x = position

circle (with radius 1)

$\theta = 0$

$\theta = $ angle

$\dot{\theta} = f(\theta)$
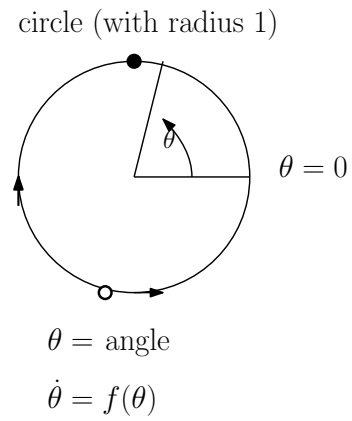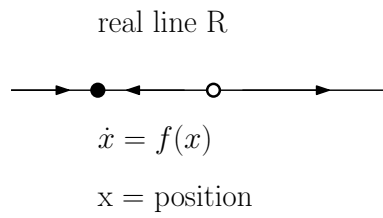
Example 13.1   i) $\dot{x} = \sin(x)$. Fixed points: $\dot{x} = 0$

$$\Longleftrightarrow \; x = \ldots, -\pi, 0, \pi, 2\pi, \ldots$$

i.e., $x = k\pi, k \in \mathbb{Z}$.
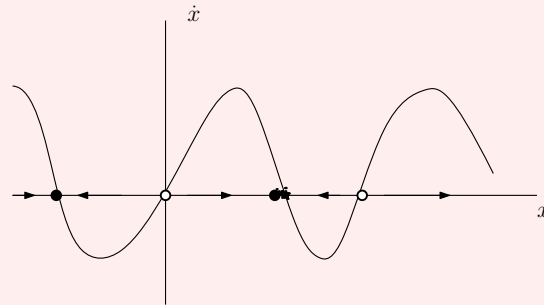
$$\dot{\theta} = \sin \theta$$
$$\dot{\theta} = 0$$
$$\Longleftrightarrow \; \theta = 0 \text{ or } \theta = \pi$$
$$\underbrace{\theta = 2\pi}_{\text{same position on circle}}$$

i.e., θ is defined up to multiples of 2π.

Note: If $f(\theta) > 0$: flow is counterclockwise, and if $f(\theta) < 0$: flow is clockwise.

$$\dot{\theta} = \sin(\theta)$$
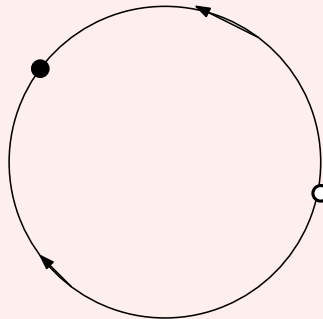
ii) $\dot{x} = x$ where $f(x) = x$ is <u>not</u> periodic.

   Thus $\dot{\theta} = \theta$ does <u>not</u> work, because $\theta = 0, \theta = 2\pi$ describe the same position on the circle but $f(\theta) = \theta$ yields different values at $\theta = 0, 2\pi$, i.e. $f(\theta)$ is <u>not</u> a vector field on the circle.

   <u>Correspondence:</u>

   $f(x)$ is 2π-periodic, i.e. $f(x + 2\pi) = f(x)$, and f is continuously differentiable $\Longleftrightarrow$ $f(\theta)$ defines a vector field on the circle.

Example 13.2 iii) $\dot{x} = c > 0$

$$x(t) = ct + x_0$$
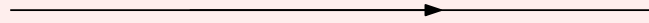
$\dot{\theta} = \omega > 0$ – <u>uniform oscillator</u>
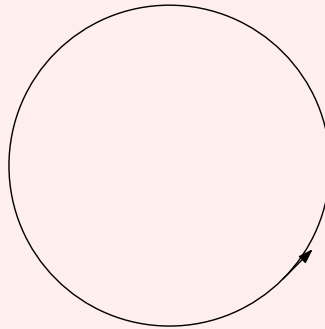
$$\theta(t) = \omega t + \theta_0$$

<u>Period T</u>:

$$\theta(T) = \theta(0) + 2\pi$$
$$\omega T + \theta_0 = \theta_0 + 2\pi$$
$$T = \frac{2\pi}{\omega}$$

In particular, <u>periodic solutions</u> are possible.

Example 13.3

Two runners are on a circular track, running in the same direction, with constant speed:

- Runner 1: period $T_1 = \frac{2\pi}{\omega_1}$, angle θ_1

- Runner 2: period $T_2 = \frac{2\pi}{\omega_2}$, angle θ_2

Runner 1, 2 start at the same position. Suppose $T_1 < T_2$, i.e. Runner 1 is faster than runner 2.

Question 13.1. How long does it take runner 1 to lap runner 2?

<u>Ans</u>: T_{lap} = time when <u>phase difference</u>

$$\phi = \theta_1 - \theta_2 \text{ is } 2\pi$$
$$\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2, \ \phi(0) = 0$$
$$\implies \phi(t) = (\omega_1 - \omega_2)t$$
$$\implies T_{\text{lap}} = \frac{2\pi}{\omega_1 - \omega_2} = \frac{1}{\frac{1}{T_1} - \frac{1}{T_2}} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1}$$

i.e. Runner 1,2 are <u>in phase</u> after T_{lap} again. This is called <u>beat phenomenon</u>.

§14 | Lec 13: Feb 5, 2021

§14.1 Non-uniform Oscillator

$$\dot{\theta} = \omega - a\sin\theta, \qquad \omega > 0, a > 0$$

Practical example: overdamped limit of pendulum driven by constant torque.
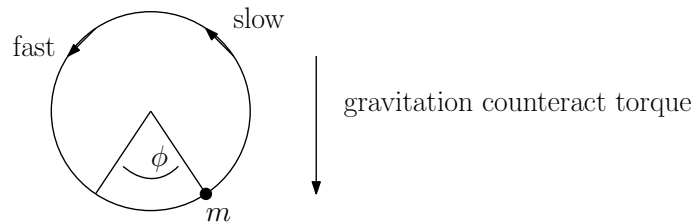
$$\dot{\phi} = \omega - a\sin\phi$$

Consider: $\dot{\theta} = \omega - a\sin\theta$
For $0 < a < \omega$:
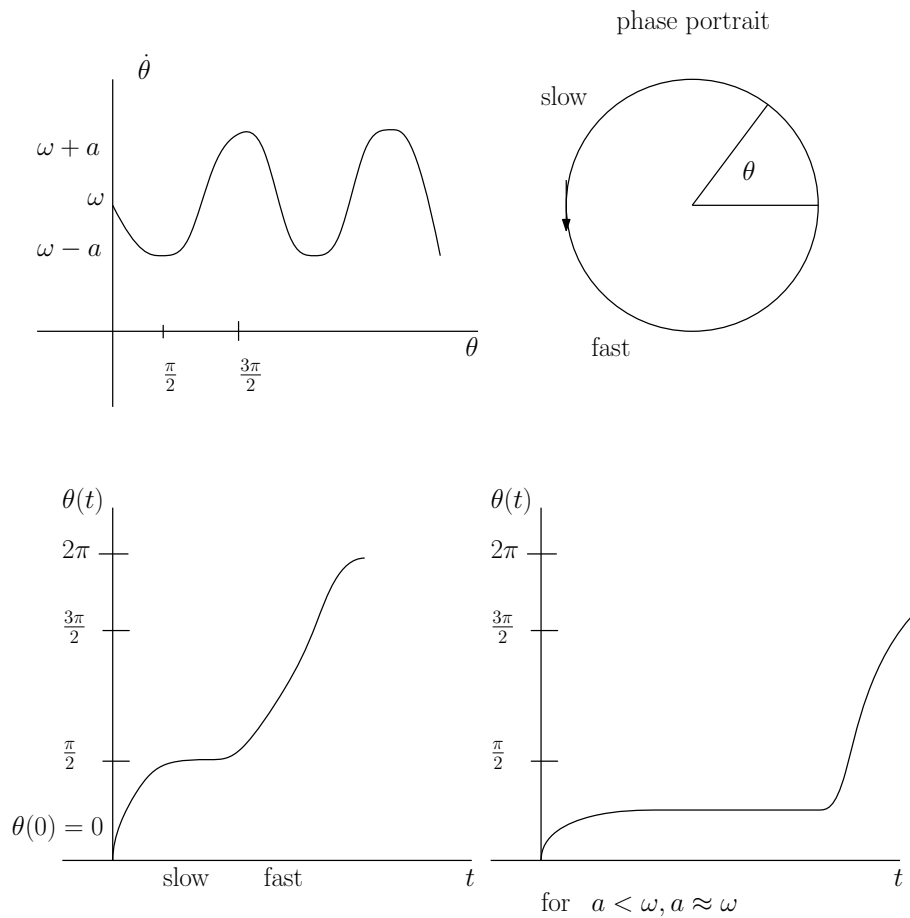
Figure 4: bottle neck remnants or "ghost" of a saddle-node bifurcation

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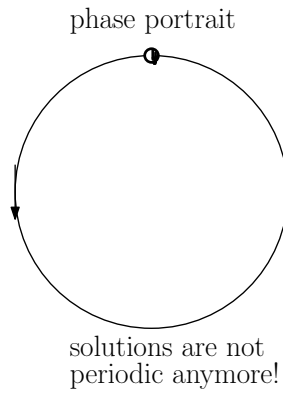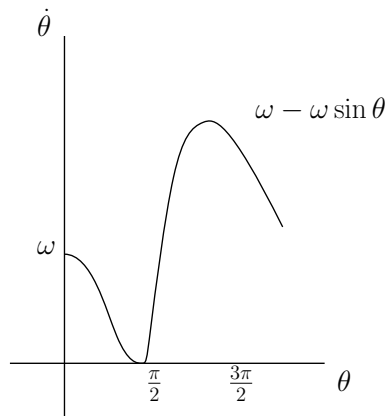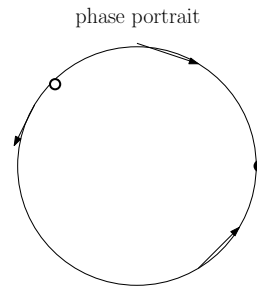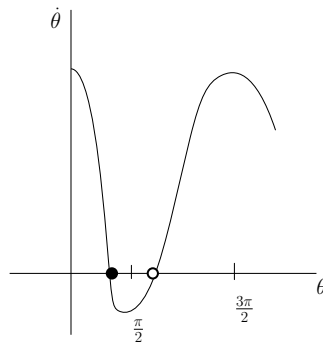
For $a = \omega$

For $a > \omega$:

Oscillation period for $a < \omega$:

$$T = \int dt = \int_0^{2\pi} \frac{dt}{d\theta} d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta}$$

$$= \ldots = \frac{2\pi}{\sqrt{\omega^2 - a^2}} = \frac{2\pi}{\sqrt{\omega + a}} \cdot \frac{1}{\sqrt{\omega - a}}$$

$$\approx \frac{2\pi}{\sqrt{2\omega}} \cdot \underbrace{\frac{1}{\sqrt{\omega - a}}}_{\text{blow up as } a \to \omega}$$
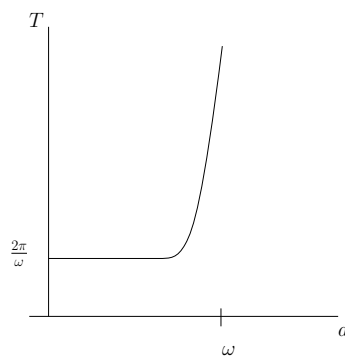
Remark 14.1. Bottlenecks/this scaling law are a general feature of saddle-node bifurcations:

$$\text{Normal form: } \frac{dx}{dt} = \dot{x} = r + x^2$$
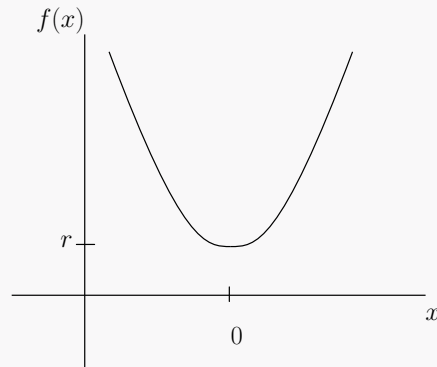
$$T_{\text{bottleneck}} \approx \int dt$$

$$= \int_{-\infty}^{\infty} \frac{dt}{dx} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{r + x^2} \, dx$$

$$T_{\text{bottleneck}} = \frac{\pi}{\sqrt{r}}$$

blows up like $\sim r^{-\frac{1}{2}} = \frac{1}{\sqrt{r}}$ as $r \to 0$ and $r > 0$.

Example 14.2

Draw all qualitatively different phase portraits of

$$\dot\theta = \omega - a\sin\theta \quad (\text{where } \omega > 0 \text{ fixed})$$

Bifurcation points: $\dot\theta = f(\theta) = 0$, $\frac{\partial f}{\partial \theta} = 0$. Thus, $0 = -a\cos\theta \implies a = 0$ or $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

$$\text{If } a = 0 : \dot\theta = \omega > 0 \text{ (no bifurcation)}$$
$$\text{If } \theta = \frac{\pi}{2} : 0 = \dot\theta = \omega - a \implies a = \omega$$
$$\text{If } \theta = \frac{3\pi}{2} : 0 = \dot\theta = \omega + a \implies a = -\omega$$

Bifurcation points $(a_*, \theta_*) = \left(\omega, \frac{\pi}{2}\right), \left(-\omega, \frac{3\pi}{2}\right)$.
$\dot\theta = \omega - a\sin\theta$
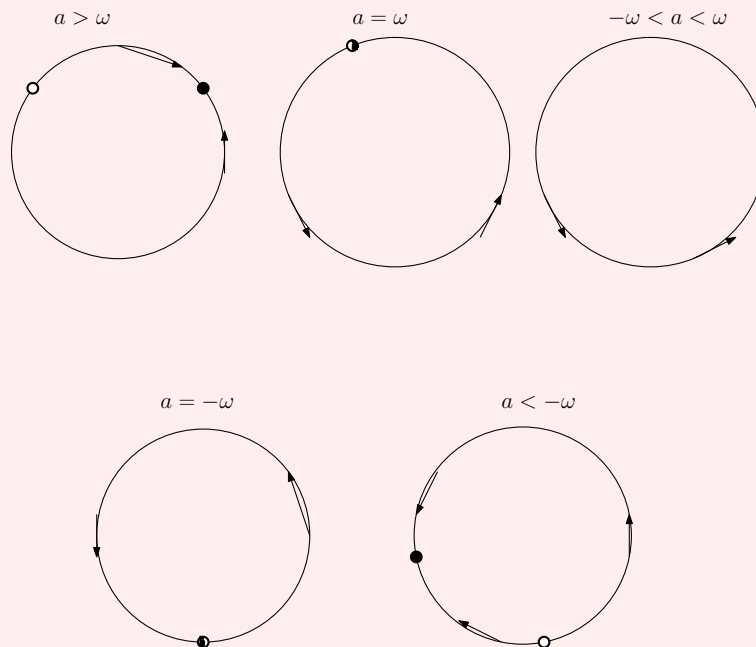
§14.2   2D Dynamical Systems

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

Introduction & Linear Systems:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ i.e. } \dot{x} = Ax$$

Harmonic Oscillator: $m\ddot{x} + kx = 0$
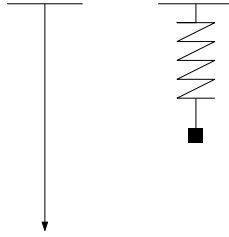
$$\ddot{x} + \omega^2 x = 0, \ \omega^2 = \frac{k}{m}$$

where k : spring constant and m : mass, x : position, v : velocity.

$$\dot{x} = v$$
$$\dot{v} = \ddot{x} = -\omega^2 x$$
$$\frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2 x \end{pmatrix}$$

<u>*Note*</u>: the last matrix defines vector field on phase plane.

phase portrait
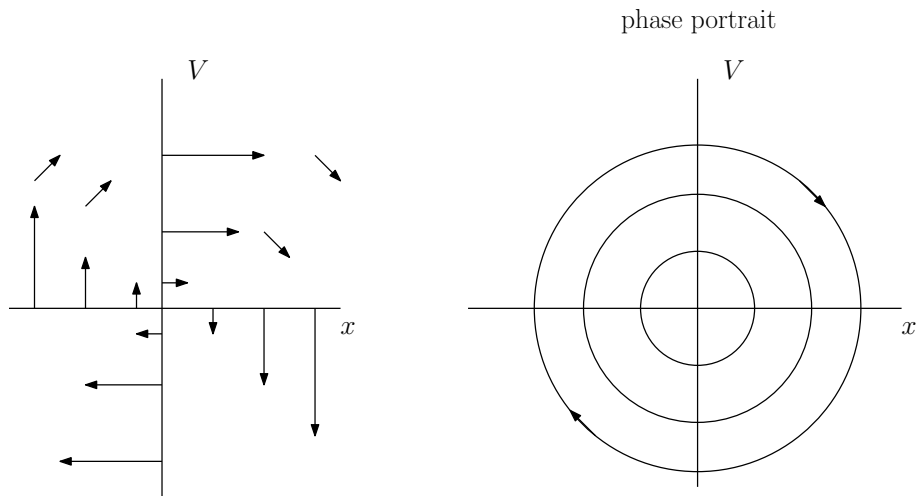
Harmonic oscillator:
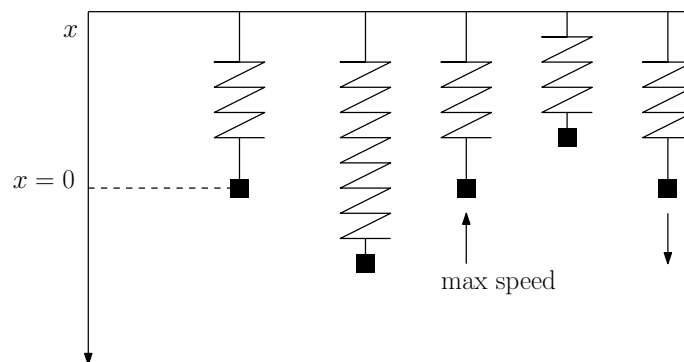
Remark 14.3. Have:

$$\frac{d}{dt}(\omega^2 x^2 + v^2) = 2\omega^2 x\dot{x} + 2v\dot{v}$$
$$= 2\omega^2 xv - 2\omega^2 vx = 0$$
$$\implies \omega^2 x^2 + v^2 = \text{ const}$$

$\implies$ trajectories $\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$ describe ellipses, in particular, they are <u>closed orbits</u> i.e. correspond to <u>periodic solutions</u>.

§15 | Lec 14: Feb 8, 2021

§15.1   Classification of Linear Systems

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{i.e. } \dot{x} = Ax$$

Question 15.1. What is the stability type of $x_* = 0$?

> **Definition 15.1** (Eigenvector) — $v \neq 0$ is an eigenvector of A if
>
> $$Av = \lambda v$$
>
> for some $\lambda \in \mathbb{C}$

$\lambda \in \mathbb{C}$ is an eigenvalue

$$\begin{aligned}
&\iff \Lambda_\lambda(A) = \det(A - \lambda I) = 0 \\
&= \det\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \\
&= (a - \lambda)(d - \lambda) - bc \\
&= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) \\
&= 0 \\
&\iff \lambda_{1,2} = \frac{1}{2}\left(\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}\right)
\end{aligned}$$

3 cases:

i) $\lambda_1 \neq \lambda_2$ real valued $\iff \operatorname{tr}(A)^2 > 4\det(A)$

ii) $\lambda_1 = \lambda_2$ real valued $\iff \operatorname{tr}(A)^2 = 4\det(A)$

iii) $\lambda_1 = \overline{\lambda_2}$ complex conjugate $\iff \operatorname{tr}(A)^2 < 4\det(A)$

1. $\lambda_1 \neq \lambda_2 \implies$ there are linearly independent eigenvectors v_i :

$$Av_i = \lambda_i v_i \quad \text{for } i = 1, 2$$

A is diagonalizable.

Coordinate change:

$$\begin{aligned}
C &= (v_1 | v_2) \\
B &= C^{-1}AC = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\
y &= C^{-1}x
\end{aligned}$$

Then $\dot{y} = C^{-1}\dot{x} = C^{-1}Ax = C^{-1}ACy = By$ i.e. $\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} =$
$\begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{pmatrix}$ i.e. the ODE decouples

$$\dot{y}_i = \lambda_i y_i \quad \text{for } i = 1, 2$$

So

$$\implies y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

$$\implies x(t) = Cy(t) = c_1 e^{\lambda_1 t} C \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} C \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If $\lambda_1 \neq \lambda_2$:

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

Phase portraits:

$$\lambda_1 < 0 < \lambda_2 \text{ (saddle)}$$
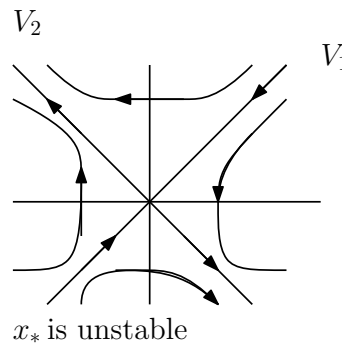
x_* is unstable

Definition 15.2 (Hyperbolic Fixed Point) — x_* is a hyperbolic fixed point if $\text{Re}(\lambda_i) \neq 0$ for $i = 1, 2$ otherwise <u>non-hyperbolic</u>.

$\lambda_1 = 0 < \lambda_2 : x(t) = c_1 v_1 + c_2 e^{\lambda_2 t} v_2$
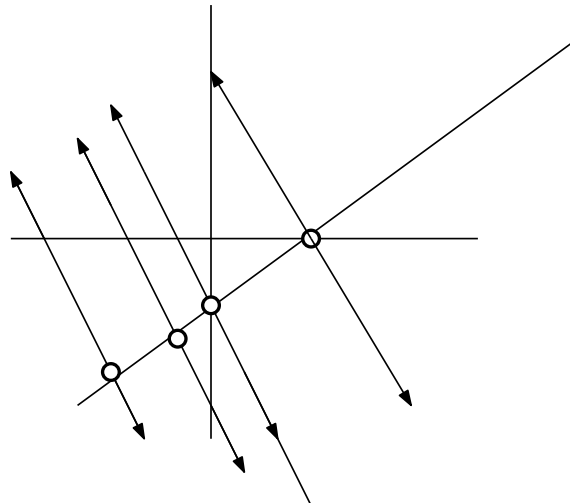
x_* is unstable and v_1 axis consists of fixed points $x_* = 0$ is a non-isolated fixed point.

$\lambda_1 < 0 = \lambda_2$: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 v_2$
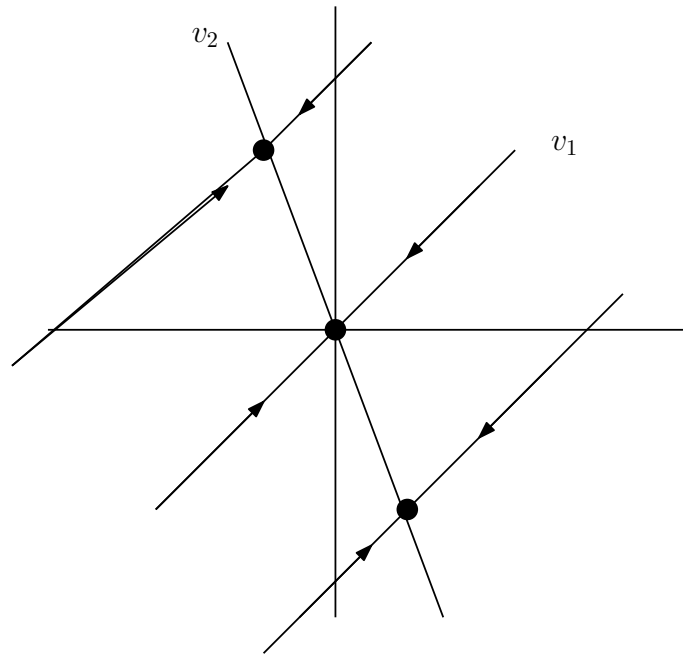
v_2 axis consists of fixed points.

