

# Math 131AH – Honors Real Analysis I

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This is math 131AH – Honors Real Analysis I taught by Professor Greene, and our TA is Haiyu Huang. We meet weekly on MWF from 1:00pm – 2:00pm for lectures. There are two textbooks used for the class, *Principles of Mathematical Analysis* by Rudin and *Metric Spaces* by Copson. You can find other lecture notes at my [github](#) site. Please let me know through my [email](#) if you spot any mathematical errors/typos.

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# §1 | Lec 1: Oct 2, 2020

Overview:

- Hmwrk: 30 %
- Midterm 1: 20 %
- Midterm 2: 20 %
- Final: 30 %

## §1.1 Introduction

functions  $\rightarrow 1, 2, 3, 4, 5, 6, 7 \dots$

functions defined on  $\mathbb{Q}$  with value in  $\mathbb{Q}$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

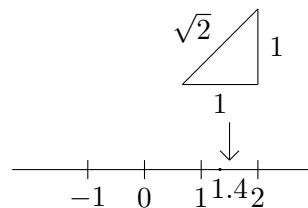
$a_i \in \mathbb{Q}$   $f(x) \in \mathbb{Q}$  if  $x \in \mathbb{Q}$ . Continuity makes sense.

$$x_0, x \text{ close to } x_0 \implies f(x) \text{ close } f(x_0)$$

polynomials are continuous.

Something wrong:  $\sqrt{2}$  is missing. What are these numbers that are not  $\in \mathbb{Q}$ ? Choice:

1. Assume everything works and isolate what you need about "real numbers" (most of Rudin chap 1).
2. Construct the real numbers from rational numbers.



Classical argument:

$$x^2 \neq 2 \text{ if } x = \frac{p}{q} \in \mathbb{Q}$$

*Proof.* Suppose  $\left(\frac{p}{q}\right)^2 = 2$

*Note:* wolog (without loss of generality)

can take  $\frac{p}{q} > 0$   $p > 0$   $q > 0$

$$\left(\frac{p}{q}\right)^2 = 2$$

$$\frac{p^2}{q^2} = 2$$

$$p^2 = 2q^2$$

Now also wolog, can assume  $p$  and  $q$  are not both even numbers. But  $p^2 = 2q^2$  means  $p$  has to be even ( $p^2$  odd if  $p$  is odd).

$$\begin{aligned} p &= 2n \\ p^2 &= 2q^2 \\ 4n^2 &= 2q^2 \end{aligned}$$

So  $q^2 = 2n^2$ ,  $q$  is even. But it contradicts the initial assumption,  $p$  and  $q$  not both even  $\square$

Related to: Why functions  $\mathbb{Q}$  to  $\mathbb{Q}$  not ideal for analysis?  
– INFINITE DECIMAL

## §2 | Lec 2: Oct 5, 2020

### §2.1 Mathematical Induction and More on Real Numbers

$P(n) \rightarrow 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ , where  $n$  is positive numbers.

Math induction: Proof by two steps:

1. Check  $P(1)$  is true  $\checkmark$
2. Assume  $P(n)$  is true for all  $n \leq N$ . Check that

$$P(N+1) \text{ is true}$$

Assume  $1 + \dots + N = \frac{N(N+1)}{2}$ . Check

$$1 + \dots + N + (N+1) = \frac{(N+1)(N+1+1)}{2}$$

Induction on  $k$  :

$$1^k + 2^k + \dots + n^k$$

2<sup>nd</sup> illustration:

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad r \neq 1$$

$$r = 1 \implies 1 + r = \frac{1 - r^2}{1 - r}$$

$$\begin{aligned} 1 + r + r^2 + \dots + r^n + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \\ &= \frac{1 - r^{n+1} + r^{n+1} - r^{n+2}}{1 - r} \\ &= \frac{1 - r^{n+2}}{1 - r} \end{aligned}$$

$$(1 - r)(1 + r + \dots + r^n) = 1 - r^{n+1} \quad \text{Inspection}$$

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r}, \quad r \neq 1$$

$|r| < 1$  get infinite sum  $\frac{1}{1-r}$

**Example 2.1**

Prime factors, prime = positive integers ( $> 1$ ) with no factors except itself and 1,  
 $p = ab$ ,  $a > 1$ ,  $b > 1$

2 3 5 7 11 13 17 19 ...

Thin out as go along

**Theorem 2.2** (Fundamental Theorem of Arithmetic)

Every positive integer  $> 1$  is a product of primes.

*Proof.* Induction:  $P(n)$   $n = 2, 3, \dots$

$$P(2) = 2\checkmark$$

Assume  $P(n) \dots n \leq N$  ( $N > 2$ ). Every integer greater than 1 but smaller than or equal to  $N$  as a product of primes. We try to prove:  $N + 1$  is a product of primes.

1.  $N + 1$  is prime: Done  $N + 1 = N + 1$

2.  $N + 1$  is not a prime

$$N + 1 = a \cdot b \quad a > 1 \quad b > 1$$

Induction assumption ( $a < N + 1$  since  $b > 1$ ),  $a$  is a product of primes  $a > 1 \implies b < N + 1$ ,  $b$  also a product of primes. So,  $N + 1 = ab$  is a product of primes.

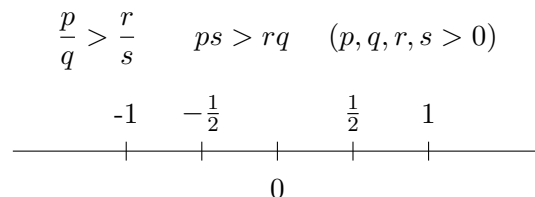
$N + 1 = ab$  is a product of prime. □

Why does induction work? If  $P(n)$  not always true,  $P(n)$  look at smallest  $n$  where  $P(n)$  is false.

$n = 1$  not there  $P(1)$  is supposed true (checked already).  $N_0$  smallest one where  $P(N_0)$  false  $N_0 > 1$ . Induction step says that  $P(n)$  is true for all  $n \leq \underbrace{N_0 - 1}_{>0} \implies P(N_0)$  true ( $\times$ ).

Let's go back to real numbers.

Last time: talked about  $\sqrt{2}$  is irrational but  $\sqrt{2}$  exists, so we need to enlarge our number system:  $\mathbb{Q}$  rational numbers.



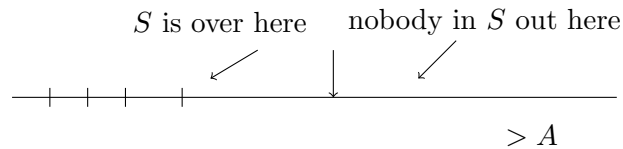
$x, y$  rational  $x, y > 0$ ,  $x + y > 0$ ,  $xy > 0$

$x^2 = 2$  no answer in  $\mathbb{Q}$ . Enlarge number system,  $\mathbb{Q} \subset \mathbb{R}$ . What should  $\mathbb{R}$  be like?

1.  $\mathbb{R}$  ought to have arithmetic like  $\mathbb{Q}$

$$x + y \quad xy \quad \frac{x}{y} \quad 0 \quad 1$$

2.  $\mathbb{Q} \subset \mathbb{R}$ , arithmetic in  $\mathbb{R}$  restricted to  $\mathbb{Q}$ ,  $\frac{1}{2} + \frac{1}{3}$  in  $\mathbb{Q}$  ought to be  $\frac{5}{6}$  in  $\mathbb{R}$ .
3. Order should positive in  $\mathbb{Q} \implies$  in  $\mathbb{R}$ .  $\mathbb{R}$  should have an order of its own too,  $x > y$  positive then  $x + y$  pos and  $xy$  pos.
4. want to fill in the holes in  $\mathbb{Q}$ . Want to have **Least Upper Bound Property**  
 $S \subset \mathbb{R}$  : An upper bound for  $S$  is a number  $A$  with property  $A \geq x$  if  $x \in S$



$1, 2, 3, 4, \dots$  have no upper bound.

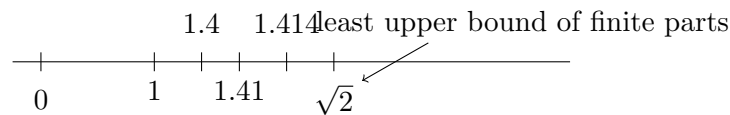
$S$  is bounded above means that some upper bound  $A$  exists.

## §2.2 Least Upper Bound Property

If  $S$  is bounded above ( $S \neq \emptyset$ ) then it has a “least upper bound” where a number  $A_0$  is called the least upper bound of  $S$  if  $A_0$  is an upper bound for  $S$  & if  $A$  is an upper bound for  $S$  then  $A_0 \leq A$ .



Motivation: Think about  $\sqrt{2}$



Denote: l.u.b(or supremum)(sequence) =  $\sqrt{2}$

Means can define an infinite decimals: least upper bound of successive truncation.

$$0.99999 \dots \rightarrow 1.0$$

## §3 | Lec 3: Oct 7, 2020

### §3.1 Cauchy Sequence

$\{x_n\}$   $x_1, x_2, x_3, \dots$  values  $x_j \in \mathbb{Q}$   $x_j \in \mathbb{R}$   
 $S$   $x_1, x_i \dots x_j \in S$

**Definition 3.1 (Sequence)** — A sequence with values in a set  $S$  is a function from positive integers  $\{1, 2, 3 \dots\}$  into  $S$ .

**Definition 3.2 (Cauchy Sequence)** — A Cauchy sequence is ( $\mathbb{Q}$  valued or  $\mathbb{R}$  valued)  $\{x_i\}$  is sequence s.t. for every  $\epsilon > 0$  there is a positive integer  $N_\epsilon$  s.t.

$$|x_i - x_j| < \epsilon \quad \text{if } i, j > N_\epsilon$$

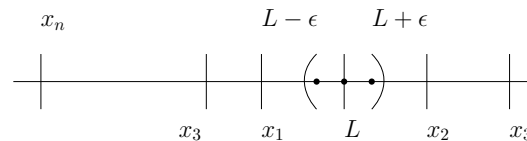


$\epsilon$  rational or real (same idea).

### Lemma 3.3

If  $\{x_j\}$  has a finite limit then it's a Cauchy sequence.

$\{x_i\}$  has  $L$  as a limit  $\lim x_j = L$  means for every  $\epsilon > 0$  then there is an  $N_\epsilon$  such that  $j \geq N_\epsilon, |x_j - L| < \epsilon$

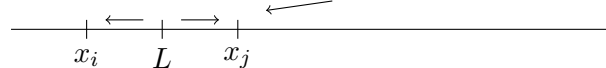


Everybody in  $(L - \epsilon, L + \epsilon)$  except a finite number

*Proof.* Given  $\epsilon > 0$ , want to find  $N$  so that  $i, j \geq N \implies |x_i - x_j| < \epsilon$   
 $|x_i - L|$  small,  $|x_j - L|$  small and  $\lim x_j = L$ .

$$|x_i - x_j| \leq |x_i - L| + |x_j - L|$$

$$|x_i - x_j| = |L - x_i| + |L - x_j|$$



$i, j \geq N_{\frac{\epsilon}{2}}$  :

$$|x_i - x_j| \leq \underbrace{|x_i - L|}_{< \frac{\epsilon}{2}} + \underbrace{|x_j - L|}_{< \frac{\epsilon}{2}}$$

Because  $\lim x_n = L$ , there is an  $N_{\frac{\epsilon}{2}}$  s.t.  $|L - x_n| < \frac{\epsilon}{2}$  if  $n \geq N_{\frac{\epsilon}{2}}$

Get  $|x_i - x_j| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$  if  $i, j \geq N$ . Cauchy sequence: there exists number  $N$  s.t.

$$|x_i - x_j| < \epsilon \quad \text{if } i, j \geq N$$

□

Cauchy sequence  $\implies$  the existence of limit? Yes, for  $\mathbb{R}$  valued sequences but NO for  $\mathbb{Q}$  valued things.

$\underbrace{\{x_n\}}_{\text{rational numbers}}$  can be Cauchy seq without there being a rational number  $L$  such that  $\lim x_j = L$

But allow real  $L$  then  $\exists L$  s.t.  $\lim x_j = L$  if  $\{x_j\}$  is Cauchy sequence (no rational limit – since  $\sqrt{2}$  is irrational). Because  $\mathbb{Q}$  has holes in it! (intuitive idea).



**Example 3.4**

1, 1.4, 1.41, 1.414, 1.4142... (decimal approx of  $\sqrt{2}$ ) – Cauchy sequence. No – since  $\sqrt{2}$  is irrational.

**§3.2 Cauchy Completeness of  $\mathbb{R}$** 

If  $\{x_j\}, x_j \in \mathbb{R}$  is Cauchy sequence, then  $\exists L \in \mathbb{R}$  s.t.  $\lim x_j = L$ .

“ $\mathbb{Q}$  is not Cauchy complete” but  $\mathbb{R}$  is. Why does this work?

Need: Least upper bound property. Assume L.U.B Property proof.

*Proof.* (Cauchy completeness from L.U.B Property)

Hypothesis:  $\{x_i\}$  Cauchy seq

1. Prove that  $\{x_i\}$  bounded  $\iff \exists M > 0$  s.t.  $|x_i| \leq M$  all  $i$ .

Clear if take  $\epsilon = 1$  in def. of Cauchy seq  $\exists N$  s.t.  $|x_i - x_j| < 1$  if  $i, j \geq N \implies |x_N - x_j| < 1$  if  $j \geq N \implies |x_j| \leq |x_N| + 1 \quad j \geq N$

So,  $M = \max(|x_N| + 1, |x_1|, \dots, |x_{N-1}|)$  then  $|x_j| \leq M$  all  $j$  !

