

# Math 115AH – Honors Linear Algebra

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This is math 115AH – Honors Linear Algebra, a traditional first upper-division course that UCLA math students usually take. It's taught by Professor Elman, and our TA is Harris Khan. We meet weekly on MWF at 2:00pm – 2:50pm for lectures, and our discussion is on TR at 2:00pm – 2:50pm. With regard to book, we use *Linear Algebra 2<sup>nd</sup>* by *Hoffman and Kunze* for the class. Note that some of the theorems' name are not necessarily the official name of the theorem; it's just a way to assign meaning to a theorem (easier for reference) instead of a tedious section number. Other course notes can be found through my [github](#) site. Please contact me at [ducvu2718@ucla.edu](mailto:ducvu2718@ucla.edu) if you find any concerning mathematical errors/typos.

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## §1 | Lec 1: Oct 2, 2020

**Remark 1.1.** To know a definition, theorem, lemma, proposition, corollary, etc., you must

1. Know its precise statement and what it means without any mistake
2. Know explicit example of the statement and specific examples that do not satisfy it
3. Know consequences of the statement
4. Know how to compute using the statement
5. At least have an idea why you need the hypotheses – e.g., know counter-examples,...
6. Know the proof of the statement
7. Know the important (key) steps of in the proof, separate from the formal part of the proof – i.e., the main idea(s) of the proof

**THIS IS NOT EASY AND TAKES TIME – EVEN WHEN YOU THINK THAT YOU HAVE MASTERED THINGS.**

### §1.1 Field

What are the properties of the REAL NUMBERS?

$$\mathbb{R} := \{x | x \text{ is a real no.}\}$$

– at least algebraically?

There are two FUNCTIONS (or MAPS)

- $+: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  called ADDITION write  $a + b := +(a, b)$
- $\cdot: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  called MULTIPLICATION write  $a \cdot b := \cdot(a, b)$

that satisfy certain rule e.g., associativity, commutativity,...

**Definition 1.2 (Field)** — A set  $F$  is called a FIELD if there are two functions

- Addition:  $+: F \times F \rightarrow F$ , write  $a + b := +(a, b)$
- Multiplication:  $\cdot: F \times F \rightarrow F$ , write  $a \cdot b := \cdot(a, b)$

satisfying the following AXIOMS(A: addition, M: multiplication, D: distributive)

A1  $(a + b) + c = a + (b + c)$  Associativity

A2  $\exists$  an element  $0 \in F \ni a + 0 = a = 0 + a$  Existence of a Zero

A3  $\forall x \in F \exists y \in F \ni x + y = 0 = y + x$  Existence of an Additive Inverse

A4  $a + b = b + a$  Commutativity

M1  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M2 (A2) holds and  $\exists$  an element  $1 \in F$  with  $1 \neq 0 \ni a \cdot 1 = a = 1 \cdot a$  Existence of a One

M3 (M2) holds and  $\forall 0 \neq x \in F \exists y \in F \ni xy = 1 = yx$  Existence of a Multiplicative Inverse

M4  $x \cdot y = y \cdot x$

D1  $a \cdot (b + c) = a \cdot b + a \cdot c$  Distributive Law

D2  $(a + b) \cdot c = a \cdot c + b \cdot c$

Comments: Let  $F$  be a field,  $a, b \in F$ . Then the following are true

1.  $F \neq \emptyset$  ( $F$  at least has 2 elements)
2. 0 and 1 are unique
3. If  $a + b = 0$ , then  $b$  is unique write  $b$  as  $-a$  :  
if  $a + b = a + c$ , then

$$\begin{aligned} b &= b + 0 \\ &= b + (a + c) \\ &= (b + a) + c \\ &= (a + b) + c \\ &= 0 + c \\ &= c \end{aligned}$$

4. if  $a + b = a + c$ , then  $b = c$
5. if  $a \neq 0$  and  $ab = 1 = ba$ , then  $b$  is unique write  $a^{-1}$  for  $b$ .
6.  $0 \cdot a = 0 \forall a \in F$

$$0 \cdot a + 0 \cdot a = (0 + 0) \cdot a = 0 \cdot a = 0 \cdot a + 0$$



so  $0 \cdot a = 0$  by 3.

7. if  $a \cdot b = 0$ , then  $a = 0$  or  $b = 0$ . If  $a \neq 0$ , then  $0 = a^{-1}(ab) = (a^{-1}a)b = 1b = b$

8. if  $a \cdot b = a \cdot c$ ,  $a \neq 0$ , then  $b = c$

9.  $(-a)(-b) = ab$

10.  $-(-a) = a$

11. if  $a \neq 0$ , then  $a^{-1} \neq 0$  and  $(a^{-1})^{-1} = a$

### Example 1.3

$$\mathbb{Q} := \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

$\mathbb{R} :=$  set of real no.

$\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$  with

$$\begin{aligned} (a + b\sqrt{-1}) + (c + d\sqrt{-1}) &= (a + c) + (b + d)\sqrt{-1} \\ (a + b\sqrt{-1}) \cdot (c + d\sqrt{-1}) &= (ac - bd) + (ad + bc)\sqrt{-1} \end{aligned}$$

$\forall a, b, c, d \in \mathbb{R}$

Under usual  $+$ ,  $\cdot$  of  $\mathbb{C}$

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

are all field and we say  $\mathbb{Q}$  is a **subfield** of  $\mathbb{R}$ ,  $\mathbb{Q}, \mathbb{R}$  **subfield** of  $\mathbb{C}$ , i.e., they have the same  $+$ ,  $\cdot$ ,  $0, 1$ .

$\mathbb{Z}$  is not a field as  $\nexists n \in \mathbb{Z} \ni 2n = 1$ , so  $\mathbb{Z}$  do not satisfy (M3).

Note: To show something is FALSE, we need only one COUNTER-EXAMPLE. To show something is TRUE, one needs to show true for all elements – not just example.

## §2 | Lec 2: Oct 5, 2020

### §2.1 Field(Cont'd)

Note:  $\mathbb{Z}$  does satisfy the weaker properly if  $a, b \in \mathbb{Z}$  then

(M3') if  $ab = 0$  in  $\mathbb{Z}$ , then  $a = 0$  or  $b = 0$  and all other axioms except M3 hold

1. Let  $F = \{0, 1\}$ ,  $0 \neq 1$ . Define  $+$ ,  $\cdot$  by following table Then  $F$  is a field.

Table 1: ADDITION

$+$	0	1
0	0	1
1	1	0

Table 2: MULTIPLICATION

$\cdot$	0	1
0	0	0
1	0	1

2.  $\exists$  fields with  $n$  elements for

$$n = 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, \dots$$

[conjecture?]

3. Let  $F$  be a field

$$F[t] := \{(\text{formal polynomial in one variable})\}$$

with  $t$ , given by

$$(a_0 + a_1t + a_2t^2 + \dots) + (b_0 + b_1t + b_2t^2 + \dots) := (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \dots$$

$$(a_0 + a_1t + a_2t^2 + \dots) \cdot (b_0 + b_1t + b_2t^2 + \dots) := a_0b_0 + (a_0b_1 + a_1b_0)t + \dots$$

Note:  $f, g \in F[t]$  are EQUAL iff they have the same COEFFICIENTS (coeffs) for each  $t^i$  (if  $t^i$  does not occur we assume its coeff is 0.)  $F[t]$  is not a field but satisfy all axioms except (M3) but it does satisfy (M3') (compare  $\mathbb{Z}$ ). Let

$$F(t) := \left\{ \frac{f}{g} \mid f, g \in F[t], g \neq 0 \right\} \quad \text{with}$$

- $\frac{f}{g} = \frac{h}{k}$  if  $fk = gh$
- $\frac{f}{g} + \frac{h}{k} := \frac{fk+gh}{gk} \quad \forall f, g, h, k \in F[t]$
- $\frac{f}{g} \cdot \frac{h}{k} := \frac{fh}{gk} \quad g \neq 0, k \neq 0$

is a field, the FIELD of RATIONAL POLYS over  $F$ .

Note: the 0 in  $F[t]$  is  $\frac{0}{f}$ ,  $f \neq 0$ , and 1 in  $F[t]$  is  $\frac{f}{f}$ ,  $f \neq 0$ .

4. let  $F$  be a field.

$$M_n F := \{A \mid A \text{ is } n \times n \text{ matrix entries in } F\}$$

usual  $+$ ,  $\cdot$  of matrices, i.e. for  $A, B \in M_n F$ , let

$$A_{ij} := i \text{th entry of } A, \text{ etc}$$

Then

$$(A+B)_{ij} := A_{ij} + B_{ij}$$

$$(AB)_{ij} := C_{ij} := \sum_{k=1}^n A_{ik} B_{kj} \quad \forall i, j$$

Note:  $A = B$  iff  $A_{ij} = B_{ij} \quad \forall i, j$ .

If  $n = 1$ , then

$F$  and  $M_1F$  and the “same” so  $M_1F$  is a field. If  $n > 1$  then  $M_nF$  is not a field nor does it satisfy (M3), (M4), (M3’). It does satisfy other axioms with

$$I = I_n := \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}, \quad 0 = 0_n := \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

## §2.2 Vector Space

$\mathbb{R}^2 := \{(x, y) | x, y \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R}$  Vector in  $\mathbb{R}^2$  are added as above and if  $v \in \mathbb{R}^2$  is a vector,

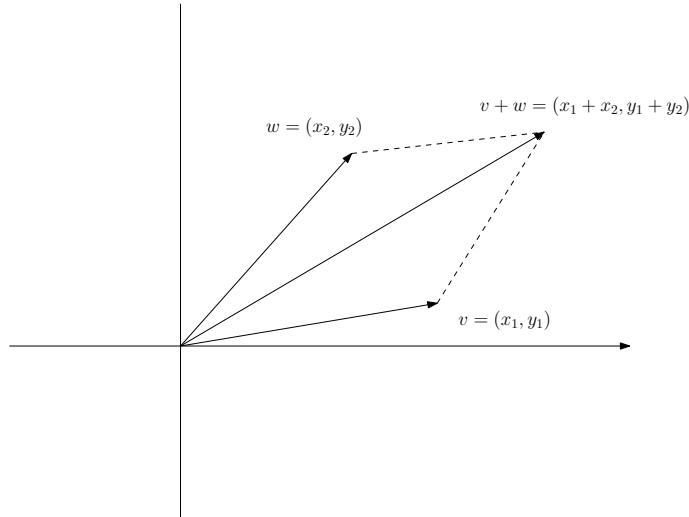


Figure 1: Geometry in  $\mathbb{R}^2$

$\alpha v$  makes sense  $\forall \alpha \in F$  by  $\alpha(x, y) = (\alpha x, \alpha y)$  called SCALAR MULTIPLICATION. For  $+$ , scalar mult and  $(0, 0)$  is the ZERO VECTOR satisfying various axioms. e.g., assoc, comm, “distributive law...”. To abstractify this

**Definition 2.1 (Vector Space)** —  $V$  is a vector space over  $F$ , via  $+$ ,  $\cdot$  or  $(V, +, \cdot)$  is a vector space over  $F$  where

$$\begin{array}{ll} + : V \times V \rightarrow V & \cdot : F \times V \rightarrow V \\ \text{Addition} & \text{Scalar Multiplication} \\ \text{write: } v + w := +(v, w) & \text{write: } \alpha \cdot v := \cdot(\alpha, v) \text{ or } \alpha v \end{array}$$

if the following axioms are satisfied

$$\forall v, v_1, v_2, v_3 \in V, \quad \forall \alpha, \beta \in F$$

1.  $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$
2.  $\exists$  an element  $0 \in V \ni v + 0 = v = 0 + v$
3. (2) holds and the element  $(-1)v$  in  $V$  satisfies

$$v + (-1)v = 0 = (-1)v + v$$

or (2) holds and  $\forall v \in V \exists w \in V \ni v + w = 0 = w + v$

4.  $v_1 + v_2 = v_2 + v_1$
5.  $1 \cdot v = v$
6.  $(\alpha \cdot \beta) \cdot v = \alpha(\beta \cdot v)$
7.  $(\alpha + \beta)v = \alpha v + \beta v$
8.  $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$

Elements of  $V$  are called **vector**, elements of  $F$  **scalars**.

Comments:  $V$ : a vector space over  $F$

1. The zero of  $F$  is unique and is a scalar. The zero of  $V$  is unique and is a vector. They are different (unless  $V = F$ ) even if we write 0 for both – should write  $0_F, 0_V$  for the zero of  $F, V$  respectively.
2. if  $v, w \in V, \alpha \in F$  then

$$\begin{array}{ll} \alpha v + w & \text{makes sense} \\ v\alpha, vw & \text{do not make sense} \end{array}$$

3. We usually write  
vector using Roman letter  
scalar using Greek letter  
exception things like  $(x_1, \dots, x_n) \in \mathbb{R}^n, x_i \in \mathbb{R} \forall i$

4.  $+: V \times V \rightarrow V$  says

$$\text{if } v, w \in V, \text{ then } v + w \in V$$

write  $v, w \in V \xrightarrow{\text{implies}} v + w \in V$ . We say  $V$  is CLOSED under  $+$

5.  $\cdot : F \times V \rightarrow V$  says  $\alpha \in F, v \in V \rightarrow \alpha v \in V$ . We say  $V$  is CLOSED under SCALAR MULTIPLICATION.

### Example 2.2

$F$  a field, e.g.,  $\mathbb{R}$  or  $\mathbb{C}$

1.  $F$  is a vector space over  $F$  with  $+, \cdot$  of a field, i.e., the field operation are the vector space operation with  $0_F = 0_V$ .
2.  $F^n := \{\alpha_1, \dots, \alpha_n\} | \alpha_i \in F \forall i$  is a vector space over  $F$  under COMPONENT-WISE OPERATION and

$$0_{F^n} := (0, \dots, 0)$$

Even have

$$F_{\text{finite}}^\infty = \{(\alpha_1, \dots, \alpha_n, \dots) | \alpha_i \in F \forall i \text{ with only FINITELY MANY } \alpha_i \neq 0\}$$

3. Let  $\alpha < \beta$  in  $\mathbb{R}$

$$I = [\alpha, \beta], \quad (\alpha, \beta), \quad [\alpha, \beta), \quad (\alpha, \beta]$$

including  $(\alpha = -\infty, \beta = \infty)$ . Let  $\text{fxn } I := \{f : I \rightarrow \mathbb{R} | f \text{ a fxn}\}$  called the SET of REAL VALUE FXNS on  $I$ .

Define  $+, \cdot$  as follows:  $\forall f, g \in \text{Fxn } I$ ,

$$f + g \quad \text{by } (f + g)(x) := f(x) + g(x)$$

$$\alpha f \quad \text{by } (\alpha f)(x) := \alpha f(x) \quad \forall \alpha \in \mathbb{R}$$

and 0 by  $0(\alpha) = 0 \forall \alpha \in F$ . Then  $\text{Fxn } I$  is a vector space over  $\mathbb{R}$ .

## §3 | Lec 3: Oct 7, 2020

### §3.1 Vector Space(Cont'd)

#### Example 3.1

$F$  is a field, e.g.  $\mathbb{R}$  or  $\mathbb{C}$

1.  $F$  is a vector space over  $F$  with  $+, \cdot$  of a field, i.e. the field operation are the vector space operation with  $0_F = 0_V$ .
2.  $F^n := \{(\alpha_1, \dots, \alpha_n) | \alpha_i \in F \forall i\}$  is a vector space over  $F$  under COMPONENT-WISE OPERATIONS

$$(\alpha_1, \dots, \alpha_n) + (\beta_1, \dots, \beta_n) := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$$

$$\beta(\alpha_1, \dots, \alpha_n) := (\beta\alpha_1, \dots, \beta\alpha_n)$$

with  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in F$  and  $0_{F^n} := (0, \dots, 0)$ .

Even have:

$$F^\infty = F_{\text{this}}^\infty : \{(\alpha_1, \dots, \alpha_n, \dots) | \alpha_i \in F \forall i \text{ with only FINITELY MANY } \alpha_i \neq 0\}$$

3. Let  $\alpha < \beta$  in  $\mathbb{R}$

$$I = [\alpha, \beta], \quad (\alpha, \beta), \quad [\alpha, \beta), \quad (\alpha, \beta]$$

(including  $\alpha = -\infty, \beta = \infty$ . Let function  $I := \{f : I \rightarrow \mathbb{R} | f \text{ a function}\}$

Define  $+, \cdot$  as follows:  $\forall f, g \in \text{Fxn } I$ ,

$$\begin{aligned} f + g & \text{ by } (f + g)(x) := f(x) + g(x) \\ \alpha f & \text{ by } (\alpha f)(x) := \alpha f(x) \quad \forall \alpha \in \mathbb{R} \end{aligned}$$

and 0 by  $0(\alpha) = 0 \forall \alpha \in F$ . Then  $\text{Fxn } I$  is a vector space over  $\mathbb{R}$ .

Using this, we get subsets which are also vector space over  $\mathbb{R}$  with same  $+, \cdot, 0$ .

- $C(I) := \{f \in \text{fxn } I | f \text{ continuous on } I\}$
- $\text{Diff } (I) := \{f \in \text{fxn } I | f \text{ differentiable on } I\}$
- $C^n(I) := \{f \in \text{fxn } I | f(n) \text{ then}^{\text{th}} \text{ derivative of } f \text{ and } f \text{ exists on } I \text{ and is cont on } I\}$
- $C^\infty(I) := \{f \in \text{fxn } I | f(n) \text{ exists } \forall n \geq 0 \text{ on } I \text{ and is cont}\}$
- $C^\omega(I) := \{f \in \text{fxn } I | f \text{ converges to its Taylor Series}\}$   
(in a neighborhood of every  $x \in I$  – be careful at boundary points)
- $\text{Int } (I) := \{f \in \text{fxn } I | f \text{ is integrable on } I\}$

4.  $F[t]$  the set of polys, coeffs in  $F$  old  $+, \cdot$  with scalar mult

$$\alpha(\alpha_0 + \alpha_1 t + \dots + \alpha_n t^n) := \alpha \alpha_0 + \alpha \alpha_1 t + \dots + \alpha \alpha_n t^n$$

5.  $\underbrace{F[t]_n}_{\text{truncating } F[t]} := \{0 \in F[t]\} \cup \{f \in F[t] | \deg f \leq n\}$  (not closed under  $\cdot$  of polys)  
where  $\deg f$  = the highest power of  $t$  occurring non-trivially in  $f$  if  $f \neq 0$  is a vector space over  $F$  with  $+$ , scalar mult, 0.

**Example 3.2** 1.  $F^{m \times n} :=$  set of  $m \times n$  matrices entries in  $F$  where  $A \in F^{m \times n}$ ,  $A_{ij} =$   $ij^{\text{th}}$  entry of  $A$

$$\begin{aligned} (A + B)_{ij} &:= A_{ij} + B_{ij} \in F & \forall A, B \in F^{m \times n} \\ (\alpha A)_{ij} &:= \alpha A_{ij} \in F & \forall \alpha \in F \end{aligned}$$

$$0 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \text{ ( m rows and n columns)}$$

COMPONENTWISE OPERATION! Then  $F^{m \times n}$  is a vector space over  $F$ , e.g.

$M_n F$  is a vector space over  $F$ .

### Example to GENERALIZE

Let  $V$  be a vector space over  $F$ ,  $\emptyset \neq S$  a set. Set  $W := \{f : S \rightarrow V \mid f \text{ a map}\}$ . Define  $+$ ,  $\cdot$  on  $W$  by

$$\begin{aligned} f + g & \quad (f + g)(s) := f(s) + g(s) \in V \\ \alpha f & \quad (\alpha f)(s) := \alpha(f(s)) \in V \\ 0_W & \quad 0(s) = 0_V \quad \text{ZERO FUNCTION} \end{aligned}$$

$\forall f, g \in W; \alpha \in F; s \in S$ . Then  $W$  is a vector space over  $F$ . (of componentwise operation)

2. Let  $F \subset K$  be a fields under  $+$ ,  $\cdot$  on  $K$ . Same 0,1, i.e.  $F$  is a SUBFIELD of  $K$  e.g.  $\mathbb{R} \subset \mathbb{C}$ . Then  $K$  is a vector space over  $F$  by RESTRICTION of SCALARS. i.e.,  $+$  is  $+$  on  $K$ . With scalar mult,  $F \times K \rightarrow K$  by

$$\underbrace{\alpha v}_{\text{in } K \text{ as a vector space over } F} = \underbrace{\alpha v}_{\text{in } K \text{ as a field}} \quad \forall \alpha \in F \quad \forall v \in V$$

e.g.  $\mathbb{R}$  is a vector space over  $\mathbb{Q}$  by  $\frac{m}{n}r = \frac{mr}{n}$ ,  $m, n \in \mathbb{Z}, n \neq 0, r \in \mathbb{R}$ . More generally, let  $V$  be a vector space over  $K$ ,  $F \subset K$  subfield, then it is a vector space over  $F$  by RESTRICTION of SCALARS.

$$\cdot|_{F \times V} : F \times V \rightarrow V$$

e.g.,  $K^n$  is a vector space over  $F$  (e.g.  $\mathbb{C}^n$  is a vector space over  $\mathbb{R}$ ).

Properties of Vector Space: Let  $V$  be a vector space over  $F$ . Then  $\forall \alpha, \beta \in F, \quad \forall v, w \in V$ , we have

1. The zero vector is unique write 0 or  $0_V$ .
2.  $(-1)v$  is the unique vector  $w \ni w + v = 0 = v + w$  write  $-v$ .
3.  $0 \cdot v = 0$
4.  $\alpha \cdot 0 = 0$
5.  $(-\alpha)v = -(\alpha v) = \alpha(-v)$
6. if  $\alpha v = 0$ , then either  $\alpha = 0$  or  $v = 0$
7. if  $\alpha v = \alpha w, \alpha \neq 0$ , then  $v = w$
8. if  $\alpha v = \beta v, v \neq 0$ , then  $\alpha = \beta$
9.  $-(v + w) = (-v) + (-w) = -v - w$
10. can ignore parentheses in  $+$

### §3.2 Subspace

**Definition 3.3 (Subspace)** — Let  $V$  be a vector space over  $F$ ,  $W \subset V$  a subset. We say  $W$  is a **subspace** of  $V$  if  $W$  is a vector space over  $F$  with the operation  $+, \cdot$  on  $V$ , i.e.,  $(V, +, \cdot)$  is a vector space over  $F$ , via  $+: V \times V \rightarrow V$  and  $\cdot: F \times V \rightarrow V$  then  $W$  is a vector space over  $F$  via

- $+=+|_{W \times W}: W \rightarrow W$ : restrict the domain to  $W \times W$
  - $\cdot=\cdot|_{F \times W}: F \times W \rightarrow W$ : restrict the domain to  $F \times W$
- i.e.  $W$  is closed under  $+, \cdot$  from  $V$ ,  $\forall w_1, w_2 \in W \quad \forall \alpha \in F, \quad w_1 + w_2 \in W$  and  $\alpha w_1 \in W$  and  $0_W = 0_V$ .

#### Theorem 3.4 (Subspace)

Let  $V$  be a vector space over  $F$ ,  $\emptyset \neq W \subset V$  a subset. Then the following are equivalent:

1.  $W$  is a subspace for  $V$
2.  $W$  is closed under  $+$  and scalar mult from  $V$
3.  $\forall w_1, w_2 \in W, \forall \alpha \in F, \alpha w_1 + w_2 \in W$

*Proof.* Some of the implication are essentially ??

1)  $\rightarrow$  2) : by def.  $W$  is a subspace of  $V$  under  $+, \cdot$  on  $V$  (and satisfies the axioms of a vector space over  $F$ ) as  $0_V = 0_W$ .

2)  $\rightarrow$  1) claim:  $0_V \in W$  and  $0_W = 0_V$ : As  $\emptyset \neq W \exists w \in W$

By 2)  $(-1)w \in W$ , hence  $0_V = w + (-w) \in W$ . Since  $0_V + w' = w' = w' + 0_V$  in  $V \quad \forall w' \in W$ , the claim follows. The other axioms hold for elements of  $V$  hence for  $W \subset V$ .

2)  $\rightarrow$  3) : let  $\alpha \in F, w_1, w_2 \in W$ . As 2) holds,  $\alpha w_1 \in W$  hence also  $\alpha w_1 + w_2 \in W$

3)  $\rightarrow$  2) Let  $\alpha \in F, w_1, w_2 \in W$ . As above and 3)

$$0_V = w_1 + (-w_1) \in W \quad \text{and} \quad 0_V = 0_W$$

Therefore,

$$w_1 + w_2 = 1 \cdot w_1 + w_2 \in W \quad \text{and} \quad \alpha w_1 + \alpha w_1 + 0_V \in W$$

by 3). □

Note: Usually 3) is the easiest condition to check. WARNING: must subsets of a vector space over  $F$  are NOT subspace.

#### Example 3.5

$V$  a vector space over  $F$ .

1.  $0 := \{0_V\}$  and  $V$  are subspace of  $V$



2. Let  $I \subset \mathbb{R}$  be an interval (not a point) then

$$C^\omega(I) < C^\infty(I) < \dots < C^n(I) < \dots < C'(I) \\ < \text{Diff } I < C(I) < \text{Int } I < \text{Fxn } I$$

are subspaces of the vector space containing them... where we write

$$A < B \quad \text{if} \quad A \subset B \quad \text{and} \quad A \neq B$$

3. Let  $F$  be a field, e.g.  $\mathbb{R}$ . Then  $F = F[t]_0 < F[t]_1 < \dots < F[t_n] < \dots < F[t]$  are vector spaces over  $F$  each a subspace of the vector space over  $F$  containing it.

4. If  $W_1 \subset W_2 \subset V$ ,  $W_1, W_2$  subspaces of  $V$ , then  $W_1 \subset W_2$  is a subspace.

5. If  $W_1 \subset W_2$  is a subspace and  $W_2 \subset V$  is a subspace, then  $W_1 \subset V$  is a subspace.

6. Let  $W := \{(0, \alpha_1, \dots, \alpha_n) \mid \alpha_i \in F, \quad 2 \leq i \leq n\} \subset F^n$  is a subspace, but  $\{(1, \alpha_2, \dots, \alpha_n) \mid \alpha_i \in F, \quad 2 \leq i \leq n\}$  is not. Why?

7. Every line or plane through the origin in  $\mathbb{R}^3$  is a subspace.

## §4 | Lec 4: Oct 9, 2020

### §4.1 Span & Subspace

**Definition 4.1 (Linear Combination)** — Let  $V$  be a vector space over  $F$ ,  $v_1, \dots, v_n \in V$  we say  $v \in V$  is a LINEAR COMBINATION of  $v_1, \dots, v_n$  if  $\exists \alpha_1, \dots, \alpha_n \in F \ni v = \alpha v_1 + \dots + \alpha_n v_n$ .

Let

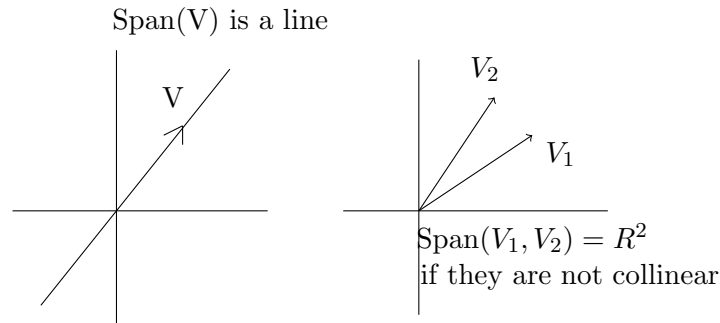
$$\text{Span}(v_1, \dots, v_n) := \{ \text{all linear combos of } v_1, \dots, v_n \}$$

Let  $v_1, \dots, v_n \in V$ . Then

$$\text{Span}(v_1, \dots, v_n) = \left\{ \sum_{i=1}^n \alpha_i v_i \mid \alpha_1, \dots, \alpha_n \in F \right\}$$

is a subspace of  $V$  (by the Subspace Theorem) called the SPAN of  $v_1, \dots, v_n$ . It is the (unique) smallest subspace of  $V$  containing  $v_1, \dots, v_n$ .

i.e., if  $W \subset V$  is a subspace and  $v_1, \dots, v_n \in W$  then  $\text{Span}(v_1, \dots, v_n) \subset W$ . We also let  $\text{Span } \emptyset := \{0_V\} = 0$ , the smallest vector space containing no vectors.



Question: If we view  $\mathbb{C}$  as a vector space over  $\mathbb{R}$ , then  $\mathbb{R}$  is a subspace of  $\mathbb{C}$ , but if we view  $\mathbb{C}$  as a vector space over  $\mathbb{C}$ , then  $\mathbb{R}$  is not a subspace of  $\mathbb{C}$  (why? What's going on?) – not closed under operation(s).

**Definition 4.2 (Span)** — Let  $V$  be a vector space over  $F$ ,  $\emptyset \neq S \subset V$  a subset. Then,  $\text{Span } S :=$  the set of all FINITE linear combos of vectors in  $S$ . i.e., if  $V \in \text{Span } S$ , then

$$\exists v_1, \dots, v_n \in S, \quad \alpha_1, \dots, \alpha_n \in F \ni v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

$\text{Span } S \subset V$  is a subspace. What is  $\text{Span } V$ ?

**Example 4.3** 1. Let  $V = \mathbb{R}^3$ .

$$\text{Span}(i + j, i - j, k) = \text{Span} V = \text{Span}(i, j, i + j, k) = \text{Span}(i + j, i - j, k + i)$$

2. Define

$$\text{Symm}_n F := \left\{ A \in M_n F \mid A = A^\top \right\}$$

Recall:  $A^\top$  is the transpose of  $A$ , i.e.,

$$(A^\top)_{ij} := A_{ji} \quad \forall i, j$$

is a subspace of  $M_n F$

3.

$$V = \left\{ \begin{pmatrix} a & c + di \\ c - di & b \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\} \subset M_2 \mathbb{C}$$

is NOT a subspace as a vector space over  $\mathbb{C}$ , eg,

$$i \begin{pmatrix} a & c + di \\ c - di & b \end{pmatrix} = \begin{pmatrix} ai & -d + ci \\ d + ci & bi \end{pmatrix}$$

does not lie in  $V$  if either  $a \neq 0$  or  $b \neq 0$  (cannot be imaginary). Also  $V$  is not a subspace of  $M_2 \mathbb{R}$  as a vector space over  $\mathbb{R}$  as  $V \not\subset M_2 \mathbb{R}$ .  $V \subset M_2 \mathbb{C}$  is a subspace as a vector space over  $\mathbb{R}$ .

4. (Important computational example) Fix  $A \in F^{m \times n}$ . Let

$$\ker A := \left\{ x \in F^{n \times 1} \mid Ax = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \text{ in } F^{m \times 1} \right\}$$

called the **KERNEL** or **NULL SPACE** of  $A$ .  $\ker A \subset F^{n \times 1}$  is a subspace and it is the **SOLUTION SPACE** of the system of  $m$  linear equations in  $n$  unknowns. – which we can compute by Gaussian elimination.

5. Let  $W_i \subset V, i \in \underbrace{I}_{\text{indexing set}}$  be subspaces. Then  $\bigcap_I W = \bigcap_{i \in I} W_i := \{x \in V \mid x \in W_i \ \forall i \in I\}$  is a subspaces of  $V$  (why?)

6. In general, if  $W_1, W_2 \subset V$  are subspaces,  $W_1 \cup W_2$  is NOT a subspace.

e.g.,  $\text{Span}(i) \cup \text{Span}(j) = \{(x, 0) \mid x \in \mathbb{R}\} \cup \{(0, y) \mid y \in \mathbb{R}\}$  is not a subspace

$$(x, y) = (x, 0) + (0, y) \notin \text{Span}(i) \cup \text{Span}(j)$$

if  $x \neq 0$  and  $y \neq 0$

**Definition 4.4 (Subspace & Span)** — Let  $W_1, W_2 \subset V$  be subspaces. Define

$$\begin{aligned} W_1 + W_2 &:= \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\} \\ &= \text{Span}(W_1 \cup W_2) \end{aligned}$$

So  $w_1 + w_2 \subset V$  is a subspace and the smallest subspace of  $V$  containing  $W_1$  and  $W_2$ .

More generally, if  $W_i \subset V$  is a subspace  $\forall i \in I$  let

$$\sum_I W_i = \sum_{i \in I} W_i := +W_i := \text{Span}\left(\bigcup_I W_i\right)$$

the smallest subspace of  $V$  containing  $W_i \forall i \in I$ . What do elements in  $\sum_I W_i$  look like?

Determine the span of vector  $v_1, \dots, v_n$  in  $\mathbb{R}^n$

Suppose  $v_i = (a_{i1}, \dots, a_{in}), i = 1, \dots, n$ . To determine when  $w \in \mathbb{R}^n$  lies in  $\text{Span}(u_1, \dots, u_n)$  i.e., if  $w = (b_1, \dots, b_n) \in \mathbb{R}^n$  when does

$$w = \alpha_1 v_1 + \dots + \alpha_n v_n, \quad \alpha_1, \dots, \alpha_n \in \mathbb{R}$$

What  $v_i$  is an  $n \times 1$  column matrix  $\begin{pmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{ni} \end{pmatrix}$

$$A = (a_{ij}), \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

view  $w$  as  $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ . To solve

$$Ax = B, \quad X = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

is equivalent to finding all the  $n \times 1$  matrices  $B$  (actually  $B^\top$ ) s.t.

$$Ax = B$$

when the columns of  $A$  are the  $v_i(v_i^\top)$ .

Note: If  $m = n$  an  $A$  is invertible then all  $B$  work.

## §4.2 Linear Independence

We know that  $\mathbb{R}^n$  is an  $n$ -dimensional vector space over  $\mathbb{R}$ . Since we need  $n$  coordinates (axes) to describe all vector in  $\mathbb{R}^n$  but no fewer will do.

We want something like the following:

Let  $V$  be a vector space over  $F$  with  $V \neq \emptyset$ . Can we find distinct vectors  $v_1, \dots, v_n \in V$ , some  $n$  with following properties

1.  $V = \text{Span}(v_1, \dots, v_n)$
2. No  $v_i$  is a linear combos of  $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$  (i.e. we need them all)

Then we want to call  $V$  an  $n$ -DIMENSIONAL VECTOR SPACE OVER  $F$ .

### Lemma 4.5

Let  $V$  be a vector space over  $F$ ,  $n > 1$ . Suppose  $v_1, \dots, v_n$  are distinct. Then (2) is equivalent to

$$\text{If } \alpha_1 v_1 + \dots + \alpha_n v_n = \beta_1 v_1 + \dots + \beta_n v_n, \quad \alpha_i, \beta_i \in F \forall i, j$$

i.e. the “coordinates” are unique.

*Proof.* ( $->$ ) If not, relabelling the  $v_i$ 's, we may assume that  $\alpha_1 \neq \beta_1$  in (\*), then

$$(\alpha_1 - \beta_1)v_1 = \sum_{i=2}^n (\beta_i - \alpha_i)v_i$$

As  $\alpha_1 - \beta_1 \neq 0$  in  $F$ , a field,  $(\alpha_1 - \beta_1)^{-1}$  exists, so

$$v_1 = \sum_{i=2}^n (\alpha_1 - \beta_1)^{-1}(\beta_i - \alpha_i)v_i \in \text{Span}(v_1, \dots, v_n)$$

a contradiction.

(< -) Relabelling, we may assume that

$$v_1 = \alpha_2 v_2 + \dots + \alpha_n v_n, \quad \text{some } \alpha_i \in F$$

Then,

$$1 \cdot v_1 + 0v_2 + \dots + 0v_n = v_1 = 0 \cdot v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

so  $1 = 0$ , a contradiction.  $\square$

**Remark 4.6.** The case  $n = 1$  is special because there are two possibilities

**Case 1:**  $v \neq 0$  : then  $\alpha v = \beta v \rightarrow \alpha = \beta$

**Case 2:**  $v = 0$  : then  $\alpha v = \beta v \forall \alpha, \beta \in F$

So the only time the above lemma is false is when  $n = 1$  and  $v = 0$ . We do not want to say this, so we use another definition.

## §5 | Lec 5: Oct 12, 2020

### §5.1 Linear Independence(Cont'd)

**Definition 5.1 (Linear Independence & Dependence)** — Let  $V$  be a vector space over  $F$ ,  $v_1, \dots, v_n$  in  $V$  all distinct. We say  $\{v_1, \dots, v_n\}$  is LINEARLY DEPENDENT if  $\exists \alpha_1, \dots, \alpha_n \in F$  not all zero  $\ni$

$$\alpha_1 v_1 + \dots + \alpha_n v_n = 0$$

and  $\{v_1, \dots, v_n\}$  is LINEARLY INDEPENDENT if it is NOT linearly dependent, i.e., if for any eqn

$$0 = \alpha v_1 + \dots + \alpha_n v_n, \quad \alpha_1, \dots, \alpha_n \in F,$$

then  $\alpha_i = 0 \forall i$ , i.e., the only linear comb of  $v_1, \dots, v_n$  – the zero vector is the TRIVIAL linear combo (we shall also say that distinct  $v_1, \dots, v_n$  are linearly independent if  $\{v_1, \dots, v_n\}$  is. More generally, a set  $\emptyset \neq S \subset V$  is called LINEARLY DEPENDENT if for some FINITE subset (of distinct elements of  $S$ ) of  $S$  is linearly dependent and it is called LINEARLY INDEPENDENT if every FINITE subset of  $S$  (of distinct elements) is linearly independent.