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 v_2$$

$x_* = 0$ is Lypunov stable but not attracting (neutrally stable)

$\lambda_1 < \lambda_2 < 0:$ $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$
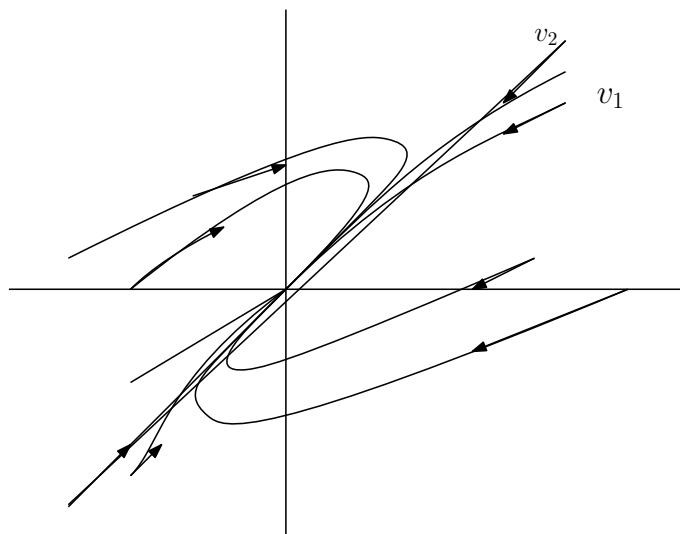
Trajectories approach x_* tangent to "slower" v_2 direction (note $|\lambda_1| > |\lambda_2| > 0$) – <u>stable node</u>.

$0 < \lambda_1 < \lambda_2$: trajectories quickly appear parallel to "faster" v_2 direction.

unstable node
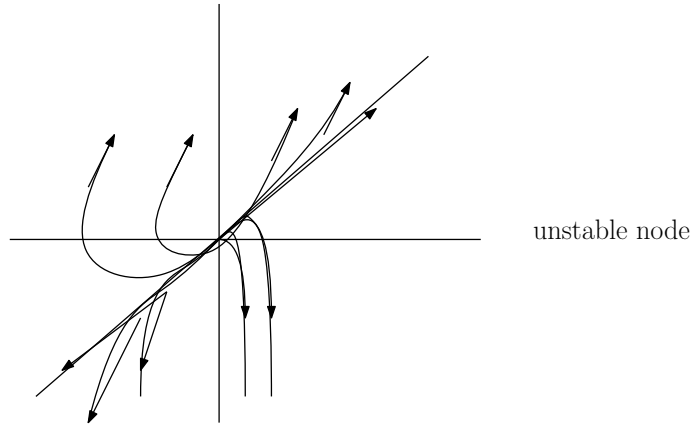
Case ii) $\lambda = \lambda_1 = \lambda_2$, real valued
1. There are v_1, v_2 linearly independent eigenvectors $Av_i = \lambda v_i$ for $i = 1, 2$

$$\implies \quad \text{For } v \in \mathbb{R}^2 : Av = \lambda v \implies A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \lambda I$$

So, $\dot{x} = Ax$ is solved by

$$x(t) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t}$$

Phase portraits:

$\lambda > 0$

$\lambda < 0$

$\lambda = 0 : A = 0$

every point is a fixed point

$x(t) = x(0)$

$(x_* = 0$
is stable non-hyperbolic,
non-isolated)

x_* is stable (stable star)
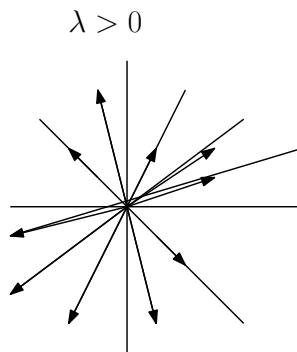
x_* is unstable

unstable star

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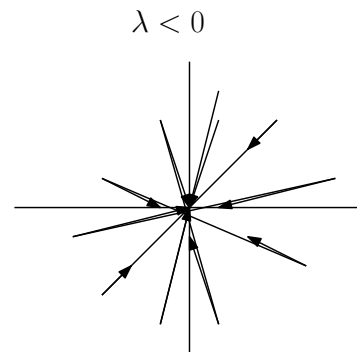
§16 | Lec 15: Feb 10, 2021

§16.1   Classification (Cont'd)

Case ii) $\lambda = \lambda_1 = \lambda_2$
  2. Eigenspace $\text{Eig}_\lambda(A) = \text{span}(v)$, $v \neq 0$ A is not diagonalizable.

$$\implies x(t) = [(c_1 + c_2 t)v + c_2 \omega] e^{\lambda t}$$

where λ s.t. $(A - \lambda I)\omega = v$. Note $\frac{x(t)}{|x(t)|} \to \frac{v}{|v|}$ as $t \to \pm\infty$ i.e. $x(t)$ tangent/parallel to v-direction as $t \to \pm\infty$.
  Recall: $\lambda_1 < \lambda_2 < 0$:

unstable node
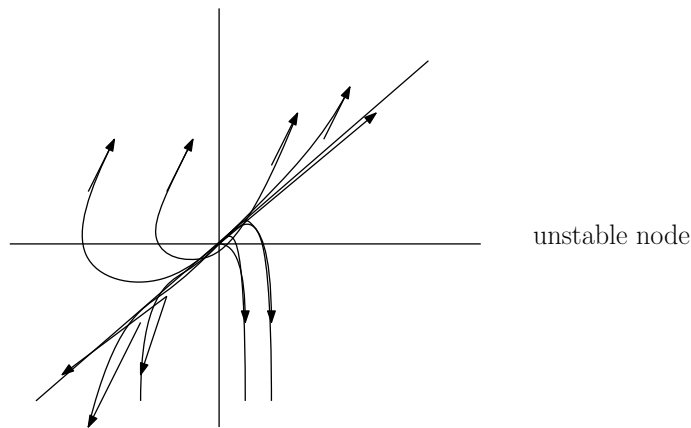
  intuitively as $\lambda_1 \to \lambda_2$ and $v_1 \to v_2$.
$\lambda < 0$: stable degenerate node
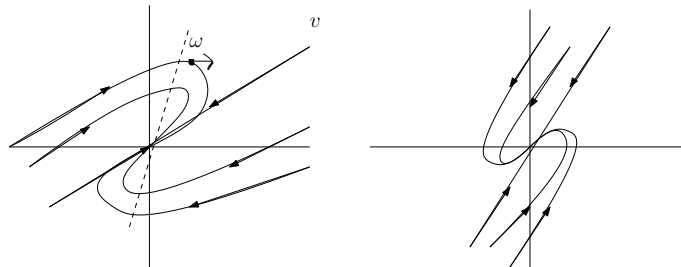
> **Remark 16.1.** Instead of solving for ω explicitly, calculate Az for some vector z to determine which way the solution "curls".

$\lambda > 0$
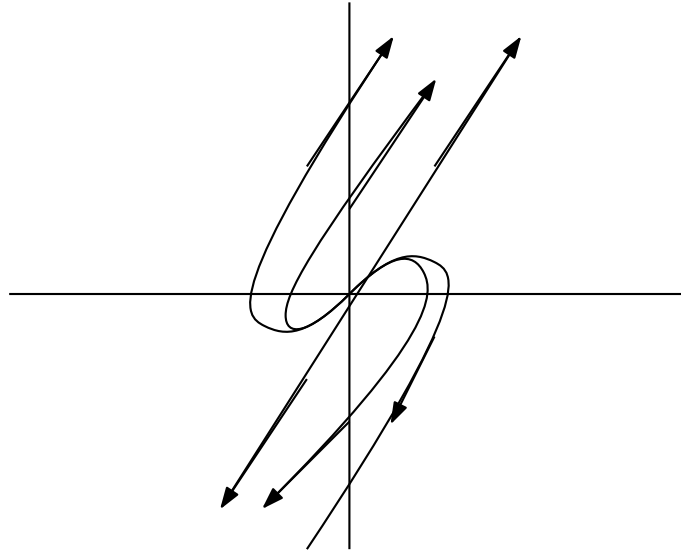
$\lambda = 0 : x(t) = (c_1 + c_2 t)v + c_2 \omega$
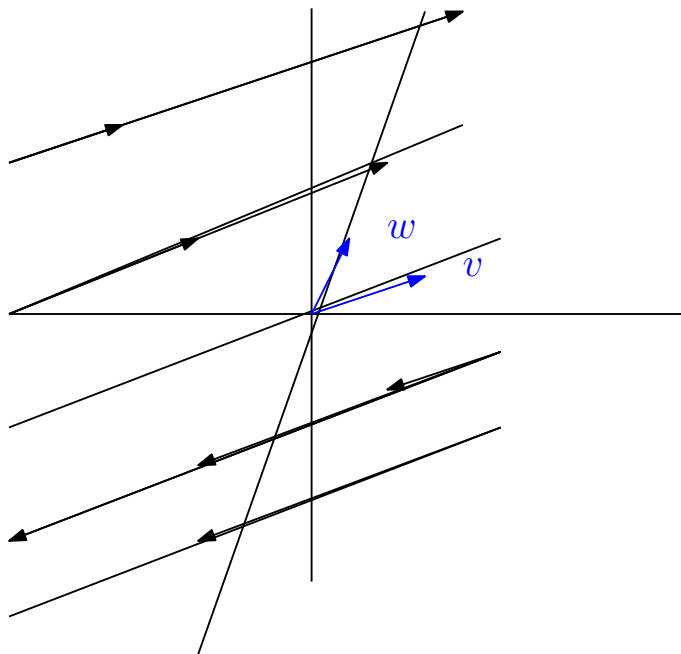
Note: $x(0) = c_1 v \implies x(t) = c_1 v$ for all t i.e. the $v-$axis consists of fixed points (non-isolated fixed points, $x_* = 0$ unstable).

Remark 16.2. If $\lambda = \lambda_1 = \lambda_2$, $\text{Eig}_\lambda(A) = \text{span}(v)$. Then there is ω s.t.

$$(A - \lambda I)\omega = v$$
$$\implies v_1 \omega \text{ lin. indep}$$
$$\implies v_1 \omega \text{ form a basis of } \mathbb{R}^2$$

Coordinate change:

Set

$$C = (v|w)$$

$$B = C^{-1}AC = \underbrace{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}}_{\text{Jordan normal form}}$$

$$y = C^{-1}x: \quad \dot{y} = By$$

So

$$\dot{y}_2 = \lambda y_2 \implies y_2(t) = c_2 e^{\lambda t}$$

$$\dot{y}_1 = \lambda y_1 + y_2 \implies y_1(t) = (c_1 + c_2 t) \, e^{\lambda t}$$

$$\implies x = Cy = [(c_1 + c_2 t) + c_2 \omega] \, e^{\lambda t}$$

Case iii)

$$\begin{cases} \lambda_1 = \lambda = \alpha + i\beta \\ \lambda_2 = \overline{\lambda} = \alpha - i\beta \end{cases} \quad (\beta > 0)$$

$\implies A$ is diagonalizable over $\mathbb{C}$, in particular there is $v \in \mathbb{C}^2, v \neq 0$, s.t. $Av = \lambda v$.
Let $v = a - ib$, $a, b \in \mathbb{R}^2$. Assume $a \perp b$. General solution:

$$x(t) = (a|b) \underbrace{\begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix}}_{\text{rotation} R(\beta t) \text{ period } \frac{2\pi}{\beta}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \underbrace{e^{\lambda t}}_{\text{stretching factor}}$$

In particular, $x(t) = [a\cos(\beta t) + b\sin(\beta t)] \, e^{\lambda t}$ is the solution with $x(0) = a$ and $x\left(\frac{\pi}{2\beta} = be^{\alpha t}\right)$ $\left[\text{set } \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$.
Phase portraits:
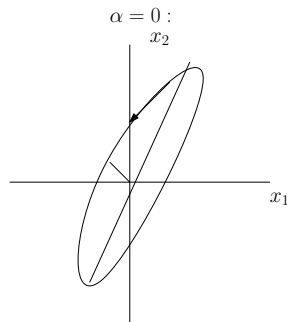
center
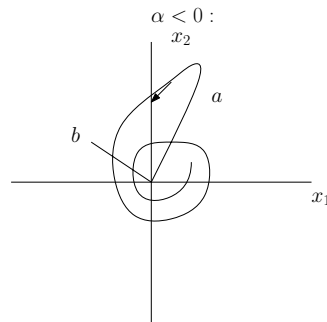
stable spiral
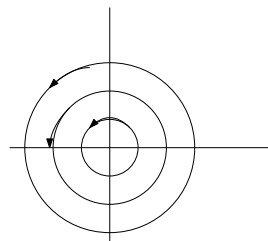
$x_* = 0$ is Lyapunov stable
but not attracting

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$\alpha > 0$: unstable spiral

$\alpha > 0$:   unstable spiral
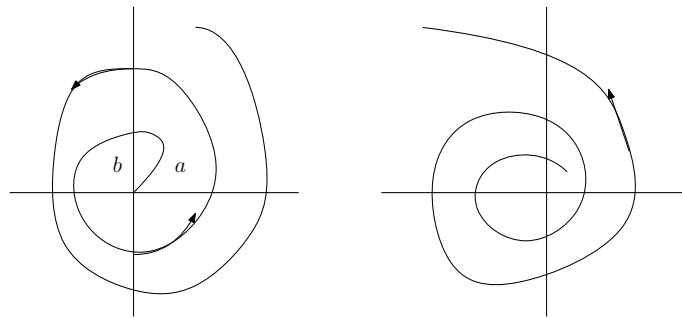
Remark 16.3. i) If $\alpha = 0$, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies x(t) = \cos(\beta t) \cdot a + \sin(\beta t) \cdot b$. Then since $a \perp b$:

$$\frac{1}{|a|^2} \langle x(t), \frac{a}{|a|} \rangle^2 + \frac{1}{|b|^2} \langle x(t), \frac{b}{|b|} \rangle^2 = \frac{1}{|a|^2} \left(\frac{a \cdot a}{|a|} \cdot \cos(\beta t) \right)^2 + \frac{1}{|b|^2} \left(\frac{b \cdot b}{|b|} \cdot \sin(\beta t) \right)^2$$
$$= (\cos(\beta t))^2 + (\sin(\beta t))^2 = 1$$

$\implies x(t)$ is on an ellipse with axes $\frac{a}{|a|}, \frac{b}{|b|}$.

ii) $\lambda = \alpha + i\beta$, $v = a - ib$. If a is not orthogonal to b, then replace v by

$$w = (\gamma + i\delta) v$$

with $\gamma = -2ab$

$$\delta = \left(|a|^2 - |b|^2 \right) \pm \sqrt{\left(|a|^2 - |b|^2 \right)^2 + 4(ab)^2}$$

Then $A\omega = \lambda\omega$ and $\mathrm{Re}\,\omega \perp \mathrm{Im}\,\omega$.

Assume $Av = \lambda v$, $v = a - ib$, $a \perp b$.

$$Aa - iAb = A(a - ib) = Av = \lambda v = (\alpha + i\beta)(a - ib)$$
$$= (\alpha a + \beta b) + i(\beta a - \alpha b)$$

So

$$Aa = \alpha a + \beta b$$
$$Ab = -\beta a + \alpha b$$

Set $C = (a|b)$. Then

$$AC = C \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$B = C^{-1}AC = \underbrace{\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}}_{\text{normal form}}$$

Set $y = C^{-1}x$, $\dot{y} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} y$ with solution:

$$y(t) = \begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\alpha t}$$
$$\implies x(t) = C \cdot y(t)$$

§17 │ Lec 16: Feb 12, 2021

§17.1   Linear Systems – Harmonic Oscillator

Example 17.1 (Harmoinc oscillator)

$$m\ddot{x} + kx = 0$$

where k : spring constant.
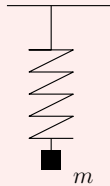
$$\implies \ddot{x} + \omega^2 x = 0 \text{ where } \omega^2 = \frac{k}{m}. \text{ Set}$$

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega^2 x_1 \end{cases}$$

i.e.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

eigenvalues:

$$0 = \det(A - \lambda I)$$
$$= \det\begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{pmatrix}$$
$$= \lambda^2 + \omega^2$$

$\implies \lambda_{1,2} = \pm i\omega \implies$ center
Phase portrait:

   i) in practice: compute $\dot{x} = Ax$ for a specific vector to determine which way
      solutions turn

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} x$$

e.g. $\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\omega^2 \end{pmatrix}.$

Example 17.2 (Cont'd of example 17.1)

Then,
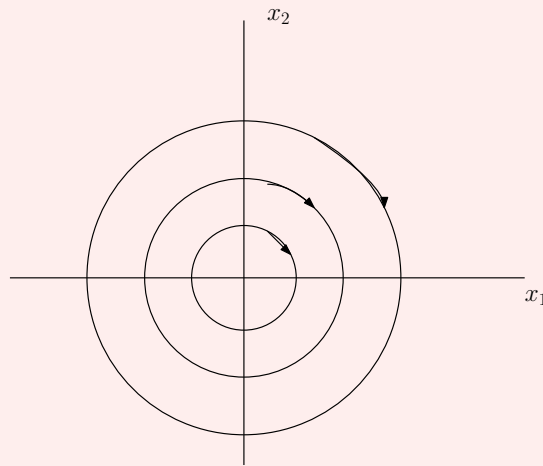
ii) more precise quantitative analysis, eigenvectors solutions of $(A - \lambda I)v = 0$

$$A - i\omega I = \begin{pmatrix} -i\omega & 1 \\ -\omega^2 & -i\omega \end{pmatrix} \rightarrow \begin{pmatrix} -i\omega & 1 \\ 0 & 0 \end{pmatrix}$$

eigenvector $v = \begin{pmatrix} -i \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ \omega \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$
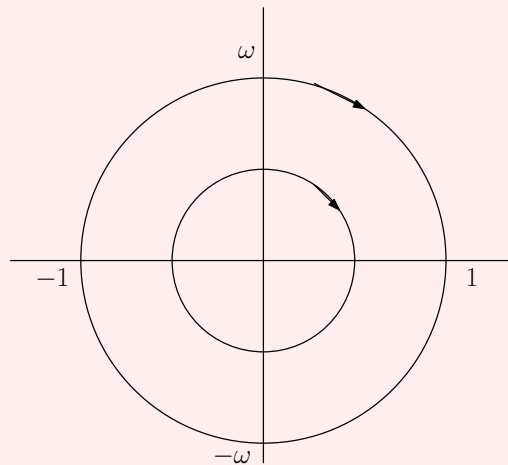
Recall:

$$x(t) = C \cdot \left[\begin{pmatrix} 0 \\ \omega \end{pmatrix} \cos(\omega t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(\omega t) \right]$$

Example 17.3

$\dot{x} = Ax \quad A = \begin{pmatrix} 8 & -1 \\ 4 & 4 \end{pmatrix}$. Eigenvalues:

$$0 = \det(A - \lambda I)$$
$$= \det \begin{pmatrix} 8 - \lambda & -1 \\ 4 & 4 - \lambda \end{pmatrix}$$
$$= (8 - \lambda)(4 - \lambda) - 4(-1)$$
$$= \lambda^2 - 12\lambda + 36 = 0$$
$$\implies \lambda = 6$$

$A \neq \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$, we have an unstable degenerate node. Eigenvector: $A - \lambda I = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \to$
$\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$, so $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector. Note $A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$.

Phase portrait:
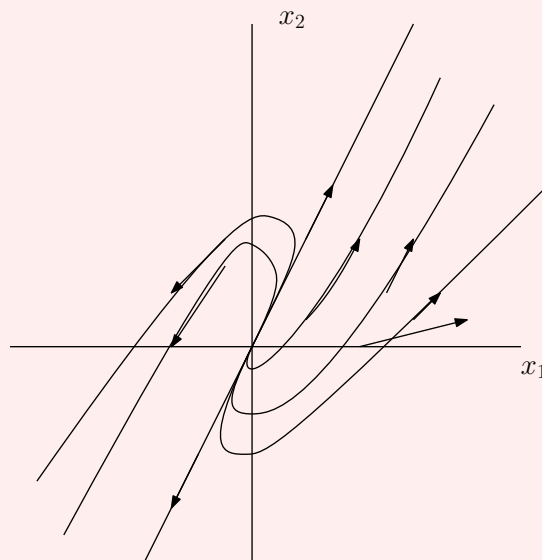
Summary:

Recall $\lambda_{1,2} = \frac{1}{2}\left(\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}\right)$

$tr(A) = \lambda_1 + \lambda_2$

unstable nodes

$tr(A)^2 - 4\det(A) = 0$

unstable spirals

saddle   centers

points $\det(A) = \lambda_1 \cdot \lambda_2$

stable spirals

degenerate nodes, stars

stable nodes

non-hyperbolic fixed points

non-isolated fixed points
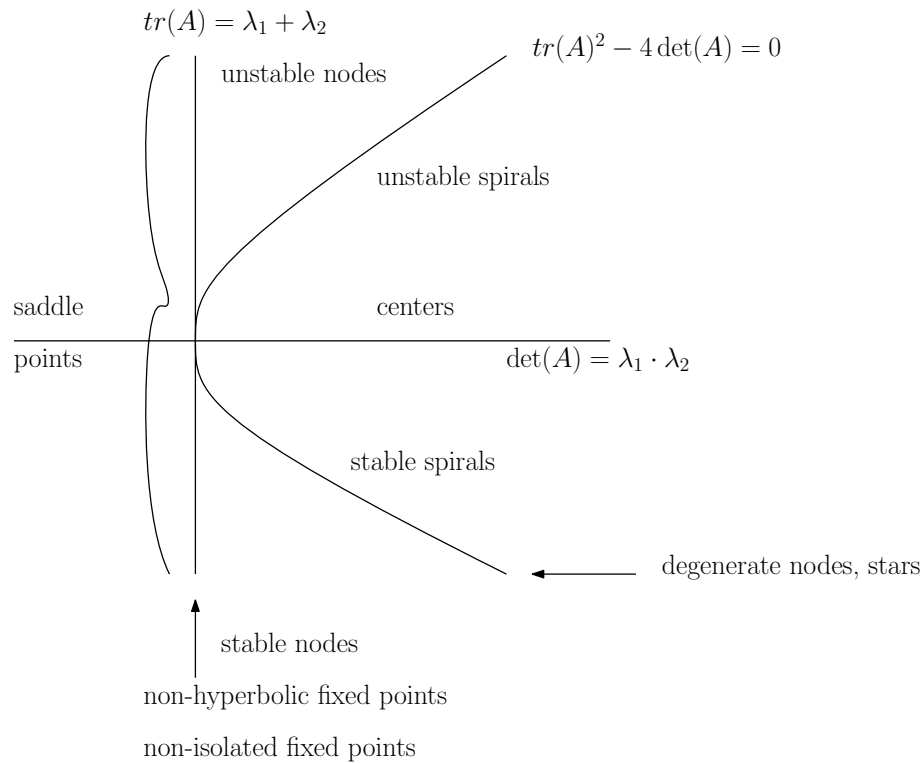
§17.2   Nonlinear Systems – Existence and Uniqueness

$$\dot{x} = f(x) \qquad \text{i.e.} \qquad \dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$
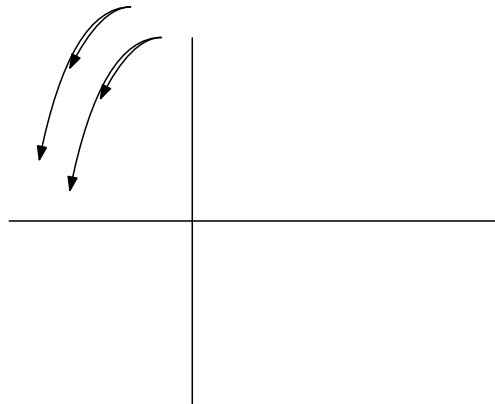
Theorem 17.4 (Existence & Uniqueness of Systems)

Let $D \subseteq \mathbb{R}^n$ be open, $f : D \to \mathbb{R}^n$ s.t. $\frac{\partial f_i}{\partial x_j}$ exist and are continuous, that is $f \in C^1(D)$. Then for every $x_0 \in D$ there $\tau > 0$ s.t. $\dot{x} = f(x)$, $x(t_0) = x_0$ has a unique solution $\phi : (t_0 - \tau, t_0 + \tau) \to \mathbb{R}^n$ i.e. $\dot{\phi}(t) = f(\phi(t))$, $\phi(t_0) = x_0$.

