

# Math 131AH – Honors Real Analysis I

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This is math 131AH – Honors Real Analysis I taught by Professor Greene, and our TA is Haiyu Huang. We meet weekly on MWF from 1:00pm – 2:00pm for lectures. There are two textbooks used for the class, *Principles of Mathematical Analysis* by Rudin and *Metric Spaces* by Copson. You can find other lecture notes at my [github](#) site. Please let me know through my [email](#) if you spot any mathematical errors/typos.

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# §1 | Lec 1: Oct 2, 2020

Overview:

- Hmwrk: 30 %
- Midterm 1: 20 %
- Midterm 2: 20 %
- Final: 30 %

## §1.1 Introduction

functions  $\rightarrow 1, 2, 3, 4, 5, 6, 7 \dots$

functions defined on  $\mathbb{Q}$  with value in  $\mathbb{Q}$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

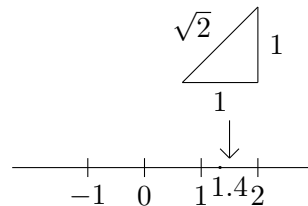
$a_i \in \mathbb{Q}$   $f(x) \in \mathbb{Q}$  if  $x \in \mathbb{Q}$ . Continuity makes sense.

$$x_0, x \text{ close to } x_0 \implies f(x) \text{ close } f(x_0)$$

polynomials are continuous.

Something wrong:  $\sqrt{2}$  is missing. What are these numbers that are not  $\in \mathbb{Q}$ ? Choice:

1. Assume everything works and isolate what you need about "real numbers" (most of Rudin chap 1).
2. Construct the real numbers from rational numbers.



Classical argument:

$$x^2 \neq 2 \text{ if } x = \frac{p}{q} \in \mathbb{Q}$$

*Proof.* Suppose  $\left(\frac{p}{q}\right)^2 = 2$

*Note:* wolog (without loss of generality)

can take  $\frac{p}{q} > 0$   $p > 0$   $q > 0$

$$\begin{aligned} \left(\frac{p}{q}\right)^2 &= 2 \\ \frac{p^2}{q^2} &= 2 \\ p^2 &= 2q^2 \end{aligned}$$

Now also wolog, can assume  $p$  and  $q$  are not both even numbers. But  $p^2 = 2q^2$  means  $p$  has to be even ( $p^2$  odd if  $p$  is odd).

$$\begin{aligned} p &= 2n \\ p^2 &= 2q^2 \\ 4n^2 &= 2q^2 \end{aligned}$$

So  $q^2 = 2n^2$ ,  $q$  is even. But it contradicts the initial assumption,  $p$  and  $q$  not both even  $\square$

Related to: Why functions  $\mathbb{Q}$  to  $\mathbb{Q}$  not ideal for analysis?  
– INFINITE DECIMAL

## §2 | Lec 2: Oct 5, 2020

### §2.1 Mathematical Induction and More on Real Numbers

$P(n) \rightarrow 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ , where  $n$  is positive numbers.

Math induction: Proof by two steps:

1. Check  $P(1)$  is true  $\checkmark$
2. Assume  $P(n)$  is true for all  $n \leq N$ . Check that

$$P(N+1) \text{ is true}$$

Assume  $1 + \dots + N = \frac{N(N+1)}{2}$ . Check

$$1 + \dots + N + (N+1) = \frac{(N+1)(N+1+1)}{2}$$

Induction on  $k$  :

$$1^k + 2^k + \dots + n^k$$

2<sup>nd</sup> illustration:

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad r \neq 1$$

$$r = 1 \implies 1 + r = \frac{1 - r^2}{1 - r}$$

$$\begin{aligned} 1 + r + r^2 + \dots + r^n + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \\ &= \frac{1 - r^{n+1} + r^{n+1} - r^{n+2}}{1 - r} \\ &= \frac{1 - r^{n+2}}{1 - r} \end{aligned}$$

$$(1 - r)(1 + r + \dots + r^n) = 1 - r^{n+1} \quad \text{Inspection}$$

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r}, \quad r \neq 1$$

$|r| < 1$  get infinite sum  $\frac{1}{1-r}$

**Example 2.1**

Prime factors, prime = positive integers ( $> 1$ ) with no factors except itself and 1,  
 $p = ab$ ,  $a > 1$ ,  $b > 1$

2 3 5 7 11 13 17 19 ...

Thin out as go along

**Theorem 2.2** (Fundamental Theorem of Arithmetic)

Every positive integer  $> 1$  is a product of primes.

*Proof.* Induction:  $P(n)$   $n = 2, 3, \dots$

$$P(2) = 2\checkmark$$

Assume  $P(n) \dots n \leq N$  ( $N > 2$ ). Every integer greater than 1 but smaller than or equal to  $N$  as a product of primes. We try to prove:  $N + 1$  is a product of primes.

1.  $N + 1$  is prime: Done  $N + 1 = N + 1$

2.  $N + 1$  is not a prime

$$N + 1 = a \cdot b \quad a > 1 \quad b > 1$$

Induction assumption ( $a < N + 1$  since  $b > 1$ ),  $a$  is a product of primes  $a > 1 \implies b < N + 1$ ,  $b$  also a product of primes. So,  $N + 1 = ab$  is a product of primes.

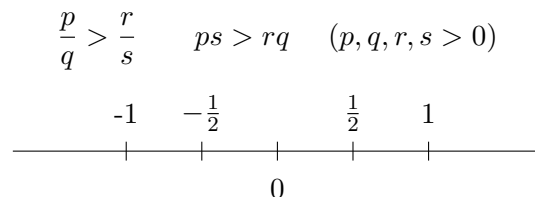
$N + 1 = ab$  is a product of prime.  $\square$

Why does induction work? If  $P(n)$  not always true,  $P(n)$  look at smallest  $n$  where  $P(n)$  is false.

$n = 1$  not there  $P(1)$  is supposed true (checked already).  $N_0$  smallest one where  $P(N_0)$  false  $N_0 > 1$ . Induction step says that  $P(n)$  is true for all  $n \leq \underbrace{N_0 - 1}_{>0} \implies P(N_0)$  true ( $\times$ ).

Let's go back to real numbers.

Last time: talked about  $\sqrt{2}$  is irrational but  $\sqrt{2}$  exists, so we need to enlarge our number system:  $\mathbb{Q}$  rational numbers.



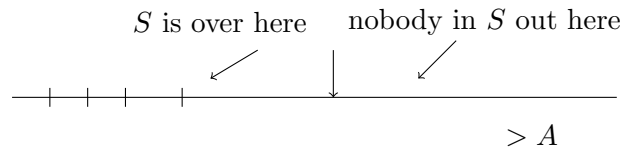
$x, y$  rational  $x, y > 0$ ,  $x + y > 0$ ,  $xy > 0$

$x^2 = 2$  no answer in  $\mathbb{Q}$ . Enlarge number system,  $\mathbb{Q} \subset \mathbb{R}$ . What should  $\mathbb{R}$  be like?

1.  $\mathbb{R}$  ought to have arithmetic like  $\mathbb{Q}$

$$x + y \quad xy \quad \frac{x}{y} \quad 0 \quad 1$$

2.  $\mathbb{Q} \subset \mathbb{R}$ , arithmetic in  $\mathbb{R}$  restricted to  $\mathbb{Q}$ ,  $\frac{1}{2} + \frac{1}{3}$  in  $\mathbb{Q}$  ought to be  $\frac{5}{6}$  in  $\mathbb{R}$ .
3. Order should positive in  $\mathbb{Q} \implies$  in  $\mathbb{R}$ .  $\mathbb{R}$  should have an order of its own too,  $x > y$  positive then  $x + y$  pos and  $xy$  pos.
4. want to fill in the holes in  $\mathbb{Q}$ . Want to have **Least Upper Bound Property**  
 $S \subset \mathbb{R}$  : An upper bound for  $S$  is a number  $A$  with property  $A \geq x$  if  $x \in S$



$1, 2, 3, 4, \dots$  have no upper bound.

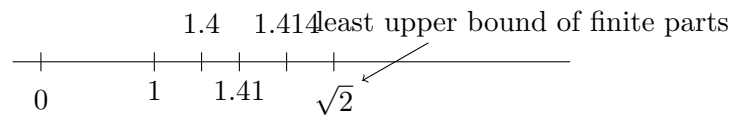
$S$  is bounded above means that some upper bound  $A$  exists.

## §2.2 Least Upper Bound Property

If  $S$  is bounded above ( $S \neq \emptyset$ ) then it has a “least upper bound” where a number  $A_0$  is called the least upper bound of  $S$  if  $A_0$  is an upper bound for  $S$  & if  $A$  is an upper bound for  $S$  then  $A_0 \leq A$ .