We say  $v_i, i \in I$ , all distinct are LINEARLY INDEPENDENT if  $\{v_i\}_{i \in I}$  is linearly independent and  $v_i \neq v_j \forall i, j \in I, i \neq j$ .

**Remark 5.2.** Let  $V$  be a vector space over  $F$ ,  $\emptyset \neq S \subset V$  a subset

1. If  $0 \in S$ , then  $S$  is linearly dependent as  $1 \cdot 0 = 0$

2. distinct:  $v_1, \dots, v_n$  in  $V$  are linearly independent iff

- no  $v_i = 0$
- $\alpha_1 v_1 + \dots + \alpha_n v_n = \beta_1 v_1 + \dots + \beta_n v_n, \quad \alpha_i, \beta_i \in F$  implies  $\alpha_i = \beta_i \forall i$

Note:  $v, v$  are linearly dependent if we allow repetitions – and  $\{v, v\} = \{v\}$ .

For homework, make sure to show this:

Suppose  $v_1, \dots, v_n$  are distinct,  $n > 2$ , no  $v_i = 0$ . Suppose no  $v_i$  is a scalar multiple of another  $v_j, j \neq i$ . It does not follow that  $v_1, \dots, v_n$  are linearly independent (in general).

**Example 5.3** (counter-example)

$(1, 0), (0, 1), (1, 1)$  in  $V = \mathbb{R}^2$   
 $(1, 0), (0, 1)$  are linearly indep. but not  $(1, 0), (0, 1)$ , and  $(1, 1)$ .

**Remark 5.4.** Let  $\emptyset \neq T \subset S$  be a subset. If  $T$  is linearly dependent, so is  $S$ . Then the contraposition is also true: if  $S$  is linearly indep., so is  $T$ .

More remarks:

1. Let  $0 \neq v \in V$ . Then  $\{v\}$  is linearly independent and

$$Fv := \text{Span}(v)$$

is called a LINE in  $V$ :

$$\alpha v = 0 \rightarrow \alpha = 0$$

2.  $u, v, w \in V \setminus \{0\}$  and  $v \notin \text{Span}(w)$  (equivalently,  $w \notin \text{Span}(v)$ ), then  $\{v, w\}$  is linearly indep. and  $\text{span}(v, w)$  is called a PLANE in  $V$ .
3.  $(1, 1), (-2, -2)$  are linearly dep. in  $\mathbb{R}^2$ .
4.  $(1, 1), (2, -2)$  are linearly indep. in  $\mathbb{R}^2$  (show coefficients are equal to each other and to 0).
5. More generally,

$$v_i = (a_{i1}, \dots, a_{in}) \text{ in } \mathbb{R}^n, \quad i = 1, \dots, m \text{ (distinct)}$$

Then

$$\exists \alpha_1, \dots, \alpha_m \in \mathbb{R} \text{ not all } 0 \ni \alpha_1 v_1 + \dots + \alpha_m v_m = 0$$

iff  $v_1, \dots, v_m$  are linearly dep – iff  $\exists \alpha_1, \dots, \alpha_m \in \mathbb{R}$  not all 0 s.t.

$$\alpha_1(a_{11}, \dots, a_{1m}) + \dots + \alpha_m(a_{m1}, \dots, a_{mn}) = 0$$

iff the matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

with rows  $v_i$  row reduced to echelon form with a zero row. Also,

$$B = A^\top = \begin{pmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \\ a_{1m} & \dots & a_{mn} \end{pmatrix}$$

i.e., write the vectors  $v_i$  as columns then

$$\underbrace{B}_{n \times m} \underbrace{X}_{m \times 1} = 0$$

has a NON-TRIVIAL solution, i.e.,

$$\ker B \neq 0$$

where

$$\ker B := \{X \in F^{m \times 1} \mid BX = 0\}$$

the kernel of  $B$ .

6. Let  $f_1, \dots, f_n \in C^{n-1}(I)$ ,  $I = (\alpha, \beta)$ ,  $\alpha < \beta$  in  $\mathbb{R}$  and

$$\alpha_1 f_1 + \dots + \alpha_n f_n = \underbrace{0}_{\text{the zero func}}$$

i.e.,  $(\alpha_1 f_1 + \dots + \alpha_n f_n)(x) = 0 \quad \forall x \in (\alpha, \beta)$ . Taking the derivatives  $(n-1)$  times and put them in matrix form, we have

$$\begin{pmatrix} f_1 & \dots & f_n \\ f_1' & \dots & f_n' \\ \vdots & \dots & \vdots \\ f_1^{n-1} & \dots & f_n^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

In particular, the Wronskian of  $f_1, \dots, f_n$  is not the zero func, i.e.,  $\exists x \in (\alpha, \beta) \ni W(f_1, \dots, f_n)(x) \neq 0$ . This means that the matrix above is invertible for some  $x \in (\alpha, \beta)$ . Then,  $\alpha_1 = 0, \dots, \alpha_n = 0$  by Cramer's rule – only the trivial soln.

Conclusion:  $W(f_1, \dots, f_n) \neq 0 \rightarrow \{f_1, \dots, f_n\}$  is linearly indep.

WARNING: the converse is false.

### Example 5.5 (of the conclusion)

Let  $\alpha < \beta$  in  $\mathbb{R}$ .

1.  $\sin x, \cos x$  are linearly indep. on  $(\alpha, \beta)$ .
2. We need some (sub) defns for this example.

For  $x \in \mathbb{R}$ , define the map

$$e_x : \mathbb{R}[t] \rightarrow \mathbb{R} \text{ by}$$

$$g = \sum a_i t^i \mapsto g(x) := \sum a_i x^i \text{ called EVALUATION at } x.$$

We call a map  $f : \mathbb{R} \rightarrow \mathbb{R}$  (or some  $f : I \rightarrow \mathbb{R} (I \subset \mathbb{R})$ ) a POLYNOMIAL FUNCTION if

$$\exists P_f = \sum_{i=1}^n a_i t^i \in \mathbb{R}[t]$$

and

$$f(x) = e_x P_f = P_f(x) = \sum_{i=1}^n a_i x^i \quad \forall x \in \mathbb{R}$$

i.e., the function arising from a (formal) polynomial by evaluation at each  $x$ . We let

$$\mathbb{R}[x] := \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ a poly fcn} \}$$

Note: Polynomial fcn's are defined on all of  $\mathbb{R}$ .  $\mathbb{R}[x]$  is a vector space over  $\mathbb{R}$ .

Warning: if we replace  $\mathbb{R}$  by  $F$ ,  $F[t]$  may be “very different” from  $F[x]$ , e.g., let  $F = \{0, 1\}$ . Then

$$t, t^2 \in F[t], \quad t \neq t^2 \quad \text{but} \quad P_t = P_{t^2}$$

Now we can give our example using Wronskians

$$\{1, x, \dots, x^n\}$$

is linearly indep. on  $(\alpha, \beta)$  assuming  $\alpha < \beta$ .

HOMEWORK: Let  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  be distinct, then

$$e^{\alpha_1 t}, \dots, e^{\alpha_n t}$$

are linearly indep. on  $(\alpha, \beta)$ . THINK OVER IT!

### Theorem 5.6 (Toss In)

Let  $V$  be a vector space over  $F$ ,  $\emptyset \neq S \subset V$  a linearly indep. subset. Suppose that  $v \in V \setminus \text{Span } S$ . Then  $S \cup \{v\}$  is linearly indep.

*Proof.* Suppose this is false which is  $S \cup \{v\}$  is linearly dep. Then  $\exists v_1, \dots, v_n \in S$  and  $\alpha, \alpha_1, \dots, \alpha_n \in F$  some  $n$  not all zero s.t.

$$\alpha v + \alpha_1 v_1 + \dots + \alpha_n v_n = 0$$

**Case 1:**  $\alpha = 0$

Then  $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$  not all  $\alpha_1, \dots, \alpha_n$  zero so  $\{v_1, \dots, v_n\}$  is linearly dep., a contradiction.

**Case 2:**  $\alpha \neq 0$

Then  $\alpha^{-1}$  exists.

$$v = -\alpha^{-1}\alpha_1 v_1 - \dots - \alpha^{-1}\alpha_n v_n$$

is a linear combo of  $v_1, \dots, v_n$ , i.e.,  $v \in \text{Span}(v_1, \dots, v_n)$  – a contradiction. Therefore,  $S \cup \{v\}$  is linearly indep.  $\square$

### Corollary 5.7

Let  $V$  be a vector space over  $F$  and  $v_1, \dots, v_n \in V$  linearly indep. if

$$\text{Span}(v_1, \dots, v_n) < V$$

then  $\exists v_{n+1} \in V \ni v_1, \dots, v_n, v_{n+1}$  are linearly indep. and

$$\text{Span}(v_1, \dots, v_n) < \text{Span}(v_1, \dots, v_{n+1}) \subset V$$



**Question 5.1.** Why can't we get a linearly indep. set spanning any vector space over  $F$  using this theorem?

Ans: Certainly we may not get a finite set. We shall only be interested in the case, much of the time, when such a finite linearly indep. set spans our vector space over  $F$ .

**Example 5.8**

$(1, 3, 1) \in \mathbb{R}^3$  is linearly indep. but  $\text{Span}(1, 3, 1) < \mathbb{R}^3$ .  
 $(1, 1, 0) \notin \text{Span}(1, 3, 1)$  so  $(1, 3, 1), (1, 1, 0)$  are linearly indep. Similarly for  $(0, 0, 1)$ .  
 $\mathbb{R}^3 = \text{Span}((1, 3, 1), (1, 1, 0), (0, 0, 1))$

## §6 | Lec 6: Oct 14, 2020

### §6.1 Bases

**Definition 6.1 (Basis)** — Let  $\emptyset \neq V$  be a vector space over  $F$ . A **BASIS**  $B$  for  $V$  is a linearly indep. set in  $V$  and spans  $V$ . i.e.,

1.  $V = \text{Span } B$ .
2.  $B$  is linearly indep.

We say  $V$  is a **FINITE DIMENSIONAL VECTOR SPACE OVER  $F$**  if there exists  $B$  for  $V$  with finitely many elements, i.e.,  $|B| < \infty$ .

Notation: If  $V = 0$ , we say  $V$  is a finite dimensional vector space over  $F$  of **DIMENSION ZERO**.

Goal: To show if  $V$  is finite dimensional vector space over  $F$  with bases  $B$  and  $b$  then  $|B| = |b| < \infty$ . This common integer is called the **DIMENSION** of  $V$ .

**Example 6.2**

Let  $V$  be a vector space over  $F$ ,  $S \subset V$  a linearly indep. set. Then  $S$  is a basis for  $\text{Span } S$ .

Warning:  $S$  is not a subspace just a subset.

**Definition 6.3 (Ordered Basis)** — If  $V$  is a finite dimensional vector space over  $F$  with a basis  $B = \{v_1, \dots, v_n\}$  we called it an **ORDERED BASIS** if the given order of  $v_1, \dots, v_n$  is to be used, i.e., the  $i^{\text{th}}$  vector in  $B$  is the  $i^{\text{th}}$  in the written list, e.g.,  $\{v_1, v_2, v_4, v_3, \dots\}$  then  $v_4$  is the 3<sup>rd</sup> element in the ordered list if we want  $B$  to be ordered in this way.

**Theorem 6.4 (Coordinate)**

Let  $V$  be a finite dimensional vector space over  $F$  with basis  $B = \{v_1, \dots, v_n\}$  and  $v \in V$ . Then  $\exists! \alpha_1, \dots, \alpha_n \in F \ni v = \alpha_1 v_1 + \dots + \alpha_n v_n$ . We call  $\alpha_1, \dots, \alpha_n$  the COORDINATE of  $v$  relative to the basis  $B$  and call  $\alpha_i$  the  $i^{\text{th}}$  coordinate relative to  $B$ .

*Proof.* Existence: By defn,  $V = \text{Span } B$ , so if  $v \in V$

$$\exists \alpha_1, \dots, \alpha_n \in F \ni v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

Uniqueness: Let  $v \in V$  and suppose that  $\alpha_1 v_1 + \dots + \alpha_n v_n = \beta_1 v_1 + \dots + \beta_n v_n$ , for some  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in F$ . Then

$$(\alpha_1 - \beta_1)v_1 + \dots + (\alpha_n - \beta_n)v_n = 0$$

Since  $B$  is linearly indep,

$$\alpha_i - \beta_i = 0 \quad \text{for } i = 1, \dots, n \quad \square$$

**Question 6.1.** Does the above theorem hold if the basis  $B$  is not necessarily finite? If so prove it!

**Exercise 6.1.** Let  $V$  be a vector space over  $F$ ,  $v_1, \dots, v_n \in V$  then

$$\text{Span}(v_1, \dots, v_n) = \text{Span}(v_2, \dots, v_n) \iff v_1 \in \text{Span}(v_2, \dots, v_n)$$

Make sure to PROVE THIS

Note: For induction, you CAN'T assume  $n$  in the induction hypothesis is special in any way except it is greater than 1. Also, you can start induction at  $n = 0$ , i.e., show  $P(0)$  true (or at any  $n \in \mathbb{Z}$ ).

**Theorem 6.5 (Toss Out)**

Let  $V$  be a vector space over  $F$ . If  $V$  can be spanned by finitely many vector then  $V$  is a finite dimensional vector space over  $F$ . More precisely, if

$$V = \text{Span}(v_1, \dots, v_n)$$

then a subset of  $\{v_1, \dots, v_n\}$  is a basis for  $V$ .

*Proof.* If  $V = 0$ , there is nothing to prove. So we may assume that  $V \neq 0$ . Suppose that  $V = \text{Span}(v_1, \dots, v_n)$ . We can use induction on  $n$  and show a subset of  $\{v_1, \dots, v_n\}$  is a basis.

- $n = 1$  :  $V = \text{Span}(v_1) \neq 0$  as  $V \neq 0$ , so  $v_1 \neq 0$ . Hence  $\{v_1\}$  is linearly indep and it is the basis.

- Assume  $V = \text{Span}(w_1, \dots, w_n)$  – the induction hypothesis – to be true. Then a subset of  $w_1, \dots, w_n$  is a basis for  $V$ . Now suppose that  $v = \text{Span}(v_1, \dots, v_{n+1})$ . To show a subset of  $\{v_1, \dots, v_{n+1}\}$  is a basis for  $V$ , we need to show if  $\{v_1, \dots, v_{n+1}\}$  is linearly indep., then it is a basis for  $V$  and it spans  $V$  and we are done. So let us assume that  $\{v_1, \dots, v_{n+1}\}$  is linearly dep. Hence,

$$\exists \alpha_1, \dots, \alpha_{n+1} \in F \text{ not all zero } \ni$$

$$\alpha_1 v_1 + \dots + \alpha_{n+1} v_{n+1} = 0$$

Assume  $\alpha_{n+1} \neq 0$ , then

$$v_{n+1} = -\alpha_{n+1}^{-1} \alpha_1 v_1 - \dots - \alpha_{n+1}^{-1} \alpha_n v_n$$

lies in  $\text{Span}(v_1, \dots, v_n)$ . By the Exercise above,

$$V = \text{Span}(v_1, \dots, v_{n+1}) = \text{Span}(v_1, \dots, v_n)$$

By the induction hypo, a subset of  $\{v_1, \dots, v_n\}$  is a basis for  $V$ .

□

**Example 6.6** 1. Let  $e_i = \{(0, \dots, 0, 1, 0, \dots)\} \in F^n$

$$s = s_n := \{e_1, \dots, e_n\} \subset F^n$$

If  $v \in F^n$ , then

$$v = (\alpha_1, \dots, \alpha_n) = \alpha_1 e_1 + \dots + \alpha_n e_n$$

since  $\alpha_i \in F$ , so  $F^n = \text{Span } s$ . If  $0 = \alpha_1 e_1 + \dots + \alpha_n e_n = (\alpha_1, \dots, \alpha_n) = (0, \dots, 0)$ , then  $\alpha_i = 0 \forall i$ . So  $s$  is linearly indep. Hence  $s$  is a basis for  $F^n$  called the standard basis. More generally, let

$e_{ij} \in F^{m \times n}$  be the  $m \times n$  matrix with all entries 0 except in the  $i$ th place.

Then  $s_{mn} := \{e_{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$  is a basis for  $F^{m \times n}$  called the STANDARD BASIS for  $F^{m \times n}$  – same proof – everything is done componentwise.

2.  $V = F[t] := \{\text{polys in } t, \text{ coeffs in } F\}$  ( $F = \mathbb{R}$ ). Let  $f \in V$ . Then, there exists  $n \geq 0$  in  $\mathbb{Z}$  and  $\alpha_0, \dots, \alpha_n$  in  $F$  s.t.

$$f = \alpha_0 + \alpha_1 t + \dots + \alpha_n t^n$$

So  $B = \{t^n | n \geq 0\} = \{1, t, t^2, \dots\}$  spans  $V$  and by defn if

$$\alpha_0 + \alpha_1 t + \dots + \alpha_n t^n = \underbrace{0}_{\text{zero poly}}$$

then  $\alpha_i = 0$  for all  $i$  so  $B$  is linearly indep. Hence  $B$  is a basis for  $F[t]$ .  $B$  is not a finite set. We shall see that  $F[t]$  is not a finite dimensional vector space over  $F$ .

How?

3.  $F[t]_n := \{f \in F[t] \mid f = 0 \text{ or } \deg f \leq n\} \subset F[t]$  is spanned by  $\{1, t, t^2, \dots, t^n\}$ . It is a subset of linearly indep. set.  $\{1, t, t^2, \dots\} = \{t^n \mid n \geq 0\}$  so also linearly indep. and therefore a basis.
4.  $\{1, \sqrt{-1}\}$  is a basis for  $\mathbb{C}$  as a vector space over  $\mathbb{R}$ .  $\{1\}$  is a basis for  $C$  as a vector space over  $\mathbb{C}$  (indeed, if  $F$  is a field,  $F$  is a vector space over  $F$  and if  $0 \neq \alpha \in F$ , then  $\alpha^{-1}$  exists and  $x = x\alpha^{-1}\alpha \in \text{Span } F$  so  $\{\alpha\}$  is a basis. e.g.,  $\{\pi\}$  is a basis for  $\mathbb{R}$  as a vector space over  $\mathbb{R}$ ).
5.  $\{e^{-x}, e^{3x}\}$  is a basis for

$$V := \{f \in \mathbb{C}^2(-\infty, \infty) \mid f'' - 2f' - 3f = 0\}$$

a vector space over  $\mathbb{R}$ .

6. Given  $v_1, \dots, v_n \in F^n$ , you know how to find  $W = \text{Span}(v_1, \dots, v_n)$ . Note: If  $m > n$  then rows reducing  $A^\top$  must lead to a zero row so  $v_1, \dots, v_m$  cannot be linearly indep. If  $m = n$  we can see if

$$\det A^\top = 0 \quad (\text{or } \det A = 0)$$

then linearly dep. And if

$$\det A^\top \neq 0 \quad (\text{or } \det A \neq 0)$$

then linearly indep.

## §7 | Lec 7: Oct 16, 2020

### §7.1 Replacement Theorem

#### Theorem 7.1 (Replacement)

Let  $V$  be a vector space over  $F$ ,  $\{v_1, \dots, v_n\}$  a basis for  $V$ . Suppose that  $v \in V$  satisfies

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n, \quad \alpha_1, \dots, \alpha_n \in F, \alpha_i \neq 0$$

Then

$$\{v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_n\}$$

is also a basis for  $V$ .

*Proof.* Changing notation, we may assume  $\alpha_1 \neq 0$ . To show  $\{v_1, v_2, \dots, v_n\}$  is a basis for  $V$ , we have to show  $\{v, v_2, \dots, v_n\}$  spans  $V$ . Since

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n, \quad \alpha_1 \neq 0$$

$\alpha_1^{-1}$  exists, so

$$v_1 = \alpha_1^{-1} v - \alpha_1^{-1} \alpha_2 v_2 - \dots - \alpha_1^{-1} \alpha_n v_n$$

lies in  $\text{Span}(v, v_2, \dots, v_n)$ . By Exercise ...,

$$V = \text{Span}(v, v_1, \dots, v_n) = \text{Span}(v, v_2, \dots, v_n)$$

So  $\{v, v_2, \dots, v_n\}$  spans  $V$ . Thus,  $\{v, v_2, \dots, v_n\}$  is linearly indep.

Suppose  $\exists \beta_1, \beta_2, \dots, \beta_n \in F$  not all 0  $\ni$

$$\beta v + \beta_2 v_2 + \dots + \beta_n v_n = 0$$

**Case 1:**  $\beta = 0$

Then  $\beta_2 v_2 + \dots + \beta_n v_n = 0$  not all  $\beta_i = 0$ . So  $\{v_2, \dots, v_n\}$  is linearly dep., a contradiction.

**Case 2:**  $\beta \neq 0$ , so  $\beta^{-1}$  exists.

Then using (\*), we see

$$v = 0 \cdot v_1 - \beta^{-1} \beta_2 v_2 - \dots - \beta^{-1} \beta_n v_n = \alpha_1 v_1 + \dots + \alpha_n v_n$$

As  $\{v_2, \dots, v_n\}$  is a basis, by the Coordinate Theorem, we have

$$\alpha_1 = 0 \quad \text{and} \quad \alpha_i = \beta^{-1} \beta_i$$

a contradiction. □

**Question 7.1.** In the Replacement Theorem, do we need the basis to be finite?

Ans: I think it can be infinite ...

## §7.2 Main Theorem

### Theorem 7.2 (Main)

Suppose  $V$  is a vector space over  $F$  with  $V = \text{Span}(v_1, \dots, v_n)$ . Then any linearly indep. subset of  $V$  has at most  $n$  elements.

*Proof.* We know that a subset of  $B = \{v_1, \dots, v_n\}$  is a basis for  $V$  by Toss Out Theorem. So we may assume  $B$  is a basis for  $V$ . It suffices to show any linearly indep. set in  $V$  has at most  $|B| = n$  elements where  $B$  is a basis. Let  $\{w_1, \dots, w_m\} \subset V$  be linearly indep. where no  $w_i = 0$ . To show  $m \leq n$ , the idea is to use Toss In and Toss out in conjunction with the Replacement Theorem.

**Claim 7.1.** After changing notation, if necessary, for each  $k \leq n$

$$\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$$

is a basis for  $V$ .

Suppose we have shown the above claim for  $k = n$ . Apply the claim to  $k = n$  if  $m > k$ , then  $\{w_1, \dots, w_{n+1}\}$  is linearly dep., a contradiction as  $\{w_1, \dots, w_n\}$  is a basis. Thus, we prove the claim for  $m \leq n$  as needed. We prove it by induction on  $k$ . BY the argument above, we may assume  $k \leq n$ .

- $k = 1$  : As  $w_1 \in \text{Span } B = \text{Span } (v_1, \dots, v_n)$  and  $w_1 \neq 0$ ,  $\exists \alpha_1, \dots, \alpha_n \in F$  not all 0  
 $\ni$

$$w_1 = \alpha_1 v_1 + \dots + \alpha_n v_n$$

Changing notation, we may assume  $\alpha_1 \neq 0$ . By the Replacement Theorem,

$$\{w_1, v_2, \dots, v_n\} \text{ is a basis for } V$$

- Assume the claim hold for  $k(k < n)$ .
- We must show the claim holds for  $k + 1$ ,

$$\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\} \text{ is a basis for } V$$

We can write

$$0 \neq w_{k+1} = \beta_1 w_1 + \dots + \beta_k w_k + \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n$$

for some (new)  $\beta_1, \dots, \beta_k, \alpha_{k+1}, \dots, \alpha_n \in F$  not all 0

**Case 1:**  $\alpha_{k+1} = \alpha_{k+2} = \dots = \alpha_n = 0$

Then  $w_{k+1} \in \text{Span}(w_1, \dots, w_k)$ , hence  $\{w_1, \dots, w_{k+1}\}$  is linearly dep., a contradiction.

**Case 2:**  $\exists i \ni \alpha_i \neq 0$  :

Changing notation, we may assume  $\alpha_{k+1} \neq 0$ . By the Replacement Theorem

$$\{w_1, \dots, w_{k+1}, v_{k+2}, \dots, v_n\}$$

is a basis for  $V$ . This completes the induction step thus prove the claim and establish the theorem.  $\square$

### §7.3 A Glance at Dimension

#### Corollary 7.3

Let  $V$  be a finite dimensional vector space over  $F$ ,  $B_1, B_2$  two bases for  $V$ . Then  $|B_1| = |B_2| < \infty$ . We call  $|B_1|$  the dimension of  $V$ , write  $\dim V = \dim_F V = |B_1|$  (dropping  $F$  if  $F$  is clear).

*Proof.* By defn of finite dimensional vector space over  $F$ ,  $\exists$  a basis  $b$  for  $V$  with  $|b| < \infty$ . By the Main Theorem,  $|B| \leq |b|$ , if  $B$  is a basis for  $V$ , so  $B$  is finite. Again by the Main Theorem,  $|b| \leq |B|$  if  $B$  is a basis for  $V$ , so  $|b| = |B|$  for any basis  $B$  of  $V$ .  $\square$

The corollary above says  $\dim V$  is well-defined for all finite dimensional vector space over  $F$ , i.e., “ $\dim$ ” : {finite dimensional vector space over  $F$ }  $\rightarrow \mathbb{Z}^+ \cup \{0\}$  is a function.

Warning:  $F$  makes a difference.

**Example 7.4**

$$\begin{array}{ll} \dim_{\mathbb{C}} \mathbb{C} = 1 & \text{basis } \{1\} \\ \dim_{\mathbb{R}} \mathbb{C} = 2 & \text{basis } \{1, \sqrt{-1}\} \\ \dim_{\mathbb{Q}} \mathbb{C} = ? & \end{array}$$

**Corollary 7.5**

$$\dim_F F^n = n.$$

**Corollary 7.6**

$$\dim_F F^{m \times n} = mn.$$

**Corollary 7.7**

$$\dim_F F[t]_n = 1 + n.$$

*Note:* If  $V$  is a finite dimensional vector space over  $F$  with bases  $B$ , then the Replacement Theorem allows us to find many other bases.

**Corollary 7.8**

Let  $V$  be a finite dimensional vector space over  $F$ ,  $n = \dim V$ ,  $\emptyset \neq S \subset V$  a subset. Then

- If  $|S| > n$ , then  $S$  is linearly dep.
- If  $|S| < n$ , then  $\text{Span } S < V$ .

*Proof.* • First bullet point: The Main Theorem says:

A maximal linearly indep. set in  $V$  is a basis and can have at most  $n$  elements by Toss In Theorem.

- Second bullet point: By Toss Out Theorem, we can assume that  $S$  is linearly indep., so it cannot be a basis by Corollary ?.

□

**Question 7.2.** What is  $\dim_{\mathbb{R}} M_n(\mathbb{C})$ ?

## §8 | Lec 8: Oct 19, 2020

### §8.1 Extension and Counting Theorem

**Theorem 8.1** (Extension)

Let  $V$  be a finite dimensional vector space over  $F$ ,  $W \subset V$  a subspace. Then every linearly independent subset  $S$  in  $W$  is finite and part of a basis for  $W$  which is a finite dimensional vector space over  $F$ .

*Proof.* Any linearly indep. set in  $W$  is linearly indep. subset  $S$  in  $V$  so  $|S| \leq \dim V < \infty$  by the Main Theorem. In particular,

$$\dim \text{Span} S \leq \dim V$$

if  $W = \text{Span } S$ , we are done.

If not,  $\exists w_1 \in W \setminus \text{Span } S$ , and hence  $S_1 = S \cup \{w_1\}$  is linearly indep. by Theorem and

$$|S_1| = |S \cup \{w_1\}| = |S| + 1 \leq \dim V$$

if  $\text{Span } S_1 < W$ , then  $\exists w_2 \in W \setminus \text{Span } S_1$ , so  $S_2 = S \cup \{w_1, w_2\} \subset W$  is linearly indep., hence

$$|S_2| = |S| + 2 \leq \dim V$$

Continuing in this manner, we must stop when  $n \leq \dim V - \dim \text{Span } S$  as  $\dim V < \infty$ . So  $S$  is a part of a basis for  $W$  and  $W$  is a finite dimensional vector space over  $F$ .  $\square$

Think about the proof for this

**Corollary 8.2**

Let  $V$  be a finite dimensional vector space over  $F$ . Then any linearly indep. set in  $V$  can be EXTENDED to a basis for  $V$ , i.e., is part of a basis for  $V$ . We often call this special case the **Extension Theorem**.

**Corollary 8.3**

Let  $V$  be a finite dimensional vector space over  $F$ ,  $W \subset V$  a subspace. Then  $W$  is a finite dimensional vector space over  $F$  and  $\dim W \leq \dim V$  with equality iff  $W = V$ .

*Proof.* Left as exercise.  $\square$

**Theorem 8.4** (Counting)

Let  $V$  be a finite dimensional vector space over  $F$ ,  $W_1, W_2 \subset V$  subspaces. Suppose that both  $W_1$  and  $W_2$  are finite dimensional vector space over  $F$ . Then

1.  $W_1 \cap W_2$  is a finite dimensional vector space over  $F$ .
2.  $W_1 + W_2$  is a finite dimensional vector space over  $F$ .
3.  $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$ .



*Proof.* 1.  $W_1 \cap W_2 \subset W_i, i = 1, 2$ , so it is a finite dimensional vector space over  $F$  by corollary 8.2.

2. Let  $B_i$  be a basis for  $W_i, i = 1, 2, \dots$ . Then  $W_1 + W_2 = \text{Span}(B_1 \cup B_2)$  and

$$|B_1 \cup B_2| \leq |B_1| + |B_2| < \infty$$

So  $W_1 + W_2$  is a finite dimensional vector space over  $F$  by Toss Out.

3. Let  $B = \{v_1, \dots, v_n\}$  be a basis for  $W_1 \cap W_2$ . Extend  $B$  to a basis

$$b_1 = \{v_1, \dots, v_n, y_1, \dots, y_r\} \text{ for } W_1$$

$$b_2 = \{v_1, \dots, v_n, z_1, \dots, z_s\} \text{ for } W_2$$

using the Extension Theorem.

**Claim 8.1.**  $b_1 \cup b_2 = \{v_1, \dots, v_n, y_1, \dots, y_r, z_1, \dots, z_s\}$  is a basis for  $W_1 + W_2$  and has  $n + r + s$  elements. So if we show the claim, the result will follow.

Certainly,

$$\text{Span}(b_1 \cup b_2) = \text{Span } b_1 + \text{Span } b_2 = W_1 + W_2$$

So we need only to show  $b_1 \cup b_2$  is linearly indep. Suppose this is false. Then

$$0 = \alpha_1 v_1 + \dots + \alpha_n v_n + \beta_1 y_1 + \dots + \beta_r y_r + \gamma_1 z_1 + \dots + \gamma_s z_s \quad (*)$$

for some  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_r, \gamma_1, \dots, \gamma_s$  in  $F$  not all zero.

**Case 1:** All the  $\gamma_i = 0$ . Since  $b_1$  is linearly indep., this is a contradiction.

**Case 2:** Some  $\gamma_i \neq 0$ .

Changing notation, we may assume  $\gamma_1 \neq 0$ . Since  $b_2$  is a basis, (\*) leads to an equation

$$0 \neq z = \gamma_1 z_1 + \dots + \gamma_s z_s = -\alpha_1 v_1 - \dots - \alpha_n v_n - \beta_1 y_1 - \dots - \beta_r y_r$$

Therefore,  $0 \neq z$  lies in  $\text{Span } b_2 \cap \text{Span } b_1 = W_2 \cap W_1$ . So we can write  $z_i \in W_1 \cap W_2$  using basis  $B$  as

$$0 \neq z = \delta_1 v_1 + \dots + \delta_n v_n \quad \text{some } \delta_1, \dots, \delta_n \in F$$

Thus  $W_2 = \text{Span } b_2$ , we have

$$\delta_1 v_1 + \dots + \delta_n v_n - 0z_1 + \dots + 0z_s = z = 0v_1 + \dots + 0v_n + \gamma_1 z_1 + \dots + \gamma_s z_s$$

By the Coordinate Theorem,  $\gamma_1 = 0$ , a contradiction.

□

### Corollary 8.5

Let  $V$  be a vector space over  $F$ ,  $W_1, W_2 \subset V$  finite dimensional subspaces of  $V$ . Then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2$$

iff

$$W_1 \cap W_2 = \emptyset$$

In this case, we write  $W_1 + W_2 = W_1 \oplus W_2$  called the DIRECT SUM.

## §8.2 Linear Transformation

In mathematics, whenever you have a collection of objects, one studies maps between them that preserves any special properties of the objects in the collection and tries to see what information can be gained from such maps.

**Definition 8.6 (Linear Transformation)** — Let  $V, W$  be a vector space over  $F$ . A map  $T : V \rightarrow W$  is called a Linear Transformation, write  $T : V \rightarrow W$  is linear if  $\forall v_1, v_2 \in V, \forall \alpha \in F$

- $T(v_1 + v_2) = T(v_1) + T(v_2)$ .
- $T(\alpha v_1) = \alpha T(v_1)$ .
- $T(0_V) = 0_W$ .

Notation: We write  $Tv$  for  $T(v)$ .

**Remark 8.7.** Let  $V, W$  be a vector space over  $F$ ,  $T : V \rightarrow W$  a map.

1. If  $T$  satisfies 1) and 2), then it satisfies 3):

$$0_W + T(0_V) = T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$$

$$\text{so } 0_W = T(0_V).$$

2.  $T$  is linear iff  $T(\alpha v_1 + v_2) = \alpha T v_1 + T v_2 \quad \forall v_1, v_2 \in V, \forall \alpha \in F$ .
3. If  $T$  is linear,  $\alpha_1, \dots, \alpha_n \in F, v_1, \dots, v_n \in V$ , then

$$T\left(\sum_{i=1}^n \alpha_i v_i\right) = \sum_{i=1}^n \alpha_i T v_i$$

We leave a proof of 2) and 3) as exercises.

### Example 8.8

Let  $V, W$  be a vector space over  $F$ . The followings are linear transformations

1.  $0_{V,W} : V \rightarrow W$  by  $v \mapsto 0_W$ .
2.  $V = W$ ,  $1_V : V \rightarrow V$  by  $v \mapsto v$ .

A linear transformation  $T : V \rightarrow V$  is called a **Linear Operator**.

3. If  $\emptyset \neq Z \subset W$  is a subset, then we have a map

$$\text{inc} : Z \rightarrow W$$

given by  $z \mapsto z$  called the **Inclusion Map**. Then,  $Z$  is a subspace of  $V$  iff  $\text{inc} : Z \hookrightarrow W$  is linear.

Note:  $\text{inc} = \underbrace{1_W|_Z}_{\text{Restriction map}}.$

This is the Subspace Theorem.

4.  $T : F^n \rightarrow F^{n-1}$  by  $(\alpha_1, \dots, \alpha_n) \mapsto (\alpha_1, \dots, \overbrace{i}^{\text{omit}}, \dots, \alpha_n)$  for a fixed  $i$ .
5.  $T : F^n \rightarrow F$  by  $(\alpha_1, \dots, \alpha_n) \mapsto \alpha_i$  for a fixed  $i$ .
6.  $T : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$  by  $(\alpha_1, \dots, \alpha_{n-1}) \mapsto (\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_i, \dots, \alpha_n)$  for fixed  $i$ .
7.  $T : \mathbb{R} \rightarrow \mathbb{R}^n$  by  $\alpha \mapsto (0, 0, \dots, \alpha, 0, \dots, 0)$  for fixed  $i$ .
8. If  $\alpha < \beta$  in  $\mathbb{R}$ ,  $D : C'(\alpha, \beta) \rightarrow C(\alpha, \beta)$  by  $f \mapsto f'$ .
9. If  $\alpha < \beta$  in  $\mathbb{R}$ ,  $\text{Int} : C(\alpha, \beta) \rightarrow C'(\alpha, \beta)$  by  $f \mapsto \int f$  where  $\int f$  is the antiderivative – constant of integration 0.
10. Fix  $\alpha \in F$ , then  $\lambda\alpha : V \rightarrow V$  by  $v \mapsto \alpha v$ . Left translation by  $\alpha$ .
11. Let  $A \in F^{m \times n}$ . Define

$$T : F^{n \times 1} \rightarrow F^{m \times 1} \text{ by } T \cdot X = A \cdot X$$

$$\text{i.e. } \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \mapsto A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Matrices can be viewed as linear transformation. We should see the converse is true IF  $V$  is a finite dimensional vector space over  $F$ . It is not true in general.

