Math 134 – Nonlinear ODE

University of California, Los Angeles

Duc Vu

Winter 2021

This is math 134 – Linear and Nonlinear System of Differential Equations taught by Professor Wink. The class lecture is prerecorded, and we have live session every Monday and Friday at 3:00 pm – 3:50 pm for Q & A. We use *Nonlinear Dynamics and Chaos* 2^{nd} by *Steven Strogatz* as our main book for the class. Other course notes can be found through my github. Any error spotted in the notes is my responsibility, and please let me know through my email at ducvu2718@ucla.edu.

Contents

List of Theorems

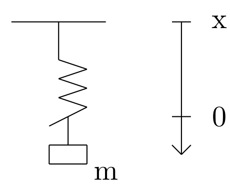
List of Definitions

§1 Lec 1: Jan 4, 2021

§1.1 Intro to Dynamical Systems

There are two types of dynamical systems:

1. Discrete in time:
   - Difference equation
   - Iterated map: $a_{n+1} = f(a_n)$
2. Continuous in time: differential equation
   - Partial Differential Equation (PDE):
     e.g. heat equation
     $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$
     wave equation
     $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$
     where the derivatives w.r.t time and space.
   - Ordinary Differential Equation (ODE):
     i) Harmonic oscillator



m: mass
k: spring constant

$$m\dot{x} + kx = 0$$

If $\omega^2 = \frac{k}{m}$, then
$$x(t) = x_0 \cos(\omega t) + x_1 \sin(\omega t)$$

ii) Damped harmonic oscillator
$$m\ddot{x} + b\dot{x} + kx = 0, \qquad b\text{: damping constant}$$

iii) Forced, damped harmonic oscillator
$$m\ddot{x} + b\dot{x} + kx = F\cos(t), \qquad F\text{: force}$$

so derivatives w.r.t time only.

Definition 1.1 (Order of ODE) — Highest occurring derivative is defined as the order of the ODE.

Remark 1.2. We can always write an ODE of n^{th} order as a system of ODEs of 1^{st} order.

Trick: Consider the damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

Then,

$$\begin{aligned} \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x \\ &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{aligned}$$

i.e.,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{aligned}$$

General framework: $\dot{x} = f(t, x)$

$$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

i.e.,

$$\begin{aligned} \dot{x}_1 &= f_1(t_1, x_1, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, \ldots, x_n) \end{aligned} \tag{1}$$

which is 1^{st} order n-dimensional ODE.

Definition 1.3 (Linear ODE) — The ODE (1) is called linear if $f(t, x) = A(t) \cdot x$ for a time dependent matrix $A(t)$, otherwise we call it non-linear.

Example 1.4

The damped harmonic oscillator is linear.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Question 1.1. Why are linear equations special?

They satisfy the principle of superposition. If ϕ, ψ solve $\dot{x} = A(t)x$, then $y(t) = c \cdot \phi(t) + \psi(t), c \in \mathbb{R}$ also solves $\dot{x} = A(t)x$. This is valid because $\dot{y} = c\dot{\phi} + \dot{\psi} = cA\phi + A\psi = A(c\phi + \psi) = Ay$. For non-linear ODEs, the principle of superposition fails.

Definition 1.5 (Autonomous ODE) — The ODE (1) is called autonomous if f does not depend on t, i.e., $f(t,x) = f(x)$.

Example 1.6

$$m\ddot{x} + b\dot{x} + kx = F\cos(t)$$

is non-autonomous.

However, we can always consider an autonomous system instead. Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= t \end{aligned}$$

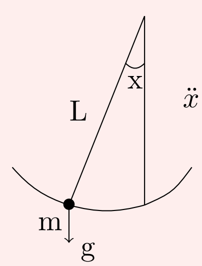
Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + F\cos(x_3) \\ \dot{x}_3 &= 1 \end{aligned}$$

We will primarily study autonomous 1^{st} order system in 1 or 2 variables.

Example 1.7 (Swinging Pendulum)

Consider a swinging pendulum



$$\ddot{x} + \tfrac{g}{L}\sin(x) = 0$$

Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L}\sin(x_1) \end{aligned}$$

1^{st} order, non-linear autonomous ODE in 2 variables.

Question 1.2. What can we say about the behavior of a solution $x_1(t), x_2(t)$ for larger time t? How does it depend on $\frac{g}{L}$?

<u>Idea</u>: Use geometric methods, without solving $\dot{x} = f(x)$ explicitly, to make qualitative statements about the long time behavior of the solution.

§2 Lec 2: Jan 6, 2021

§2.1 Phase Portraits

We want to study 1D autonomous dynamical systems

$$\dot{x} = f(x), \qquad f : \mathbb{R} \to \mathbb{R}$$

Remark 2.1. $x(t)$ is the solution to $\dot{x} = f(x)$ with $x(0) = x_0$. Find the solution $y(t)$ with $y(t_0) = x_0$.

Ans: $y(t) = x(t - t_0)$ because $y(t_0) = x(0) = x_0$ and $\dot{y}(t) = \dot{x}(t - t_0) = f(x(t - t_0)) = f(y(t))$.

Example 2.2

$x = \sin(x)$. Suppose $x_0 = \frac{\pi}{4}, x(t)$ solution with $x(0) = x_0$. Answer the followings

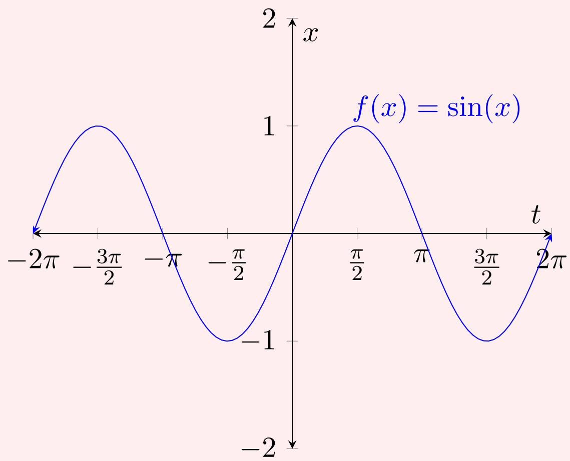
- Describe the long time behaviors of $x(t)$ as $t \to \infty$.
- How does the long time behavior depend on $x_0 \in \mathbb{R}$?

Attemp 1: Find explicit solution

$$\begin{aligned}\frac{dx}{dt} &= \sin(x)\\ dt &= \frac{dx}{\sin(x)}\\ t &= -\ln\left|\frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)}\right| + c\end{aligned}$$

We know $x(0) = x_0$, so $c = \ln\left|\frac{1+\cos(x_0)}{\sin(x_0)}\right|$. But what is $x(t) =$? This approach fails!

Attempt 2: Draw a phase portrait/diagram. We want to interpret the velocity $\dot{x} = f(x)$ as a vector field on the real line.

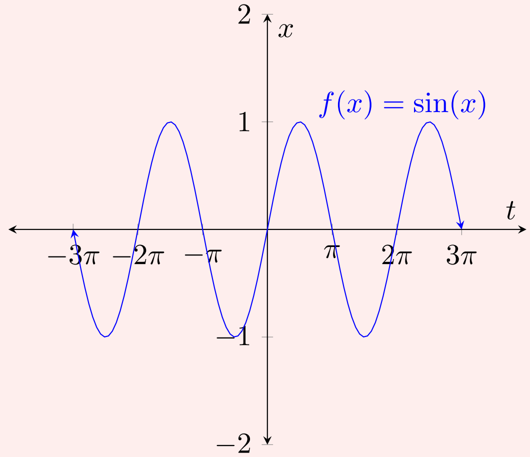


Idea:

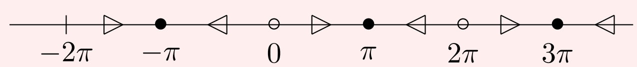
- If $f(x_0) > 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ increase near x_0.
- If $f(x_0) < 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ decrease near x_0.
- If $f(x_0) = 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ is $x(t) = x_0$ for all $t \in \mathbb{R}$, i.e., we have a fixed point/equilibrium point.

Example 2.3

$\dot{x} = f(x) = \sin(x)$



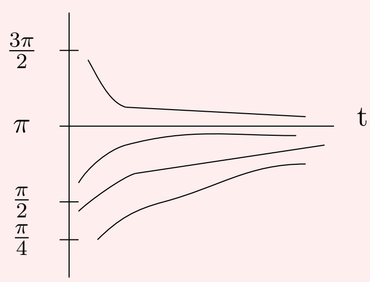
Phase portrait:



stable fixed point (sink, attractors)

unstable fixed point (source,)

Qualitative plot of solution:

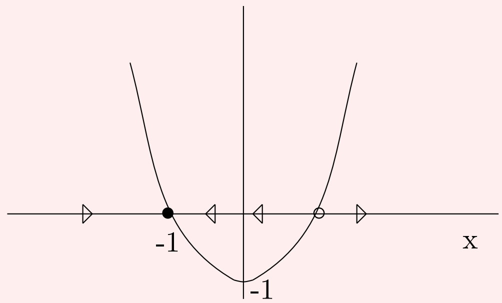


$x(t)$ is concave up if $f(x(t))$ is increasing

$x(t)$ is concave down if $f(x(t))$ is decreasing

Example 2.4

$\dot{x} = x^2 - 1$. Fixed points: $f(x) = x^2 - 1 = 0 \implies x = \pm 1$



<u>*Note*</u>: If $x_0 > 1$, then solution $x(t)$ with $x(0) = x_0 > 1$ is unbounded. In fact, $x(t) \to \infty$ in finite time.

§3 Lec 3: Jan 8, 2021

§3.1 Stability Types of Fixed Points

Definition 3.1 (Stability Types) — Consider the ODE $\dot{x} = f(x)$ and suppose that $f(x_*) = 0$. The fixed point x_* is called

1. Lyapunov stable if every solution $x(t)$ with $x(0) = x_0$ closed to x_* remain close to x_* for all $t \geq 0$, otherwise unstable.

2. Attracting if every solution $x(t)$ with $x(0) = x_0$ close to x_* satisfies $x(t) \to x_*$ as $t \to \infty$.

3. (asymptotically) stable if x_* is both Lyapunov stable and attracting.

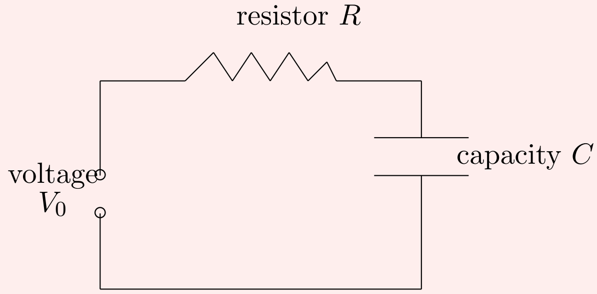
Example 3.2

Let $\alpha \in \mathbb{R}, \dot{x} = \alpha x$. General solution $x(t) = x_0 e^{\alpha t}$.

- $x_* = 0$ is always an equilibrium solution.
- $x_* = 0$ is
    1. attracting if $\alpha < 0$
    2. Lyapunov stable if $\alpha \leq 0$
    3. unstable if $\alpha > 0$

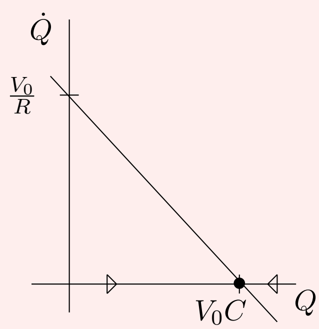
Example 3.3 (RC circuit)

We have the following circuit

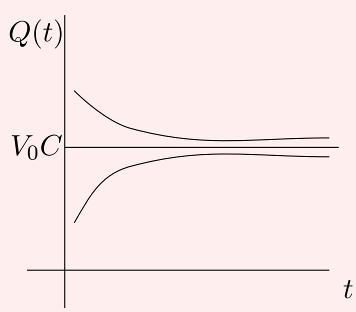


$$V_0 = RI + \frac{Q}{C}$$
$$I: \text{ current}, \ Q: \text{ change}$$
$$I = \dot{Q}$$
$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$$

Phase portrait



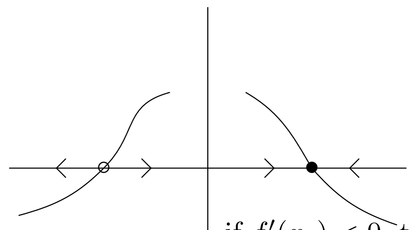
$Q_* = V_0C$ globally stable because every $Q(t)$ approaches Q_* as $t \to \infty$.



§3.2 Linear Stability Analysis

We have $\dot{x} = f(x), f(x_*) = 0$. Our task is to find an analytic criterion to decide if a fixed point x_* is stable/unstable.

Picture:



If $f'(x*) > 0$, then x_* is unstable. On the other hand, if $f'(x_*) < 0$, then x_* is a stable fixed point.
<u>The linearization</u>:
Consider: $\eta(t) = x(t) - x_*$ where $x(t)$ is the solution of $\dot{x} = f(x)$ with $x(0)$ close to x_*, $f(x_*) = 0$.
<u>Note</u>: $\dot{\eta}(t) = \dot{x}(t) = f\left(x(t)\right) = f\left(x(t) - x_* + x_*\right) = f\left(\eta(t) + x_*\right)$.
Taylor's Theorem:

$$f(x_* + \eta) = \underbrace{f(x_*)}_{=0} + f'(x_*)\eta + \underbrace{\mathcal{O}(\eta^2)}_{\text{error term and negligible if} f'(x_*)\neq 0 \text{ and } \eta \text{ is small}}$$

$\implies \quad \dot{\eta}(t) \approx f'(x_*)\eta(t)$ (as long as $\eta(t)$ is small) which is called the linearization of $\dot{x} = f(x)$ about x_*. The general solution is

$$\eta(t) = \eta_0 e^{f'(x_*)\cdot t}$$

In particular, η grows exponentially if $f'(x_*) > 0$ or decreases exponentially if $f'(x_*) < 0$.

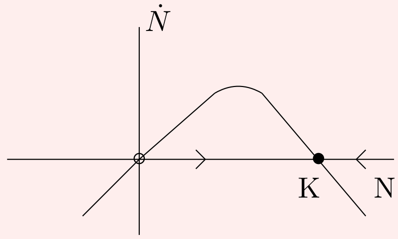
Definition 3.4 (Characteristics Time Scale) — $\frac{1}{|f'(x_*)|}$ is called the characteristics time scale.