Remark 17.5. $f \in C^2(D)$ if $\frac{\partial^2 f_i}{\partial x_k \partial x_l}$ exist and continuous.

Consequence: Different trajectories in the phase portrait cannot intersect

impossible
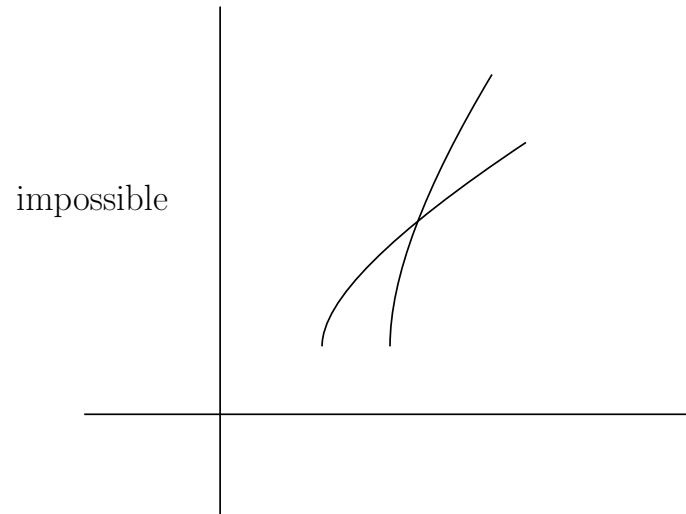
§18 | Lec 17: Feb 15, 2021

§18.1   Nonlinear Systems – Nullclines

$\dot{x} = f(x)$ and $\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$

Example 18.1

Consider
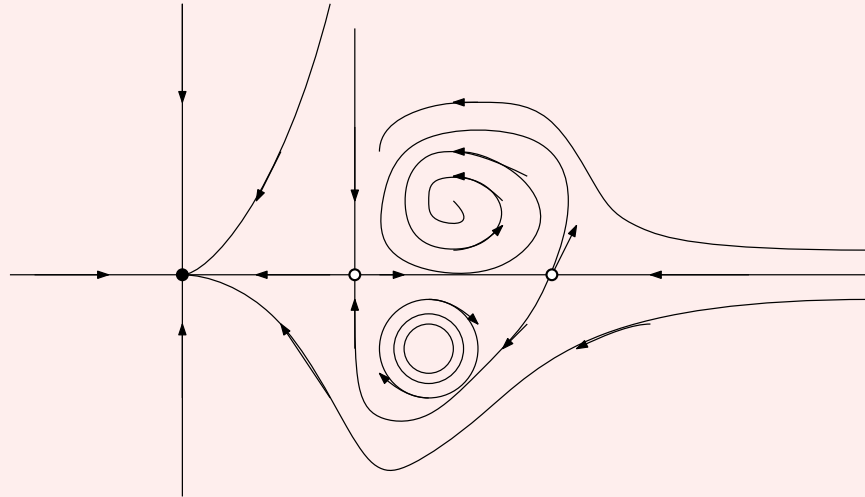
Definition 18.2 (Isocline and Nullcline) — Let $c \in \mathbb{R}$. The curves $\{(x_1, x_2) | f_i(x_1, x_2) = c\}$ $i = 1, 2$ are called isoclines. Specifically, if $c = 0$

- $f_1(x_1, x_2) = 0$ is called vertical nullcline.

- $f_2(x_1, x_2) = 0$ is called horizontal nullcline.

Example 18.3

Consider:

$$\dot{x}_1 = x_1 + e^{-x_2}$$
$$\dot{x}_2 = -x_2$$

Fixed points: $\dot{x} = f(x) = 0 \iff (x_1, x_2) = (-1, 0)$.

Nullclines:

$$\dot{x}_1 = 0 : x_1 = -e^{-x_2} \text{ (vertical nullcline)}$$
$$\dot{x}_2 = 0 : x_2 = 0 \text{ (horizontal nullcline)}$$
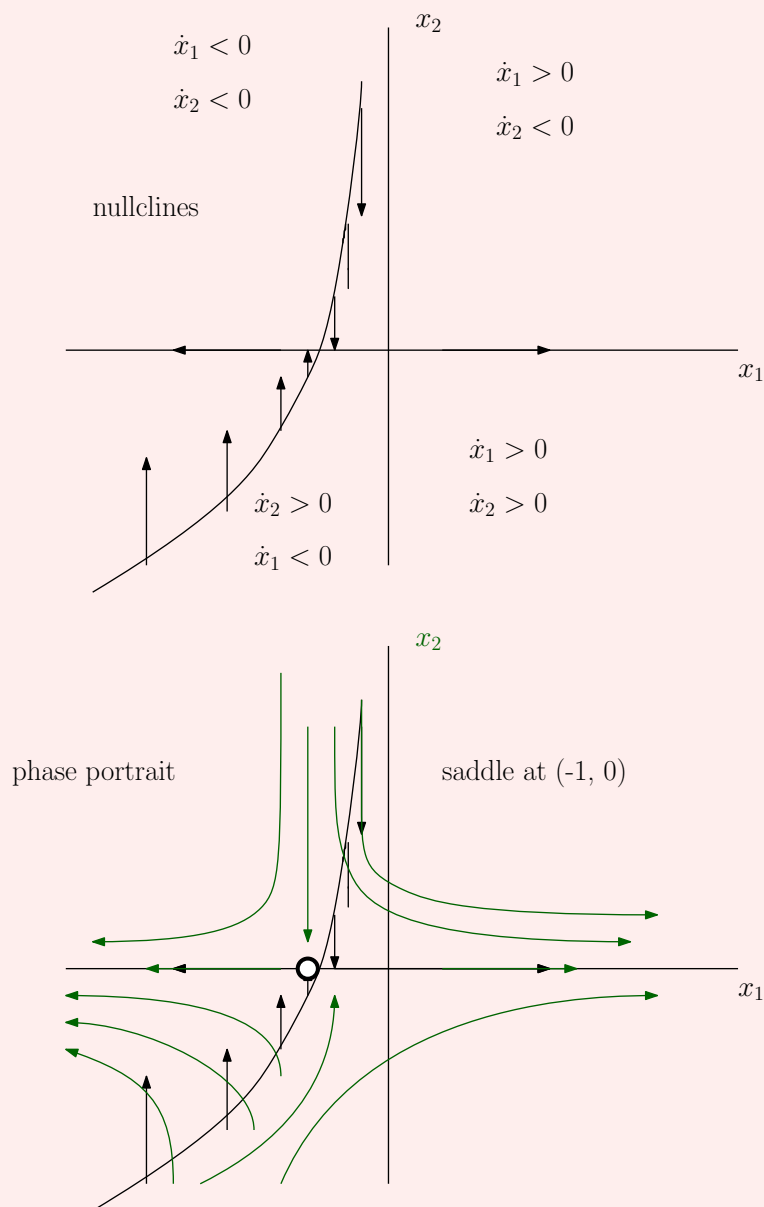
> **Remark 18.4.** A nullclines typically are <u>not</u>/do not consist of trajectories. Vertical(horizontal) nullclines consist of trajectories if it is exactly vertical(horizontal.

§18.2   Principle of Linear Stability

$\dot{x} = f(x)$, $f \in C^1(D)$, $f(x_*) = 0$. We want to approximate the nonlinear DE near the fixed point.

$$\frac{d}{dt}(x - x_*) = \dot{x} = f(x) = f(x - x_* + x_*)$$
$$\stackrel{\text{Taylor}}{=} \underbrace{f(x_*)}_{=0} + Df(x_*)(x - x_*) + \mathcal{O}(|x - x_*|^2)$$

i.e. $y = x - x_*$ approximately solves the linear ODE

$$\dot{y} = Df(x_*)y$$

where

$$Df(x_*) \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

Let λ_1, λ_2 denote the eigenvalues of $Df(x_*)$.

Theorem 18.5 (Linear Stability)

Similar to the linear systems,

i) If $\text{Re}(\lambda_1) < 0$, $\text{Re}(\lambda_2) < 0$ then x_* is asymptotically stable, i.e. x_* is Lyapunov stable and attracting.

ii) If $\text{Re}(\lambda_i) > 0$ for $i = 1$ or $i = 2$ then x_* is unstable.

§19 | Lec 18: Feb 19, 2021

§19.1   The Stable/Unstable Manifold Theorem

$f \in C^1$, $\dot{x} = f(x)$, $f(x_*) = 0$ i.e. x_* fixed point, λ_1, λ_2 eigenvalues of $Df(x_*)$.
Let x_* be a hyperbolic fixed point and $x(t, x_0)$ be the solution of

$$\dot{x} = f(x), \quad x(0) = x_0$$

Set

$$\mathcal{M}_s := \left\{ x_0 \in D \,\middle|\, x(t, x_0) \text{ defined for all } t \geq 0 \text{ and } \lim_{t \to \infty} x(t, x_0) = x_* \right\} \text{ (stable manifold)}$$

$$\mathcal{M}_u := \left\{ x_0 \in D \,\middle|\, x(t, x_0) \text{ defined for all } t \leq 0 \text{ and } \lim_{t \to -\infty} x(t, x_0) = x_* \right\} \text{ (unstable manifold)}$$

Example 19.1

Linear stable node
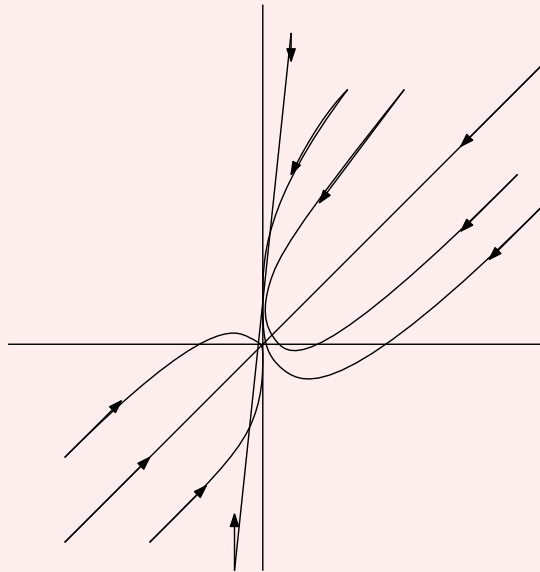
$$\mathcal{M}_s = \mathbb{R}^2$$
$$\mathcal{M}_u = \{x_*\} = \{0\}$$

Linear saddle
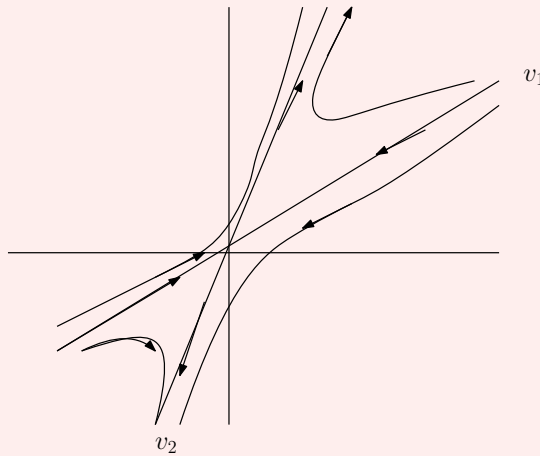
$$\mathcal{M}_s = \text{span}(v_1)$$
$$\quad = \text{ line through } v_1(\lambda_1 < 0) \text{ (trajectories that approach } x_*)$$
$$\mathcal{M}_u = \text{ span}(v_2)$$
$$\quad = \text{line through } v_2(\lambda_2 > 0) \text{ (trajectories that emanate from } x_*)$$

Theorem 19.2 (Stable/Unstable Manifold)

Let $f \in C^1$, x_* is a hyperbolic fixed point.

  i) If $\mathrm{Re}(\lambda_i) < 0$ for $i = 1, 2$, then $\mathcal{M}_s$ contains an open neighborhood of x_* and $\mathcal{M}_u = \{x_*\}$.

  ii) If $\mathrm{Re}(\lambda_i) > 0$ for $i = 1, 2$, then $\mathcal{M}_s = \{x_*\}$ and $\mathcal{M}_u$ contains an open neighborhood of x_*.

  iii) If $\mathrm{Re}(\lambda_1) < 0 < \mathrm{Re}(\lambda_2)$, then $\mathcal{M}_s, \mathcal{M}_u$ are C^1-curves through x_*. $\mathcal{M}_s$ tangent to v_1 at x_*, $Df(x_*)v_1 = \lambda_1 v_1$, and $\mathcal{M}_u$ tangent to v_2 at x_*, $Df(x_*)v_2 = \lambda_2 v_2$

Theorem 19.3

Suppose x_* is a <u>hyperbolic</u> fixed points of $\dot{x} = f(x)$. Then the phase portrait of $\dot{y} = Df(x_*)y$ near $y_* = 0$ gives a qualitatively accurate picture of the phase portrait of $\dot{x} = f(x)$ near x_* if

  a) $f \in C^2$ i.e. $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exists and are continuous.

  or

  b) $f \in C^1$ and $\lambda_1 \neq \lambda_2$.

Example 19.4

Consider:

$$\dot{x}_1 = x_1 + e^{-x_2}$$
$$\dot{x}_2 = -x_2$$

only fixed point: $(x_1, x_2) = (-1, 0)$ and note that $f(x_1, x_2) = \begin{pmatrix} x_1 + e^{-x_2} \\ -x_2 \end{pmatrix}$.

$$Df = \begin{pmatrix} 1 & -e^{-x_2} \\ 0 & -1 \end{pmatrix}$$

$$Df(x_*) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = -1, \lambda_2 = 1 \implies (-1, 0)$ is unstable (by Theorem 18.5)

Eigenvectors:

$$A - (-1)I = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \implies v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A - (1)I = \begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \implies v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where v_1 is the tangent direction of stable manifold at $x_* = (-1, 0)$ and v_2 is the tangent direction of unstable manifold at $x_* = (-1, 0)$.

unstable manifold

stable manifold
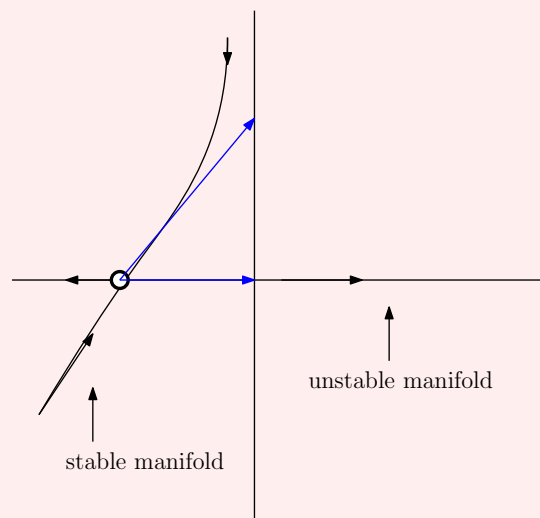
Example 19.5 (Cont'd from above)

<u>Note</u>: $f(x_1, x_2) = \begin{pmatrix} x_1 + e^{-x_2} \\ -x_2 \end{pmatrix}$ is infinitely often differentiable, in particular, $f \in C^2$
(or $f \in C^1$), thus the phase portrait of

$$\dot{y} = Df(x_*) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \text{ near } y_* = 0$$

is an accurate picture of the phase portrait of $\dot{x} = f(x)$ near x_*.
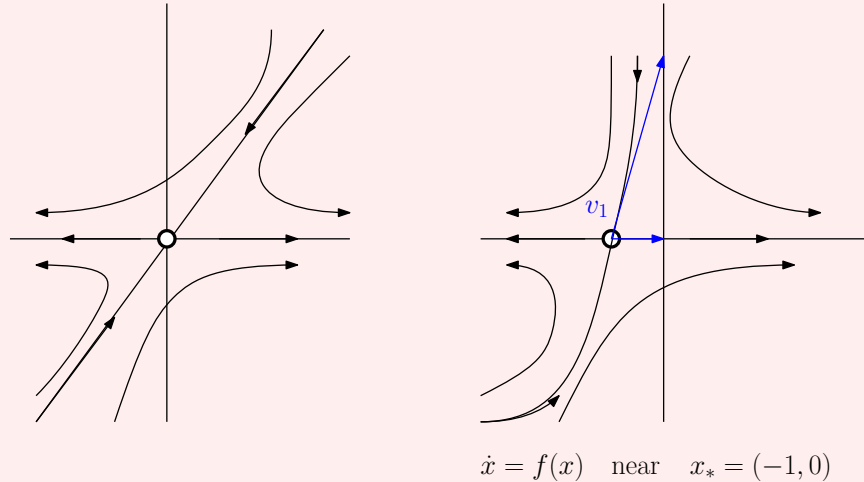
$$\dot{x} = f(x) \quad \text{near} \quad x_* = (-1, 0)$$

where the left figure denote the approximation $\dot{y}$.

Theorem 19.6 (Hartman – Grobman)

Let $f \in C^1$, x_* a hyperbolic fixed point of $\dot{x} = f(x)$. Then the phase portrait of $\dot{x} = f(x)$ near x_* and $\dot{y} = Df(x_*)y$ near $y_* = 0$ are topologically equivalent i.e. the same up to continuous deformation (homeomorphisms).

Morally: hyperbolic fixed points are <u>structurally stable</u>.

§19.2   Lotka Volterra Model

Example 19.7 (Lotka Volterra model for competition of two species for limited resources)

Recall: logistic model

$$\dot{x} = rx\left(1 - \frac{x}{k}\right)$$

Consider:

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

fixed points (x_*, y_*)	eigenvalues/eigendirections of $\quad Df(x_*, y_*)$			
	λ_1	v_1	λ_2	v_2
$(0,0)$	3	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	2	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$(0,2)$	-1	$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$	-2	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$(3,0)$	-3	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	-1	$\begin{pmatrix} 3 \\ -1 \end{pmatrix}$
$(1,1)$	$-1+\sqrt{2}$	$\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$	$-1-\sqrt{2}$	$\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$

where all the fixed points above are hyperbolic fixed points.

Example 19.8 (Cont'd from above)

Phase portrait: tangent directions of stable/unstable manifolds

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$
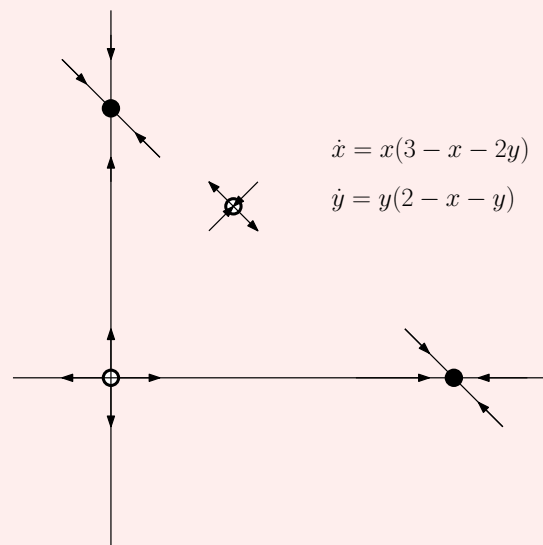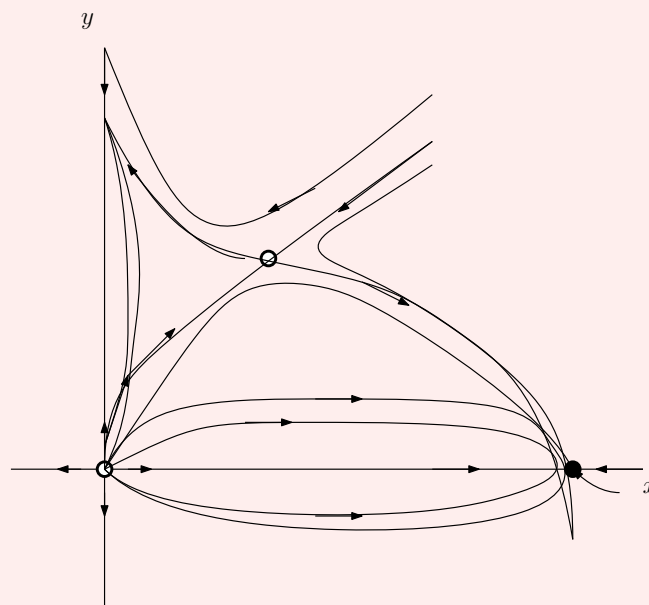
Phase portrait:

Conclusion: Only one species survives.

§20 | Lec 19: Feb 22, 2021

§20.1   Non-Hyperbolic Fixed Points

Example 20.1 (Sheet 7, Ex A)

The phase portrait of a non-linear ODE near a hyperbolic fixed point can be very different from the phase portrait of the linearization at the fixed point.

Example 20.2 (Centers)

For $a \in \mathbb{R}$, consider

$$\dot{x} = -y + ax(x^2 + y^2)$$
$$\dot{y} = x + ay(x^2 + y^2)$$

$(0,0)$ is the only fixed point.

$$Df(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \implies \text{eigenvalues: } \lambda = \pm i$$

$$\implies \text{phase portrait of linearization is center around origin}$$

In polar coordinates, (r, θ)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$
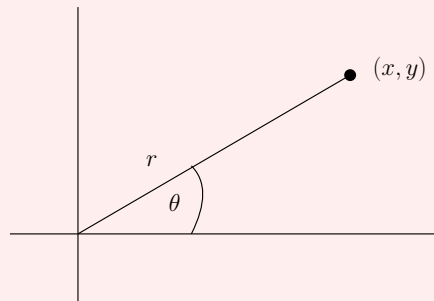
Have

$$\dot{r} = \frac{1}{r}(x\dot{x} + y\dot{y}) = ar^3$$
$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = 1$$

Thus phase portrait of non-linear ODE:
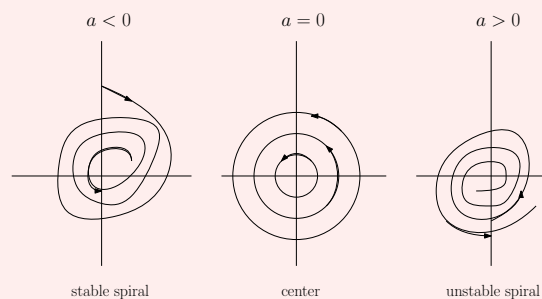
i.e. we have qualitatively different phase portraits (linearization compared to non-linear ODE) for $a \neq 0$.

§20.2 Conservative Systems

Consider Newton's Law: $m\ddot{x} = F(x)$. The force F is called <u>conservative</u> if there is $V(x)$ s.t. $F(x) = -\frac{dV}{dx}$. V is called <u>potential energy</u>. In this case,

$$m\ddot{x} + \frac{dV}{dx} = 0 \tag{*}$$

Proposition 20.3

The total energy $E = \frac{1}{2}m\dot{x}^2 + V(x)$ is preserved, i.e. if $x(t)$ solves $(*)$ then $E\left(x(t)\right) = $ const.

Proof. Observe

$$\frac{d}{dt}E\left(x(t)\right) = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + +V(x)\right)$$
$$= \frac{1}{2}m \cdot 2 \cdot \dot{x}\ddot{x} + V'\left(x(t)\right)\dot{x}$$
$$= \dot{x}\left(m\ddot{x} + V'(x)\right) = 0 \qquad \square$$

Definition 20.4 (Conserved Quantity/First Integral) — Suppose $f : D \to \mathbb{R}^2, D \subseteq \mathbb{R}^2$. A <u>conserved quantity/first integral</u> for $\dot{x} = f(x)$ is a function $E : D \to \mathbb{R}$ s.t.

 i) $\frac{d}{dt}E\left(x(t)\right) = 0$ for every solution $x(t)$ of $\dot{x} = f(x)$.

 ii) E is non-constant on every ball $B_r(x_0) \subset D$.

Remark 20.5. If E is a first integral of $\dot{x} = f(x)$ then $\dot{x} = f(x)$ cannot have attracting fixed points.