## §9 | Lec 9: Oct 21, 2020

### §9.1 Kernel, Image, and Dimension Theorem

**Definition 9.1** (Kernel(Nullspace)) — Let  $V, W$  be a vector space over  $F$ ,  $T : V \rightarrow W$  linear set

$$N(T) = \ker T := \{v \in V | Tv = 0_W\}$$

called the nullspace or kernel of  $T$ .

**Definition 9.2 (Range(Image))** — Let  $V, W$  be a vector space over  $F$ ,  $T : V \rightarrow W$  linear set

$$\begin{aligned}\text{im } T = T(V) &:= \{w \in W \mid \exists v \in V \ni Tv = w\} \\ &= \{Tv \mid v \in V\}\end{aligned}$$

called the range or image of  $T$ .

**Proposition 9.3**

Let  $T : V \rightarrow W$  be linear. Then

1.  $\ker T \subset V$  is a subspace.
2.  $\text{im } T \subset W$  is a subspace.

*Proof.* Left as exercise. □

**Theorem 9.4 (Dimension)**

Let  $T : V \rightarrow W$  be linear with  $V$  is a finite dimensional vector space over  $F$ . Then

1.  $\text{im } T$  and  $\ker T$  are finite dimensional vector space over  $F$ .
2.  $\dim V = \dim \ker T + \dim \text{im } T$ .

Note:  $\dim \ker T$  is also called the NULLITY of  $T$  and  $\dim \text{im } T$  is also called the RANK of  $T$ .

*Proof.* Let  $n = \dim V$ .

$\ker T \subset V$  is a subspace,  $V$  is a finite dimensional vector space over  $F$  so  $\ker T$  is a finite dimensional vector space over  $F$  and  $\dim \ker T \leq \dim V = n$ . Say  $m = \dim \ker T$ .

Let  $\mathcal{B}_0 = \{v_1, \dots, v_m\}$  be a basis for  $\ker T$ . By the Extension Theorem  $\exists \mathcal{B} = \{v_1, \dots, v_m, \dots, v_n\}$  a basis for  $V$ .

**Claim 9.1.**  $Tv_{m+1}, \dots, Tv_n$  are linearly indep. (in particular, distinct) and

$$\mathcal{C} = \{Tv_{m+1}, \dots, Tv_n\}$$

is a basis for  $\text{im } T$ .

If we prove the claim above, then  $\text{im } T$  is a finite dimensional vector space over  $F$  of dimension  $n - m$  and we are done.

**Step 1:**  $\mathcal{C}$  spans  $\text{im } T$ :

Let  $w \in \text{im } T$ . By definition,  $\exists v \in V \ni Tv = w$ . As  $\mathcal{B}$  is a basis for  $V$   $\exists \alpha_1, \dots, \alpha_n \in F \ni$

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

Hence

$$\begin{aligned} w = T(v) &= T(\alpha_1 v_1 + \dots + \alpha_n v_n) = \alpha_1 T v_1 + \dots + \alpha_n T v_n \\ &= \alpha_1 0_W + \dots + \alpha_m 0_W + \alpha_{m+1} T v_{m+1} + \dots + \alpha_n T v_n \end{aligned}$$

lies w  $\text{Span}(\mathcal{C})$  (as  $v_1, \dots, v_m \in \ker T$ ).

**Case 2:**  $\mathcal{C}$  is linearly indep.

Suppose  $\alpha_{m+1}, \dots, \alpha_n \in F$  and

$$\alpha_{m+1} T v_{m+1} + \dots + \alpha_n T v_n = 0_W$$

Then

$$0_W = T(\alpha_{m+1} v_{m+1} + \dots + \alpha_n v_n)$$

So  $\alpha_{m+1} v_{m+1} + \dots + \alpha_n v_n \in \ker T$ . By defn,  $\mathcal{B}_0$  is a basis for  $\ker T$ . So  $\exists \beta_1, \dots, \beta_m \in F \ni$

$$\alpha_{m+1} v_{m+1} + \dots + \alpha_n v_n = \beta_1 v_1 + \dots + \beta_m v_m$$

Hence

$$0 = -\beta_1 v_1 - \dots - \beta_m v_m + \alpha_{m+1} v_{m+1} + \dots + \alpha_n v_n$$

As  $\mathcal{B}$  is a basis for  $V$ , it is linearly indep, so  $\beta_1 = 0, \dots, \beta_m = 0, \alpha_{m+1} = 0, \dots, \alpha_n = 0$  (Coordinate Theorem) and the claim follows.  $\square$

*Note:* Let  $V$  be a finite dimensional vector space over  $F$ ,  $W \subset V$  a subspace,  $V/W$  the quotient space, then  $- : V \rightarrow V/W, v \mapsto \bar{v} = v + W$  and  $\dim V/W = \dim V - \dim W$ .

## §9.2 Algebra of Linear Transformation

We want to study the set of all linear transformation from a vector space over  $F$   $V$  to a vector space over  $F$   $W$ . Let  $V, W$  be a vector space over  $F$ . Set

$$L(V, W) := \{T : V \rightarrow W \mid T \text{ is linear}\}$$

Check: if  $T, S \in L(V, W), \alpha \in F$ , then  $\alpha T + S \in L(V, W)$ . Since we know  $\mathcal{F}(V, W) = \{f : V \rightarrow W \mid f \text{ a map}\}$  is a vector space over  $F$ , by the Subspace Theorem,  $L(V, W) \subset \mathcal{F}(V, W)$  is a subspace.

### Proposition 9.5

Let  $V, W$  be a vector space over  $F$ , then  $L(V, W) \subset \mathcal{F}(V, W)$  is a subspace.

Now we know if we have maps

$$f : X \rightarrow Y \quad \text{and} \quad g : Y \rightarrow Z,$$

we have the COMPOSITE MAP

$$g \circ f : X \rightarrow Z \quad \text{by} \quad (g \circ f)(x) = g(f(x)) \forall x \in X$$

where  $\circ$  is called the COMPOSITION (and often omitted when clear). Then we have

need  
recheck

**Proposition 9.6**

Let  $V, W, X, U$  be vector space over  $F$ ,  $T, T' : V \rightarrow W$ ,  $S, S' : W \rightarrow X$ ,  $R : X \rightarrow U$  all be linear. Then,

1.  $S \circ T : V \rightarrow W$  is linear. (the composition of linear transformations is linear).
2.  $R \circ (S \circ T) = (R \circ S) \circ T$  and linear.
3.  $S \circ (T + T') = S \circ T + S \circ T'$  and linear.
4.  $(S + S') \circ T = S \circ T + S' \circ T$  and linear.

*Proof.*

$$\begin{aligned}(S \circ T)(\alpha v_1 + v_2) &= S(T(\alpha v_1 + v_2)) = S(\alpha T v_1 + T v_2) \\ &= \alpha S \circ T(v_1) + S \circ T(v_2)\end{aligned}$$

$\forall v_1, v_2 \in V, \alpha \in F$ .

The rest are left as exercises. □

**Definition 9.7 (Linear Operator)** — Let  $V$  be a vector space over  $F$ ,  $T : V \rightarrow V$  linear, so a linear operator is defined as

$$\begin{aligned}T^n &:= \underbrace{T \circ \dots \circ T}_n \quad \text{if } n \in \mathbb{Z}^+ \\ T^0 &= 1_V\end{aligned}$$

**Proposition 9.8**

Let  $V$  be a vector space over  $F$ . Then  $L(V, V)$  under  $+$  and  $\circ$  of functions  $V \rightarrow V$  satisfies all the axioms of a field except possibly (M3) and (M4) with

$$\begin{aligned}\text{one} &= 1_V : V \rightarrow V \quad \text{by } v \mapsto v \\ \text{zero} &= 0_V : v \mapsto 0\end{aligned}$$

We say  $L(V, V)$  is a (non-commutative) ring of  $M_n F$ .

### §9.3 Linear Transformation Theorems

**Definition 9.9** (Properties/Consequences of Linear Transformation) — Let  $T : V \rightarrow W$  be linear. We say that  $T$  is

1. a **MONOMORPHISM** (write mono or monic) or **NONSINGULAR** if  $T$  is 1 – 1. (i.e., injective).
2. an **EPIMORPHISM** (write epi or epic) if  $T$  is onto (i.e., surjective).
3. an **ISOMORPHISM** (write iso) or **INVERTIBLE** if  $T$  is bijective and  $T^{-1} : W \rightarrow V$  is linear. We say  $V, W$  vector spaces over  $F$  are **ISOMORPHIC** (write  $V \cong W$ ) if  $\exists$  an isomorphism  $S : V \rightarrow W$ , we also write an isomorphism  $S : V \rightarrow W$  as  $S : V \xrightarrow{\sim} W$

**Remark 9.10.**  $V \cong W$  vector space over  $F$  means that we cannot take  $V$  and  $W$  apart algebraically.

**Example 9.11**

$F^{n+1} \cong F[t]_n$  as  $F^{n+1} \rightarrow F[t]_n$  by  $(\alpha_0, \dots, \alpha_n) \mapsto \alpha_0 + \alpha_1 t_1 + \dots + \alpha_n t^n$  is an isomorphism with inverse  $F[t]_n \rightarrow F^{n+1}$  by  $\alpha_0 + \alpha_1 t + \dots + \alpha_n t^n \mapsto (\alpha_0, \dots, \alpha_n)$

$$\begin{aligned} T^{-1}(\alpha w_1 + w_2) &= T^{-1}(\alpha T v_1 + T v_2) = T^{-1}(T(\alpha v_1 + v_2)) \\ &= T^{-1}T(\alpha v_1 + v_2) \\ &= \alpha v_1 + v_2 \\ &= \alpha T^{-1} w_1 + T^{-1} w_2 \quad \square \end{aligned}$$

**Corollary 9.12**

Let  $T : V \rightarrow W$  be a monomorphism. Then  $V \cong \text{im} T$  via  $T$ .

**Remark 9.13.** If  $V, W, X$  are vector space over  $F$ , then

1.  $V \cong V$
2.  $V \cong W \rightarrow W \cong V$
3.  $V \cong W$  and  $W \cong X$  then  $V \cong X$

In algebra, isomorphisms are usually easier to check than are one might assume, because the following result is often true.

**Proposition 9.14**

Let  $T : V \rightarrow W$  be linear. Then  $T$  is an isomorphism iff  $T$  is bijective.

*Proof.*  $(\rightarrow)$  immediate.

( $\leftarrow$ ) Let  $T^{-1} : W \rightarrow V$  be the set inverse of  $T : V \rightarrow W$ , so

$$T \circ T^{-1} = 1_W \quad \text{and} \quad T^{-1} \circ T = 1_V$$

In particular, if  $v \in V$  and  $w \in W$ ,

$$w = Tv \quad \text{iff} \quad T^{-1}w = v$$

Let  $w_1, w_2 \in W$ ,  $\alpha \in F$ . To show

$$T^{-1}(\alpha w_1 + w_2) = \alpha T^{-1}w_1 + T^{-1}w_2$$

$T$  is onto so

$$\exists v_i \in V \ni Tv_i = w_i, i = 1, \dots$$

Hence, we have

$$\begin{aligned} T^{-1}(\alpha w_1 + w_2) &= T^{-1}(\alpha Tv_1 + Tv_2) = T^{-1}(T(\alpha v_1 + v_2)) \\ &= T^{-1}T(\alpha v_1 + v_2) = \alpha v_1 + v_2 \\ &= \alpha T^{-1}w_1 + T^{-1}w_2 \end{aligned}$$

□

## §10 | Lec 10: Oct 23, 2020

### §10.1 Monomorphism, Epimorphism, and Isomorphism

#### Corollary 10.1

Let  $T : V \rightarrow W$  be a monomorphism. Then  $V \cong \text{im } T$  via  $T$ .

**Definition 10.2 (Linear Map)** — Let  $T : V \rightarrow W$  be linear. We say  $T$  takes linearly independent sets to linearly independent sets if  $v_i, i \in I$  are linearly independent in  $V$  (in particular, distinct). Then,  $Tv_i, i \in I$  are linearly indep. in  $W$ . ( $Tv_i \neq Tv_j$  if  $i \neq j$  in  $I$ )

#### Theorem 10.3 (Monomorphism)

Let  $T : V \rightarrow W$  be linear. Then the followings are true

1.  $T$  is 1-1, so it's monomorphism.
2.  $T$  takes linearly indep. sets in  $V$  to linearly indep. sets in  $W$ .
3.  $\ker T = 0 := \{0_V\}$ .
4.  $\dim \ker T = 0$ .

*Proof.* • 3) iff 4) is the defn of the 0-space.



- 1)  $\rightarrow$  2) It suffices to show that  $T$  takes finite linearly indep. sets in  $V$  to linearly indep. sets in  $W$ .

Suppose that  $v_1, \dots, v_n \in V$  are linearly indep. and  $\alpha_1, \dots, \alpha_n \in F$  satisfy

$$0_W = \alpha_1 T v_1 + \dots + \alpha_n T v_n$$

Then

$$T(0_V) = 0_W = T(\alpha_1 v_1 + \dots + \alpha_n v_n)$$

As  $T$  is  $1 - 1$

$$0_V = \alpha_1 v_1 + \dots + \alpha_n v_n$$

Since  $v_1, \dots, v_n$  are linearly indep.  $\alpha_i = 0, i = 1, \dots, n$  as needed.

- 2)  $\rightarrow$  3) Let  $v \in \ker T$ . Then  $Tv = 0_W$ . If  $v \neq 0$ , then  $\{v\}$  is linearly indep. By 2)  $Tv \neq 0_W$  as then  $\{Tv\}$  is linearly indep. So  $v = 0$ .
- 3)  $\rightarrow$  1) If  $Tv_1 = Tv_2, v_1, v_2 \in V$ , then

$$0_W = Tv_1 - Tv_2 = T(v_1 - v_2)$$

So  $v_1 - v_2 = 0_V$  by 3), i.e.,  $v_1 = v_2$  □

**Remark 10.4.** The Monomorphism Theorem says  $\ker T$  measures the deviation of  $T$  from being  $1 - 1$ .

Note: In the Monomorphism Theorem, we do not assume that  $V$  or  $W$  is a finite dimensional vector space over  $F$ .

### Theorem 10.5 (Isomorphism)

Suppose  $T : V \rightarrow W$  is linear with  $\dim V = \dim W < \infty$ , i.e.,  $V, W$  are finite dimensional vector space over  $F$  of the same dimension. Then the followings are true

1.  $T$  is an isomorphism.
2.  $T$  is a monomorphism.
3.  $T$  is an epimorphism.
4. If  $\mathcal{B} = \{v_1, \dots, v_n\}$  is a basis for  $V$ , then  $\{Tv_1, \dots, Tv_n\}$  is a basis for  $W$  (so  $Tv_1, \dots, Tv_n$  are distinct), i.e.,  $T$  takes basis of  $V$  to basis of  $W$ .
5. There exists a basis  $\mathcal{B}$  of  $V$  that maps to a basis of  $W$ .

**Remark 10.6.** 1. The condition that  $\dim V = \dim W < \infty$  is crucial

Come up with a counter example

2. Let  $V \cong W$  with  $V, W$  be finite dimensional vector space over  $F$ . So  $\dim V = \dim W$ . Let  $S : V \rightarrow W$  be linear. Then  $S$  may or may not be an isomorphism, e.g., if  $S$  is the zero map then it is not an isomorphism unless  $V = 0$ . The theorem only says that  $\exists$  an isomorphism and any such satisfies the theorem.
3. Let  $f : A \rightarrow B$  be a map of finite sets with  $|A| = |B|$ . Then  $f$  is a bijection iff  $f$  is an injection iff  $f$  is a surjection.

*Proof.* (of Theorem)

- 1)  $\rightarrow$  2) follows by defn.
- 2)  $\rightarrow$  3) By the Dimension Theorem

$$\dim W = \dim V = \dim \ker T + \dim \operatorname{im} T$$

Thus,  $T$  is onto iff  $\operatorname{im} T = W$  iff  $\dim W = \dim \operatorname{im} T$  (by the Corollary to the Existence Theorem) iff  $\dim \ker T = 0$  iff  $T$  is 1-1.

- 3)  $\rightarrow$  1) as 3)  $\rightarrow$  2) and 1) = 2) + 3) by the Proposition ?
- 2)  $\rightarrow$  4) Let  $\{v_1, \dots, v_n\}$  be a basis for  $V$ . By the Monomorphism Theorem,  $Tv_1, \dots, Tv_n$  are linearly indep. in  $W$ , so

$$n \leq \dim W = \dim V = n$$

Hence  $\{Tv_1, \dots, Tv_n\}$  also spans as  $\dim W = \dim V$ .

- 4)  $\rightarrow$  5)  $\rightarrow$  3) are clear.

□

## §10.2 Existence of Linear Transformation

The next result is really the defining property of finite dimensional vector space and linear transformation.

**Theorem 10.7** (Existence of Linear Transformation (UPVS))

– (Universal Property of Vector Space) Let  $V$  be a finite dimensional vector space over  $F$ ,  $\mathcal{B} = \{v_1, \dots, v_n\}$  a basis for  $V$  and  $W$  an arbitrary vector space over  $F$ . Let  $w_1, \dots, w_n \in W$ , not necessarily distinct. Then

$$\exists! T : V \rightarrow W \text{ linear } \ni Tv_i = w_i \forall i$$

We can write this in an other way as follows:

Let  $B \hookrightarrow V$  be a basis for  $V$ ,  $V$  a finite dimensional vector space over  $F$  and  $W$  a vector space over  $F$ . Given a diagram,

$$\begin{array}{ccc} B \hookrightarrow V & & \\ \searrow f & \text{of sets and} & \\ & \text{set maps} & \\ & W & \end{array}$$

then  $\exists! T : V \rightarrow W$  linear  $\ni$

$$\begin{array}{ccc} B \hookrightarrow V & & \\ \searrow f & \text{inc} & \downarrow T \\ & W & \end{array}$$

commutes, i.e.,  $T \circ \text{inc} = f$ .

*Proof.* Define  $T : V \rightarrow W$  as follows: let  $v \in V$ . The  $\exists! \alpha_1, \dots, \alpha_n \in F \ni v = \alpha_1 v_1 + \dots + \alpha_n v_n$  by the Coordinate Theorem. Define

$$Tv = T(\alpha_1 v_1 + \dots + \alpha_n v_n) := \alpha_1 w_1 + \dots + \alpha_n w_n$$

Since the  $\alpha_i$  ARE UNIQUE, this defines a map – we say  $T : V \rightarrow W$  is WELL – DEFINED. Certainly,  $Tv_i = w_i, i = 1, \dots, n$ . To show  $T$  is linear, let  $v = \sum_{i=1}^n \alpha_i v_i, v' = \sum_{i=1}^n \beta_i v_i, \alpha, \alpha_i, \beta_j \in F \forall i, j$ . Then

$$\begin{aligned} T(\alpha v + v') &= T\left(\alpha \sum_{i=1}^n \alpha_i v_i + \sum_{i=1}^n \beta_i v_i\right) \\ &= T\left(\sum_{i=1}^n (\alpha \alpha_i + \beta_i) v_i\right) = \sum_{i=1}^n (\alpha \alpha_i + \beta_i) w_i \\ &= \alpha \sum_{i=1}^n \alpha_i w_i + \sum_{i=1}^n \beta_i w_i = \alpha T v + T v' \end{aligned}$$

as needed. This shows existence.

Uniqueness: Let  $T : V \rightarrow W$  by (\*) and  $S : V \rightarrow W$  linear s.t.  $Sv_i = w_i \forall i$ . To show  $S = T$ , let  $v = \sum_{i=1}^n \alpha_i v_i, \alpha_i \in F$  unique,  $i = 1, \dots, n$ . Then  $Tv = \sum_{i=1}^n \alpha_i T v_i = \sum_{i=1}^n \alpha_i w_i$  which is equivalent to

$$= \sum_{i=1}^n \alpha_i S v_i = S\left(\sum_{i=1}^n \alpha_i v_i\right) = S v$$

So  $S = T$  and we have proven uniqueness.  $\square$

**Remark 10.8.** The theorem says a linear transformation from a finite dimensional vector space over  $F$  is completely determined by what it does to a fixed basis. i.e., as there are no non-trivial RELATIONS on linear combos of elements in  $\mathcal{B}$ , the only relation in  $\text{im } T$  will arise from the kernel of  $T$ .

## §11 | Lec 11: Oct 26, 2020

### §11.1 Lec 10 (Cont'd)

**Remark 11.1.** 1. In the above, given  $fv_i = w_i \forall i$ , we say that  $T : V \rightarrow W$  by  $\sum \alpha_i v_i \mapsto \sum \alpha_i w_i$  EXTENDS  $f$  linearly.

2. Let  $V$  be any vector space over  $F$  (not necessarily finite dimensional). Suppose  $V$  has a basis  $\mathcal{B}$ , then every  $v \in V$  is a finite linear combo elements in  $\mathcal{B}$ . Using the same proof of UPVS, shows

if  $W$  is a vector space over  $F$ , then given a diagram

$$\begin{array}{ccc} B \hookrightarrow V & & \\ \searrow f & \text{of sets and} & \\ & \text{set maps} & \\ & W & \end{array}$$

of set and set maps.  $\exists! T : V \rightarrow W$  linear s.t.

$$\begin{array}{ccc} B \hookrightarrow V & & \\ \searrow f & \text{inc} \downarrow T & \\ & W & \end{array}$$

commutes. I.E., if  $\mathcal{B} = \{v_i\}_I$  is a basis for  $V$ ,  $w_i \in W$ ,  $i \in I$  (not necessarily distinct),  $f : V \rightarrow W$  by  $v_i \mapsto w_i \forall i \in I$ . Then  $\exists! T : V \rightarrow W$  linear s.t.  $Tv_i = w_i \forall i \in I$ . So any linear transformation from a vector space over  $F$   $V$  having a basis is completely determined by what it does to that basis.

3. Axiom: Every vector space over  $F$  has a basis. This is equivalent to the Axiom of Choice.

### Theorem 11.2 (Classification of Finite Dimensional Vector Space)

Let  $V, W$  be finite dimensional vector space over  $F$ . Then

$$V \cong W \iff \dim V = \dim W$$

*Proof.* ( $\rightarrow$ ) Let  $T : V \rightarrow W$  be an isomorphism,  $\mathcal{B} = \{v_1, \dots, v_n\}$  a basis for  $V$  (so  $\dim V = n$ ). By the Monomorphism Theorem,

$$\mathcal{C} = \{Tv_1, \dots, Tv_n\}$$

is linearly indep. in  $W$ . Since  $|\mathcal{C}| = n$  and  $\text{span}(\mathcal{C}) = W$  (as  $T$  is onto),  $\mathcal{C}$  is a basis for  $W$  and  $\dim W = \dim V$ .

( $\leftarrow$ ) Suppose  $n = \dim V = \dim W$ . Let  $\mathcal{B} = \{v_1, \dots, v_n\}$  be a basis for  $V$ ,  $\mathcal{C} = \{w_1, \dots, w_n\}$  a basis for  $W$ . By the UPVS,  $\exists! T : V \rightarrow W$  linear  $v_i \mapsto w_i \forall i$ , i.e.,  $T$  takes the basis  $\mathcal{B}$  of  $V$  to the basis  $\mathcal{C}$  of  $W$ . By the Isomorphism Theorem,  $T$  is an isomorphism.  $\square$

**Example 11.3** 1.  $F^{n \times m} \cong F^{m \times n} \cong F^{mn}$

2.  $M_n F \cong F^{n^2}$

3.  $F[t]_n \cong F^{n+1}$

Let  $T : V \rightarrow W$  be linear with  $V, W$  arbitrary. Since  $T$  only tells us about  $\text{im } T$ , we replace the target  $W$  by  $\text{im } T = T(V)$ , i.e., view  $T : V \rightarrow W$  surjective linear. Let  $\mathcal{B}_0$  be a basis for  $\ker T \subset V$  subspace. Then Extension Theorem holds even when  $V$  is not finite dimensional. Extend  $\mathcal{B}_0$  to a basis  $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{C}$  so  $\mathcal{C} \cap \mathcal{B}_0 = \emptyset$  and  $V = \text{span } \mathcal{B}$ . By the argument proving the Dimension Theorem,

$$T(\mathcal{C}) = \{T(y) | y \in \mathcal{C}\}$$

is linearly indep. and since  $T$  is onto  $T(\mathcal{C})$  is a basis for  $W = \text{im } T$  comes from

$$Tx = 0, x \in \mathcal{B}_0$$

In the extra section (3), we showed

$$V/\ker T = \{\bar{v} | v \in V\}$$

where

$$\bar{v} = v + \ker T = \{v + z | z \in \ker T\}$$

is a vector space over  $F$ . In fact,  $\{\bar{y} | y \in \mathcal{C}\}$  is a basis for  $V/\ker T$ . By the UPVS,  $\exists!$  linear transformation

$$\bar{T} : V/\ker T \rightarrow W$$

given by  $\bar{0} = \bar{x} \mapsto 0, x \in \mathcal{B}_0, \bar{y} \mapsto Ty, y \in \mathcal{C}$ .  $\bar{T}$  is clearly onto and  $\bar{T}$  is 1-1,

$$\bar{T}(\bar{v}) = T(v) \quad \forall v \in V$$

So

$$\bar{T} : V/\ker T \rightarrow W = \text{im } T$$

is an isomorphism.

As  $- : V \rightarrow V/\ker T$  by  $v \mapsto \bar{v}$  is a surjective linear transformation, by definition,

$$\overline{\alpha v + v'} = \alpha \bar{v} + \bar{v'}$$

Note:  $\ker - = \ker T$ .

We have a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & \text{im } T \\ \downarrow - & \nearrow \bar{T} & \\ V/\ker T & & \end{array} \quad \text{commutes}$$

with  $-$  an epimorphism  
 $\bar{T}$  an isomorphism

Notice if  $W \neq \text{im } T, \bar{T}$  is only a monomorphism.

We shall show that all of this is true without using bases (or the Extension Theorem in the Extra Lecture). In particular,

$$V/\ker T \cong \text{im } T$$

## §11.2 Matrices and Linear Transformations

Goal: Let  $V, W$  be finite dimensional vector spaces over  $F$ . Reduce the study of linear transformations  $T : V \rightarrow W$  to matrix theory, hence often to computation (Deabstractify).

**Remark 11.4.** In this section, all bases are ORDERED.

Set up and Notation: Let  $V, W$  be finite dimensional vector space over  $F$ .  $\mathcal{B} = \{v_1, \dots, v_n\}$  an ordered basis for  $V$ , so  $\dim V = n$ .  $\mathcal{C} = \{w_1, \dots, w_m\}$  an ordered basis for  $W$ , so  $\dim W = m$ .

Step 1: If  $v \in V$ , write

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

i.e.,  $\alpha_1, \dots, \alpha_n$  are the unique coordinate of  $v$  relative to  $\mathcal{B}$ . Then let

$$[v]_{\mathcal{B}} := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in F^{n \times 1}$$

the coordinate matrix of  $v$  relative to the ordered basis  $\mathcal{B}$ . E.g.,

$$[v_i]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{pmatrix} i^{\text{th}}$$

and set

$$v_{\mathcal{B}} := \{[v]_{\mathcal{B}} | v \in V\} = F^{n \times 1}$$

Then

$$v \rightarrow v_{\mathcal{B}} \quad \text{by } v \mapsto [v]_{\mathcal{B}} \quad \text{isomorphism}$$

as

$$v_i \mapsto e_i := \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} i^{\text{th}}, f_{n,1} = \{e_1, \dots, e_n\}$$

the standard basis for  $F^{n \times 1}$ .

Step 2: Let  $T : V \rightarrow W$  be linear, then

$$Tv_i \in W = \text{Span } \mathcal{C} = \text{Span}(w_1, \dots, w_m)$$

as  $\mathcal{C}$  is a basis for  $W$ . Therefore,

$$\begin{aligned} \exists! \alpha_{ij} \in F, 1 \leq i \leq m, 1 \leq j \leq n \ni \\ Tv_j = \sum_{i=1}^m \alpha_{ij} w_i, \quad j = 1, \dots, n \end{aligned}$$

Let  $A = (\alpha_{ij} \in F^{m \times n})$ , i.e.,  $A_{ij} = \alpha_{ij} \forall i, j$ . Then the  $j^{\text{th}}$  COLUMN of A is

$$\begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{pmatrix} = [Tv_j]_{\mathcal{C}} \in W_{\mathcal{C}} = F^{m \times 1}$$

**Step 3:** Let

$$A : V_{\mathcal{B}} \rightarrow W_{\mathcal{C}} \text{ by } A([v]_{\mathcal{B}}) = A \cdot [v]_{\mathcal{B}}$$

This is a linear transformation.

$$A : F^{n \times 1} \rightarrow F^{m \times 1}$$

Since

$$A([v_j]_{\mathcal{B}}) = [Tv_j]_{\mathcal{C}}, j = 1, \dots, n$$

A is the unique linear transformation s.t.

$$A[v_j]_{\mathcal{B}} = [Tv_j]_{\mathcal{C}}$$

So by UPVS,

$$A[v]_{\mathcal{B}} = [Tv]_{\mathcal{C}} \quad \forall v \in V \quad (*)$$

**Definition 11.5 (Matrix Representation)** — The unique matrix  $A \in F^{m \times n}$  in  $(*)$  is called the **matrix representation of  $T$  relative to the ordered bases,  $\mathcal{B}, \mathcal{C}$** . We denote A by  $[T]_{\mathcal{B}, \mathcal{C}}$ .

Notation: if  $V = W$ ,  $\mathcal{B} = \mathcal{C}$ , we usually write  $[T]_{\mathcal{B}}$  for  $[T]_{\mathcal{B}, \mathcal{B}}$ .

## §12 | Lec 12: Oct 28, 2020

### §12.1 Lec 11 (Cont'd)

Summary: Let  $T : V \rightarrow W$  be linear with  $V, W$  finite dimensional vector space over  $F$

$\mathcal{B} = \{v_1, \dots, v_n\}$  an ordered basis for  $V, \dim V = n$

$\mathcal{C} = \{w_1, \dots, w_m\}$  an ordered basis for  $W, \dim W = m$

Then  $\exists!$   $A = [T]_{\mathcal{B}, \mathcal{C}} \in F^{m \times n}$  satisfying

$$A[v]_{\mathcal{B}} = [T]_{\mathcal{B}, \mathcal{C}}[v]_{\mathcal{B}} = [Tv]_{\mathcal{C}} \forall v \in V$$

Moreover, if

$$Tv_j = \sum_{i=1}^m \alpha_{ij} w_i, \quad j = 1, \dots, n$$

then the  $j^{\text{th}}$  column of  $A = [T]_{\mathcal{B}, \mathcal{C}}$  is precisely

$$[Tv_j]_{\mathcal{C}} = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{pmatrix} \in F^{m \times 1}$$

i.e.,

$$[T]_{\mathcal{B}, \mathcal{C}} = \begin{pmatrix} \underbrace{[Tv_1]_{\mathcal{C}} \dots [Tv_n]_{\mathcal{C}}}_{\text{columns}} \end{pmatrix}$$

Warning: If  $\mathcal{B}', \mathcal{C}'$  are two other ordered bases for  $V, W$  respectively (even the same vectors in  $\mathcal{B}, \mathcal{C}$  written in a different order), then in general

$$[T]_{\mathcal{B}, \mathcal{C}} \neq [T]_{\mathcal{B}', \mathcal{C}'}$$

**Example 12.1** 1. Let  $\mathcal{B} = \{v_1, \dots, v_n\}, \mathcal{C} = \{w_1, \dots, w_n\}$  be two ordered bases for  $V$ . Let

$$T : V \rightarrow V \text{ linear by } v_i \mapsto w_i, i = 1, \dots, n$$

Then  $[T]_{\mathcal{B}, \mathcal{C}} = I$ , the identity matrix. Moreover, if

$$Tv_j = w_j = \sum_{i=1}^n \alpha_{ij} v_i$$

then

$$[T]_{\mathcal{B}} = [T]_{\mathcal{B}, \mathcal{B}} = (\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & & \alpha_{nn} \end{pmatrix}$$

2.  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $(\alpha, \beta) \mapsto (\beta, \alpha)$ ,  $\mathcal{S} = \mathcal{S}_2 = \{e_1, e_2\}$ , the standard ordered basis for  $\mathbb{R}^2$ . Then

$$[T]_{\mathcal{S}} = ([Te_1]_{\mathcal{S}}, [Te_2]_{\mathcal{S}}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and if  $\mathcal{B}$  is the ordered bases  $\mathcal{B} = \{e_2, e_1\}$  then

$$[T]_{\mathcal{S}, \mathcal{B}} = ([Te_1]_{\mathcal{B}}, [Te_2]_{\mathcal{B}}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3. Let  $\mathcal{B} = \{1, x, x^2, x^3\}$  be a basis for  $\mathbb{R}[x]_3$ , the polynomial functions of degree  $\leq 3$  (and 0), and

$$D : \mathbb{R}[x]_3 \rightarrow \mathbb{R}[x]_3 \text{ differentiation}$$



Find  $[D]_{\mathcal{B}}$

$$D \cdot 1 = 0 \text{ so } [D \cdot 1]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Dx = 1 \text{ so } [Dx]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Dx^2 = 2x \text{ so } [Dx^2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$Dx^3 = 3x^2 \text{ so } [Dx^3]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

Hence,

$$[D]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Some more examples

**Example 12.2** 1. Let  $T_{\theta} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be counterclockwise rotation by an  $\angle \theta$

$$T_{\theta}e_1 = \cos \theta e_1 + \sin \theta e_2$$

$$T_{\theta}e_2 = (-\sin \theta)e_1 + \cos \theta e_2$$

So

$$[T_{\theta}]_{\mathcal{S}} = ([T_{\theta}e_1]_{\mathcal{S}} [T_{\theta}e_2]_{\mathcal{S}}) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

2. Let  $\mathcal{B} = \{v_1, v_2\}$  be an ordered basis for  $V$  and  $\mathcal{C} = \{w_1, w_2, w_3\}$  an ordered basis for  $W$ . Suppose

$$T : V \rightarrow W \text{ by } \begin{cases} Tv_1 = 3w_1 + w_3 \\ Tv_2 = w_1 + 6w_2 + w_3 \end{cases}$$

$$\text{then } [T]_{\mathcal{B}, \mathcal{C}} = \begin{pmatrix} 3 & 1 \\ 0 & 6 \\ 1 & 1 \end{pmatrix}$$

3. Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the reflection about the  $e_1, e_2$  plane. What is  $[T]_{\mathcal{S}}$ ?

$$e_1 \mapsto e_1$$

$$e_2 \mapsto e_2$$

$$e_3 \mapsto -e_3$$

$$\text{So } [T]_{\mathcal{S}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

### Theorem 12.3 (Matrix Theory)

(MTT) Let  $V, W$  be finite dimensional vector space  $F$ ,  $\dim V = n$ ,  $\dim W = m$ , and  $\mathcal{B}, \mathcal{C}$  ordered bases for  $V, W$ . Then the map

$$\phi : L(V, W) \rightarrow F^{m \times n} \text{ by } T \mapsto [T]_{\mathcal{B}, \mathcal{C}}$$

is an isomorphism. In particular

$$\dim L(V, W) = mn$$

*Proof.* Left as exercise (Homework). □

Using the fact that  $W \rightarrow W_{\mathcal{C}}$  is an isomorphism if  $w \mapsto [w]_{\mathcal{C}}$  show that

1.  $\phi$  is linear
2.  $\phi$  is onto
3.  $\phi$  is 1-1
4.  $\dim L(V, W) = mn$

### Theorem 12.4

Let  $V, W, U$  be finite dimensional vector space over  $F$  with ordered bases  $\mathcal{B}, \mathcal{C}, \mathcal{D}$  respectively,  $T : V \rightarrow W$ ,  $S : W \rightarrow U$  linear. Then

$$[S \circ T]_{\mathcal{B}, \mathcal{D}} = [S]_{\mathcal{C}, \mathcal{D}} \cdot [T]_{\mathcal{B}, \mathcal{C}}$$

*Proof.*

$$\begin{aligned} [S]_{\mathcal{C}, \mathcal{D}} [T]_{\mathcal{B}, \mathcal{C}} [v]_{\mathcal{B}} &= [S]_{\mathcal{C}, \mathcal{D}} [Tv]_{\mathcal{C}} \\ &= [S(Tv)]_{\mathcal{D}} \\ &= [(S \circ T)(v)]_{\mathcal{D}} \\ &= [S \circ T]_{\mathcal{B}, \mathcal{D}} [v]_{\mathcal{B}} \end{aligned}$$

□

Exercise: Let  $V, W$  be finite dimensional vector space over  $F$  with  $\dim V = \dim W$ ,  $\mathcal{B}, \mathcal{C}$  ordered bases of  $V, W$  respectively,  $T : V \rightarrow W$  linear. Then,  $T$  is an isomorphism iff  $[T]_{\mathcal{B}, \mathcal{C}}$  is invertible.