Example 3.5 (Logstics Equation)

$N \geq 0$ population size, $r > 0$ growth rate, $K > 0$ carrying capacity

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

Fixed points: $\dot{N} = 0 \implies N_* = 0$ or $N_* = K$.
Let $f(N) = rN\left(1 - \frac{N}{K}\right) \implies f'(N) = r - 2\frac{r}{K}N$. In particular, $f'(0) = r > 0 \implies N_* = 0$ is an unstable fixed point and $f'(K) = r - 2r = -r < 0 \implies N_* = K$ is stable.
Phase portrait:



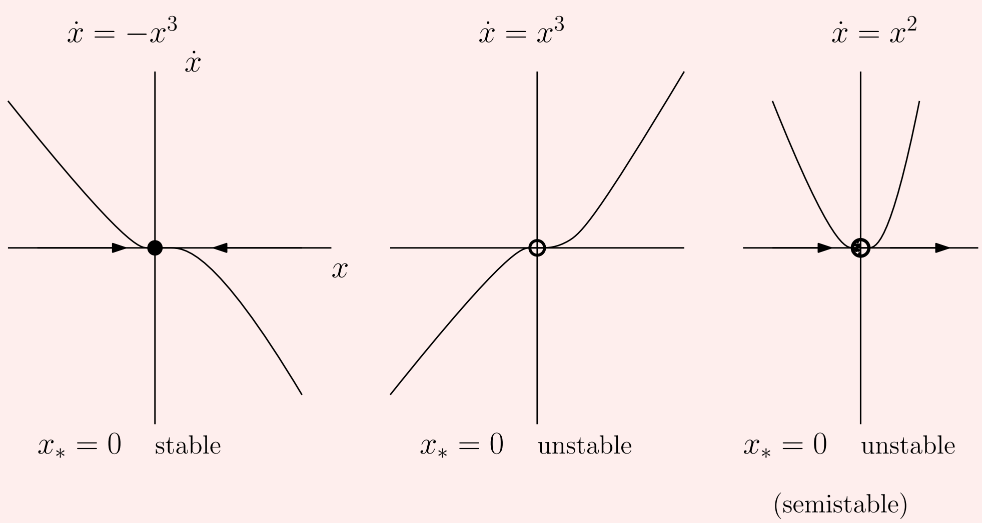
Thus, if $N(t)$ is the population with

$$N(0) = N_0 > 0 \implies N(t) \to K \text{ as } t \to \infty$$
$$N(0) = 0 \to N(t) = 0 \quad \forall t \text{ (no spontaneous outbreak)}$$

Characteristics time scale: $\frac{1}{|f'(N_*)|} = \frac{1}{r}$ for both $N_* = 0, K$.

Example 3.6

What if $f'(x_*) = 0$? Then we can't tell.

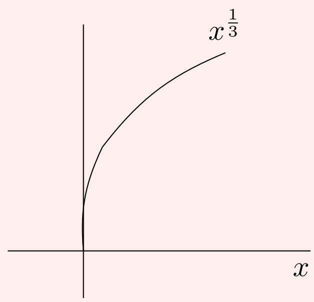


§4 Lec 4: Jan 11, 2021

§4.1 Existence and Uniqueness

Example 4.1 (Non-uniqueness)

$\dot{x} = x^{\frac{1}{3}} \implies x_1(t) \equiv 0$ (for all t) is a solution with $x_1(0) = 0$ but $x_2(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$ is also a solution with $x_2(0) = 0$



Is $x_0 = 0$ really a fixed point? No, it's unclear how it would behave (according to $x(t) = 0$ or $x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$).

Theorem 4.2 (Picard's)

Let $I = (a, b) \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ differentiable and f' continuous. Let $x_0 \in I$. Then there is $\tau > 0$ s.t. the initial value problem

$$\dot{x} = f(x), x(0) = x_0$$

has a unique solution $x : (-\tau, \tau) \to \mathbb{R}$.

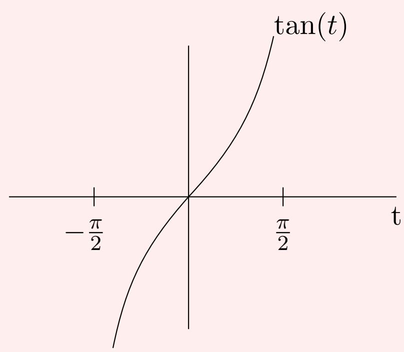
Example 4.3

(The solution might not exist for all times) Consider

$$\frac{dx}{dt} = \dot{x} = 1 + x^2, \quad x(0) = 0$$

So,

$$\begin{aligned} dt &= \frac{dx}{1+x^2} \\ t &= \int \frac{dx}{1+x^2} = \arctan x + C \\ 0 &= 0 + C \implies C = 0 \\ x(t) &= \tan(t) \end{aligned}$$



In particular,

$$\begin{aligned} x(t) &\to +\infty \text{ as } t \to \frac{\pi}{2} \\ x(t) &\to -\infty \text{ as } t \to \frac{-\pi}{2} \end{aligned}$$

i.e., $x(t)$ reaches infinity in finite time, i.e., the solution $x(t)$ blows up in finite time.

Remark 4.4. (Hw 1) If $x_0 > 0$, then the solution to $\dot{x} = x^2, x(0) = x_0 > 0$ blows up in finite time. In fact, if $\alpha > 1$, then the solution to $\dot{x} = x^\alpha$, $x(0) = x_0 > 0$ blows up in finite time.

Theorem 4.5 (ODE Comparison)

If $x_1(t)$ solves $\dot{x} = f(x)$, $x_2(t)$ solves $\dot{x} = q(x)$ and $x_1(0) \le x_2(0)$, $f(x) < q(x)$, then $x_1(t) \le x_2(t)$ for all $t > 0$.
In particular, if $x_1(t) \to \infty$ in finite time, then $x_2(t) \to \infty$ in finite time.

Example 4.6

The solution to $\dot{x} = 1 + x^2 + x^3, x(0) = 0$ blows up in finite time.
<u>*Note*</u>: For $x \geq 0$:

$$1 + x^2 \leq 1 + x^2 + x^3$$

Recall: $\tan(t)$ solves $\dot{x} = 1 + x^2, x(0) = 0$. By comparison: the solution $x(t)$ to $\dot{x} = 1 + x^2 + x^3, x(0) = 0$ satisfies $x(t) \geq \tan(t)$. Thus, $x(t)$ blows up in finite time. We may indeed assume that $x(t) > 0$. Since $\dot{x}(0) = 1$, it follows that $x(t) > 0$ for $t > 0$ small. In fact, $\dot{x} = 1 + x^2 + x^3 > 0$ for $x(t)$ small, i.e., whenever $x(t)$ is close to zero, it must increase $\implies x(t) > 0$ for $t > 0$.

Example 4.7 (No Oscillating Solution in 1D)

Let $f \in C^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} | f \text{ differentiable}, f' \text{ continuous}\}$. Suppose $f(x_*) = 0, x(t)$ solution of $\dot{x} = f(x)$. If $x(t_0) = x_*$ for some t_0. Then $x(t) = x_*$ for all time t. Geometrically this says that a solution can never reach/cross a fixed point (unless it is a fixed point).

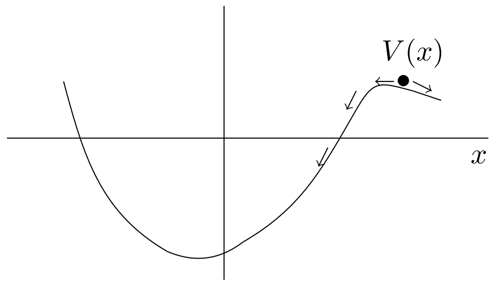
- $f(x(t)) > 0$ and $\dot{x}(t) > 0$, i.e., $x(t)$ increases.
- $f(x(t)) = 0$ and $x(t) = \text{constant}$ for all t.
- $f(x(t)) < 0$ and $\dot{x}(t) < 0$ i.e., $x(t)$ decreases.

In particular, there is no oscillating solution.

§5 Lec 5: Jan 13, 2021

§5.1 Potential

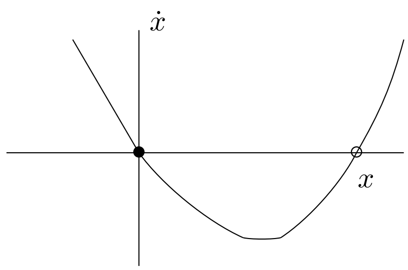
Consider the movement of a particle (with lots of friction) in a potential.



Notice:

- Particle approaches the local minimum of $V(x)$ (minimum energy level) no fixed point.
- Local minima of $V(x)$ are stable fixed points.
- Local maxima of $V(x)$ are unstable fixed points.

$\implies$ $\dot{x} = f(x) = -\frac{dV}{dx} = -V'(x)$.



Expect $t \to V(x(t))$ is non-increasing for a solution $x(t)$ of $\dot{x} = -V'(x)$.
Indeed:

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= V'(x(t))\frac{d}{dt}x(t)\\ &= V'(x(t))\left(-V'(x(t))\right)\\ &= -\left(V'(x(t))\right)^2 \le 0\end{aligned}$$

$\implies$ particle always moves towards a lower energy level.

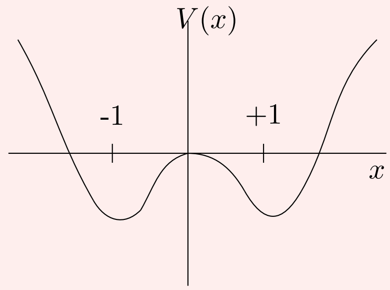
Definition 5.1 (Potential) — A function $V(x)$ s.t. $\dot{x} = f(x) = -\frac{dV}{dx}$ is called a potential.

Example 5.2

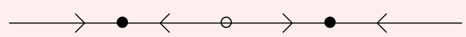
Graph potential for $\dot{x} = x - x^3$. Find/characterize equilibria (fixed points).

$$\dot{x} = f(x) = x - x^3 = -\frac{dV}{dx} \overset{\int}{\Longrightarrow} V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$$

$\Longrightarrow$ V is only defined up to a constant, we may choose any $C \in \mathbb{R}$, e.g., choose $C = 0$.



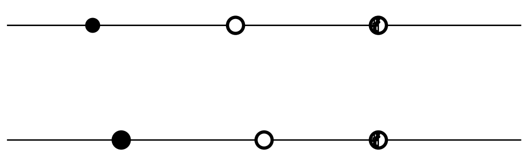
Local minima of V correspond to stable fixed points $\Longrightarrow$ $0 = -\frac{dV}{dx} = f(x) = x - x^3$, i.e., $x = \pm 1$.
Local maximum of V corresponds to an unstable fixed point at $x = 0$.
Phase portrait:



Remark 5.3. This system is often called <u>bistable</u> because it has two stable fixed points.

§5.2 Bifurcations

The qualitative behavior of 1D dynamical systems $\dot{x} = f(x)$ is determined by fixed points.



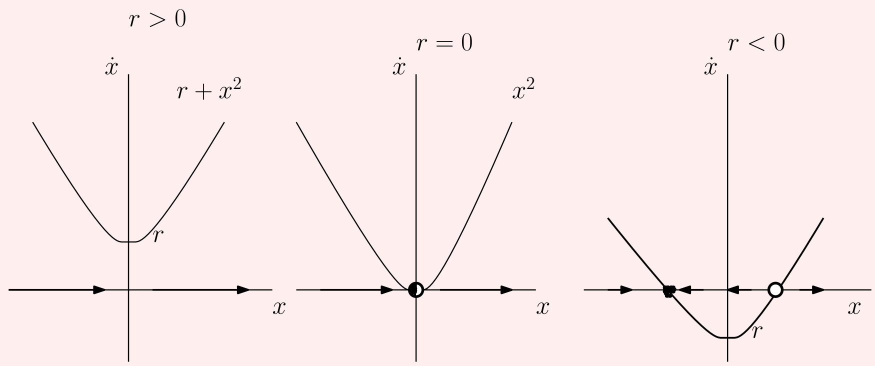
qualitative the same

(dynamics)

If $\dot{x} = f(r, x)$ depends on a parameter r, then the numbers of fixed points and their stability may change as r varies. This is called bifurcation.

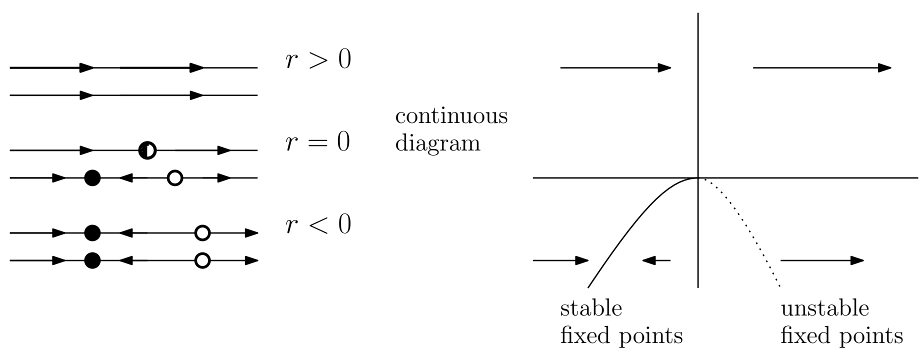
Example 5.4 (Saddle-node, blue sky bifurcation)

$\dot{x} = r + x^2, \quad r \in \mathbb{R}.$

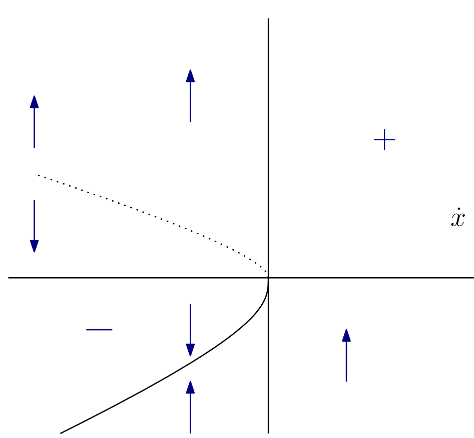


Hence, the qualitative behavior changes at $r_* = 0$, i.e., $r_* = 0$ is called a <u>bifurcation point</u>.

Ways to plot the dependence on the parameter:



Most common: <u>bifurcation diagram</u>



$\dot{x} = r + x^2$

$\dot{x} = 0 \to x = \pm\sqrt{-r}$

§6 Lec 6: Jan 15, 2021

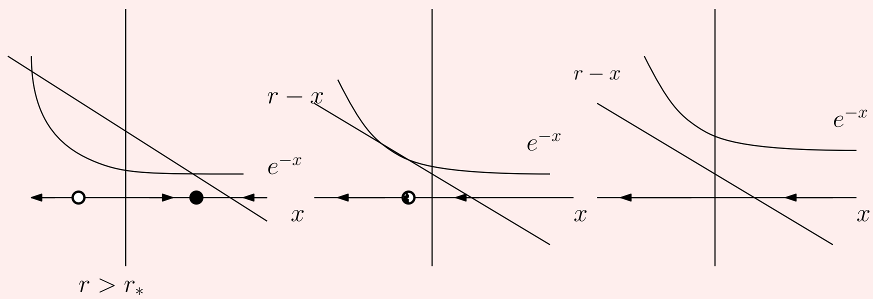
§6.1 Saddle-Node Example

Example 6.1

Argue geometrically that the ODE

$$\dot{x} = r - x - e^{-x}$$

undergoes a saddle-node bifurcation. Furthermore, find the bifurcation point.
Note: Fixed points of $\dot{x} = r - x - e^{-x}$ correspond to intersection points of the functions $r - x, e^{-x}$ because $r - x - e^{-x} = 0 \iff r - x = e^{-x}$.