Motivation: Think about  $\sqrt{2}$



Denote: l.u.b(or supremum)(sequence) =  $\sqrt{2}$

Means can define an infinite decimals: least upper bound of successive truncations

$$0.99999 \dots \rightarrow 1.0$$

## §3 | Lec 3: Oct 7, 2020

### §3.1 Cauchy Sequence

$\{x_n\}$   $x_1, x_2, x_3, \dots$  values  $x_j \in \mathbb{Q}$   $x_j \in \mathbb{R}$   
 $S$   $x_1, x_i \dots x_j \in S$

**Definition 3.1 (Sequence)** — A sequence with values in a set  $S$  is a function from positive integers  $\{1, 2, 3 \dots\}$  into  $S$ .

**Definition 3.2 (Cauchy Sequence)** — A Cauchy sequence is ( $\mathbb{Q}$  valued or  $\mathbb{R}$  valued)  $\{x_i\}$  is sequence s.t. for every  $\epsilon > 0$  there is a positive integer  $N_\epsilon$  s.t.

$$|x_i - x_j| < \epsilon \quad \text{if } i, j > N_\epsilon$$

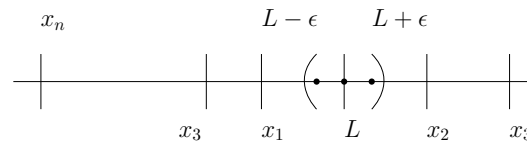


$\epsilon$  rational or real (same idea).

### Lemma 3.3

If  $\{x_j\}$  has a finite limit then it's a Cauchy sequence.

$\{x_i\}$  has  $L$  as a limit  $\lim x_j = L$  means for every  $\epsilon > 0$  then there is an  $N_\epsilon$  such that  $j \geq N_\epsilon, |x_j - L| < \epsilon$

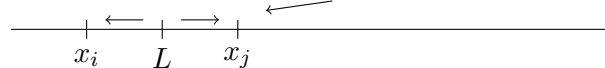


Everybody in  $(L - \epsilon, L + \epsilon)$  except a finite number

*Proof.* Given  $\epsilon > 0$ , want to find  $N$  so that  $i, j \geq N \implies |x_i - x_j| < \epsilon$   
 $|x_i - L|$  small,  $|x_j - L|$  small and  $\lim x_j = L$ .

$$|x_i - x_j| \leq |x_i - L| + |x_j - L|$$

$$|x_i - x_j| = |L - x_i| + |L - x_j|$$



$i, j \geq N_{\frac{\epsilon}{2}} :$

$$|x_i - x_j| \leq \underbrace{|x_i - L|}_{< \frac{\epsilon}{2}} + \underbrace{|x_j - L|}_{< \frac{\epsilon}{2}}$$

Because  $\lim x_n = L$ , there is an  $N_{\frac{\epsilon}{2}}$  s.t.  $|L - x_n| < \frac{\epsilon}{2}$  if  $n \geq N_{\frac{\epsilon}{2}}$

Get  $|x_i - x_j| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$  if  $i, j \geq N$ . Cauchy sequence: there exists number  $N$  s.t.

$$|x_i - x_j| < \epsilon \quad \text{if } i, j \geq N$$

□

Cauchy sequence  $\implies$  the existence of limit? Yes, for  $\mathbb{R}$  valued sequences but NO for  $\mathbb{Q}$  valued things.

$\underbrace{\{x_n\}}_{\text{rational numbers}}$  can be Cauchy seq without there being a rational number  $L$  such that  $\lim x_j = L$

But allow real  $L$  then  $\exists L$  s.t.  $\lim x_j = L$  if  $\{x_j\}$  is Cauchy sequence (no rational limit – since  $\sqrt{2}$  is irrational). Because  $\mathbb{Q}$  has holes in it! (intuitive idea).

**Example 3.4**

1, 1.4, 1.41, 1.414, 1.4142... (decimal approx of  $\sqrt{2}$ ) – Cauchy sequence. No – since  $\sqrt{2}$  is irrational.

**§3.2 Cauchy Completeness of  $\mathbb{R}$** 

If  $\{x_j\}, x_j \in \mathbb{R}$  is Cauchy sequence, then  $\exists L \in \mathbb{R}$  s.t.  $\lim x_j = L$ .

“ $\mathbb{Q}$  is not Cauchy complete” but  $\mathbb{R}$  is. Why does this work?

Need: Least upper bound property. Assume L.U.B Property proof.

*Proof.* (Cauchy completeness from L.U.B Property)

Hypothesis:  $\{x_i\}$  Cauchy seq

1. Prove that  $\{x_i\}$  bounded  $\iff \exists M > 0$  s.t.  $|x_i| \leq M$  all  $i$ .

Clear if take  $\epsilon = 1$  in def. of Cauchy seq  $\exists N$  s.t.  $|x_i - x_j| < 1$  if  $i, j \geq N \implies |x_N - x_j| < 1$  if  $j \geq N \implies |x_j| \leq |x_N| + 1 \quad j \geq N$

So,  $M = \max(|x_N| + 1, |x_1|, \dots, |x_{N-1}|)$  then  $|x_j| \leq M$  all  $j$  !

Next stage is to show that a bounded sequence always has a subsequence(tricky!) with a limit. Then if a Cauchy seq has a subseq with limit  $L$ , then  $L$  is limit of whole seq. (Bolzano – Weierstrass Theorem)

□

**§4 | Lec 4: Oct 9, 2020****§4.1 Bolzano – Weierstrass Theorem**

– implied by Least Upper Bound Property

**Theorem 4.1 (Bolzano – Weierstrass)**

If  $\{x_n\}$  sequence  $(x_1, x_2, x_3 \dots)$  that is bounded (means:  $\exists M > 0 \ni |x_n| \leq M \forall n$ ), then  $\exists L$  and a subsequence  $\{x_{n_i}\}$  s.t.  $\lim x_{n_i} = L$ .

Slogan: Every bounded sequence has a convergent subsequence.

**Example 4.2**

1, 2, 1, 2, 1, 2, ...

The subsequence of the above sequence has either 1 or 2 as the limit.

1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, ...

Unbounded sequence – subsequence (limit 1, limit 2, limit 3...)



**No claim of uniqueness of anything.**

Proof – Summer 2008 Analysis Lec 4

*Proof.* So either  $[-M, 0]$  or  $[0, M]$  (maybe both) contains  $x_n$  for infinitely many  $n$  values. If each contained  $x_n$  for only finitely many  $n$  values  $X$ .

$$\begin{array}{c} -M \qquad \qquad \qquad 0 \qquad \qquad \qquad M \\ |-----| \\ \text{Every } x_n \text{ is in } [-M, M] - \{x_n\} \text{ is bounded} \end{array}$$

$$[-M, M] = [-M, 0] \cup [0, M]$$

$$I_1 = [-M, 0] \quad \text{or} \quad [0, M]$$

where chosen interval has  $x_n$  for infinitely many  $n$  values.

Do this again!

$$\begin{array}{c} I_1 = [a_1, b_1] \qquad |b_1 - a_1| = M \\ I_1 \longleftarrow \text{length} \\ |-----| \end{array}$$

left half of  $I_1$ , right half of  $I$ . Let  $I_2 =$  one of halves that contains  $x_n$  for infinitely many  $n$  values.

$$I_2 = [a_2, b_2] \qquad a_2 < b_2, \quad b_2 - a_2 = \frac{M}{2}$$

Continue

$$I_3 = [a_3, b_3] \qquad a_3 < b_3, \quad b_3 - a_3 = \frac{M}{4}$$

$$\vdots$$

$$I_k = [a_k, b_k] \qquad b_k - a_k = \frac{M}{2^{k-1}}$$

Each  $I_k$  contains  $x_n$  for infinitely many  $n$  values.

$$\begin{array}{c} \text{Nested Intervals} \\ a_1 \qquad \qquad I_1 \qquad \qquad b_1 = b_2 \\ |-----| \\ \qquad \qquad \nearrow \qquad \qquad \nwarrow \\ \qquad a_3 \qquad \qquad b_3 \\ I_{k+1} \subset I_k \subset \dots \subset I_1 \subset [-M, M] \\ a_{k+1} \geq a_k \dots \qquad b_{k+1} \leq b_k \dots \end{array}$$

Claim  $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$

Reason:  $\sup a_k \in \bigcap_{k=1}^{\infty} I_k$  where  $\sup =$  sup of left hand endpoint (=greatest lower bound of bs). l.u.b of  $a$ 's  $\leq b_k$ ,  $b_k$  bigger than or  $\geq$  all  $a$ 's.

$$\alpha = \text{lub } a\text{'s}$$

$$\alpha \geq a_k \quad \forall k$$

$$\alpha \leq b_k \quad \forall k$$

$$\alpha \in [a_k, b_k]$$

Goal:  $\alpha \in \bigcap_{k=1}^{\infty} I_k$ . Find a subsequence of  $\{x_n\}$  converges to  $\alpha$ .