Let  $V$  be a finite dimensional vector space over  $F$ ,  $\dim V = n$ ,  $\mathcal{B}$  an ordered basis for  $V$ . Then

$$\phi : L(V, V) \rightarrow M_n F \text{ by } T \mapsto [T]_{\mathcal{B}}$$

satisfies all of the following:  $\forall T, S \in L(V, V)$

- (i)  $\phi(T + S) = \phi(T) + \phi(S)$
- (ii)  $\phi(T \circ S) = \phi(T)\phi(S)$
- (iii)  $\phi(0_V) = 0_{F^{n \times 1}}$
- (iv)  $\phi(1_V) = 1_{F^{n \times 1}}$

By the exercise,  $\phi$  is bijection linear transformation. Both  $L(V, V)$  and  $M_n F$  satisfy all the axioms of a field except (M3) and (M4). We call them (NON COMMUTATIVE) rings and since  $\phi$  preserves all the structure i) – iv) as does its inverse(?), we say  $\phi$  is an ISOMORPHISM of rings

**Definition 12.5 (Change of Basis Matrix)** — Let  $V$  be a finite dimensional vector space over  $F$  with ordered bases  $\mathcal{B}, \mathcal{C}$ . Then the invertible matrix  $[1_V]_{\mathcal{B}, \mathcal{C}}$  is called a CHANGE OF BASIS MATRIX.

**Example 12.6** 1.  $\mathcal{S} = \{e_1, e_2\}, \mathcal{B} = \{(1, 1), (2, 1)\}, \mathcal{C} = \{(3, 4), (6, 1)\}$  ordered bases for  $\mathbb{R}^2$ .

$$\begin{aligned} [1_{\mathbb{R}^2}]_{\mathcal{B}, \mathcal{S}} &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad [1_{\mathbb{R}^2}]_{\mathcal{S}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ [1_{\mathbb{R}^2}]_{\mathcal{C}, \mathcal{S}} &= \begin{pmatrix} 3 & 6 \\ 4 & 1 \end{pmatrix}, \quad [1_{\mathbb{R}^2}]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

- 2.  $\mathcal{B}$  an ordered basis for  $V$ , a finite dimensional vector space over  $F$ ,  $\dim V = n$ , then  $[1_V]_{\mathcal{B}} = I \in M_n F$
- 3.  $V$  a finite dimensional vector space over  $F$ ,  $\mathcal{B}, \mathcal{C}$  ordered bases for  $V$ , then  $[1_V]_{\mathcal{B}, \mathcal{C}}$  is invertible and

$$\begin{aligned} [1_V]_{\mathcal{B}, \mathcal{C}}^{-1} &= [1_V]_{\mathcal{C}, \mathcal{B}} \\ [1_V]_{\mathcal{B}, \mathcal{C}} [1_V]_{\mathcal{C}, \mathcal{B}} &= [1_V]_{\mathcal{C}} \\ &= I \\ &= [1_V]_{\mathcal{C}, \mathcal{B}} [1_V]_{\mathcal{B}, \mathcal{C}} \end{aligned}$$

4. Apply 3) to 1)

$$\begin{aligned} [1_V]_{\mathcal{S},\mathcal{C}} &= [1_V]_{\mathcal{C},\mathcal{S}}^{-1} = \begin{pmatrix} 3 & 6 \\ 4 & 1 \end{pmatrix}^{-1} = -\frac{1}{21} \begin{pmatrix} 1 & -6 \\ -4 & 3 \end{pmatrix} \\ [1_V]_{\mathcal{B},\mathcal{C}} &= [1_V]_{\mathcal{S},\mathcal{C}} [1_V]_{\mathcal{B},\mathcal{S}} \\ &= -\frac{1}{21} \begin{pmatrix} 1 & -6 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \\ &= -\frac{1}{21} \begin{pmatrix} -5 & -4 \\ -1 & -5 \end{pmatrix} \end{aligned}$$

Some more examples

**Example 12.7** 1. Any invertible matrix  $A \in M_n F$  is a change of basis matrix for some ordered bases  $\mathcal{B}, \mathcal{C}$  for  $F^n$ : if  $A = (\alpha_{ij})$  is invertible, define

$$v_j = \sum_{i=1}^n \alpha_{ij} e_i, \quad \mathcal{B} = \{v_1, \dots, v_n\}$$

Then  $A = [A]_{\mathcal{B},\mathcal{S}}$  since  $A$  is invertible, so  $\mathcal{B}$  is linearly indep., hence a basis by counting and  $A = [\mathcal{F}_v]_{\mathcal{B},\mathcal{S}}$ .

2. The  $j^{\text{th}}$  column of  $[1_v]_{\mathcal{B},\mathcal{C}}$ ,  $V$  a finite dimensional vector space over  $F$  is the  $j^{\text{th}}$  vector of  $\mathcal{B}$  expressed as a linear combo of vectors in  $\mathcal{C}$ .
3. Generalizing (1), (3) from above example, we get the following crucial computational device: if  $V = F^n$ ,  $\mathcal{B}, \mathcal{C}$  ordered bases for  $V$ , then

$$[1_v]_{\mathcal{B},\mathcal{C}} = [1_v]_{\mathcal{S},\mathcal{C}} [1_v]_{\mathcal{B},\mathcal{S}} = [1_v]_{\mathcal{C},\mathcal{S}}^{-1} [1_v]_{\mathcal{B},\mathcal{S}}$$

if we only have  $V \cong F^n$ , then we have to use an isomorphism  $V \rightarrow F^n$  – how?

Since  $[1_v]_{\mathcal{B},\mathcal{S}}$  and  $[1_v]_{\mathcal{C},\mathcal{S}}$  are usually (often?) easy to write down, this is quite useful. What if  $V = F^{m \times n}$ ?

### Theorem 12.8 (Change of Basis)

Let  $V, W$  be finite dimensional vector space over  $F$  with ordered bases  $\mathcal{B}, \mathcal{B}'$  for  $V$  and  $\mathcal{C}, \mathcal{C}'$  for  $W$ . Let  $T: V \rightarrow W$  be linear. Then

$$\begin{aligned} [T]_{\mathcal{B},\mathcal{C}} &= [1_W]_{\mathcal{C}',\mathcal{C}} [T]_{\mathcal{B}',\mathcal{C}'} [1_V]_{\mathcal{B},\mathcal{B}'} \\ &= [1_W]_{\mathcal{C},\mathcal{C}'}^{-1} [T]_{\mathcal{B}',\mathcal{C}'} [1_V]_{\mathcal{B},\mathcal{B}'} \\ &= [1_W]_{\mathcal{C}',\mathcal{C}} [T]_{\mathcal{B}',\mathcal{C}'} [1_V]_{\mathcal{B}',\mathcal{B}}^{-1} \end{aligned}$$

*Proof.* We have

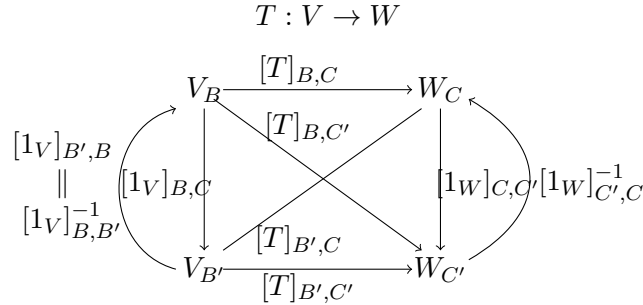
$$[1_W]_{\mathcal{C},\mathcal{C}'}^{-1} = [1_W]_{\mathcal{C}',\mathcal{C}} \text{ and } [1_V]_{\mathcal{B},\mathcal{B}'} = [1_V]_{\mathcal{B}',\mathcal{B}}^{-1}$$

Since

$$\begin{aligned} [1_W]_{\mathcal{C}',\mathcal{C}}[T]_{\mathcal{B}',\mathcal{C}'}[1_V]_{\mathcal{B},\mathcal{B}'} &= [1_W \circ T]_{\mathcal{B}',\mathcal{C}}[1_V]_{\mathcal{B},\mathcal{B}'} \\ &= [1_W \circ T \circ 1_V]_{\mathcal{B},\mathcal{C}} \\ &= [T]_{\mathcal{B},\mathcal{C}} \end{aligned}$$

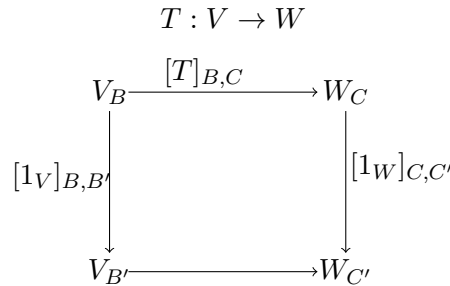
the result follows.  $\square$

To use (and remember) this, do it as follows – to let the notation help you:



COMMUTES, i.e., can compose along any allowable arrows in the correct direction if we arrive at the same place in different way starting at the same place we get the same answer.

Warning: You can only reverse direction if the arrow is an isomorphism and then you can take the inverse. To remember the theorem, we write



and fill in arrows you can find in the diagram before.

## §13 | Lec 13: Oct 30, 2020

### §13.1 Some Examples of Change of Basis

If  $V, W$  are finite dimensional vector space over  $F$  with ordered bases  $\mathcal{B}, \mathcal{C}$  respectively and if  $T : V \rightarrow W$  is linear

$$[Tv]_{\mathcal{C}} = [T]_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}} \forall v \in V$$

Note: There is nothing about the bases in which  $v$  was written.

1.  $V = \mathbb{R}^2$ ,  $\mathcal{S} = \{e_1, e_2\}$ ,  $\mathcal{B} = \{v_1 = (1, 1), v_2 = (2, 1)\}$  ordered bases. Find  $[T]_{\mathcal{S}}$  in the following (equivalently,  $[T]_{\mathcal{S}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{\mathcal{S}} \leftrightarrow T(\alpha, \beta)$ )

- (i)  $T(1, 1) = (2, 1)$  and  $T(2, 1) = (1, 1)$

$$\begin{array}{ccc} V_B & \xrightarrow{[T]_B} & V_B \\ [1_V]_{B,S} \downarrow & & \downarrow [1_V]_{B,S} \\ V_S & \xrightarrow{[T]_S} & V_S \end{array} \quad [1_V]_{B,S} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

So

$$\begin{aligned} [T]_{\mathcal{S}} &= [1_V]_{\mathcal{B},\mathcal{S}} [T]_{\mathcal{B}} [1_V]_{\mathcal{B},\mathcal{S}}^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

So  $T(\alpha, \beta) = (-\alpha + 3\beta, \beta)$

(ii)  $T(1, 1) = 6(1, 1) + (2, 1)$  and  $T(2, 1) = -2(1, 1) + (2, 1)$

$$\begin{array}{ccc} V_B & \xrightarrow{[T]_B} & V_B \\ [1_V]_{B,S} \downarrow & & \downarrow [1_V]_{B,S} \\ V_S & \xrightarrow{[T]_S} & V_S \end{array} \quad [1_V]_{B,S} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

So

$$[T]_{\mathcal{S}} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -8 & 16 \\ -8 & 15 \end{pmatrix}$$

(iii)  $T(1, 1) = (3, 1)$  and  $T(2, 1) = (5, 1)$

$$\begin{array}{ccc} V_B & \xrightarrow{\quad} & V_B \\ \downarrow & \searrow [T]_{B,S} & \downarrow [1]_{B,S} \\ V_S & \xrightarrow{\quad} & V_S \end{array}$$

$$[T]_{\mathcal{B},\mathcal{S}} = ([T(1, 1)]_{\mathcal{S}} [T(2, 1)]_{\mathcal{S}}) = ([ (3, 1) ] [ (5, 1) ]_{\mathcal{S}})$$

So  $[T]_{\mathcal{S}} = [T]_{\mathcal{B},\mathcal{S}} [1_V]_{\mathcal{B},\mathcal{S}}^{-1}$  which is equal to  $\begin{pmatrix} 3 & 5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}$

2. Let  $T$  be a rotation about the axis  $(1, 1, 1) \in V = \mathbb{R}^3$  of an  $\angle \theta$  in the counter-clockwise direction with  $(1, 1, 1)$  up. We will use stuff from 33A – dot product. Normalize  $(1, 1, 1)$  to

$$v_1 = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) = \frac{(1, 1, 1)}{\|(1, 1, 1)\|}$$

a unit vector in the DIRECTION of  $v_1$ . Find a vector  $\perp$  to  $v_1$ , say

$$v'_2 = (0, 1, -1)$$

and normalize it to

$$v_2 = \left( 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

Let  $v_3 = v_1 \times v_2$  the cross product of  $v_1, v_2$ . It is orthogonal to  $v_1$  and  $v_2$  and by the right hand rule in the correct orientation

$$v_3 = \begin{pmatrix} i & j & k \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \left( -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$$

a unit vector (or use Gram – Schmidt and check you have  $v_3 = v_1 \times v_2$  and not  $-(v_1 \times v_2)$ )

## §13.2 Orthonormal Basis

**Definition 13.1 (Orthonormal Basis)** — Let  $\mathcal{B} = \{v_1, v_2, v_3\}$  an ordered bases of vectors of length 1 and each  $\perp$  to the others, called an ORTHONORMAL BASIS.

$$\begin{aligned} T v_1 &= v_1 \\ T v_2 &= \cos \theta v_2 + \sin \theta v_3 \\ T v_3 &= -\sin \theta v_2 + \cos \theta v_3 \\ [T]_{\mathcal{B}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \\ [1_V]_{\mathcal{B}, \mathcal{S}} &= \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \end{aligned}$$

$$\begin{array}{ccc} V_B & \xrightarrow{[T]_B} & V_B \\ [1_V]_{B, S} \downarrow & & \downarrow [1_V]_{B, S} \\ V_S & \xrightarrow{[T]_S} & V_S \end{array}$$

$$[T]_{\mathcal{S}} = [1_V]_{\mathcal{B}, \mathcal{S}} [T]_{\mathcal{B}} [1_V]_{\mathcal{B}, \mathcal{S}}^{-1} = [1_V]_{\mathcal{B}, \mathcal{S}} [T]_{\mathcal{B}} [1_V]_{\mathcal{S}, \mathcal{B}}$$

Since both  $\mathcal{S}$  and  $\mathcal{B}$  are orthonormal bases and  $F = \mathbb{R}$ , it turns out that

$$[1_V]_{\mathcal{B}, \mathcal{S}}^{-1} = [1_V]_{\mathcal{B}, \mathcal{S}}^{\top}$$

This is, however, not true in general.

3.  $V = \mathbb{R}^3$ ,  $T : V \rightarrow V$  as in 2) and  $S : V \rightarrow V$  a reflection about the plane  $\perp (1, 2, 3)$ . Find  $[S]_{\mathcal{S}}$  and  $[S \circ T]_{\mathcal{S}}$ .

Find an orthonormal basis with  $(1, 2, 3)$  direction of the first vector

$$(1, 2, 3), (0, 3, -2), (-13, 2, 3)$$

then normalize as follows:

$$\begin{aligned} w_1 &= \left( \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) \\ w_2 &= \left( 0, \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \right) \\ w_3 &= \left( \frac{-13}{\sqrt{182}}, \frac{2}{\sqrt{182}}, \frac{3}{\sqrt{182}} \right) \end{aligned}$$

So  $\mathcal{C} = \{w_1, w_2, w_3\}$  is an orthonormal basis and

$$[S]_{\mathcal{C}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ccccc} V_B & \xrightarrow{[T]_B} & V_B & & \\ \downarrow [1_V]_{B,S} & & \downarrow & & \\ V_S & \xrightarrow{[T]_S} & V_S & \xrightarrow{[S]_S} & V_S \\ & & \uparrow & & \uparrow [1_V]_{C,S} \\ & & V_C & \xrightarrow{[S]_C} & V_C \end{array}$$

$$[1_V]_{\mathcal{C}, \mathcal{S}} = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & \frac{13}{\sqrt{182}} \\ \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{182}} \\ \frac{3}{\sqrt{14}} & -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{182}} \end{pmatrix}$$

$$\begin{aligned} [S]_{\mathcal{S}} &= [1_V]_{\mathcal{C}, \mathcal{S}} [S]_{\mathcal{C}} [1_V]_{\mathcal{C}, \mathcal{S}}^{-1} \\ [S \circ T]_{\mathcal{S}} &= [1_V]_{\mathcal{C}, \mathcal{S}} [S]_{\mathcal{C}} [1_V]_{\mathcal{B}, \mathcal{S}} [T]_{\mathcal{B}, \mathcal{S}} [1_V]_{\mathcal{B}, \mathcal{S}}^{-1} \end{aligned}$$

The only reason to normalize  $\mathcal{C}$  to an orthonormal basis is

$$[1_V]_{\mathcal{C}, \mathcal{S}}^{-1} = [1_V]_{\mathcal{C}, \mathcal{S}}^{\top}$$

### §13.3 Similarity

**Definition 13.2 (Similar Matrices)** — Let  $A, B \in M_n F$ . We say  $A$  is SIMILAR to  $B$  if  $\exists C \in M_n F$  invertible  $\ni$

$$A = C^{-1}BC$$



**Remark 13.3.**  $A, B \in M_n F$  :

1.  $A \sim B \rightarrow B \sim A$  :

$$A = C^{-1}BC, C \text{ invertible} \rightarrow B = (C^{-1})^{-1}AC^{-1} \text{ as } CC^{-1} = I = C^{-1}C$$

2. If  $A \sim B$ , then  $\det A = \det B$ . If  $A = C^{-1}BC$ , invertible, then

$$\begin{aligned} \det A &= \det (C^{-1}BC) = \det(C^{-1}) \det B \det C \\ &= (\det C)^{-1} \det B \det C = \det B \end{aligned}$$

3.  $\sim$  is an equivalence relation.

### Theorem 13.4 (Similar Matrices)

Let  $A, B \in M_n F$ . Then  $A \sim B$  iff  $\exists V$  a vector space over  $F$ ,  $\dim V = n$ ,  $T : V \rightarrow V$  linear and ordered bases  $\mathcal{B}, \mathcal{C}$  for  $V$  s.t

$$A = [T]_{\mathcal{B}} \quad \text{and} \quad B = [T]_{\mathcal{C}}$$

i.e.,  $A \sim B$  iff they represent the same linear transformation relative to (possibly) different ordered bases.

## §14 | Lec 14: Nov 2, 2020

### §14.1 Lec 13 (Cont'd)

*Proof.* (Of Similar Matrices Theorem) ( $\leftarrow$ ) If  $A = [T]_{\mathcal{B}}, B = [T]_{\mathcal{C}}$ , then  $C = [1_V]_{\mathcal{B}, \mathcal{C}} \in M_n F$  is invertible with  $A = C^{-1}BC$  by the Change of Basis Theorem.

( $\rightarrow$ ) Suppose  $C \in M_n F$  is invertible,  $A = C^{-1}BC$ . Define  $V = F^n$ ,  $T : V \rightarrow V$  by

$$T_{ij} = \sum_{i=1}^n A_{ij} e_i$$

with  $\mathcal{S} = \{e_1, \dots, e_n\}$  the standard basis

$$[T]_{\mathcal{S}} = A = C^{-1}BC$$

Let  $w_j := \sum_{i=1}^n (C^{-1})_{ij} e_i$ , i.e.,  $(C^{-1})_{ij}$  is the  $ij^{\text{th}}$  entry of  $C^{-1}$ . As  $C$  is invertible,  $C^{-1}$  exists and is invertible. Then

$$\mathcal{B} = \{w_1, \dots, w_n\}$$

is a basis for  $V$  and  $[1_V]_{\mathcal{B}, \mathcal{S}} = C^{-1}$  figure here so  $A = C^{-1}[T]_{\mathcal{B}}C$  and  $B = [T]_{\mathcal{B}}$  works.  $\square$

### §14.2 Eigenvalues and Eigenvectors

**Definition 14.1** (Eigenvalues, Eigenvectors & Eigenspace) — Let  $0 \neq V$  be a vector space over  $F$ ,  $T : V \rightarrow V$  a linear operator and  $\lambda \in F$ . Set

$$S_\lambda := T - \lambda 1_V : V \rightarrow V,$$

a linear operator, so

$$S_\lambda(v) = Tv - \lambda v \forall v \in V$$

We say  $\lambda$  is an EIGENVALUE of  $T$  if  $S_\lambda$  is not 1-1, i.e.,  $\ker S_\lambda \neq 0$ . Let

$$\begin{aligned} E_T(\lambda) &:= \ker S_\lambda = \{v \in V \mid Tv - \lambda v = 0\} \\ &= \{v \in V \mid Tv = \lambda v\} \end{aligned}$$

if  $E_T(\lambda) \neq 0$ , we call  $E_T(\lambda)$  an EIGENSPACE of  $V$  relative  $T, \lambda$  and any  $v \in E_T(\lambda)$  an EIGENVECTOR of  $T$  relative to  $\lambda$ . So if  $T : V \rightarrow V$  is linear,  $\lambda \in F$  is an eigenvalue of  $T$  iff

$$\exists 0 \neq v \in V \ni Tv = \lambda v$$

**Remark 14.2.** Let  $0 \neq V$  be a vector space over  $F$  and  $T : V \rightarrow V$  linear

1. Eigenvalues occur as measured quantities in science and engineering, e.g., resonance, quantum number – measurable values.
2. If  $\lambda \in F$  is an eigenvalue of  $T$ , then

$$0 \neq E_T(\lambda) \subset V \text{ is a subspace}$$

3. If  $\lambda \in F$  an eigenvalue, any  $v \in E_T(\lambda)$  is an eigenvector. In particular, any basis for  $E_T(\lambda)$  consists of eigenvectors of  $T$  relative to  $\lambda$ . Hence

$$T|_{E_T(\lambda)} = \lambda 1_{E_T(\lambda)}$$

(the notation above means we restrict the domain to  $E_T(\lambda)$ . In particular, if  $V = E_T(\lambda)$ , then  $T = \lambda 1_V$ ).

4. If  $T = 0$ , then  $V = E_T(\lambda)$  with eigenvalue  $\lambda = 0$  ( $\lambda = 1$ ).

**Example 14.3** 5. Let  $V = \mathbb{R}^3$ ,  $T : V \rightarrow V$  a counterclockwise rotation by an  $\angle \theta, 0 < \theta < 2\pi$  around the axis determined by  $0 \neq v \in V$ . Then

$$T(\alpha v) = \alpha Tv = \alpha v \forall \alpha \in F$$

So  $\text{Span}(v) \subset E_T(1)$ . Note if  $0 \neq v$  is an eigenvector with eigenvalue  $\mu$  of linear  $S : V \rightarrow V$ , then

$$Sv \in \text{Span}(v) = Fv \text{ so } \text{Span}(v) \subset E_S(\mu)$$

Do there exist other eigenvalues of  $T$ ? Ever? So the only other possibilities would

be

$$\theta = \pi, \lambda = -1$$

In that case

$$E_T(-1) = \text{Span}(w_1, w_2)$$

where  $w_1, w_2$  are linearly indep. with  $w_i \perp v, i = 1, 2$ . (of course, if one allows  $\theta = 0, T = 1_V$ .)

6. Let  $0 \neq v \in V$ . Suppose that

$$\mu v = Tv = \lambda v, \quad \lambda, \mu \in F$$

Then  $\mu = \lambda$  so  $0 \neq v \in V$  is an eigenvector of at most one eigenvalue of  $T$  – usually none. In particular,

$$E_T(\lambda) \cap E_T(\mu) = 0 \text{ if } \lambda \neq \mu$$

and we write

$$E_T(\lambda) \oplus E_T(\mu) = E_T(\lambda) + E_T(\mu)$$

and call it the DIRECT SUM of the subspace  $E_T(\lambda)$  and  $E_T(\mu)$ .

What do you think is  $W_1 \oplus W_2 \oplus W_3$ ?

7. Suppose  $\dim V = n$ ,  $\mathcal{B} = \{v_1, \dots, v_n\}$  is an ordered basis for  $V$ . Suppose that that

$$Tv_i = \alpha_i v_i, \quad i = 0, \dots, n$$

$\lambda_1, \dots, \lambda_n \in F$  not necessarily distinct. Then

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix}$$

is a DIAGONAL MATRIX, i.e., all non-diagonal entries 0. We say  $T$  is DIAGONALIZABLE if  $\exists$  an ordered bases  $\mathcal{C}$  for  $V$   $\ni [T]_{\mathcal{C}}$  is diagonal.

8. Suppose  $\dim V = n (< \infty)$  and  $T$  is diagonalizable, i.e.,  $\exists$  an ordered basis  $\mathcal{C} = \{w_1, \dots, w_n\}$  for  $V$  s.t.

$$[T]_{\mathcal{C}} = \begin{pmatrix} \mu_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mu_n \end{pmatrix}$$

Then  $Tw_i = \mu_i w_i, i = 1, \dots, n$  and  $\mathcal{C}$  is an ordered basis for  $V$  consisting of eigenvectors for  $T$ .

Conclusion: Let  $V$  be a finite dimensional vector space over  $F$ ,  $T : V \rightarrow V$  linear. Then  $T$  is diagonalizable iff  $\exists$  a basis for  $V$  consisting of eigenvectors of  $T$ .

Note: If  $T$  is diagonalizable,  $T : V \rightarrow V$  linear,  $V$  a finite dimensional vector space over  $F$ , ordered basis  $\mathcal{B}$  for  $V$ . Then  $\exists C \in M_n F$ , invertible,  $n = \dim V$   $\ni C^{-1}[T]_{\mathcal{B}}C$  is diagonal by the Change of Basis Theorem.

**Example 14.4** 9. Let  $V$  be a finite dimensional vector space over  $F$ ,  $n = \dim V$ ,  $\mathcal{B}$  an ordered basis for  $V$ ,  $S : V \rightarrow V$  linear. Then by the Isomorphism Theorem,  $S$  is 1-1 iff  $S$  is onto. Apply this to

$$S_\lambda = T - \lambda 1_V : V \rightarrow V$$

to conclude:

$\lambda$  is an eigenvalue of  $T$  iff  $S_\lambda = T - \lambda 1_V$  is singular (i.e.,  $S_\lambda$  is not 1-1)

iff

$$[S_\lambda]_{\mathcal{B}} = [T - \lambda 1_V]_{\mathcal{B}} \text{ is not invertible}$$

iff

$$\det[T - \lambda 1_V]_{\mathcal{B}} = 0 \text{ (by properties of det)}$$

iff

$$\det([T]_{\mathcal{B}} - \lambda[1_V]_{\mathcal{B}}) = 0$$

iff

$$\det([T]_{\mathcal{B}} - \lambda I) = 0$$

iff

$$\det(\lambda I - [T]_{\mathcal{B}}) = 0$$

Summary: Let  $V$  be a finite dimensional vector space over  $F$ ,  $\dim V = n$ ,  $T : V \rightarrow V$  linear,  $\mathcal{B}$  an ordered basis for  $V$ ,  $\lambda \in F$ . Then,  $\lambda$  is an eigenvalue of  $T$  iff  $\det(\lambda I - [T]_{\mathcal{B}}) = 0$ .

**Definition 14.5 (Characteristics Polynomial)** — Let  $A \in M_n F$ . Define

$$f_A := \det(tI - A) \in F[t]$$

called the Characteristics Polynomial of  $A$ .

The properties of the determinant on  $F[t]$  is the same as on  $F$  except that  $A \in M_n F[t]$  is invertible iff  $\det A \in F \setminus \{0\}$  and we assume these properties.

**Proposition 14.6**

If  $A, B \in M_n F$  are similar, then  $f_A = f_B$

*Proof.* If  $A = C^{-1}BC$ ,  $C \in M_n F$  in

$$\begin{aligned} f_A &= \det(C^{-1}(tI - B)C) = \det C^{-1} \det(tI - B) \det C \\ &= \det(tI - B) = f_B \end{aligned}$$

□

Warning: Let  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ . Then,  $A$  and  $B$  are not similar, but  $f_A = f_B$ , i.e., the converse is false.

**Corollary 14.7**

Let  $V$  be a finite dimensional vector space over  $F$ ,  $T : V \rightarrow V$  linear,  $\mathcal{B}, \mathcal{C}$  ordered bases for  $V$ . Then

$$f_{[T]_{\mathcal{B}}} = f_{[T]_{\mathcal{C}}}$$

*Proof.* Change of Basis Theorem. □

**Definition 14.8 (Characteristics Polynomial)** — Let  $V$  be a finite dimensional vector space over  $F$ ,  $T : V \rightarrow V$  linear,  $\mathcal{B}$  ordered basis for  $V$ . We call  $f[t]_{\mathcal{B}}$  the characteristics polynomial of  $T$ . By the corollary, it is independent of  $\mathcal{B}$ , so we denote it by  $f_T (= f_{[T]_{\mathcal{B}}})$  and write  $f_T = \det(tI_V - T) := \det(tI - [T]_{\mathcal{B}})$

**Theorem 14.9 (Eigenvalue – Root of  $f_T$ )**

Let  $V$  be a finite dimensional vector space over  $F$ ,  $T : V \rightarrow V$  linear. Then, the eigenvalues of  $T$  are precisely, the roots of  $f_T$ , i.e., those  $\alpha \in F \ni f_T(\alpha) = 0$ .

*Proof.* Let  $\lambda \in F$ ,  $\mathcal{B}$  an ordered basis for  $V$ . Set  $A = [T]_{\mathcal{B}}$ , so  $f_T = \det(tI - A)$ . Then  $\lambda$  is a root of  $f_T$  iff evaluating  $f_T$  at  $\lambda$ , i.e.,  $f_T(\lambda)$ , we have

$$f_T(\lambda) = \det(tI - A) \Big|_{t=\lambda} = 0 \iff \lambda \text{ is an eigenvalue of } T$$

i.e., expanding the polynomial  $\det(tI - A)$  and plugging  $\lambda$  for  $t$  gives 0. □

We cannot use the following theorem if we fully prove it.

**Theorem 14.10 (Cayley – Hamilton)**

Let  $A \in M_n F$ . Then

$$f_A(A) = 0$$

plugging  $A$  into the expansion of the determinant  $f_A$ , you get 0.

**Remark 14.11.** By HW, we have  $\{I, A, A^2, \dots, A^{n^2}\} \subset M_n F$  is linearly dep., i.e.,  $\{I, A, \dots, A^N\}$  is linearly dep. for some  $N > 0$ . This means  $\exists 0 \neq g \in F[t]$  with  $\deg g \leq N$  and  $g(A) = 0$  – why?

So Cayley – Hamilton’s Theorem says  $\{I, A, \dots, A^n\}$  in  $M_n F$  is always linearly dep. in  $M_n F$  with  $f_A(A)$  giving a dependence relation.

Note: If you know Cramer’s Rule in determinant theory, one can prove Cayley – Hamilton follows from it. In fact, it is essentially Cramer’s Rule.

**Remark 14.12.** Let  $V$  be a finite dimensional vector space over  $F$ ,  $T : V \rightarrow V$  linear. You will show in your Take home Exam. There exists a polynomial  $q \in F[t]$  satisfying

1.  $q \neq 0$
2.  $q(A) = 0$
3.  $\deg q$  is the minimal degree for a poly  $g \neq 0$  in  $F[t]$  to satisfy  $g(A) = 0$
4.  $q$  is MONIC, i.e., leading coeff is 1.

Moreover,  $q$  is unique and called the MINIMAL POLYNOMIAL of  $A$  and denoted  $q_T$ . Using it we shows a stronger form of the Cayley – Hamilton Theorem.

## §15 | Lec 15: Nov 4, 2020

### §15.1 Lec 14 (Cont'd)

Cayley – Hamilton (Stronger Form): Let  $V$  be a finite dimensional vector space over  $F$ ,  $T : V \rightarrow V$  linear, then

$$q_T | f_T \text{ in } F[t]$$

(where  $q_T = q[T]_{\mathcal{B}}$ ,  $\mathcal{B}$  an ordered basis and  $q_T$  is indep. of  $\mathcal{B}$ ). Why does this show the other form?

Computation: Let  $V$  be a finite dimensional vector space over  $F$ ,  $T : V \rightarrow V$  linear. To find eigenvalues and eigenvectors of  $T$ , you must solve

$$Tv = \alpha v$$

By Matrix Theory Theorem, this is equivalent to

$$[T]_{\mathcal{B}}[v]_{\mathcal{B}} = \lambda[v]_{\mathcal{B}} \quad (*)$$

$\mathcal{B}$  an ordered basis for  $V$ . To find eigenvalues, we find the roots of  $f_T$ . To find the eigenvectors, we solve (\*).

#### Theorem 15.1

Let  $T : V \rightarrow V$  be linear and  $\lambda_1, \dots, \lambda_n$  in  $F$  distinct eigenvalues of  $T$ ,  $0 \neq v_i \in E_T(\lambda_i)$ ,  $i = 1, \dots, n$ . Then  $\{v_1, \dots, v_n\}$  is linearly indep.

*Proof.* We induct on  $n$ .

- $n = 1$  :  $v_1 \neq 0$  so  $\{v\}$  is linearly indep.
- $n > 1$  – Induction Hypothesis (IH) : If  $\lambda_1, \dots, \lambda_{n-1}$  are distinct eigenvalues of  $T$ ,  $0 \neq v_i \in E_T(\lambda_i)$ ,  $i = 1, \dots, n-1$  then  $\{v_1, \dots, v_{n-1}\}$  is linearly indep. Suppose that

$$0 = \alpha_1 v_1 + \dots + \alpha_n v_n, \alpha_1, \dots, \alpha_n \in F \quad (*)$$

Apply the linear operator  $S_{\lambda_n} = T - \lambda_n 1_V$  to (\*). As

$$S_{\lambda_n}(v_i) = Tv_i - \lambda_n v_i = \lambda_i v_i - \lambda_n v_i = (\lambda_i - \lambda_n)v_i$$

We get

$$\begin{aligned} S_{\lambda_n}(\alpha_1 v_1 + \dots + \lambda_n v_n) &= \alpha_1 S_{\lambda_n} v_1 + \dots + \alpha_n S_{\lambda_n} v_n \\ 0 &= \alpha_1(\alpha_1 - \alpha_n)v_1 + \dots + \alpha_{n-1}(\lambda_{n-1} - \lambda_n)v_{n-1} \end{aligned}$$

By the IH,  $\alpha_i(\lambda_i - \lambda_n) = 0, i = 1, \dots, n-1$

As  $\lambda_i - \lambda_n \neq 0, i = 1, \dots, n-1, \alpha_i = 0, i = 1, \dots, n-1$ . So  $0 = \alpha_n v_n$ . As  $v_n \neq 0$ ,  $\alpha_n = 0$  also. □

*Proof.* (Alternative) Take  $T$  of (\*) to get an eqn 1). Multiply (\*) by  $\lambda_n$  to get an eqn 2). Subtract eqn 2) from eqn 1). The proof that if  $\alpha_1, \dots, \alpha_n$  are distinct then  $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$  are linearly indep. □

### Corollary 15.2

Let  $V$  be a finite dimensional vector space over  $F$ ,  $\dim V = n$  if  $T : V \rightarrow V$  linear has  $n$  distinct eigenvalues, then  $T$  is diagonalizable. The converse is false, e.g.,  $T = 1_V$ .

### Corollary 15.3

If  $V$  is a finite dimensional space over  $F$ ,  $\dim V = n$ ,  $T : V \rightarrow V$  linear, then  $T$  has at most  $n$  distinct eigenvalues. This also follows as any  $0 \neq f \in F[t]$  has at most  $\deg f$  roots.

### Corollary 15.4

Let  $V$  be a vector space over  $F$ ,  $T : V \rightarrow V$  linear,  $\lambda_1, \dots, \lambda_n$  distinct eigenvalues of  $T$ . Set

$$w = E_T(\lambda_1) + \dots + E_T(\lambda_n)$$

if  $v_i \in E_T(\lambda_i), i = 1, \dots, n$  satisfy

$$v_1 + \dots + v_n = 0$$

then  $v_i = 0, i = 1, \dots, n$ . We write this as

$$W = E_T(\lambda_1) \oplus \dots \oplus E_T(\lambda_n)$$

**Exercise 15.1.** Let  $V$  be a vector space over  $F$ ,  $W_1, \dots, W_n \subset V$  subspaces. Let  $W = W_1 + \dots + W_n$ . Then the followings are equivalent

1. If  $w_i \in W_i, i = 1, \dots, n$  satisfy  $w_1 + \dots + w_n = 0$  then  $w_i = 0 \forall i$ . We say  $W_i$  are indep.
2. If  $v \in W \exists! w_i \in W_i \ni v = w_1 + \dots + w_n$
3.  $W_i \cap \sum_{j \neq i, j=1}^n W_j = 0 \forall i = 1, \dots, n$

4. If  $\mathcal{B}_i$  is a basis for  $W_i, i = 1, \dots, n$  then  $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$  is a basis for  $W$ .