Indeed we have a saddle-node bifurcation.
Note: At $r = r_*$, the graph of $r - x$ and e^{-x} intersect tangentially. Thus, for the bifurcation point we require:

$$0 = \dot{x} = r - x - e^{-x} \implies r - x = e^{-x}$$
$$0 = \frac{d}{dx}(r - x - e^{-x}) \implies \frac{d}{dx}(r - x) = \frac{d}{dx}e^{-x}$$

So,

$$\begin{aligned} -1 &= -e^{-x} \\ e^{-x} &= 1 \\ x &= 0 \\ r_* &= x_* + e^{-x_*} = 0 + 1 = 1 \end{aligned}$$

Thus the bifurcation point is $(r_*, x_*) = (1, 0)$.

Note:

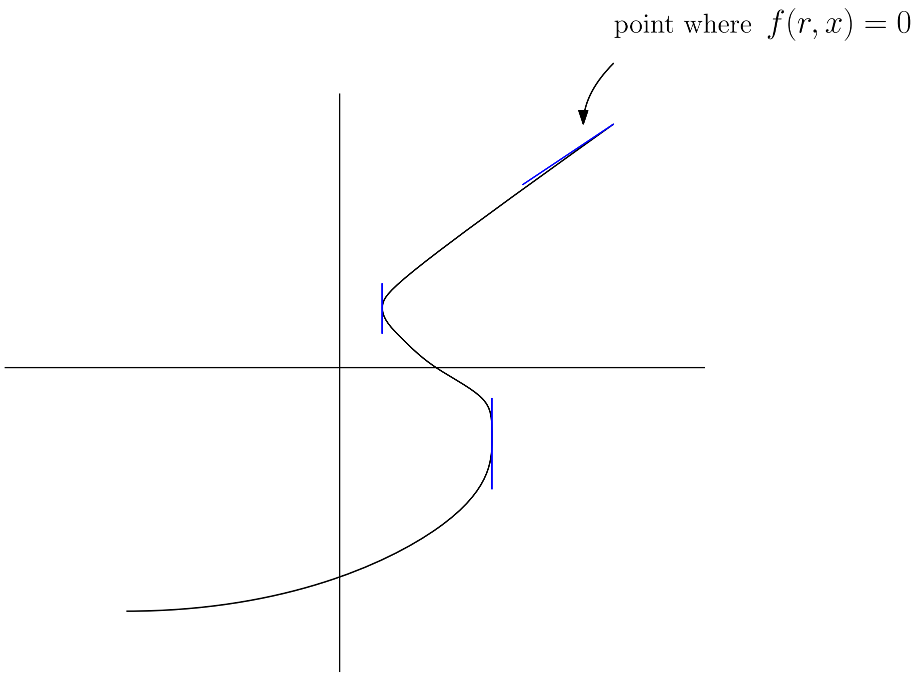
$$\begin{aligned} \dot{x} = r - x - e^{-x} &= r - x - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right) \\ &= r - 1 - \frac{1}{2}x^2 + \frac{x^3}{6} - \dots \\ &\approx (r - 1) - \frac{1}{2}x^2 \text{ for x near } x_* = 0 \end{aligned}$$

Set $R = r - 1$, then $\dot{x} \approx R - \frac{1}{2}x^2$.
Upshot: Up to appropriate rescalings/coordinate changes, every saddle-node bifurcation looks like its normal form

$$\dot{x} = r - x^2 \qquad (\text{or } \dot{x} = r + x^2)$$

close to the bifurcation point $(r_*, x_*) = (0, 0)$.

§6.2 Normal Forms



Recall:

- Normal vector: $\begin{pmatrix} \partial_r f \\ \partial_x f \end{pmatrix}$
- Tangent vector: $\begin{pmatrix} -\partial_x f \\ \partial_r f \end{pmatrix}$

<u>*Note*</u>: Bifurcation points have vertical tangent vectors, i.e., $\partial_x f = 0, \partial_r f \neq 0$.

Theorem 6.2 (Taylor's)
Suppose $f(r_*, x_*) = 0$.

$$f(r,x) = f(r_*, x_*) + \underbrace{\frac{\partial f}{\partial r}(r_*, x_*)}_{p_1}(r - r_*) + \underbrace{\frac{\partial f}{\partial x}(r_*, x_*)}_{q_1}(x - x_*)$$
$$+ \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial r^2}(r_*, x_*)}_{p_2}(r - r_*)^2 + \underbrace{\frac{\partial^2 f}{\partial r \partial x}(r_*, x_*)}_{R}(r - r_*)(x - x_*) + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial x^2}(r_*, x_*)}_{q_2}(x - x_*)^2 + \dots$$

Remark 6.3. If $q_1 \neq 0$, then there is no bifurcation at (r_*, x_*), linear stability (sign of q_1) determines if (r_*, x_*) is (un)stable.

Theorem 6.4

Suppose that $f(r_*, x_*) = 0, q_1 = 0, p_1 \neq 0, q_2 \neq 0$, then $\dot{x} = f(r, x)$ undergoes a saddle node bifurcation at (r_*, x_*) and

$$\dot{x} = \frac{\partial f}{\partial r}(r^*, x^*)(r - r^*) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

for $|r - r_*| < \epsilon^2, \quad |x - x_*| < \epsilon$.

Remark 6.5.

i) Note that the constant $(r - r_*)(x - x_*)$ is $\mathcal{O}(\epsilon^3)$

ii) With a coordinate change $(t, x, r) \mapsto (s, y, R)$ we can arrange that ODE looks like

$$\frac{d}{ds}y = R + y^2$$

near $(0, 0) = (R(r_*), y(x_*))$

Example 6.6

$\dot{x} = e^r - x - e^{-x}$ undergoes a saddle-node bifurcation near $(r_*, x_*) = (0, 0)$. Apply the theorem 6.4,

$$\begin{aligned}
f(r,x) &= e^r - x - e^{-x} \\
f(0,0) &= 1 - 0 - 1 = 0 \\
\frac{\partial f}{\partial x}(r,x) &= -1 + e^{-x} \implies \frac{\partial f}{\partial x}(0,0) = 0 \\
\frac{\partial f}{\partial r}(r,x) &= e^r \implies \frac{\partial f}{\partial r}(0,0) = 1 \neq 0 \\
\frac{\partial^2 f}{\partial x^2}(r,x) &= -e^{-x} \implies \frac{\partial^2 f}{\partial x^2}(0,0) = -1 \neq 0
\end{aligned}$$

Therefore, by theorem 6.4, $(r_*, x_*) = (0, 0)$ is a bifurcation point of a saddle-node bifurcation.
Normal form near $(r_*, x_*) = (0, 0)$:

$$\begin{aligned}
\dot{x} &= e^r - x - e^{-x} \\
&= 1 + r + \frac{r^2}{2} + \mathcal{O}(r^3) - x - \left(1 - x + \frac{x^2}{2} + \mathcal{O}(x^3)\right) \\
&= r + \underbrace{\frac{r^2}{2}}_{\mathcal{O}(\epsilon^4)} - \frac{x^2}{2} + \mathcal{O}(r^3) + \mathcal{O}(x^3) \\
&= \underbrace{r - \frac{x^2}{2}}_{\mathcal{O}(\epsilon^2)} + \mathcal{O}(\epsilon^3) \text{ if } |r - r_*| = |r| < \epsilon^2 \\
&\qquad\qquad \text{if } |x - x_*| = |x| < \epsilon
\end{aligned}$$

Set $y = \frac{x}{2}$, then

$$\dot{y} = \frac{1}{2}\dot{x} = \frac{r}{2} - \frac{x^2}{4} + \mathcal{O}(\epsilon^3) = \frac{r}{2} - y^2 + \mathcal{O}(\epsilon^3)$$

Set $s = -t$, then

$$\frac{d}{ds}y = -\frac{d}{dt}y = -\frac{r}{2} + y^2 + \mathcal{O}(\epsilon^3)$$

Set $R = -\frac{r}{2}$, then

$$\underbrace{\frac{d}{ds}y = R + y^2}_{\text{normal form of a saddle-node bifurcation}} \qquad +\mathcal{O}(\epsilon^3)$$

§7 Lec 7: Jan 20, 2021

§7.1 Classification of Bifurcations

Let's rewrite $\dot{x}$ in theorem 6.4 as

$$\dot{x} = p(r - r_*) + \frac{c}{2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

if $|r - r_*| < \epsilon^2, |x - x_*| < \epsilon$. After a coordinate change $(t, x, r) \mapsto (s, y, R)$ such that

$$\begin{aligned} s &= t \\ y &= \frac{c}{2}(x - x_*) \\ R &= p\frac{c}{2}(r - r_*) \end{aligned}$$

the ODE is represented by the normal form.

$$\frac{d}{ds}y = \dot{y} = R + y^2 + \mathcal{O}(\epsilon^3)$$

for $|R| < \epsilon^2, |y| < \epsilon$.
If $f(x_*, r_*) = 0$, and also $\frac{\partial f}{\partial x}(x_*, r_*) = 0 = \frac{\partial f}{\partial r}(x_*, r_*)$, then the second derivatives determines the bifurcation type.

$$\text{Hessian} \quad \text{Hessf} = \begin{pmatrix} \frac{\partial^2 f}{\partial r^2} & \frac{\partial^2 f}{\partial r \partial x} \\ \frac{\partial^2 f}{\partial r \partial x} & \frac{\partial^2 f}{\partial x^2} \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Second test: if $AC - B^2 > 0$, (r_*, x_*) is a local maximum/minimum. In particular, (r_*, x_*) is an isolated fixed point. (irrelevant case)
Practically relevant case: If $AC - B^2 < 0$: (r_*, x_*) is a saddle. If also $C \neq 0$: <u>transcritical bifurcation</u>.

$$\dot{y} = Ry - y^2 + \mathcal{O}(\epsilon^2)$$

for $|R| < \epsilon, |y| < \epsilon$ (after an appropriate coordinate change)

$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

If also $C = 0$: <u>Pitchfork bifurcation</u>

- <u>Supercritical</u> Pitchfork bifurcation:

$$y' = Ry - y^3 + \mathcal{O}(\epsilon^3)$$

- <u>Subcritical</u> Pitchfork bifurcation

$$y' = Ry + y^3 + \mathcal{O}(\epsilon^3)$$

  for $|R| < \epsilon^2, |y| < \epsilon$

Again,

$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

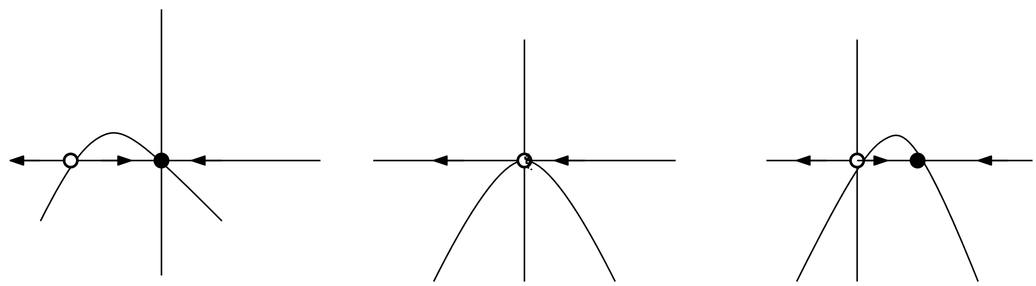
§7.2 Transcritical Bifurcation

Normal form:

$$\dot{x} = rx - x^2 = x(r-x)$$

In particular, $x_* = 0$ is always a fixed point but it changes stability.

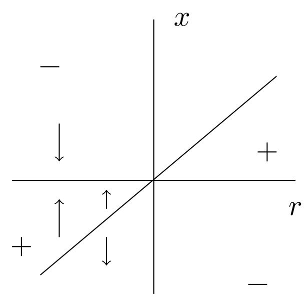
$r < 0$ $\qquad$ $r = 0$ $\qquad$ $r > 0$



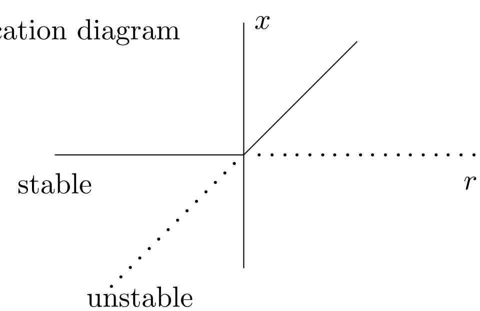
Bifurcation diagram: $\dot{x} = x(r-x) = rx - x^2 = f(x)$. Fixed points:

$$x_* = 0, \quad x_* = r \quad r \in \mathbb{R}$$

intermediate step:
draw fixed points
(without stability)



bifurcation diagram



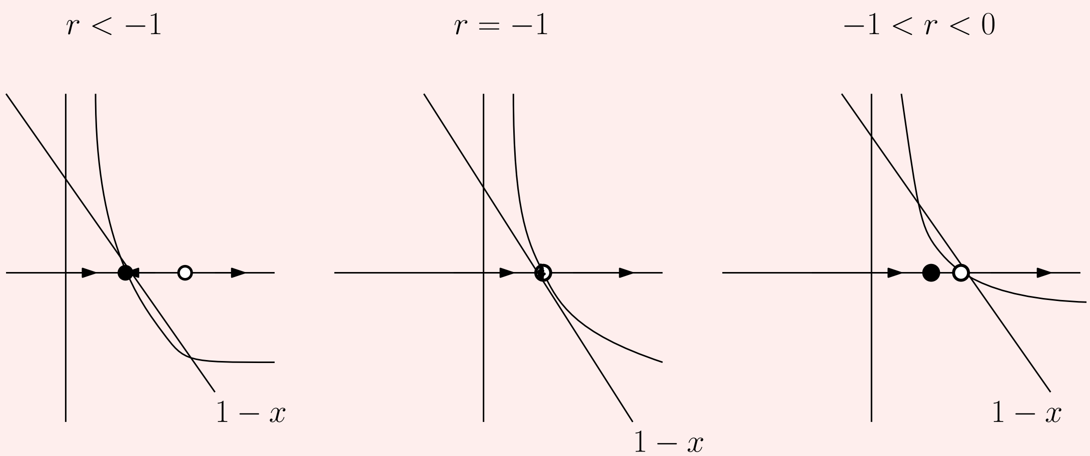
§8 Lec 8: Jan 22, 2021

§8.1 Example of Transcritical Bifurcation

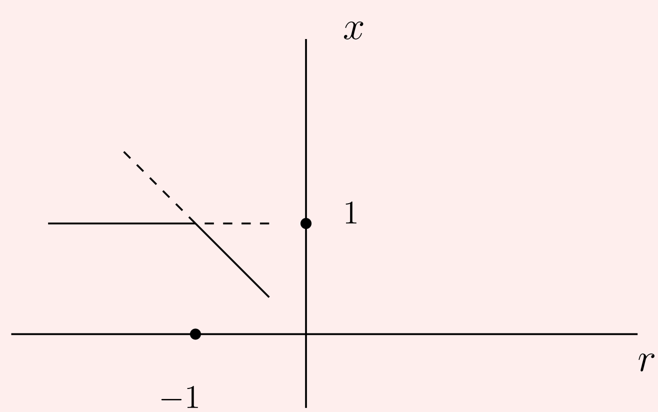
Example 8.1

$\dot{x} = r\ln(x) + x - 1$ has a transcritical bifurcation at $(r_*, x_*) = (-1, 1)$.
Geometric approach:

$$\dot{x} = 0 \iff r\ln(x) = 1 - x$$



Bifurcation near $(r_*, x_*) = (-1, 1)$



Normal form: $\dot{x} = r\ln(x) + x - 1$.