Choose  $x_k = x_n$  that belongs to  $I_k$ . Can also arrange successively:

$$n_1 < n_2 < n_3 < n_4$$

$x_{n_1} \in I_1$   $x_{n_2} \in I_2$  can make  $n_2 > n_1$  because infinitely possible  $x'_n$ s in  $I_2$  n value.  
Continue to get subsequence,  $\{x_{n_k}\}$  subsequence. Claim:

$$\lim_{k \rightarrow \infty} x_{n_k} = \alpha$$

Reason:

$$\text{dis}(x_{n_k}, \alpha) \leq \text{length of } I_k \quad \alpha \in I_k, \quad x_{n_k} \in I_k$$

which is equivalent to

$$|x_{n_k} - \alpha| \leq \frac{M}{2^{k-1}} \quad \text{given } \epsilon > 0$$

When  $k$  is large,

$$\frac{M}{2^{k-1}} < \epsilon$$

So  $|x_{n_k} - \alpha| < \epsilon$

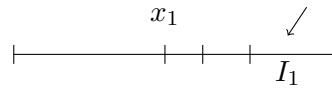
□

This argument (or a variant) shows something else:

If  $\{x_n\}$  sequence in  $[0, 1]$  then there's an  $\alpha \in [0, 1]$  with it never happening that

$$x_n = \alpha$$

“The real numbers in  $[0, 1]$  are uncountable.” (come from the least upper bound property)



$I_1$  one of  $[0, \frac{1}{3}]$   $[\frac{1}{3}, \frac{2}{3}]$   $[\frac{2}{3}, 1]$  such that  $x_1 \notin I_1$ ,

$$[0, \frac{1}{3}] \cap [\frac{1}{3}, \frac{2}{3}] \cap [\frac{2}{3}, 1] = \emptyset$$

$x_1 \notin I_2$   $I_2 \subset I_1$ , &  $x_1 \notin I_1$ . Continue. Get

$$I_1 \supset I_2 \supset I_3 \supset \dots$$

length  $I_k = \frac{1}{3^k}$  and  $I_k$  is such that  $x_1, x_2, x_3 \dots x_k$  are none of the ?n? in  $I_k$ . Same as before

$$\exists \alpha \in \bigcap_{k=1}^{\infty} I_k$$

$\alpha = \sup$  of set of left hand endpoints of  $I_k$ . Claim  $\alpha$  cannot be an  $x_N$  value. Clear:  $x_N \notin I_N$  but  $\alpha \in I_n$   $\alpha \in \bigcap_{n=1}^{\infty} I_n$ . But contrast:

There is a list of rational numbers in  $[0, 1]$

	$\frac{p}{q}$	$p < q$				
	2	3	4	5	6	...
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$			
2	-	$\frac{2}{3}$	$\frac{2}{4}$			
3	-	-	$\frac{3}{4}$			
$\vdots$	-	-	$\frac{\sqrt{2}}{2} \in [0, 1] \rightarrow$	irrational - no exist		
			$[0, 1]$	<div> <div>not</div> <div>countable</div> </div>		
Q is countable						

## §5 | Lec 5: Oct 12, 2020

### §5.1 Equivalence Relation

(p.10, Copson – Metric Space)

$R$  set, relation of  $A$  and  $B$  ( $A \times B$ )  $(a, b) \in R \implies aRb$

Functions: one  $b$  given  $a$  – exact one. ( $A \rightarrow B$ )

#### Example 5.1

$A = B = Q$

$aRb$  or  $(a, b) \in R$  if  $a > b$

(mother, child)

- $(\text{Sara}, \text{Sebastian}) \in R$
- $(\text{Sara}, \text{Alita}) \in R$

Equivalence is a special kind of relation: (on a set  $A; B \subseteq A \times A$ )

Properties:

1.  $aRa \implies A = Q$
2.  $aRb \implies bRa$
3.  $aRb \ \& \ bRc$  then  $aRc$

Example:  $\mathbb{Z}$   $a \sim b$  means  $a - b$  is divisible by 5

$$1 \sim 6 \quad 0 \sim 5 \dots$$

$$a \sim a \quad a - b \text{ div } 5 \implies b - a \text{ div. by } 5.$$

If  $a - b$  div. by 5, and  $b - c$  div by 5, then is  $a - c$  div. by 5 true?

$$\text{Sure, } a - b = 5k, \quad b - c = 5l \implies a - c = 5(k + l)$$

“Equivalence classes”: set  $[a] = \{ \text{all } b \text{ such that } aRb \}$

In the example above,  $[a] = \{ \text{all } b \text{ such that } a - b \text{ div. by } 5 \}$

$$[2] = \{2, 7, -3, 12, -8, \dots\}$$

$\mathbb{Z}_5$  : integer mod 5.

1.  $[a] \cap [p]$  either equal or have nothing in common.
2.  $a \in [a]$  so is in some equivalence class.

A equivalence relation  $\sim$  on  $A \leftrightarrow$  a partition of  $A$  into subsets which are pairwise disjoint.

$\mathbb{Q}$  Cauchy seq. of rational numbers

$$\{x_n\} \sim \{y_n\}$$

means  $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$ . Equivalence relation:

1.  $\{x_n\} \sim \{x_n\}$  ( $\lim(x_n - x_n) = 0$ )
2.  $\{x_n\} \sim \{y_n\} \implies \{y_n\} \sim \{x_n\}$
3.  $\{x_n\} \sim \{y_n\} \& \{y_n\} \sim \{z_n\} \implies \{x_n\} \sim \{z_n\}$

Idea: Define a real number to be a (Cauchy seq. of rationals) equivalence class.

Homework: want to check that arithmetic extends to “real numbers”

$$[\{x_n\}] + [\{z_n\}] = [\{x_n + z_n\}]$$

Check that

1.  $\{x_n + z_n\}$  is a Cauchy seq.
2. Only depends on equivalence classes.

Want

$$\{x_n\} \sim \{y_n\} \quad \{z_n\} \sim \{w_n\}$$

then  $\{x_n + z_n\} \sim \{y_n + w_n\}$ . So,

$$[\{x_n + z_n\}] = [\{y_n + w_n\}]$$

**Example 5.2**

$$[2] + [11] = [2 + 11] = [13]$$

So,  $[2 + 1] \sim [13]([11] = [1])$ . Arithmetic (addition) in  $\mathbb{Z}_5$  thus makes sense. How about multiplication?  $\frac{[1]}{[a]} \leftarrow$  exists  $[a] \neq 0$ .

$$\frac{[1]}{[2]} = [3] \quad [2][3] = [6] = [1]$$

Thus,  $\mathbb{Z}_5$  is a field.

$\frac{p}{q} \sim \frac{r}{s}$ ,  $q, s \neq 0$  means  $ps = rq$  (when talking about fractions – associate it with equivalence relation).  $Q$  = set of equivalence classes.  $(\frac{p}{q})$  : equivalence classes).

Last time, we proved that Cauchy seq. of real numbers have limits (lub property). Also, no sequence  $\{x_n\}$  such that it hits all real numbers in  $[0, 1]$  – this is important. Contrast with  $Q \cap [0, 1]$ , then there is a sequence that hits them all. Refer to the last figure in Lec 4 or [math.ucla.edu/~greene](http://math.ucla.edu/~greene) – Summer 2008.

## §6 | Lec 6: Oct 14, 2020

Bolzano - Weierstrass:

Every bounded sequence has a convergent subsequence.

And we know about the Least Upper Bound Prop.

### §6.1 Continuous Functions on Closed Interval

$$f : S \rightarrow \mathbb{R}, \quad S \subset \mathbb{R}$$

**Example 6.1**

$$S = [a, b]$$

$$S = \mathbb{R}$$

**Definition 6.2** (Continuity) —  $s_0 \in S$ ,  $f$  is continuous at  $s_0$  if given  $\epsilon > 0$ ,  $\exists \delta > 0$  s.t.