Example 20.6 (Particle of mass $m = 1$ in a double-well potential)

Consider the following:

$$V(x) = \frac{-1}{2}x^2 + \frac{1}{4}x^4$$

ODE: $\ddot{x} + V'(x) = 0$
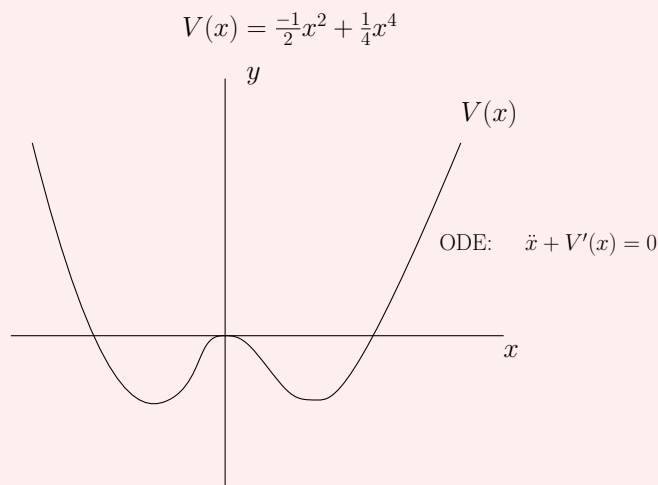
The ODE is equivalent to

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -V'(x_1) = x - x^3 = x_1(1 - x_1^2)$$

Fixed points: $(-1, 0), (0, 0), (1, 0)$

$$Df = \begin{pmatrix} 0 & 1 \\ 1 - 3x_1^2 & 0 \end{pmatrix}$$

$$Df(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \implies \text{ eigenvalues } \lambda = \pm 1$$

$$\implies (0, 0) \text{ is saddle for both linear and nonlinear ODE}$$

$$Df(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \implies \text{ eigenvalues: } \lambda^2 + 2 = 0 \implies \lambda = \pm i\sqrt{2}$$

$$\implies (-1, 0), (1, 0) \text{ are linear centers}$$

Theorem 20.7

$f \in C^1(D)$. Suppose E is a preserved quantity for $\dot{x} = f(x)$. Suppose x_* is an isolated fixed point. If x_* is a local minimum (or maximum) of E, then all trajectories sufficiently close to x_* are closed trajectories. In particular, x_* is a center for the ODE $\dot{x} = f(x)$.

Example 20.8

Recall from the previous example, $\ddot{x} + V'(x) = 0$, $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ i.e. equivalently for $x_1 = x$ and $x_2 = \dot{x}$:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3$$

By example, $E = \frac{1}{2}\dot{x}^2 + V(x) = \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$ is a preserved quantity.
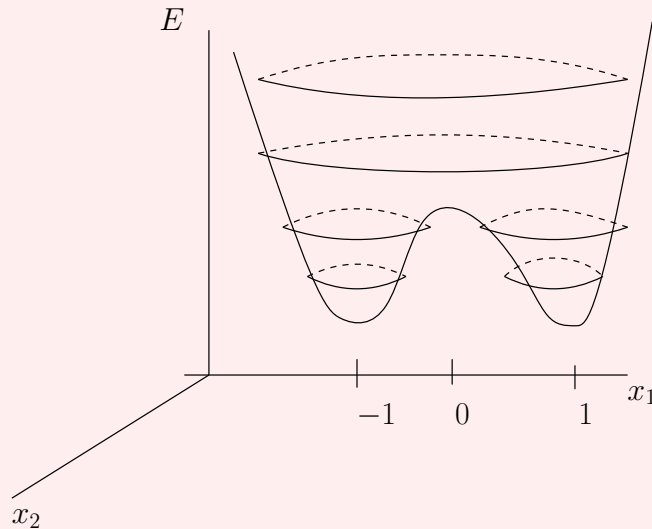
Look at level sets: $E = \text{const}$

$$x_1 \text{ large} : E \approx \frac{x_2^2}{2} + \frac{x_1^4}{4} = \text{const}$$
$$x_1 \text{ small} : E \approx \frac{x_2^2}{2} - \frac{x_1^2}{2} = \text{const}$$

Recall if $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ solves $\dot{x} = f(x)$, then $E\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \text{const}$ i.e. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is on level set.

§21 | Lec 20: Feb 24, 2021

§21.1 Conservative System (Cont'd)

Example 21.1 (Cont'd from the last example in Lec 19)

Phase portrait:

homoclinic

orbits (not periodic)
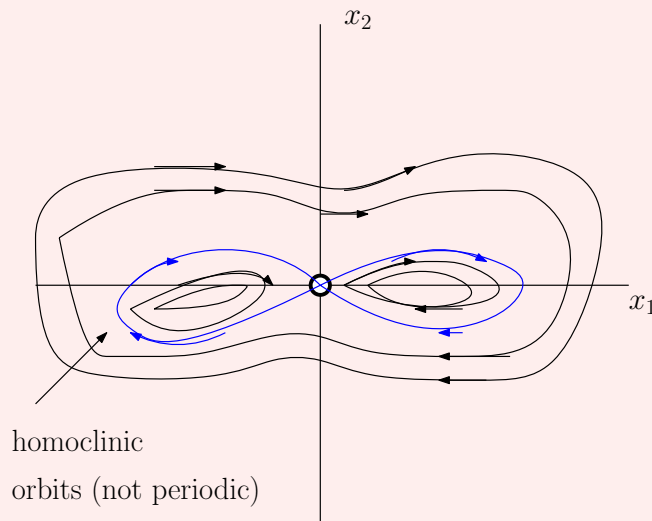
Remark 21.2. The assumption that x_* is isolated is necessary:

$$\dot{x} = xy$$
$$\dot{y} = -x^2$$

has the preserved quantity $E = x^2 + y^2$ ($\frac{d}{dt}E = 2x\dot{x} + 2y\dot{y} = 2x^2y - 2yx^2 = 0$), E has a minimum at $(x, y) = (0, 0)$, but $\{(0, y) | y \in \mathbb{R}\} = y$-axis is a line of fixed points.

fixed points
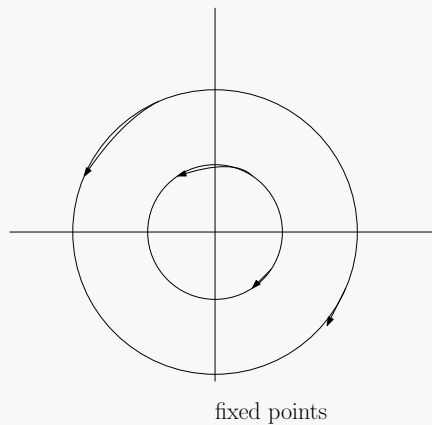
and in particular, the ODE has no closed orbit (around $(0, 0)$).

<u>Recall</u>: Suppose $E : \mathbb{R}^2 \to \mathbb{R}$,

$$DE = \left(\frac{\partial E}{\partial x_1}, \frac{\partial E}{\partial x_2}\right) = 0 \text{ at } x_*$$

If

$$\text{Hess } E = \begin{pmatrix} \frac{\partial^2 E}{\partial x_1^2} & \frac{\partial^2 E}{\partial x_1 \partial x_2} \\ \frac{\partial^2 E}{\partial x_1 \partial x_2} & \frac{\partial^2 E}{\partial x_2^2} \end{pmatrix}$$

has only negative (positive) eigenvalues, then x_* is a local maximum (minimum) of E (alternatively, if $\det \text{Hess } E > 0$, then E has either a local minimum or local maximum at x_*).
If Hess E has eigenvalues $\lambda_1 < 0 < \lambda_2$ (i.e. $\det \text{Hess } E < 0$), these x_* is a saddle.

Example 21.3

Consider:

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$
$$= \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$$
$$DE = \left(-x_1 + x_1^3, x_2\right) = 0$$
$$\iff (x_1, x_2) = (-1, 0), (0, 0), (1, 0)$$
$$\text{Hess } E = \begin{pmatrix} -1 + 3x_1^2 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\text{Hess E}(\pm 1, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \implies (\pm 1, 0) \text{ are local minima}$$
$$\text{Hess } E(0, 0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \implies (0, 0) \text{ is a saddle}$$

Remark 21.4. If E is a preserved quantity, then the trajectories are on the level sets, $a, b > 0$.

$$\text{If } E \approx ax_1^2 + bx_2^2 = 1 \leftrightarrow \text{ ellipse}$$
$$\text{If } E \approx ax_1^2 - bx_2^2 = 1 \leftrightarrow \text{ saddle}$$

§21.2   Reversible Systems

Definition 21.5 (Involution) — A map $R : \mathbb{R}^2 \to \mathbb{R}^2$ is an involution if $R^2(x) = R(R(x)) = x$.

Example 21.6   i) $R(x, y) = (x, y)$ identity

  ii) R is a reflection ,e.g. $R(x, y) = (x, -y)$ reflection along x-axis.

  iii) $R(x, y) = (-x, -y)$ antipodal map
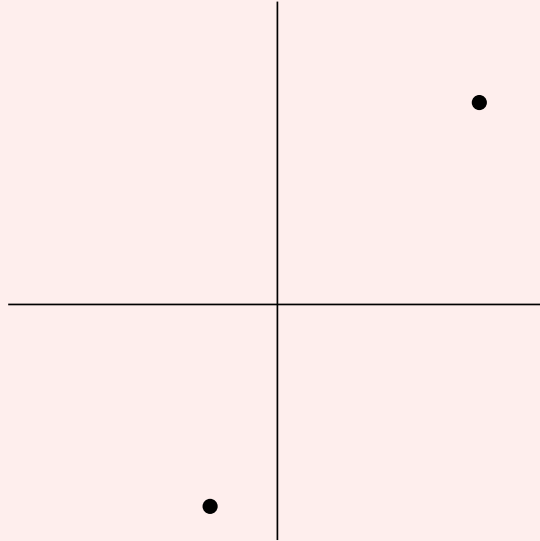
Definition 21.7 (Time-Reversible) — Let R be an involution. The ODE $\dot{x} = f(x)$ is
time – reversible with respect to R if for every solution $x(t)$ of $\dot{x} = f(x)$, $R\left(x(-t)\right)$ is
also a solution.

Example 21.8

$m\ddot{x} = F(x)$ i.e.

$$(*) \begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{m}F(x) \end{cases}$$

Consider: $R(x,v) = (x,-v)$. Let

$$(X,V)(t) = R\left(x(-t), v(-t)\right) = (x(-t), -v(-t))$$

Then

$$\frac{d}{dt}(X,V)(t) = (-\dot{x}(-t), \dot{v}(-t))$$

$$= \left(-v(-t), \frac{1}{m}F\left(x(-t)\right)\right)$$

$$= \left(V(t), \frac{1}{m}F\left(X(t)\right)\right)$$

i.e. $(X,V)(t)$ indeed solves the ODE (*) geometrically:
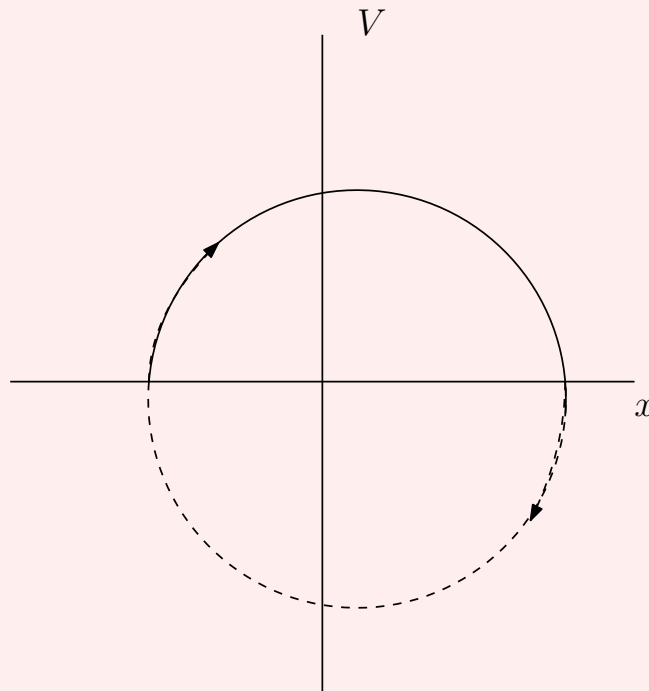
harmonic oscillator: $F(x) = -kx$ with spring constant k. Recall: conservation of energy

$$\left(\frac{k}{m}\right)^2 x^2 + v^2 = \text{const}$$

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Remark 21.9. Reversible systems may not be conservative, e.g.

$$\dot{x} = -2\cos(x) - \cos(y)$$
$$\dot{y} = -2\cos(y) - \cos(x)$$

has a sink at $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. On the other hand, the ODE is time-reversible with respect to $R(x,y) = (-x, -y)$ – more details: Strogatz example 6.6.

§22 | Midterm 2: Feb 26, 2021

§23 | Lec 21: Mar 1, 2021

§23.1   Reversible Systems (Cont'd)

> **Theorem 23.1**
>
> Let $f \in C^1(\mathbb{R}^2)$, $f(x_*) = 0$ and suppose that x_* is a center for the linearization $\dot{y} = Df(x_*) = y$. If $\dot{x} = f(x)$ is time-reversible with respect to a reflection through x_*, then x_* is a center for $\dot{x} = f(x)$, i.e. all trajectories close to x_* are closed orbits.

Idea:
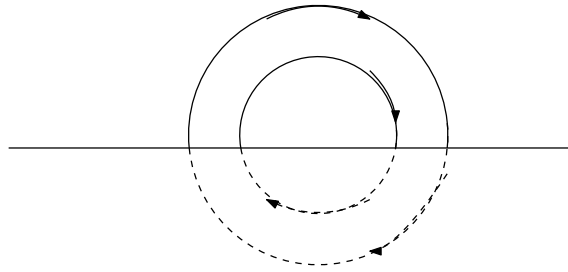
linear centers induces rotational behavior, hence yields intersections with reflection axis, thus closed trajectory.

Example 23.2
Consider:

$$\dot{x} = y$$
$$\dot{y} = x - x^2 = x(1-x)$$

Fixed points: $y = 0$, $x = 0$ or $x = 1$.

$$Df = \begin{pmatrix} 0 & 1 \\ 1 - 2x & 0 \end{pmatrix}$$

$$\implies \text{Eigenvectors} : \lambda = \pm 1 : \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{Eigenvalues} : \lambda = \pm 1$$

$\implies (0,0)$ is a saddle for both the linear and non-linear ODE.

$$Df(1,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ eigenvalues} : \lambda = \pm i$$

$\implies (1,0)$ is a linear center. ODE time-reversible wrt the reflection $R(x,y) = (x,-y)$.
Check: Suppose $(x(t), y(t))$ is a solution. Then

$$(X(t), Y(t)) = R\left(x(-t), y(-t)\right) = (x(-t), -y(-t))$$

satisfies

$$\frac{d}{dt}\left(X(t), Y(t)\right) = (-\dot{x}(-t), \dot{y}(-t))$$
$$= (-y(-t), x(-t)(1 - x(-t)))$$
$$= (Y(t), X(t)(1 - X(t)))$$

i.e. $(X(t), Y(t))$ is a solution $\overset{\text{theorem}}{\implies} (1,0)$ is also a non-linear center.

Example 23.3 (Cont'd from above)

Phase portrait:

$$\dot{x} = y$$
$$\dot{y} = x(1-x)$$
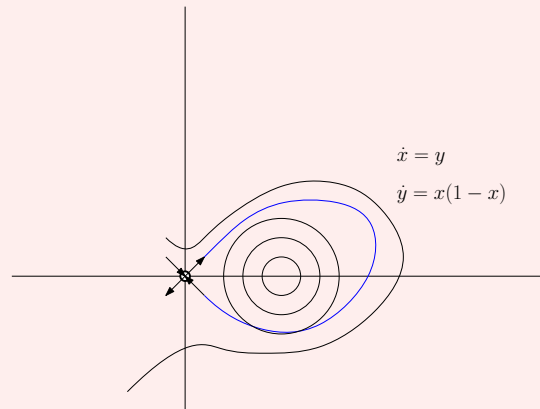
Note:

$$\dot{x} > 0 \iff y > 0$$
$$\dot{y} > 0 \iff 0 < x < 1$$

$\implies$ solution (x, y) with $x > 0$, $y > 0$ in the unstable manifold of $(0,0)$ satisfies $x = 1$ for TBA, then $\dot{y} < 0$ as long as $x > 1$, hence it must cross the x-axis; time reversibility yields a homoclinic orbit.

Remark 23.4. The ODE

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

is time reversible wrt $R(x, y) = (x, -y)$

$$\iff \begin{cases} f \text{ is odd in } y, \ f(x, -y) = -f(x, y) \\ g \text{ is even in } y, \ g(x, -y) = g(x, y) \end{cases}$$

§23.2   Index Theory

$\dot{x} = f(x)$
Phase plane:
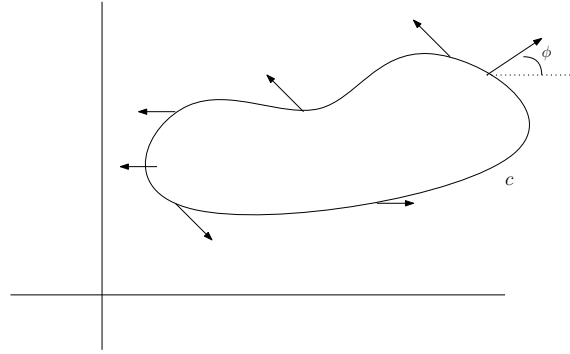
unless stated explicitly otherwise C is a simple (=no self-intersections) closed curve, no fixed points on C, oriented counterclockwise.

Remark 23.5. Usually C is <u>not</u> a trajectory.
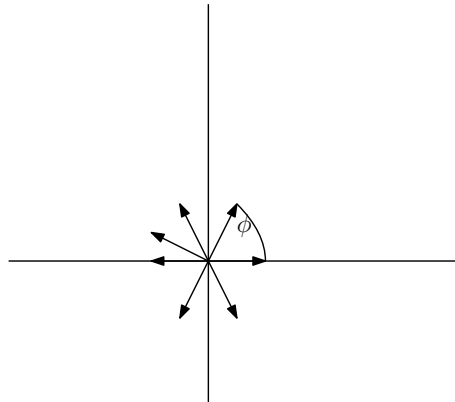
> **Definition 23.6** (Index of a Curve) — Index of $C : I_C(f) = I_C =$ net numbers of counter-clockwise rotations of the vector field f along $C = \frac{1}{2\pi}$ (change of angle).

> **Theorem 23.7**
>
> If C can be continuously deformed into C' without passing through fixed points, then $I_C = I_{C'}$

<u>Idea</u>: C changes continuously, I_C is an integer, hence it cannot jump.

Example 23.8

Consider:
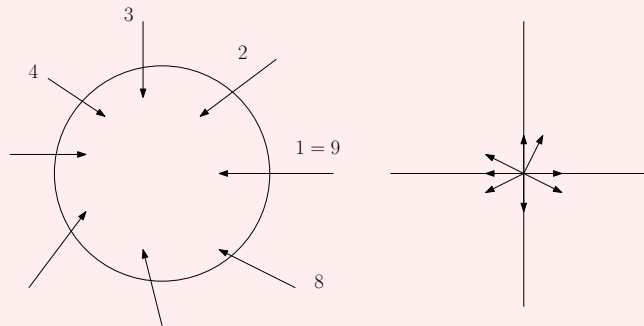
i) $\implies$ $I_C = 1$. In particular, if C encloses a stable node (and no other fixed points), then $I_C = 1$.
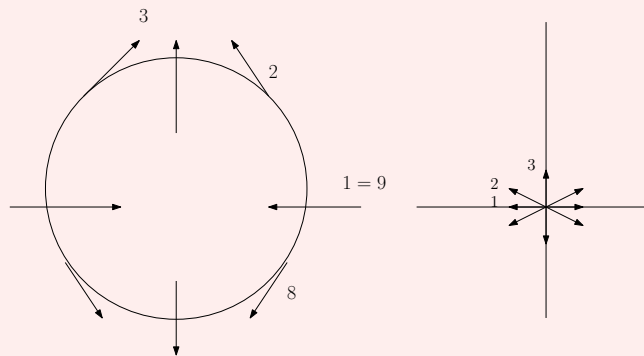
ii) $\implies$ $I_C = -1$. In particular, if C encloses a saddle (and no other fixed points), then $I_C = -1$.

Proposition 23.9

If C does not enclose a fixed point, then $I_C = 0$.

<u>Idea</u>:

shrink

tiny curve C^1
f is continuous, hence
approach const along C^1
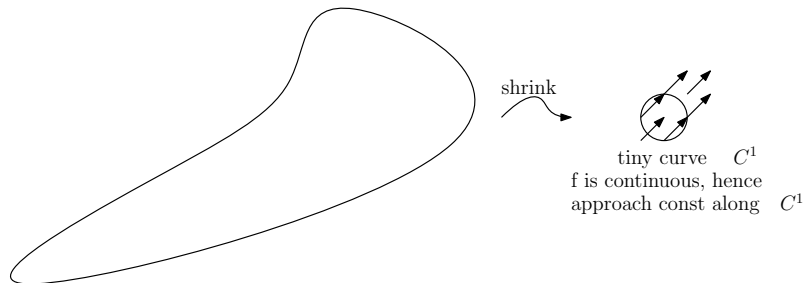
Proposition 23.10

$I_C(f) = I_C(-f)$ i.e. the index does not change when reversing all arrows.
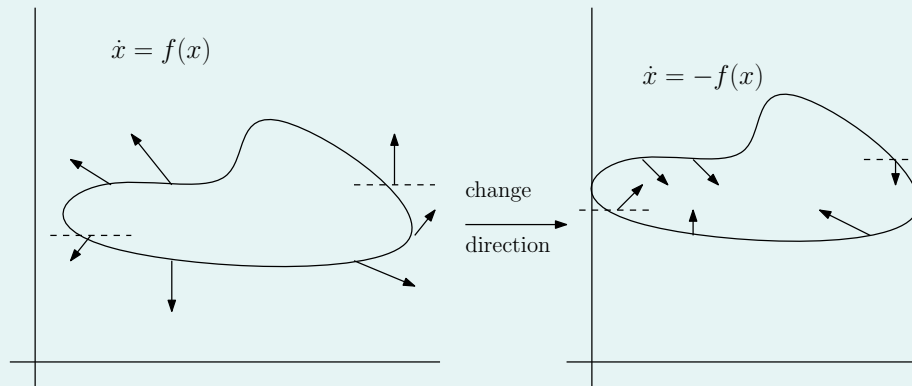
Idea: angle changes from ϕ to $\phi + \pi$, hence the difference stays the same.

Proposition 23.11

If C is a trajectory, i.e. a closed orbit of $\dot{x} = f(x)$, then $I_C = 1$. Intuition:
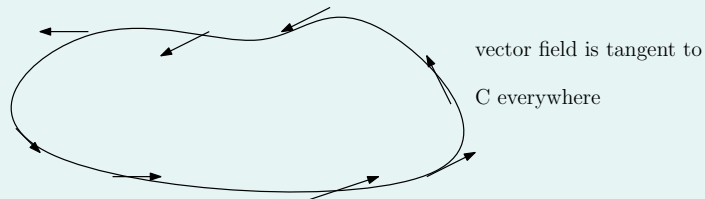
precise result: Hopfscher Umlaufsatz.

<u>Note</u>: Closed orbits precisely correspond to periodic solutions.

§24 | Lec 22: Mar 3, 2021

§24.1 Index Theory (Cont'd)

> **Definition 24.1** (Index of a Fixed Point) — Let x_* be a fixed point of $\dot{x} = f(x)$, $f(x_*) = 0$. The index of x_* is $I_{x_*} = I_C$ where C encloses x_* and no other fixed point.

> **Proposition 24.2**
> If x_* is a hyperbolic fixed point, then $I_* = \text{sign} \det Df(x_*)$. In particular, $I_* = -1 \iff x_*$ saddle.

> **Proposition 24.3**
> If C encloses the fixed points $x_1^*, \ldots, x_n^*$ then
> $$I_C = \sum_{i=1}^{n} I_{x_1^*}$$

Idea:
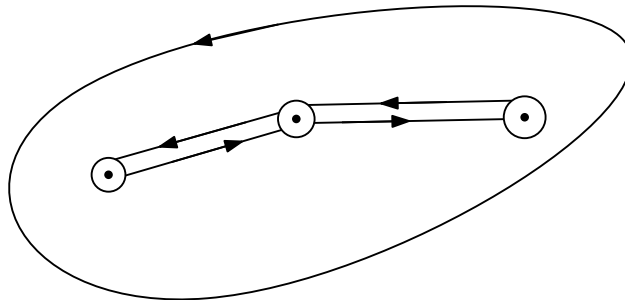
> **Theorem 24.4**
> Any closed orbit in $\mathbb{R}^2$ must enclose fixed point(s) whose indices sum up to $+1$. In particular, every closed orbit encloses a fixed point.