Next stage is to show that a bounded sequence always has a subsequence(tricky!) with a limit. Then if a Cauchy seq has a subseq with limit  $L$ , then  $L$  is limit of whole seq. (Bolzano – Weierstrass Theorem)

□

**§4 | Lec 4: Oct 9, 2020****§4.1 Bolzano – Weierstrass Theorem**

– implied by Least Upper Bound Property

**Theorem 4.1 (Bolzano – Weierstrass)**

If  $\{x_n\}$  sequence  $(x_1, x_2, x_3 \dots)$  that is bounded (means:  $\exists M > 0 \ni |x_n| \leq M \forall n$ ), then  $\exists L$  and a subsequence  $\{x_{n_i}\}$  s.t.  $\lim x_{n_i} = L$ .

Slogan: Every bounded sequence has a convergent subsequence.

**Example 4.2**

1, 2, 1, 2, 1, 2, ...

The subsequence of the above sequence has either 1 or 2 as the limit.

1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, ...

Unbounded sequence – subsequence (limit 1, limit 2, limit 3...)

**No claim of uniqueness of anything.**

Proof – Summer 2008 Analysis Lec 4

*Proof.* So either  $[-M, 0]$  or  $[0, M]$  (maybe both) contains  $x_n$  for infinitely many  $n$  values. If each contained  $x_n$  for only finitely many  $n$  values  $X$ .

$$\begin{array}{c} -M \qquad \qquad \qquad 0 \qquad \qquad \qquad M \\ | \qquad \qquad \qquad | \qquad \qquad \qquad | \\ \hline \end{array}$$

Every  $x_n$  is in  $[-M, M] - \{x_n\}$  is bounded

$$[-M, M] = [-M, 0] \cup [0, M]$$

$$I_1 = [-M, 0] \quad \text{or} \quad [0, M]$$

where chosen interval has  $x_n$  for infinitely many  $n$  values.

Do this again!

$$\begin{array}{c} I_1 = [a_1, b_1] \qquad |b_1 - a_1| = M \\ I_1 \longleftarrow \text{length} \\ | \qquad \qquad \qquad | \qquad \qquad \qquad | \\ \hline \end{array}$$

left half of  $I_1$ , right half of  $I$ . Let  $I_2 =$  one of halves that contains  $x_n$  for infinitely many  $n$  values.

$$I_2 = [a_2, b_2] \qquad a_2 < b_2, \quad b_2 - a_2 = \frac{M}{2}$$

Continue

$$I_3 = [a_3, b_3] \qquad a_3 < b_3, \quad b_3 - a_3 = \frac{M}{4}$$

$$\vdots$$

$$I_k = [a_k, b_k] \qquad b_k - a_k = \frac{M}{2^{k-1}}$$

Each  $I_k$  contains  $x_n$  for infinitely many  $n$  values.

$$\begin{array}{c} \text{Nested Intervals} \\ a_1 \qquad \qquad I_1 \qquad \qquad b_1 = b_2 \\ | \qquad \qquad | \qquad | \qquad | \qquad | \\ \hline \qquad \qquad \nearrow \qquad \qquad \nwarrow \\ \qquad a_3 \qquad \qquad b_3 \\ I_{k+1} \subset I_k \subset \dots \subset I_1 \subset [-M, M] \\ a_{k+1} \geq a_k \dots \qquad b_{k+1} \leq b_k \dots \end{array}$$

Claim  $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$

Reason:  $\sup a_k \in \bigcap_{k=1}^{\infty} I_k$  where  $\sup =$  sup of left hand endpoint (=greatest lower bound of bs). l.u.b of  $a$ 's  $\leq b_k$ ,  $b_k$  bigger than or  $\geq$  all  $a$ 's.

$$\alpha = \text{lub } a\text{'s}$$

$$\alpha \geq a_k \quad \forall k$$

$$\alpha \leq b_k \quad \forall k$$

$$\alpha \in [a_k, b_k]$$

Goal:  $\alpha \in \bigcap_{k=1}^{\infty} I_k$ . Find a subsequence of  $\{x_n\}$  converges to  $\alpha$ .

Choose  $x_k = x_n$  that belongs to  $I_k$ . Can also arrange successively:

$$n_1 < n_2 < n_3 < n_4$$

$x_{n_1} \in I_1$   $x_{n_2} \in I_2$  can make  $n_2 > n_1$  because infinitely possible  $x'_n$ s in  $I_2$  n value.  
Continue to get subsequence,  $\{x_{n_k}\}$  subsequence. Claim:

$$\lim_{k \rightarrow \infty} x_{n_k} = \alpha$$

Reason:

$$\text{dis}(x_{n_k}, \alpha) \leq \text{length of } I_k \quad \alpha \in I_k, \quad x_{n_k} \in I_k$$

which is equivalent to

$$|x_{n_k} - \alpha| \leq \frac{M}{2^{k-1}} \quad \text{given } \epsilon > 0$$

When  $k$  is large,

$$\frac{M}{2^{k-1}} < \epsilon$$

So  $|x_{n_k} - \alpha| < \epsilon$

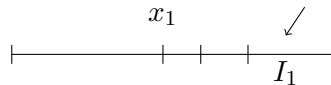
□

This argument (or a variant) shows something else:

If  $\{x_n\}$  sequence in  $[0, 1]$  then there's an  $\alpha \in [0, 1]$  with it never happening that

$$x_n = \alpha$$

“The real numbers in  $[0, 1]$  are uncountable.” (come from the least upper bound property)



$I_1$  one of  $[0, \frac{1}{3}]$   $[\frac{1}{3}, \frac{2}{3}]$   $[\frac{2}{3}, 1]$  such that  $x_1 \notin I_1$ ,

$$[0, \frac{1}{3}] \cap [\frac{1}{3}, \frac{2}{3}] \cap [\frac{2}{3}, 1] = \emptyset$$

$x_1 \notin I_2$   $I_2 \subset I_1$ , &  $x_1 \notin I_1$ . Continue. Get

$$I_1 \supset I_2 \supset I_3 \supset \dots$$

length  $I_k = \frac{1}{3^k}$  and  $I_k$  is such that  $x_1, x_2, x_3 \dots x_k$  are none of the ?n? in  $I_k$ . Same as before

$$\exists \alpha \in \bigcap_{k=1}^{\infty} I_k$$

$\alpha = \sup$  of set of left hand endpoints of  $I_k$ . Claim  $\alpha$  cannot be an  $x_N$  value. Clear:  $x_N \notin I_N$  but  $\alpha \in I_n$   $\alpha \in \bigcap_{n=1}^{\infty} I_n$ . But contrast:

There is a list of rational numbers in  $[0, 1]$

	$\frac{p}{q}$	$p < q$				
	2	3	4	5	6	...
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$			
2	-	$\frac{2}{3}$	$\frac{2}{4}$			
3	-	-	$\frac{3}{4}$			
$\vdots$	-	-	$\frac{\sqrt{2}}{2} \in [0, 1] \rightarrow$	irrational - no exist		
			$[0, 1]$	<div> <div>not</div> <div>countable</div> </div>		
Q is countable						

## §5 | Lec 5: Oct 12, 2020

### §5.1 Equivalence Relation

(p.10, Copson – Metric Space)

$R$  set, relation of  $A$  and  $B$  ( $A \times B$ )  $(a, b) \in R \implies aRb$

Functions: one  $b$  given  $a$  – exact one. ( $A \rightarrow B$ )

#### Example 5.1

$A = B = Q$

$aRb$  or  $(a, b) \in R$  if  $a > b$

(mother, child)

- $(\text{Sara}, \text{Sebastian}) \in R$
- $(\text{Sara}, \text{Alita}) \in R$

Equivalence is a special kind of relation: (on a set  $A; B \subseteq A \times A$ )

Properties:

1.  $aRa \implies A = Q$
2.  $aRb \implies bRa$
3.  $aRb \ \& \ bRc$  then  $aRc$

Example:  $\mathbb{Z}$   $a \sim b$  means  $a - b$  is divisible by 5

$$1 \sim 6 \quad 0 \sim 5 \dots$$

$$a \sim a \quad a - b \text{ div } 5 \implies b - a \text{ div. by } 5.$$

If  $a - b$  div. by 5, and  $b - c$  div by 5, then is  $a - c$  div. by 5 true?

$$\text{Sure, } a - b = 5k, \quad b - c = 5l \implies a - c = 5(k + l)$$

“Equivalence classes”: set  $[a] = \{ \text{all } b \text{ such that } aRb \}$

In the example above,  $[a] = \{ \text{all } b \text{ such that } a - b \text{ div. by } 5 \}$

$$[2] = \{2, 7, -3, 12, -8, \dots\}$$

$\mathbb{Z}_5$  : integer mod 5.

1.  $[a] \cap [p]$  either equal or have nothing in common.
2.  $a \in [a]$  so is in some equivalence class.

A equivalence relation  $\sim$  on  $A \leftrightarrow$  a partition of  $A$  into subsets which are pairwise disjoint.

$\mathbb{Q}$  Cauchy seq. of rational numbers

$$\{x_n\} \sim \{y_n\}$$

means  $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$ . Equivalence relation:

1.  $\{x_n\} \sim \{x_n\}$  ( $\lim(x_n - x_n) = 0$ )
2.  $\{x_n\} \sim \{y_n\} \implies \{y_n\} \sim \{x_n\}$
3.  $\{x_n\} \sim \{y_n\} \& \{y_n\} \sim \{z_n\} \implies \{x_n\} \sim \{z_n\}$

Idea: Define a real number to be a (Cauchy seq. of rationals) equivalence class.

Homework: want to check that arithmetic extends to “real numbers”

$$[\{x_n\}] + [\{z_n\}] = [\{x_n + z_n\}]$$

Check that

1.  $\{x_n + z_n\}$  is a Cauchy seq.
2. Only depends on equivalence classes.

Want

$$\{x_n\} \sim \{y_n\} \quad \{z_n\} \sim \{w_n\}$$

then  $\{x_n + z_n\} \sim \{y_n + w_n\}$ . So,

$$[\{x_n + z_n\}] = [\{y_n + w_n\}]$$

**Example 5.2**

$$[2] + [11] = [2 + 11] = [13]$$

So,  $[2 + 1] \sim [13]([11] = [1])$ . Arithmetic (addition) in  $\mathbb{Z}_5$  thus makes sense. How about multiplication?  $\frac{[1]}{[a]} \leftarrow$  exists  $[a] \neq 0$ .

$$\frac{[1]}{[2]} = [3] \quad [2][3] = [6] = [1]$$

Thus,  $\mathbb{Z}_5$  is a field.

$\frac{p}{q} \sim \frac{r}{s}$ ,  $q, s \neq 0$  means  $ps = rq$  (when talking about fractions – associate it with equivalence relation).  $Q$  = set of equivalence classes.  $(\frac{p}{q})$  : equivalence classes).

Last time, we proved that Cauchy seq. of real numbers have limits (lub property). Also, no sequence  $\{x_n\}$  such that it hits all real numbers in  $[0, 1]$  – this is important. Contrast with  $Q \cap [0, 1]$ , then there is a sequence that hits them all. Refer to the last figure in Lec 4 or [math.ucla.edu/~greene](http://math.ucla.edu/~greene) – Summer 2008.

## §6 | Lec 6: Oct 14, 2020

Bolzano - Weierstrass:

Every bounded sequence has a convergent subsequence.

And we know about the Least Upper Bound Prop.

### §6.1 Continuous Functions on Closed Interval

$$f : S \rightarrow \mathbb{R}, \quad S \subset \mathbb{R}$$

**Example 6.1**

$$S = [a, b]$$

$$S = \mathbb{R}$$

**Definition 6.2** (Continuity) —  $s_0 \in S$ ,  $f$  is continuous at  $s_0$  if given  $\epsilon > 0$ ,  $\exists \delta > 0$  s.t.

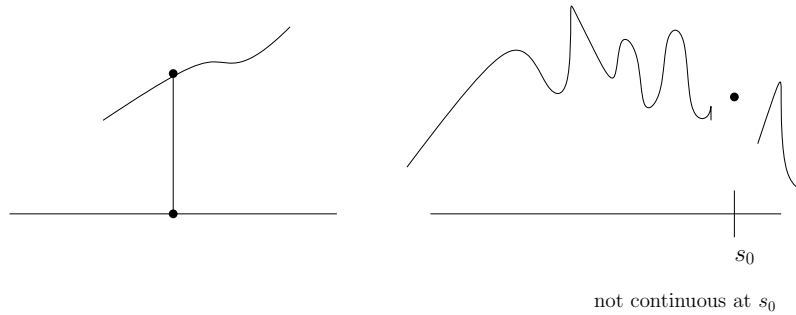
$$|s - s_0| < \delta_\epsilon \implies |f(s) - f(s_0)| < \epsilon$$

Three properties:

$$f : [a, b] \rightarrow \mathbb{R}$$

$f$  continuous

1.  $f$  is bounded on  $[a, b]$  means  $\exists M$  s.t. for all  $x \in [a, b]$ ,  $|f(x)| \leq M$



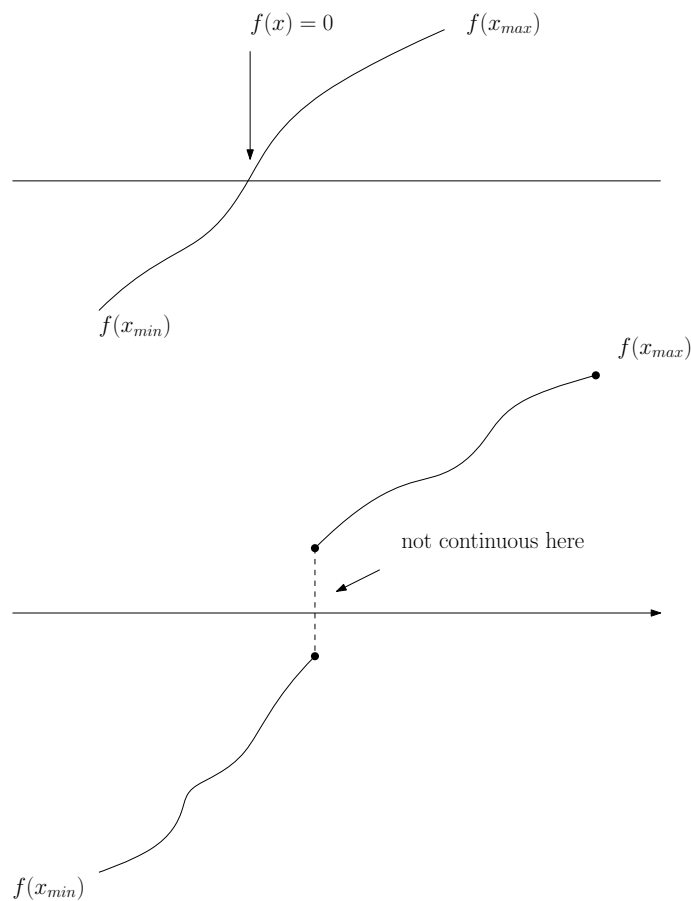
2. There exists  $x_{\min}, x_{\max} \in [a, b]$  such that for all  $x \in [a, b]$

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

Slogan:  $f$  attains its maximum and minimum.