Remark 8.2. $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \quad |x| < 1$

So,

$$\begin{aligned}
\dot{x} &= r\ln(x) + x - 1 \\
&= r(x - 1 - \frac{1}{2}(x-1)^2 + \mathcal{O}((x-1)^3) + x - 1 \\
&= (r+1)(x-1) - \frac{1}{2}\left((r+1)-1\right)(x-1)^2 + \mathcal{O}\left(r(x-1)^3\right) \\
&= (r+1)(x-1) + \frac{1}{2}(x-1)^2 + \mathcal{O}(\epsilon^3)
\end{aligned}$$

if $|r-(-1)| < \epsilon$ and $|x - 1 < \epsilon|$.
Now, set $R = r+1, y = c \cdot (x-1)$. Then,

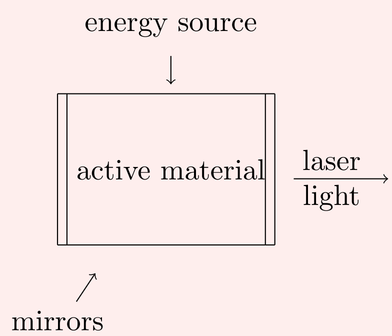
$$\begin{aligned}
\dot{y} &= c\dot{x} \\
&= (r+1)c(x-1) + \frac{1}{2}c(x-1)^2 + \mathcal{O}(\epsilon^3) \\
&= Ry + \frac{1}{2c}\left(c(x-1)\right)^2 + \mathcal{O}(\epsilon^3) \\
&= Ry + \underbrace{\frac{1}{2c}}_{=1} y^2 = Ry + y^2
\end{aligned}$$

for $c = \frac{1}{2}$.

§8.2 Application of Transcritical Bifurcations

Example 8.3 (Laser Threshold)

Consider



Simple model:

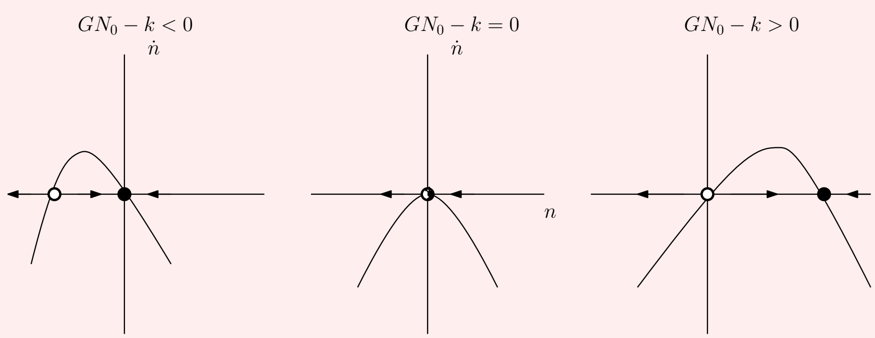
$$n = n(t) = \#\text{ photons in the laser}$$

Then

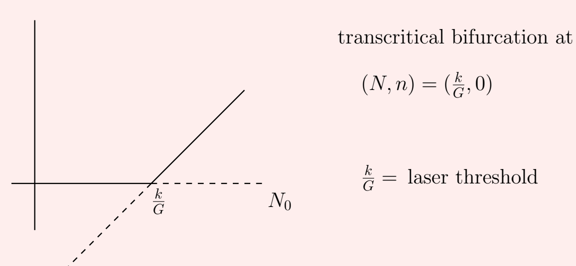
$$\begin{aligned}\dot{n} &= G \cdot \underbrace{N}_{\#\text{ excited atoms}} \cdot n - kn \\ &= N_0 - \alpha \cdot n \\ &= G(N_0 - \alpha n)n - kn \\ &= (GN_0 - k)n - \alpha G n^2\end{aligned}$$

where $G, k, \alpha > 0$. Fixed points:

$$\dot{n} = 0 \iff n = 0 \text{ or } n = \frac{GN_0 - k}{\alpha G}$$



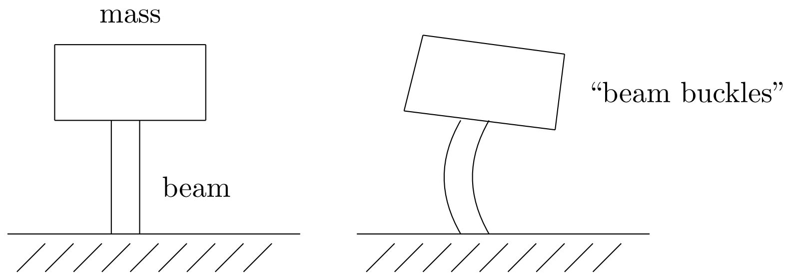
Bifurcation diagram



§9 Lec 9: Jan 25, 2021

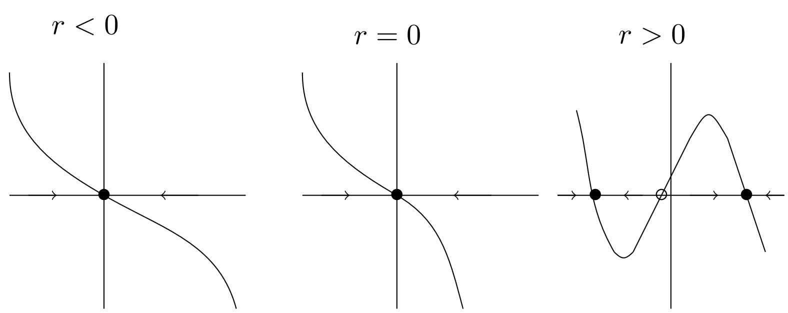
§9.1 Supercritical Pitchfork Bifurcation

Fixed points appear/disappear in symmetric pairs



Supercritical Pitchfork Bifurcation:

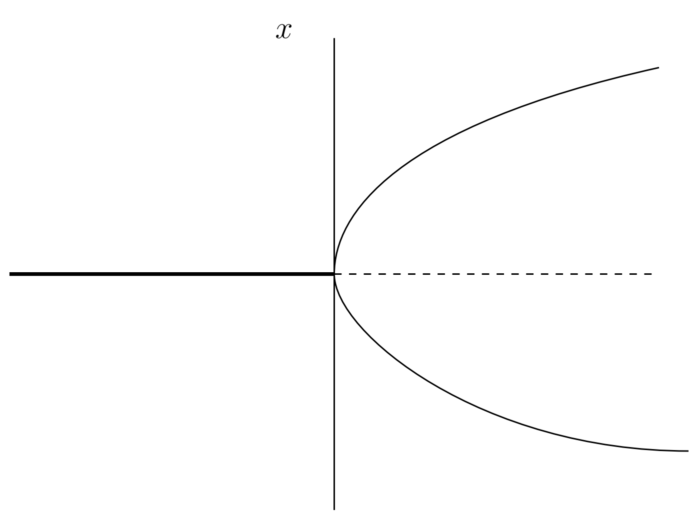
$$\dot{x} = rx - x^3$$



> **Remark 9.1.** Decay towards $x_* = 0$ is not exponential in time for $r = 0$.

Bifurcation diagram:

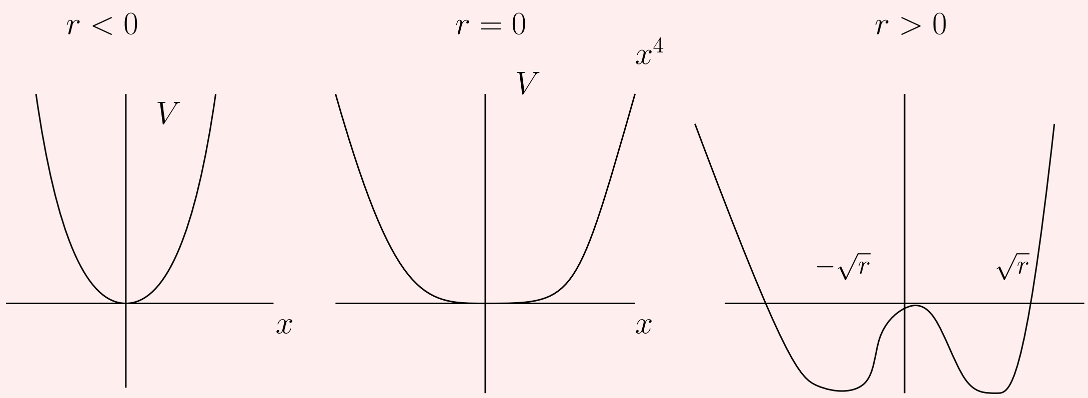
$$\dot{x} = rx - x^3 = 0$$
$$\implies x = 0, \qquad x = \pm\sqrt{r}, \quad r > 0$$



Example 9.2

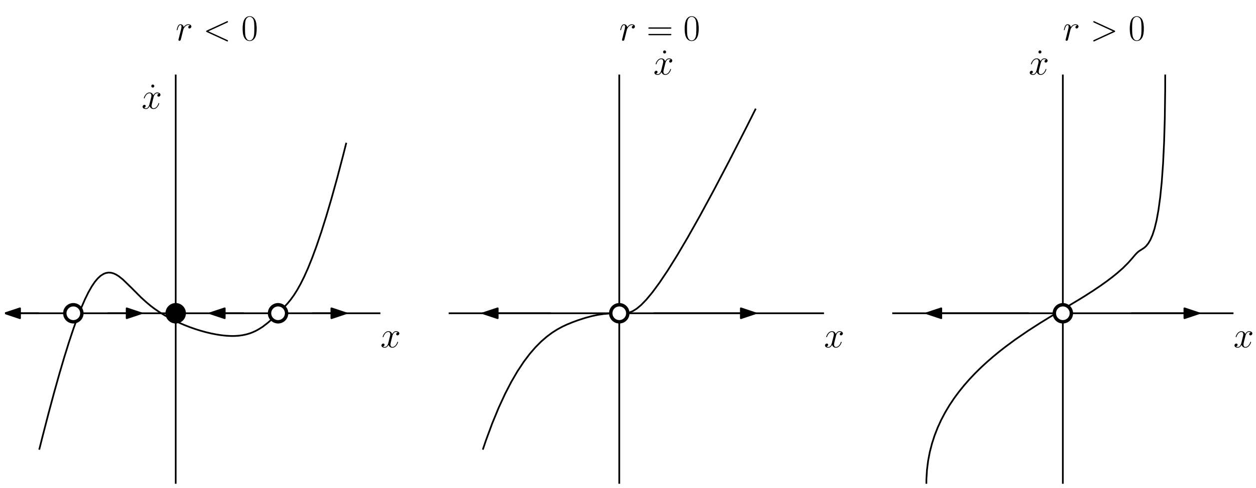
Potential for $\dot{x} = rx - x^3 = -\frac{dV}{dx}$

$$\implies V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4 + \underbrace{C}_{=0}$$



§9.2 Subcritical Pitchfork Bifurcation

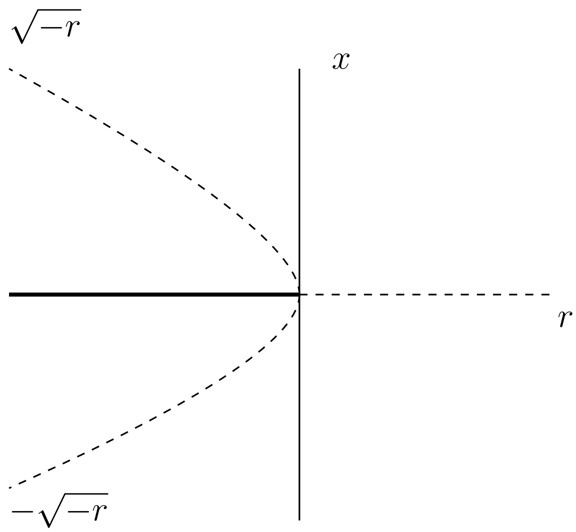
$$\dot{x} = rx + x^3$$



Fixed points:

$$\dot{x} = rx + x^3 = 0$$
$$\implies x = 0, \qquad x = \pm\sqrt{-r}, \quad r < 0$$

Bifurcation Diagram:



Remark 9.3. If $r > 0, x_0 > 0$, then the solution $x(t)$ with $x(0) = x_0 > 0$ blows up in finite time (cf. homework). Interpretation: $+x^3$ is destabilizing.