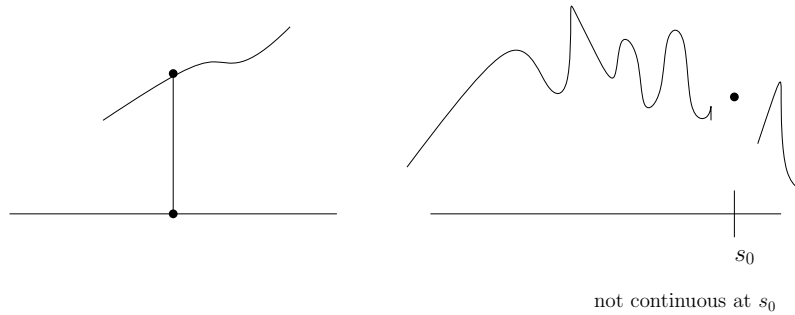
$$|s - s_0| < \delta_\epsilon \implies |f(s) - f(s_0)| < \epsilon$$

Three properties:

$$f : [a, b] \rightarrow \mathbb{R}$$

$f$  continuous

1.  $f$  is bounded on  $[a, b]$  means  $\exists M$  s.t. for all  $x \in [a, b]$ ,  $|f(x)| \leq M$



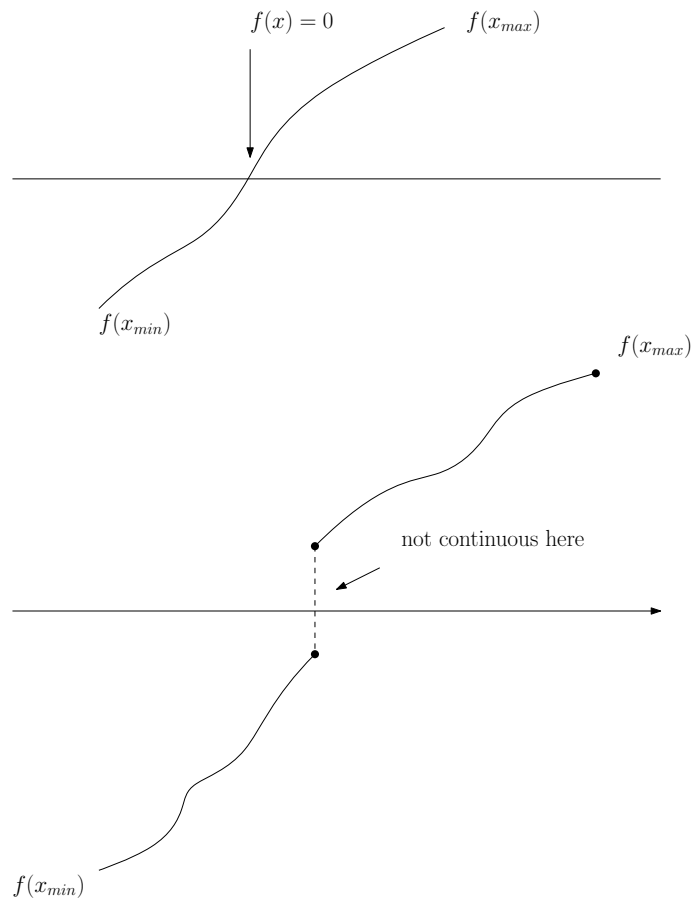
2. There exists  $x_{\min}, x_{\max} \in [a, b]$  such that for all  $x \in [a, b]$

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

Slogan:  $f$  attains its maximum and minimum.

3. If  $\alpha, f(x_{\min}) < \alpha < f(x_{\max})$ , then  $\exists x \in S = [a, b]$  s.t.  $f(x) = \alpha$ .

“Intermediate Value Theorem” Need the least upper bound prop – “completeness of



real numbers”

Exercise: def of continuity  $\{s_n\}$  converges to  $s_0 \iff$  if  $s_n \rightarrow s_0, s_n \in S, s_0 \in S$  then  $\{f(s_n)\}$  converges to  $f(s_0)$ .

**Example 6.3**

For (3),

$$f(x) = x^2 - 2 \quad \text{on } \mathbb{Q} \cap [1, 2]$$

Then  $f(1) = -1$ ,  $f(2) = 2$ , but no rational  $x \in [1, 2]$  s.t.  $f(x) = 0$ .

Back to the properties:

1.  $f$  is bounded – Think about  $|f| \leftarrow$  continuous if  $f$  is (exercise).

$\exists M$  such  $|f(x)| \leq M$  all  $x \in [a, b]$ . Suppose no such  $M$  exists.

Try  $M = 1, 2, 3, 4, 5, 6, \dots$  So  $\exists x_1 \quad |f(x_1)| > 1$

$$|f(x_2)| > 2$$

$\vdots$

$$|f(x_n)| > n$$

But Bolzano – Weierstrass: subsequence  $\{x_{n_j}\}$  that converges to  $x_0$  say  $|f(x_0)| \leftarrow$



finite number. So  $\exists N \ni |f(x_0)| \leq N$ .

Now for  $j$  large enough

$$|f(x_{n_j}) - f(x_0)| < 1$$

$x_{n_j}$  converges to  $x_0$

$$|f(x_{n_j})| < |f(x_0)| + |f(x_{n_j} - f(x_0))|$$

So  $j$  is large enough that

$$\underbrace{|f(x_{n_j})|}_{\geq |f(x_0)|} \leq N + \text{something less than } 1 \leq N$$

2. Attains max and min

Similar:  $\{f(x) : x \in [a, b]\}$  bounded set, has sup where

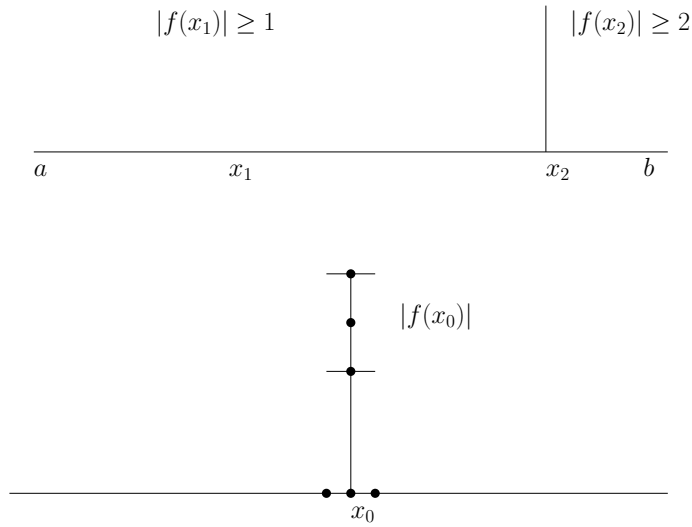
$$\sup \{f(x) : x \in [a, b]\}$$

either in the set of  $f$ -values (done if that's true),  $\sup f = f(x_0)$ .

OR:  $\sup f$  actually not in the set  $\{f(x) : x \in [a, b]\}$

Now  $\{x_{n_j}\}$  converges to  $x_0 \in [a, b]$

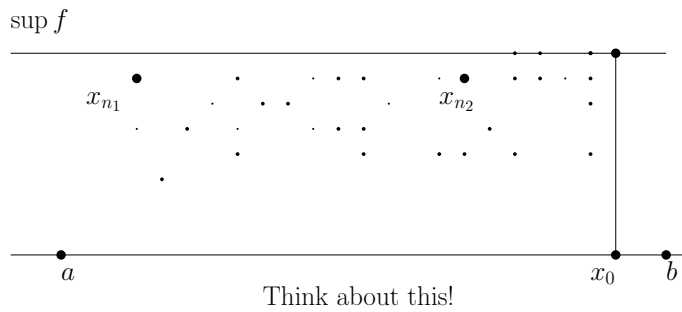
**Claim 6.1.**  $f(x_0) = \sup \{f(x) : x \in [a, b]\}$



$$f(x_{n_j}) \leq \sup \{f(x) : x \in [a, b]\}$$

and  $\lim f(x_{n_j}) = f(x_0) = f(\lim x_{n_j})$ . So

$$f(x_0) = \sup f$$



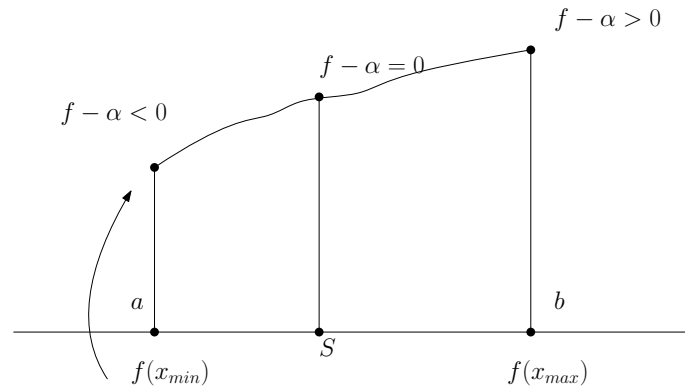
3.  $\alpha \in [f(x_{\min}), f(x_{\max})]$  then  $x$  such that  $f(x) = \alpha$ .