> **Corollary 24.5**
> If $\dot{x} = f(x)$, $f \in C^1(\mathbb{R}^2)$, does not have any fixed points, then it does not have a closed orbit.

Example 24.6

The ODE:

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

does not have closed orbit, $(0,0)$ – unstable node, $(0,2), (3,0)$ – stable nodes, $(1,1)$ – saddle.

Phase portrait:

$I_{C_1} = 0$

(C_1 does not enclose any f.p.)

$I_{C_2} = -1$

(C_2 encloses a saddle)
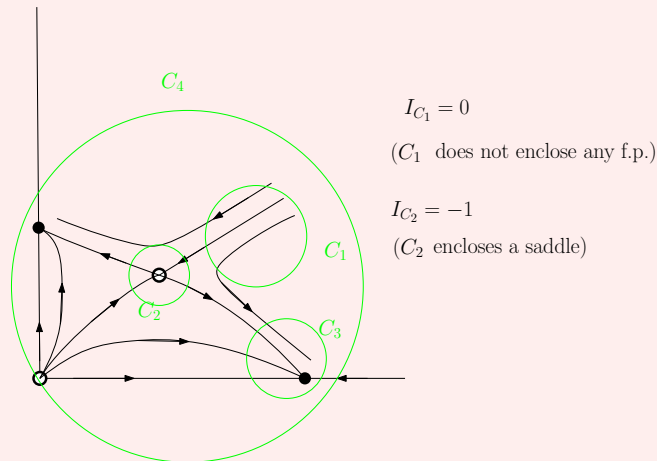
Any closed has index $= +1$, so C_1, C_2 cannot be closed orbit. $I_{C_3} = I_{C_4} = 1$ but C_3, C_4 intersect the x- or y-axis. However, the x-axis, y-axis consist of trajectories. By uniqueness, trajectories cannot intersect, hence C_3, C_4 cannot be trajectories.

The same argument applies to any other curve with index $+1$ since all f.p. with index $+1$ are on the x- or y-axis.

§24.2 Limit Cycles

Definition 24.7 (Limit Cycles) — Limit cycles are isolated closed trajectories.

stable

unstable

unstable
(semi-stable/half stable)
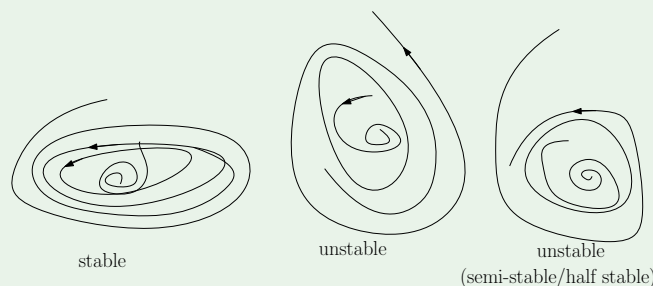
Remark 24.8. Limit cycles are a non-linear phenomenon.

Example 24.9

$\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$ (counter-clockwise rotation with speed $+1$)

$$\dot{r} = 0 : r = 0, \, r = 1$$

Phase portrait for radius:

$$r = 1$$
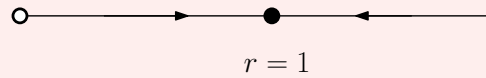
Phase portrait of ODE:
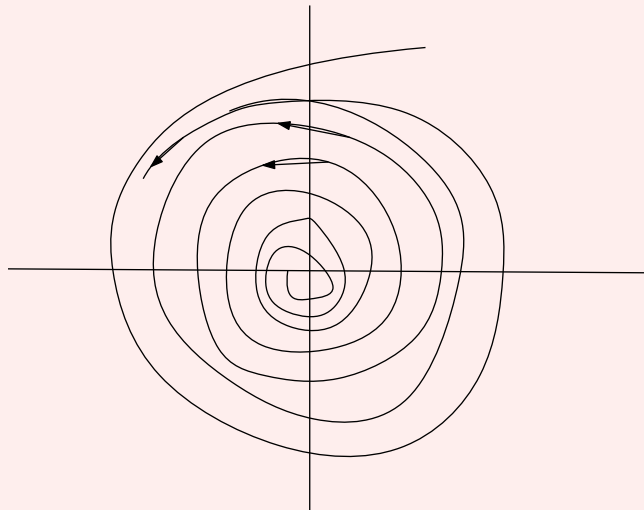
at $r = 1$ we have a stable limit cycle.

§24.3   Gradient Systems

Definition 24.10 (Gradient) — $\dot{x} = f(x)$ is gradient if $f(x) = -\nabla V = -\begin{pmatrix} \partial_{x_1} V \\ \partial_{x_2} V \end{pmatrix}$ for a scalar function $V(x_1, x_2)$. V is called potential.

Example 24.11

Consider:

  i) $V = x^2 + y^2$ where

$$\dot{x} = -\partial_x V = -2x$$
$$\dot{y} = -\partial_y V = -2y$$

    i.e. $(0,0)$ is a stable star. Level sets of $V = x^2 + y^2 = r^2$
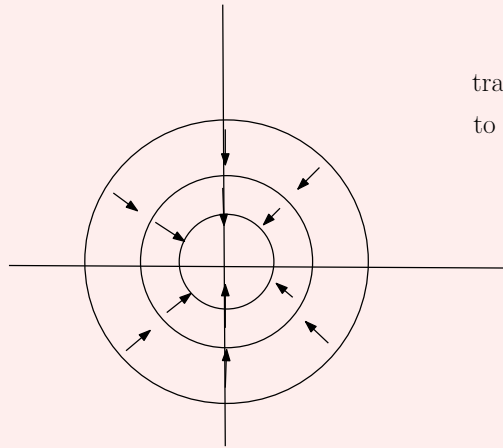
trajectories are orthogonal
to the level sets of V

  ii) $V = x^2 - y^2$ where

$$\dot{x} = -\partial_x V = -2x$$
$$\dot{y} = -\partial_y V = 2y$$

    i.e. $(0,0)$ is a saddle.
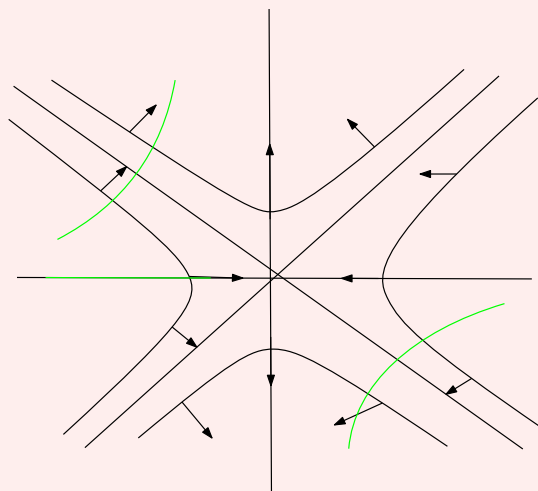
> **Theorem 24.12**
>
> Gradient systems cannot have closed orbits.

Proof. Otherwise, let $x(t)$, $t \in [0,t]$ be a closed orbit. Then

$$
\begin{aligned}
0 &= V\left(x(T)\right) - V\left(x(x(0))\right) \\
&= \int_0^T \frac{d}{dt} V\left(x(t)\right) \, dt \\
&= \int_0^T \langle \nabla V\left(x(t)\right), \dot{x}(t) \rangle \, dt \\
&= - \int_0^T \|\dot{x}(t)\|^2 \, dt < 0
\end{aligned}
$$

unless $\dot{x} = 0$ i.e. $x(t) = \text{const}$ is a fixed point. Contradiction. $\qquad \square$

§25 | Lec 23: Mar 5, 2021

§25.1   Gradient Systems (Cont'd)

Remark 25.1. If $\dot{x} = f(x)$ is gradient, i.e.

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = - \begin{pmatrix} \partial_{x_1} V \\ \partial_{x_2} V \end{pmatrix}$$

then $\frac{\partial f}{\partial x_2} = -\partial_{x_2}\partial_{x_1}V = -\partial_{x_1}\partial_{x_2}V = \frac{\partial f_2}{\partial x_1}$ i.e. $\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} = 0$ and f is curl-free.

Theorem 25.2

Suppose f is curl-free. Then $\dot{x} = f(x)$ is gradient provided that the domain of f does not contain any holes e.g

$$\mathbb{R}^2 \text{ or } B_r\left((x_0, y_0)\right) = \left\{ (x, y)|\ \sqrt{(x - x_0)^2 + (y - y_0)^2} < r \right\}$$

In this case

$$V(x_1, x_2) = - \int_{\gamma_{x_0}} \langle f(x), dx \rangle = - \left(\text{line integral from } x_0 \text{ to } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ along a path } \gamma \right)$$

$$= - \int_a^b f\left(\gamma_{x_0}(t)\right) \cdot \dot{\gamma}_{x_0}(t)\, dt$$

where $\gamma_{x_0}(a) = x_0, \gamma_{x_0}(b) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and also

$$V(x_1, x_2) = - \int_0^1 \left(f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2 \right)\, dt$$

for $\gamma : [0, 1] \to \mathbb{R}^2$ and $\gamma(t) = t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Example 25.3

Consider:

$$\dot{x} = \sin(y)$$
$$\dot{y} = x\cos(y)$$

Then $\frac{\partial f_1}{\partial y} = \cos(y) = \frac{\partial f_2}{\partial x}$ i.e. f is curl-free. Thus, the ODE is gradient. Then the potential

$$V(x,y) = -\int_0^1 (f_1(tx, ty)x + f_2(tx, ty)y) \, dt$$
$$= -\int_0^1 (x\sin(ty) + tx\cos(ty)y) \, dt$$
$$= -x\sin(y)$$

§25.2   Lyapunov Functions

Definition 25.4 (Lyapunov Function) — Let S be a set. A function $L(x)$ is a <u>Lyapunov function</u> for $\dot{x} = f(x)$ if

   i) $L(x) \geq 0$ and $L(x) = 0 \iff x \in S$.

   ii) $\frac{d}{dt}L(x(t)) < 0$ for every solution $x(t)$ of $\dot{x} = f(x)$, $x(t) \notin S$

      $\frac{d}{dt}L(x(t)) = 0 \iff$ for every solution $x(t)$ of $\dot{x} = f(x)$, $x(t) \in S$.

Theorem 25.5

If $\dot{x} = f(x)$ has a Lyapunov function $L(x)$ with $L(x) = 0 \iff x = x_*$, then x_* is a globally stable fixed point. In particular, there is not closed orbit.

Example 25.6

The ODE:

$$\dot{x} = -x + 4y$$
$$\dot{y} = -x - y^3$$

does not have closed orbits, moreover $(0,0)$ is a globally stable fixed point.

Proof. The function $L(x,y) = x^2 + 4y^2$ is a Lyapunov function cont $S = \{(0,0)\}$

- $L(x,y) = x^2 + 4y^2 \geq 0$, $L(x,y) = 0 \iff x = y = 0$

- Consider:

$$\frac{d}{dt}L(x,y) = 2x\dot{x} + 8y\dot{y}$$
$$= -2x^2 + 8xy - 8xy - 8y^4$$
$$= -2(x^2 + 4y^4) \leq 0$$
$$\frac{d}{dt}L(x,y) = 0 \iff x = y = 0$$

Thus, the theorem applies.

$\square$

Example 25.7

Consider:

$$\dot{x} = x\left(1 - 4x^2 - y^2\right) - \frac{1}{2}y(1+x)$$
$$\dot{y} = y\left(1 - 4x^2 - y^2\right) + 2x(1+x)$$

linear stability analysis: $(0,0)$ is an unstable spiral. Consider $L(x,y) = \left(1 - 4x^2 - y^2\right)^2$

- $L(x,y) \geq 0$ and $L(x,y) = 0 \iff 4x^2 + y^2 = 1$

- Have:

$$\frac{d}{dt}L(x,y) = 2\left(1 - 4x^2 - y^2\right)\left(-8x\dot{x} - 2y\dot{y}\right)$$
$$= \dots$$
$$= -4\left(1 - 4x^2 - y^2\right)^2\left(4x^2 + y^2\right)$$
$$\leq 0$$

and $\frac{d}{dt}L(x,y) = 0 \iff x = y = 0$ or $4x^2 + y^2 = 1$.
Consequence: $4x^2 + y^2 = 1$ is a limit cycle because

- ODE does not have a f.p. on $4x^2 + y^2 = 1$. Note: if $4x^2 + y^2 = 1$:

$$\dot{x} = \frac{1}{2}y(1+x)$$
$$\dot{y} = 2x(1+x)$$

thus if $\dot{x} = 0$, then $(x,y) = \left(\pm\frac{1}{2}, 0\right)$ is the only option on $4x^2 + y^2 = 1$ and $\dot{y} \neq 0$. Similarly, if $\dot{y} = 0$, then $(x,y) = (0, \pm1)$ and $\dot{x} \neq 0$

- Trajectories approach the minimum level set $4x^2 + y^2 = 1$ unless $(x(t), y(t)) = (0,0)$.

§26 | Dis 1: Jan 7, 2021

§26.1 Fixed points and Stability

$\dot{x} = f(x)$

Example 26.1
$\dot{x} = -x^3 + x^2$

a) Sketch the vector field, classify the fixed points.

"vector fields" = x-axis with arrows
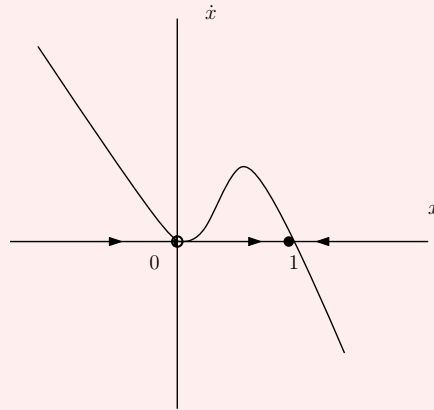
so:

- $\dot{x} > 0 \implies x(t)$ increasing
- $\dot{x} < 0 \implies x(t)$ decreasing

"Fixed point" $\iff$ x_* s.t. $f(x_*) = 0$ $\iff$ x_* s.t. the constant function $x(t) = x_*$ is a solution.

We have 2 fixed points:

- $x_* = 0$ is semi-stable.
- $x_* = 1$ is stable.

b) Sketch various solutions of $x(t)$.
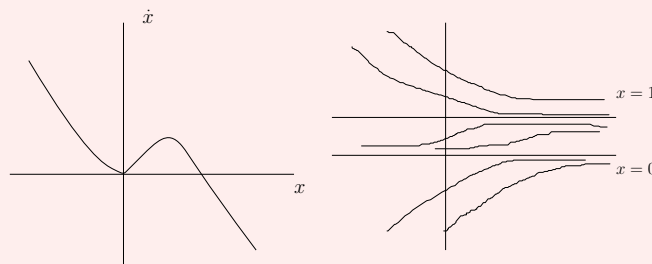
- $\dot{x} = 0$ for $x = 0, 1 \implies x(t) = 0, 1$ are solutions.
- $\dot{x} > 0$ for $x < 0 \implies x(t)$ increasing.
- $\dot{x} > 0$ for $0 < x < 1 \implies x(t)$ increasing.
- $\dot{x} < 0$ for $x > 1 \implies x(t)$ decreasing.

Example 26.2

$\dot{x} = -1 + 4\sin x$

  a) Sketch vector field, classify fixed points.
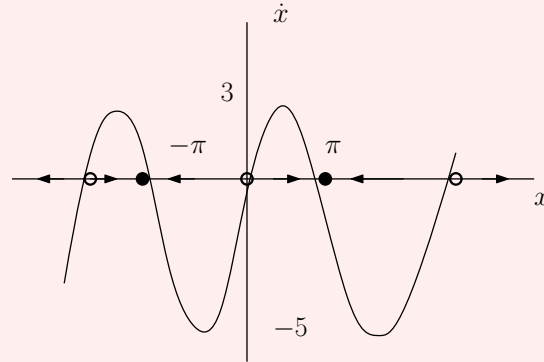
    Fixed points:

$$\sin(x_*) = \frac{1}{4}$$

- $x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$ for $n = 0, \pm 1, \ldots$ are unstable.
- $x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$ for $n = 0, \pm 1, \ldots$ are stable.

  b) Sketch various solutions $x(t)$.

vertically
2π -periodic
because f(x) is

§26.2   First Order Autonomous System

$\vec{\dot{x}} = \vec{f}(\vec{x})$ – first order and autonomous.

Example 26.3

A unit mass with displacement $x(t)$ attached to a spring with spring constant 6 obeys:

$$\ddot{x} = -6x - b(t)\dot{x}$$

where $b(t) \geq 0$ is the friction coefficient.

a) Show that this can expressed as a first order autonomous system

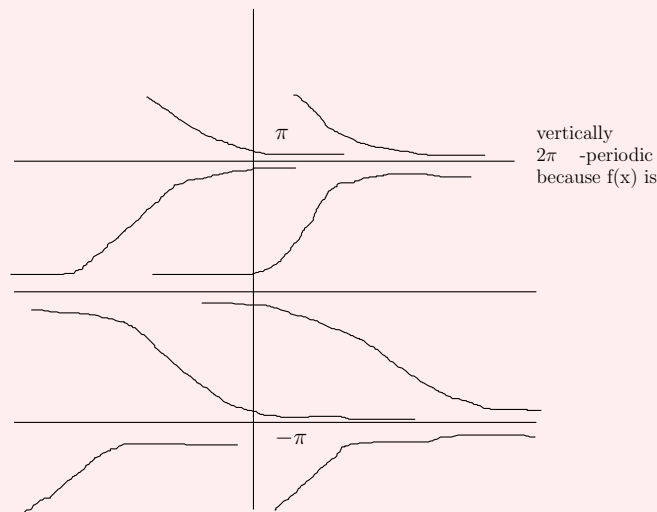$$x_1 = x, \quad x_2 = \dot{x}$$
$$\dot{x}_2 = \ddot{x} = -6x_1 - b(t)x_2$$
$$\vec{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\dot{\vec{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -6x_1 - b(x_3)x_2 \end{pmatrix}$$

where $x_3 = t \implies \dot{x}_3 = 1$.

b) In the case $b(t) = 5$, find the explicit solution for $x(0) = x_0, \dot{x}(0) = v_0$.

$$\ddot{x} = -6x - 5\dot{x} \implies \ddot{x} + 5\dot{x} + 6x = 0$$

Try $x(t) = e^{kt}$:

$$0 = \ddot{x} + 5\dot{x} + 6x = k^2 e^{kt} + 5k e^{kt} + 6e^{kt}$$
$$= e^{kt}(k^2 + 5k + 6) \implies k = -3, -2$$

Now, $x(t) = c_1 e^{-3t} + c_2 e^{-2t}, c_1, c_2 \in \mathbb{R}$. Using the initial conditions, we obtain

$$x(t) = (-2x_0 - v_0)e^{-3t} + (3x_0 + v_0)e^{-2t}$$

§27 | Dis 2: Jan 14, 2021

§27.1   Linearization and Potentials

Example 27.1

$\dot{x} = -x^3 + x^2$

a) Use linear stability analysis to classify the fixed points. If it fails, use a graphical argument.

Idea: For x near a fixed point x_*, $\dot{x} = f(x) \approx f(x_*)(= 0) + f'(x_*)(x - x_*) = \ldots$

$$f(x) = -x^3 + x^2, \quad f'(x) = -3x^2 + 2x$$
$$0 = f(x_*) = -x_*^2(x_* - 1) \implies x_* = 0, 1$$

- $x_* = 1 : f'(1) = -1 < 0 \implies$ stable
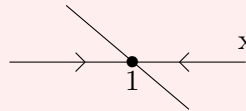
- $x_* = 0 : f'(0) = 0 :$ inconclusive
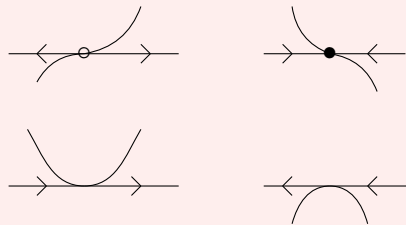
b) Find and plot a potential function.

"Potential function" $\iff V(x)$ s.t. $\dot{x} = -\frac{dV}{dx}$

$$\dot{x} = -x^3 + x^2 = -V'(x) \implies V(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 + C \text{ (choose 0)}$$

- fixed points
  $\iff f(x_*) = 0$
  $\iff V'(x_*) = 0$
  $\iff$ critical point

- Trajectories move toward decreasing V, like a ball rolling down the graph of V

Example 27.2

$\dot{x} = 4\sin x - 1$.

a) Use linear stability analysis to classify the fixed points.

$$f(x) = 4\sin x - 1, \quad f'(x) = 4\cos x$$

Last time: Fixed points are

- $x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \ldots$

$$f'(x_*) = 4\cos\left(\sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) > 0$$

   $\implies$ unstable

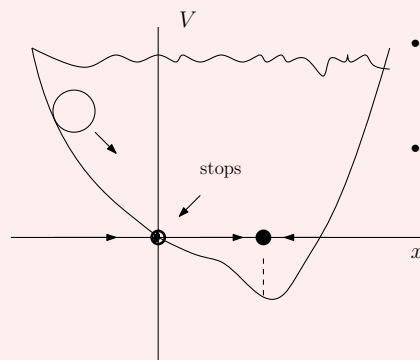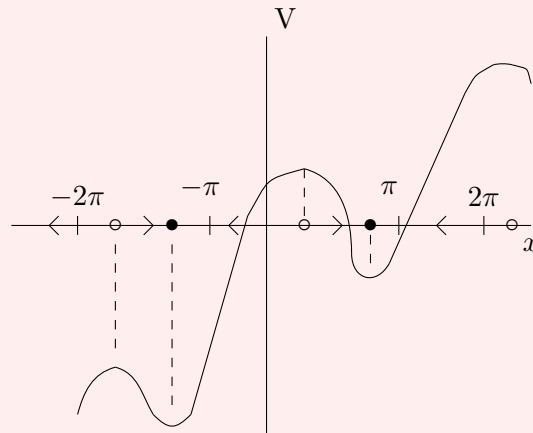- $x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \ldots$

$$f'(x_*) = 4\cos\left(\pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) < 0$$

   $\implies$ Stable.

b) Plot potential $-1 + 4\sin x = -V'(x) \implies V(x) = x + 4\cos x$

§27.2   Existence of Solutions

Example 27.3   a) Let $a > 0$ be a constant. Show that the solution of

$$\begin{cases} \dot{x} = ax^2 \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

$$\frac{dx}{dt} = ax^2 \implies \int \frac{dx}{x^2} = \int a\,dt$$

$$\implies -\frac{1}{x} = at + c$$

$$\implies x(t) = \frac{1}{c - at} \forall c \in \mathbb{R}$$

$$x(0) > 0 \implies c > 0 \implies \lim_{t \to T} x(t) = +\infty$$

for some $T > 0$. In fact, $c = \frac{1}{x_0}$ and so $T = \frac{c}{a} = \frac{1}{ax_0}$.

b) Let $0 < \epsilon < 1$ be a constant. Show that the solution of

$$\begin{cases} \dot{x} = x^2\,(1 + \epsilon \sin x) \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

Idea: $\dot{x} \geq ax^2$ for some $a > 0$, so our solution grows faster than a function which blows up, and must blow up too.

$$|\sin x| \leq 1 \implies 1 + \epsilon \sin x \geq 1 - \epsilon$$
$$\implies \dot{x} = x^2\,(1 + \epsilon \sin x) \geq \underbrace{1 - \epsilon}_{>0}\, x^2$$

Let $x(t)$ be the solution to

$$\begin{cases} \dot{x} = x^2\,(1 + \epsilon \sin x) \\ x(0) = x_0 \end{cases}$$

Let $y(t)$ be the solution to

$$\begin{cases} \dot{x} = (1 - \epsilon)x^2 \\ x(0) = x_0 \end{cases}$$

By part a), $y(t)$ blows up at some time $T > 0$. Since $x(0) = y(0)$ and $\dot{x} \geq \dot{y}$, then $x(t) \geq y(t)$ for all $t \geq 0$ (ODE Comparison Lec 4). Therefore, $x(t)$ must blow up in finite time. In fact, blow up time must be $\leq T = \frac{1}{(1-\epsilon)x_0}$.
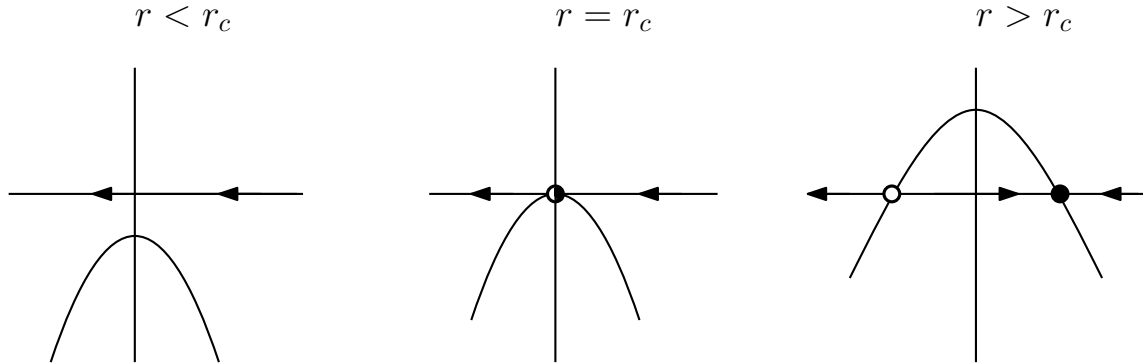
§28 | Dis 3: Jan 21, 2021

§28.1 Bifurcations

$\dot{x} = f(x, r)$, r parameter

"Bifurcation" $\iff$ change in the number or stability of fixed points.