3. If  $\alpha, f(x_{\min}) < \alpha < f(x_{\max})$ , then  $\exists x \in S = [a, b]$  s.t.  $f(x) = \alpha$ .

“Intermediate Value Theorem” Need the least upper bound prop – “completeness of



real numbers”

Exercise: def of continuity  $\{s_n\}$  converges to  $s_0 \iff$  if  $s_n \rightarrow s_0, s_n \in S, s_0 \in S$  then  $\{f(s_n)\}$  converges to  $f(s_0)$ .

**Example 6.3**

For (3),

$$f(x) = x^2 - 2 \quad \text{on } \mathbb{Q} \cap [1, 2]$$

Then  $f(1) = -1$ ,  $f(2) = 2$ , but no rational  $x \in [1, 2]$  s.t.  $f(x) = 0$ .

Back to the properties:

1.  $f$  is bounded – Think about  $|f| \leftarrow$  continuous if  $f$  is (exercise).

$\exists M$  such  $|f(x)| \leq M$  all  $x \in [a, b]$ . Suppose no such  $M$  exists.

Try  $M = 1, 2, 3, 4, 5, 6, \dots$  So  $\exists x_1 \quad |f(x_1)| > 1$

$$|f(x_2)| > 2$$

$\vdots$

$$|f(x_n)| > n$$

But Bolzano – Weierstrass: subsequence  $\{x_{n_j}\}$  that converges to  $x_0$  say  $|f(x_0)| \leftarrow$



finite number. So  $\exists N \ni |f(x_0)| \leq N$ .

Now for  $j$  large enough

$$|f(x_{n_j}) - f(x_0)| < 1$$

$x_{n_j}$  converges to  $x_0$

$$|f(x_{n_j})| < |f(x_0)| + |f(x_{n_j} - f(x_0))|$$

So  $j$  is large enough that

$$\underbrace{|f(x_{n_j})|}_{\geq |f(x_0)|} \leq N + \text{something less than } 1 \leq N$$

2. Attains max and min

Similar:  $\{f(x) : x \in [a, b]\}$  bounded set, has sup where

$$\sup \{f(x) : x \in [a, b]\}$$

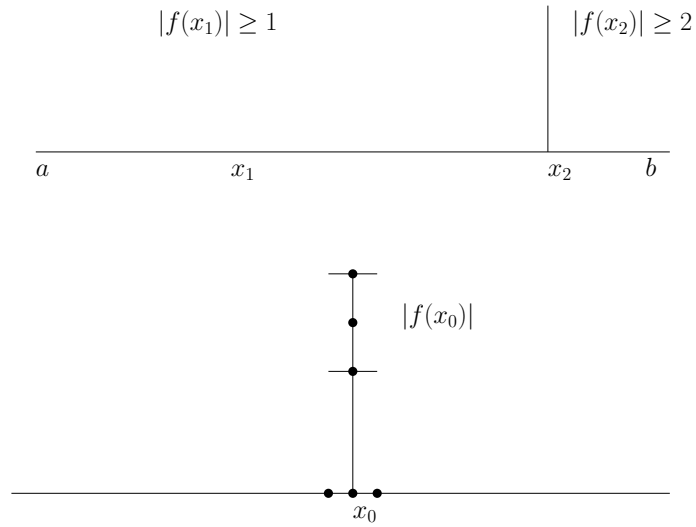
either in the set of  $f$ -values (done if that's true),  $\sup f = f(x_0)$ .

OR:  $\sup f$  actually not in the set  $\{f(x) : x \in [a, b]\}$

Now  $\{x_{n_j}\}$  converges to  $x_0 \in [a, b]$

**Claim 6.1.**  $f(x_0) = \sup \{f(x) : x \in [a, b]\}$

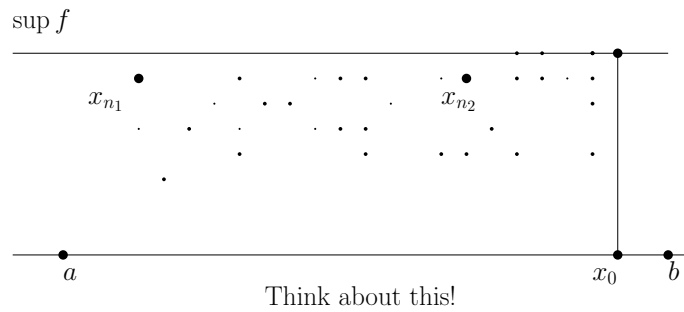




$$f(x_{n_j}) \leq \sup \{f(x) : x \in [a, b]\}$$

and  $\lim f(x_{n_j}) = f(x_0) = f(\lim x_{n_j})$ . So

$$f(x_0) = \sup f$$

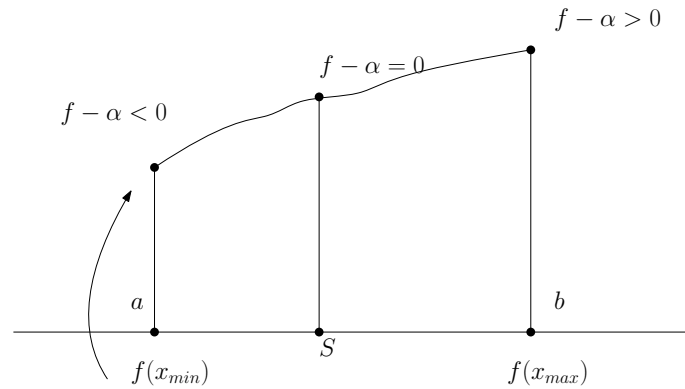


3.  $\alpha \in [f(x_{\min}), f(x_{\max})]$  then  $x$  such that  $f(x) = \alpha$ .

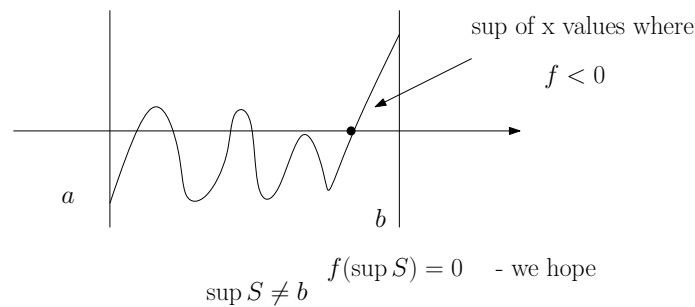
*Proof.* Wolog:

$$f(a) < 0 \quad \text{and} \quad f(b) > 0$$

then  $\exists x \in [a, b]$  with  $f(x) = 0$ .

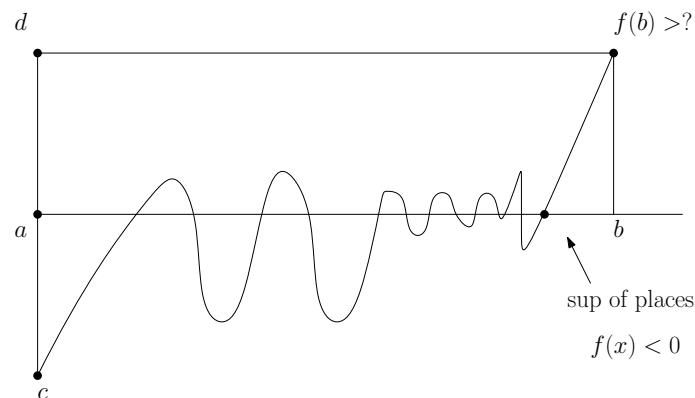


Use l.u.b: Look at  $S : \{x : f(x) < 0\}$  and  $S \neq \emptyset$  because  $f(a) \in S$ . Also,  $S$  is bounded above  $\rightarrow \exists$  l.u.b for  $S$ ,  $\sup S \in [a, b]$ . Hope that  $f(\sup S) = 0$ .



$\sup S \neq b$  is clear because  $f(b) > 0$  so  $f(b - \epsilon) > 0$  for small  $\epsilon$ .

So  $\sup S = x_0$ ,  $a < x_0 < b$ . What is  $f(x_0)$ ? If it's negative, then there are slightly bigger  $x \in [a_0, b] \ni f(x) < 0$  (continuity). In addition,  $x_0$  cannot be a limit of  $x$  with  $f(x) < 0 \rightarrow x_0 = \sup$  places where  $f < 0$ .  $\square$



$f$  continuous on  $[a, b]$  if it is

1. bounded.
2. attains max and min.
3. attains every value between max value and min value.

$f([a, b]) = [c, d]$  where  $c$  is min of  $f$  and  $d$  is max of  $f$ .

## §7 | Lec 7: Oct 16, 2020

### §7.1 Uniform Continuity

**Definition 7.1 (Uniform Continuity)** —  $S \subset \mathbb{R}$ ,  $f : S \rightarrow \mathbb{R}$ .  $f$  is uniformly continuous on  $S$  if given  $\epsilon > 0$  there is a  $\delta > 0$  s.t.  $|f(x) - f(y)| < \epsilon$  if  $x, y \in S$  and  $|x - y| < \delta_\epsilon$

#### Example 7.2

$f : S \rightarrow \mathbb{R}$ ,  $S = \mathbb{R}$ ,  $f(x) = x^2$ . Continuous on  $\mathbb{R}$  but it is not uniformly continuous on  $\mathbb{R}$ .

Continuity: Given fixed  $x$ , and  $\epsilon > 0$  want  $\delta$  so that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

$|x^2 - y^2| = |x - y||x + y|$  and want it smaller than  $\epsilon$ . Assume  $\delta \leq 1$ .

$$|x + y| \leq |x| + |y|$$

$$|y| < |x| + 1 \text{ if } |x - y| < \delta (\leq 1)$$

So, if  $|x - y| < \delta (\leq 1)$ ,

$$\begin{aligned} |x^2 - y^2| &= |x - y||x + y| \\ &\leq |x - y|(2|x| + 1) \end{aligned}$$

Choose  $\delta < \frac{\epsilon}{2|x|+1}$  (ok since  $x$  is fixed)

$$\begin{aligned} |x^2 - y^2| &< \frac{\epsilon}{2|x|+1} (2|x|+1) \\ &= \epsilon \text{ if } |x - y| < \min \left\{ 1, \frac{1}{2|x|+1} \right\} \end{aligned}$$

Uniform continuity does not work on  $\mathbb{R}$ .

**Claim 7.1.**  $\epsilon = 1 > 0$ , there is no  $\delta > 0$  s.t.  $|x^2 - y^2| < 1 = \epsilon$  for all  $x, y$  with  $|x - y| < \delta$ .