Physically more realistic scenario:

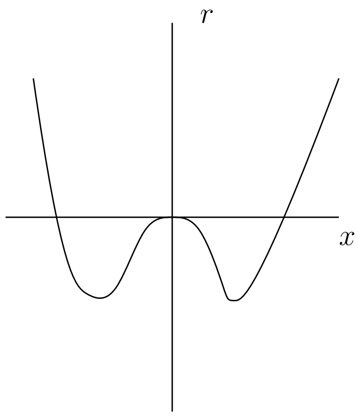
$$\dot{x} = rx + x^3 - x^5$$

where x^5 is the stabilizing higher order term.
Fixed points:

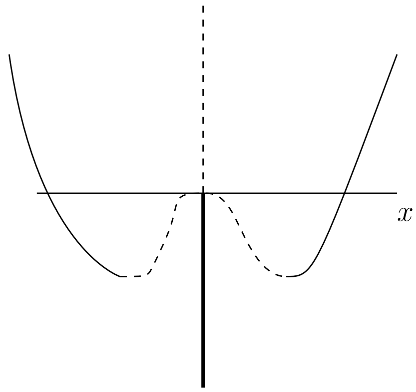
$$\dot{x} = 0 \iff x = 0, \quad r = -x^2 + x^4$$

Bifurcation diagram:

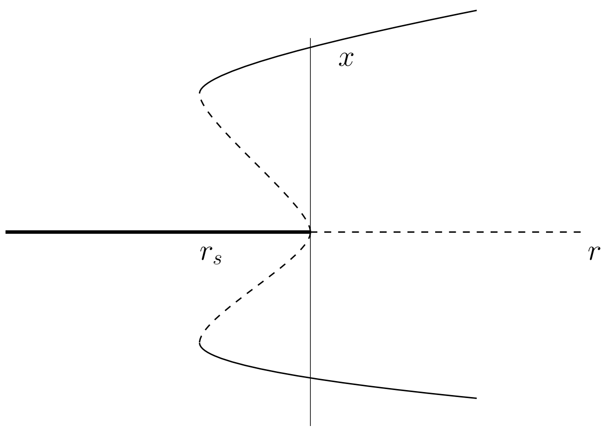
1. Intermediate step



2. Stability Types

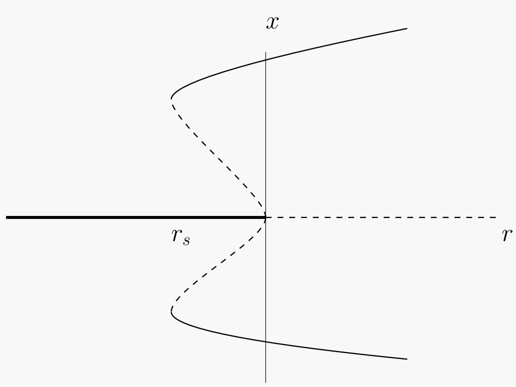


3. Change axes: bifurcation diagram



$$\dot{x} = rx + x^3 - x^5$$

Remark 9.4. i) Subcritical pitchfork bifurcation at $(r_*, x_*) = (0, 0)$ and saddle node bifurcation at $(r_s, x_*) = \left(-\frac{1}{4}, \pm\sqrt{2}\right)$.



$$\dot{x} = r(t)x + x^3 - x^5$$

ii) jump at $r_* = 0$: A small perturbation of a stable fixed point at $(0, r)$ with $r < 0$ jumps to the stable large amplitude branch as r becomes positive, but does not jump back until $r < r_s$.

This non-reversibility is called hysteresis.

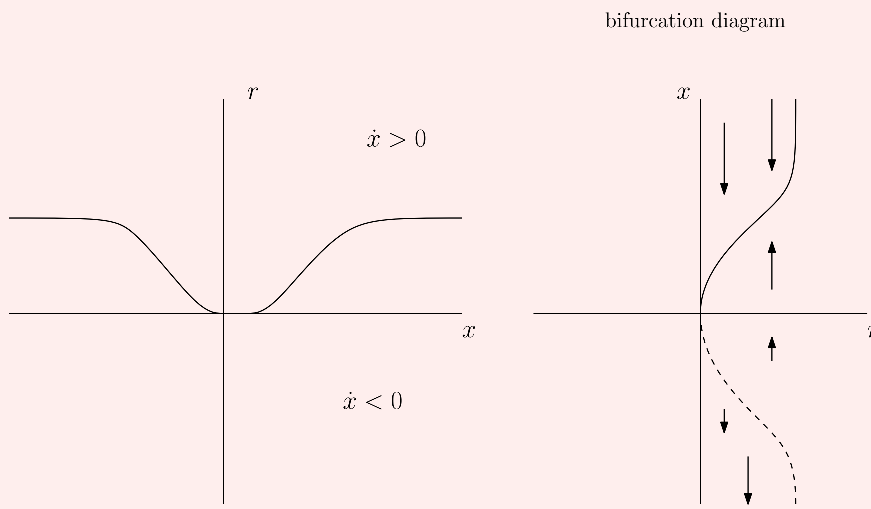
§10 Lec 10: Jan 27, 2021

§10.1 Bifurcation at Infinity

Example 10.1

$\dot{x} = r - \frac{x^2}{1+x^2}$

Fixed points: $\dot{x} = 0 \iff r = \frac{x^2}{1+x^2}$



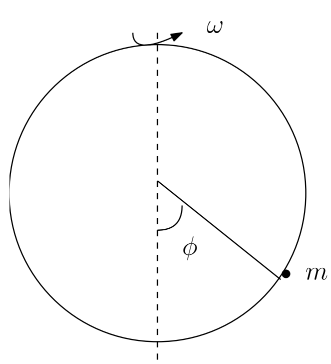
Note:

- At (r_*, x_*) we have a saddle node bifurcation.
- If $r \in (0, 1)$ we have two fixed points.
- For $r \geq 1$ we have no fixed points.

Thus, we have a bifurcation at (spatial) infinity.

§10.2 Dimensional Analysis and Scaling

Over-damped bead over a hoop:



forces: gravitation: $-mg\vec{e_2}$

centrifugal: $mr\sin\phi\omega^2\vec{e_x}$

damping: $-b\phi\vec{e_\phi}$

Physics: $mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$
Experiment: Provided ω large enough, bead slides slowly towards a fixed angle, after an initial acceleration phase.

Question 10.1. When we can neglect second order term $\ddot{\phi}$?

Problem 10.1. We're working with different dimensions, e.g.

$$[m] = kg$$
$$[b] = \frac{kg\cdot m}{s}$$

What is small – what quantity is actually small so we can neglect the second order term?

Idea: Non-dimensionalize

- small means $\ll 1$
- reduce the numbers of parameters
- no general algorithm

Quantity ω large, time scale T.
Set $\tau = \frac{t}{T} \implies d\tau = \frac{1}{T}dt$, where T is the characteristics time scale.
$\dot{\phi} = \frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d\phi}{d\tau}$
Similarly, $\ddot{\phi} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}$

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi \tag{1}$$

So

$$\implies \frac{mr}{T^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{T}\frac{d\phi}{d\tau} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi \qquad \text{(unit force)}$$
$$\implies \frac{r}{gT^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{mgT}\frac{d\phi}{d\tau} - \sin\phi + \frac{r\omega^2}{g}\sin\phi\cos\phi \qquad \text{(dimensionless)}$$

Thus 1st order term $\frac{d\phi}{d\tau}$ dominates $\frac{d^2\phi}{d\tau^2}$ if $\frac{r}{gT^2} \ll 1$ and $\frac{b}{mgT} \approx \mathcal{O}(1)$, i.e., $\frac{b}{mgT} = 1$ and $\epsilon = \frac{r}{gT^2}$

$$\implies T = \frac{b}{mg}$$
$$\implies \epsilon = \frac{rgm^2}{b^2} \ll 1$$

Set $\gamma = \frac{r\omega^2}{g}$. Then the non-dimensionalize equation becomes

$$\epsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma\sin\phi\cos\phi$$

Overdamped limit: $\epsilon \to 0$

$$\frac{d\phi}{d\tau} = -\sin\phi + \gamma\sin\phi\cos\phi$$
$$= \sin\phi(\gamma\cos\phi - 1)$$

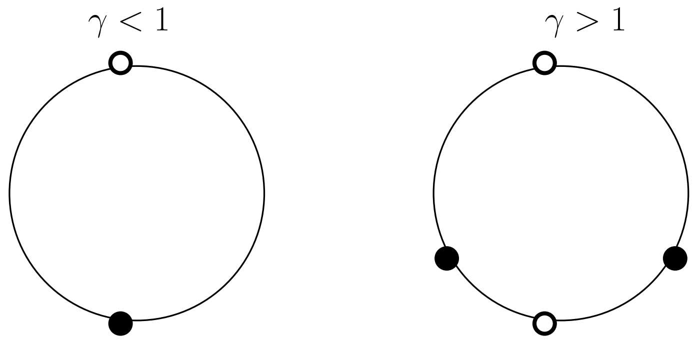
<u>Dynamics:</u> $\frac{d\phi}{d\tau} = 0$ (fixed points)

$$\implies \sin\phi = 0 \iff \phi = 0, \pi \text{ (bottom/top of hoop)}$$

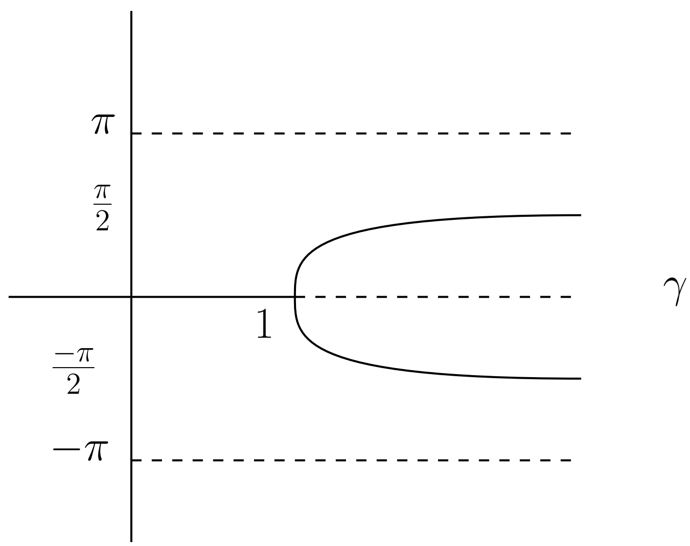
or

$$\cos\phi = \frac{1}{\gamma} \in (0, 1] \implies \gamma \geq 1$$

Fixed points:



Bifurcation Diagram:



In particular, we have a supercritical pitchfork bifurcation at $\gamma = 1$.

§11 Lec 11: Jan 29, 2021

§11.1 Imperfect Bifurcation and Catastrophes

$\dot{x} = h + rx - x^3$

- If $h = 0$: symmetry, if $x(t)$ is a solution then $-x(t)$ is also a solution (supercritical pitchfork bifurcation).
- If $h \neq 0$: imperfect parameter, breaks symmetry.

Aim: Study qualitative behavior of ODE as parameters vary.
Strategy: keep h fixed and vary r

- $h = 0$: supercritical pitchfork bifurcation

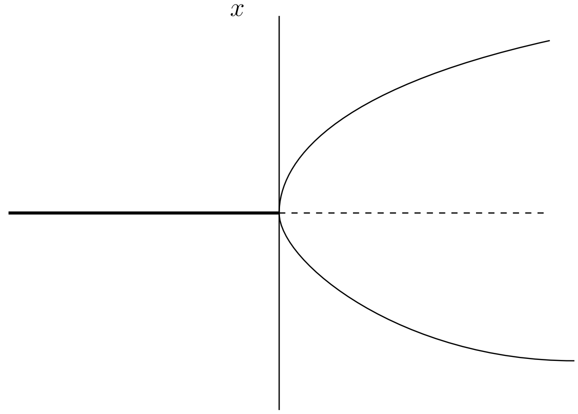


Figure 1: Bifurcation Diagram

- $h > 0$: fixed points: $\dot{x} = 0 \iff x^3 = h + rx$

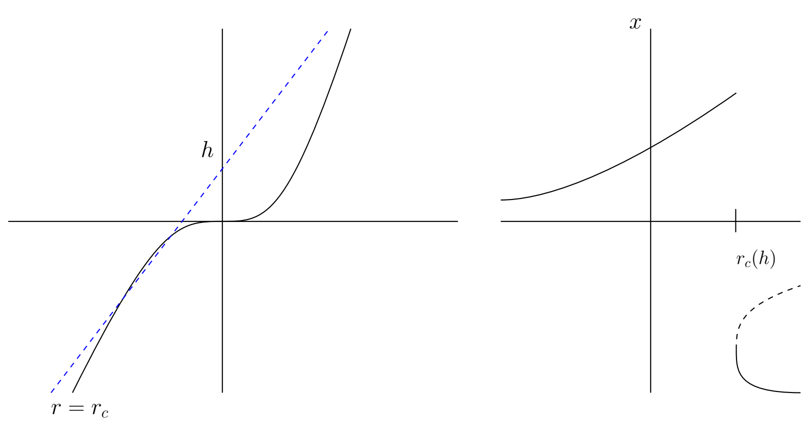


Figure 2: Bifurcation Diagram

- $h < 0$: Fixed points: $x^3 = h + rx$

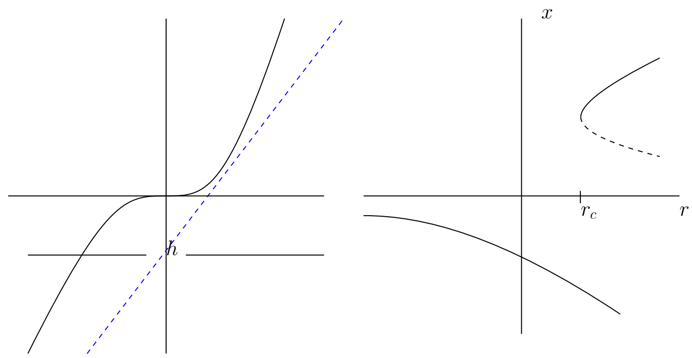


Figure 3: Bifurcation Diagram

Note:We have saddle node bifurcation at $r_c = r(h)$

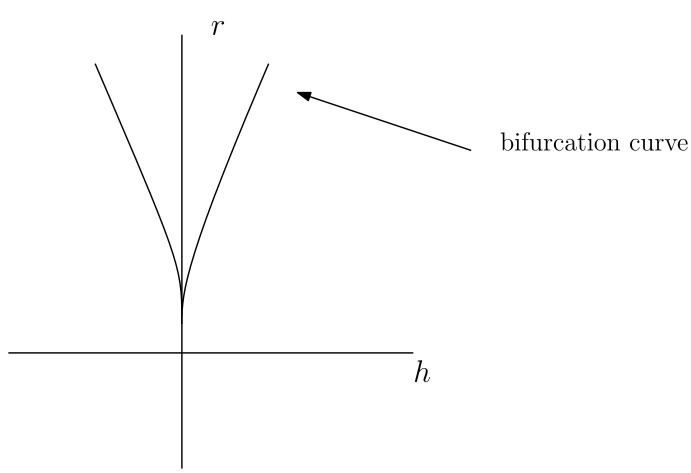
Bifurcation Curves

$$\left\{(h, r)|(h, r, x) \text{ solves } f = 0, \frac{\partial f}{\partial x} = 0\right\}$$

in our example $\dot{x} = h + rx - x^3$

$$\begin{aligned}
0 = \frac{\partial f}{\partial x} = r - 3x^2 &\implies x = \pm\sqrt{\frac{r}{3}} \\
0 = f = h + rx - x^3 &\implies h = x^3 - rx \\
&\implies h = x^3 - rx = \pm\frac{2\sqrt{3}}{9}r^3 \\
h = h_c(r) &= \pm\frac{2\sqrt{3}}{9}r^{\frac{3}{2}} \\
&\implies r = r_c(h) = \left(\frac{9}{2\sqrt{3}}|h|\right)^{\frac{2}{3}}
\end{aligned}$$

Stability Diagram:
Plot the bifurcation curves in the parameters space ($= (h, r)$ plane).