There are two types:

- Saddle-node: $0 \to 1 \to 2$ fixed points

$$r < r_c \qquad\qquad r = r_c \qquad\qquad r > r_c$$

- Transcritical: $2 \to 1 \to 2$ fixed points

$$r < r_c \qquad\qquad r = r_c \qquad\qquad r > r_c$$
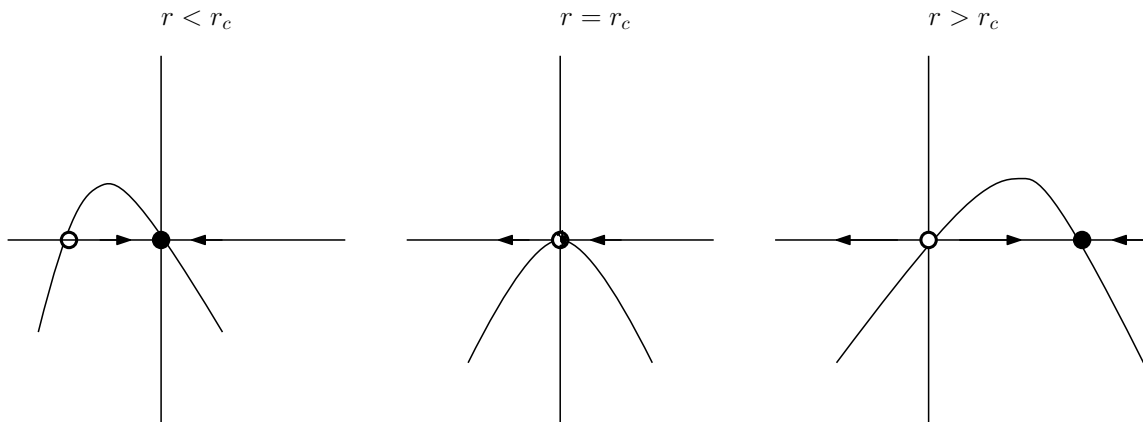
Example 28.1

$\dot{x} = rx - \log(1+x)$

(a) Sketch all qualitatively different vector fields, sketch bifurcation diagram, find and classify bifurcations.
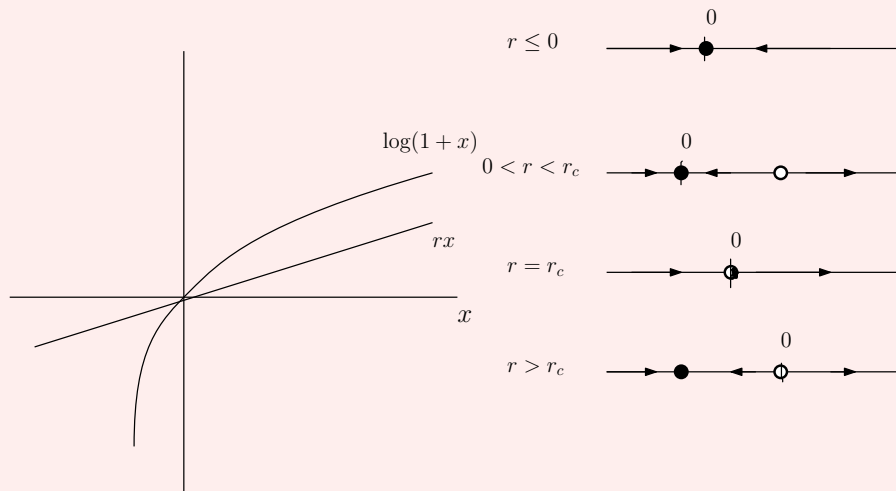
Transcritical bifurcation at $r = r_c$

"Bifurcation diagram" $\iff$ Plot of the fixed points x_* as a function of r.

$$0 = rx_* - \log(1 + x_*)$$

$$r = \frac{1}{x_*} \log(1 + x_*) \quad \text{or} \quad x_* = 0$$
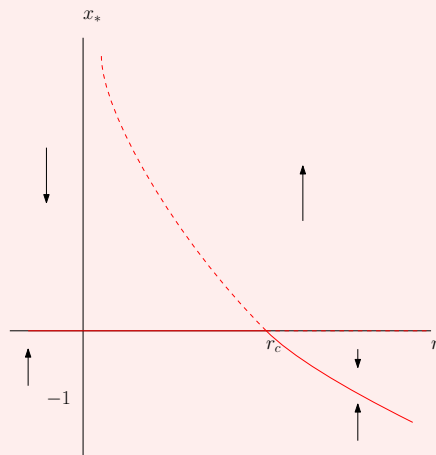
- Vertical slices of constant r are vector fields
- Whole regions have same arrow direction

Bifurcation point: $(r_c, x_c) = (1, 0)$

Example 28.2 (Cont'd of example 28.1)

From the above example,

(b) Show that there is a transcritical bifurcation at $(r_c, x_c) = (1, 0)$ using normal forms. Taylor expand about $r = 1, x = 0$

$$\dot{x} = rx - \log(1 + x)$$
$$= (r - 1)x + x - (x - \frac{1}{2}x^2 + \mathcal{O}(x^3))$$
$$= (r - 1)x + \frac{1}{2}x^2 + \mathcal{O}(x^3)$$

$\dot{x} = (r - 1)x + \frac{1}{2}x^2 + \mathcal{O}(\epsilon^3)$ for $|x| < \epsilon, |r - 1| < \epsilon$

This is normal form for transcritical bifurcation.

Example 28.3

$\dot{x} = r + x - x^3$

(a) Sketch all vector field, sketch bifurcation diagram, find and classify bifurcations.
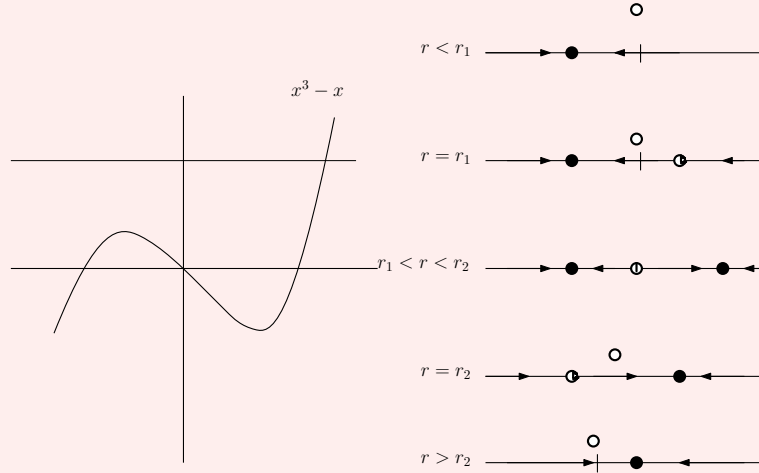
2 saddle node bifurcations at $r = r_1, r_2$

$$0 = r + x_* - x_*^3 \implies r = x_*^3 - x_*$$
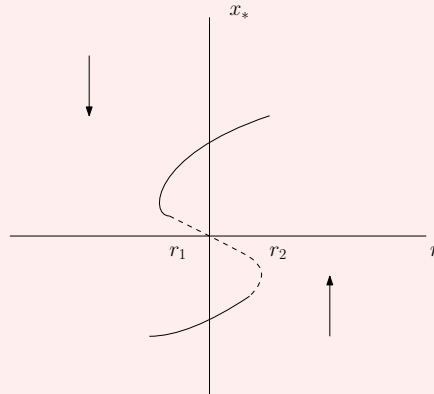
Bifurcation point (x_c, r_c) satisfies:

- Fixed point: $0 = f(x_c, r_c) = r_c + x_c - x_c^3$
- $0 = \frac{\partial f}{\partial x}(x_c, r_c) = 1 - 3x_c^2$

$$0 = 1 - 3x_c^2 \implies x_c = \pm\frac{1}{\sqrt{3}}$$

$$0 = r_c + x_c - x_c^3 \implies r_1 = -\frac{2}{3\sqrt{3}}, r_2 = \pm\frac{2}{3\sqrt{3}}$$

$$(r_c, x_c) = \left(\frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

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Example 28.4 (Cont'd of example 28.3) (b) Show that there is a saddle-node bifurcation at $(r_c, x_c) = \left(\frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ using normal forms.

Taylor expand about $r = \frac{2}{3\sqrt{3}}, x = -\frac{1}{\sqrt{3}}$.

$$\dot{x} = r + x - x^3, \quad r = \frac{2}{3\sqrt{3}} + (r - \frac{2}{3\sqrt{3}})$$

$$x - x^3 = -\frac{2}{3\sqrt{3}} + 0(x + \frac{1}{\sqrt{3}}) + \frac{1}{2} \cdot \frac{6}{\sqrt{3}}(x + \frac{1}{\sqrt{3}})^2 + \mathcal{O}\left(\left(x + \frac{1}{\sqrt{3}}\right)^3\right)$$

Plug these in, get

$$\dot{x} = \left(r - \frac{2}{3\sqrt{3}}\right) + \sqrt{3}\left(x + \frac{1}{\sqrt{3}}\right)^2 + \mathcal{O}(\epsilon^3)$$

for $\left|r - \frac{2}{3\sqrt{3}}\right| < \epsilon^2, \left|x + \frac{1}{\sqrt{3}}\right| < \epsilon$. This is normal form for a saddle-node bifurcation $(\dot{y} = R + y^2)$.

§29 | Dis 4: Jan 28, 2021

§29.1   Review of Bifurcations

We've seen 3 types of bifurcations so far

- saddle-node: $0 \to 1 \to 2$ fixed points

- transcritical: $2 \to 1 \to 2$ swap stability

- sub-/supercritical pitchfork: $1 \to 1 \to 3$

supercritical
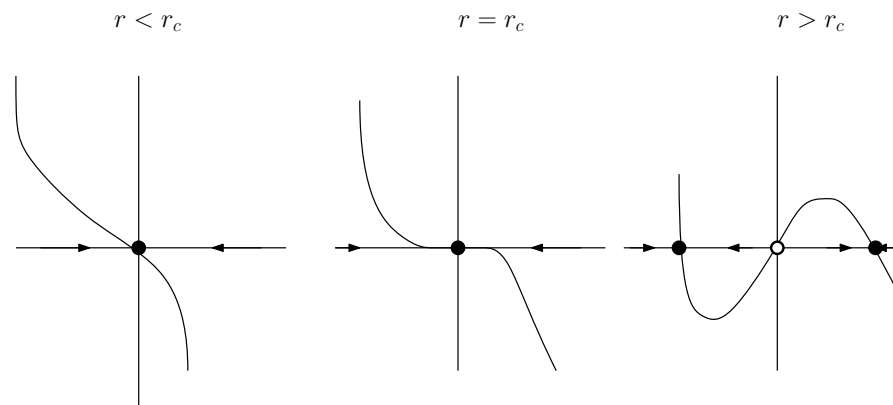
subcritical
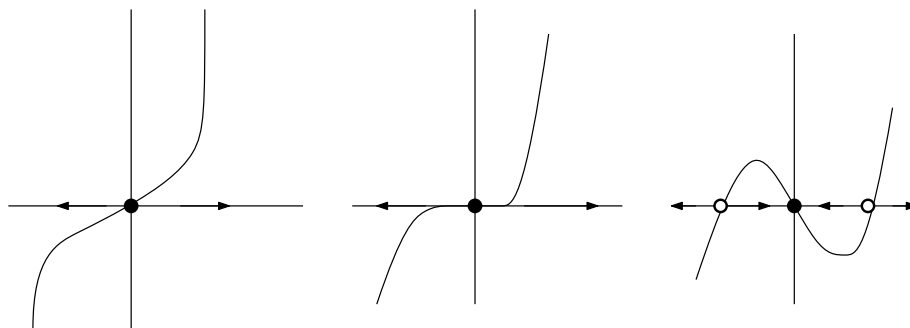
Example 29.1

$\dot{x} = r \tanh x - x$. Sketch all qualitatively different phase portraits, sketch bifurcation diagram, find any classify bifurcations. Recall

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$
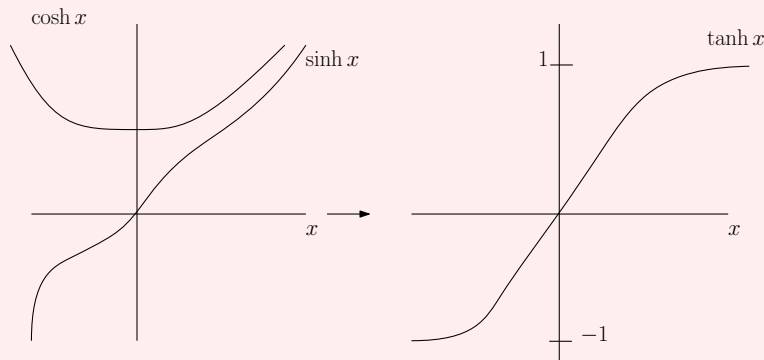
$\dot{x} = r \tanh x - x$
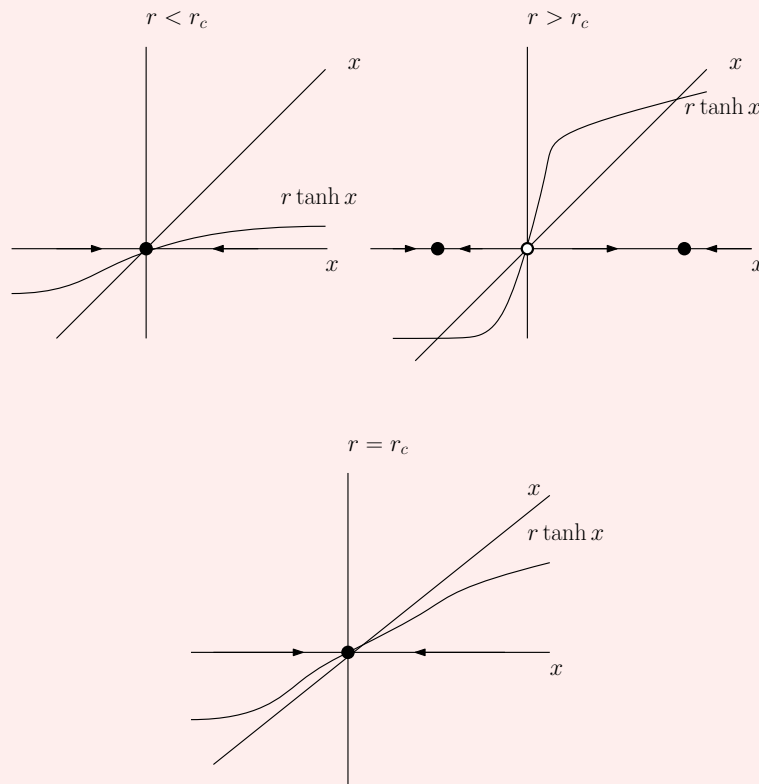
Figure 5: Supercritical pitchfork at $r = r_c$

Example 29.2 (Cont'd from above)

$$0 = r \tanh x_* - x_* \implies r = \frac{x_*}{\tanh x_*}, \quad x_* = 0$$
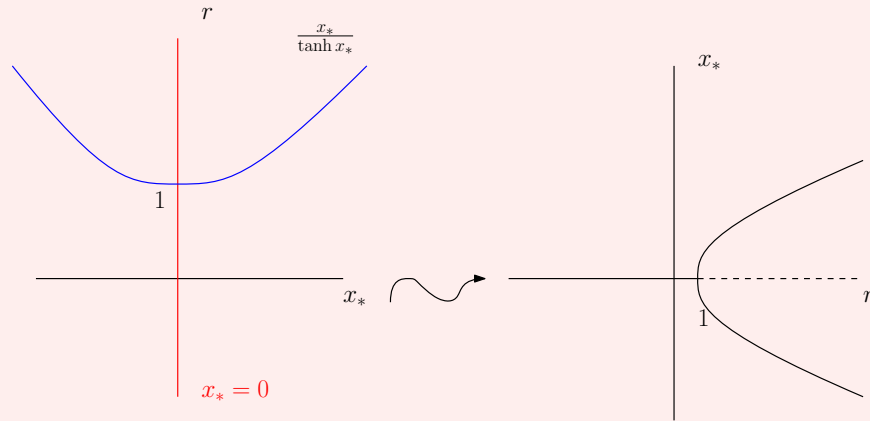
Near $x = 0, \tanh x = \frac{x + \cdots}{1 + \cdots} = x + \ldots \implies \frac{x}{\tanh x} = 1 + \ldots$
Bifurcation point: $(r_c, x_c) = (1, 0)$.

Example 29.3

Same for $\dot{x} = x - x(x - r)^2$

$$0 = x_* \left(1 - (x_* - r)^2 \right) \implies x_* = 0, \quad x_* = r \pm 1$$
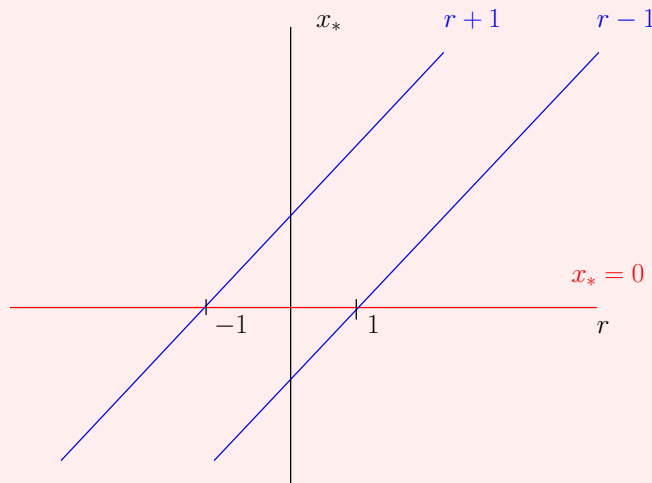
2 transcritical bifurcations at $r_c = \pm 1, x_c = 0$.

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Example 29.4 (Cont'd of example 29.3)

$$\dot{x} = x\left(1 - (x-r)^2\right) = -x\left(x - (r+1)\right)\left(x - (r-1)\right)$$
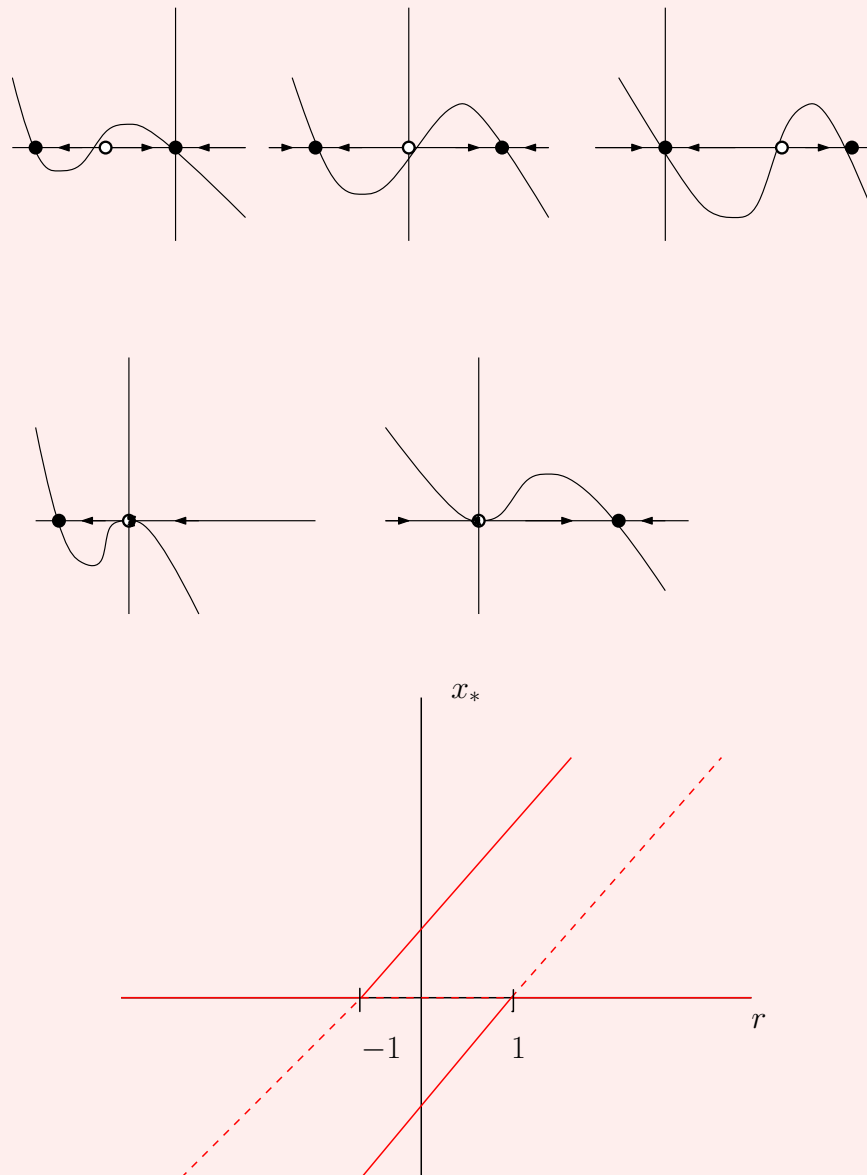
Figure 6: Bifurcation Diagram

Example 29.5

Same for $\dot{x} = rx - \frac{x}{1+x^2}$

$$r \geq r_c :$$

$$0 < r < r_c :$$

$$r \leq 0 :$$
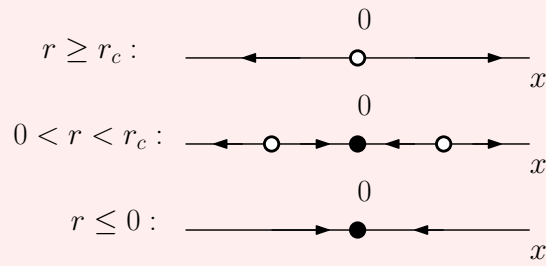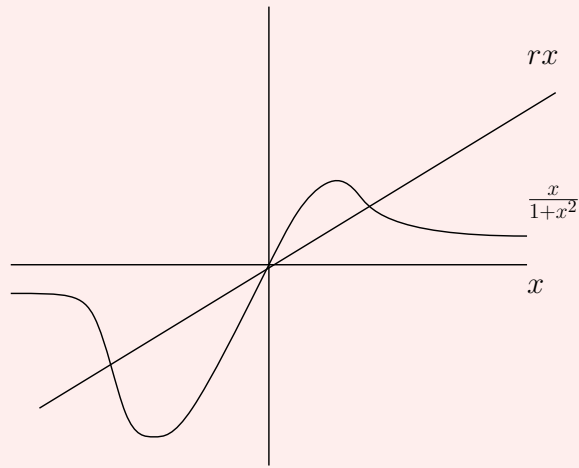
$$0 = x_* \left(r - \frac{1}{1 + x_*^2} \right) \implies x_* = 0, \quad r = \frac{1}{1 + x_*^2}$$
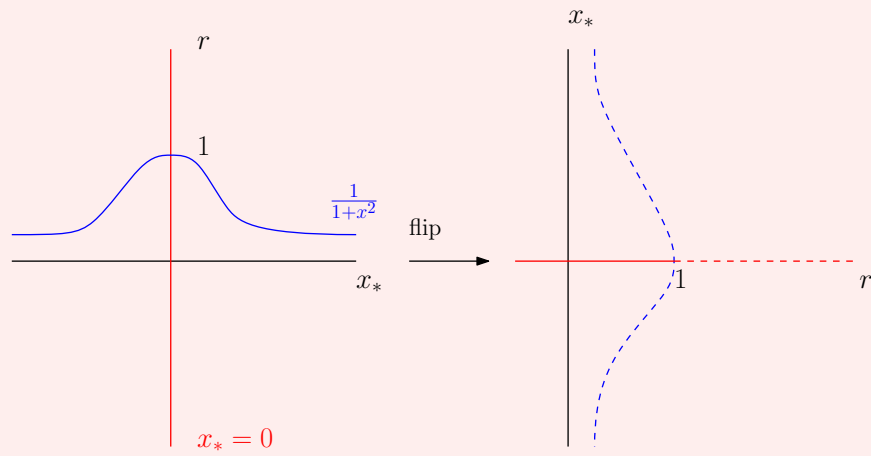
Subcritical pitchfork at $(r_c, x_c) = (1, 0)$.