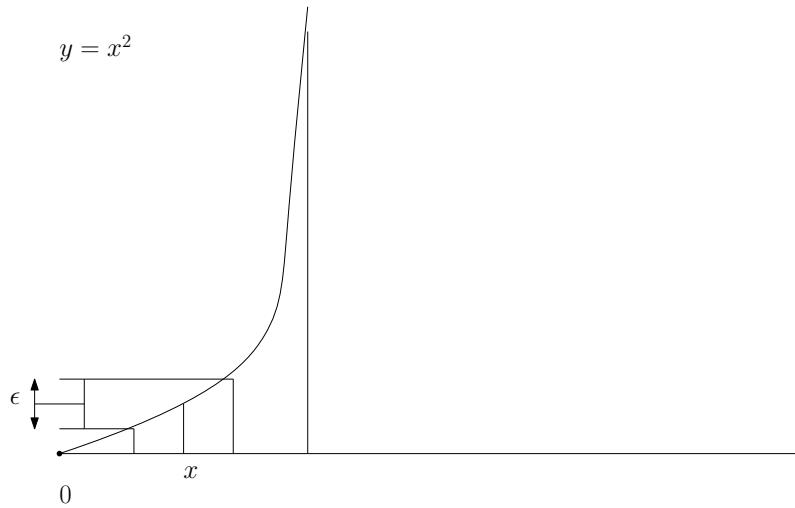
Why? Look at for  $\delta > 0$ , consider  $y = \frac{1}{\delta} + \frac{\delta}{2}$ ,  $x = \frac{1}{\delta}$

$$|x - y| < \delta$$

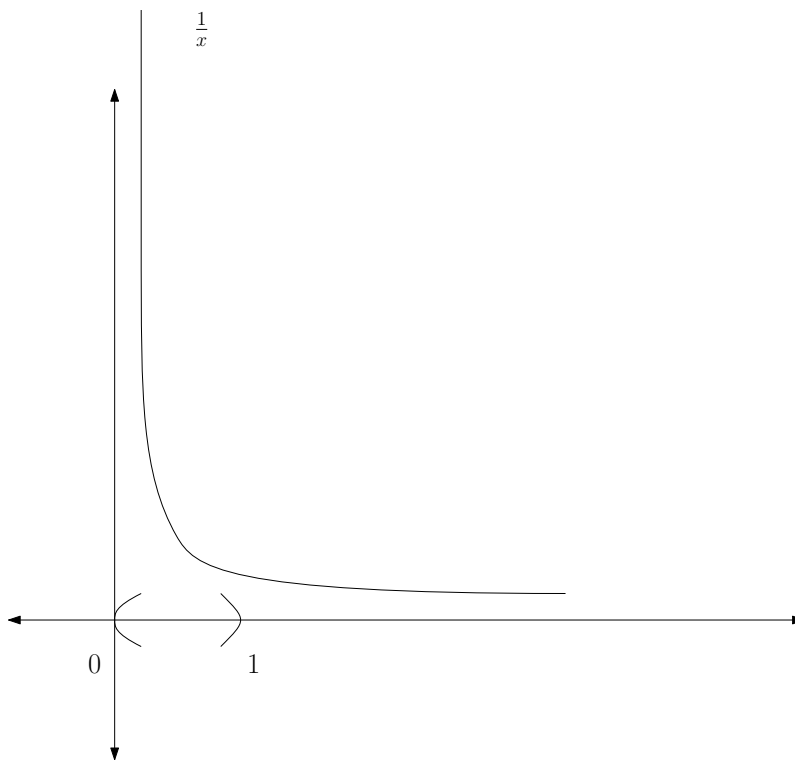
Also,

$$\begin{aligned} &\left| \left( \frac{1}{\delta} + \frac{\delta}{2} \right)^2 - \left( \frac{1}{\delta} \right)^2 \right| \\ &= \left| \frac{1}{\delta^2} + 2 \left( \frac{1}{\delta} \right) \left( \frac{\delta}{2} \right) + \left( \frac{\delta}{2} \right)^2 - \frac{1}{\delta^2} \right| \\ &= 1 + \left( \frac{\delta}{2} \right)^2 > 1 \end{aligned}$$

which is a contradiction.



**Exercise 7.1.**  $\frac{1}{x}$  on  $(0, 1)$  is continuous but not uniformly continuous. Suggest plausibly  $f$



continuous on  $[a, b]$  then it's uniformly continuous on  $[a, b]$  where  $a, b$  are finite.

**Theorem 7.3 (Heine – Cantor (Uniformly Continuous))**

A continuous function  $f$  on a closed interval is uniformly continuous.

*Proof.* (By contradiction) Suppose not. Then  $\epsilon > 0$  s.t. no  $\delta$  “works”. In particular,  $\exists \epsilon > 0$

s.t.  $\delta = 1$  fails,  $\delta = \frac{1}{2}$  fails, etc. So  $x, y \in [a, b]$  with  $|f(x_1) - (fy_1)| \geq \epsilon$  but  $|x_1 - y_1| < 1$ .  
 $x_n, y_n \in [a, b]$  with  $|f(x_n) - f(y_n)| \geq \epsilon$  but  $|x_n - y_n| < \frac{1}{n}$ . Hope this is impossible.  
 Bolzano - Weierstrass  $\implies \{n_j\}$  s.t.  $\{x_{n_j}\}$  has a limit

$$x_0 = \lim, \quad x_0 \in [a, b]$$

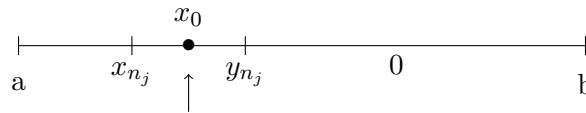
Now, claim  $\{y_{n_j}\}$  also has limit  $x_0$ .

$$|x_{n_j} - y_{n_j}| < \frac{1}{n_j}$$

small when  $n_j$  large ( $j$  large).

$$\begin{aligned} \lim x_{n_j} &= x_0 \\ \lim y_{n_j} &= x_0 \\ \lim f(x_{n_j}) &= f(x_0) \\ \lim f(y_{n_j}) &= f(x_0) \end{aligned}$$

So,  $\lim f(x_{n_j}) - f(y_{n_j}) = 0$ , but it contradicts  $|f(x_{n_j}) - f(y_{n_j})| \geq \epsilon$  for all  $j$ .  $\square$



$$f(x_0) \leq |f(x_{n_j}) - f(x_0)| + |f(x_0) - f(y_{n_j})| \rightarrow 0$$

Ideas of continuity and uniform continuity and Bolzano - Weierstrass Theorem – all have reasons in metric spaces.

## §8 | Lec 8: Oct 19, 2020

### §8.1 Convergence of Series

Series is “formal sum”, an infinite sum

$$a_0 + a_1 + a_2 + \dots = \sum_{j=1}^{\infty} a_j$$

A series  $\iff$  sequence  $a_1, a_2, a_3, \dots$  add together. Associated to  $a_1 + a_2 + a_3 + a_4 \dots$  is a sequence of partial sum

$$S_N = \sum_{n=1}^N a_n, \quad N = 1, 2, 3, 4, 5, \dots$$

number valued sequence.

**Definition 8.1 (Convergence of Series)** — Series converges if sequence associated  $\{S_N\}$  converges (has a limit).

Lots of things are defined by series such as ( $x \in \mathbb{R}$ ),

$$e^x = \lim_{N \rightarrow \infty} \left( 1 + x + \frac{x^2}{2!} + \dots + \frac{x^N}{N!} \right)$$

Given series  $a_0 + a_1 + a_2 + a_3 + \dots$ , when does it converge?

$$1 - 2 + 3 - 4 + 5 - 6 + 7 \dots$$

$$S_1 = 1, \quad S_2 = -1, \quad S_3 = 2 \dots$$

NO LIMIT! Series do not necessarily have to converge then it's okay to write

$$\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n$$

First thing to look at – Case where  $a_j \geq 0$

$$S_N \leq S_{N+1}, \quad N = 1, 2, 3, \dots$$

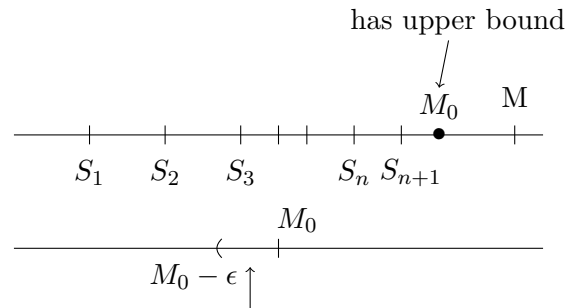
$S_{N+1} = S_N + a_{N+1}$  so  $a_{N+1} \geq 0$  means  $S_{N+1} \geq S_N$ . Two cases:

**Case 1:**  $\{S_n\}$  not bounded above.

$\lim S_N$  does not exist  $\rightarrow$  Series diverges (sequences with limits are always bounded above and below).

**Case 2:**  $\{S_n\}$  bounded above.

$\lim_{n \rightarrow \infty} S_n$  always exists. Namely, it is the least upper bound of set of values of  $S_n$ .



There is an  $S_{n_0}$  in this interval  $(M_0 - \epsilon, M_0]$ ,  $M_0$  is lub

From that  $n_0$  on,

$$S_n \geq S_{n_0}, \quad S_n \leq M$$

$S_n$  satisfies  $|S_n - M_0| < \epsilon$  if  $n \geq n_0$ . So  $\lim S_n = M_0$ . This implies that  $S_n$  is a Cauchy

sequence (it has a limit). Given  $\epsilon > 0$ ,  $\exists N_\epsilon$  s.t.  $\left| \sum_{1}^{n_1} a_n - \sum_{1}^{n_2} a_n \right| < \epsilon$  if  $n_1, n_2 \geq N_\epsilon$ .

Suppose  $n_1 > n_2 \geq N_\epsilon$

$$\sum_{1}^{n_1} a_n - \sum_{1}^{n_2} a_n = \sum_{n_2+1}^{n_1} a_n$$

Note:  $S_7 - S_5 = a_6 + a_7$  which explains the above expression.

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} \dots$$

converges, but so does the following series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots < 2$$

This works for arbitrary choices of  $+$  or  $-$ .

### Theorem 8.2 (Absolute Convergence)

If  $|b_1| + |b_2| + |b_3| + \dots$  converges, then

$$b_1 + b_2 + b_3 + \dots \text{ converges}$$

“Absolute convergence”  $\implies$  convergence (but not necessarily the same limit).

*Proof.* Assume  $\underbrace{\{S_n^A\}}_{A \text{ for absolute}}$  for absolute series has limit. So

$$\sum_1^\infty |b_n| \text{ converges}$$

$\implies \{S_n^A\}$  Cauchy sequence.

We hope it  $\implies \{S_n\} = \left\{ \sum_{j=1}^n b_j \right\}$  is a Cauchy sequence.

$$S_{n_1}^A - S_{n_2}^A = |b_{n_2+1}| + |b_{n_2+2}| + \dots + |b_{n_1}|$$

But

$$|b_{n_2+1} + \dots + b_{n_1}| \leq |b_{n_2+1}| + \dots + |b_{n_1}| (= S_{n_1}^A - S_{n_2}^A)$$

So,

$$|S_{n_1} - S_{n_2}| \leq S_{n_1}^A - S_{n_2}^A < \epsilon \text{ for } n_1, n_2 \geq N_\epsilon$$

Then  $|S_{n_1} - S_{n_2}| < \epsilon$  for  $n_1, n_2 \geq N_\epsilon$ . □

This is IMPORTANT – Better understand it thoroughly.

### Corollary 8.3 (Root Test)

$|b_n| \leq Cr^n, 0 < r < 1, C, r$  fixed, then  $\sum b_n$  converges.

Reason:  $\sum_{n=0}^\infty Cr^n = C \frac{1}{1-r}$  (geometric series).

**Exercise 8.1.**  $\sum_{n=0}^N Cr^n = C \frac{r^{N+1}-1}{r-1}, 0 < r < 1$  has limit  $\frac{C}{1-r}$ . Prove by induction.

Detail: Hypothesis:

$$|b_n| \leq Cr^n$$

$$\sum_1^\infty |b_n| \leq \sum_1^\infty Cr^n < \infty$$

$$\sum_b^N |b_n| \leq \sum_0^N Cr^n \leq M < \infty$$

So  $\sum_0^N |b_n|$  converges and bounded by  $Cr$ , and  $b_1 + b_2 + \dots$  converges absolutely.

## §9 | Lec 9: Oct 21, 2020

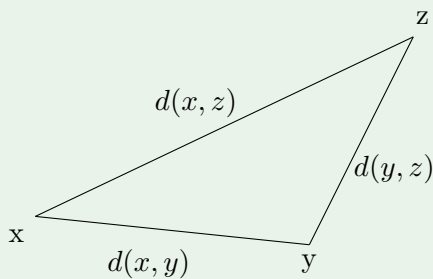
### §9.1 Metric Spaces

**Definition 9.1 (Metric Spaces)** — A set  $X$ , elements are “points”, together with a function on  $\underbrace{X \times X}_{\text{ordered pairs } (x,y)}$ ,  $x \in X, y \in Y$ ,  $\underbrace{d(x,y)}_{\text{distance}}$  with the following properties:

1.  $d(x, y) \geq 0$  for all  $x, y$ .  
 $d(x, y) = 0 \iff x = y$ . Or  $d(x, x) = 0$ .
2.  $d(x, y) = d(y, x)$ .
3.  $\triangle$  inequality:

$$d(x, y) + d(y, z) \geq d(x, z)$$

$$d(x, z) \leq d(x, y) + d(y, z)$$





**Example 9.2** 1.  $X$  set. Can you define a  $d : X \times X \rightarrow \mathbb{R}$  to make  $(X, d)$  a metric space?

YES! Define given set  $X$ ,  $d(x_1, x_2) = 0$  if  $x_1 = x_2$ , or  $d(x_1, x_2) = 1$  if  $x_1 \neq x_2$ . “discrete”.

- $d(x, y) \geq 0$ .
- $d(x, y) = d(y, x)$ .  
 $x = y$  both are 0.  
 $x \neq y$  both are 1.
- $d(x, z) \leq d(x, y) + d(y, z)$   
 $x = z \implies d = 0$ .  
 $x \neq z \implies d(x, z) = 1$ .  
 If  $x = y$  then  $y \neq z$  so  $1 \leq 0 + 1$   
 $\dots$

2. (INTERESTING)  $d(x, y) = |x - y|$  for  $\mathbb{R}$ .

$$d\left(\frac{p}{q}, \frac{r}{s}\right) = \left|\frac{p}{q} - \frac{r}{s}\right| \text{ for } \mathbb{Q}.$$

Note:  $X$  is a metric space  $Y \subset X$  then  $\left(Y, d|_{Y \times Y}\right)$  is a metric space.

Motivation: Stuff about  $\mathbb{R}$  involving e.g., continuity and limits can be transferred to metric space.