<u>*Note*</u>: qualitative behavior of ode changes as (h, r) cross bifurcation curve.
In example:

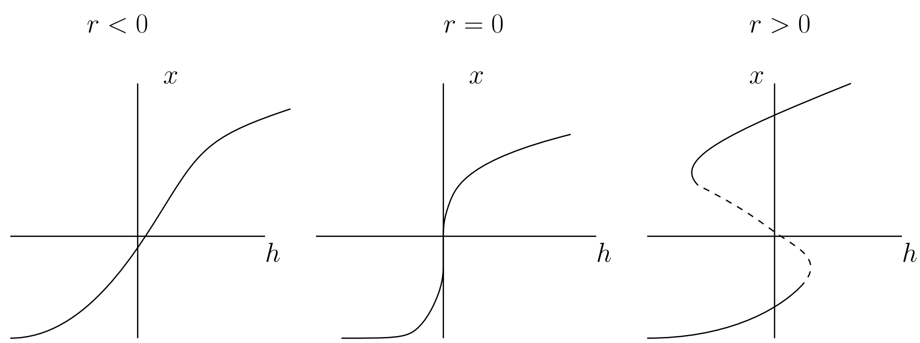
- "below" bifurcation curve: ODE has one (stable) fixed point.
- "on" bifurcation curve: two fixed points.
- "above" bifurcation curve: three fixed points.

Remark 11.1.

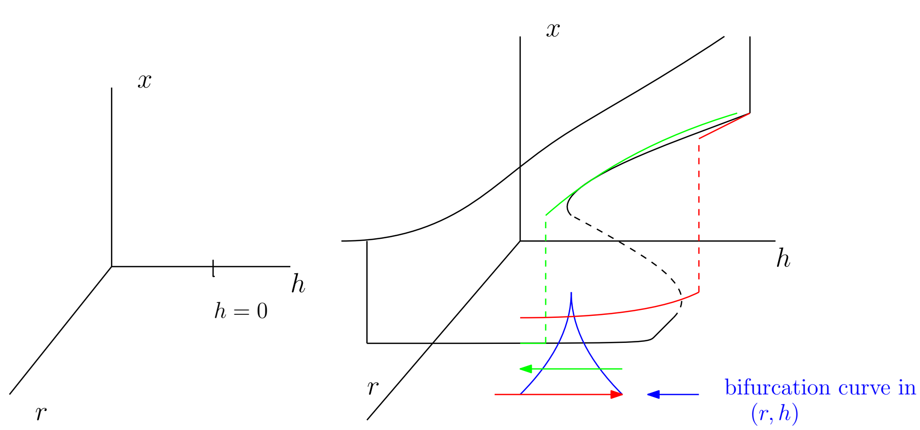
- Saddle-node bifurcation occurs along bifurcation curve for $(h, r) \neq (0, 0)$
- At $(h, r) = (0, 0)$, the branches $r_c(h) = \left(\frac{9}{2\sqrt{3}}|h|\right)^{\frac{2}{3}}$ for $h > 0$ and $h < 0$ meet tangentially, and we have a <u>cusp point</u> at $(h, r) = (0, 0)$. This is an example of a <u>codimension 2 bifurcation</u> (i.e., we need two parameters to model this type of bifurcation).

Bifurcation diagrams for fixed $r \in \mathbb{R}$.

$$\dot{x} = h + rx - x^3 = 0 \iff h = x^3 - rx$$



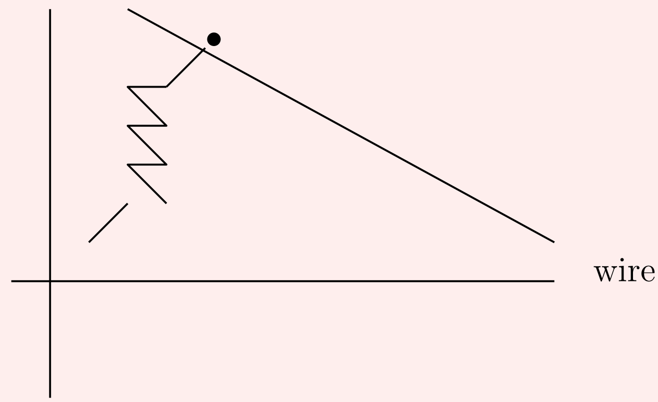
3D plot(h, r, fixed points x)



Picture/surface of cusp catastrophe solutions close to "upper" stable fixed points drop to "lower" stable fixed points as (r, h) vary (and vice versa).

Example 11.2 (practical)

Details in the book, page 74



§12 Dis 1: Jan 7, 2021

§12.1 Fixed points and Stability

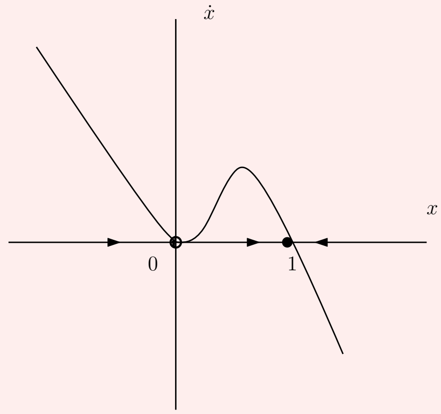
$\dot{x} = f(x)$

Example 12.1

$\dot{x} = -x^3 + x^2$

a) Sketch the vector field, classify the fixed points.

"vector fields" = x-axis with arrows



so:

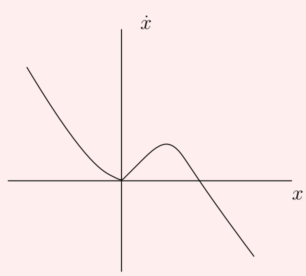
- $\dot{x} > 0 \implies x(t)$ increasing
- $\dot{x} < 0 \implies x(t)$ decreasing

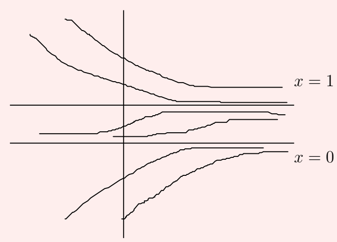
"Fixed point" $\iff$ x_* s.t. $f(x_*) = 0$ $\iff$ x_* s.t. the constant function $x(t) = x_*$ is a solution.

We have 2 fixed points:

- $x_* = 0$ is semi-stable.
- $x_* = 1$ is stable.

b) Sketch various solutions of $x(t)$.



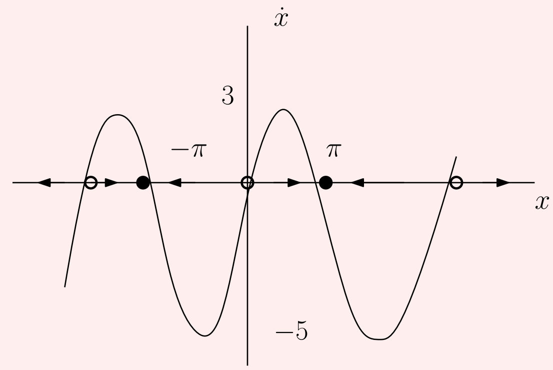


- $\dot{x} = 0$ for $x = 0, 1 \implies x(t) = 0, 1$ are solutions.
- $\dot{x} > 0$ for $x < 0 \implies x(t)$ increasing.
- $\dot{x} > 0$ for $0 < x < 1 \implies x(t)$ increasing.
- $\dot{x} < 0$ for $x > 1 \implies x(t)$ decreasing.

Example 12.2

$\dot{x} = -1 + 4\sin x$

a) Sketch vector field, classify fixed points.

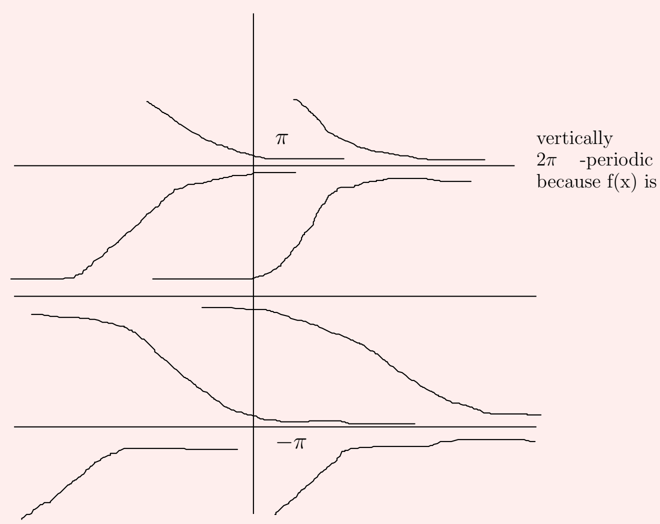


Fixed points:

$$\sin(x_*) = \frac{1}{4}$$

- $x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$ for $n = 0, \pm 1, \ldots$ are unstable.
- $x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$ for $n = 0, \pm 1, \ldots$ are stable.

b) Sketch various solutions $x(t)$.



§12.2 First Order Autonomous System

$\dot{\vec{x}} = \vec{f}(\vec{x})$ – first order and autonomous.

Example 12.3

A unit mass with displacement $x(t)$ attached to a spring with spring constant 6 obeys:

$$\ddot{x} = -6x - b(t)\dot{x}$$

where $b(t) \geq 0$ is the friction coefficient.

a) Show that this can expressed as a first order autonomous system

$$\begin{aligned} x_1 &= x, \quad x_2 = \dot{x} \\ \dot{x}_2 &= \ddot{x} = -6x_1 - b(t)x_2 \\ \vec{x} &:= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \dot{\vec{x}} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -6x_1 - b(x_3)x_2 \end{pmatrix} \end{aligned}$$

where $x_3 = t \implies \dot{x}_3 = 1$.

b) In the case $b(t) = 5$, find the explicit solution for $x(0) = x_0, \dot{x}(0) = v_0$.

$$\ddot{x} = -6x - 5\dot{x} \implies \ddot{x} + 5\dot{x} + 6x = 0$$

Try $x(t) = e^{kt}$:

$$\begin{aligned} 0 &= \ddot{x} + 5\dot{x} + 6x = k^2e^{kt} + 5ke^{kt} + 6e^{kt} \\ &= e^{kt}(k^2 + 5k + 6) \implies k = -3, -2 \end{aligned}$$

Now, $x(t) = c_1e^{-3t} + c_2e^{-2t}, c_1, c_2 \in \mathbb{R}$. Using the initial conditions, we obtain

$$x(t) = (-2x_0 - v_0)e^{-3t} + (3x_0 + v_0)e^{-2t}$$

§13 Dis 2: Jan 14, 2021

§13.1 Linearization and Potentials

Example 13.1

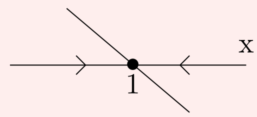
$\dot{x} = -x^3 + x^2$

a) Use linear stability analysis to classify the fixed points. If it fails, use a graphical argument.

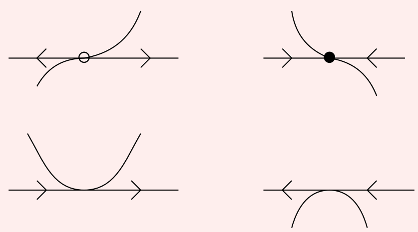
Idea: For x near a fixed point x_*, $\dot{x} = f(x) \approx f(x_*)(= 0) + f'(x_*)(x - x_*) = \dots$

$$f(x) = -x^3 + x^2, \quad f'(x) = -3x^2 + 2x$$
$$0 = f(x_*) = -x_*^2(x_* - 1) \implies x_* = 0, 1$$

- $x_* = 1 : f'(1) = -1 < 0 \implies$ stable



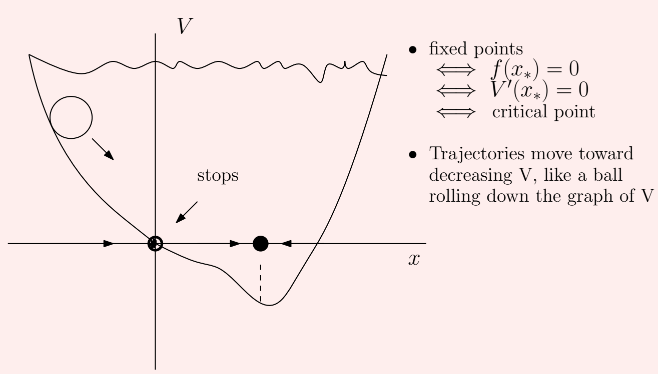
- $x_* = 0 : f'(0) = 0 :$ inconclusive



b) Find and plot a potential function.

"Potential function" $\iff V(x)$ s.t. $\dot{x} = -\frac{dV}{dx}$

$$\dot{x} = -x^3 + x^2 = -V'(x) \implies V(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 + C \text{ (choose 0)}$$



Example 13.2

$\dot{x} = 4\sin x - 1$.

a) Use linear stability analysis to classify the fixed points.

$$f(x) = 4\sin x - 1, \quad f'(x) = 4\cos x$$

Last time: Fixed points are

- $x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \dots$

$$f'(x_*) = 4\cos\left(\sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) > 0$$

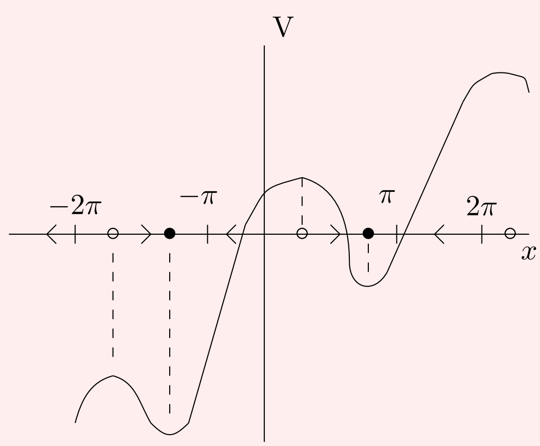
$\implies$ unstable

- $x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \dots$

$$f'(x_*) = 4\cos\left(\pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) < 0$$

$\implies$ Stable.

b) Plot potential $-1 + 4\sin x = -V'(x) \implies V(x) = x + 4\cos x$



§13.2 Existence of Solutions

Example 13.3 a) Let $a > 0$ be a constant. Show that the solution of

$$\begin{cases} \dot{x} = ax^2 \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

$$\begin{aligned} \frac{dx}{dt} = ax^2 \implies \int \frac{dx}{x^2} &= \int a\,dt \\ \implies -\frac{1}{x} &= at + c \\ \implies x(t) &= \frac{1}{c - at} \forall c \in \mathbb{R} \\ x(0) > 0 \implies c > 0 \implies \lim_{t \to T} x(t) &= +\infty \end{aligned}$$

for some $T > 0$. In fact, $c = \frac{1}{x_0}$ and so $T = \frac{c}{a} = \frac{1}{ax_0}$.

b) Let $0 < \epsilon < 1$ be a constant. Show that the solution of

$$\begin{cases} \dot{x} = x^2 (1 + \epsilon \sin x) \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

Idea: $\dot{x} \geq ax^2$ for some $a > 0$, so our solution grows faster than a function which blows up, and must blow up too.