§29.2　　Other Bifurcations

Example 29.6

$\dot{x} = r^2 - \frac{x^2}{1+x^2}$. Sketch all qualitatively different phase portraits, sketch bifurcation diagrams, find bifurcation point.
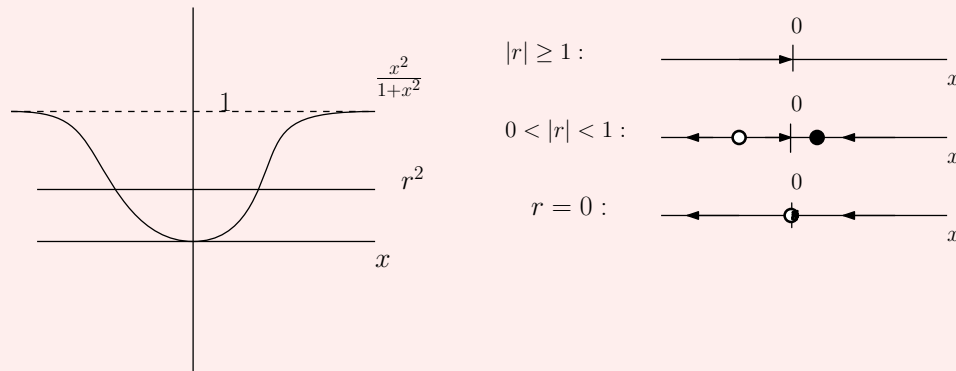
Bifurcation point $(r_c, x_c) = (0,0)$ and still satisfies $f(r_c, x_c) = 0, \frac{\partial f}{\partial x}(r_c, x_c) = 0$. Bifurcation diagram:

$$0 = r^2 - \frac{x_*^2}{1+x_*^2} \implies r = \pm\sqrt{\frac{x_*^2}{1+x_*^2}} = \pm\frac{x_*}{\sqrt{1+x_*^2}}$$
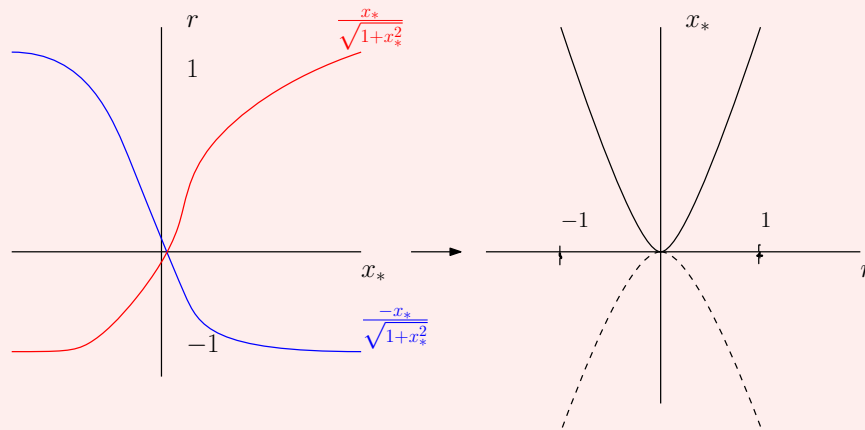
This is not one of our 3 types of bifurcations

- Graphically, it's not transcritical because both fixed points are moving.

- Analytically, we can check that the Taylor expansion at $(r, x) = (0,0)$:

$$f(r,x) = r^2 - x^2 + \mathcal{O}(x^3)$$

doesn't match one of the 3 normal forms we know.

---

§30 | Dis 5: Feb 4, 2021

§30.1   Bifurcations (Cont'd)

Example 30.1

$\dot{N} = rN - aN(N-b)^2$ population model, $r > 0, a > 0, b \in \mathbb{R}$ parameters. Show that this can be rewritten in the dimensionless form $\frac{dx}{d\tau} = x - x(x-c)^2$. What are x, τ, c in terms of the original quantities?

"dimensionless" $\iff$ change of variables s.t. new variables have no units

N has units pop (population) and $\dot{N} = \frac{dN}{dt}$ has units pop/time.

$$\underbrace{\dot{N}}_{\frac{\text{pop}}{\text{time}}} = r \underbrace{N}_{\text{pop}} - a \underbrace{N^3}_{\text{pop}^3} + 2ab \underbrace{N^2}_{\text{pop}^2} - ab^2 \underbrace{N}_{\text{pop}}$$

$$\implies r \sim \frac{1}{\text{time}}, \ a \sim \frac{1}{\text{pop}^2 \cdot \text{time}}, \ b \sim \text{pop}$$

Rescale: $N = nx$, $t = T\tau$ where n, T are constant has units and τ is new variable with no units.

$$\frac{dN}{dt} = \underbrace{\frac{dN}{dx}}_{n} \frac{dx}{dt} = n \frac{dx}{d\tau} \underbrace{\frac{d\tau}{dt}}_{\frac{1}{T}} = \frac{n}{T} \frac{dx}{d\tau}$$

$$\frac{n}{T} \frac{dx}{d\tau} = rnx - anx(nx - b)^2$$

$$\frac{dx}{d\tau} = rTx - aTx(nx - b)^2$$

$$= \underbrace{rT}_{=1} x - \underbrace{aTn^2}_{=1} x(x - \underbrace{\frac{b}{n}}_{=c})^2$$

$\implies T = \frac{1}{r}, n = \frac{1}{\sqrt{aT}} = \sqrt{\frac{r}{a}}$.

Units: $T \sim \frac{1}{\frac{1}{\text{time}}} = \text{time}, n \sim \sqrt{\frac{\frac{1}{\text{time}}}{\frac{1}{\text{pop}^2 \cdot \text{time}}}} = \text{pop}$ where τ no units and x no units.

$\implies \frac{dx}{d\tau} = x - x(x-c)^2$ with $x = \sqrt{\frac{a}{r}}N$, $\tau = rt$, $c = b\sqrt{\frac{a}{r}}$ which is defined since $a, r > 0$.

Example 30.2

Population model $\dot{x} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$ where $r, k > 0$.

(a) Sketch the 2 qualitatively different bifurcation diagrams (r, x_*) for $k \leq k_*$ and $k > k_*$.

$$\dot{x} = x\left[r\left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2}\right]$$

$0 < k \leq k_*$:

$r > 0$:

$$0 = x_*\left[r\left(1 - \frac{x_*}{k}\right) - \frac{x_*}{1+x_*^2}\right]$$

$$\implies x_* = 0 \text{ or } 0 = r\left(1 - \frac{x_*}{k}\right) - \frac{x_*}{1+x_*^2} \text{ (with 1 solution)}$$

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Example 30.3 (Cont'd of example 30.2)

$k > k_*$:
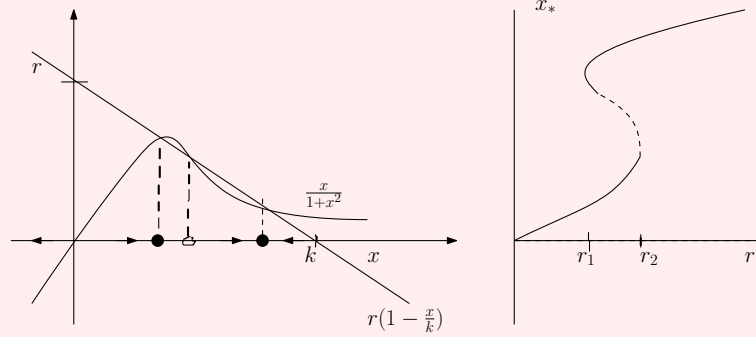
$x_* = 0$ or $0 = r\left(1 - \frac{x_*}{k}\right) - \frac{x_*}{1+x_*^2}$ (with 1-3) solutions.

(b) Sketch regions in (k, r) plane with qualitatively different vector fields, classify bifurcations on boundaries.

Fixed points:

$$0 = x\left[r\left(1 - \frac{x}{k}\right) - \frac{x}{1 + x^2}\right]$$

$$\implies x = 0(\text{bifurcation}), \quad r\left(1 - \frac{x}{k}\right) = \frac{x}{1 + x^2}(\text{includes 2 saddle-node bif})$$

$$(1)$$

Tangent:

$$0 = \frac{\partial}{\partial x}\left\{x\left[r\left(1 - \frac{x}{k}\right) - \frac{x}{1 + x^2}\right]\right\}$$

$$= \underbrace{r\left(1 - \frac{x}{k}\right) - \frac{x}{1 + x^2}}_{=0 \text{ by (1)}} \underbrace{x}_{\neq 0}\underbrace{\left[-\frac{r}{k} - \frac{1}{1 + x^2} + \frac{2x^2}{(1 + x^2)^2}\right]}_{=0}$$

$$\implies -\frac{r}{k} = \frac{1}{1 + x^2} - \frac{2x^2}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2} \qquad (2)$$

$$\implies \frac{x}{1 + x^2} \overset{(1)}{=} r + \left(-\frac{r}{k}\right)x \overset{(2)}{=} r + \frac{x - x^3}{1 + x^2}$$

$$\implies r = \frac{2x^3}{(1 + x^2)^2} \overset{(2)}{\implies} k = \frac{2x^3}{x^2 - 1}$$

for $x > 1$ (so $r, k > 0$).

Example 30.4 (Cont'd of example 30.2)

So
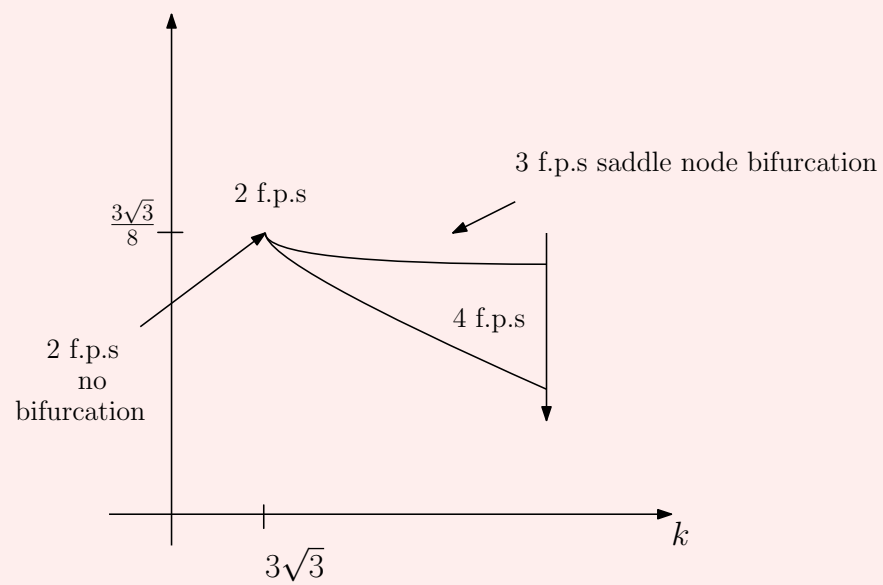
§31 | Dis 6: Feb 11, 2021

§31.1   Flows on the Circle

$\dot{\theta} = f(\theta)$, $\theta + 2\pi = \theta$ for all θ, so $f(\theta)$ is $2\pi-$periodic

Example 31.1

$\dot{\theta} = \mu + \sin^2\theta$. Plot all of the qualitatively different vectors fields on the circle, sketch the bifurcation diagrams, classify bifurcations.
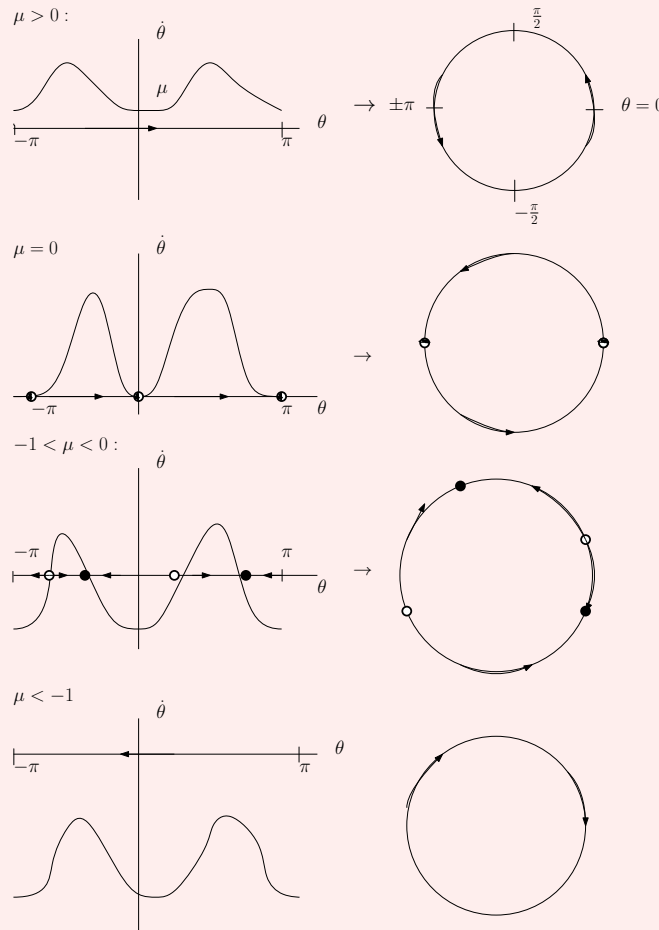
$$0 = \mu + \sin^2(\theta_*) \implies \mu = -\sin^2\theta_*$$
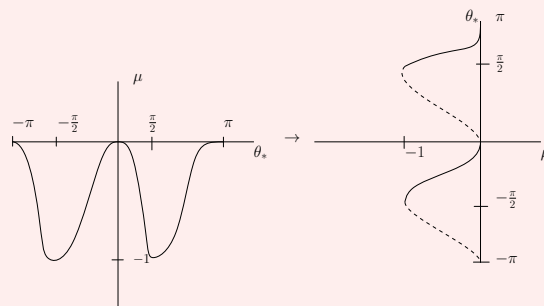
4 saddle node bifurcations at: $(\mu_*, \theta_*) = (0,0), (0,\pi), (-1, \frac{\pi}{2}), (-1, \frac{-\pi}{2})$

§31.2   2-Dim Linear Systems

$\dot{x} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = Ax.$ v eigenvector (evec) with eigenvalue (eval), $\lambda : Av = \lambda v$

$$\implies x(t) = e^{\lambda t}v \text{ is a solution}$$

Example 31.2

$\dot{x} = -4x, \dot{y} = -3x - y.$ Sketch phase portrait, classify fixed point $(0, 0)$.

$$\dot{x} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = Ax$$

Step 1: Find evals.

$$
\begin{aligned}
Av &= \lambda v \text{ for some } v \neq 0 \\
&\Longleftrightarrow 0 = Av - \lambda v = (A - \lambda I)v \text{ for some } v \neq 0 \\
&\Longleftrightarrow A - \lambda I \text{ not invertible} \\
&\Longleftrightarrow \det(A - \lambda I) = 0 \\
0 = \det(A - \lambda I) &= \det \begin{pmatrix} -4 - \lambda & 0 \\ -3 & -1 - \lambda \end{pmatrix} \\
&= (-4 - \lambda)(-1 - \lambda) - 0 = (\lambda + 1)(\lambda + 4)
\end{aligned}
$$

$\implies \lambda_1 = -4, \lambda_2 = -1$

Step 2: Find evecs. Want $Av_1 = \lambda v_1$ for $v_1 = \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

$$
\begin{aligned}
\begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= -4 \begin{pmatrix} a \\ b \end{pmatrix} \\
\implies -4a = -4b, &\quad -3a - b = -4b \\
\implies a &= b
\end{aligned}
$$

Any $a = b$ works, pick one $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$ Similarly, $Av_2 = \lambda_2 v_2 \implies \ldots \implies v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

Step 3: Graph.

$$\text{vector field} = (x, y)\text{-plane w/ vectors in the dimension of } \dot{x}$$
$$\text{phase portrait} = (x, y)\text{-plane w/ trajectories } (x(t), y(t)) \text{ and arrows}$$

General soln: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$ for $c_1, c_2 \in \mathbb{R}.$

Example 31.3 (Cont'd of example 31.2)

So
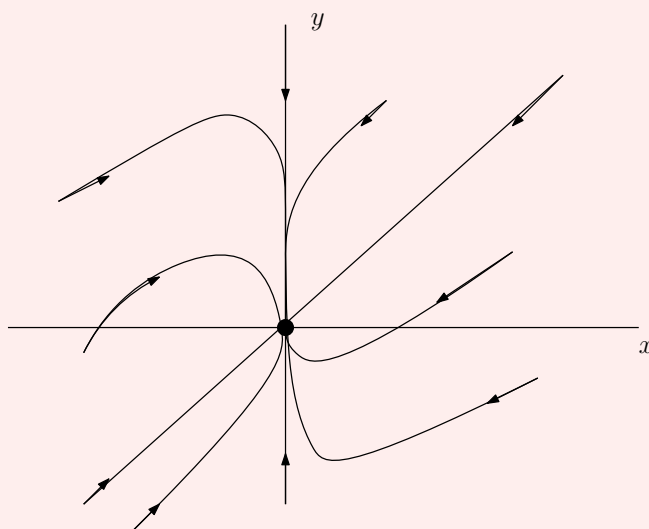
- $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- $c_1 e^{\lambda_1 t} v_1$ solutions straight line towards $(0,0)$.

- Solution is combination, v_1 coefficient decays faster $\implies$ soln approach v_2 as $t \to \infty$ (v_1 as $t \to -\infty$)

$x_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a stable node.

Example 31.4

Same for $\dot{x} = -5x + 2y$ and $\dot{y} = -9x + 6y$

$$\dot{x} = Ax, \quad A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}$$

$$\lambda_1 = -3, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

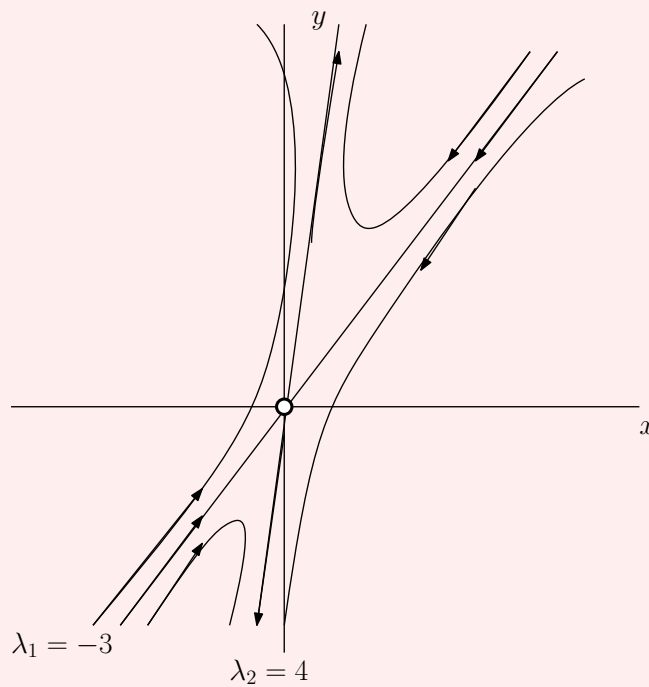$$\lambda_2 = 4, \quad v_2 = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$$

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

v_1 coeff decays, v_2 coeff grows.

$x_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a saddle node.

Example 31.5

Same for $\dot{x} = 3x - 4y$, $\dot{y} = x - y$.

$$\dot{x} = Ax, \quad A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

Evals: $0 = \det(A - \lambda I) = (\lambda - 1)^2 \implies \lambda = 1$

Evec: $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \implies \ldots \implies a = 2b$. Only 1 evec, take $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
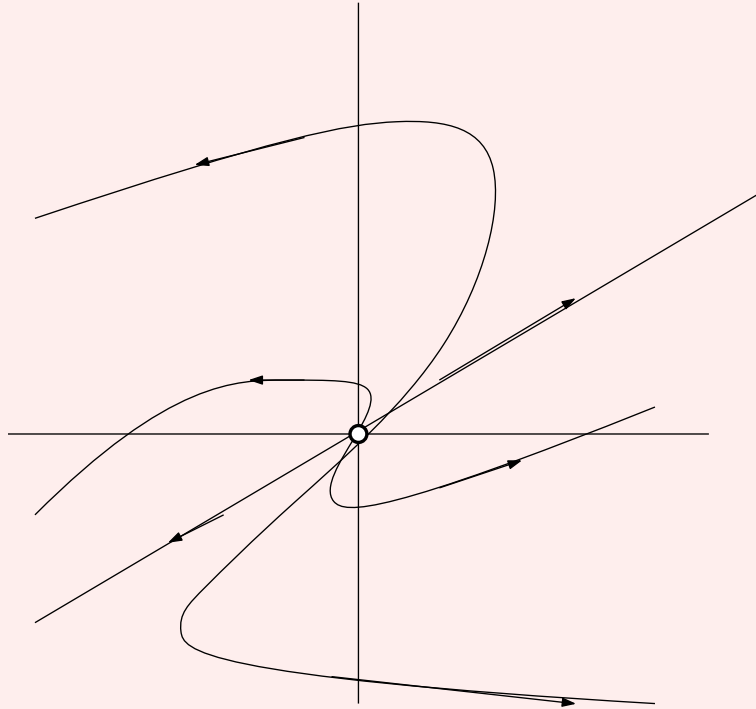
- $ce^{\lambda t}v$ are solutions.

- $A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$ determines rotation direction.

$x_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an unstable degen. node.

§32 | Dis 7: Feb 18, 2021

§32.1   2D – Linear System

Example 32.1

$\dot{x} = 3x - 13y$, $\dot{y} = 5x + y$. Plot phase portrait, classify the fixed point $x_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

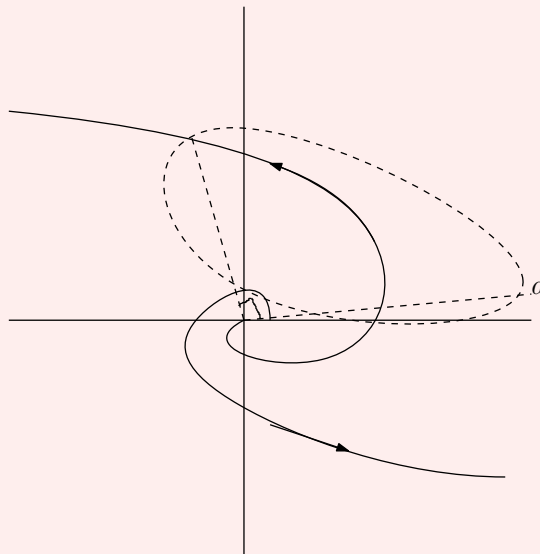$$\dot{x} = Ax, \quad \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}$$

Evals: $\lambda = 2 \pm 8i$. There exists evec $v = a - ib$ for $2 + 8i$ s.t. a, b real are perpendicular. General solution:

$$x(t) = e^{2t} \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \cos(8t) & -\sin(8t) \\ \sin(8t) & \cos(8t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

In particular, taking $c_2 = 0$, then

$$x(t)e^{2t} \left[c_1 \cos(8t)a + c_1 \sin(8t)b \right]$$

is a solution. Rotate $a \to b \to -a \to -b$.