### Example 9.3

$\{x_n\}$  is a sequence in a metric space  $(X, d)$  (or  $X$ ) has limit  $x_0 \in X$  if for every  $\epsilon > 0$ , there is an  $N_\epsilon$  s.t.  $d(x, x_0) < \epsilon$  if  $n \geq N_\epsilon$ . (If  $X = \mathbb{R}$ ,  $d(x, y) = |x - y|$  same as before)

### Example 9.4

Function:  $f : (X, d_1) \rightarrow (Y, d_2)$ . Continuity at  $x_0 \in X$ ?

Real case:  $f$  cont at  $x_0$  means given  $\epsilon > 0 \exists \delta > 0$  s.t.  $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$ .

Metric space case:  $f$  cont at  $x_0$  means given  $\epsilon > 0 \exists \delta > 0$  s.t.  $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \epsilon$ .

More examples:

### Example 9.5

$$\mathbb{R}^2 = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$$

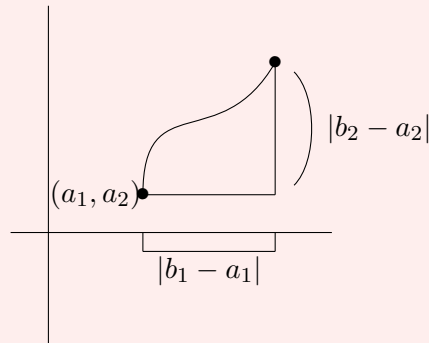
$$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}\}$$

$\vdots$

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \dots, x_n \in \mathbb{R}\}$$

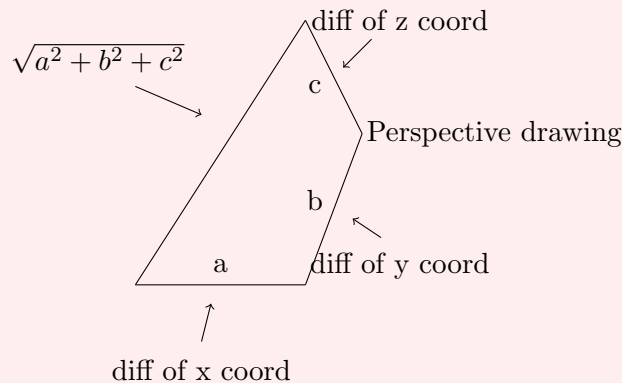
Interesting metric on  $\mathbb{R}^2$   $d((a_1, a_2), (b_1, b_2))$

$$d = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$$



$\mathbb{R}^n(x_1, x_2, \dots, x_n), (y_1, \dots, y_n)$

$$d := \sqrt{(y_1 - x_1)^2 + \dots + (y_n - x_n)^2}$$



Is this function on  $\mathbb{R}^n$  a metric?

1.  $d(x, y) \geq 0, = 0 \iff x = y$  where  $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$  and

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

2.  $d(x, y) = d(y, x)$

3. BUT BUT BUT  $\triangle$  inequality is not so easy.

$$\sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \leq \sqrt{(x_1 - z_1)^2 + \dots + (x_n - z_n)^2} + \sqrt{(z_1 - y_1)^2 + \dots + (z_n - y_n)^2}???$$

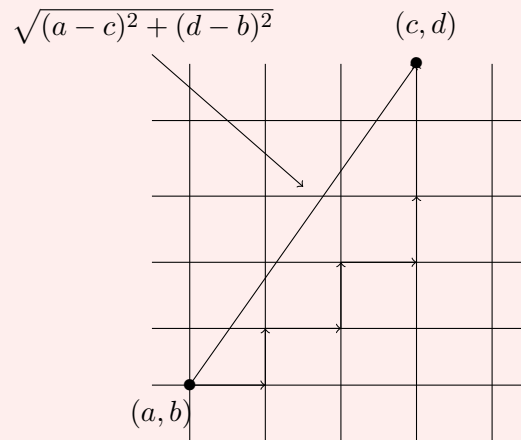
Does  $d(x, y) \leq d(x, z) + d(z, y)$  work?

YES but proof later :(

Realize that it's okay to assume  $z = (0, 0, \dots, 0)$

**Example 9.6**

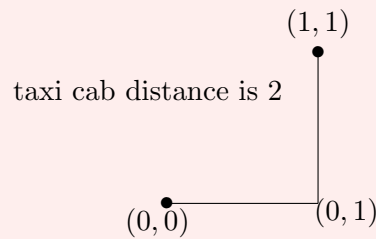
Try another metric  
 $\mathbb{R}^2$  – taxicab



$$|c-a| + |d-b| = d((a, b), (c, d))$$

← min of length of taxi car

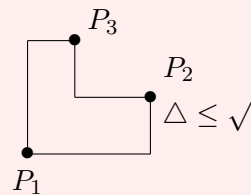
Easy to see that this  $d$  is really a metric.  $\triangle$  inequality is easy!



$$\text{Euclidean distance} = \sqrt{2}$$

$$\text{diff of } x\text{'s} \leq \text{Euc dis}$$

$$\text{diff of } y\text{'s} \leq \text{Euc dis}$$

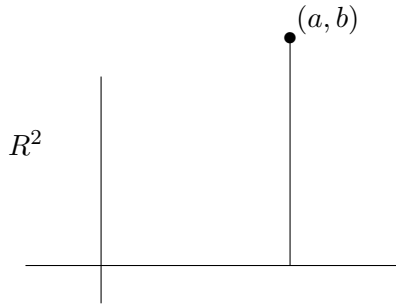


$$d(P_1, P_2) + d(P_2, P_3) \geq d(P_1, P_3)$$

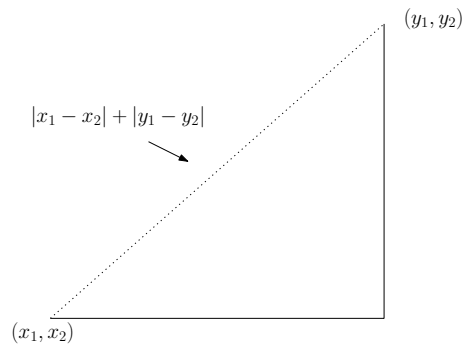
## §10 | Lec 10: Oct 23, 2020

### §10.1 Metric on $\mathbb{R}^n$

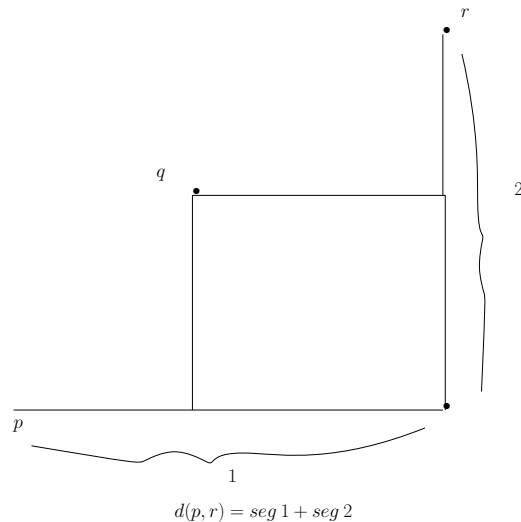
$$\mathbb{R}^n : \{(x_1, \dots, x_n) : x_j \in \mathbb{R}\}$$



We want to make  $\mathbb{R}^n$  a metric space. Last time, we defined “taxi cab metric”,  $d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{i=1}^n |x_i - y_i|$ . Verify  $d(\vec{x}, \vec{y}) \geq 0$  or  $= 0$  if  $\vec{x} = \vec{y}$  and  $\triangle$  inequality,



$$d(p, q) + d(q, r) \geq d(p, r)$$

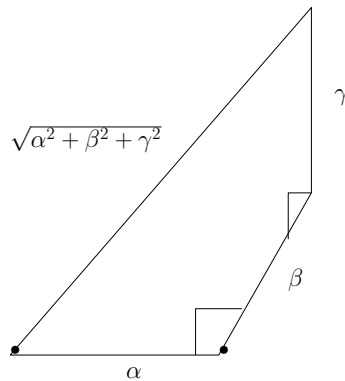
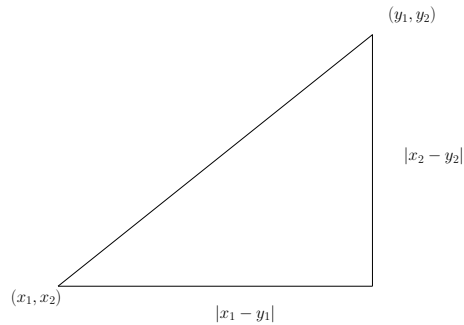


## §10.2 Triangle Inequality in Euclidean Space

New idea: Euclidean distance (or Pythagorean distance)

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$\text{For } \mathbb{R}^n : d((a_1, \dots, a_n), (b_1, \dots, b_n)) := \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \dots + (a_n - b_n)^2}.$$



We need to know:

1.  $d(\vec{a}, \vec{b}) \geq 0$
2.  $d(\vec{a}, \vec{a}) = 0$  so  $d(\vec{a}, \vec{b}) = 0 \implies \vec{a} = \vec{b}$
3.  $d(\vec{a}, \vec{b}) = d(\vec{b}, \vec{a})$
4. ?  $\triangle \leq 0, \vec{a}, \vec{b}, \vec{c}$

$$d(\vec{a}, \vec{c}) \leq d(\vec{a}, \vec{b}) + d(\vec{b}, \vec{c})$$

For  $\mathbb{R}^n$ ,

$$\sqrt{(a_1 - c_1)^2 + \dots + (a_n - c_n)^2} \leq \sqrt{(a_1 - b_1)^2 + \dots + (a_n - b_n)^2} + \sqrt{(b_1 - c_1)^2 + \dots + (b_n - c_n)^2}$$

We certainly need proof for  $\triangle$  inequality: Copson( $p > 1$ ) – for case  $p = 2$

First step:  $\alpha\beta \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2$  for all real  $\alpha, \beta$ .

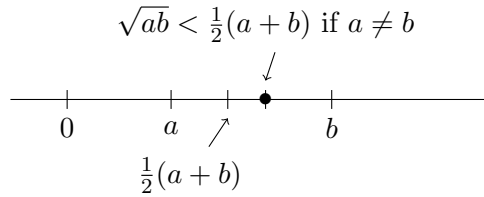
Reason:

$$\begin{aligned} 2\alpha\beta &\leq \alpha^2 + \beta^2 \\ \alpha^2 + \beta^2 - 2\alpha\beta &\geq 0 \\ (\alpha - \beta)^2 &\geq 0 \checkmark \end{aligned}$$

“Geometric mean  $\leq$  Arithmetic mean”

Let  $\alpha = \sqrt{a}, \beta = \sqrt{b}, a, b \geq 0$

$$\underbrace{\sqrt{ab}}_{\text{geometric mean of a,b}} \leq \frac{1}{2}(a) + \frac{1}{2}(b) = \underbrace{\frac{1}{2}(a+b)}_{\text{arithmetic mean}}$$



Second step:

$$\vec{a} = (a_1, \dots, a_n)$$

$$\vec{b} = (b_1, \dots, b_n)$$

and we know

$$a_i b_i \leq \frac{1}{2} a_i^2 + \frac{1}{2} b_i^2$$

Then,

$$\sum_{i=1}^n a_i b_i \leq \frac{1}{2} \sum_{i=1}^n a_i^2 + \frac{1}{2} \sum_{i=1}^n b_i^2$$

So,  $\sum a_i^2 = 1$ ,  $\sum b_i^2 = 1$ ,  $\sum a_i b_i \leq 1$

**Claim 10.1.**

$$\sum a_i b_i \leq \left( \sum a_i^2 \right)^{\frac{1}{2}} \left( \sum b_i^2 \right)^{\frac{1}{2}}$$

But

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$$

So it's okay to define  $\theta$ ,

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \in [-1, 1]$$

Verification of claim:  $\vec{a}, \vec{b} \neq \vec{0}$

$$A_i = \frac{a_i}{\sqrt{\sum a_i^2}}, \quad B_i = \frac{b_i}{\sqrt{\sum b_i^2}}$$

And  $\sum A_i^2 = 1$ ,  $\sum B_i^2 = 1$  – also  $\sum_{i=1}^n A_i B_i \leq 1$  which is equivalent to  $\frac{\sum a_i b_i}{\sqrt{\sum a_i^2} \sqrt{\sum b_i^2}} \leq 1$ .

So  $|\sum a_i b_i| \leq \sqrt{\sum a_i^2} \sqrt{\sum b_i^2}$ .

BIG DEAL: “Cauchy Schwarz inequality” What does this have to do with  $\triangle$  inequality for Euclidean metric. Consider:  $\vec{a}, \vec{b}$

$$\sum_{j=1}^n (a_j + b_j)^2 = \sum_{j=1}^n a_j (a_j + b_j) + \sum_{j=1}^n b_j (a_j + b_j)$$

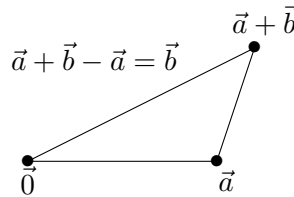
Now apply Cauchy – Schwarz

$$\begin{aligned} \sum_{j=1}^n (a_j + b_j)^2 &\leq \left( \sum_{j=1}^n a_j^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^n (a_j + b_j)^2 \right)^{\frac{1}{2}} \\ &\quad + \left( \sum_{j=1}^n b_j^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^n (a_j + b_j)^2 \right)^{\frac{1}{2}} \end{aligned}$$

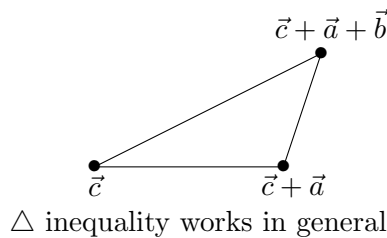
Divide through by  $(\sum (a_j + b_j)^2)^{\frac{1}{2}}$

$$\left(\sum (a_j + b_j)^2\right)^{\frac{1}{2}} \leq \left(\sum a_j^2\right)^{\frac{1}{2}} + \left(\sum b_j^2\right)^{\frac{1}{2}}$$

The above inequality is indeed the triangle inequality for  $\vec{0}, \vec{a}, \vec{a} + \vec{b}$



But of course this gives you the triangle inequality in general.