*Proof.* Wolog:

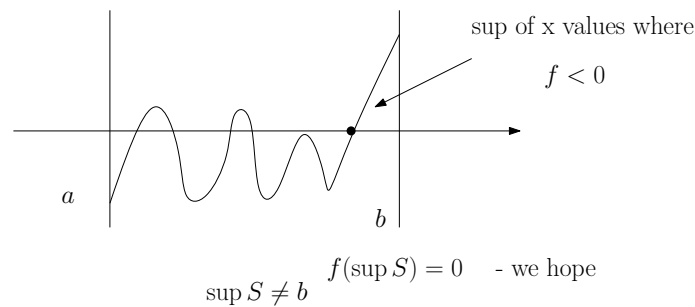
$$f(a) < 0 \quad \text{and} \quad f(b) > 0$$

then  $\exists x \in [a, b]$  with  $f(x) = 0$ .



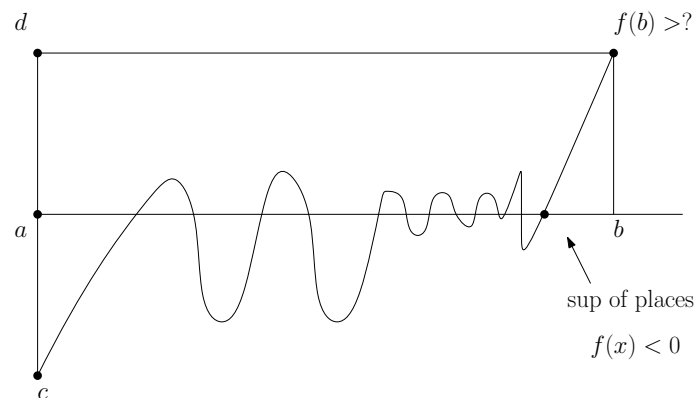


Use l.u.b: Look at  $S : \{x : f(x) < 0\}$  and  $S \neq \emptyset$  because  $f(a) \in S$ . Also,  $S$  is bounded above  $\rightarrow \exists$  l.u.b for  $S$ ,  $\sup S \in [a, b]$ . Hope that  $f(\sup S) = 0$ .



$\sup S \neq b$  is clear because  $f(b) > 0$  so  $f(b - \epsilon) > 0$  for small  $\epsilon$ .

So  $\sup S = x_0$ ,  $a < x_0 < b$ . What is  $f(x_0)$ ? If it's negative, then there are slightly bigger  $x \in [a_0, b] \ni f(x) < 0$  (continuity). In addition,  $x_0$  cannot be a limit of  $x$  with  $f(x) < 0 \rightarrow x_0 = \sup$  places where  $f < 0$ .  $\square$



$f$  continuous on  $[a, b]$  if it is

1. bounded.
2. attains max and min.
3. attains every value between max value and min value.

$f([a, b]) = [c, d]$  where  $c$  is min of  $f$  and  $d$  is max of  $f$ .

## §7 | Lec 7: Oct 16, 2020

### §7.1 Uniform Continuity

**Definition 7.1 (Uniform Continuity)** —  $S \subset \mathbb{R}$ ,  $f : S \rightarrow \mathbb{R}$ .  $f$  is uniformly continuous on  $S$  if given  $\epsilon > 0$  there is a  $\delta > 0$  s.t.  $|f(x) - f(y)| < \epsilon$  if  $x, y \in S$  and  $|x - y| < \delta_\epsilon$

#### Example 7.2

$f : S \rightarrow \mathbb{R}$ ,  $S = \mathbb{R}$ ,  $f(x) = x^2$ . Continuous on  $\mathbb{R}$  but it is not uniformly continuous on  $\mathbb{R}$ .

Continuity: Given fixed  $x$ , and  $\epsilon > 0$  want  $\delta$  so that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

$|x^2 - y^2| = |x - y||x + y|$  and want it smaller than  $\epsilon$ . Assume  $\delta \leq 1$ .

$$\begin{aligned} |x + y| &\leq |x| + |y| \\ |y| &< |x| + 1 \quad \text{if } |x - y| < \delta (\leq 1) \end{aligned}$$

So, if  $|x - y| < \delta (\leq 1)$ ,

$$\begin{aligned} |x^2 - y^2| &= |x - y||x + y| \\ &\leq |x - y|(2|x| + 1) \end{aligned}$$

Choose  $\delta < \frac{\epsilon}{2|x|+1}$  (ok since  $x$  is fixed)

$$\begin{aligned} |x^2 - y^2| &< \frac{\epsilon}{2|x|+1}(2|x|+1) \\ &= \epsilon \quad \text{if } |x - y| < \min \left\{ 1, \frac{1}{2|x|+1} \right\} \end{aligned}$$

Uniform continuity does not work on  $\mathbb{R}$ .

**Claim 7.1.**  $\epsilon = 1 > 0$ , there is no  $\delta > 0$  s.t.  $|x^2 - y^2| < 1 = \epsilon$  for all  $x, y$  with  $|x - y| < \delta$ .

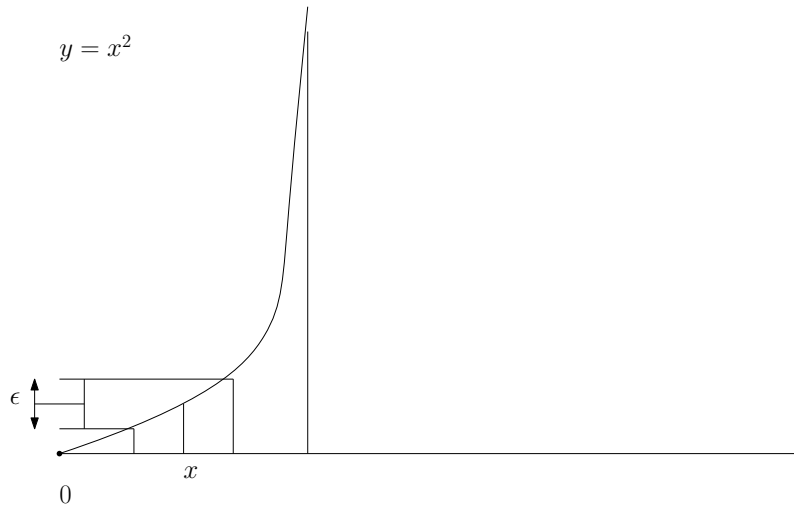
Why? Look at for  $\delta > 0$ , consider  $y = \frac{1}{\delta} + \frac{\delta}{2}$ ,  $x = \frac{1}{\delta}$

$$|x - y| < \delta$$

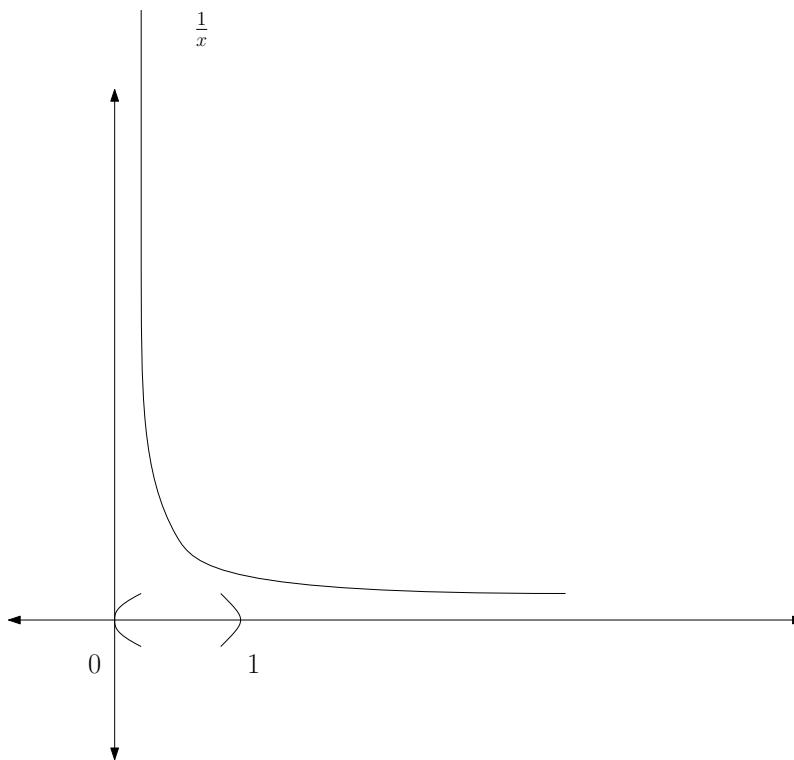
Also,

$$\begin{aligned} &\left| \left( \frac{1}{\delta} + \frac{\delta}{2} \right)^2 - \left( \frac{1}{\delta} \right)^2 \right| \\ &= \left| \frac{1}{\delta^2} + 2 \left( \frac{1}{\delta} \right) \left( \frac{\delta}{2} \right) + \left( \frac{\delta}{2} \right)^2 - \frac{1}{\delta^2} \right| \\ &= 1 + \left( \frac{\delta}{2} \right)^2 > 1 \end{aligned}$$

which is a contradiction.