$x_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an unstable spiral.

To find a, b (optional)

- Find one evec:

$$Av = (2 + 8i)v \implies \ldots \implies \text{pick } v = \begin{pmatrix} 13 \\ 9 \end{pmatrix} - i \begin{pmatrix} -13 \\ 7 \end{pmatrix}$$

- Use Lec 15-2 Rmk (ii) to find a new evec $(\gamma + i\delta)v$ with $a \perp b$.

$$(\gamma + i\delta)v = \begin{pmatrix} 5,127.2\ldots \\ 631.1 \end{pmatrix} - i \begin{pmatrix} -384.7 \\ 3,125.6 \end{pmatrix}$$

Remark 32.2.   • Not to scale: from $t = 0$ to $t = 2\pi$, Re $\lambda = 2 \implies$ grow by $e^{2 \cdot 2\pi} \approx$
300,000 and Im $\lambda = 8 \implies$ 8 rotations.

• For spirals on hw, just need correct stability and direction, for which we can find evals
and one vector Ax.

§32.2   2-Dim Linearization

Example 32.3

$\dot{x} = x^2 - 1, \dot{y} = -y$.

a) Find fixed points.

$$0 = \dot{x} = x^2 - 1 \implies x = \pm 1$$
$$0 = \dot{y} = -y \implies y = 0$$
$$\implies x_* = (-1, 0), (1, 0)$$

b) Find and classify linearized systems.

x_* is hyperbolic $=$ all evals of linearized systems at x_* have Re $\lambda \neq 0$. If x_*
hyperbolic $\implies$ phase portrait near x_* looks like linearized system.

$$\dot{x} = f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = \begin{pmatrix} x^2 - 1 \\ -y \end{pmatrix}$$

$$Df(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}$$

We have

$$x_* = (-1, 0) : \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \text{ evals: } -2, -1, \text{ evecs: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
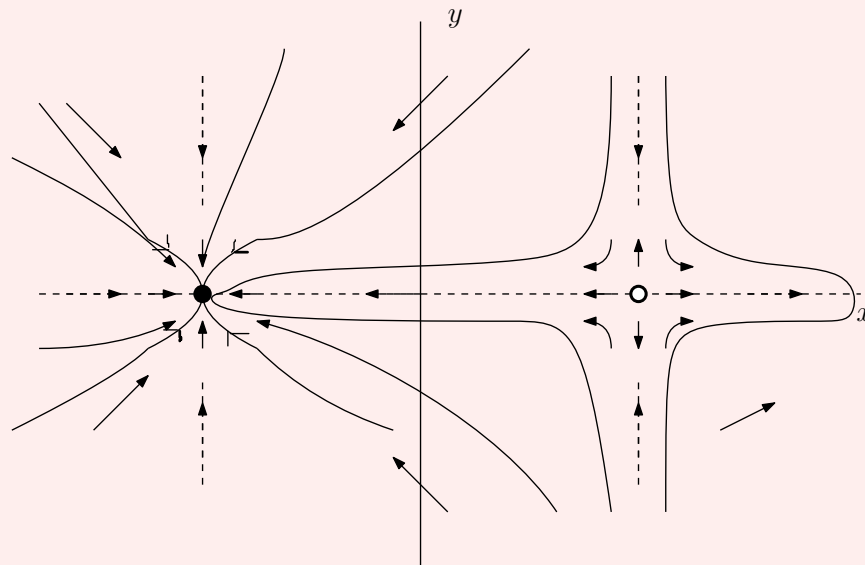
stable node, hyperbolic

$$x_* = (1, 0) : \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \text{ evals: } 2, -1, \text{ evecs: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

saddle node, hyperbolic

Example 32.4 (Cont'd from above)   c) Sketch the phase portrait, determined the fixed point stability

"nullclines" = curves where vector field is horizontal or vertical

$$\text{vertical} : 0 = \dot{x} = x^2 - 1 \implies x = \pm 1$$
$$\text{horizontal} : 0 = \dot{y} = -y \implies y = 0$$

Asymptotically stable = both

- attracting: $x(0)$ near x_* $\implies$ $x(t) \to x_*$ as $t \to \infty$.
- Lyapunov stable: $x(0)$ near x_* $\implies$ $x(t)$ stays near x_*.

Unstable = neither.

$x_* = (-1, 0)$ asymptotically stable.

$x_* = (1, 0)$ unstable.

Remark 32.5. Can be either attracting or Lyapunov stable, too.

Example 32.6

$\dot{x} = x(x - y), \dot{y} = y(2x - y)$

  a) Find fixed points.

$$0 = \dot{x} = x(x - y) \implies x = 0 \text{ or } y = x$$
$$0 = \dot{y} = y(2x - y) \implies y = 0 \text{ or } 2x = y$$

  $\implies x_* = (0,0)$ only fixed point.

  b) Find and classify linearized system.

$$f(x, y) = \begin{pmatrix} x^2 - xy \\ 2xy - y^2 \end{pmatrix}$$

$$Df(x, y) = \begin{pmatrix} 2x - y & -x \\ 2y & 2x - 2y \end{pmatrix}$$
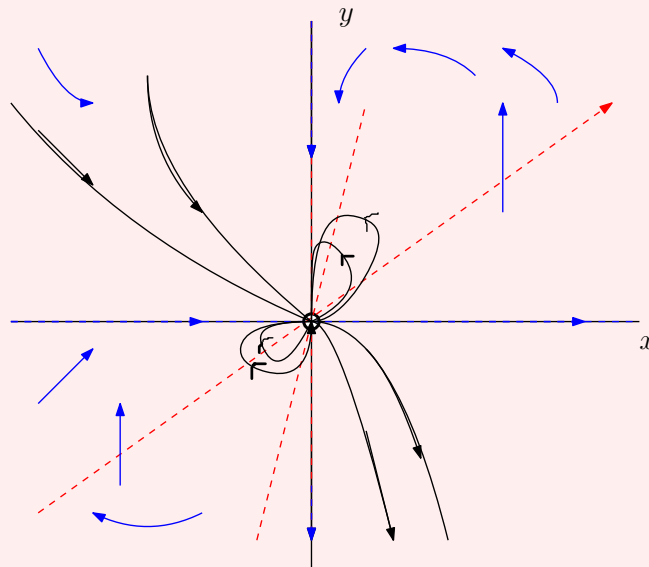
$$Df(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

  Non-isolated fixed point, nonhyperbolic.

  c) Sketch the phase portrait and determine fixed point stability.

$$\dot{x} = x(x - y)$$
$$\dot{y} = y(2x - y)$$

Example 32.7 (Cont'd from above)

Nullclines:

- $x = 0$:

$$\dot{x} = 0$$
$$\dot{y} = -y^2$$

- $y = 0$:

$$\dot{y} = 0$$
$$\dot{x} = x^2$$

- $y = x$:

$$\dot{x} = 0$$
$$\dot{y} = x^2$$

- $y = 2x$:

$$\dot{x} = -x^2$$
$$\dot{y} = 0$$

$x_* = (0, 0)$ is unstable.

§33 | Dis 8: Feb 25, 2021

§33.1   2D-Dim Nonlinear Systems

> **Example 33.1**
>
> $\dot{x} = -4x + y^3, \dot{y} = -3x - y + y^3$
>
> a) Find fixed points
>
> $$0 = \dot{x} = -4x + y^3 \implies x = \frac{1}{4}y^3$$
>
> $$0 = \dot{y} = -3x - y + y^3 \implies x = \frac{1}{3}(y^3 - y)$$
>
> $$\implies x_* = (-2, -2), (0, 0), (2, 2).$$
>
> b) Find and classify linearized systems
>
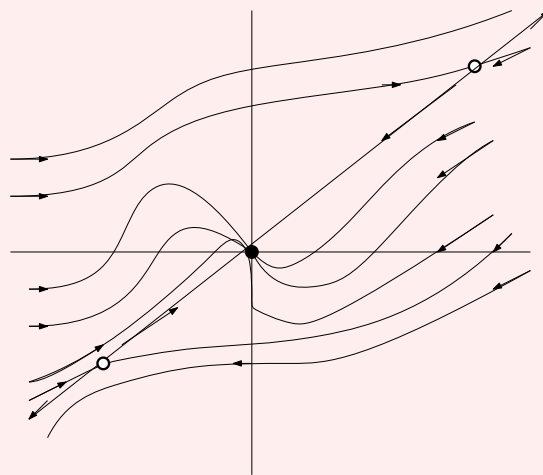> $$f(x, y) = \begin{pmatrix} -4x + y^3 \\ -3x - y + y^3 \end{pmatrix}$$
>
> $$Df(x, y) = \begin{pmatrix} -4 & 3y^2 \\ -3 & -1 + 3y^2 \end{pmatrix}$$
>
> $$Df(-2, -2) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix} \implies \text{ saddle node, hyperbolic}$$
>
> $$Df(0, 0) = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix} \implies \text{ stable node, hyperbolic}$$
>
> $$Df(2, 2) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix} \implies \text{ saddle node, hyperbolic}$$
>
> c) Sketch phase portrait, determine fixed point stabilities.
>
>
>
> $x_* = (-2, -2), (2, 2)$ unstable, $x_* = (0, 0)$ stable.

> **Remark 33.2.** 4 choices for stability:
>
> - stable (both)
> - unstable (either)
> - just attracting
> - just Lyapunov stable (neutrally stable)

Example 33.3

$$\dot{x} = x \left(-3 + \frac{4}{1 + x^2 + y^2} \right) - 4y \left(1 - x^2 - y^2 \right)$$

$$\dot{y} = y \left(-3 + \frac{4}{1 + x^2 + y^2} \right) + 4x(1 - x^2 - y^2)$$

Write in polar coords. Let $x = r \cos \theta$, $y = r \sin \theta$ and $\dot{x} = \frac{\partial x}{\partial r} \dot{r} + \frac{\partial x}{\partial \theta} \dot{\theta}$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}}_{\det = r} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix}$$
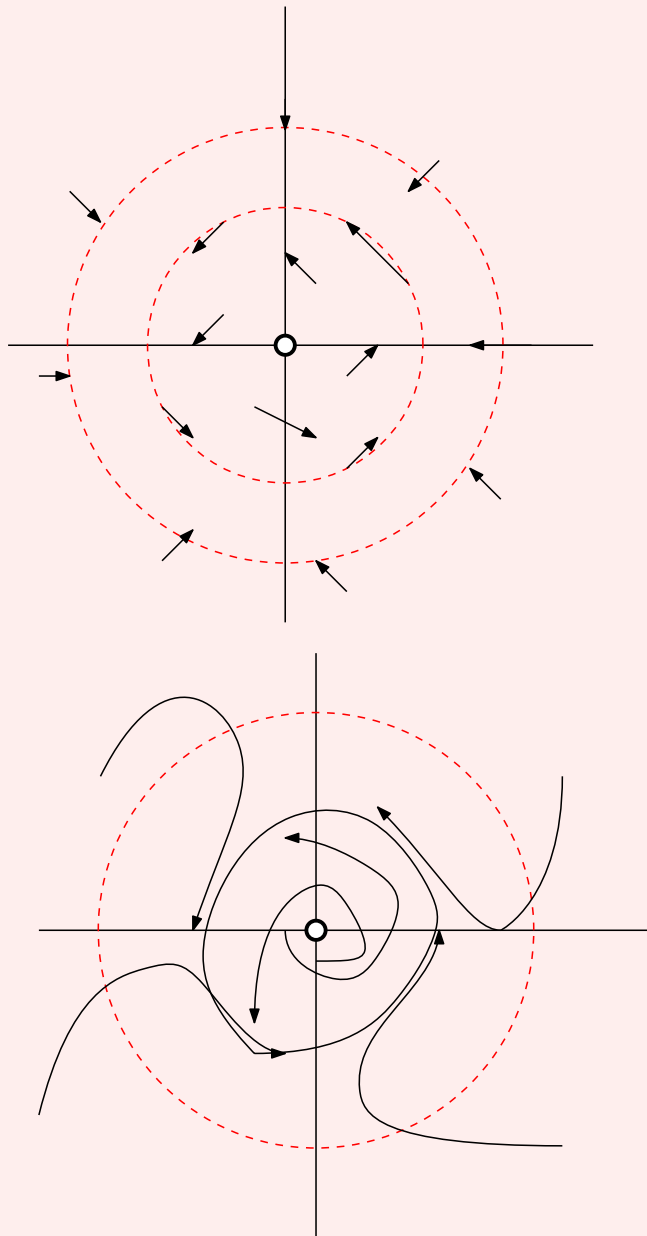
$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \end{pmatrix}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$= \frac{1}{r} \begin{pmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

So,

$$\dot{r} = \cos \theta \dot{x} + \sin \theta \dot{y}$$

$$= \ldots = r \left(-3 + \frac{4}{1 + r^2} \right)$$

$$\dot{\theta} = -\frac{1}{r} \sin \theta \dot{x} + \frac{1}{r} \cos \theta \dot{y}$$

$$= \ldots = 4(1 - r^2)$$

Example 33.4 (Con'td from above)

b) Sketch phase portrait

§34 | Dis 9: Mar 4, 2021

§34.1   Conservative System

There is a non-constant function $E(x)$ that is conserved (i.e. $\frac{d}{dt}E(x(t)) = 0$. Then:

- a) Trajectories lie in level sets of E

- No attracting fixed points

- Strict local min/max of E are centers

Example 34.1

The system $\ddot{x} = -\frac{k}{x^2} + \frac{k^2}{x^3}$ for $x > 0$ describes a point at a distance $x(t)$ from a mass with gravitational constant $k > 0$ and angular momentum $k > 0$
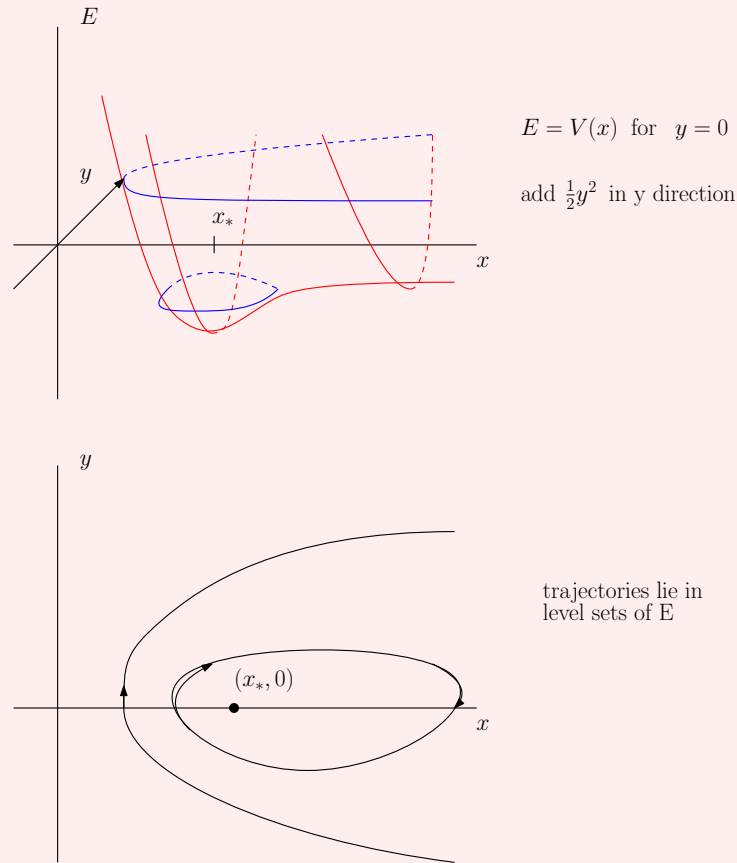
a) Show this system is conservative.

$$\ddot{x} = -\frac{dV}{dx}, \quad V(x) = -\frac{k}{x} + \frac{k^2}{2x^2}$$

Set $y = \dot{x}$, $E(x, y) := \frac{1}{2}y^2 + V(x)$. E is conserved:

$$\frac{d}{dt}E = y\dot{y} + \frac{dV}{dx}\dot{x} = y\left(\frac{-dV}{dx}\right) + \frac{dV}{dx}y = 0$$

b) Sketch the phase portrait. Show that trajectories near the origin are closed.

$E = V(x)$ for $y = 0$

add $\frac{1}{2}y^2$ in y direction

trajectories lie in level sets of E

x_* min of $V(x) \implies (x_*, 0)$ min of $E(x, y) \implies (x_*, 0)$ nonlinear center $\implies$ trajectories near $(x_*, 0)$ are closed. Specifically, if $E_0 = E(x(0))$, then

- $V(x_*) \leq E_0 < 0 \implies x(t)$ is closed orbit (= periodic trajectory)
- $E_0 \geq 0 \implies x(t)$ is unbounded.
- x_* hyperbolic and isolated, i.e. there are no other fixed points nearby.
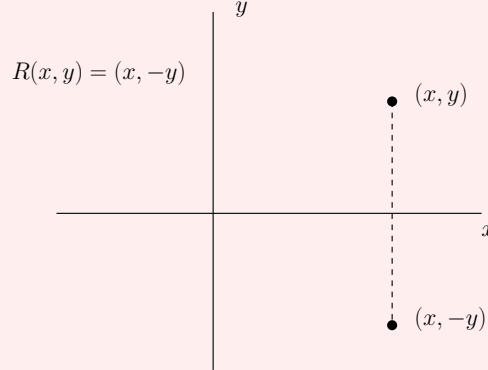
§34.2   Reversible System

There is an involution $R(x)$ such that the system is the same after changing variables $t \to -t$, $x \to \mathbb{R}(x)$. Then

- $R(x)$ reflection $\implies$ trajectories symmetric across line of reflection, arrows reversed

- x_* linear center on line of reflection $\implies$ x_* is nonlinear center.

Example 34.2

$\dot{x} = y - y^3$, $\dot{y} = -x - y^2$

   a) Show the system is reversible with respect to reflection across x-axis.

   Change variables:

$$T = -t, \ X = x, \ Y = -y$$

$$\frac{dX}{dT} = (-1)\frac{dx}{dt} = (-1)(y - y^3) = Y - Y^3$$

$$\frac{dY}{dT} = (-1)^2\frac{dy}{dt} = -x - y^2 = -X - Y^2$$

System for X, Y, T is same as system for $x, y, t \implies$ system is reversible.

   b) Find any classify linearizations.

$$0 = \dot{x} = y - y^3 = y(1 - y^2) \implies y = 0, -1, 1$$

$$0 = \dot{y} = -x - y^2 \implies x = -y^2$$

3 fixed points: $x_* = (0,0), (-1,-1), (-1,1)$

$$Df(x,y) = \begin{pmatrix} 0 & 1 - 3y^2 \\ -1 & -2y \end{pmatrix}$$

$$x_* = (0,0) : \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \text{ evals } \pm i \implies \text{ center}$$

$$x_* = (-1,-1) : \begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix} \implies \text{ saddle node}$$

$$x_* = (-1,1) : \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix} \implies \text{ saddle node}$$

Example 34.3   c) Sketch phase portrait. Identify any closed orbits or heteroclinic orbits
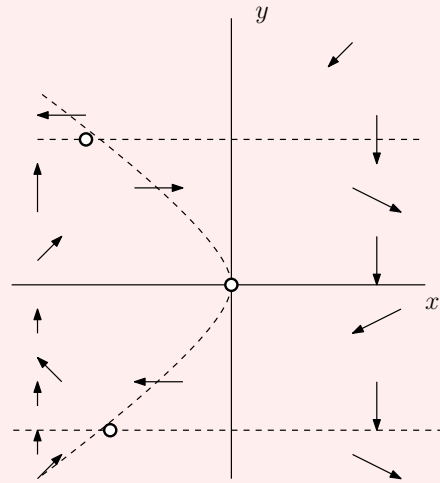
Nullclnes:

$0 = \dot{x} \to y = 0, -1, 1$

$\dot{y} = -x - y^2$

$0 = \dot{y} \to x = -y^2$
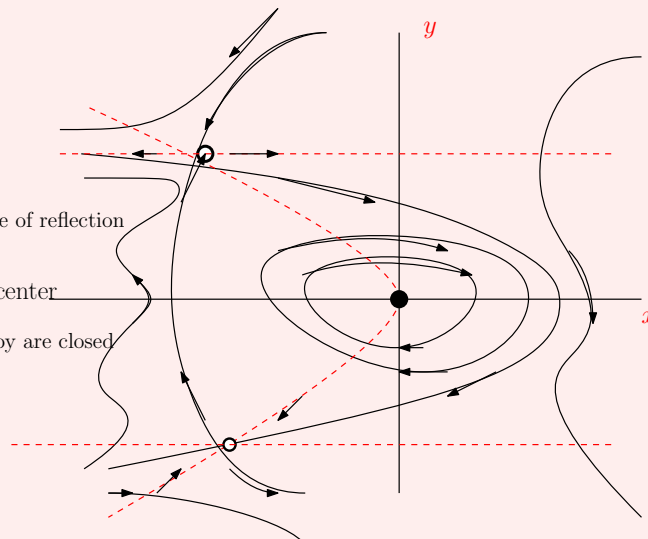
$\dot{x} = y - y^3$

$x_* = (0,0)$

linear center, on line of reflection

$\to$ nonlinear center

$\to$ trajectories nearby are closed

Trajectories are symmetric across x-axis $\implies$ 2 heteroclinic orbits.

§34.3   Index

Index:

- of a closed curve C : go around C (ccv) and count number of (ccw) rotations of vector fields

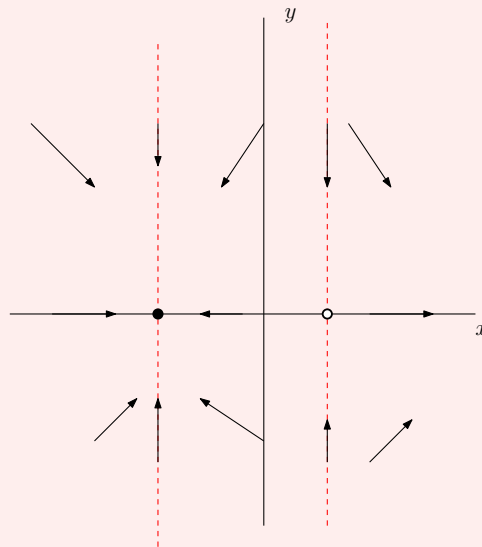- of a fixed point x_* : find closed curve containing only x_*, compute its index.

Example 34.4

Show $\dot{x} = x^2 - 1$, $\dot{y} = -y$ has no closed orbits.
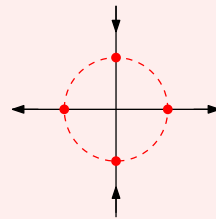
Nullclines:

$0 = \dot{x} = x^2 - 1 \rightarrow x = \pm 1$

$0 = \dot{y} = -y \rightarrow y = 0$

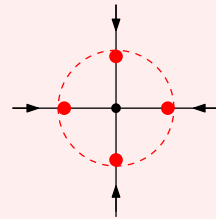From discussion 7: $x_*(1,0)$ saddle node, $x_* = (-1,0)$ stable node.

Index of $(1,0) = $ -1

1 cw rotation:

Index of $(-1,0) = + 1$

1 cw rotation:

Suppose there were a closed orbit $C \implies$ index of $C = +1 = $ sum of enclosed fixed points $\implies C$ encloses $(-1,0)$ and not $(1,0) \implies C$ intersects $x = -1$, a trajectory can't make a full rotation. Contradiction, so C doesn't exist.

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