Last step:  $\vec{p}, \vec{q}, \vec{r}$

Triangle inequality:

$$d(0, \vec{p} - \vec{r}) \leq d(0, \vec{q} - \vec{p}) + d(\vec{q} - \vec{p}, \vec{r} - \vec{p})$$

Same as  $\triangle$  ineq for  $0, \vec{q} - \vec{p}, (\vec{r} - \vec{q}) + (\vec{q} - \vec{p})$  or  
 $0, \vec{a}, \vec{a} + \vec{b}$  if  $\vec{a} = \vec{q} - \vec{p}, \vec{b} = \vec{r} - \vec{q}$ .

## §11 | Lec 11: Oct 26, 2020

### §11.1 Metric Spaces Examples

Last time, we prove  $\triangle$  ineq. proof, taxi-cab metric, and sup norm metric. This gives rise to same “convergence idea”. Namely  $x_n \in X(X, d)$  converges to  $L \in X$  means

$$\lim_{n \rightarrow \infty} (x_n - L) = 0$$

In all three metrics

$$\vec{x}_j \rightarrow L \quad \lim \vec{x}_j = L$$

means (is same as)  $i$ th coordinate of  $\vec{x}_j$  converges to  $i$ th coord of  $L$  for each  $i = 1, 2, \dots, n$ .  
 $\{x_n\}$  Cauchy if given  $\epsilon > 0 \exists N_\epsilon \ni n_1, n_2 \geq N_\epsilon$

$$d(x_{n_1}, x_{n_2}) < \epsilon$$

**Exercise 11.1.**  $\{x_n\}$  Cauchy in  $\mathbb{R}^n$  (any one of three metrics – Cauchy is the same idea in all three metrics) then  $\{x_n\}$  has limit  $L$ , some  $L$ .

$$\sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \leq \sqrt{n} \max |x_j - y_j|, j = 1, \dots, n$$

So,

$$\begin{aligned} |x_j - y_j| &\leq \max |x_l - y_l| \\ |x_j - y_j|^2 &\leq \max^2 |x_l - y_l|, l = 1, \dots, n \\ (x_1 - y_1)^2 + \dots + (x_n - y_n)^2 &\leq n \max^2 |x_l - y_l| \\ \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} &\leq \sqrt{n} \max \end{aligned}$$

$l_2 : \{x_j\}$  infinite sequences  $j = 1, 2, 3, \dots$  where  $\left\{ \sum_{j=1}^{\infty} x_j^2 < \infty \right\}$  which means

$$\exists M \ni \sum_{j=1}^M x_j^2 \leq M$$

$$\begin{aligned} (1, \frac{1}{2}, \frac{1}{3}, \dots) &\in l_2 \\ (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \dots) &\notin l_2 \end{aligned}$$

because  $1 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \dots \rightarrow \infty$  vector space:

$$\begin{aligned} c\{x_j\} &= \{cx_j\} \\ \{x_j\} \in l_2 &\implies \in l_2 \\ \sum c^2 x_j^2 &= c^2 \sum x_j^2 \end{aligned}$$

Also,

$$\begin{aligned} \{x_j\} + \{y_j\} &= \{x_j + y_j\} \\ (x_j + y_j)^2 &\leq 2(x_j^2 + y_j^2) \\ x_j y_j &\leq \frac{1}{2}(x_j^2 + y_j^2) \end{aligned}$$

$\{x_j\}, \{y_j\} \in l_2$  then

$$d(\{x_j\}, \{y_j\}) = \left[ \sum (x_j - y_j)^2 \right]^{\frac{1}{2}}$$

makes sense.  $(l_2, d)$  is a metric space obvious except  $\triangle$  ineq. It's enough to check

$$d(0, \vec{x}) + d(\vec{x}, \vec{x} + \vec{y}) \geq d(0, \vec{x} + \vec{y})$$

which follows by taking limits of  $\triangle$  ineq. for truncation up to level  $N$ .

$$d(\vec{0}, (x_1, \dots, x_N)) + d((y_1, \dots, y_N), (x + y) \text{ up to } N) \geq d(\vec{0}, (x + y)_N)$$

$l_2$  is metric space

$l_2$  is complete – Cauchy sequences have some limits.



**Example 11.1**

$C([0, 1]) := \text{cont: } \mathbb{R} - \text{valued function } [0, 1]$

$$\begin{aligned} d(f, g) &= \max |f(x) - g(x)| \\ &= \sup |f(x) - g(x)| \end{aligned}$$

“sup norm” All properties clear. “ $L^2$  norm” – distance on  $C[0, 1]$  :

$$d_2(f, g) = \left( \int_0^1 (f(x) - g(x))^2 \right)^{\frac{1}{2}}$$

where  $d_2 \geq 0$ ,  $f, g, h \in C[0, 1]$ .

Imitate argument for  $\triangle$  ineq. on  $\mathbb{R}^n$ : Cauchy Schwarz ineq.

$$\int_0^1 fg \leq \left( \int_0^1 f^2 \right)^{\frac{1}{2}} \left( \int_0^1 g^2 \right)^{\frac{1}{2}}$$

So,

$$\begin{aligned} f(x)g(x) &\leq \frac{1}{2} (f^2(x) + g^2(x)) \\ \int_0^1 f(x)g(x) &\leq \frac{1}{2} \int_0^1 f^2(x) + \frac{1}{2} \int_0^1 g^2(x) \end{aligned}$$

Apply these,  $F = \frac{f(x)}{\sqrt{\int_0^1 f^2}}$ ,  $G = \frac{g}{\sqrt{\int_0^1 g^2}}$ ,  $\int F^2 = 1$ ,  $\int G^2 = 1$ . Also, we know  $\int fg \leq 1$  if  $\int f^2 = 1$ ,  $\int g^2 = 1$ .

Remainder argument for  $\triangle$  ineq. is same as before

$$\int (f + g)^2 = \int f(f + g) + \int g(f + g)$$

Apply Cauchy – Schwartz,

$$\begin{aligned} \int (f + g)^2 &\leq \left( \int f^2 \right)^{\frac{1}{2}} \left( \int (f + g)^2 \right)^{\frac{1}{2}} + \left( \int g^2 \right)^{\frac{1}{2}} \left( \int (f + g)^2 \right)^{\frac{1}{2}} \\ \left( \int (f + g)^2 \right)^{\frac{1}{2}} &\leq \left( \int f^2 \right)^{\frac{1}{2}} + \left( \int g^2 \right)^{\frac{1}{2}} \end{aligned}$$

**§11.2 A Glance at Complex Number**

Special case of  $\mathbb{R}^n$ , Euclidean norm

$$\begin{aligned} \mathbb{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} &= d((x_1, x_2), (y_1, y_2)) \\ \mathbb{C} : \{(a + bi)\} &- \text{Complex numbers} \end{aligned}$$

$(x_1, x_2) \leftrightarrow x_1 + ix_2$ . Metric on  $\mathbb{C}$ ,  $z, w \in \mathbb{C}$

$$|z - w| = d(z, w) \quad \text{as pts in } \mathbb{R}^2$$

$$z = a + bi$$

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

We also define multiplication in  $\mathbb{C}$  as follows

$$(a + bi)(c + di) := (ac - bd) + (bc + ad)i$$

### Example 11.2

$$\frac{1}{c + di} = \frac{c}{c^2 + d^2} - \frac{d}{c^2 + d^2}i$$

For  $z = a + bi, w = c + di$  we define

$$\begin{aligned} |zw| &= |z||w| \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} \\ &= \sqrt{(ac - bd)^2 + (bc + ad)^2} \end{aligned}$$

verify if the above step is actually equal

## §12 | Lec 12: Oct 28, 2020

### §12.1 Midterm Announcement

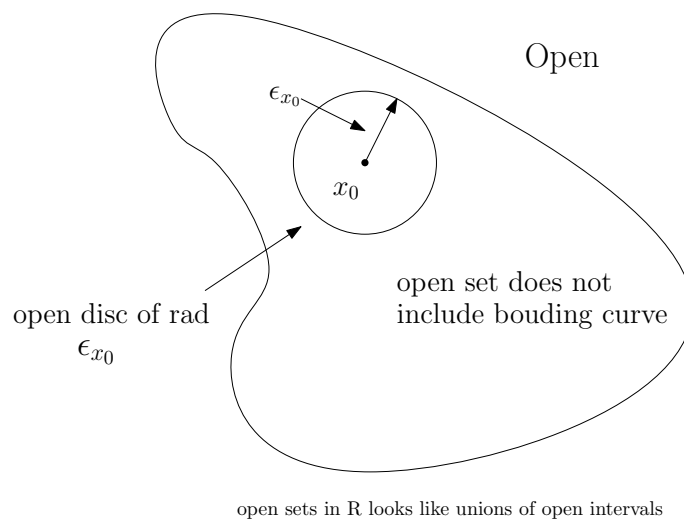
Midterm – Given out at Fri, Nov 6 at 3:00 pm. and due by Sat, Nov 7 at 11:00 pm.

### §12.2 Open sets in Metric Space

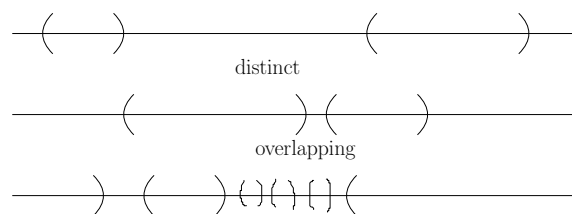
Beginning of “topology”:  $(X, d)$  metric space

**Definition 12.1** (Open sets) —  $U \subset X$  open if for every  $x_0 \in U$  there is an  $\epsilon_{x_0} > 0$  s.t.

$$\underbrace{\{x \in X : d(x, x_0) < \epsilon_{x_0}\}}_{B(x_0, \epsilon_{x_0})\text{- open ball}} \subset U$$

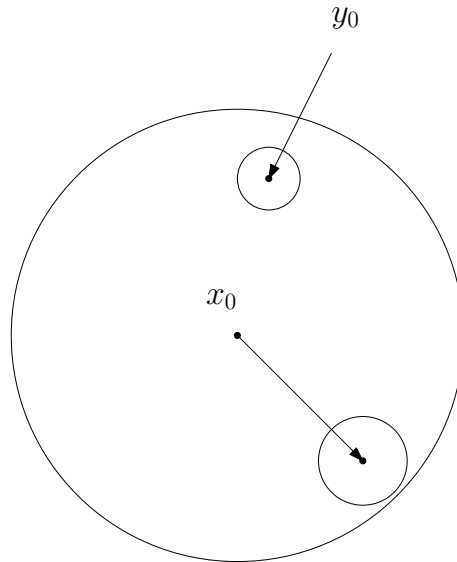


Open set in  $\mathbb{R}$



### Lemma 12.2

$B(x_0, \epsilon), \epsilon > 0$  open ball is open set.



*Proof.* Need given  $y \in B(x_0, \epsilon)$ ,  $\lambda_y > 0$  s.t.  $B(y, \lambda) \subset B(x_0, \epsilon)$ .

Try  $\lambda = \epsilon - d(x_0, y_0)$ .

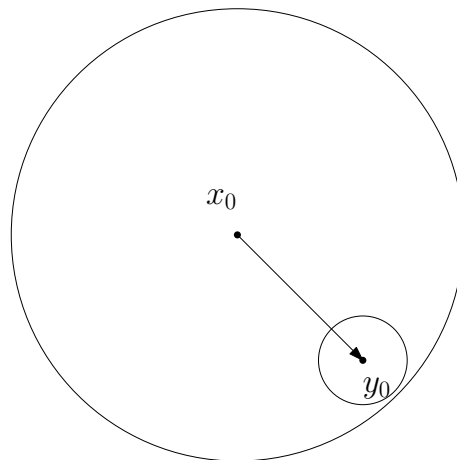
Suppose  $y \in B(y_0, \epsilon) \iff d(y_0, y) < \epsilon - d(x_0, y_0)$

$$d(y_0, y) + d(x_0, y_0) < \epsilon$$

So,

$$d(x_0, y) \leq d(x_0, y_0) + d(y_0, y) < \epsilon$$

So  $y \in B(x_0, \epsilon)$ .



□

Reason why people care about open sets:

Remember:  $f(X, d) \rightarrow (Y, d)$  continuous means given  $\epsilon > 0, x_0 \in X$  there exists  $\delta > 0$  s.t.

$$d_X(x_0, x) < \delta \implies d_Y(f(x_0), f(x)) < \epsilon$$

$\rightarrow$  Direct transcription of number – continuity of  $f$  can be described in terms of open sets in  $X$  and in  $Y$ . For this:  $f : X \rightarrow Y$  and  $V \subset Y$ , then  $f^{-1}(V) = \{x \in X : f(x) \in V\}$   $f$  does not need to be invertible.