If these hold for  $W$ , we say  $W$  is an (internal) direct sum of the  $W_i$  and write

$$W = W_1 \oplus \dots \oplus W_n$$

**Remark 15.5.** This generalizes to  $W = \oplus W_i$ , general  $I$  – How. What is the proof?

**Exercise 15.2.** Let  $V$  be a vector space over  $F$ ,  $W_1, \dots, W_n \subset V$  subspaces  $\ni V = W_1 + \dots + W_n$ . Let

$$W = W_1 \times \dots \times W_n = \{(W_1, \dots, W_n) | w_i \in W_i \forall i\}$$

a vector space over  $F$  via component wise operations. Show

$$v = W_1 \oplus \dots \oplus W_n \iff T : W_1 \times \dots \times W_n \rightarrow V$$

by  $(w_1, \dots, w_n) \mapsto w_1 + \dots + w_n$  is an isomorphism. We call  $W$  the **external direct sum** of the  $W_i$ .

Consequences: Let  $V$  be a finite dimensional vector space over  $F$ ,  $\lambda_1, \dots, \lambda_n$  distinct eigenvalues of  $T : V \rightarrow V$  linear,  $r_i = \dim E_T(\lambda_i)$ ,  $\mathcal{B}_i$  ordered basis for  $E_T(\lambda_i), i = 1, \dots, n$  if

$$V = E_T(\lambda_1) + \dots + E_T(\lambda_n)$$

then

$$V = E_T(\lambda_1) \oplus \dots \oplus E_T(\lambda_n)$$

and  $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$  is an ordered basis for  $V$  and

$$[T]_{\mathcal{B}} = \begin{pmatrix} [\lambda_1 1_{E_T(\lambda_1)}]_{\mathcal{B}_1} & & \\ & \ddots & \\ & & [\lambda_n 1_{E_T(\lambda_n)}]_{\mathcal{B}_n} \end{pmatrix}$$

(Block form) is a diagonal matrix. In particular,

$$f_T = \det(T 1_V - T) = (t - \lambda_1)^{r_1} \dots (t - \lambda_n)^{r_n}$$

By determinant theory,

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det A \det B$$

$A, B$  square matrices and  $T$  is diagonalizable.

**Remark 15.6.**  $T : V \rightarrow V$  linear may or may not have eigenvalues

1.  $V = \mathbb{R}^2, f_T = t^2 + 1$ , then  $T$  has not eigenvalues.
2. If  $V$  is a finite dimensional vector space over  $\mathbb{C}$ , then  $T$  has an eigenvalue as  $f_T$  has a root by the FUNDAMENTAL THEOREM OF ALGEBRA (which we shall always assume to be true).



## §15.2 Inner Product Space

We know that the dot product of vectors in  $\mathbb{R}^3$  allows us to define  $\perp$ ,  $\angle$ , distance, etc. We want to generalize this to “inner product spaces”. When we talk about inner product spaces, we shall always assume that OUR FIELD  $F$  LIES in  $\mathbb{C}$  (e.g.,  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ ) as a subfield. Let  $- : \mathbb{C} \rightarrow \mathbb{C}$  by  $\alpha + \beta\sqrt{-1} \mapsto \alpha - \beta\sqrt{-1} \forall \alpha, \beta \in \mathbb{R}$  denoted **complex conjugation**.

Note: Let  $a = \alpha + \beta\sqrt{-1}$  in  $\mathbb{C}$ ,  $\alpha, \beta \in \mathbb{R}$ . Then

1.  $a = \bar{a}$  iff  $a \in \mathbb{R}$
2.  $\overline{\bar{a}}$
3.  $|a|^2 := a\bar{a} \geq 0$  in  $\mathbb{R}$  as  $a\bar{a} = \alpha^2 + \beta^2$  and  $= 0$  iff  $a = 0$ .

As we shall assume  $F \subset \mathbb{C}$ , we define:

$$\bar{F} := \{\bar{z} \in \mathbb{C} | z \in F\}$$

and we shall also assume that

$$F = \bar{F}$$

This is true if  $F \subset \mathbb{R}$  or  $F = \mathbb{C}$ , but does not always hold UNLESS we only consider those  $F$  that do which we will.

**Definition 15.7 (Inner Product Space)** — Let  $F \subset \mathbb{C}$  be a subfield satisfying  $F = \bar{F}$ ,  $V$  a vector space over  $F$ . We call  $V$  an inner product space over  $F$ , write  $V$  is an ips /  $F$ , under the map

$$\langle, \rangle := \langle, \rangle_V : V \times V \rightarrow F$$

Write:  $\langle v, w \rangle$  for  $\langle, \rangle(v, w)$  if  $\langle, \rangle$  satisfies  $\forall v_1, v_2, v_3, v \in V, \forall \alpha \in F$

1.  $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$
2.  $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$
3.  $\langle \alpha v_1, v_2 \rangle = \alpha \langle v_1, v_2 \rangle = \langle v_1, \bar{\alpha} v_2 \rangle$
4.  $\langle v, v \rangle \in \mathbb{R}$  and  $\langle v, v \rangle \geq 0$  with  $\langle v, v \rangle = 0$  iff  $v = 0$ .

If  $V$  is an inner product space over  $F$  (under  $\langle, \rangle$ ), the LENGTH (or NORM or MAGNITUDE) of  $v \in V$  is given by

$$\|v\| := \sqrt{\langle v, v \rangle} \geq 0 \in \mathbb{R}$$

Note: If  $F \subset \mathbb{C}$ ,  $\|v\|^2 \in F$ , but it is possible that  $\|v\| \notin F$ , e.g., if  $V = \mathbb{Q}^2$  a vector space over  $\mathbb{Q}$  and an inner product space over  $\mathbb{Q}$  under the dot product  $\|(1, 1)\| = \sqrt{2} \notin \mathbb{Q}$ . This is a reason to work only with  $F = \mathbb{R}$  or  $\mathbb{C}$ .

## §16 | Lec 16: Nov 6, 2020

### §16.1 Lec 15 (Cont'd)

Properties: Let  $V$  be an inner product space over  $F$ ,  $\alpha \in F, v_1, v_2, v_3 \in V$ .

1.  $\langle 0, v \rangle = 0 = \langle w, 0 \rangle, \forall v, w \in V$ .
2.
  - $\langle \alpha v_1 + v_2, v_3 \rangle = \alpha \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$
  - $\langle v_1, \alpha v_2 + v_3 \rangle = \bar{\alpha} \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle$
3. If  $F \subset \mathbb{R}$  define the ANGLE  $\theta, 0 \leq \theta \leq 2\pi$  between  $v_1 \neq 0$  and  $v_2 \neq 0$  in  $V$  by

$$\cos \theta := \frac{\langle v_1, v_2 \rangle}{\|v_1\| \|v_2\|}$$

and if  $F \not\subset \mathbb{R}$  define  $\theta$  by

$$\cos \theta := \frac{|\langle v_1, v_2 \rangle|}{\|v_1\| \|v_2\|}$$

Note: This does not make sense yet, and will not until we show

$$\frac{|\langle v_1, v_2 \rangle|}{\|v_1\| \|v_2\|} \leq 1 \quad \text{for } v_1 \neq 0, v_2 \neq 0$$

4. (very useful prop) Let  $v \in V$ . If  $\langle v, w \rangle = 0, \forall w \in V$  (or  $\langle w, v \rangle = 0 \forall w \in W$ ), then  $v = 0$ .
5. Let  $0 \neq x \in V$ . Then

$$\langle, x \rangle : V \rightarrow F \text{ by } v \mapsto \langle v, x \rangle$$

is a linear transformation, i.e., linear functional, i.e.,  $\langle, x \rangle \in V^*$ . However,

$$\langle x, \rangle : V \rightarrow F \text{ by } v \mapsto \langle x, v \rangle$$

is linear iff  $F \subset \mathbb{R}$ . In general, we say that  $\langle x, \rangle$  is **SESQUILINEAR** as  $\forall \alpha \in F, \forall v_1, v_2 \in V$

$$\langle x, \alpha v_1 + v_2 \rangle = \bar{\alpha} \langle x, v_1 \rangle + \langle x, v_2 \rangle$$

Of course if  $x = 0, \langle 0, \rangle \langle, 0 \rangle \in V^*$ .

### Example 16.1

Let  $F \subset \mathbb{C}, F = \bar{F} = \{\bar{\alpha} | \alpha \in F\}$ . The following  $V$  vector space over  $F$  are inner product space over  $F$  under the given  $\langle, \rangle$ :

1.  $V = F^n$  and  $\langle, \rangle = \underbrace{\cdot}_{\text{dot product}}$ , i.e., if

$$v = (\alpha_1, \dots, \alpha_n), w = (\beta_1, \dots, \beta_n), \alpha_i, \beta_i \in F, \forall i, j$$

Then,

$$\langle v, w \rangle = \sum_{i=1}^n \alpha_i \bar{\beta}_i$$

Note: If  $F \subset \mathbb{R}$ , then

$$\langle v, w \rangle = \sum_{i=1}^n \alpha_i \beta_i$$

2. Let  $I = [\alpha, \beta]$ ,  $\alpha < \beta$  in  $\mathbb{R}$ ,  $V = C(I)$  with  $C(I) = \{f : I \rightarrow \mathbb{R} | f \text{ cont}\}$  then

$$\langle f, g \rangle := \int_{\alpha}^{\beta} fg$$

Think about what if  $C_{\mathbb{C}} := \{f : I \rightarrow \mathbb{C} | f \text{ cont}\}$ .

3. In 2), let  $h \in C(I)$  satisfy  $h(x) > 0 \forall x \in I$ . Then

$$\langle f, g \rangle_h := \int_{\alpha}^{\beta} hfg$$

the WEIGHTED INNER PRODUCT SPACE via  $h$ .

4. Let  $A \in M_n F$ . Define the adjoint of  $A$  to be  $A^*$  where

$$(A^*)_{ij} := \overline{A_{ji}}, \quad \forall i, j$$

the conjugate transpose of  $A$ , i.e.,  $A^* = \overline{A}^{\top}$ . So if  $F \subset \mathbb{R}$ ,  $A^* = A^{\top}$ .

**Remark 16.2.** If  $A = F^{m \times n}$ , then  $A^*$  defined by  $(A^*)_{ij} = \overline{A_{ji}}$  still makes sense and is called the ADJOINT of  $A$ . What can you say about  $AA^*$  and  $A^*A$ ?

Let  $V = M_n F$  under

$$\langle A, B \rangle := \text{tr}(AB^*)$$

where  $\text{tr } C = \sum_{i=1}^n C_{ii}$ . So if  $F \subset \mathbb{R}$ ,  $\langle A, B \rangle = \text{tr}(AB^{\top})$ .

tr=trace

**Example 16.3** 5. Let  $F = \mathbb{R}$

$$l_2 := \left\{ (a_0, a_1, \dots, a_n, \dots) \mid a_i \in \mathbb{R} \forall i - \text{infinite seq with } \sum a_i^2 < \infty \right\}$$

a vector space over  $F$  by component wise operation ( a subspace of  $\mathbb{R}_{\text{inf}}^{\infty}$  – see below) and an inner product space over  $\mathbb{R}$  via

$$\langle v, w \rangle := \sum_{i=0}^{\infty} a_i b_i \in \mathbb{R}$$

if  $v = (a_0, a_1, \dots), w = (b_0, b_1, \dots)$

$$0 \leq (a_i \pm b_i)^2 = a_i^2 \pm 2a_i b_i + b_i^2, \forall i \text{ so}$$

$$\mp 2 \sum_{i=0}^{\infty} a_i b_i \leq \sum_{i=0}^{\infty} a_i^2 + \sum_{i=0}^{\infty} b_i^2 < \infty$$

**Theorem 16.4**

Let  $V$  be an inner product space over  $F$ . Then  $\forall v_1, v_2 \in V, \forall \alpha \in F$ , we have

1.  $\|v_1\| \in \mathbb{R}$  with  $\|v_1\| \geq 0$  and  $\|v_1\| = 0$  iff  $v_1 = 0$ .
2.  $\|\alpha v_1\| = |\alpha| \|v_1\|$ .
3. Cauchy – Schwarz Inequality

$$|\langle v_1, v_2 \rangle| \leq \|v_1\| \|v_2\|$$

4. Minkowski Inequality(special case)

$$\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$$

*Proof.* 1) and 2) are left as exercise.

3) If  $v_1 = 0$  or  $v_2 = 0$ , the result is immediate, so we may assume that  $v_1 \neq 0, v_2 \neq 0$ . We use the following important trick. Take the orthogonal projection. Let

$$v = v_2 - \underbrace{\frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1}_{\text{orthogonal projection on } v_1}$$

**Claim 16.1.**  $\langle v, \alpha v_1 \rangle = 0 \forall \alpha \in F$  (i.e.,  $v \perp \alpha v_1$ )

$$\begin{aligned} \langle v, \alpha v_1 \rangle &= \langle v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1, \alpha v_1 \rangle \\ &= \langle v_2, \alpha v_1 \rangle + \langle -\frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1, \alpha v_1 \rangle \\ &= \overline{\alpha} \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \langle v_1, \alpha v_1 \rangle \\ &= \overline{\alpha} \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \overline{\alpha} \|v_1\|^2 = 0 \end{aligned}$$

establishing the claim. Therefore, we have

$$\begin{aligned} 0 &\leq \langle v, v \rangle = \langle v, v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1 \rangle \\ &= \langle v, v_2 \rangle + \langle v_1 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1 \rangle = \langle v, v_2 \rangle \\ &= \langle v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1, v_2 \rangle = \langle v_2, v_2 \rangle - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \langle v_1, v_2 \rangle \\ &= \|v_2\|^2 - \frac{\overline{\langle v_1, v_2 \rangle}}{\|v_1\|^2} \langle v_1, v_2 \rangle = \|v_2\|^2 - \frac{|\langle v_1, v_2 \rangle|^2}{\|v_1\|^2} \end{aligned}$$

So

$$|\langle v_1, v_2 \rangle|^2 \leq \|v_1\|^2 \|v_2\|^2$$

or

$$|\langle v_1, v_2 \rangle| \leq \|v_1\| \|v_2\|$$

as required. □

*Proof.* 4.

$$\begin{aligned} \|v_1 + v_2\|^2 &= \langle v_1 + v_2, v_1 + v_2 \rangle \\ &= \|v_1\|^2 + \langle v_1, v_2 \rangle + \langle v_2, v_1 \rangle + \|v_2\|^2 \\ &= \|v_1\|^2 + \langle v_1, v_2 \rangle + \overline{\langle v_1, v_2 \rangle} + \|v_2\|^2 \end{aligned}$$

Let  $\langle v_1, v_2 \rangle = \alpha + \beta\sqrt{-1}$ ,  $\alpha, \beta \in \mathbb{R}$ . Then

$$\begin{aligned} \|v_1 + v_2\|^2 &= \|v_1\|^2 + 2\alpha + \|v_2\|^2 \\ &\leq \|v_1\|^2 + 2\sqrt{\alpha^2 + \beta^2} + \|v_2\|^2 \\ &= \|v_1\|^2 + 2|\langle v_1, v_2 \rangle| + \|v_2\|^2 \\ &\leq (\|v_1\| + \|v_2\|)^2 \end{aligned}$$

So,  $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ . □

## §17 | Lec 17: Nov 9, 2020

### §17.1 Lec 16 (Cont'd)

#### Example 17.1

Let  $V$  be an inner product space over  $F$

1.  $|\alpha_1\beta_1 + \dots + \alpha_n\beta_n| \leq \sqrt{\sum_{i=1}^n \alpha_i^2} \sqrt{\sum_{i=1}^n \beta_i^2}, \forall \alpha_i, \beta_i \in \mathbb{R}.$
2.  $\int_{\alpha}^{\beta} fg \leq \sqrt{\int_{\alpha}^{\beta} f^2} \sqrt{\int_{\alpha}^{\beta} g^2}, \forall f, g \in C[\alpha, \beta].$
3.  $\angle$  between nonzero vectors in  $V$  makes sense.

4. Distance between (end pts) vectors makes sense by the following:

If  $V$  is an inner product space over  $F$ , define the distance between  $v_1, v_2 \in V$  by

$$d(v_1, v_2) := \|v_1 - v_2\| \geq 0 \in \mathbb{R}$$

Then  $d$  satisfies  $\forall v, w, x \in V$

- $d(v, w) \geq 0 \in \mathbb{R}$  and  $d(v, w) = 0$  iff  $v = w$ .
- $d(v, w) = d(w, v)$
- Triangle inequality

$$d(v, x) \leq d(v, w) + d(w, x)$$

We call  $V$  a METRIC SPACE under  $d$ .

**Example 17.2 (Metric Space)**

If  $v = (\alpha_1, \dots, \alpha_n)$ ,  $w = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$  under the dot product, then

$$d(v, w) = \sqrt{(\alpha_1 - \beta_1)^2 + \dots + (\alpha_n - \beta_n)^2}$$

**§17.2 Orthogonal Bases**

Motivation: in  $\mathbb{R}^n$  (or  $\mathbb{C}^n$ ),  $\mathcal{S} = \mathcal{S}_n = \{e_1, \dots, e_n\}$  the standard basis satisfies

$$e_i \cdot e_j = \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \forall i, j \\ 0, & \text{if } i \neq j \end{cases}$$

Goal: Let  $V$  be a finite dimensional inner product space over  $F$ ,  $F = \mathbb{R}$  or  $\mathbb{C}$ . Find a basis  $\mathcal{B} = \{v_1, \dots, v_n\}$  for  $V \ni$

$$\langle v_i, v_j \rangle = \delta_{ij}, \forall i, j \quad (*)$$

if we only want bases  $\mathcal{C} = \{w_1, \dots, w_n\}$  for  $V \ni$

$$\langle w_i, w_j \rangle = 0 \forall i \neq j,$$

we can work with any subfield  $F \subset \mathbb{C}$  with  $F = \overline{F}$ , since we do not need  $\|w_i\| \in F$  for such a  $\mathcal{C}$ .

**Example 17.3**

In  $\mathbb{R}^2$ , let  $0 \leq \theta < 2\pi$  be fixed. Then

$$\mathcal{C}_\theta = \{(\cos \theta, \sin \theta), (-\sin \theta, \cos \theta)\}$$

satisfies (\*)

**Definition 17.4 (Orthonormal/Orthogonal)** — Let  $V$  be an inner product space over  $F$ ,  $\emptyset \neq S \subset V$  a subset. We say

1.  $S$  is ORTHOGONAL (or OR) if

$$\langle v, w \rangle = 0 \forall v \neq w \in S$$

2. If  $S$  is an OR set, we call it ORTHONORMAL (or ON) if, in addition  $\|v\| = 1 \forall v \in S$ .
3. An OR set is called an OR basis if, in addition, it is a basis for  $V$ .
4. If  $v, w \in V$ , we say  $v, w$  are orthogonal or perpendicular if  $\langle v, w \rangle = 0$  write  $v \perp w$ . (equivalently  $\langle w, v \rangle = 0$ )

Goal: If  $F \subset \mathbb{C}$  is a subfield (and  $F = \overline{F}$ ),  $V$  a finite dimensional inner product space over  $F$ , then  $V$  has an OR bases and an ON bases if  $F = \mathbb{R}$  or  $\mathbb{C}$ .

**Remark 17.5.** Let  $V$  be an inner product space over  $F$ ,  $x, y \in V$ .

1.  $0 \perp x$
2.  $x \perp y$  iff  $y \perp x$
3.  $0$  is the only vector perpendicular to all  $z \in V$ .

### Theorem 17.6

Let  $V$  be an inner product space over  $F$ ,  $S \subset V$  an OR set. Suppose that  $0 \neq S$ , then  $S$  is linearly indep. If, in addition,  $V$  is a finite dimensional inner product space over  $F$  and  $|S| = \dim V$ , then  $S$  is an OR basis for  $V$ .

*Proof.* Let  $v \in \text{Span}(S)$ . Then  $\exists$  (distinct)  $v_1, \dots, v_n \in S, \alpha_1, \dots, \alpha_n \in F \ni$

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

We have

$$\begin{aligned} \langle v, v_j \rangle &= \langle \alpha_1 v_1 + \dots + \alpha_n v_n \rangle \\ &= \sum_{i=1}^n \alpha_i \langle v_i, v_j \rangle \\ &= \sum_{i=1}^n \alpha_i \delta_{ij} \|v_j\|^2 = \alpha_j \|v_j\|^2 \end{aligned}$$

This is so useful, we record it as

Crucial Equation: If  $\{v_1, \dots, v_n\}, \alpha_1, \dots, \alpha_n \in F$  then

$$\alpha_j = \frac{\langle v, v_j \rangle}{\|v_j\|^2}, j = 1, \dots, n$$

Note: If  $V$  is not necessarily finite dimensional and  $S$  is an OR set not containing  $0$ , the same holds.

Now, suppose that  $v = 0$ , i.e.,

$$0 = \alpha_1 v_1 + \dots + \alpha_n v_n$$

so

$$\alpha_j = \frac{\langle v, v_j \rangle}{\|v_j\|^2} = \frac{\langle 0, v_j \rangle}{\|v_j\|^2} = 0, j = 1, \dots, n$$

and the result follows. □

Note: If  $\mathcal{B} = \{v_1, \dots, v_n\}$  is an OR set,  $v_i \neq 0 \forall i, V = \text{Span} \mathcal{B}$ , hence a basis for  $V$  then

$$\frac{\langle v, v_j \rangle}{\|v_j\|^2}$$

is the  $j$ th coordinate of  $v$  on  $v_j$  and

$$v = \sum_{j=1}^n \frac{\langle v, v_j \rangle}{\|v_j\|^2}$$

If, in addition,  $\|v_j\| \in F \forall j$ , then

$$\mathcal{C} = \left\{ \frac{v_1}{\|v_1\|}, \dots, \frac{v_n}{\|v_n\|} \right\}$$

is an ON basis and  $\forall v \in V$ .

$$v = \sum_{j=1}^n \frac{\langle v, v_j \rangle}{\|v_j\|^2} v_j = \sum_{j=1}^n \langle v, \frac{v_j}{\|v_j\|} \rangle \frac{v_j}{\|v_j\|}$$

Hence if  $w_i = \frac{v_i}{\|v_i\|}$ ,  $i = 1, \dots, n$ ,  $\mathcal{C} = \{w_1, \dots, w_n\}$  is an ON basis and

$$v = \sum_{i=1}^n \langle v, w_i \rangle w_i$$

i.e.,  $\langle v, w_i \rangle$  is the coordinate of  $v$  and  $w_i$  for each  $i$ .

**Remark 17.7.** Does this look familiar?

1. Look at the proof of the Cauchy – Schwarz Inequality
2. Let  $\mathcal{B} = \{v_1, \dots, v_n\}$  be an OR basis for  $V$  a finite dimensional inner product space over  $F$  and

$$\mathcal{B}^* = \{f_1, \dots, f_n\}$$

the dual basis for  $V^* = L(V, F)$ . So,  $f_i(v_j) = \delta_{ij}$ ,  $\forall i, j$ . Then  $f_i : V \rightarrow F$  is  $f_i(v) = \frac{\langle v, v_i \rangle}{\|v_i\|^2}$ ,  $i = 1, \dots, n$  by Crucial Equation:

$$f_i = \langle -, \frac{v_i}{\|v_i\|^2} \rangle : V \rightarrow F$$

and if  $\mathcal{C} = \{w_1, \dots, w_n\}$  is an ON basis then

$$\begin{aligned} f_i &= \langle -, w_i \rangle \in \mathcal{C}^* \\ f_i(v) &= \langle v, w_i \rangle \end{aligned}$$

i.e., we can associate a vector in  $V$  to a linear functional.

### Theorem 17.8

Let  $V$  be an inner product space over  $F$ ,  $\mathcal{B}$  an OR basis for  $V$ ,  $v \in V$ . Then  $\langle v, w \rangle = 0$  for all but finitely many  $w \in \mathcal{B}$  and

$$v = \sum_{\mathcal{B}} \frac{\langle v, w \rangle}{\|w\|^2} w$$

is a finite sum. If, in addition,  $\mathcal{B}$  is ON, then this becomes

$$v = \sum_{\mathcal{B}} \langle v, w \rangle w$$



**Corollary 17.9** (Parseval's Equation)

Let  $V$  be a finite dimensional inner product space over  $F$  with ON basis  $\{v_1, \dots, v_n\}$  and  $v, w \in V$ . Then

$$\langle v, w \rangle = \sum_{i=1}^n \langle v, v_i \rangle \overline{\langle w, v_i \rangle}$$

In particular,

$$\|v\|^2 = \sum_{i=1}^n |\langle v, v_i \rangle|^2, \quad (\text{Pythagorean Theorem})$$

*Proof.* Hw – Take home. □

## §18 | Veterans Day: Nov 11, 2020

No class :D

## §19 | Lec 18: Nov 16, 2020

### §19.1 Lec 17 (Cont'd)

**Example 19.1**

Let  $V = C[0, 2\pi]$  an inner product space over  $\mathbb{R}$  via

$$\langle f, g \rangle := \int_0^{2\pi} fg$$

Let  $u_0 = \frac{1}{\sqrt{2\pi}}, u_{2n} = \frac{1}{\sqrt{\pi}} \sin nx, u_{2n+1} = \frac{1}{\sqrt{\pi}} \cos nx$  for all  $n \in \mathbb{Z}^+$  and set

$$S = \{u_i | i \geq 0\}$$

By calculus

$$\langle u_i, u_j \rangle = \int_0^{2\pi} u_i u_j = \delta_{ij}, \forall i, j$$

So  $S$  is ON hence linearly indep ( $0 \notin S$ ) and a ON basis for  $\text{Span } S$ .

Note: Vectors in span  $S$  are finite linear combos of vectors in  $S$ . In particular,  $C[0, 2\pi]$  is infinite dimensional (and  $\text{Span } S < C[0, 2\pi]$  is a subspace). In calculus, you studied convergent series, a convergent series

$$\sum_{i=0}^{\infty} \alpha_i u_i \tag{*}$$

is called a FOURIER SERIES, the  $\alpha_i$  Fourier coefficients.

Warning:  $S = \mathcal{B} = \cup \mathcal{B}_n, \mathcal{B}_n = \{u_i | i = 0, \dots, 2n+1\}$  is ON but not a basis for  $C[0, 2\pi]$  or even

$$V = \{f \in C[0, 2\pi] | f \text{ converges to its Fourier series}\}$$

It can be shown that  $C'[0, 2\pi] \subset V$ .

Note: No one knows a precise basis for  $C[0, 2\pi]$  although it exists by axioms.

- Remark 19.2.**
1. One can modify the interval  $[0, 2\pi]$  in the above with appropriate changes to the  $u_i$ .
  2. Infinite ON sets are very useful.

To solve our goal about finite dimensional inner product space over  $F$ , we know show:

### Theorem 19.3 (Gram-Schmidt)

Let  $V$  be an inner product space over  $F$  and  $\emptyset \neq S_n = \{v_1, \dots, v_n\} \subset V$  a linearly indep. set. Then  $\exists y_1, \dots, y_n \in V \ni$

- $y_1 = v_1$
- $T_n = \{y_1, \dots, y_n\}$  is an OR set and linearly indep.
- $\text{Span } T_n = \text{Span } S_n$

*Proof.* We construct  $T_n$  from  $S_n$ . This construction is called the Gram – Schmidt process.  $n = 1$  is clear. We proceed by induction. We may assume we have done the  $S_n$  case, i.e.,

1.  $y_1, \dots, y_n \in V, y_1 = v_1, y_i \neq 0, i = 1, \dots, n$
2.  $T_n = \{y_1, \dots, y_n\}$  is OR. (hence linearly indep. as  $0 \notin T_n$ )
3.  $\text{Span } S_n = \text{Span}\{y_1, \dots, y_n\}$
4. Must extend this to the case of  $n + 1$ .

As in the proof of GS (where we threw away one orthogonal complement), we subtract an ORTHOGONAL PROJECTION figure here Define:

$$y_{n+1} = v_{n+1} - \sum_{k=1}^n \frac{\langle v_{n+1}, y_k \rangle}{\|y_k\|^2} y_k \quad (*)$$

**Claim 19.1.**  $y_{n+1} \neq 0$  : if  $y_{n+1} = 0$ , then  $v_{n+1} \in \text{Span } T_n = \text{Span}(v_1, \dots, v_n)$  contradicting  $S$ ?, is linearly indep. So  $y_{n+1} \neq 0$

**Claim 19.2.**  $\langle y_{n+1}, y_j \rangle = 0, j = 1, \dots, n$

$$\begin{aligned}
 \langle y_{n+1}, y_j \rangle &= \langle v_{n+1} - \sum_{k=1}^n \frac{\langle v_{n+1}, y_k \rangle}{\|y_k\|^2} y_k, y_j \rangle \\
 &= \langle v_{n+1}, y_j \rangle - \sum_{k=1}^n \frac{\langle v_{n+1}, y_k \rangle}{\|y_k\|^2} \langle y_k, y_j \rangle \\
 &= \langle v_{n+1}, y_j \rangle - \sum_{k=1}^n \frac{\langle v_{n+1}, y_k \rangle}{\|y_k\|^2} \delta_{kj} \|y_j\|^2 \\
 &= \langle v_{n+1}, y_j \rangle - \langle v_{n+1}, y_j \rangle = 0
 \end{aligned}$$

This prove the above claim.

Since  $0 \notin T_{n+1} = \{y_1, \dots, y_{n+1}\}$  and  $T_{n+1}$  is OR, it is linearly indep. As  $\text{Span } T_n = \text{Span}\{v_1, \dots, v_n\}$  and  $\{v_1, \dots, v_{n+1}\}$  is linearly indep.

$$\text{Span } T_{n+1} = \text{Span}(v_{n+1}, y_1, \dots, y_n) = \text{Span}(v_1, \dots, v_{n+1})$$

by the Replacement Theorem and (\*). The theorem follows by induction.  $\square$

#### Theorem 19.4 (Orthogonal)

Let  $V$  be a finite dimensional inner product space over  $F$ . Then  $V$  has an OR basis. If  $F = \mathbb{R}$  or  $\mathbb{C}$ , then  $V$  has an ON basis.

*Proof.* Any basis for  $V$  can be converted to an OR basis  $\mathcal{C}$  for  $V$  by the GS process if  $V$  is finite dimensional if  $F = \mathbb{R}$  or  $\mathbb{C}$ , then  $\left\{ \frac{v}{\|v\|} \mid v \in \mathcal{C} \right\}$  is an ON basis for  $V$  as  $\|v\| \in \mathbb{R} \forall v \in \mathcal{C}$   $\square$

**Remark 19.5.** Let  $V = \mathbb{Q}^2$  a finite dimensional inner product space over  $\mathbb{Q}$  with inner product defined by

$$\langle (\alpha_1, \alpha_2), (\beta_1, \beta_2) \rangle_{\frac{1}{3}} := \frac{1}{3}(\alpha_1\beta_1 + \alpha_2\beta_2)$$

i.e., WEIGHTED DOT PRODUCT by  $\frac{1}{3}$ . Then  $V$  has an OR basis but not any ON basis

$\left\| \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right\|_{\frac{1}{3}} \notin \mathbb{Q}$  as  $3b_1^2b_2^2 = a_1^2b_2^2 + b_1^2a_2^2$  has no solution in  $\mathbb{Z}$ .

## §19.2 Examples – Computation

**Example 19.6** 1.  $V = \mathbb{R}^3$  under  $\langle, \rangle = \text{dot product}$  with  $v_1 = (1, 1, 1), v_2 = (1, 1, 0), v_3 = (1, 0, 1)$ . GS  $v_1, v_2, v_3$  to an OR basis and then to an ON ba-

sis:

$$y_1 = (1, 1, 1)$$

$$y_2 = v_2 - \frac{v_2 \cdot y_1}{\|y_1\|^2} y_1$$

... some boring calculation – can refer online notes/textbook

Note:

1. It is easier to guess.
2. If instead of  $F = \mathbb{R}$ , we had  $F = \mathbb{Q}$ , we could not get an ON basis after GS-ing.

### Example 19.7

$V = \mathbb{R}[x]$  (polynomial function) via

$$\langle f, g \rangle := \int_{-1}^1 fg$$

$\mathcal{B}_n = \{x^i | 0 \leq i \leq n\}$  is a basis for  $\mathbb{R}[x]_n$ . GS,  $\mathcal{B}_n$  to an OR basis, at least start

$$g_0 = 1$$

$$g_1 = x - \frac{\langle x, 1 \rangle}{\|1\|^2} 1 = x - \frac{\int_{-1}^1 x}{\int_{-1}^1 1} = x$$

$$g_2 = x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2} 1 - \frac{\langle x^2, x \rangle}{\|x\|^2} x$$

$$= x^2 - \frac{\int_{-1}^1 x^2}{\int_{-1}^1 1} - \frac{\int_{-1}^1 x^3}{\int_{-1}^1 x^2} x = x^2 - \frac{1}{3}$$

$$\vdots$$

The  $g_i$  are called LEGENDRE POLYNOMIALS. You can normalize them, i.e., form  $\frac{g_i}{\|g_i\|}$  to get an ON set.

These are important polynomials,  $g_n$  satisfies the ODE

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

These occur in physics, e.g., converting Laplace's Equation  $\nabla^2 g = 0$  into spherical coordinates in some cases in quantum mechanics in the solution of Schrodinger's Eqn for the hydrogen atom.

Flow of an (ideal fluid) past a sphere. Determination of the electric fluid due to a charged sphere. Determination of the temperature distribution in a sphere given its surface temperature. Computing  $g'_n$ s by GS is too difficult. There are many formulas to determine the  $g'_n$ s. Many arise by proving the following recurrence relation:

Rodriguez Representation:

$$g_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Some of these are, using the appropriate ? of the binomial coefficient

$$\binom{n}{m} := \frac{n!}{m!(n-m)!}, 0 \leq m \leq n :$$

let  $M = \frac{n}{2}$  or  $\frac{n-1}{2}$  whichever one is an integer, i.e.,  $\lfloor \frac{n}{2} \rfloor = \text{greatest integer} \leq \frac{n}{2}$ .

$$\begin{aligned} g_n &= 2^{\frac{1}{n}} \sum_{m=0}^M (-1)^m \frac{(2n-2m)!}{m!(n-m)!(n-2m)!} x^{n-2m} \\ &= 2^n \sum_{k=0}^n \binom{n}{k}^2 (x-1)^{n-k} (x+1)^k \\ &= \sum_{k=0}^n \binom{n}{k} \binom{-n-1}{k} \left(\frac{1-x}{2}\right)^k \end{aligned}$$

## §20 | Lec 19: Nov 18, 2020

### §20.1 Lec 18(Cont'd)

Note: Gamma function:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

where  $z$  is complex and  $\text{Re}(z) > 0$  and  $\Gamma(n) = (n-1)!, \forall n > 1,$ .

3. GS  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$  in  $M_2(\mathbb{R})$  under

$$\langle A, B \rangle = \text{tr } AB^*$$

$$\begin{aligned} y_1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ y_2 &= \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} - \frac{\text{tr} \left( \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^* \right)}{\text{tr} \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^* \right)} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ y_2 &= \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} - \frac{\text{tr} \left( \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right)}{\text{tr} \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right)} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

4.  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  rotation counterclockwise by  $\angle\theta$  about a vector  $0 \neq v_1$  as axis. Find  $T(\alpha, \beta, \gamma)$  i.e.,  $[T]_{\mathcal{S}}$  complete  $v_1$  to a basis  $\{v_1, v_2, v_3\}$  for  $\mathbb{R}^3$ . GS it to an OR basis, then an ON basis  $\mathcal{C}$ . Compute  $[T]_{\mathcal{C}}$ . Then use Change of Basis to compute  $[T]_l$  or guess  $v_2$ , normalize  $v_1, v_2$  to  $v'_1, v'_2$  then  $v_3 \subset v'_1 \times v'_2$ .

Note: If you have a basis with vectors of different lengths, it is hard to compute in this basis. If each vector in your OR basis has the same length  $r$ , you can compute.

## §20.2 Orthogonal Polynomials

There are many interesting infinite sets of orthogonal polys  $\{f_n\}_{n \in \mathbb{Z}^+}$ . They often arise as relate  $\alpha$  to the HYPERGEOMETRIC ODE

$$z(1-z)\frac{d^2y}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{dy}{dz} - \alpha\beta y = 0$$

where  $z$  is a complex variable,  $y = y(z)$ ,  $\alpha, \beta, \gamma \in \mathbb{C}$ . They arise as OR sets or weighted inner product space over  $\mathbb{R}$  ( or  $\mathbb{C}$  on an interval  $[a, b]$  (or variant).

$$\int_a^b f g w = \langle f, g \rangle_w$$

where  $w > 0$  in  $[a, b]$ .