**Exercise 7.1.**  $\frac{1}{x}$  on  $(0, 1)$  is continuous but not uniformly continuous. Suggest plausibly  $f$



continuous on  $[a, b]$  then it's uniformly continuous on  $[a, b]$  where  $a, b$  are finite.

**Theorem 7.3 (Heine – Cantor (Uniformly Continuous))**

A continuous function  $f$  on a closed interval is uniformly continuous.

*Proof.* (By contradiction) Suppose not. Then  $\epsilon > 0$  s.t. no  $\delta$  “works”. In particular,  $\exists \epsilon > 0$

s.t.  $\delta = 1$  fails,  $\delta = \frac{1}{2}$  fails, etc. So  $x, y \in [a, b]$  with  $|f(x_1) - (fy_1)| \geq \epsilon$  but  $|x_1 - y_1| < 1$ .  
 $x_n, y_n \in [a, b]$  with  $|f(x_n) - f(y_n)| \geq \epsilon$  but  $|x_n - y_n| < \frac{1}{n}$ . Hope this is impossible.  
 Bolzano - Weierstrass  $\implies \{n_j\}$  s.t.  $\{x_{n_j}\}$  has a limit

$$x_0 = \lim, \quad x_0 \in [a, b]$$

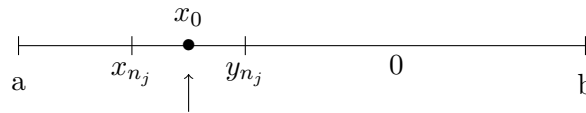
Now, claim  $\{y_{n_j}\}$  also has limit  $x_0$ .

$$|x_{n_j} - y_{n_j}| < \frac{1}{n_j}$$

small when  $n_j$  large ( $j$  large).

$$\begin{aligned} \lim x_{n_j} &= x_0 \\ \lim y_{n_j} &= x_0 \\ \lim f(x_{n_j}) &= f(x_0) \\ \lim f(y_{n_j}) &= f(x_0) \end{aligned}$$

So,  $\lim f(x_{n_j}) - f(y_{n_j}) = 0$ , but it contradicts  $|f(x_{n_j}) - f(y_{n_j})| \geq \epsilon$  for all  $j$ .  $\square$



$$f(x_0) \leq |f(x_{n_j}) - f(x_0)| + |f(x_0) - f(y_{n_j})| \rightarrow 0$$

Ideas of continuity and uniform continuity and Bolzano - Weierstrass Theorem – all have reasons in metric spaces.

## §8 | Lec 8: Oct 19, 2020

### §8.1 Convergence of Series

Series is “formal sum”, an infinite sum

$$a_0 + a_1 + a_2 + \dots = \sum_{j=1}^{\infty} a_j$$

A series  $\iff$  sequence  $a_1, a_2, a_3, \dots$  add together. Associated to  $a_1 + a_2 + a_3 + a_4 \dots$  is a sequence of partial sum

$$S_N = \sum_{n=1}^N a_n, \quad N = 1, 2, 3, 4, 5, \dots$$

number valued sequence.

**Definition 8.1 (Convergence of Series)** — Series converges if sequence associated  $\{S_N\}$  converges (has a limit).

Lots of things are defined by series such as ( $x \in \mathbb{R}$ ),

$$e^x = \lim_{N \rightarrow \infty} \left( 1 + x + \frac{x^2}{2!} + \dots + \frac{x^N}{N!} \right)$$

Given series  $a_0 + a_1 + a_2 + a_3 + \dots$ , when does it converge?

$$1 - 2 + 3 - 4 + 5 - 6 + 7 \dots$$

$$S_1 = 1, \quad S_2 = -1, \quad S_3 = 2 \dots$$

NO LIMIT! Series do not necessarily have to converge then it's okay to write

$$\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n$$

First thing to look at – Case where  $a_j \geq 0$

$$S_N \leq S_{N+1}, \quad N = 1, 2, 3, \dots$$

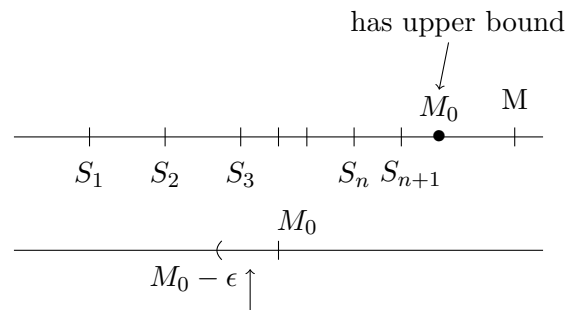
$S_{N+1} = S_N + a_{N+1}$  so  $a_{N+1} \geq 0$  means  $S_{N+1} \geq S_N$ . Two cases:

**Case 1:**  $\{S_n\}$  not bounded above.

$\lim S_N$  does not exist  $\rightarrow$  Series diverges (sequences with limits are always bounded above and below).

**Case 2:**  $\{S_n\}$  bounded above.

$\lim_{n \rightarrow \infty} S_n$  always exists. Namely, it is the least upper bound of set of values of  $S_n$ .



There is an  $S_{n_0}$  in this interval  $(M_0 - \epsilon, M_0]$ ,  $M_0$  is lub

From that  $n_0$  on,

$$S_n \geq S_{n_0}, \quad S_n \leq M$$

$S_n$  satisfies  $|S_n - M_0| < \epsilon$  if  $n \geq n_0$ . So  $\lim S_n = M_0$ . This implies that  $S_n$  is a Cauchy

sequence (it has a limit). Given  $\epsilon > 0, \exists N_\epsilon$  s.t.  $\left| \sum_{1}^{n_1} a_n - \sum_{1}^{n_2} a_n \right| < \epsilon$  if  $n_1, n_2 \geq N_\epsilon$ .

Suppose  $n_1 > n_2 \geq N_\epsilon$

$$\sum_1^{n_1} a_n - \sum_1^{n_2} a_n = \sum_{n_2+1}^{n_1} a_n$$

Note:  $S_7 - S_5 = a_6 + a_7$  which explains the above expression.

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} \dots$$

converges, but so does the following series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots < 2$$

This works for arbitrary choices of  $+$  or  $-$ .

### Theorem 8.2 (Basic)

If  $|b_1| + |b_2| + |b_3| + \dots$  converges, then

$$b_1 + b_2 + b_3 + \dots \quad \text{converges}$$

“Absolute convergence”  $\implies$  convergence (but not necessarily the same limit).

*Proof.* Assume  $\underbrace{\{S_n^A\}}_{A \text{ for absolute}}$  for absolved series has limit. So

$$\sum_1^\infty |b_n| \quad \text{converges}$$

$\implies \{S_n^A\}$  Cauchy sequence.

We hope it  $\implies \{S_n\} = \left\{ \sum_{j=1}^n b_j \right\}$  is a Cauchy sequence.

$$S_{n_1}^A - S_{n_2}^A = |b_{n_2+1}| + |b_{n_2+2}| + \dots + |b_{n_1}|$$

But

$$|b_{n_2+1} + \dots + b_{n_1}| \leq |b_{n_2+1}| + \dots + |b_{n_1}| (= S_{n_1}^A - S_{n_2}^A)$$

So,

$$|S_{n_1} - S_{n_2}| \leq S_{n_1}^A - S_{n_2}^A < \epsilon \quad \text{for } n_1, n_2 \geq N_\epsilon$$

Then  $|S_{n_1} - S_{n_2}| < \epsilon$  for  $n_1, n_2 \geq N_\epsilon$ . □

This is IMPORTANT – Better understand it thoroughly.

### Corollary 8.3 (Root Test)

$|b_n| \leq Cr^n, 0 < r < 1, C, r$  fixed, then  $\sum b_n$  converges.

Reason:  $\sum_{n=0}^\infty Cr^n = C \frac{1}{1-r}$  (geometric series).

**Exercise 8.1.**  $\sum_{n=0}^N Cr^n = C \frac{r^{N+1}-1}{r-1}, 0 < r < 1$  has limit  $\frac{C}{1-r}$ . Prove by induction.

Detail: Hypothesis:

$$|b_n| \leq Cr^n$$

$$\sum_1^\infty |b_n| \leq \sum_1^\infty Cr^n < \infty$$

$$\sum_b^N |b_n| \leq \sum_0^N Cr^n \leq M < \infty$$

So  $\sum_0^N |b_n|$  converges and bounded by  $Cr$ , and  $b_1 + b_2 + \dots$  converges absolutely.