**Example 12.3**

$$f : \underbrace{X}_{\text{people}} \rightarrow \mathbb{Z}, \quad f(x) = \text{integer age of } x$$

$$f^{-1}(\{20, 21, 22\}) = \text{everybody that's age 20, 21, or 22}$$

**Theorem 12.4** (Continuity – Open Sets)

$f : (X, d_x) \rightarrow (Y, d_y)$  is continuous if and only if (in  $\delta, \epsilon$  sense)  $f^{-1}(V)$  is open in  $X$  for every  $V$  open in  $Y$ .

Slogan: continuity means inverses of open sets are open.

$f : X \rightarrow Y, g : Y \rightarrow Z \rightarrow g(f(x))$  compositions of  $f$  and  $g$ .

**Claim 12.1.** If  $f, g$  continuous then the composition is continuous

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

*Proof.* (of Theorem) Suppose  $f^{-1}(V)$  is open when  $V$  is open. Given  $x_0 \in X, \epsilon > 0$  want  $\delta > 0 \ni x \in B(x_0, \delta) \implies d(f(x), f(x_0)) < \epsilon$

$$\underbrace{x \in B(x_0, \delta)}_{d(x, x_0) < \delta}$$

$$\{y : d(y, f(x_0)) < \epsilon\} = B(f(x_0), \epsilon)$$

Know that it's open by the above lemma. So,

$$f^{-1}(B(f(x_0), \epsilon)) \text{ open}$$

and  $x_0 \in (B(f(x_0), \epsilon))$ . So  $f^{-1}(B(f(x_0), \epsilon))$  being open

$$\implies \delta > 0 \quad B(x_0, \delta) \subset f^{-1}(B(f(x_0), \epsilon))$$

says  $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \epsilon \checkmark$

Took care of  $f^{-1}(\text{open})$  is open  $\implies$  continuity. Now,

Does continuity ( $\epsilon, \delta$  sense)  $\implies f^{-1}(\text{open})$  is open?

This also works: Suppose  $V$  open, and  $x_0 \in f^{-1}(V)$ . Need  $\delta > 0$  s.t.  $B(x_0, \delta) \subset f^{-1}(V)$ .  $f(x_0) \in V$  (meaning of  $x_0 \in f^{-1}(V)$ )  $\exists \epsilon$  s.t.  $B(f(x_0), \epsilon) \subset V$  ( $V$  is open). Then  $\epsilon, \delta$  defn of continuity  $\exists \delta$  s.t.  $f(B(x_0, \delta)) \subset B(f(x_0), \epsilon) \subset V$ . So  $B(x_0, \delta) \subset f^{-1}(V)$ .  $\checkmark$   $\square$

Forward continuous images of open sets are not necessarily open.

**Example 12.5**

$f(x) = x^2, \quad f((-1, 1)) = [0, 1)$  which is not open.

Note: A notion to help understand the concept of open sets is thinking about how a map sends a point to a point but its inverse can send a point to a set.

## §13 | Dis 1: Oct 1, 2020

Notation:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$$

$$\mathbb{R} = \text{real numbers}$$

$$\mathbb{C} = \{a + bi, \ a, b \in \mathbb{R}\}$$

Set theory:

- $A \subset B$  (or  $A \subseteq B$ ) means  $x \in A \implies x \in B$
- $x \in A \cap B$  means  $x \in A$  and  $x \in B$
- $x \in A \cup B$  means  $x \in A$  or  $x \in B$
- $x \in A \setminus B \iff x \in A$  and  $x \notin B$
- $A = B \iff A \subset B$  and  $B \subset A$

### §13.1 Induction

Given a sequence of mathematical statement  $P(n)$  indexed by  $\mathbb{N}$ . If  $P(1)$  is true and  $P(k) \implies P(k+1)$  is true  $\forall k \in \mathbb{N}$ , then  $P(n)$  is true  $\forall n \in \mathbb{N}$ .

#### Example 13.1

Prove  $\sum_{k=1}^n (2k-1) = n^2$  (\*) using induction.

Base case  $n = 1 : 1 = 1^2$  ✓

Induction step: assume as induction hypothesis that (\*) holds

$$\begin{aligned} \sum_{k=1}^{n+1} (2k-1) &= \sum_{k=1}^n (2k-1) + 2(n+1) - 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

Or we can prove it the following way

$$\begin{aligned} S &= 1 + 3 + 5 + \dots + (2n-1) \\ S &= (2n-1) + (2n-3) + \dots + 3 + 1 \\ 2S &= 2n \cdot n \\ S &= n^2 \end{aligned}$$

**Example 13.2**

$a_{n+1} = \sqrt{2 + a_n}$ ,  $a_1 = 1$ . Prove  $a_n > 0$  and  $a_n$  increasing.  
 $a_1 > 0$  assume  $a_n > 0$ ,  $a_{n+1} = \sqrt{2 + a_n} > 0$

$$a_2 = \sqrt{3} \approx 1.732 > 1 = a_1$$

Assume  $a_n \leq a_{n+1}$ , want to show  $a_{n+1} \leq a_{n+2} \iff \sqrt{a_n + 2} \leq \sqrt{a_{n+1} + 2} \iff a_n \leq a_{n+1}$

**Example 13.3**

$(1+x)^n \geq 1+nx$  : Bernoulli Inequality

$$x \geq -1, \quad n \geq 0$$

base case  $1 \geq 1$

Assume  $(1+x)^n \geq 1+nx$

$$\begin{aligned} (1+x)^{n+1} &= (1+x)^n(1+x) \geq (1+nx)(1+x) = 1 + (n+1)x + nx^2 \\ &= 1 + (n+1)x \end{aligned}$$

Strong Induction:

If  $P(1)$  true and  $P(1), P(2), \dots, P(k) \implies P(k+1)$  true  $\forall k \in \mathbb{N}$  then  $P(n)$  holds for all  $n \in \mathbb{N}$

**Remark 13.4.** Induction  $\iff$  strong induction

**Example 13.5**

Every integer greater than 1 is a product of primes.

Assume  $2, 3, \dots, n$  is a product of primes.  $n+1$  is either a prime or a composite, in which case  $n+1 = ab$ ,  $1 < a, b < n+1$ .

By strong induction hypothesis, both  $a$  and  $b$  are product of primes, hence so is  $n+1 = ab$ .

**Exercise 13.1.** Every integer greater than 1 has a prime divisor.

Proof of infinitude of primes by Euclid:

*Proof.* Assume on the contrary there are finitely many primes  $\{p_1, p_2, \dots, p_k\}$ . Define  $N = p_1 \dots p_k + 1 > 1$  and (by above exercise) let  $p$  be a prime divisor of  $N$  but  $p \neq p_j$  for any  $1 \leq j \leq k$  otherwise if  $p = p_j$  then  $p|p_2 \dots p_k$  also  $p|N \implies p|N - p_1 \dots p_k \implies p|1$ , a contradiction. (no primes divide 1)  $\square$

### §14.1 Number System

- $(\mathbb{N}, +, \cdot, <)$  :  $+$  :  $\mathbb{N} \times \mathbb{N} = \mathbb{N}^2 \rightarrow \mathbb{N}$  satisfies commutativity and associativity. Note that 0 is the identity with respect to addition, but  $\mathbb{N}$  has no additive inverse.
- $(\mathbb{Z}, +, \cdot, <)$  :  $(\mathbb{Z}, +)$  is a commutative group (associativity, identity, inverse).  $(\mathbb{Z}, \cdot)$  satisfies commutativity, associativity with 1 as mult identity but 2 has no mult inverse.
- $(\mathbb{Q}, +, \cdot, <)$  :  $(\mathbb{Q}, +)$  and  $(\mathbb{Q}, \cdot)$  are commutative group(i).  $+$  and  $\cdot$  are compatible with distributive law:  $a(b + c) = ab + ac$  (ii). Both (i) and (ii) mean  $(\mathbb{Q}, +, \cdot)$  is a FIELD.  $(\mathbb{Q}, <)$  is an ordered set with  $<$  satisfying trichotomy and transitivity.  $+$ ,  $\cdot$  are compatible :  $y < z \implies x + y < x + z \forall x, x > 0, y > 0 \implies xy > 0$ . With the above compatibility,  $(\mathbb{Q}, +, \cdot, <)$  is an **ordered field**. Even though  $\mathbb{Q}$  is additivity and multiplicatively complete,  $\mathbb{Q}$  is not satisfying in that

1.  $\mathbb{Q}$  is not algebraically closed,  $x^2 - 2$  is a polynomial with no root in  $\mathbb{Q}$ .
2.  $\mathbb{Q}$  is not complete in a metric space: there exists subsets of  $\mathbb{Q}$  bounded above but with no least upper bound (supremum), e.g.  $A := \{p \in \mathbb{Q} : p < 0 \text{ or } p^2 < 2\}$  and  $B = \mathbb{Q} \setminus A$ .  $A$  contains no largest number and  $B$  contains no smallest.

$$\forall p \in A \exists q \in A \quad q > p$$

Let  $p \in A$ . Define  $q := p - \frac{p^2 - 2}{p + 2} > p$

$$q^2 - 2 = \left( \frac{2p + 2}{p + 2} \right)^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2} < 0 \implies q^2 < 2$$

If  $A$  has an upper bound  $\alpha$ ,  $\alpha \notin A$  : then  $\alpha \in B$ . It follows that  $B$  is the set of all upper bounds for  $A$ . Since  $B$  contains no smallest number,  $A$  has no least upper bound in  $\mathbb{Q}$ .

**Definition 14.1** (Least Upper Bound Property) —  $S$  has the least-upper-bound property if  $\forall E \subset S$  nonempty, bounded above  $\sup E \in S$ .

**Remark 14.2.**  $\mathbb{Q}$  does not satisfy the least-upper-bound property.

$(\mathbb{R}, +, \cdot, <)$  there exists an ordered field with the l.u.b property that contains an isomorphic copy of  $\mathbb{Q}$ .

### §14.2 Equivalence Relation

An equivalence relation given  $\sim$  on  $A \times A$  satisfies

- $x \sim x$  reflexivity
- $x \sim y \iff y \sim x$  symmetry
- $x \sim y \cdot y \sim z \implies x \sim z$  transitivity



**Example 14.3**

$\mathbb{Q}$  Define  $\sim$  on  $\{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$  by  $(a, b) \sim (c, d)$  if  $ad = bc$

$$A = \mathbb{Z}^2 \setminus \{(a, 0) : a \in \mathbb{Z}\}$$

$\mathbb{Q} =$  the set of all equivalence classes of  $A$  write  $\sim$   
 $= A / \sim = \{[x] : x \in A\}$

In this construction,  $\mathbb{Z} \rightarrow \mathbb{Q}, \quad n \rightarrow [(n, 1)]$   
 $+$  and  $\cdot : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$  : note that  $+$  and  $\cdot$  need to be well-defined on  $\mathbb{Q}^2$ . (need to show  
 $\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}$  if  $\frac{a}{b} \sim \frac{a'}{b'}$  and  $\frac{c}{d} \sim \frac{c'}{d'}$ ).

**Example 14.4**

$$S' = [0, 1] / 0_m$$

**Definition 14.5 (Convergent Sequences)** —  $\{a_n\}_{n \geq 1} \subseteq \mathbb{R}$  is said to be convergent to  $l$   
 if  $\forall \epsilon > 0 \quad \exists N(\epsilon) > 0$  s.t.  $\forall n \geq N, \quad |a_n - l| < \epsilon$

## §15 | Dis 3: Oct 13, 2020

### §15.1 Equivalence Relation (Cont'd)

**Example 15.1**

Define  $\sim p$  on  $\mathbb{Z}$  by  $a \sim pb$  if  $a - b \in p\mathbb{Z}$  ( $p|a - b$ ).

$$\forall a \exists ! b \in \mathbb{Z}, \quad 0 \leq r < p \text{ s.t. } a = bp + r.$$

$$F_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z} / \sim p = \{[0]_p, [1]_p, [2]_p, \dots, [p-1]_p\}$$

$$[a]_p + [b]_p = [a + b]_p \quad \& \quad [a]_p [b]_p = [ab]_p$$

**Remark 15.2.**  $(F_p, +, \cdot)$  is a finite field.  $F_p$  cannot be ordered:  $1 > 0, 1 + 1 > 0, \dots, p - 1 > 0$   
 but  $p - 1 = -1$

**Example 15.3**

$$T = \mathbb{R}/\mathbb{Z} \quad a \sim b \text{ if } ab \in \mathbb{Z}$$

$$[0, 1] / 0 \sim 1$$

$$\forall a \in \mathbb{R}, \quad \exists b = \underbrace{\{a\}}_{\text{fractional part of } a} \in [0, 1) \text{ s.t. } a \sim b$$

## §15.2 Construction of $\mathbb{R}$ via Cauchy Sequences (Cantor)

$S$  = set of rational Cauchy sequences.