- A very general such is the OR set of JACOBI POLYNOMIALS  $\{P_n^{\alpha, \beta}\}$  under the weighted inner product space

$$\langle f, g \rangle_w = \int_{-1}^1 f g w$$

and

$$w = \frac{(1-x)^\alpha (1+x)^\beta}{\langle \alpha, \beta \rangle - 1}$$

Often such OR sets are not orthonormalized but rather normalized “by dividing by  $P_n^{\alpha, \beta}(1)$ ”. In this case,  $P_n^{\alpha, \beta}(1) = \binom{n+\alpha}{n}$ . The  $P_n^{\alpha, \beta}$  are solutions to the ODE.

$$0 = (1-x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + n(n + \alpha + \beta - 1)y$$

used in Wigner d-matrix theory in quantum mechanics. There are many special cases of Jacobi polys.

1. Gegenbauer polys (ultra-symmetric) polynomials,  $C_n^{(\alpha)}$  where

$$\begin{aligned} w &= (1-x^2)^{\alpha-\frac{1}{2}} \\ C_n^{(\alpha)} &= P_n^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})} \\ (1-x^2)y'' - (2\alpha+1)xy' + n(n+2\alpha)y &= 0 \end{aligned}$$

potential theory, harmonics analysis, Newtonian’s potential.

2. Legendre polys. There are a special case of Gegenbauer polys, namely

$$\begin{aligned} w &= 1 \\ C_n^{\frac{1}{2}} &= P_n^{\frac{1}{2}, \frac{1}{2}} \\ ((1-x^2)y')' + n(n+1)y &= 0 \end{aligned}$$

3. Chebychev polys come in two kinds:  $T_n, U_n$

$$w = \frac{1}{\sqrt{1-x^2}}$$

$$T_n = P_n^{(-\frac{1}{2}, -\frac{1}{2})}$$

$$U_n = P_n^{(\frac{1}{2}, \frac{1}{2})}$$

$$(1-x^2)y'' - xy' + n^2y = 0$$

$$(1-x^2)y'' - 3xy' + n(n+2)y = 0$$

Least square fit, optimal control, numerical analysis.

- Laguerre polys  $L_n^{(\alpha)}$  OR set with  $w_\alpha(x) = x^\alpha e^{-x}$ ,  $\alpha > -1$  in  $\mathbb{R}$  on  $[0, \infty)$

$$xy'' + (\alpha + 1 - x)y' + ny = 0, 0 \neq n \in \mathbb{Z}$$

quantum mechanics, plasma physics.

- HERMITE polys.  $H_n, He_n$

$$w = e^{-x^2}, \text{ for } H_n \text{ on } (-\infty, \infty)$$

$$= e^{-\frac{x^2}{2}}, \text{ for } He_n \text{ on } (-\infty, \infty)$$

( $H_n$  is called physicist Hermite polys and  $He_n$  probabilists Hermite polys).

$$0 = (e^{-\frac{1}{2}x^2}y')' + ne^{-\frac{1}{2}x^2}y = 0$$

probability, numerical analysis, physics.

**Remark 20.1.** Let

$$D = \text{diff} = \frac{d}{dx}, \quad p, q \text{ functions, } w > 0$$

$$L = -\frac{1}{w} (D(pD) + q), \quad \text{a linear operator}$$

Then one wants to solve

$$Lf = \lambda f$$

The solutions are called eigenfunctions in the above they are the eigenfunctions for the given ODEs.

## §20.3 Orthogonal Complement

Notation:  $F \subset \mathbb{C}$  a field satisfying  $F = \overline{F}$ .

**Definition 20.2** (Distance from a Vector to a Set) — Let  $V$  be an inner product space over  $F$ ,  $v_1, v_2 \in V$ . We know that the DISTANCE between  $v_1, v_2$  is defined to be

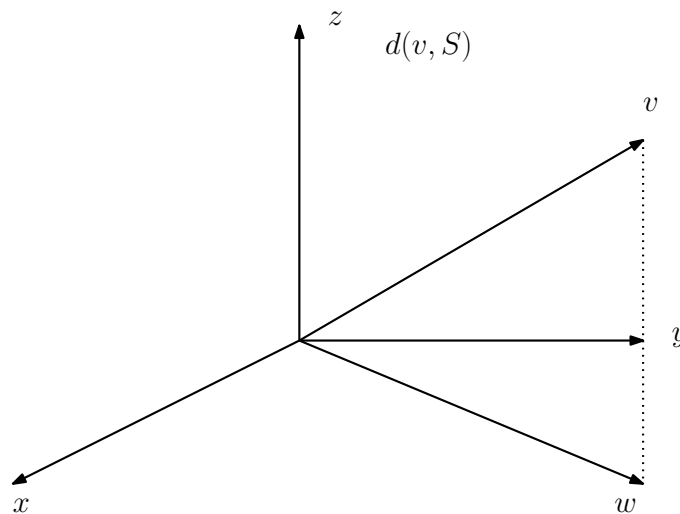
$$d(v_1, v_2) := \|v_1 - v_2\| \geq 0$$

More generally, let  $\emptyset \neq S \subset V$  be a subset and  $v \in V$ . Define the DISTANCE of  $v$  to  $S$  by

$$d(v, S) := \inf \{d(v, w) | w \in S\}$$

if it exists and hence finite.

**Problem 20.1.** Let  $V$  be an inner product space over  $F$ ,  $S \subset V$  a finite dimensional subspaces,  $v \in V$ . Determine



Solution take the orthogonal projection of  $v$  to  $w$  in  $S$

**Definition 20.3** (Orthogonal Complement) — Let  $V$  be an inner product space over  $F$ ,  $\emptyset \neq S \subset V$  a subset of,  $v \in V$ . We say  $v$  is ORTHOGONAL to  $S$ , write  $v \perp S$ , if

$$\langle s, v \rangle = 0, \forall s \in S$$

Set:

$$S^\perp := \{v \in V | v \perp S\}$$

called the ORTHOGONAL COMPLEMENT of  $S$  in  $V$ .



**Remark 20.4.** 1. Compare  $S^\perp$  to  $S^\circ \subset V^*$ , if  $V$  is an arbitrary vector space over  $F$ .

2. In  $\mathbb{R}^3$  (under the dot product)

$$(\text{Span} e_1)^\perp = \text{Span}(e_2, e_3)$$

3. Let  $V$  be an inner product space over  $F$ ,  $\emptyset \neq S \subset V$  a subset, not necessarily a subspace. Then  $S^\perp \subset V$  is a subspace (if  $\emptyset \neq S \subset V$  a subset with  $V$  a vector space over  $F$ ,  $F$  arbitrary, then  $S^\circ \subset V^*$  is a subspace).

*Proof.* Hw. □

4. In 3),  $S \subset S^{\perp\perp} := (S^\perp)^\perp$ :  $S^\perp \subset S^{\perp\perp}$  so  $S \subset S^{\perp\perp}$ . If, in addition,  $S \subset V$  is a subspace and  $V$  is a finite dimensional inner product space over  $F$ , then  $S = S^{\perp\perp}$  (if  $V$  is a finite dimensional vector space over  $F$ ,  $F$  arbitrary  $W \subset V$  a subspace, then  $W = W^{\circ\circ} = (W^\circ)^\circ$ ).

5. Let  $V$  be a finite dimensional inner product space over  $F$ ,  $S = \{v_1, \dots, v_n\}$  an OR basis for  $V$ . Then

$$(\text{Span}(v_1, \dots, v_r))^\perp = \text{Span}(v_{r+1}, \dots, v_n)$$

6. Let  $V$  be an inner product space over  $F$ ,  $S \subset V$  a subspace. Then

$$S \cap S^\perp = 0$$

if  $v \in S \cap S^\perp$ , then  $\langle v, v \rangle = \|v\|^2 = 0$ , so  $v = 0$ . In particular,

$$S + S^\perp = S \oplus S^\perp$$

We write:  $S \oplus S^\perp$  as  $S \perp S^\perp$  to show it is also orthogonal. The key result ( and most important result for use about general inner product space over  $F$  ) is:

### Theorem 20.5 (Orthogonal Decomposition)

Let  $V$  be an inner product space over  $F$ ,  $S \subset V$  a finite dimensional subspace,  $v \in V$ . Then

$$\exists! s \in S, s^\perp \in S^\perp \ni v = s + s^\perp \quad (*)$$

In particular,  $V = S + S^\perp$ ,  $S \cap S^\perp = 0$ , so  $V = S \perp S^\perp$ . Moreover, if

$$v = s + s^\perp, s \in S, s^\perp \in S^\perp$$

then

$$\|v\|^2 = \|s\|^2 + \|s^\perp\|^2, \quad (\text{Pythagorean Theorem})$$

In addition, if  $V$  is a finite dimensional inner product space over  $F$ , then

$$\dim V = \dim S + \dim S^\perp$$

## §21.1 Lec 19 (Cont'd)

*Proof.* By the OR Theorem,  $\exists$  an OR basis  $\mathcal{B} = \{v_1, \dots, v_n\}$  for the finite dimensional inner product space over  $F$   $S$ .

Existence: Let  $v \in V$ . Define  $s \in S = \text{Span } \mathcal{B}$  by

$$s = \sum_{i=1}^n \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i$$

and set

$$s^\perp = v - s$$

Suppose we have shown  $s^\perp \in S^\perp$ . Then  $v = s + s^\perp$  giving existence as well as  $V = S + S^\perp$  and  $S \cap S^\perp = 0$ , i.e.,  $V = S \oplus S^\perp$ . Repeating the previous computation, we have if  $j = 1, \dots, n$  then

$$\begin{aligned} \langle s^\perp, v_j \rangle &= \langle v - s, v_j \rangle = \langle v, v_j \rangle - \langle s, v_j \rangle \\ &= \langle v, v_j \rangle - \sum_{i=1}^n \frac{\langle v, v_i \rangle}{\|v_i\|^2} \langle v_i, v_j \rangle \\ &= \langle v, v_j \rangle - \sum_{i=1}^n \frac{\langle v, v_i \rangle}{\|v_i\|^2} \delta_{ij} \|v_j\|^2 = 0 \end{aligned}$$

Since  $s^\perp \perp v_j$ ,  $j = 1, \dots, n$  i.e.,  $\forall v_j \in \mathcal{B}$ , if  $\sum_{i=1}^n \alpha_i v_i \in S$ , then

$$\langle s^\perp, \sum_{i=1}^n \alpha_i v_i \rangle = \sum_{i=1}^n \overline{\alpha_i} \langle s^\perp, v_i \rangle = 0$$

Thus,  $s^\perp \in S^\perp$  as needed.

Uniqueness: If

$$s + s^\perp = v = r + r^\perp, r \in S, r^\perp \in S^\perp$$

( $s \in S, s^\perp \in S^\perp$ ) as both  $S, S^\perp$  are subspaces

$$s - r = r^\perp - s^\perp \in S \cap S^\perp = 0$$

So  $s = r$  and  $s^\perp = r^\perp$ . □

### Theorem 21.1 (Pythagorean)

Let  $v = s + s^\perp$ ,  $s \in S, s^\perp \in S^\perp$ . Then

$$\begin{aligned} \|v\|^2 &= \langle s + s^\perp, s + s^\perp \rangle = \langle s, s \rangle + \langle s, s^\perp \rangle + \langle s^\perp, s \rangle + \langle s^\perp, s^\perp \rangle \\ &= \|s\|^2 + \|s^\perp\|^2 \end{aligned}$$

**Corollary 21.2 (Bessel's Inequality)**

Let  $V$  be an inner product space over  $F$ ,  $\mathcal{B} = \{v_1, \dots, v_n\}$  an OR set in  $V$  with  $0 \notin \mathcal{B}$ . Let  $v \in V$ . Then

$$\sum_{i=1}^n \frac{|\langle v, v_i \rangle|^2}{\|v_i\|^2} \leq \|v\|^2$$

with equality iff

$$v = \sum_{i=1}^n \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i$$

*Proof.* Hw. □

**Remark 21.3.** Let  $V$  be an inner product space over  $F$ ,  $S \subset V$  a finite subspace. Then by the OR Decomposition Theorem,  $\forall v \in V \exists! s \in S, s^\perp \in S^\perp \implies v = s + s^\perp$ . We call  $s$  the orthogonal projection of  $v$  on  $S$  and denote it by  $v_S$ . By the proof of the OR Decomposition Theorem, if  $\mathcal{B} = \{v_1, \dots, v_n\}$  is ANY OR basis for  $S$ , then the uniqueness of  $v_S$  means

$$v_S = \sum_{i=1}^n \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i$$

i.e., is INDEPENDENT of OR basis. So the ORTHOGONAL PROJECTION of  $v$  onto  $S$ .

**Theorem 21.4 (Approximation)**

Let  $V$  be an inner product space over  $F$ ,  $S \subset V$  a finite dimensional subspace, and  $v \in V$ . Then  $v_S$  is closer to  $v$  than any other vector in  $S$ , i.e.,

$$d(v, v_S) = \|v - v_S\| \leq \|v - r\| = d(v, r)$$

in  $\mathbb{R}, \forall r \in S$ . Equivalently,

$$d(v, S) = d(v, v_S)$$

Moreover, if  $r \in S$ , then

$$\|v - v_S\| = \|v - r\| \in \mathbb{R} \iff r = v_S$$

We say  $v_S$  gives the BEST APPROXIMATION.

*Proof.* By the OR Decomposition Theorem (and its proof),  $v = s + s^\perp$  with  $s = v_S, s^\perp = v - s = v - v_S, s^\perp \in S^\perp$ . Let  $r \in S$ . Then

$$v - r = (v - v_S) + (v_S - r) = s^\perp + (v_S - r)$$

$S \subset V$  is a subspace, so  $v_S - r \in S$ , hence  $s^\perp \perp v_S - r$ , i.e.,

$$0 = \langle s^\perp, v_S - r \rangle = \langle v - v_S, v_S - r \rangle$$

By the Pythagorean Theorem,

$$\|v - r\|^2 = \|v - v_S\|^2 + \|v_S - r\|^2 \geq \|v - v_S\|^2$$

with equality iff

$$\|v_S - r\| = 0 \iff v_S = r$$

□

**Definition 21.5 (Error)** — Let  $V$  be an inner product space over  $F$ ,  $S \subset V$  a finite dimensional subspace and  $v \in S$ . Then,  $\|v - v_S\|$  is called the error of  $v$  not being  $v_S$ .

**Problem 21.1.** Let  $V, X$  be inner product space over  $F$ ,  $S \subset V$  a finite dimensional subspace  $v \in V$ , and  $T : X \rightarrow V$  linear. Find  $x \in X$  with  $\|x\|$  minimal s.t.  $Tx$  is the best approximation to  $v \in V$  in  $S$ , i.e., find  $x \in X$ ,  $\|x\|$  minimal  $\ni Tx = v_S$ .

## §21.2 Examples of Best Approximation

**Example 21.6 (Fourier Coefficient)**

Let  $V = C[0, \pi]$  an inner product space over  $\mathbb{R}$  via  $\langle f, g \rangle = \int_0^{2\pi} fg$ ,  $u_0 = \frac{1}{\sqrt{2\pi}}$ ,  $u_{2n-1} = \frac{\cos nx}{\sqrt{\pi}}$ ,  $u_{2n} = \frac{\sin nx}{\sqrt{\pi}}$ ,  $n > 0$ . Set

$$S = \{u_0, \dots, u_n, \dots\}$$

an ON set (as we have seen) and let

$$\begin{aligned}\mathcal{B}_n &:= \{u_0, \dots, u_{2n+1}\} \\ V_n &:= \text{Span}(\mathcal{B}_n)\end{aligned}$$

if  $f \in V$ , then

$$f_n := f_{v_n} = f_{\text{span } \mathcal{B}_n},$$

the function in  $V_n$  closest to  $f$ , i.e., the orthogonal projection of  $f$  onto  $V_n$ . So

$$f_n = \sum_{i=0}^{2n+1} \langle f, u_i \rangle u_i$$

where

$$\langle f, u_i \rangle = \int_0^{2\pi} f u_i, \quad \forall i \leq 2n$$

called the  $i^{\text{th}}$  FOURIER COEFFICIENT. The ERROR to the actual  $f$  is

$$d(f, f_n) = \|f - f_n\| = \sqrt{\int_0^{2\pi} (f - f_n)^2}$$

One checks:

$$f_n = \frac{1}{2}a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx)$$

with

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx \\a_k &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx \\b_k &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx\end{aligned}$$

is the BEST APPROXIMATION of  $f$  by such functions. If  $\lim_{n \rightarrow \infty} \|f - f_n\| = 0$ , i.e.,  $f = \sum_{i=0}^{\infty} \langle f, u_i \rangle u_i$  converges, we say  $f$  converges to its Fourier expansion (similar results with modest change work for  $([0, L])$ ).

### Example 21.7

Let  $V = C[-1, 1]$  with  $\langle f, g \rangle = \int_{-1}^1 fg$ . Let  $f(x) = e^x$ . Find a linear polynomial nearest  $f$  and compute  $d(f, g)$  (=error) for such a  $g$  and we let  $W = \text{span}(1, x) \subset V$  a finite dimensional subspace. We want  $f_W$ . To do this, we compute ON (or OR) basis for  $W$  i.e., GS  $\{1, x\}$  and normalize. GS yields  $1, x$  (as before) and ON it to  $\frac{1}{\|1\|}, \frac{x}{\|x\|}$ , i.e.,  $\frac{1}{\sqrt{\int_{-1}^1 1}}$ ,  $\frac{x}{\sqrt{\int_{-1}^1 x^2}}$  which is

$$\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x$$

Let  $f = e^x$ . Then

$$\begin{aligned}f_W &= \langle f, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} + \langle f, \frac{\sqrt{3}}{2}x \rangle \frac{\sqrt{3}}{2}x \\&= \frac{1}{2} \int_{-1}^1 e^z dz + \frac{3}{2}x \int_{-1}^1 ze^z dz \\&= \dots \\&= \frac{1}{2}(e - \frac{1}{e}) + \frac{3}{e}x\end{aligned}$$

So,  $f_W = \frac{1}{2}(e - \frac{1}{e}) + \frac{3}{e}x$ . Let  $\alpha = \frac{1}{2}(e - \frac{1}{e}), \beta = \frac{3}{e}x$ . So  $g = f_W = \alpha + \beta x$  and

$$\begin{aligned}\|f - f_W\|^2 &= \|f - g\|^2 = \int_{-1}^1 (f - g)^2 dz \\&= \int_{-1}^1 (f^2 - 2fg + g^2) dz \\&= \int_{-1}^1 [(e^{2x} - 2e^x(\alpha + \beta x) + \alpha^2 + 2\alpha\beta x + \beta^2 x^2)] dx \\&= \dots \text{(boring algebra)} \\&= 1 - \frac{7}{e^2}\end{aligned}$$

So

$$d(f, g) = d(f, f_W) = \sqrt{1 - \frac{7}{e^2}} \approx .05625$$

### §21.3 Hermitian Operators

**Definition 21.8 (Hermitian/Self-Adjoint)** — Let  $V$  be an inner product space over  $F$ ,  $T : V \rightarrow V$  linear. We say  $T$  is HERMITIAN or SELF-ADJOINT if

$$\langle Tv, w \rangle = \langle v, Tw \rangle, \forall v, w \in V$$

if  $F \subset \mathbb{R}$  is an hermitian operator, it is also called a SYMMETRIC OPERATOR.

**Example 21.9** 1. Let  $V = F^{n \times 1}$  be an inner product space over  $F$  via the dot product, i.e.,

$$\left\langle \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \right\rangle := \sum_{i=1}^n \alpha_i \bar{\beta}_i$$

remember we always assume  $F = \bar{F} \subset \mathbb{C}$ . Note that some people write the dot product  $v * w$  – they do not like columns.

Let  $A \in M_n(F)$ . As usual, we view  $A$  as a linear operator,

$$A : F^{n \times 1} \rightarrow F^{n \times 1} \text{ by } X \mapsto A \cdot X$$

By HW,  $A$  is hermitian iff  $A = A^*$  (so if  $F \subset \mathbb{R} \iff A = A^t$ ). In fact, you will prove on the takehome the following theorem

#### Theorem 21.10

Let  $V, W$  be finite dimensional inner product space over  $F$  with ON bases,  $T : V \rightarrow W$  linear. Then,  $\exists! T^* : V \rightarrow W$  linear s.t.

$$\langle Tv, w \rangle_W = \langle v, T^*w \rangle_V, \forall v \in V, \forall w \in W$$

$T^*$  is called the ADJOINT of  $T$ . Hence if  $T : V \rightarrow V$  is a linear operator, then  $T$  is hermitian iff  $T = T^*$  and  $T^*$  exists.

#### Example 21.11

Let  $\alpha < \beta$  in  $\mathbb{R}$  and  $V = C[\alpha, \beta] := \{f : [\alpha, \beta] \rightarrow \mathbb{R}/\text{cont}\}$  an inner product space over  $\mathbb{R}$  by

$$\langle f, g \rangle := \int_{\alpha}^{\beta} fg$$

If  $T : V \rightarrow V$  linear, then  $T$  is hermitian iff

$$\int_{\alpha}^{\beta} (fTg - gTf) = 0, \forall f, g \in V \quad (*)$$

Note:  $V$  is not finite dimensional and  $(*)$  is a commutativity type of condition.

**Example 21.12 (fancy)**

$V = C^{\infty}[\alpha, \beta]$ ,  $\alpha < \beta$  in  $\mathbb{R}$ . (often  $C^{\infty}[\alpha, \beta]$  vector space of convergent power series in some neighborhood of every point of  $(\alpha, \beta)$  and ? open neighborhood at  $\alpha, \beta$ ). Again  $V$  is not finite dimensional and is an inner product space over  $\mathbb{R}$  as in the above example. Let  $p \in V$  be fixed,  $p(x) > 0$ , and

$$W = \{f \in V | p(\alpha)f(\alpha) = 0 = p(\beta)f(\beta)\}$$

an inner product space as in the above example (e.g.,  $p(\alpha) = 0, p(\beta) = 0$ ). Fix  $q \in W$  and let

$$T_{p,q} = T : W \rightarrow W \text{ the linear operator}$$

defined by

$$Tf := (pf')' + qf$$

called a STURM LIOUVILLE operator. Then  $T$  is hermitian. Check  $T$  satisfies  $(*)$  in the above example using integration by parts.

**Example 21.13**

More generally, let  $V = C^{\infty}[\alpha, \beta]$ ,  $\alpha < \beta \in \mathbb{R}$  an inner product space over  $\mathbb{R}$  as in the above. Let  $p, q, w \in V$ ,  $p(x) > 0, w(x) > 0, \forall x \in [\alpha, \beta]$ . Fix  $a, b, c, d \in \mathbb{R} \ni$  both  $a = 0 = b$  and  $c = 0 = d$  are excluded. Let

$$W = \{f \in V | af(\alpha) + bf'(\alpha) = 0 = cf(\beta) + df'(\beta)\}$$

where  $f$  satisfies the boundary condition. Let  $W$  be an inner product space over  $\mathbb{R}$  by the weighted inner product

$$\langle f, g \rangle_w = \int_{\alpha}^{\beta} wfg$$

Define the STURM LIOUVILLE OPERATOR:

$$T = T_{p,q,w} : W \rightarrow W \text{ by}$$

$f \mapsto -\frac{1}{w}((pf')' + qf)$ . Then  $T$  is hermitian. This arises from finding eigenvalues of  $T_{p,q,w}$ , i.e., solutions to the ODE

$$\frac{d}{dx} \left( p \frac{dy}{dx} \right) + q(x)y = -\lambda wy$$

which have as special cases – Legendre ODE

$$(1 - x^2)y'' + 2xy' + n(n + 1) = 0$$

arising in spherical harmonic problems. Bessel's ODE:

$$x^2y'' + xy' + (x^2 - a^2)y = 0$$

$\alpha \in \mathbb{C}$  (often in  $\mathbb{Z}$  or  $2\alpha \in \mathbb{Z}$ ), i.e., one wants to find the eigenvalues of  $f = y$ ,  $\lambda$  in (\*) for which there is a solution and  $f \in E_T(\lambda)$ . Eigenvectors in function spaces are called EIGENFUNCTIONS.

## §22 | Lec 21: Nov 23, 2020

### §22.1 Lec 20 (Cont'd)

Goal: Spectral Theorem for Hermitian Operator: Let  $V$  be a finite dimensional inner product space over  $F$ ,  $F = \mathbb{R}$  or  $\mathbb{C}$ ,  $T : V \rightarrow V$  hermitian. Then  $T$  is diagonalizable, i.e.,  $\exists$  a basis  $\mathcal{B}$  for  $V$  consisting of eigenvectors of  $T$ , and in fact, such a  $\mathcal{B}$  is ON.

Calculus Application: Let  $S \subset \mathbb{R}^n$  be “nice” (open + nice boundary + ...),  $x_1, \dots, x_n$  the rectilinear coordinate functions relative to the standard basis and

$$(\cdot)f : S \rightarrow \mathbb{R} \text{ a } C^2 - \text{ a function}$$

Calculus Theorem if  $f$  satisfies  $(\cdot)$ , then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a), \forall i, j, \forall a \in S$$

For each  $a \in S$ , associate the symmetric matrix

$$Hf(a) := \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right)$$

called the HESSIAN at  $f$  at  $a$ . Suppose  $a \in S$  is a critical point of  $f$ , i.e.,

$$Df(a) := \left( \frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right) = (0, \dots, 0)$$

Equivalently,  $\nabla f(a) = 0$ . Recall the TOTAL DERIVATIVE of  $f$  at  $a$  is the linear transformation

$$f'(a, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R} \text{ given by}$$

$f'(a, v) = Df(a) \cdot v$ . Now, let  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  be the eigenvalues of  $Hf(a)$ , so the roots of  $f_{Hf(a)}$  counted with multiplicity. Since  $Hf(a)$  is symmetric, by the Spectral Theorem,  $m = n$  and

$$Hf(a) \sim \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \text{ in } M_n \mathbb{R}$$

$\lambda_1, \dots, \lambda_n$  not necessarily distinct. Then, we have the 2<sup>nd</sup> Derivative Test under the above conditions at the critical point  $a$ .



1.  $a$  is a relative minimum for  $f$  at  $a$  if  $\lambda_i > 0 \forall i$ .
2.  $a$  is a relative maximum for  $f$  at  $a$  if  $\lambda_i < 0 \forall i$ .
3.  $a$  is a saddle point for  $f$  at  $a$  if  $\exists i, j \ni \lambda_i > 0, \lambda_j < 0$ .
4. No info if  $\lambda_i = 0 \forall i$  or  $\exists i \ni \lambda_i = 0$ .

The total derivative  $f'(a, -) : \mathbb{R}^n \rightarrow \mathbb{R}$  can be defined at  $a \in S$  if it exists as the following: it is a linear transformation

$$Ta : \mathbb{R}^n \rightarrow \mathbb{R} \ni$$

$\exists$  a scalar valued function satisfying

$$f(a + v) = f(a) + \|v\|E(a, v)$$

for some  $r, \ni$  if  $\|v\| < r$  then

$$E(a, v) \rightarrow 0 \text{ as } \|v\| \rightarrow 0$$

**Question 22.1.** What is the total derivative

$$f'(a, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ if } f : S \rightarrow \mathbb{R}^m?$$

### Theorem 22.1

Let  $V$  be an inner product space over  $F, T : V \rightarrow V$  linear,  $\lambda$  an eigenvalue of  $T, 0 \neq v \in E_T(\lambda)$ . Then

$$\lambda = \frac{\langle Tv, v \rangle}{\|v\|^2} \text{ and } \bar{\lambda} = \frac{\langle v, Tv \rangle}{\|v\|^2}$$

In particular,  $\lambda \in \mathbb{R}$  iff

$$\langle Tv, v \rangle = \langle v, Tv \rangle$$

*Proof.* By assumption,  $Tv = \lambda v, \|v\| \neq 0$ . So  $\langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle = \lambda \|v\|^2$  and  $\langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle = \bar{\lambda} \|v\|^2$ . As  $\|v\| \neq 0$ , the first statement follows. Hence,

$$\lambda = \bar{\lambda} \iff \langle Tv, v \rangle = \langle v, Tv \rangle$$

□

### Corollary 22.2 (Hermitian)

Let  $V$  be an inner product space over  $F, T : V \rightarrow V$  linear. Suppose that  $T$  is hermitian. Then any eigenvalues of  $T$  is real, i.e., lies in  $F \cap \mathbb{R}$ .

**Theorem 22.3** (Fundamental Theorem of Algebra)

Let  $f \in \mathbb{C}[t] \setminus \mathbb{C}$ . Then  $f$  has a root in  $\mathbb{C}$ , i.e.,  $\exists \alpha \in \mathbb{C} \ni f(\alpha) = 0$

Addendum: Let  $f \in \mathbb{R}[t] \setminus \mathbb{R}$ . As  $\mathbb{R} \subset \mathbb{C}$ ,  $\mathbb{R}[t] \subset \mathbb{C}[t]$ . So we can view  $f \in \mathbb{C}[t]$ . Then  $f$  has a root  $\beta \in \mathbb{C}$ . Of course,  $\beta$  may not lie in  $\mathbb{R}$ .

Suppose  $\beta$  is real, i.e.,  $\beta \in \mathbb{R}$ . As  $\beta$  is a root of  $f \in \mathbb{C}$

$$f = (t - \beta)g, g \in \mathbb{C}[t], \beta \in \mathbb{R}$$

Then

$$f = (t - \beta)(h), h \in \mathbb{R}[t] \text{ (if } \beta \in \mathbb{R})$$

*Proof.* 1. If  $f = \sum_{i=0}^n \alpha_i t^i$ ,  $\alpha_i \in \mathbb{R} \forall i$  and  $\sum_{i=1}^n \alpha_i \beta^i = 0$  in  $\mathbb{C}$  with  $\beta \in \mathbb{R}$ , then every term in  $\sum \alpha_i \beta^i$  lies in  $\mathbb{R}$ , so  $\beta$  is a root of  $f$  when viewed in  $\mathbb{R}[t]$ .

2. (Generalization) Let  $F \subset K$ ,  $K$  a field,  $F$  a subfield of  $K$  so same  $+, \cdot, 0, 1$  as in  $K$  (e.g.,  $\mathbb{R} \subset \mathbb{C}$ ). Let  $f \in F[t]$ ,  $\alpha \in F$ . By the DIVISION ALGORITHM,

$$f = f(t - \alpha)g + r, \quad r, g \in F[t] \text{ unique with } r = 0 \text{ or } \deg r < \deg(t - \alpha) \quad (*)$$

But  $\deg(t - \alpha) = 1$ , so  $r \in F$  (a constant). Evaluate  $(*)$  at  $t = \alpha$ , so  $(e_\alpha : F[t] \rightarrow F$  by  $h \mapsto h(\alpha)$  a ring homomorphism)

$$f(\alpha) = (\alpha - \alpha)g(\alpha) + r = r$$

i.e.,

$$f = (t - \alpha)g + f(\alpha)$$

So

$$\alpha \in F \text{ is a root in } F \iff$$

$(\star) f = (t - \alpha)g$  in  $F[t]$  some  $g \in F[t]$ . So we have, viewing  $F[t] \subset K[t]$ . If  $\beta \in K$ , then

$$f = (t - \beta)h + f(\beta), h \in K[t]$$

and if  $\beta \in K$  is a root of  $f$  in  $K$ , then

$$f = (t - \beta)h \in K[t]$$

So if  $\beta \in K$  is a root of  $f$  with  $\beta \in F$ , then

$$f(\beta) = 0_K = 0_F,$$

so  $(\star)$  holds. □

**Remark 22.4.** 1. By the Addendum and induction, FTA says if  $f \in \mathbb{C}[t] \setminus \mathbb{C}$ , says  $n = \deg f \geq 1$ , then  $\exists! \alpha_1, \dots, \alpha_n \in \mathbb{C}$ , not necessarily distinct and  $\beta \in \mathbb{C} \ni$

$$f = \beta(t - \alpha_1) \dots (t - \alpha_n)$$

i.e.,  $f$  factors into a product of linear polys. We say  $f$  splits in  $\mathbb{C}$  and  $\alpha_1, \dots, \alpha_n$  are the unique roots (up to multiplicity) of  $f$  in  $\mathbb{C}$ .

2. FTA is proven in Math 132 and math 110C. The essential analysis fact used in math 132 is if  $f \in \mathbb{C}[t] \setminus \mathbb{C}$ , then  $|f(z)| \rightarrow \infty$  as  $|z| \rightarrow \infty$  and the essential analysis fact used in math 110C is the Intermediate Value Theorem in the special case that says if  $f \in \mathbb{R}[t]$  is of odd degree, then  $f$  has a real root.
3. The following fact is true: If  $V$  is a finite dimensional vector space over  $F$ ,  $F$  an arbitrary field,  $T : V \rightarrow V$  linear, then  $\exists$  an ordered basis  $\mathcal{B}$  for  $V \ni [T]_{\mathcal{B}}$  is UPPER TRIANGULAR (i.e.  $([T]_{\mathcal{B}})_{ij} = 0 \forall i > j$ ) iff  $f_T \in F[t]$  splits, i.e., factors into a product of linear terms. If this occurs, we say  $T$  is TRIANGULARIZABLE. Can you prove that if  $F = \mathbb{C}$ , then every such  $T$  is triangularizable? ( $T$  is diagonalizable iff  $q_T$  of the HW7/Midterm splits and has no multiple roots)

## §23 | Lec 22: Nov 25, 2020

### §23.1 Lec 21 (Cont'd)

**Definition 23.1 (T-invariant)** — Let  $F$  be an arbitrary field,  $V$  a vector space over  $F$ ,  $W \subset V$  a subspace,  $T : V \rightarrow V$  linear. We say  $W$  is T-INVARIANT (or INVARIANT under  $T$ ) if

$$Tw \in W, \forall w \in W, \text{ i.e., } T(W) \subset W$$

if  $W$  is T-invariant, then we can (and do) view

$$T|_W : W \rightarrow W \text{ linear}$$

**Example 23.2** 1. Any subspace of an eigenspace of  $T$  (if any) is T-invariant.

2.  $\ker T \subset V$  is T-invariant.

3.  $\text{im } T \subset V$  is T-invariant.

**Lemma 23.3 (Hermitian Operator (Key Lemma))**

Let  $V$  be an inner product space over  $F$ ,  $T : V \rightarrow V$  hermitian,  $S \subset V$  a T-invariant subspaces. Then

1.  $S^\perp$  is T-invariant, i.e.,  $T(S^\perp) \subset S^\perp$ .

2.  $T|_{S^\perp} : S^\perp \rightarrow S^\perp$  is hermitian.

*Proof.* 1. Let  $w \in S^\perp$ . To show  $Tw \in S^\perp$ , if  $v \in S$ , then  $Tv \in S$  as  $S$  is  $T$ -invariant. So

$$\langle v, Tw \rangle = \langle Tv, w \rangle = 0$$

So,  $Tw \in S^\perp$ .

2. By 1),  $T|_{S^\perp} : S^\perp \rightarrow S^\perp$  is linear. As  $\langle Tv, w \rangle = \langle v, Tw \rangle, \forall v, w \in V$ , this is certainly true  $\forall v, w \in S^\perp$ .

□

**Remark 23.4.** Let  $F = \mathbb{R}$  or  $\mathbb{C}$ ,  $V$  a finite dimensional inner product space over  $F$ ,  $T : V \rightarrow V$  hermitian. By the Hermitian Corollary, if  $T$  has an eigenvalue, it is real and  $\alpha \in F$  is a root of  $f_T$  in  $F$  iff eigenvalue of  $T$ . We know  $f_T$  has a root in  $\mathbb{C}[t]$  by the FTA. The key lemma should allow us to induct on  $\dim V$ .

Subtle Difficulty: Let  $V$  be a finite dimensional inner product space over  $\mathbb{R}$ ,  $T : V \rightarrow V$  hermitian. We know  $f_T \in \mathbb{R}[t]$  has a root in  $\mathbb{C}$ , but we do not know a priori that  $f_T$  is the characteristics polynomial of an hermitian operator over an inner product space over  $\mathbb{C}$ , so we do not know that the roots of  $f_T$  are real.

Unfortunately, to overcome this, we have to use bases. There is an abstract way to do it but we cannot do it.