## §9 | Dis 1: Oct 1, 2020

Notation:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$$

$$\mathbb{R} = \text{real numbers}$$

$$\mathbb{C} = \{a + bi, \quad a, b \in \mathbb{R}\}$$

Set theory:

- $A \subset B$  (or  $A \subseteq B$ ) means  $x \in A \implies x \in B$
- $x \in A \cap B$  means  $x \in A$  and  $x \in B$
- $x \in A \cup B$  means  $x \in A$  or  $x \in B$
- $x \in A \setminus B \iff x \in A$  and  $x \notin B$
- $A = B \iff A \subset B$  and  $B \subset A$

### §9.1 Induction

Given a sequence of mathematical statement  $P(n)$  indexed by  $\mathbb{N}$ . If  $P(1)$  is true and  $P(k) \implies P(k+1)$  is true  $\forall k \in \mathbb{N}$ , then  $P(n)$  is true  $\forall n \in \mathbb{N}$ .

#### Example 9.1

Prove  $\sum_{k=1}^n (2k-1) = n^2$  (\*) using induction.

Base case  $n = 1 : 1 = 1^2$  ✓

Induction step: assume as induction hypothesis that (\*) holds

$$\begin{aligned} \sum_{k=1}^{n+1} (2k-1) &= \sum_{k=1}^n (2k-1) + 2(n+1) - 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

Or we can prove it the following way

$$\begin{aligned} S &= 1 + 3 + 5 + \dots + (2n - 1) \\ S &= (2n - 1) + (2n - 3) + \dots + 3 + 1 \\ 2S &= 2n \cdot n \\ S &= n^2 \end{aligned}$$

### Example 9.2

$a_{n+1} = \sqrt{2 + a_n}$ ,  $a_1 = 1$ . Prove  $a_n > 0$  and  $a_n$  increasing.

$a_1 > 0$  assume  $a_n > 0$ ,  $a_{n+1} = \sqrt{2 + a_n} > 0$

$$a_2 = \sqrt{3} \approx 1.732 > 1 = a_1$$

Assume  $a_n \leq a_{n+1}$ , want to show  $a_{n+1} \leq a_{n+2} \iff \sqrt{a_n + 2} \leq \sqrt{a_{n+1} + 2} \iff a_n \leq a_{n+1}$

### Example 9.3

$(1 + x)^n \geq 1 + nx$  : Bernoulli Inequality

$$x \geq -1, \quad n \geq 0$$

base case  $1 \geq 1$

Assume  $(1 + x)^n \geq 1 + nx$

$$\begin{aligned} (1 + x)^{n+1} &= (1 + x)^n(1 + x) \geq (1 + nx)(1 + x) = 1 + (n + 1)x + nx^2 \\ &= 1 + (n + 1)x \end{aligned}$$

Strong Induction:

If  $P(1)$  true and  $P(1), P(2), \dots, P(k) \implies P(k + 1)$  true  $\forall k \in \mathbb{N}$  then  $P(n)$  holds for all  $n \in \mathbb{N}$

**Remark 9.4.** Induction  $\iff$  strong induction

### Example 9.5

Every integer greater than 1 is a product of primes.

Assume  $2, 3, \dots, n$  is a product of primes.  $n+1$  is either a prime or a composite, in which case  $n + 1 = ab$ ,  $1 < a, b < n + 1$ .

By strong induction hypothesis, both  $a$  and  $b$  are product of primes, hence so is  $n + 1 = ab$ .

**Exercise 9.1.** Every integer greater than 1 has a prime divisor.

Proof of infinitude of primes by Euclid:

*Proof.* Assume on the contrary there are finitely many primes  $\{p_1, p_2, \dots, p_k\}$ . Define  $N = p_1 \dots p_k + 1 > 1$  and (by above exercise) let  $p$  be a prime divisor of  $N$  but  $p \neq p_j$  for



any  $1 \leq j \leq k$  otherwise if  $p = p_j$  then  $p|p_2 \dots p_k$  also  $p|N \implies p|N - p_1 \dots p_k \implies p|1$ , a contradiction. (no primes divide 1)  $\square$

## §10 | Dis 2: Oct 8, 2020

### §10.1 Number System

- $(\mathbb{N}, +, \cdot, <)$  :  $+$  :  $\mathbb{N} \times \mathbb{N} = \mathbb{N}^2 \rightarrow \mathbb{N}$  satisfies commutativity and associativity. Note that 0 is the identity with respect to addition, but  $\mathbb{N}$  has no additive inverse.
- $(\mathbb{Z}, +, \cdot, <)$  :  $(\mathbb{Z}, +)$  is a commutative group (associativity, identity, inverse).  $(\mathbb{Z}, \cdot)$  satisfies commutativity, associativity with 1 as mult identity but 2 has no mult inverse.
- $(\mathbb{Q}, +, \cdot, <)$  :  $(\mathbb{Q}, +)$  and  $(\mathbb{Q}, \cdot)$  are commutative group(i).  $+$  and  $\cdot$  are compatible with distributive law:  $a(b + c) = ab + ac$  (ii). Both (i) and (ii) mean  $(\mathbb{Q}, +, \cdot)$  is a FIELD.  $(\mathbb{Q}, <)$  is an ordered set with  $<$  satisfying trichotomy and transitivity.  $+$ ,  $\cdot$  are compatible :  $y < z \implies x + y < x + z \forall x, x > 0, y > 0 \implies xy > 0$ . With the above compatibility,  $(\mathbb{Q}, +, \cdot, <)$  is an **ordered field**. Even though  $\mathbb{Q}$  is additivity and multiplicatively complete,  $\mathbb{Q}$  is not satisfying in that

1.  $\mathbb{Q}$  is not algebraically closed,  $x^2 - 2$  is a polynomial with no root in  $\mathbb{Q}$ .
2.  $\mathbb{Q}$  is not complete in a metric space: there exists subsets of  $\mathbb{Q}$  bounded above but with no least upper bound (supremum), e.g.  $A := \{p \in \mathbb{Q} : p < 0 \text{ or } p^2 < 2\}$  and  $B = \mathbb{Q} \setminus A$ .  $A$  contains no largest number and  $B$  contains no smallest.

$$\forall p \in A \exists q \in A \quad q > p$$

Let  $p \in A$ . Define  $q := p - \frac{p^2 - 2}{p + 2} > p$

$$q^2 - 2 = \left( \frac{2p + 2}{p + 2} \right)^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2} < 0 \implies q^2 < 2$$

If  $A$  has an upper bound  $\alpha$ ,  $\alpha \notin A$  : then  $\alpha \in B$ . It follows that  $B$  is the set of all upper bounds for  $A$ . Since  $B$  contains no smallest number,  $A$  has no least upper bound in  $\mathbb{Q}$ .

**Definition 10.1** (Least Upper Bound Property) —  $S$  has the least-upper-bound property if  $\forall E \subset S$  nonempty, bounded above  $\sup E \in S$ .

**Remark 10.2.**  $\mathbb{Q}$  does not satisfy the least-upper-bound property.

$(\mathbb{R}, +, \cdot, <)$  there exists an ordered field with the l.u.b property that contains an isomorphic copy of  $\mathbb{Q}$ .

## §10.2 Equivalence Relation

An equivalence relation given  $\sim$  on  $A \times A$  satisfies

- $x \sim x$  reflexivity
- $x \sim y \iff y \sim x$  symmetry
- $x \sim y \cdot y \sim z \implies x \sim z$  transitivity

### Example 10.3

$\mathbb{Q}$  Define  $\sim$  on  $\{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$  by  $(a, b) \sim (c, d)$  if  $ad = bc$

$$A = \mathbb{Z}^2 \setminus \{(a, 0) : a \in \mathbb{Z}\}$$

$\mathbb{Q} =$  the set of all equivalence classes of  $A$  write  $\sim$   
 $= A / \sim = \{[x] : x \in A\}$

In this construction,  $\mathbb{Z} \rightarrow \mathbb{Q}, n \rightarrow [(n, 1)]$   
 $+$  and  $\cdot : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$  : note that  $+$  and  $\cdot$  need to be well-defined on  $\mathbb{Q}^2$ . (need to show  $\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}$  if  $\frac{a}{b} \sim \frac{a'}{b'}$  and  $\frac{c}{d} \sim \frac{c'}{d'}$ ).