$\sim$  on  $S$  :  $\{x_n\} \sim \{y_n\}$  if  $\lim(x_n - y_n) = 0$  (Q3 – Homework 2)

$Q = S / \sim = \{[\{x_n\}] : \{x_n\} \in S\}$ . First we need to define arithmetic on  $Q$ .

$$\begin{aligned} [\{p_n\}] + [\{q_n\}] &= [\{p_n + q_n\}] \\ [\{p_n\}] - [\{q_n\}] &= [\{p_n - q_n\}] \\ [\{p_n\}] \cdot [\{q_n\}] &= [\{p_n q_n\}] \\ [\{p_n\}] / [\{p_n/q_n\}] &= [\{p_n/q_n\}], \quad [\{q_n\}] \neq 0, = [\{0, 0, 0, \dots\}] \end{aligned}$$

$+$  :  $Q \times Q \rightarrow Q$ . Check well-defined

- $\{x_n\} \cdot \{y_n\}$  cauchy then so is  $\{x_n + y_n\}$  (Q4)
- $\{x_n\} \sim \{y_n\}$  &  $\{z_n\} \sim \{w_n\}$  then  $\{x_n + z_n\} \sim \{y_n + w_n\}$  (Q5)  
Commutativity, assoc, identity, ( $0 = [\{0, 0, 0, \dots\}]$ ), inverse.
- Well-defined:  $\{x_n\}, \{y_n\}$  so is  $\{x_n y_n\}$  (Q4).
- $\{x_n\} \sim \{y_n\}$  &  $\{z_n\} \sim \{w_n\}$  (Q6, Q7)  
comm, assoc, iden, ( $1 = [\{1, 1, \dots, 1\}]$ )  
mult. inverse (Q9, Q10).  
<: trichotomy (Q11), transitivity  
various compatibility (distributivity, etc)  
l.u.b property (Q12)

Note: All the  $Q$  used above is assumed to be  $Q^{\text{hat}}$

### Remark 15.4.

$$\begin{aligned} Q &\rightarrow Q^{\text{hat}} \\ q &\mapsto [q^*] \\ p < q &\iff [p^*] < [q^*] \end{aligned}$$

Sequences:

- Cauchy seq. are bounded.
- Convergent seq. is Cauchy.

Theorem: in  $\mathbb{R}$ , every Cauchy seq. is convergent.

### Example 15.5

$$\begin{aligned} a_n &= \frac{1}{n} \\ \forall \epsilon > 0 \exists N \text{ s.t. } \epsilon N > 1. \\ \forall n \geq N \quad \left| \frac{1}{n} - 0 \right| &= \frac{1}{n} \leq \frac{1}{N} < \epsilon. \end{aligned}$$

□

# §16 | Dis 4: Oct 20, 2020

## §16.1 Least Upper Bound and Its Applications

**Remark 16.1** ( $\epsilon$  - Principle).  $a, b \in \mathbb{R}, \forall \epsilon > 0, a \leq b + \epsilon \implies a \leq b$ .

- $x, y \in \mathbb{R} \quad \forall \epsilon > 0, |x - y| \leq \epsilon \implies x = y$ .

Supremum:  $E \subset S$  bounded above. Suppose  $\sup E \in S$

- $e \leq \sup E \forall e \in E$ .
- $\forall \beta < \sup E, \exists e \in E$  s.t.  $\beta < e < \sup E$

OR

$\forall \epsilon > 0, \exists e \in E$  s.t.  $\sup E - \epsilon < e \leq \sup E$ .

### Example 16.2

$\sup \left\{ \frac{1}{n} \right\}_{n \geq 1} = 1, \quad \inf \left\{ \frac{1}{n} \right\} = 0$ .

- $0 \leq \frac{1}{n} \forall n \in \mathbb{N}$ .
- $\forall \epsilon > 0, \exists n \in \mathbb{N}$  s.t.  $0 \leq \frac{1}{n} < \epsilon$  by Archimedean Prop.

### Theorem 16.3 (Nested Interval)

$\{I_n = [a_n, b_n]\}_{n \geq 1} \subset \mathbb{R}, I_n \supset I_{n+1} \implies \bigcap_{n=1}^{\infty} I_n \neq \emptyset$ . Moreover, if  $|I_n| \rightarrow 0$ , then  $\bigcap I_n$  is a singleton (a set with exactly one element).

*Proof.*  $\sup a_n \in \bigcap I_n$ . □

### Theorem 16.4 ((4.1))

(Bolzano – Weierstrass): Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

*Proof.*  $I_0 = [-M, M] \supset I_1 \supset I_2 \supset \dots$

$$|I_n| = (2M) \cdot 2^{-n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

From Nested Interval Thm,  $\bigcap_{n=0}^{\infty} I_n = \{x\}$ . Choose  $x_{n_k} \in I_k, x_{n_k} \rightarrow x$ . □

**Remark 16.5.** l.u.b property of  $\mathbb{R} \implies$  Nested Interval  $\implies$  Bolzano – Weierstrass  $\xRightarrow{(*)}$  Cauchy Completeness.

(\*) Exercise:  $\{x_n\}$  Cauchy.  $x_{n_k} \rightarrow x \implies x_n \rightarrow x$ .

**Remark 16.6.** In  $\mathbb{R}$ , to check convergence, it suffices to check Cauchyness. Useful especially when you don't have a candidate for the limit. Cauchy criterion for series  $\sum_{n=1}^{\infty} a_n$  converges  $(\lim_{n \rightarrow \infty} \sum_{k=0}^n a_k)$  exists.  $\iff \sum a_n$  Cauchy  $(\forall \epsilon > 0 \exists N |\sum_{k=n}^m a_k| < \epsilon \quad \forall m \geq n \geq N)$ .

### Corollary 16.7

Absolute convergence  $\implies$  convergence.  $(\sum |a_n| \text{ converges} \implies \sum a_n \text{ converges})$ .

Monotone convergence theorem,  $\{a_n\}$  monotone. Then  $\{a_n\}$  bounded  $\iff \{a_n\}$  convergent. (HW 3 – Q1).

**Definition 16.8 (Monotone Sequence)** —  $\{a_n\}$  monotone if  $a_n \leq a_{n+1} \forall n$  or  $a_n \geq a_{n+1} \forall n$ .

### Corollary 16.9

$\sum |a_n| < \infty \iff \sum |a_n| \text{ converges}$ .

## §16.2 Continuity

**Definition 16.10 ( (6.2))** —  $f : X \rightarrow \mathbb{R}$  is continuous at  $x$  (local prop) if

1. ( $\epsilon - \delta$  def)  $\forall \epsilon > 0, \exists \delta(\epsilon, x) > 0$  s.t.  $\forall y \in X, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ .
2. (Sequential def)  $\forall \{x_n\} \subset X, x_n \rightarrow x \implies f(x_n) \rightarrow f(x)$  ( $f$  preserves sequential convergence).
3.  $\lim_{y \rightarrow x} f(y) = f(x)$

$f : X \rightarrow \mathbb{R}$  is continuous if  $f$  is continuous at all  $x \in X$ .

**Definition 16.11 ( (7.1))** —  $f$  is uniformly continuous on  $X$  (global prop) if

1. ( $\epsilon - \delta$ )  $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$  s.t.  $\forall x, y \in X |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ .
2. (Sequential)  $\forall \{x_n\} \subset X, \{x_n\}_{n \geq 1} \text{ Cauchy} \implies \{f(x_n)\}_{n \geq 1} \text{ Cauchy}$ . ( $f$  preserves Cauchy seq).

**Remark 16.12.** Uniform continuity  $\implies$  continuity.

**Example 16.13**

$f : (0, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = \frac{1}{x}$  is continuous.

$$\left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x - y|}{xy} < \frac{|x - y|}{x \cdot \frac{x}{2}} = |x - y| \cdot 2x^{-2} < \epsilon$$

$$\delta = \min \left\{ \frac{x}{2}, \frac{\epsilon x^2}{2} \right\}.$$

**Remark 16.14.**  $x \mapsto \frac{1}{x}$  is uniformly continuous on  $(a, \infty) \forall a > 0$ .  
 $x \mapsto \frac{1}{x}$  is NOT uniformly continuous on  $(0, \infty)$ .

- $x_n = \frac{1}{n}, y_n = \frac{1}{n+1}$   $|x_n - y_n| \rightarrow 0$  but  $|\frac{1}{x_n} - \frac{1}{y_n}| = 1 \forall n$ .
- $\{\frac{1}{n}\}_{n \geq 1}$  Cauchy but  $\{n\}$  is not.

## §17 | Dis 5: Oct 27, 2020

### §17.1 Metric Spaces

**Definition 17.1 ((9.1))** — A metric on a set  $X$  is a function  $d : X \times X \rightarrow [0, \infty]$  s.t.

- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

Thus  $(X, d)$  is called a metric space.

**Example 17.2** •  $(X, d), A \subset X$ .  $d|_{A \times A}$  is a metric on  $A$ .

- (Discrete metric) Given any set  $X$ , define

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

Check  $d$  is a metric on  $X$ .

**Remark 17.3** (norm). Given a vector space  $X$ . A norm on  $X$  is a function  $\|\cdot\| : x \rightarrow [0, \infty)$  s.t.

- $\|x\| = 0 \iff x = 0$
- $\|\alpha x\| = |\alpha| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

Then  $d(x, y) = \|x - y\|$  is a metric on  $X$ .

**Example 17.4** •  $\mathbb{R}^d, \|\cdot\| = \|\cdot\|_2$  where  $\|x\|_2 = \sqrt{\sum_{i=1}^d |x_i|^2}$

- On  $\mathbb{R}^d$ , define  $\|x\|_p = \left(\sum_{i=1}^d \|x_i\|^p\right)^{\frac{1}{p}}, 1 \leq p < \infty$

Inequalities:

- Young's Inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, a, b \geq 0, \frac{1}{p} + \frac{1}{q} = 1$$

- Holden's Inequality:

$$\|xy\|_1 \leq \|x\|_p \|y\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty$$

- Minkowski's Inequality (triangle inequality for  $\|\cdot\|_p$ )

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

Define  $\|x\|_\infty = \max_{i=1}^d |x_i|$ . Then

$$\begin{aligned} \|xy\|_1 &\leq \|x\|_1 \|y\|_\infty \\ \|x + y\|_\infty &\leq \|x\|_\infty + \|y\|_\infty \end{aligned}$$

Hence  $(\mathbb{R}^d, \|\cdot\|_p)$  is a metric space  $\forall 1 \leq p \leq \infty$ . Note:

- $p = 1$  : taxicab / Manhattan metric
- $p = 2$  : Euclidean metric
- $p = \infty$  : sup metric

Notation:  $\mathbb{R}^N = \{(x_i)_{i \geq 1} : x_i \in \mathbb{R}\} = \{f : \mathbb{N} \rightarrow \mathbb{R}\}$

**Definition 17.5** — Given  $x \in \mathbb{R}^N$ ,  $\|x\|_p = (\sum_{i=1}^\infty |x_i|^p)^{\frac{1}{p}}, 1 \leq p < \infty$ .  $\|x\|_\infty = \sup |x_i|$

**Example 17.6**

$l^p(\mathbb{N}) = \{f : \mathbb{N} \rightarrow \mathbb{R}, \|f\|_p < \infty\}$ ,  $1 \leq p \leq \infty$ . So  $(l^p, \|\cdot\|_p)$  is a metric space and a vector space.

**Definition 17.7 (Completeness of Metric Space)** — A metric space  $(X, d)$  is complete if every Cauchy sequence with respect to  $d$  is convergent with respect to  $d$ .

**Example 17.8** •  $(\mathbb{Q}, |\cdot|)$  is not complete;  $(\mathbb{R}, |\cdot|)$  is complete.

- $(\mathbb{R}^d, \|\cdot\|_p)$  is complete.
- $(l^p(\mathbb{N}), \|\cdot\|_p)$  is complete ( $1 \leq p \leq \infty$ ).
- $([0, 1], \mathbb{R}) = \{f : [0, 1] \rightarrow \mathbb{R}\}$  continuous

$$\|f\|_\infty = \sup_{x \in [0, 1]} |f(x)| \rightarrow \|f - g\| = \sup_{x \in [0, 1]} |f(x) - g(x)|$$

$(C([0, 1]), \|\cdot\|_\infty)$  is a complete metric space.

**Special structure when  $p = 2$** 

Inner product space:

Given vector space  $X/\mathbb{R}$  a real inner product on  $X$  is  $\langle \cdot, \cdot \rangle : x \times x \rightarrow [0, \infty]$  s.t.

- $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle, \forall a, b \in \mathbb{R}, x, y, z \in X$ .
- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x, x \rangle \in (0, \infty)$  and is 0  $\iff x = 0$ .

With the inner product:  $\|x\| = \sqrt{\langle x, x \rangle}$  is a norm, then  $(X, \|\cdot\|)$  is a metric space.

**Example 17.9**

$$\mathbb{R}^d : \langle x, y \rangle = x \cdot y = \sum x_i y_i$$

also,  $\|x\|_2 = \sqrt{\sum x_i^2} = \sqrt{\langle x, x \rangle}$

**Example 17.10**

$$l^2 : \langle f, g \rangle = \sum_{i=1}^{\infty} f(i)g(i) \text{ and } \|f\|_2 = \sqrt{\langle f, f \rangle} = \sqrt{\sum_{i=1}^{\infty} |f(i)|^2}$$

**Definition 17.11 (Orthogonality)** —  $x \perp y \iff \langle x, y \rangle = 0$

**Theorem 17.12 ( Cauchy – Schwarz)**

$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$  and equality holds  $\iff x, y$  are linearly dependent.

$\forall x, y \in X, \alpha \in \mathbb{R}$

$$\langle x - \alpha y, x - \alpha y \rangle = \|x - \alpha y\|^2 \geq 0$$

Goal: find  $\alpha$  that minimize  $\|x - \alpha y\|$

The intuition here is  $\|x - \alpha y\|$  is shortest when  $x - \alpha y \perp y$ .

$$\langle x - \alpha y, x - \alpha y \rangle = \|x\|^2 + \alpha^2 \|y\|^2 - 2\alpha \langle x, y \rangle$$

is minimal when  $\alpha = \frac{\langle x, y \rangle}{\|y\|^2}$ . Let us set  $\alpha$  to such value, so

$$\begin{aligned} &= \|x\|^2 + \frac{|\langle x, y \rangle|^2}{\|y\|^2} - \frac{2|\langle x, y \rangle|^2}{\|y\|^2} \\ &= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \geq 0 \end{aligned}$$