$$\begin{aligned} |\sin x| \leq 1 &\implies 1 + \epsilon \sin x \geq 1 - \epsilon \\ &\implies \dot{x} = x^2 (1 + \epsilon \sin x) \geq \underbrace{1 - \epsilon}_{>0} x^2 \end{aligned}$$

Let $x(t)$ be the solution to

$$\begin{cases} \dot{x} = x^2 (1 + \epsilon \sin x) \\ x(0) = x_0 \end{cases}$$

Let $y(t)$ be the solution to

$$\begin{cases} \dot{x} = (1 - \epsilon) x^2 \\ x(0) = x_0 \end{cases}$$

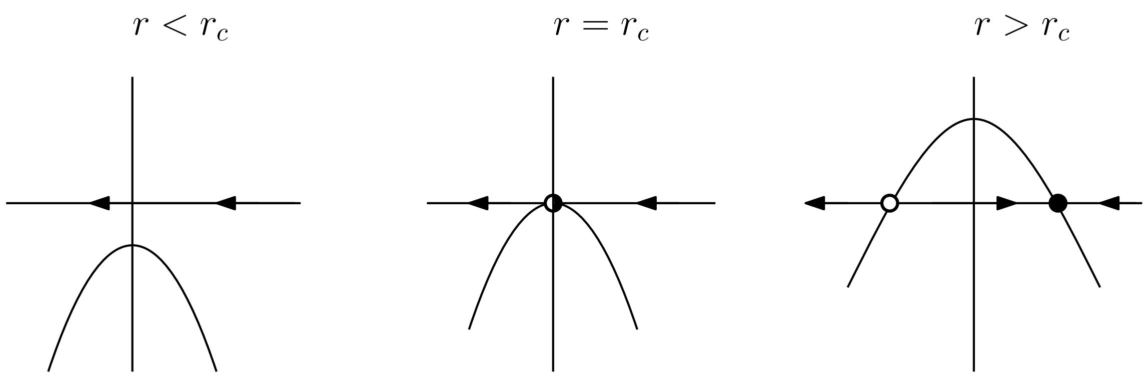
By part a), $y(t)$ blows up at some time $T > 0$. Since $x(0) = y(0)$ and $\dot{x} \geq \dot{y}$, then $x(t) \geq y(t)$ for all $t \geq 0$ (ODE Comparison Lec 4). Therefore, $x(t)$ must blow up in finite time. In fact, blow up time must be $\leq T = \frac{1}{(1-\epsilon)x_0}$.

§14 Dis 3: Jan 21, 2021

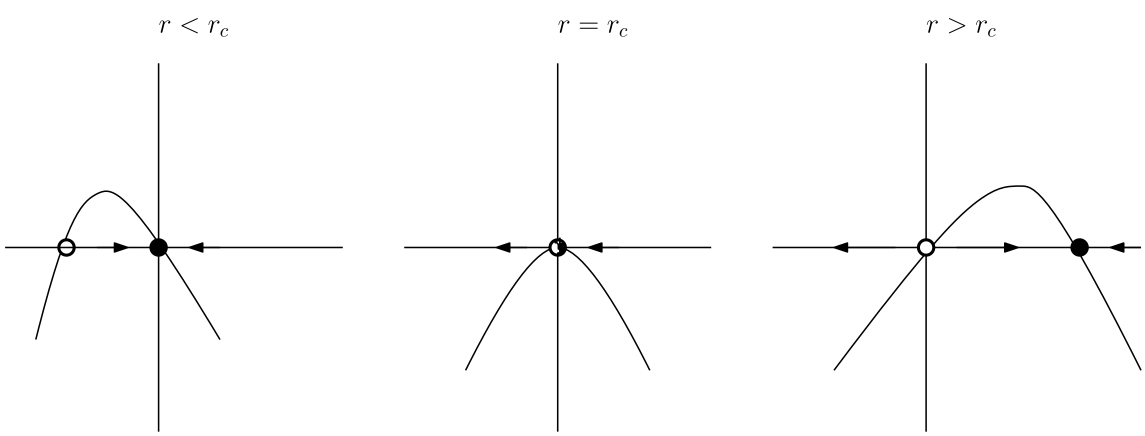
§14.1 Bifurcations

$\dot{x} = f(x, r)$, r parameter
"Bifurcation" $\iff$ change in the number or stability of fixed points.
There are two types:

- Saddle-node: $0 \to 1 \to 2$ fixed points



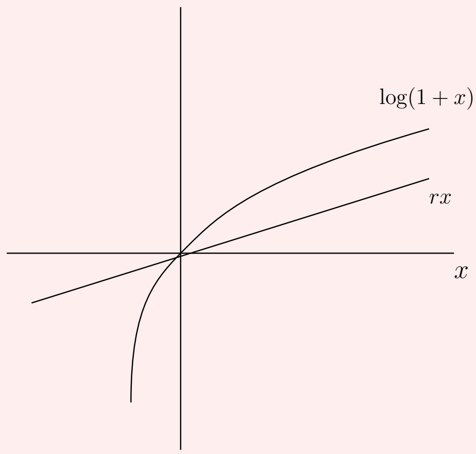
- Transcritical: $2 \to 1 \to 2$ fixed points

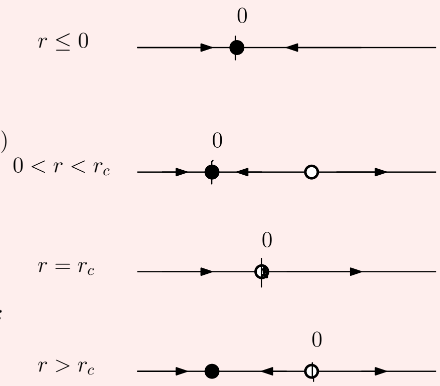


Example 14.1

$\dot{x} = rx - \log(1+x)$

(a) Sketch all qualitatively different vector fields, sketch bifurcation diagram, find and classify bifurcations.



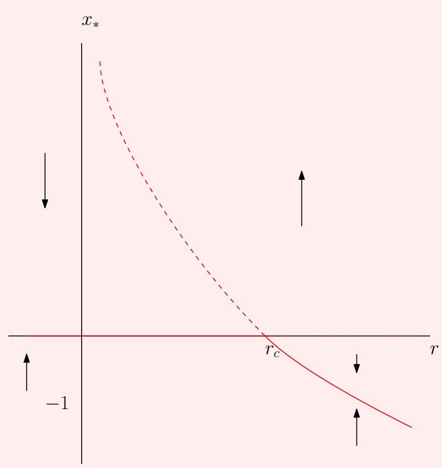


Transcritical bifurcation at $r = r_c$

"Bifurcation diagram" $\iff$ Plot of the fixed points x_* as a function of r.

$$0 = rx_* - \log(1+x_*)$$

$$r = \frac{1}{x_*}\log(1+x_*) \quad \text{or} \quad x_* = 0$$



- Vertical slices of constant r are vector fields
- Whole regions have same arrow direction

Bifurcation point: $(r_c, x_c) = (1, 0)$

Example 14.2 (Cont'd of example 14.1)

From the above example,

(b) Show that there is a transcritical bifurcation at $(r_c, x_c) = (1, 0)$ using normal forms. Taylor expand about $r = 1, x = 0$

$$\begin{aligned}\dot{x} &= rx - \log(1+x)\\ &= (r-1)x + x - (x - \frac{1}{2}x^2 + \mathcal{O}(x^3))\\ &= (r-1)x + \frac{1}{2}x^2 + \mathcal{O}(x^3)\end{aligned}$$

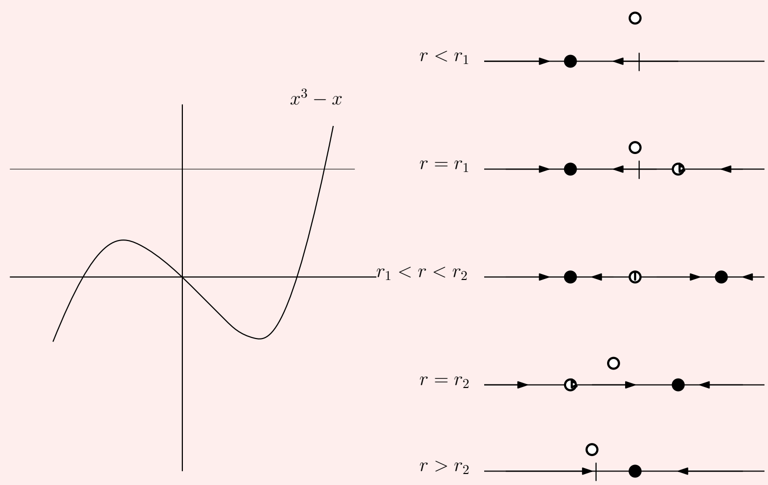
$\dot{x} = (r-1)x + \frac{1}{2}x^2 + \mathcal{O}(\epsilon^3)$ for $|x| < \epsilon, |r-1| < \epsilon$

This is normal form for transcritical bifurcation.

Example 14.3

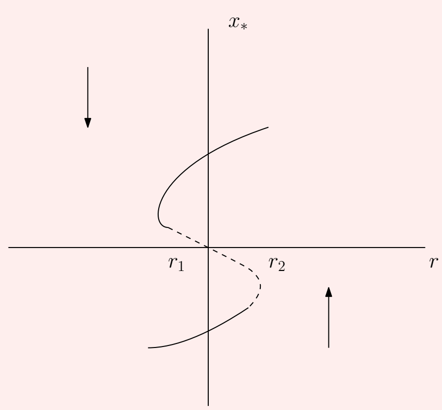
$\dot{x} = r + x - x^3$

(a) Sketch all vector field, sketch bifurcation diagram, find and classify bifurcations.



2 saddle node bifurcations at $r = r_1, r_2$

$$0 = r + x_* - x_*^3 \implies r = x_*^3 - x_*$$



Bifurcation point (x_c, r_c) satisfies:

- Fixed point: $0 = f(x_c, r_c) = r_c + x_c - x_c^3$
- $0 = \frac{\partial f}{\partial x}(x_c, r_c) = 1 - 3x_c^2$

$$0 = 1 - 3x_c^2 \implies x_c = \pm\frac{1}{\sqrt{3}}$$

$$0 = r_c + x_c - x_c^3 \implies r_1 = -\frac{2}{3\sqrt{3}}, r_2 = \pm\frac{2}{3\sqrt{3}}$$

$$(r_c, x_c) = \left(\frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Example 14.4 (Cont'd of example 14.3) (b) Show that there is a saddle-node bifurcation at $(r_c, x_c) = \left(\frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ using normal forms.

Taylor expand about $r = \frac{2}{3\sqrt{3}}, x = -\frac{1}{\sqrt{3}}$.

$$\dot{x} = r + x - x^3, \quad r = \frac{2}{3\sqrt{3}} + (r - \frac{2}{3\sqrt{3}})$$

$$x - x^3 = -\frac{2}{3\sqrt{3}} + 0(x + \frac{1}{\sqrt{3}}) + \frac{1}{2} \cdot \frac{6}{\sqrt{3}}(x + \frac{1}{\sqrt{3}})^2 + \mathcal{O}\left(\left(x + \frac{1}{\sqrt{3}}\right)^3\right)$$

Plug these in, get

$$\dot{x} = \left(r - \frac{2}{3\sqrt{3}}\right) + \sqrt{3}\left(x + \frac{1}{\sqrt{3}}\right)^2 + \mathcal{O}(\epsilon^3)$$

for $\left|r - \frac{2}{3\sqrt{3}}\right| < \epsilon^2, \left|x + \frac{1}{\sqrt{3}}\right| < \epsilon$. This is normal form for a saddle-node bifurcation ($\dot{y} = R + y^2$).

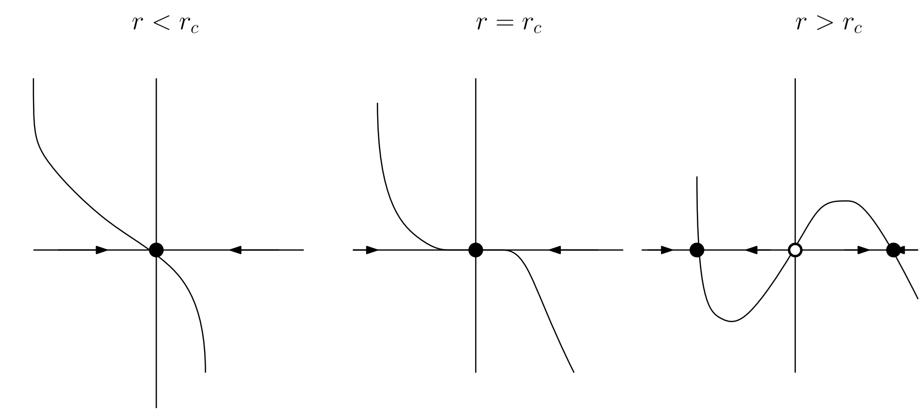
§15 Dis 4: Jan 28, 2021

§15.1 Review of Bifurcations

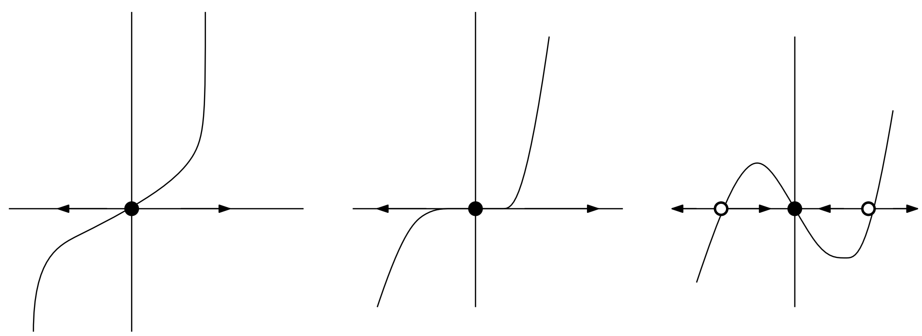
We've seen 3 types of bifurcations so far

- saddle-node: $0 \to 1 \to 2$ fixed points
- transcritical: $2 \to 1 \to 2$ swap stability
- sub-/supercritical pitchfork: $1 \to 1 \to 3$

supercritical



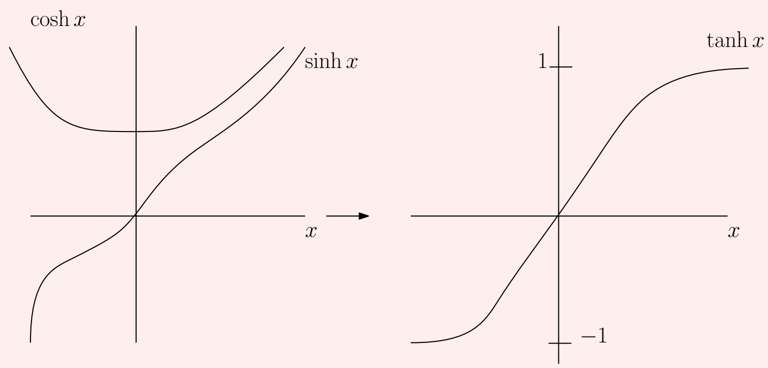
subcritical



Example 15.1

$\dot{x} = r\tanh x - x$. Sketch all qualitatively different phase portraits, sketch bifurcation diagram, find any classify bifurcations. Recall