### Example 10.4

$$S' = [0, 1] / 0_m$$

**Definition 10.5** (Convergent Sequences) —  $\{a_n\}_{n \geq 1} \subseteq \mathbb{R}$  is said to be convergent to  $l$  if  $\forall \epsilon > 0 \exists N(\epsilon) > 0$  s.t.  $\forall n \geq N, |a_n - l| < \epsilon$

## §11 | Dis 3: Oct 13, 2020

### §11.1 Equivalence Relation (Cont'd)

#### Example 11.1

Define  $\sim p$  on  $\mathbb{Z}$  by  $a \sim pb$  if  $a - b \in p\mathbb{Z}$  ( $p|a - b$ ).

$$\forall a \exists ! b \in \mathbb{Z}, 0 \leq r < p \text{ s.t. } a = bp + r.$$

$$F_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z} / \sim p = \{[0]_p, [1]_p, [2]_p, \dots, [p-1]_p\}$$

$$[a]_p + [b]_p = [a + b]_p \quad \& \quad [a]_p [b]_p = [ab]_p$$

**Remark 11.2.**  $(F_p, +, \cdot)$  is a finite field.  $F_p$  cannot be ordered:  $1 > 0, 1 + 1 > 0, \dots, p - 1 > 0$  but  $p - 1 = -1$

**Example 11.3**

$$T = \mathbb{R}/\mathbb{Z} \quad a \sim b \text{ if } ab \in \mathbb{Z}$$

$$[0, 1]/0 \sim 1$$

$$\forall a \in \mathbb{R}, \quad \exists b = \underbrace{\{a\}}_{\text{fractional part of } a} \in [0, 1) \text{ s.t. } a \sim b$$

**§11.2 Construction of  $\mathbb{R}$  via Cauchy Sequences (Cantor)**

$S$  = set of rational Cauchy sequences.

$\sim$  on  $S$  :  $\{x_n\} \sim \{y_n\}$  if  $\lim(x_n - y_n) = 0$  (Q3 – Homework 2)

$Q = S/\sim = \{[\{x_n\}] : \{x_n\} \in S\}$ . First we need to define arithmetic on  $Q$ .

$$[\{p_n\}] + [\{q_n\}] = [\{p_n + q_n\}]$$

$$[\{p_n\}] - [\{q_n\}] = [\{p_n - q_n\}]$$

$$[\{p_n\}] \cdot [\{q_n\}] = [\{p_n q_n\}]$$

$$[\{p_n\}] / [\{p_n/q_n\}] = [\{p_n/q_n\}], \quad [\{q_n\}] \neq 0, = [\{0, 0, 0, \dots\}]$$

$+$  :  $Q \times Q \rightarrow Q$ . Check well-defined

- $\{x_n\} \cdot \{y_n\}$  cauchy then so is  $\{x_n + y_n\}$  (Q4)
- $\{x_n\} \sim \{y_n\}$  &  $\{z_n\} \sim \{w_n\}$  then  $\{x_n + z_n\} \sim \{y_n + w_n\}$  (Q5)  
Commutativity, assoc, identity, ( $0 = [\{0, 0, 0, \dots\}]$ ), inverse.
- Well-defined:  $\{x_n\}, \{y_n\}$  so is  $\{x_n y_n\}$  (Q4).
- $\{x_n\} \sim \{y_n\}$  &  $\{z_n\} \sim \{w_n\}$  (Q6, Q7)  
comm, assoc, iden, ( $1 = [\{1, 1, \dots, 1\}]$ )  
mult. inverse (Q9, Q10).  
<: trichotomy (Q11), transitivity  
various compatibility (distributivity, etc)  
l.u.b property (Q12)

Note: All the  $Q$  used above is assumed to be  $Q^{\text{hat}}$

**Remark 11.4.**

$$\begin{aligned} Q &\rightarrow Q^{\text{hat}} \\ q &\mapsto [q^*] \\ p < q &\iff [p^*] < [q^*] \end{aligned}$$

Sequences:

- Cauchy seq. are bounded.
- Convergent seq. is Cauchy.

Theorem: in  $\mathbb{R}$ , every Cauchy seq. is convergent.

**Example 11.5**

$$a_n = \frac{1}{n}$$

$\forall \epsilon > 0 \exists N$  s.t.  $\epsilon N > 1$ .

$$\forall n \geq N \quad \left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

□

## §12 | Dis 4: Oct 20, 2020

### §12.1 Least Upper Bound and Its Applications

**Remark 12.1** ( $\epsilon$  – Principle).  $a, b \in \mathbb{R}, \forall \epsilon > 0, a \leq b + \epsilon \implies a \leq b$ .

- $x, y \in \mathbb{R} \quad \forall \epsilon > 0, |x - y| \leq \epsilon \implies x = y$ .

Supremum:  $E \subset S$  bounded above. Suppose  $\sup E \in S$

- $e \leq \sup E \forall e \in E$ .
- $\forall \beta < \sup E, \exists e \in E$  s.t.  $\beta < e < \sup E$

OR

$$\forall \epsilon > 0, \exists e \in E \text{ s.t. } \sup E - \epsilon < e \leq \sup E.$$

**Example 12.2**

$$\sup \left\{ \frac{1}{n} \right\}_{n \geq 1} = 1, \quad \inf \left\{ \frac{1}{n} \right\} = 0.$$

- $0 \leq \frac{1}{n} \forall n \in \mathbb{N}$ .
- $\forall \epsilon > 0, \exists n \in \mathbb{N}$  s.t.  $0 \leq \frac{1}{n} < \epsilon$  by Archimedean Prop.

**Theorem 12.3** (Nested Interval)

$\{I_n = [a_n, b_n]\}_{n \geq 1} \subset \mathbb{R}, I_n \supset I_{n+1} \implies \bigcap_{n=1}^{\infty} I_n \neq \emptyset$ . Moreover, if  $|I_n| \rightarrow 0$ , then  $\bigcap I_n$  is a singleton (a set with exactly one element).

*Proof.*  $\sup a_n \in \bigcap I_n$ .

□

**Theorem 12.4** ((4.1))

(Bolzano – Weierstrass): Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

*Proof.*  $I_0 = [-M, M] \supset I_1 \supset I_2 \supset \dots$

$$|I_n| = (2M) \cdot 2^{-n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

From Nested Interval Thm,  $\bigcap_{n=0}^{\infty} I_n = \{x\}$ . Choose  $x_{n_k} \in I_k, x_{n_k} \rightarrow x$ .

□

**Remark 12.5.** l.u.b property of  $\mathbb{R} \implies$  Nested Interval  $\implies$  Bolzano – Weierstrass  $\xRightarrow{(*)}$  Cauchy Completeness.

(\*) Exercise:  $\{x_n\}$  Cauchy.  $x_{n_k} \rightarrow x \implies x_n \rightarrow x$ .

**Remark 12.6.** In  $\mathbb{R}$ , to check convergence, it suffices to check Cauchy. Useful especially when you don't have a candidate for the limit. Cauchy criterion for TBA:  $\sum_{n=1}^{\infty} a_n$  converges  $(\lim_{n \rightarrow \infty} \sum_{k=0}^n a_k)$  exists.  $\iff \sum a_n$  Cauchy  $(\forall \epsilon > 0 \exists N |\sum_{k=n}^m a_k| < \epsilon \quad \forall m \geq n \geq N)$ .

### Corollary 12.7

Absolute convergence  $\implies$  convergence.  $(\sum |a_n| \text{ converges} \implies \sum a_n \text{ converges})$ .

Monotone convergence theorem,  $\{a_n\}$  monotone. Then  $\{a_n\}$  bounded  $\iff \{a_n\}$  convergent. (HW 3 – Q1).

**Definition 12.8 (Monotone)** —  $\{a_n\}$  monotone if  $a_n \leq a_{n+1} \forall n$  or  $a_n \geq a_{n+1} \forall n$ .

### Corollary 12.9

$\sum |a_n| < \infty \iff \sum |a_n| \text{ converges}$ .

## §12.2 Continuity

**Definition 12.10 ( (6.2) )** —  $f : X \rightarrow \mathbb{R}$  is continuous at  $x$  (local prop) if

1. ( $\epsilon - \delta$  def)  $\forall \epsilon > 0, \exists \delta(\epsilon, x) > 0$  s.t.  $\forall y \in X, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ .
2. (Sequential def)  $\forall \{x_n\} \subset X, x_n \rightarrow x \implies f(x_n) \rightarrow f(x)$  ( $f$  preserves sequential convergence).

$f : X \rightarrow \mathbb{R}$  is continuous if  $f$  is continuous at all  $x \in X$ .

**Definition 12.11 ( (7.1) )** —  $f$  is uniformly continuous on  $X$  (global prop) if

1. ( $\epsilon - \delta$ )  $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$  s.t.  $\forall x, y \in X, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ .
2. (Sequential)  $\forall \{x_n\} \subset X, \{x_n\}_{n \geq 1} \text{ Cauchy} \implies \{f(x_n)\}_{n \geq 1} \text{ Cauchy}$ . ( $f$  preserves Cauchy seq).

**Remark 12.12.** Uniform continuity  $\implies$  continuity.

**Example 12.13**

$f : (0, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = \frac{1}{x}$  is continuous.

$$\left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x - y|}{xy} < \frac{|x - y|}{x \cdot \frac{x}{2}} = |x - y| \cdot 2x^{-2} < \epsilon$$

$$\delta = \min \left\{ \frac{x}{2}, \frac{\epsilon x^2}{2} \right\}.$$