### Theorem 23.5 (Spectral – First Version)

(for Hermitian Operator) Let  $F = \mathbb{R}$  or  $\mathbb{C}$ ,  $V$  a finite dimensional inner product space over  $F$ ,  $T : V \rightarrow V$  hermitian. Then  $\exists$  an ON basis  $\mathcal{B} = \{v_1, \dots, v_n\}$  for  $V$  with each  $v_i, i = 1, \dots, n$ , an eigenvector for some eigenvalues  $\alpha_i \in \mathbb{R}, i = 1, \dots, n$  (not necessarily distinct). In particular,  $T$  is diagonalizable.

*Proof.* We prove  $\mathcal{B}$  exists by induction on  $\dim V = n$ .

$n = 1$  :  $V = \text{Span}(v)$ , any  $0 \neq u \in V$ . As  $Tv \in \text{Span}(v), \exists \alpha \in F \ni Tv = \alpha v$ , so  $v \in E_T(\alpha)$ . As  $T$  is hermitian,  $\alpha \in \mathbb{R}$  is real by Hermitian Corollary even if  $F = \mathbb{C}$ . So  $\mathcal{B} = \left\{ \frac{v}{\|v\|} \right\}$ .

$n > 1$  : Induction Hypothesis (IH): Let  $F = \mathbb{R}$  or  $\mathbb{C}$ ,  $W$  a finite dimensional inner product space over  $F$ ,  $\dim W = n - 1, T_0 : W \rightarrow W$  hermitian. Then  $\exists$  an ON basis for  $W$  of eigenvectors of  $T_0$  and every eigenvalues of  $T_0$  is real.

Let  $\mathcal{C}$  be an ON basis for  $n$ -dimensional  $V$ , which exists as  $F = \mathbb{R}$  or  $\mathbb{C}$ . Let  $A = [T]_{\mathcal{C}} \in M_n F \subset M_n \mathbb{C}$ .

$$A = A^* \text{ and } Ax \cdot y = x \cdot Ay, \forall x, y \in C^{n \times 1}$$

since  $T$  is hermitian, i.e.,

$$A : C^{n \times 1} \rightarrow C^{n \times 1} \text{ is hermitian}$$

where  $C^{n \times 1}$  is an inner product space over  $\mathbb{C}$  via the dot product. By the FTA,  $f_A$  has a root  $\alpha \in \mathbb{C}$ , hence  $\alpha$  is an eigenvalue of hermitian  $A : C^{n \times 1} \rightarrow C^{n \times 1}$ . Thus,  $\alpha \in \mathbb{R}$  by the Hermitian Corollary. But

$$f_T = f_{[T]_{\mathcal{C}}} = f_A$$

So  $f_T$  has a root  $\alpha \in \mathbb{R}$ , if  $F = \mathbb{R}$  or  $F = \mathbb{C}$  by the Addendum. Thus,  $\exists 0 \neq u \in E_T(\alpha) \subset V$  an eigenvector of  $T$ . Let  $Fv = \text{Span}(v) \subset E_T(\alpha)$ . Then  $Fv$  is  $T$ -invariant. By the OR Decomposition Theorem,

$$V = Fv \perp (Fv)^\perp$$

and

$$\dim V = \dim Fv + \dim (Fv)^\perp = 1 + \dim (Fv)^\perp$$

hence

$$\dim (Fv)^\perp = n - 1$$

By the Key Lemma, since  $Fv$  is  $T$ -invariant and  $T : V \rightarrow V$  is hermitian.  $(Fv)^\perp$  is  $T$ -invariant and

$$T|_{(Fv)^\perp} : (Fv)^\perp \rightarrow (Fv)^\perp \text{ is hermitian}$$

By the IH,  $(Fv)^\perp$  has an ON basis, say  $\{v_2, \dots, v_n\}$  of eigenvectors for  $T|_{(Fv)^\perp} : (Fv)^\perp \rightarrow (Fv)^\perp$ . But

$$T|_{(Fv)^\perp}(v_i) = Tv_i, i = 2, \dots, n$$

So,  $v_2, \dots, v_n$  are eigenvectors of  $T : V \rightarrow V$  and all the eigenvalues of the  $v_i, i = 2, \dots, n$  are real by IH. Since  $v \perp v_i, i = 2, \dots, n, 0 \neq \|v\| \in \mathbb{R} \subset F$ ,

$$\mathcal{B} = \{\|v\|, v_2, \dots, v_n\}$$

is an ON basis for  $V$  of eigenvalues for  $T$  and all the eigenvalues are real and  $T$  is diagonalizable.  $\square$

By the HW/Takehome, we know

### Theorem 23.6

Let  $V$  be a finite dimensional inner product space over  $F, F = \mathbb{R}$  or  $\mathbb{C}$ . Let  $\mathcal{B}, \mathcal{C}$  be ordered ON basis for  $V$ . Then

$$[1_V]_{\mathcal{B}, \mathcal{C}} : F^{n \times 1} \rightarrow F^{n \times 1}$$

$n = \dim V$ , is an ISOMETRY. In particular,

$$[1_V]_{\mathcal{B}, \mathcal{C}}^{-1} = [1_V]_{\mathcal{B}, \mathcal{C}}^*$$

$T : V \rightarrow W$  linear is called an ISOMETRY if

- $T$  is an isomorphism.
- $\langle Tv_1, Tv_2 \rangle_W = \langle v_1, v_2 \rangle_V, \forall v_1, v_2 \in V$ .

### Theorem 23.7 (Spectral Theorem for Hermitian Operator (refined))

Let  $F = \mathbb{R}$  or  $\mathbb{C}$ ,  $V$  a finite dimensional inner product space over  $F, T : V \rightarrow V$  hermitian. Then  $\exists$  an ordered ON basis  $\mathcal{C}$  of eigenvectors for  $V$  of  $T$  and every set of  $T$  if real. Moreover, if  $\mathcal{B}$  is any ordered ON basis for  $V$ , then

$$[T]_{\mathcal{C}} = C[T]_{\mathcal{B}} C^*$$

**Remark 23.8.** The Spectral Theorem says, if  $V$  is a finite dimensional inner product space over  $F, F = \mathbb{R}$  or  $\mathbb{C}, T : V \rightarrow V$  hermitian,  $\mathcal{B}$  an ordered ON basis for  $V$ , then

$$[T]_{\mathcal{B}} \sim \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, n = \dim V, \alpha_i \in \mathbb{R}, \forall i$$

if  $V = \mathbb{R}^n$ , this is often called the PRINCIPAL AXIS THEOREM.

e.g., It means if

$$f = \sum a_{ij} t_i t_j \in \mathbb{R}[t_1, \dots, t_n]$$

with

$$a_{ij} = a_{ji}, \forall i, j$$

This can always be arranged as  $t_i t_j = t_j t_i$  and we replace  $a_{ij}, a_{ji}$  with  $\frac{a_{ij} + a_{ji}}{2}$  if necessary. Then we can change variables to make it look like

$$\lambda_1 I_1^2 + \dots + \lambda_n I_n^2$$

(How? – Confer completing the square and  $TAT^*, A = (a_{ij}), T^* = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$ . We want even more

Let  $F = \mathbb{R}$  or  $\mathbb{C}, V$  a finite dimensional inner product space over  $F, \dim V = n, T : V \rightarrow V$  hermitian,  $\mathcal{B}$  an ordered ON basis of eigenvectors of  $T$  for  $V$ . Reordering  $\mathcal{B}$  if necessary, we may assume  $\lambda_1, \dots, \lambda_k$  are all the distinct eigenvalues of  $T$ , i.e., if  $j > k$  then  $\exists i < k \ni \lambda_j = \lambda_i$ .

**Claim 23.1.** Let  $v \in E_T(\lambda_i), w \in E_T(\lambda_j), 1 \leq i, j \leq k, i \neq j$ . Then  $v \perp w$ : We may assume that  $v \neq 0, w \neq 0$ . So

$$\begin{aligned} \lambda_i \langle v, w \rangle &= \langle \lambda_i v, w \rangle = \langle Tv, w \rangle = \langle v, Tw \rangle \\ &= \langle v, \lambda_j w \rangle = \overline{\lambda_j} \langle v, w \rangle = \lambda_j \langle v, w \rangle \end{aligned}$$

as  $\lambda_i \in \mathbb{R} \forall i$ . Thus,

$$(\lambda_i - \lambda_j) \langle v, w \rangle = 0 \in F, \lambda_i \neq \lambda_j$$

so

$$\langle v, w \rangle = 0$$

**Claim 23.2.** We have

$$\begin{aligned} W &:= E_T(\lambda_1) + \dots + E_T(\lambda_k) \\ &= E_T(\lambda_1) \oplus \dots \oplus E_T(\lambda_k) \end{aligned} \quad (*)$$

if  $w_i \in E_T(\lambda_i), i = 1, \dots, k$  and

$$0 = w_1 + \dots + w_k,$$

then

$$0 = \langle w_1 + \dots + w_k, w_j \rangle = \langle w_j, w_j \rangle = \|w_j\|^2$$

by the previous claim, so  $w_j = 0$  and  $(*)$  holds.

## §24 | Lec 23: Nov 30, 2020

### §24.1 Lec 22 (Cont'd)

*Note:* Of course we already know this claim, but this proof is nice. Recall this is equivalent to  $w = E_T(\lambda_1) + \dots + E_T(\lambda_k)$  and

$$E_T(\lambda_i) \cap \sum_{j=1}^k E_T(\lambda_j) = 0, i = 1, \dots, k$$

Also by the first claim, the DIRECT SUM DECOMPOSITION (\*) of  $w$  is an ORTHOGONAL DIRECT SUM. Since  $\mathcal{B}$  is a bases for  $V$  of eigenvectors for  $T$  and  $\mathcal{B} \subset W$ , we have

$$V = E_T(\lambda_1) \perp \dots \perp E_T(\lambda_k) \quad (\star)$$

Genral Problem: Let  $V$  be a vector space over  $F$ ,  $T : V \rightarrow V$  linear operator. Can we DECOMPOSE  $V$  as

$$V = W_1 \oplus W_2 \oplus \dots \oplus W_r \oplus \dots$$

with each subspace  $W_i$   $T$ -invariant, i.e., decomposition reflects the action  $T$ . This can be done if  $V$  is finite dimensional vector space over  $F$ . Then  $V$  is a finite direct sum. If  $F = \mathbb{C}$ , the solution is called JORDAN CANONICAL FORM.

$F$  arbitrary is called RATIONAL CANONICAL FORM (done in 115B or 110BH).

By the OR Decomposition Theorem,

$$V = E_T(\lambda_i) \perp E_T(\lambda_i)^\perp, i = 1, \dots, k \quad (**)$$

So

$$E_T(\lambda_i)^\perp = E_T(\lambda_i) \perp \dots \perp E_T(\lambda_i) \perp \dots \perp E_T(\lambda_k)$$

$i = 1, \dots, k$  by uniqueness and, also by the OR Decomposition Theorem, as

$$V = E_T(\lambda_i) \perp E_T(\lambda_i)^\perp$$

means that (\*) implies if  $v \in V$ , then

$$v = v_{E_T(\lambda_1)} + \dots + v_{E_T(\lambda_k)}$$

where  $v_{E_T(\lambda_i)}$  is the ORTHOGONAL PROJECTION of  $v$  onto  $E_T(\lambda_i)$ ,  $i = 1, \dots, k$ . Define:

$$P_{\lambda_i} : V \rightarrow V \text{ by } v \mapsto v_{E_T(\lambda_i)}, i = 1, \dots, k$$

As  $P_{\lambda_i}$  is the composition

$$\begin{aligned} V &\rightarrow E_T(\lambda_i) \hookrightarrow V, \\ v &\mapsto v_{E_T(\lambda_i)} \end{aligned}$$

It is a linear operator,  $i = 1, \dots, k$ . Moreover, by (\*\*),

$$\begin{aligned} \text{im } P_{\lambda_i} &= E_T(\lambda_i) \\ \text{ker } P_{\lambda_i} &= E_T(\lambda_i)^\perp \end{aligned}$$

Since

$$P_{\lambda_j}(v_{E_T(\lambda_i)}) = \delta_{ij} v_{E_T(\lambda_i)}, i = 1, \dots, k$$

We see that

1.  $P_{\lambda_i}P_{\lambda_j} = 0$  if  $i \neq j$ .
2.  $P_{\lambda_i}P_{\lambda_i} = P_{\lambda_i}$ .

So

$$P_{\lambda_i}P_{\lambda_j} = \delta_{ij}P_{\lambda_i} : V \rightarrow V \text{ linear}$$

The  $P_{\lambda_1}, \dots, P_{\lambda_k}$  are called ORTHOGONAL IDEMPOTENTS. We now see what we have done: Let  $v \in V$ . Then

$$\begin{aligned} 1_V v = v &= v_{E_T(\lambda_1)} + \dots + v_{E_T(\lambda_k)} \\ &= P_{\lambda_1}(v) + \dots + P_{\lambda_k}(v) = (P_{\lambda_1} + \dots + P_{\lambda_k})(v) \end{aligned}$$

So

$$1_V = P_{\lambda_1} + \dots + P_{\lambda_k}$$

We also have

$$\begin{aligned} T &= T \circ 1_V = T \circ (P_{\lambda_1} + \dots + P_{\lambda_k}) \\ &= TP_{\lambda_1} + \dots + TP_{\lambda_k} \\ &= \lambda_1 P_{\lambda_1} + \dots + \lambda_k P_{\lambda_k} \end{aligned}$$

as

$$\begin{aligned} \text{im } P_{\lambda_i} &= E_T(\lambda_i) \\ T|_{E_T(\lambda_i)} &= \lambda_i 1_{E_T(\lambda_i)}, i = 1, \dots, k \end{aligned}$$

We also have

$$\begin{aligned} 1_V \circ T &= (P_{\lambda_1} + \dots + P_{\lambda_k})T \\ &= P_{\lambda_1}T + \dots + P_{\lambda_k}T \end{aligned}$$

and

$$P_{\lambda_i}T = TP_{\lambda_i}, i = 1, \dots, k$$

This is called the SPECTRAL RESOLUTION of the Hermitian operator  $T : V \rightarrow V$ . Now, appropriately reordering  $\mathcal{B}$  to  $\mathcal{B}'$ , we have, with

$$n_i = \dim E_T(\lambda_i), i = 1, \dots, k$$

$$[T]_{\mathcal{B}'} = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & 0 \\ & & & \ddots & & \\ & & & & \lambda_k & \\ & & & & & \ddots \\ 0 & & & & & & \lambda_k \end{pmatrix}$$

Summary(Spectral Theorem for Hermitian Operator – Full version):

Let  $F = \mathbb{R}$  or  $\mathbb{C}$ ,  $V$  a finite dimensional inner product space over  $F$ ,  $T : V \rightarrow V$  hermitian,  $\lambda_1, \dots, \lambda_k$  all distinct eigenvalues of  $T$ . Then  $T$  is diagonalizable and



1.  $\lambda_i \in \mathbb{R}, i = 1, \dots, k$
2. Let  $\mathcal{B}_i$  be an ordered ON basis for  $E_T(\lambda_i), i = 1, \dots, k$ . Then  $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$  is an ordered ON bases for  $V$  consisting of eigenvectors of  $T$ .
- 3.

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \lambda_1 & \\ & & & \ddots \\ 0 & & & & \lambda_k \end{pmatrix}$$

$$n_i = \dim E_T(\lambda_i)$$

$$\dim V = n = n_1 + \dots + n_k$$

4.  $f_T = (t - \lambda_1)^{n_1} \dots (t - \lambda_k)^{n_k}$
5.  $V = E_T(\lambda_1) \perp \dots \perp E_T(\lambda_k)$
6.  $1_V = P_{\lambda_1} + \dots + P_{\lambda_k} : V \rightarrow V$  where  $P_{\lambda_i} : V \rightarrow V$  linear by  $v \mapsto v$
7.  $P_{\lambda_i} P_{\lambda_j} = \delta_{ij} P_{\lambda_i}, i, j = 1, \dots, k$
8.  $T = \lambda_1 P_{\lambda_1} + \dots + \lambda_k P_{\lambda_k}$
9.  $TP_{\lambda_i} = P_{\lambda_i} T, i = 1, \dots, k$
10. If  $\mathcal{C}$  is an ON basis for  $V$ , then

$$\begin{aligned} [T]_{\mathcal{B}} &= [1_V]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{C}} [1_V]_{\mathcal{B}, \mathcal{C}} \\ &= [1_V]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{C}} [1_V]_{\mathcal{C}, \mathcal{B}}^{-1} \\ &= [1_V]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{C}} [1_V]_{\mathcal{C}, \mathcal{B}}^* \end{aligned}$$

$$\text{i.e., } [1_V]_{\mathcal{B}, \mathcal{C}}^{-1} = [1_V]_{\mathcal{B}, \mathcal{C}}^*$$

**Remark 24.1.** One can also show that the MINIMAL POLYNOMIAL  $q_T$  of the HW/Takehome in the above is

$$q_T = (t - \lambda_1) \dots (t - \lambda_k)$$

In fact this is a necessary and sufficient condition  $\iff$  to be diagonalizable.

**Remark 24.2.** The Spectral Theorem for hermitian operator for  $F = \mathbb{R}$ , e.g., symmetric matrices, has a nice generalization:

Let  $F$  be a field with  $2 \neq 0$  in  $F$  and  $A \in M_n F$  a symmetric matrix, i.e.,  $A = A^t$ . Then,  $\exists$  an invertible matrix  $P$  in  $M_n F \ni P^t A P$  is diagonal.

Note: in the above, we are not saying  $p^t = p^{-1}$

Computation: To compute: let  $V$  be a finite dimensional vector space over  $F$ ,  $F = \mathbb{R}$  or  $\mathbb{C}$ ,  $T : V \rightarrow V$  hermitian. Find all the above:

Step 1: Find a basis for  $V$  and GS it to an OR bases, then normalize to an ON bases  $\mathcal{C}$ .

Step 2: Compute:

$$f_T = f_{[T]_{\mathcal{C}}} = \det(tI - [T]_{\mathcal{C}})$$

Step 3: Factor  $f_T$ , i.e., find all the roots of  $f_T$ . There are the eigenvalues of  $T$ . Since  $T$  is hermitian  $f_T$  splits and all the roots are real.

Step 4: For each eigenvalue of  $T$ , compute  $E_T(\lambda)$  by solving

$$[T]_{\mathcal{C}}[v]_{\mathcal{C}} = \lambda[v]_{\mathcal{C}}$$

(equivalently row reduce  $[T]_{\mathcal{C}} - \lambda I$  to row echelon form and solve).

Step 5: For each eigenvalue  $\lambda$ , find a basis for  $E_T(\lambda_i)$  and GS to an ordered ON basis and normalize to an ordered ON basis  $\mathcal{B}_{\lambda}$ . Let  $\mathcal{B} = \cup \mathcal{B}_{\lambda}$  an ordered ON basis of eigenvectors of  $T$ . As  $\mathcal{C}$  is ON

$$[1_V]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{C}} [1_V]_{\mathcal{C}, \mathcal{B}}^* \text{ is diagonal}$$

## §25 | Lec 24: Dec 2, 2020

### §25.1 Normal Operators

We now need the following part of the Takehome

#### Theorem 25.1

Let  $V$  be a finite dimensional inner product space over  $F$  having an ordered ON basis  $\mathcal{B}$ ,  $T : V \rightarrow V$  linear. Then  $\exists! T^* : V \rightarrow V$  linear s.t.

$$\langle Tv, w \rangle = \langle v, T^*w \rangle, \forall v, w \in V \quad (*)$$

called the ADJOINT of  $T$ . Moreover,

$$[T]_{\mathcal{B}}^* = [T^*]_{\mathcal{B}}$$

**Remark 25.2.** Actually, to prove (\*), you do not need  $\exists$  an ON basis, only an OR basis (which you know exist) if you prove it using dual bases.

Properties: Let  $V$  be a finite dimensional inner product space over  $F$  with an ON basis  $\mathcal{B}$ ,  $S, T : V \rightarrow V$  linear,  $\lambda \in F$ . Then  $\forall v, w \in V$

- (i)  $\langle T^*v, w \rangle = \langle v, Tw \rangle$
- (ii)  $T^{**} := (T^*)^* = T$
- (iii)  $\langle v, T^*Tv \rangle = \langle Tv, Tv \rangle = \|Tv\|^2$
- (iv)  $\langle v, TT^*v \rangle = \langle T^*v, T^*v \rangle = \|T^*v\|^2$
- (v)  $(T \circ S)^* = S^* \circ T^*$
- (vi)  $(S + T)^* = S^* + T^*$

(vii)  $(\lambda T)^* = \bar{\lambda} T^*, \forall \lambda \in F$ .

*Proof.* Left as exercise. □

**Remark 25.3.** The above means: Let  $V$  be a finite dimensional inner product space over  $F$  with an ON basis. Then

$$\phi : L(V, V) \rightarrow L(V, V) \text{ by } T \rightarrow T^*$$

is a SESQUILINEAR transformation, i.e.,

$$\phi(\lambda T + S) = \bar{\lambda} T^* + S^*, \forall T, S \in L(V, V), \lambda \in F$$

and hence linear if  $F \subset \mathbb{R}$  and is also bijection with inverse sesquilinear so a sesquilinear isomorphism.

#### Lemma 25.4 (New Key)

Let  $V$  be a finite dimensional inner product space over  $F$ ,  $T : V \rightarrow V$  linear. Suppose that  $V$  has an ON basis and  $W \subset V$  is a  $T$ -invariant subspace. Then  $W^\perp \subset V$  is  $T^*$ -invariant. In particular,

$$T^*|_{W^\perp} : W^\perp \rightarrow W^\perp \text{ is linear}$$

*Proof.* Let  $w^\perp \in W^\perp$  and  $x \in W$  be arbitrary. Then

$$\langle x, T^* w^\perp \rangle = \langle Tx, w^\perp \rangle = 0,$$

as  $Tx \in W$  by hypothesis. So  $T^* w^\perp \in W^\perp$  as needed. □

**Definition 25.5 (Triangularizability)** — Let  $V$  be a finite dimensional vector space over  $F$ ,  $T : V \rightarrow V$  linear. We say  $T$  is TRIANGULARIZABLE if  $\exists$  an ordered basis  $\mathcal{B}$  for  $V$   $\ni [T]_{\mathcal{B}}$  is upper triangular, i.e.,

$$[T]_{\mathcal{B}} = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

i.e.,  $([T]_{\mathcal{B}})_{ij} = 0$  if  $i > j$ .

**Remark 25.6.** In the above,  $[T]_{\mathcal{B}}$  is upper triangular iff  $[T]_{\mathcal{B}'}$  is lower triangular where  $\mathcal{B}'$  is an ordered basis with vectors in  $\mathcal{B}$  in reverse ordered.

#### Theorem 25.7

Let  $V$  be a finite dimensional inner product space over  $\mathbb{C}$ ,  $T : V \rightarrow V$  linear. Then  $T$  is triangularizable. Moreover,  $\exists$  an ordered ON basis  $\mathcal{B}$  for  $T$   $\ni [T]_{\mathcal{B}}$  is upper triangular.

*Proof.* We induct on  $n = \dim V$ .

- $n = 1$  : is immediate: if  $\{v\}$  is a basis  $\left\{\frac{v}{\|v\|}\right\}$  works.
- $n > 1$  : By the FTA, the characteristics poly  $f_{T^*}$  for  $T^*$  has a root  $\lambda \in \mathbb{C}$ , hence  $\lambda$  is an eigenvalue of  $T^*$ . Let  $0 \neq v \in E_{T^*}(\lambda)$ . By the OR Decomposition Theorem,

$$V = \mathbb{C}v \perp (\mathbb{C}v)^\perp$$

and

$$\begin{aligned} n = \dim V &= \dim \mathbb{C}v + \dim(\mathbb{C}v)^\perp \\ &= 1 + \dim(\mathbb{C}v)^\perp \end{aligned}$$

i.e.,  $\dim(\mathbb{C}v)^\perp = n - 1$ .  $\mathbb{C}v$  is  $T^*$ -invariant as  $v \in E_{T^*}(\lambda)$ , so  $(\mathbb{C}v)^\perp$  is  $(T^*)^* = T$ -invariant by New Key Lemma. So may view

$$T|_{(\mathbb{C}v)^\perp} : (\mathbb{C}v)^\perp \rightarrow (\mathbb{C}v)^\perp \text{ linear} \quad (*)$$

By induction,  $\exists$  an ordered ON basis  $\mathcal{B}_0 = \{v_1, \dots, v_{n-1}\}$  for  $(\mathbb{C}v)^\perp \ni [T|_{(\mathbb{C}v)^\perp}]_{\mathcal{B}_0}$  is upper triangular. Let  $\mathcal{B} = \left\{v_1, \dots, v_{n-1}, \frac{v}{\|v\|}\right\}$  an ordered ON basis for  $V$ . Then by  $(*)$ , we have

$$\begin{pmatrix} [T|_{(\mathbb{C}v)^\perp}]_{\mathcal{B}_0} & * \\ & \vdots \\ 0 & \dots * \end{pmatrix} \in M_n \mathbb{C}$$

□

**Remark 25.8.** As mentioned before, if  $F$  is arbitrary,  $V$  a finite dimensional vector space over  $F$ , then  $T$  is triangularizable  $\iff f_T, T : V \rightarrow V$  linear satisfies  $f_T$  splits, i.e., factors into a product of linear polys in  $F[t]$ .

*Proof.*  $(\implies)$  is clear as  $f_T$  is independent of a matrix representation.

$(\impliedby)$  is not clear and we not prove it.

□

### Corollary 25.9

Let  $V$  be a finite dimensional inner product space over  $\mathbb{C}$ ,  $T : V \rightarrow V$  linear,  $\mathcal{C}$  an ordered ON basis for  $V$ . Then  $\exists$  an ordered ON basis  $\mathcal{B}$  for  $V \ni [T]_{\mathcal{B}}$  is upper triangular and

$$[T]_{\mathcal{B}} = [1_V]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{C}} [1_V]_{\mathcal{C}, \mathcal{B}}^*$$

with  $[1_V]_{\mathcal{C}, \mathcal{B}}^{-1} = [1_V]_{\mathcal{C}, \mathcal{B}}^*$ .

*Proof.* Theorem and HW as  $\mathcal{C}, \mathcal{B}$  are ON.

□

**Definition 25.10 (Normal Operator)** — Let  $V$  be an inner product space over  $F$ ,  $T : V \rightarrow V$  linear. Suppose that  $T^* : V \rightarrow V$  exists, i.e.,

$$\langle Tv, w \rangle = \langle v, T^*w \rangle, \forall v, w \in V$$

with  $T^* : V \rightarrow V$  linear. Then we say  $T$  is a NORMAL OPERATOR, if  $TT^* = T^*T$ .

## §26 | Lec 25: Nov 4, 2020

### §26.1 Lec 24(Cont'd)

**Example 26.1** 1. Every hermitian operator is normal as  $T = T^*$

2. Let  $T_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a rotation counterclockwise by  $\angle\theta$  with  $0 < \theta < 2\pi$  and  $\theta \neq \pi$ . Then  $T_\theta$  has no eigenvalues in  $\mathbb{R}$ . Viewing  $\mathbb{R}^2$  as an inner product space over  $\mathbb{R}$  via the dot product.

$$T_{-\theta} = T_\theta^{-1} = T_\theta^t = T_\theta^*$$

So

$$T_\theta T_\theta^* = T_\theta^* T_\theta$$

and  $T_\theta$  is normal. However,  $T_\theta$  is not diagonalizable (is not even triangularizable). We shall show that this does not happen if  $F = \mathbb{C}$ , we start with (a replacement for the Hermitian Corollary)

#### Lemma 26.2 (Crucial Property of Normal Operators)

Let  $V$  be an inner product space over  $F$ ,  $T : V \rightarrow V$  normal,  $\lambda \in F$ . Let  $0 \neq v \in V$ . Then

$$v \in E_T(\lambda) \iff v \in E_{T^*}(\bar{\lambda})$$

i.e.,  $\lambda$  is an eigenvalue of  $T$  with eigenvector  $v \iff \bar{\lambda}$  is an eigenvalue of  $T^*$  with (the same) eigenvector  $v$ . So

$$Tv = \lambda v \iff T^*v = \bar{\lambda}v$$

if  $T$  is normal.

*Proof.* Suppose  $S : V \rightarrow V$  is normal,  $v \in V$ . Then

$$\begin{aligned} \|Sv\|^2 &= \langle Sv, Sv \rangle = \langle v, S^*Sv \rangle \\ &= \langle v, SS^*v \rangle = \langle S^*v, S^*v \rangle = \|S^*v\|^2 \end{aligned}$$

Hence

$$Sv = 0 \iff S^*v = 0 \text{ when } S \text{ is normal} \quad (*)$$

Let  $S = T - \lambda 1_V : V \rightarrow V$  linear. So  $\lambda$  is an eigenvalue of  $T$  iff  $\ker S \neq 0$ . But

$$S^* = (T - \lambda 1_V)^* = T^* - \lambda 1_V$$

by properties of  $()^*$ . It follows that

$$S^*S = SS^* \text{ as } T^*T = TT^*$$

i.e.,  $S$  is also normal. The result follows by  $()^*$ .  $\square$

**Theorem 26.3 (Spectral Theorem for Normal Operator)**

Let  $V$  be a finite dimensional inner product space over  $\mathbb{C}$ ,  $T : V \rightarrow V$  normal. Then  $\exists$  an ordered ON basis  $\mathcal{C}$  for  $V$  consisting of eigenvectors of  $T$ . In particular,  $T$  is diagonalizable. Moreover, if  $\mathcal{B}$  is an ordered ON basis for  $V$ , then

$$[T]_{\mathcal{C}} = [1_V]_{\mathcal{B}, \mathcal{C}} [T]_{\mathcal{B}} [1_V]_{\mathcal{B}, \mathcal{C}}^*$$

*Proof.* We induct on  $n = \dim V$ .

- $n = 1$  is immediate.
- $n > 1$  : By the FTA,  $\exists \bar{\lambda} \in \mathbb{C}$  a root of  $f_{T^*} \in \mathbb{C}[t]$ , hence an eigenvalue of  $T^*$ . Let  $0 \neq v \in E_{T^*}(\bar{\lambda})$ . By the lemma,  $v \in E_T(\lambda)$ . Thus,  $\mathbb{C}v$  is both  $T$ - and  $T^*$ -invariant. Hence, by New Key Lemma,

$$(\mathbb{C}v)^\perp \text{ is both } T^* \text{ and } T\text{-invariant}$$

In particular,

$$\langle x, T^*y \rangle = \langle Tx, y \rangle \quad \forall x, y \in (\mathbb{C}v)^\perp$$

and  $\left(T|_{(\mathbb{C}v)^\perp}\right)^*$  is the unique linear map

$$\left(T|_{(\mathbb{C}v)^\perp}\right)^* : (\mathbb{C}v)^\perp \rightarrow (\mathbb{C}v)^\perp$$

satisfying  $\forall x, y \in (\mathbb{C}v)^\perp$

$$\begin{aligned} \langle x, \left(T|_{(\mathbb{C}v)^\perp}\right)^* y \rangle_{(\mathbb{C}v)^\perp} &= \langle T|_{(\mathbb{C}v)^\perp} x, y \rangle_{(\mathbb{C}v)^\perp} \\ &= \langle Tx, y \rangle_V \\ &= \langle x, T^*y \rangle_V \end{aligned}$$

It follows by the uniqueness of the adjoint that

$$T^*|_{(\mathbb{C}v)^\perp} = \left(T|_{(\mathbb{C}v)^\perp}\right)^*$$

Hence, we have

$$T|_{(\mathbb{C}v)^\perp} : (\mathbb{C}v)^\perp \rightarrow (\mathbb{C}v)^\perp$$

is also normal. Since

$$\dim V = \dim \mathbb{C}v + \dim(\mathbb{C}v)^\perp = 1 + \dim(\mathbb{C}v)^\perp$$

by the OR Decomposition Theorem, by induction  $\exists$  an ON basis  $\mathcal{C}_0 = \{v_2, \dots, v_n\}$  for  $(\mathbb{C}v)^+$  of eigenvectors of  $T|_{(\mathbb{C}v)^\perp}$  hence of eigenvectors of  $T$ . It follows that

$$\mathcal{C} = \left\{ \frac{v}{\|v\|}, v_2, \dots, v_n \right\}$$

is an ON basis for  $V$  consisting of eigenvectors of  $T$ . If  $\mathcal{B}$  is an ON basis for  $V$ , then  $[1_V]_{\mathcal{B}, \mathcal{C}}^* = [1_V]_{\mathcal{B}, \mathcal{C}}^{-1}$  by Hw, so

$$[T]_{\mathcal{C}} = [1_V]_{\mathcal{B}, \mathcal{C}} [T]_{\mathcal{B}} [1_V]_{\mathcal{B}, \mathcal{C}}^*$$

by the change of basis theorem.

□

In fact, the converse is also true.

#### Theorem 26.4

Let  $V$  be a finite dimensional inner product space over  $\mathbb{C}$ ,  $T : V \rightarrow V$  linear. Then  $T$  is normal iff  $\exists$  an ON basis  $\mathcal{B}$  for  $V$  consisting of eigenvectors of  $T$ . In particular,  $T$  is diagonalizable if either holds.

*Proof.* ( $\implies$ ) Has been done.

( $\impliedby$ ) Let  $\mathcal{B}$  has an ordered ON basis for  $V$  of eigenvectors of  $T$ . Then

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, n = \dim V$$

As  $\mathcal{B}$  is ON, by HW

$$[T^*]_{\mathcal{B}} = [T]_{\mathcal{B}}^* = \begin{pmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{pmatrix}$$

in  $M_n\mathbb{C}$ . So

$$\begin{aligned} [T^*T]_{\mathcal{B}} &= [T^*]_{\mathcal{B}} [T]_{\mathcal{B}} = \begin{pmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{pmatrix} \\ &= [T]_{\mathcal{B}} [T^*]_{\mathcal{B}} = [TT^*]_{\mathcal{B}} \end{aligned}$$

(as  $|\lambda_i|^2 = \lambda_i \overline{\lambda_i} = \overline{\lambda_i} \lambda_i \in \mathbb{C}$ ) By the Matrix Theory Theorem,

$$\phi : L(V, V) \rightarrow M_n\mathbb{C} \text{ by } S \mapsto [S]_{\mathcal{B}}$$

is an isomorphism, so

$$T^*T = TT^*$$

□

**Remark 26.5.** The result needs  $F = \mathbb{C}$ . Indeed if  $V = \mathbb{R}^n, n > 1$ , is an inner product space over  $\mathbb{R}$  via the dot product and  $T : V \rightarrow V$  is a rotation by an  $\angle \theta, 0 < \theta < 2\pi, \theta \neq \pi$  in some plane through the origin in  $\mathbb{R}^n$ , then  $T$  is normal and not diagonalizable.

What is true is: Let  $F = \mathbb{R}$  or  $\mathbb{C}$ ,  $V$  a finite dimensional inner product space over  $F$ ,  $T : V \rightarrow V$  linear  $\exists$  an ON basis for  $V \ni [T]_{\mathcal{B}}$  is triangularizable, then  $T$  is normal iff  $T$  is diagonalizable.

**Remark 26.6.** As in the Hermitian case, we can do more.

Extension: Let  $V$  be a finite dimensional inner product space over  $\mathbb{C}$ ,  $\dim V = n, T : V \rightarrow V$  normal,  $\mathcal{C}$  an ordered basis of  $V$  of eigenvalues for normal  $T$ . After relabeling, we may assume  $\lambda_1, \dots, \lambda_k$  are the distinct eigenvalues of  $T$ , i.e., if  $j > k \exists i, 1 \leq i \leq k \ni \lambda_i = \lambda_j$ .

**Claim 26.1.** Let  $v \in E_T(\lambda_i), w \in E_T(\lambda_j), i \neq j, i \leq 1, j \leq k$ . Then  $v \perp w$ .

*Proof.* We may assume that  $v \neq 0$  and  $w \neq 0$ . As  $w \in E_T(\lambda_j), w \in E_{T^*}(\bar{\lambda}_j)$  by the lemma, as  $T$  is normal. Hence

$$\begin{aligned} \lambda_i \langle v, w \rangle &= \langle \lambda_i v, w \rangle = \langle Tv, w \rangle = \langle v, T^* w \rangle \\ &= \langle v, \bar{\lambda}_j w \rangle = \lambda_j \langle v, w \rangle \end{aligned}$$

Since  $\lambda_i \neq \lambda_j, \langle v, w \rangle = 0$ . □

## §27 | Lec 26: Dec 7, 2020

### §27.1 Lec 25 (Cont'd)

Let  $V$  be a vector space over  $F$ ,  $W_i \subset V, i \in I$  subspace. Suppose that  $V = \sum_I W_i$ . Then  $V$  is a DIRECT SUM of the  $W_i, i \in I$  write  $V = \bigoplus_I W_i$  if one of the following equivalent condition hold

1.  $\forall v \in V \exists! w_i \in W_i \ni w_i = 0$  almost all  $i$  and  $v = \sum_I w_i$
2. If  $w_i \in W_i$ , almost all  $w_i = 0$ , and  $0 = \sum_I w_i$ , then  $w_i = 0 \forall i \in I$
3.  $\forall i \in I$

$$W_i \cap \sum_{j \in I, j \neq i} W_j = 0$$

4. If  $\mathcal{B}_i$  is a basis for  $W_i, i \in I$ , then  $\mathcal{B} = \cup \mathcal{B}_i$  is a basis for  $V$ .

If  $V$  is also an inner product space over  $F$ , and  $V = \bigoplus_I W_i$  with  $\langle w_i, w_j \rangle = 0 \forall i \neq j$  in  $I$ , we call  $V$  an orthogonal direct sum and write  $V = \bigoplus_I W_i$ .