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x}$$



$\dot{x} = r\tanh x - x$

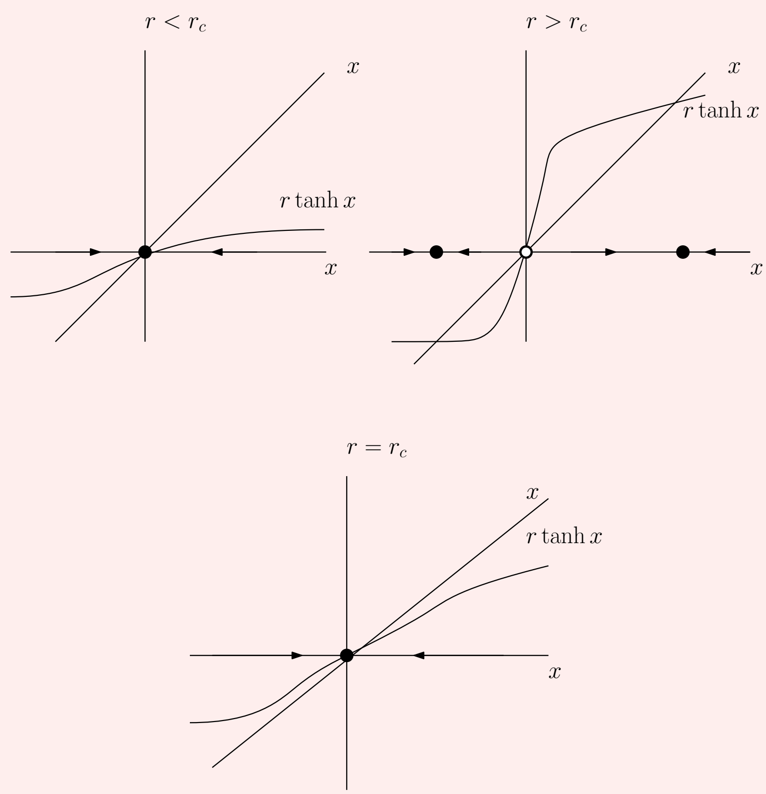
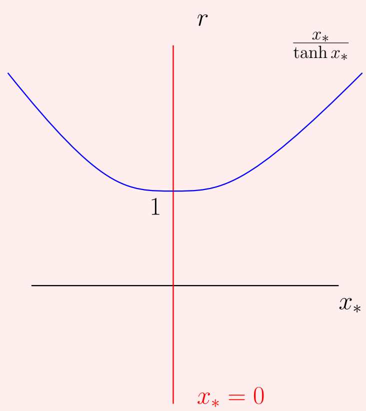
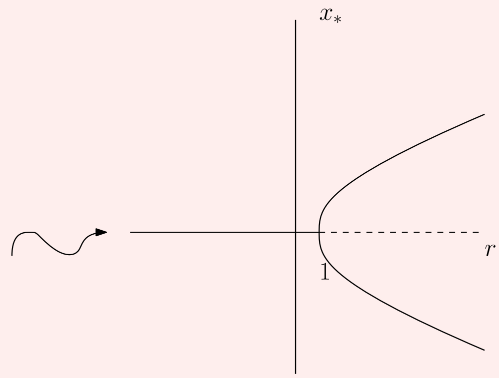


Figure 4: Supercritical pitchfork at $r = r_c$

Example 15.2 (Cont'd from above)

$$0 = r \tanh x_* - x_* \implies r = \frac{x_*}{\tanh x_*}, \quad x_* = 0$$

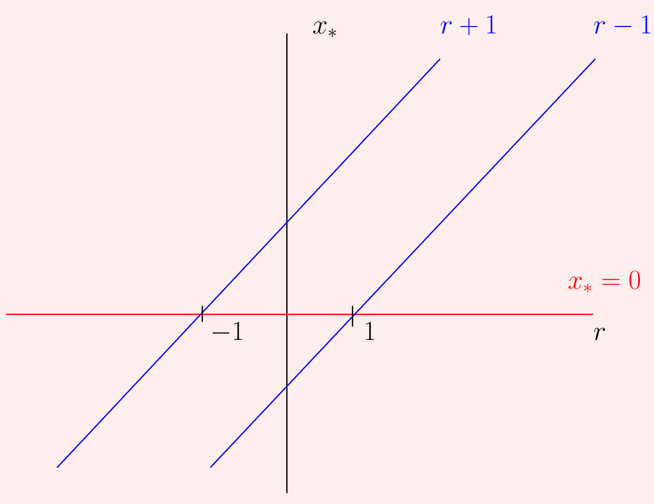




Near $x = 0, \tanh x = \frac{x+\dots}{1+\dots} = x + \dots \implies \frac{x}{\tanh x} = 1 + \dots$
Bifurcation point: $(r_c, x_c) = (1, 0)$.

Example 15.3
Same for $\dot{x} = x - x(x-r)^2$

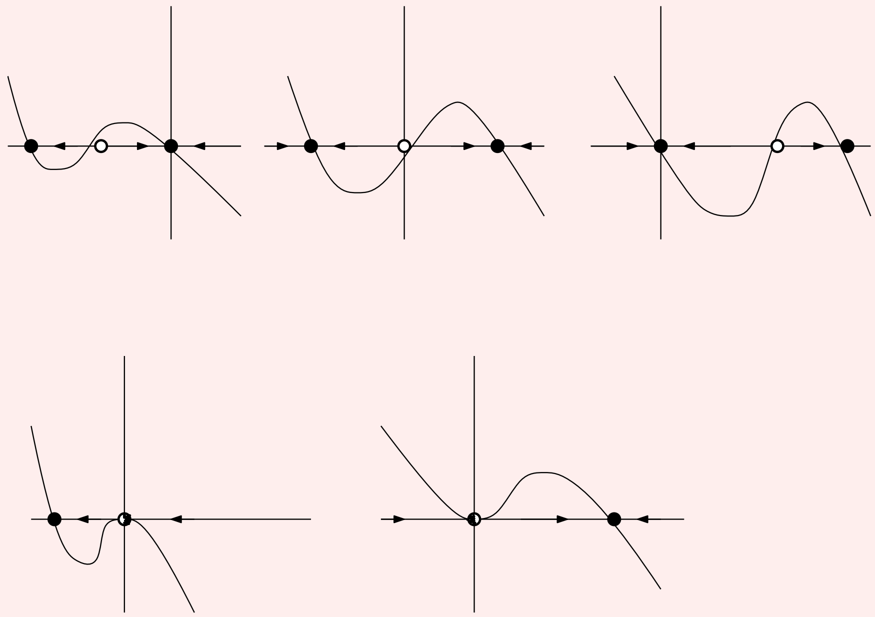
$$0 = x_* \left(1 - (x_* - r)^2\right) \implies x_* = 0, \quad x_* = r \pm 1$$



2 transcritical bifurcations at $r_c = \pm 1, x_c = 0$.

Example 15.4 (Cont'd of example 15.3)

$$\dot{x} = x\left(1 - (x-r)^2\right) = -x\left(x - (r+1)\right)\left(x - (r-1)\right)$$



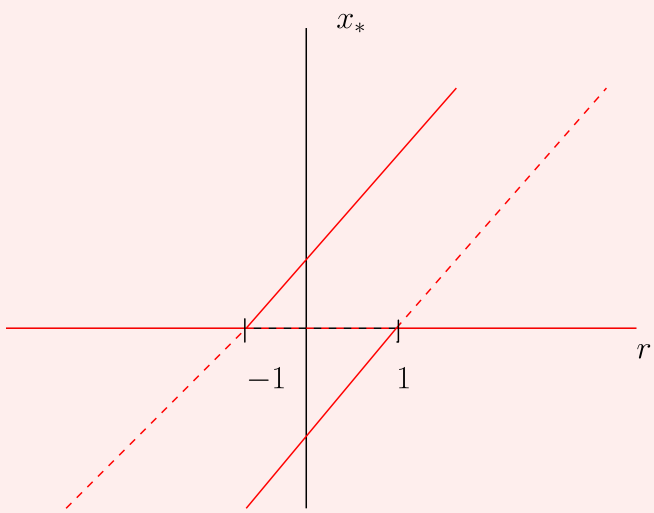
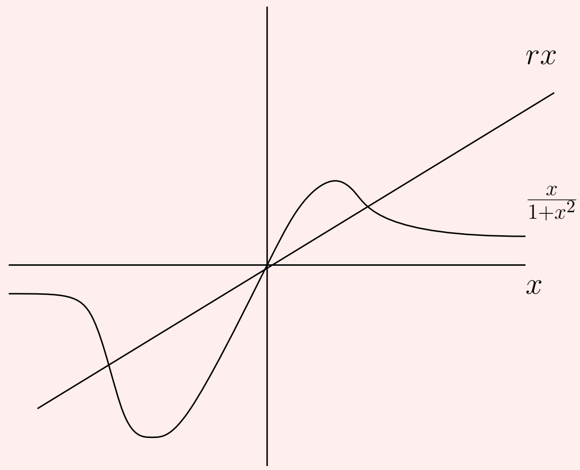
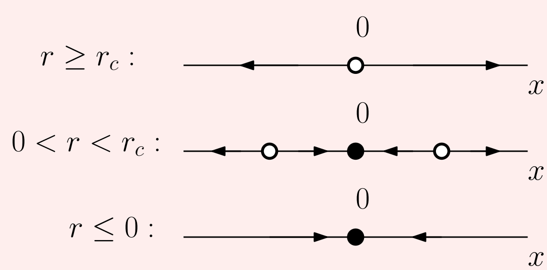


Figure 5: Bifurcation Diagram

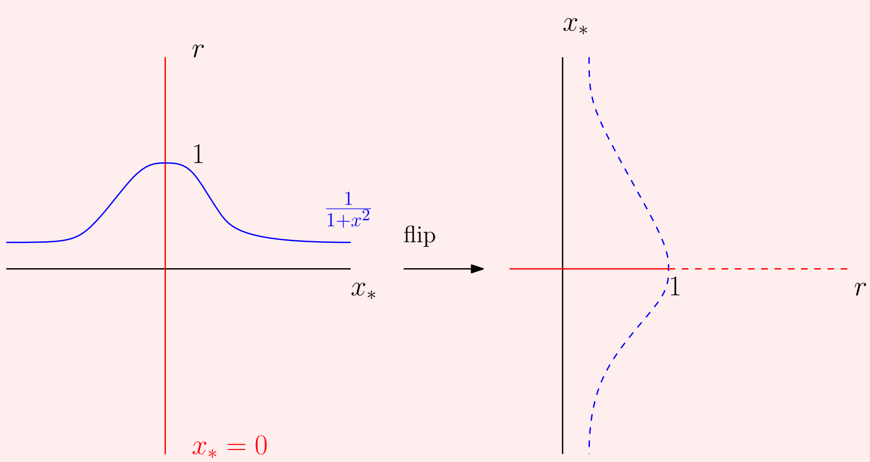
Example 15.5

Same for $\dot{x} = rx - \frac{x}{1+x^2}$





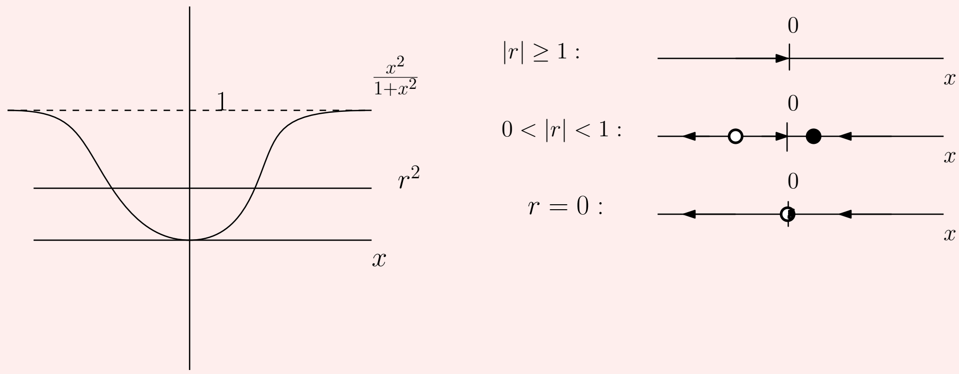
$$0 = x_* \left(r - \frac{1}{1+x_*^2} \right) \implies x_* = 0, \quad r = \frac{1}{1+x_*^2}$$



Subcritical pitchfork at $(r_c, x_c) = (1, 0)$.

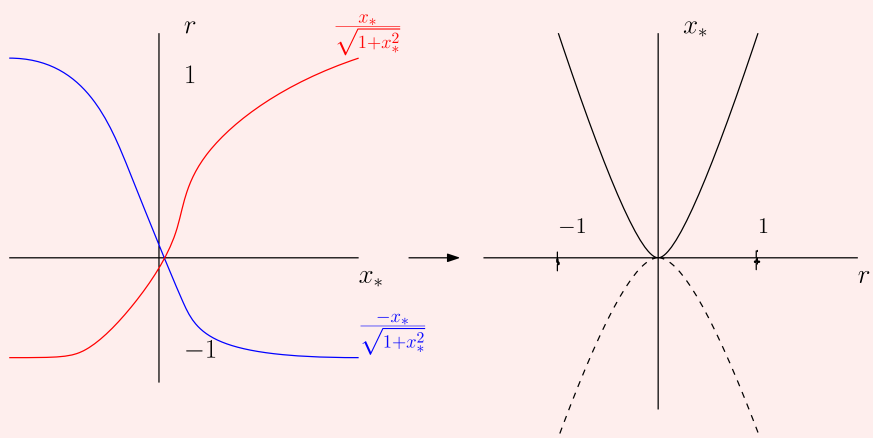
§15.2 Other Bifurcations

Example 15.6
$\dot{x} = r^2 - \frac{x^2}{1+x^2}$. Sketch all qualitatively different phase portraits, sketch bifurcation diagrams, find bifurcation point.



Bifurcation point $(r_c, x_c) = (0, 0)$ and still satisfies $f(r_c, x_c) = 0$, $\frac{\partial f}{\partial x}(r_c, x_c) = 0$.
Bifurcation diagram:

$$0 = r^2 - \frac{x_*^2}{1+x_*^2} \implies r = \pm\sqrt{\frac{x_*^2}{1+x_*^2}} = \pm\frac{x_*}{\sqrt{1+x_*^2}}$$



This is not one of our 3 types of bifurcations

- Graphically, it's not transcritical because both fixed points are moving.
- Analytically, we can check that the Taylor expansion at $(r, x) = (0, 0)$:

$$f(r, x) = r^2 - x^2 + \mathcal{O}(x^3)$$

  doesn't match one of the 3 normal forms we know.