Since  $\lambda_i \neq \lambda_j, \langle v, w \rangle = 0$ . Let

$$W = E_T(\lambda_1) + \dots + E_T(\lambda_k)$$

It is a direct OR sum for if

$$0 = w_1 + \dots + w_k, w_i \in E_T(\lambda_i), i = 1, \dots, k$$



then

$$\begin{aligned} 0 &= \langle 0, w_j \rangle = \langle w_1 + \dots + w_k, w_j \rangle = \langle w_j, w_j \rangle \\ &= \|w_j\|^2 \end{aligned}$$

$j = 1, \dots, k$ . Hence  $w_j = 0 \forall i$  and

$$W = E_T(\lambda_1) \oplus \dots \oplus E_T(\lambda_k)$$

(why – uniqueness follows immediately) and  $\mathcal{C}$  is a basis for  $V$ , so

$$V = E_T(\lambda_1) \perp \dots \perp E_T(\lambda_k)$$

By the OR Decomposition Theorem,

$$E_T(\lambda_i)^\perp = E_T(\lambda_1) \perp \dots \perp E_T(\lambda_i) \perp \dots \perp E_T(\lambda_k)$$

and if  $v \in V$

$$v = w_1 + \dots + w_k, w_i \in W_i \text{ unique}$$

So

$$w_i = v_{E_T(\lambda_i)}$$

the OR properties of  $v$  an  $E_T(\lambda_i)$  for  $i = 1, \dots, k$  by the OR Decomposition Theorem, as

$$V = E_T(\lambda_i) \perp E_T(\lambda_i)^\perp$$

Let

$$P_{\lambda_i} : V \rightarrow V \text{ by } v \mapsto v_{E_T(\lambda_i)}, i = 1, \dots, k$$

be the composition

$$\begin{aligned} V &\rightarrow E_T(\lambda_i) \hookrightarrow V \\ v &\mapsto v_{E_T(\lambda_i)} \end{aligned}$$

a linear operator

$$\begin{aligned} \text{im } P_{\lambda_i} &= E_T(\lambda_i) \\ \ker P_{\lambda_i} &= E_T(\lambda_i)^\perp \\ P_{\lambda_i} P_{\lambda_j} &= \delta_{ij} P_{\lambda_j}, \forall i, j \end{aligned}$$

i.e.,  $P_{\lambda_1}, \dots, P_{\lambda_k}$  are ORTHOGONAL IDEMPOTENTS and we see  $\forall v \in V$

$$\begin{aligned} v &= P_{\lambda_1} v + \dots + P_{\lambda_k} v \\ 1_V &= P_{\lambda_1} + \dots + P_{\lambda_k} \end{aligned}$$

So

$$\begin{aligned} T &= T \circ 1_V = T \circ P_{\lambda_1} + \dots + T \circ P_{\lambda_k} = \lambda_1 P_{\lambda_1} + \dots + \lambda_k P_{\lambda_k} \\ T &= 1_V T = P_{\lambda_1} T + \dots + P_{\lambda_k} T \\ T P_{\lambda_i} &= P_{\lambda_i} T, \forall i \end{aligned}$$

as

$$T|_{E_T(\lambda_i)} = \lambda_i 1_{E_T(\lambda_i)}, i = 1, \dots, k$$

This is the SPECTRAL RESOLUTION of  $T$  if  $n_i = \dim E_T(\lambda_i)$ ,  $\mathcal{B}_i$  an ordered ON basis for  $E_T(\lambda_i)$ ,  $\mathcal{B}_i$  an ordered ON basis for  $E_T(\lambda_i)$ ,  $i = 1, \dots, k$ . Then  $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$  is an ordered ON basis for  $V$  consisting of eigenvectors of  $T$

$$n = \dim V = n_1 + \dots + n_k$$

$$f_T = (t - \lambda_1)^{n_1} \dots (t - \lambda_k)^{n_k}$$

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_k \\ & & & & & \ddots \\ 0 & & & & & & \lambda_k \end{pmatrix}$$

**Theorem 27.1** (Spectral Theorem for Normal Operator - Full Version)

Let  $F = \mathbb{C}$ ,  $V$  a finite dimensional inner product space over  $\mathbb{C}$ ,  $T : V \rightarrow V$  normal,  $\lambda_1, \dots, \lambda_k$  all the distinct eigenvalues of  $T$ . Then  $T$  is diagonalizable and

1. Let  $\mathcal{B}_i$  be an ordered ON basis for  $E_T(\lambda_i)$ ,  $i = 1, \dots, k$ . Then  $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$  is an ordered ON basis for  $V$  (obvious order) consisting of eigenvectors of  $T$ .

2.

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_k \\ & & & & & \ddots \\ 0 & & & & & & \lambda_k \end{pmatrix}$$

where

$$n_i = \dim E_T(\lambda_i), i = 1, \dots, k$$

$$\dim V = n = n_1 + \dots + n_k$$

3.  $f_T = (t - \lambda_1)^{n_1} \dots (t - \lambda_k)^{n_k}$
4.  $V = E_T(\lambda_1) \perp \dots \perp E_T(\lambda_k)$
5.  $1_V = P_{\lambda_1} + \dots + P_{\lambda_k} : V \rightarrow V$  where  $P_{\lambda_i} : v \rightarrow v$  linear by  $v \mapsto v_{E_T(\lambda_i)}, i=1, \dots, k$  (viewed in  $V$ ).

6.  $P_{\lambda_i}P_{\lambda_j} = \delta_{ij}P_{\lambda_i}, i, j = 1, \dots, k$
7.  $T = \lambda_1 P_{\lambda_1} + \dots + \lambda_k P_{\lambda_k}$
8.  $TP_{\lambda_i} = P_{\lambda_i}T, i = 1, \dots, k$
9. If  $\mathcal{C}$  is an ON basis for  $V$  then

$$\begin{aligned} [T]_{\mathcal{B}} &= [1_V]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{C}} [1_V]_{\mathcal{B}, \mathcal{C}} \\ &= [1_V]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{C}} [1_V]_{\mathcal{C}, \mathcal{B}}^{-1} \\ &= [1_V]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{C}} [1_V]_{\mathcal{C}, \mathcal{B}}^* \end{aligned}$$

$$\text{i.e., } [1_V]_{\mathcal{B}, \mathcal{C}}^{-1} = [1_V]_{\mathcal{B}, \mathcal{C}}^*$$

$$10. q_T = (t - \lambda_1) \dots (t - \lambda_k)$$

Now  $T$  is normal so  $T^*$  is also normal with distinct eigenvalues  $\overline{\lambda_1}, \dots, \overline{\lambda_k}$  and

$$E_T(\lambda_i) = E_{T^*}(\overline{\lambda_i}), i = 1, \dots, k$$

In fact, as

$$Tv = \lambda_i v \iff T^*v = \overline{\lambda_i}v$$

the orthogonal projection

$$P_{\overline{\lambda_1}}, \dots, P_{\overline{\lambda_k}}$$

for  $T^*$  satisfy

$$P_{\lambda_i} = P_{\overline{\lambda_i}}, i = 1, \dots, k$$

as

$$v_{E_T(\lambda_i)} = v_{E_{T^*}(\overline{\lambda_i})}$$

Hence the spectral resolution for  $T^*$  is

$$\begin{aligned} T^* &= \overline{\lambda_1} P_{\overline{\lambda_1}} + \dots + \overline{\lambda_k} P_{\overline{\lambda_k}} \\ &= \overline{\lambda_1} P_{\lambda_1} + \dots + \overline{\lambda_k} P_{\lambda_k} \end{aligned}$$

## §28 | Extra Lec: Nov 2/9, 2020

### §28.1 Dual Bases – Dual Spaces

Let  $0 \neq V$  be a vector space over  $F$  with basis  $\mathcal{B}$ . For each  $v_0 \in \mathcal{B}$ , we define a map

$$f_{v_0} : V \rightarrow F \text{ linear}$$

as follows: by the UPVS (which also holds if the basis is infinite, let  $f_{v_0}$  be the unique linear transformation) s.t.

$$\begin{aligned} v_0 &\mapsto 1 \\ v &\mapsto 0 \quad \forall v_0 \neq v \in \mathcal{B} \end{aligned}$$

We have

$$0 < \text{im } f v_0 \subset F \text{ a subspace}$$

( $\text{im } f v_0 \neq 0$  as  $v_0 \neq 0$ ). As  $\dim_F F = 1$ , we must have  $\dim f v_0 = 1$ , so  $f v_0 : V \rightarrow F$  is an epimorphism and

$$\begin{aligned} \ker f v_0 &= \{w \in V \mid w \text{ has } v_0 \text{ coordinate} = 0\} \\ &= \text{Span}(\mathcal{B} \setminus \{v_0\}) \end{aligned}$$

So if  $w \in V$ ,  $w = \sum \alpha_v v$ ,  $\alpha_v \in F$  almost all 0 with  $\alpha_v$  unique.

$$f v_0(w) = \alpha_{v_0}$$

the coordinate of  $w$  on  $v_0$ . We can do this for each  $v \in \mathcal{B}$ . If  $v' \in \mathcal{B}$ ,  $f_{v'} : V \rightarrow F$  is the linear transformation determined by

$$f_{v'}(v) = \delta_{vv'} = \begin{cases} i, & \text{if } v = v' \\ 0, & \text{if } v \neq v', v \in \mathcal{B} \end{cases}, \text{ the Kronecker } \delta$$

Set

$$\mathcal{B}^* := \{f v \mid v \in \mathcal{B}\} \text{ } f_v \text{ is the coordinate function } f_v \text{ on } v$$

The vector space

$$V^* := L(V, F)$$

is called the **DUAL SPACE** of  $V$ . So by the above if  $w \in V$

$$w = \sum_{v \in \mathcal{B}} \alpha_v v, \alpha_v \in F \text{ almost all } 0$$

then

$$\alpha_v = f_v(w) \text{ the coordinate } w, v \in \mathcal{B}$$

so

$$w = \sum_{\mathcal{B}} \alpha_v v = \sum_{\mathcal{B}} f_v(w) v$$

Now by the UPVS, we have a unique linear transformation

$$D_{\mathcal{B}} : V \rightarrow V^*$$

determined by  $v \in \mathcal{B} \mapsto f_v$ . So  $\sum_{\mathcal{B}} \alpha_v v \mapsto \sum_{\mathcal{B}} \alpha_v f_v$  almost all  $\alpha_v = 0$

**Claim 28.1.**  $D_{\mathcal{B}}$  is 1-1.

Suppose  $w = \sum_{\mathcal{B}} \alpha_v v \mapsto 0$  almost all  $\alpha_v = 0$  i.e.,  $\sum_{\mathcal{B}} \alpha_v f_v = 0 \leftarrow$  in  $V^*$   
Let  $v_0 \in \mathcal{B}$ , then

$$0 = \left( \sum_{\mathcal{B}} \alpha_v f_v \right) (v_0) = \sum_{\mathcal{B}} \alpha_v f_v(v_0) = \sum_{\mathcal{B}} \alpha_v S_{vv_0} = \alpha_{v_0}$$

Hence  $\sum \alpha_v f_v = 0 \rightarrow \alpha_v = 0 \forall v \in \mathcal{B}$ , so  $w = 0$ .  $D_{\mathcal{B}}$  is therefore 1-1 as claimed.  $\square$

Warning: If  $V$  is not finite dimensional, then  $D_{\mathcal{B}}$  is not onto, i.e.,  $\mathcal{B}^*$  does not span  $V^*$ .

( $|V^*| = |F|^{|\mathcal{B}|}$  and  $|F| = |V|$  by UPVS if  $F$  is infinite)

Note:  $D_{\mathcal{B}} : V \rightarrow V^*$  depends on the choice of basis  $\mathcal{B}$ .

**Definition 28.1 (Linear Functionals)** — If  $V$  is a vector space over  $F$ , elements in  $V^* = L(V, F)$  are called LINEAR FUNCTIONALS.

**Fact 28.1.** If  $S$  is a linearly indep. set in a vector space over  $F$  (even infinite) then  $S$  is part of a basis for  $V$ , i.e., the Extension Theorem holds (This needs the Axiom of Choice).

### Example 28.2

$V$  a vector space over  $F$ . Then followings are linear functionals

1. If  $0 \neq v \in V$ , then  $\{v\}$  extend to a basis  $\mathcal{B}$  for  $V$  and  $\mathcal{B}^*$  satisfies  $\mathcal{B}^*$  is linearly indep.

$$f_v(x) = S_{vx} \forall x \in \mathcal{B}$$

Let  $w = \sum_{x \in \mathcal{B}} \alpha_x x$ ,  $\alpha_x = 0$  almost all  $x \in \mathcal{B}$ . Then  $f_x(w) = \alpha_x \in F \forall x \in \mathcal{B}$ ,  
 $w = \sum f_x(w)x$

2.  $\pi_i : F^n \rightarrow F$  by  $(\alpha_1, \dots, \alpha_n) \mapsto \alpha_i \forall i$
3. Let  $\text{Int} : C[\alpha, \beta] \rightarrow \mathbb{R}$ ,  $\alpha < \beta$  be given by

$$\text{Int } f \mapsto \int_{\alpha}^{\beta} f$$

4. trace:  $M_n F \rightarrow F$  by

$$A \mapsto \sum_{i=1}^n A_{ii}$$

The sum of the diagonal entries of  $A$  called the TRACE of  $A$ .

We can iterate our constructions as follows:

Let  $\mathcal{C}$  be a basis for  $V^* = L(V, F)$  a vector space over  $F$ , where  $V$  is a vector space over  $F$ . Then

$$D_{\mathcal{C}} : V^* \rightarrow (V^*)^* := V^{**}$$

$V^{**}$  is called the DOUBLE DUAL of  $V$ , is induced by

$$f_0 \in \mathcal{C} \mapsto G_{f_0} \in \mathcal{C}^*$$

the coordinate function on  $f_0$ , i.e.,

$$\sum_{\mathcal{C}} \alpha_f f \mapsto \sum_{\mathcal{C}^*} \alpha_f G_f$$

with

$$G_{f_0}(f) = \delta_{tf_0} = \begin{cases} 1 & \text{if } f = f_0 \forall f, f_0 \in \mathcal{C} \\ 0 & \text{if } f \neq f_0 \end{cases}$$

So we have

$$V \xrightarrow{\mathcal{D}_{\mathcal{B}}} V^* \xrightarrow{\mathcal{D}_{\mathcal{C}}} V^{**}$$

and the composition is a monomorphism.

Wonderful Result:  $\exists$  a monomorphism

$$L : V \rightarrow V^{**}$$

INDEPENDENT OF CHOICE OF BASES. We now want to show this:

For each  $v \in V$  define the following linear functionals on  $V^*$

$$L_v : V^* \rightarrow F \text{ by } L_v(f) := f(v)$$

EVALUATION at  $v$ .

Check.  $L_v : V^* \rightarrow F$  is linear, i.e.,  $L_v \in V^{**} = (V^*)^*$  :

$$\begin{aligned} L_v(\alpha f + g) &= (\alpha f + g)(v) = \alpha f(v) + g(v) \\ &= \alpha L_v f + L_v g \end{aligned}$$

$\forall t, g \in V^* \forall \alpha \in F$  as needed. Now define

$$L : V \rightarrow V^{**} \text{ by } v \mapsto L_v$$

i.e.,  $L(v) = L_v$

**Claim 28.2.**  $L$  is linear.

$\forall f \in V^*, v, v' \in V, \alpha \in F$ , we have

$$\begin{aligned} L(\alpha v + v')(f) &= L_{\alpha v + v'}(f) = f(\alpha v + v') \\ &= \alpha f(v) + f(v') = \alpha L_v f + L_{v'} f \\ &= (\alpha L_v + L_{v'})(f) \end{aligned}$$

as needed.

**Claim 28.3.**  $L : V \rightarrow V^{**}$  is monic.

Suppose  $v \neq 0$ . By Example TBA,  $\exists f \in V^* \ni L_v(f) = f(v) \neq 0$ . As  $L$  is linear,  $L$  is a monomorphism. Hence

$$L : V \rightarrow V^{**}$$

is a NATURAL or CANONICAL MONOMORPHISM, i.e., no basis is needed to define it. We now assume that  $V$  is a finite dimensional vector space over  $F$ , let

$$\begin{aligned} \mathcal{B} &= \{v_1, \dots, v_n\} \text{ be a basis for } V \\ \mathcal{B}^* &= \{f_1, \dots, f_n\} \subset V^* \text{ defined by } f_i(v_j) = \delta_{ij} \forall i, j \end{aligned}$$

i.e., the  $f_i$  are the coordinate functions relative to  $\mathcal{B}$ . Then, as before, we have a monomorphism

$$D_{\mathcal{B}} : V \rightarrow V^* \text{ induced by } v_i \mapsto f_i$$

But we also have

$$\dim V^* = \dim L(V, F) = \dim V \dim F = \dim V$$

by the Matrix Theory Theorem, so  $D_{\mathcal{B}}$  is an isomorphism by the Isomorphism Theorem with  $\mathcal{B}^*$  a basis for  $V^*$  called the **DUAL BASIS** of  $\mathcal{B}$ . We also have

$$V \cong V^* \cong V^{**}, \text{ so } V \cong V^{**}$$

and

$$\mathcal{B}^{**} := \{L_{v_1}, \dots, L_{v_n}\}$$

with

$$\begin{aligned} L_{v_i} &:= L_{f_i}, f_i \in \mathcal{B}^* \\ L_{f_i}(f_j) &= L_{v_i}(f_j) = f_j(v_i) = \delta_{ij} \end{aligned}$$

So  $\mathcal{B}^{**}$  is the DUAL BASIS of  $\mathcal{B}^*$ . We also now  $L : V \rightarrow V^{**}$  is now a natural isomorphism by the Isomorphism Theorem and even better that

$$f(v) = L_v(f) \quad \forall v \in V \quad \forall f \in V^*$$

EVALUATION at  $v$ . So when  $V$  is a finite dimensional vector space over  $F$ , we can and do identify  $L_v$  and  $v \forall v \in V$ .

Any  $v \in V$  is determined by the  $t \in V^*$  and every  $f \in V^*$  is determined by the  $L_v \in V^{\times \times}$  and

$$f(v) = L_v(f)$$

So now we have: if  $V$  is a finite dimensional vector space over  $F$

$$\begin{aligned} \mathcal{B} &= \{v_1, \dots, v_n\} \text{ a basis for } V \\ \mathcal{B}^* &= \{f_1, \dots, f_n\} : \{f_{v_1}, \dots, f_{v_n}\} \text{ the dual basis of } \mathcal{B} \\ \mathcal{B}^{**} &= \{L_{f_{v_1}}, \dots, L_{f_{v_n}}\} = \{L_{v_1}, \dots, L_{v_n}\} \text{ the dual basis of } \mathcal{B}^* \end{aligned}$$

i.e.,

$$\begin{aligned} f_i &= f_{v_i} \\ L_{f_{v_i}} &= L_{v_i} \end{aligned}$$

and these satisfy

$$f|_i(v_i) = f_{v_i}(v_i) = \delta_{ii} = L_{f_{v_i}}(v_i) = L_{v_i}(f|_i)$$

If  $v \in V$ , then

$$\begin{aligned} v &= \alpha_1 v_1 + \dots + \alpha_n v_n \text{ unique } \alpha_1, \dots, \alpha_n \in F \\ f_j(v) &= f_j(\alpha_1 v_1 + \dots + \alpha_n v_n) \\ &= \alpha_j \end{aligned}$$

So

$$v = \sum_{i=1}^n f_i(v) v_i$$

where  $f_i(v)$  is the coordinate function relative to  $\mathcal{B}$  and if  $f \in V^*$ , then

$$f = \beta_1 f_1 + \dots + \beta_n f_n \text{ unique } \beta_1, \dots, \beta_n \in F$$

As

$$\begin{aligned} L_{v_1}(f) &= (\beta_1 f_1 + \dots + \beta_n f_n)(v_j) \\ &= \beta_1 f_1(v_1) + \dots + \beta_n f_n(v_j) = \beta_j \end{aligned}$$

And

$$\begin{aligned} f &= \beta_1 f_1 + \dots + \beta_n f_n \\ &= L_{v_1}(f) f_1 + \dots + L_{v_n}(f) f_n \\ &= f(v_1) f_1 + \dots + f(v_n) f_n \end{aligned}$$

So,

$$f = \sum f(v_i) f_i$$

where  $f(v_i)$  is the coordinate function.

## §28.2 The Transpose

Let  $V, W$  be vector space over  $F$ ,  $T : V \rightarrow W$  linear if  $g \in W^* = L(W, F)$ , i.e.,  $g : W \rightarrow F$  linear, then the composition

$$V \xrightarrow{T} W \xrightarrow{g}$$

is a linear functional, i.e.,  $g \circ T \in V^*$ .

**Definition 28.3 (Transpose)** — Let  $V, W$  be vector space over  $F$ ,  $T : V \rightarrow W$  linear. Define the transpose of  $T$  by

$$T^\top : W^* \rightarrow V^* \text{ by } g \mapsto g \circ T$$

i.e.,

$$T^\top g := g \circ T \quad \forall g \in W^*$$

i.e.,

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ & \searrow T^\top g & \downarrow g \\ & & F \end{array} \text{ commutes}$$

$T^\top g := g \circ T$

So

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ V^* & \xleftarrow{T^\top} & W^* \end{array}$$



**Claim 28.4.**  $T^\top : W^* \rightarrow V^*$  is linear if  $g, g' \in W^*, \alpha \in F$ , then

$$T^\top(\alpha g + g') = (\alpha g + g') \circ T = \alpha gT + g'T = \alpha T^\top g + T^\top g'$$

$T^\top$  is called the transpose because of the followings

**Theorem 28.4**

Let  $V, W$  be finite dimensional vector space over  $F$ ,  $\mathcal{B}, \mathcal{C}$  ordered bases for  $V, W$  respectively,  $T : V \rightarrow W$  linear. Then

$$[T]_{\mathcal{B}, \mathcal{C}}^\top = [T^\top]_{\mathcal{C}^*, \mathcal{B}^*}$$

*Proof.* Let

$$\begin{aligned} \mathcal{B} &= \{v_1, \dots, v_n\}, \quad \mathcal{B}^* &&= \{f_1, \dots, f_n\} \\ \mathcal{C} &= \{w_1, \dots, w_m\}, \quad \mathcal{C}^* &&= \{g_1, \dots, g_m\} \end{aligned}$$

with  $\mathcal{B}^*, \mathcal{C}^*$  the ordered dual bases of ordered bases  $\mathcal{B}, \mathcal{C}$  of  $V, W$  respectively.

Let

$$[T]_{\mathcal{B}, \mathcal{C}} = (\alpha_{ij}) \quad \text{and} \quad [T^\top]_{\mathcal{C}^*, \mathcal{B}^*} = (\beta_{ij})$$

i.e.,

$$\begin{aligned} T_{v_k} &= \sum_{i=1}^m \alpha_{ik} w_i \in W, \quad k = 1, \dots, n \\ T^\top g_j &= \sum_{i=1}^n \beta_{ij} f_i \in V^*, \quad j = 1, \dots, m \end{aligned}$$

Then computation gives

$$\begin{aligned} (T^\top g_j)(v_k) &= g_j(T_{v_k}) = g_j\left(\sum_{i=1}^m \alpha_{ik} w_i\right) \\ &= \sum_{i=1}^m \alpha_{ik} g_j(w_i) = \sum_{i=1}^m \alpha_{ik} \delta_{ij} = \alpha_{jk} \end{aligned}$$

and

$$\begin{aligned} (T^\top g)(v_k) &= \left(\sum_{i=1}^n \beta_{ij} f_i\right)(v_k) = \sum_{i=1}^n \beta_{ij} f_i(v_k) \\ &= \sum_{i=1}^n \beta_{ij} \delta_{ik} = \beta_{kj} \end{aligned}$$

Hence,  $\alpha_{jk} = \beta_{kj} \forall j, k$  as needed. □

**Definition 28.5 (Annihilator)** — Let  $V$  be a vector space over  $F$ ,  $\emptyset \neq S \subset V$  a subset. The set

$$S^\circ := \{f \in V^* \mid f|_S = 0\} = \{f \in V^* \mid f(s) = 0 \forall s \in S\}$$

is called the annihilator of  $S$ .

**Question 28.1.** If  $V$  is an inner product space over  $F$ , can you find something analogous?

**Claim 28.5.**  $S^\circ \subset V^*$  is a subspaces (even if  $S$  is not).

*Proof.* Let  $f, g \in S^\circ, \alpha \in F$ . To show  $(\alpha f + g)|_S = 0$ , let  $s \in S$ , then

$$(\alpha f + g)(s) = \alpha f(s) + g(s) = 0$$

so  $\alpha f + g \in S^\circ$ . □

Observation: Let  $T : V \rightarrow W$  be linear. Then

$$\ker T^\top = (\operatorname{im} T)^\circ$$

$g \in \ker T^\top$  iff  $T^\top g = 0$  iff  $(T^\top g)(v) = 0 \forall v \in V$  iff  $g(Tv) = 0 \forall v \in V$  iff  $g \in (\operatorname{im} T)^\circ$ .

**Proposition 28.6**

Let  $V$  be a finite dimensional vector space over  $F$ ,  $W \subset V$  a subspace. Then

$$\dim V = \dim W + \dim W^\circ$$

**Question 28.2.** If  $V$  is a finite dimensional inner product space over  $F$ , can you find something similar?

*Proof.* Let  $\{v_1, \dots, v_k\}$  be a basis for  $W$ . Extend it to  $\mathcal{B} = \{v_1, \dots, v_n\}$  a basis for  $V$ . Let  $\mathcal{B}^* = \{f_1, \dots, f_n\}$  be the dual basis of  $\mathcal{B}$ , i.e.,

$$f_i(v_j) = \delta_{ij} \forall i, j$$

**Claim 28.6.**  $\mathcal{C} = \{f_{k+1}, \dots, f_n\}$  is a basis for  $W^\circ$ . Let  $f \in W^\circ$ . Then  $\exists \beta_1, \dots, \beta_n \in F \ni$

$$f = \sum_{i=1}^n \beta_i f_i = \sum_{i=1}^n \underbrace{f(v_i)}_{\beta_i} f_i = \sum_{i=1}^{k+1} f(v_i) f_i \in \operatorname{Span} \mathcal{C}$$

As  $\mathcal{C} \subset \mathcal{B}^*$  and  $\mathcal{B}^*$  is linearly indep., so is  $\mathcal{C}$ . This proves the claim and the result follows. □

**Corollary 28.7**

Let  $V$  be a finite dimensional vector space over  $F$ ,  $W \subset V$  a subspace. Identifying  $V$  and  $V^{**}$  via  $v \leftrightarrow L_v$ , we have

$$W = (W^\circ)^\circ := W^{\circ\circ}$$

If  $V$  is a inner product space over  $F$ , can you find something similar?

*Proof.* We have  $W^\circ \subset V^*$  and  $W^{\circ\circ} \subset V^{**} = V$  are subspaces and by the last proposition, we have

$$\begin{aligned}\dim V &= \dim W + \dim W^\circ \\ \dim V^* &= \dim W^\circ + \dim W^{\circ\circ} \\ \dim W &= \dim W^{\circ\circ}\end{aligned}$$

If  $w \in W$ , then

$$L_w f = f(w) = 0, \quad \forall f \in W^\circ$$

So

$$w = L_w \in W^{\circ\circ}$$

i.e.,  $W \subset W^{\circ\circ}$  is a subspace. As  $\dim W = \dim W^{\circ\circ}$ ,  $W = W^{\circ\circ}$ . □

**Theorem 28.8**

Let  $V, W$  be finite dimensional vector space over  $F$ ,  $T : V \rightarrow W$  linear. Then

$$\dim \operatorname{im} T = \dim \operatorname{im} T^\top$$

*Proof.* We have  $\dim W = \dim W^*$

$$\begin{aligned}\dim W &= \dim \operatorname{im} T + \dim (\operatorname{im} T)^\circ \\ \dim W^* &= \dim \operatorname{im} T^\top + \dim \ker T^\top\end{aligned}$$

by the previous proposition and the Dimension Theorem. By observation,

$$\begin{aligned}(\operatorname{im} T)^\circ &= \ker T^\top \\ \dim(\operatorname{im} T)^\circ &= \dim \ker T^\top\end{aligned}$$

Hence,

$$\dim \operatorname{im} T = \dim \operatorname{im} T^\top \quad \square$$

Application: Let  $A \in F^{m \times n}$ . The row (respectively column) RANK of  $A$  is the dimension of the subspace spanned by the rows (respectively column of  $A$  viewed as vectors in  $F^m$  (respectively  $F^{n \times 1}$ )).

Using the theorems and our previous computation, we have

**Claim 28.7.** row rank  $A = \operatorname{col} \operatorname{rank} A$ .

## §29 | Dis 1: Oct 1, 2020

Overview of the class:

- HW – 20%
- Takehome Midterm – 20(25)%
- Midterm – 20(0)%
- Final – 40(55)%

**Note:** For starred homework problems, we can resubmit these problems (if we did not get full credit for it).

Plan:

1. Proofs
2. Sets
3. Functions

### §29.1 Sets

- $\mathbb{N}$  = set of natural numbers =  $\{1, 2, 3, 4, \dots\}$
- $\mathbb{Z}$  = set of integers =  $\{\dots, -2, -1, 0, 1, 2, \dots\}$
- $\mathbb{Q}$  = set of rational numbers =  $\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$
- $\mathbb{R}$  = set of real numbers(number line)
- $\mathbb{C}$  = set of complex numbers =  $\{a + bi | a, b \in \mathbb{R}\}$
- $\mathbb{R}^2$  = (xy)-plane =  $\{(a, b) : a, b \in \mathbb{R}\}$

Notation: subset –  $\subseteq$ , proper subset –  $\subsetneq$  (subset and not equal), empty subset –  $\emptyset$ .

### §29.2 Functions

What is a set?

- A collection of elements

**Example 29.1** •  $A = \{\text{cat}, \text{dog}\}$

- $B = \{1, 2, 3\}$
- $C = \mathbb{R}^2$

So what is a function?

$$f : \underbrace{A}_{\text{set called the domain of } f} \mapsto \underbrace{B}_{\text{this set is called the codomain of } f}$$

In general, **range** and **codomain** are two different thing.

Given any element  $a \in A$ , it gives an element  $f(a) \in B$ .

**Example 29.2** •  $f : \mathbb{R} \mapsto \mathbb{R}$  given by  $f(x) = x^2$  for any  $x \in \mathbb{R}$

- $g : \mathbb{R} \mapsto \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  given by  $g(\theta) = \tan(\theta)$
- Is  $h(x) = \frac{1}{x}$  a function? No – Poorly defined. If  $\mathbb{R} \mapsto \mathbb{R}$  is included, still not defined because of  $h(0)$   
 $h : \mathbb{R} \setminus \{0\} \mapsto \mathbb{R}$  is a function
- $k : (0, 1) \mapsto \mathbb{R}$  given by  $k(x) = x^2$ . Still a function but it's different from  $f : \mathbb{R} \mapsto \mathbb{R}$  given by  $f(x) = x^2$

Note: Domain and codomain are part of the function

- $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$  given by  $T(a, b) = (a + b, a - b)$ . Yes, this is a function
- $S : \mathbb{R}^3 \mapsto \mathbb{R}^2$  given by

$$S(x, y, z) = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This is also a function. S and T are linear transformations (functions from one vector space to another)

**Definition 29.3 (Injection & Surjection)** — A function  $f : A \mapsto B$  is injective (one-to-one) if for any  $a_1, a_2 \in A$ , if  $f(a_1) = f(a_2)$  then  $a_1 = a_2$ .

A function  $f : A \mapsto B$  is surjective (onto) if for all  $b \in B$ , there is an  $a \in A$  such that  $f(a) = b$ .

**Example 29.4**

Let  $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$  be given by  $T(a, b) = (a + b, a - b)$ . Show T is injective. Show T is surjective.

Suppose  $T(x_1, y_1) = T(x_2, y_2)$ , then  $(x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2)$ . So,

$$x_1 + y_1 = x_2 + y_2$$

$$x_1 - y_1 = x_2 - y_2$$

Solve the above system of linear equations, we obtain  $(x_1, y_1) = (x_2, y_2)$ . We conclude T is injective.  $\square$

T is surjective?

Let  $(c, d) \in \mathbb{R}^2$  be arbitrary. We want to show there exists an  $(a, b) \in \mathbb{R}^2$  with

$$T(a, b) = (c, d)$$

$$a + b = c$$

$$a - b = d$$

$$a = \frac{c + d}{2}$$

$$b = \frac{c - d}{2}$$

Note:  $(a, b) \in \mathbb{R}^2$  is a valid input.

Take  $a = \frac{c+d}{2}$  and  $b = \frac{c-d}{2}$ . Then,

$$\begin{aligned} T(a, b) &= \left( \frac{c+d}{2} + \frac{c-d}{2}, \frac{c+d}{2} - \frac{c-d}{2} \right) \\ &= \left( \frac{2c}{2}, \frac{2d}{2} \right) \\ &= (c, d) \end{aligned}$$

Since  $(c, d) \in \mathbb{R}^2$  was arbitrary, we conclude  $T$  is surjective

□

## §30 | Dis 2: Oct 6, 2020

### §30.1 Field

**Definition 30.1 ((1.2))** — A field consists of a set  $F$  with two elements  $0, 1 \in F$  ( $0 \neq 1$ ) and two operations, multiplication ( $\cdot$ ) and addition ( $+$ )  
 $(F, +)$

- $+$  is associative
- $+$  is commutative
- has an additive identity ( $0$ )
- has an additive inverse

”abelian group”

$(F^*, \cdot)$  (everything except  $0$ ) —  $F \setminus \{0\} = F^*$

- assoc
- comm
- has an identity ( $1$ )
- has mult inverse

”abelian group”

Finally, distributive prop also holds

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

Linear Algebra works over any field! (Not just  $\mathbb{R}$  like we did in lower div Lin Alg class).

**Claim 30.1.** Let  $F$  be a field. Let  $\alpha \in F$  be an arbitrary element of the field. Then  $0\alpha = 0$

*Proof.* Note since  $0 + 0 = 0$

$$0\alpha = (0 + 0)\alpha$$

However, by the dist. prop,

$$(0 + 0)\alpha = 0\alpha + 0\alpha$$

Then  $0\alpha = 0\alpha + 0\alpha$ . Subtract  $0\alpha$  from both sides (i.e. add its additive inverse to both sides)

$$-(0\alpha) + (0\alpha) = -(0\alpha) + 0\alpha + 0\alpha$$

So,

$$0 = 0 + 0\alpha = 0\alpha$$

So,  $0\alpha = 0$

□

**Claim 30.2.** Let  $F$  be a field, and let  $\alpha, \beta \in F$  s.t.  $\alpha\beta = 0$ . Then either  $\alpha = 0$  or  $\beta = 0$ .

*Proof.* If  $\alpha = 0$ , there is nothing to show. Suppose  $\alpha \neq 0$ . We want to show  $\beta = 0$ . Since  $\alpha \neq 0$ ,  $\alpha \in F^*$  has a multiplicative inverse  $\alpha^{-1} \in F^*$ .

Since  $\alpha\beta = 0$ , we can mult both sides by  $\alpha^{-1}$  on the left to get  $\alpha^{-1}(\alpha\beta) = \alpha^{-1}(0) = 0$ . Moreover, by associativity,

$$\alpha^{-1}(\alpha\beta) = (\alpha^{-1}\alpha)\beta = 1\beta = \beta$$

Hence,  $\beta = 0$ . So,  $\beta = 0$  as desired.

□

**Example 30.2**

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$  are fields.  $\mathbb{Z}$  is not a field.

**Example 30.3**

$\underbrace{\mathbb{Z}/12\mathbb{Z}}_{\text{integers mod 12}} = \{0, 1, 2, 3, \dots, 11\}$

Clock arithmetic. Addition is clock addition:

$$2 + 11 = 1$$

Multiplication is "clock mult"

$$2 \cdot 11 = 10$$

Multiply and add like normal but then subtract multiples of 12 until you get an element of the set.

- Additive identity: 0
- Multiplicative identity: 1

Is  $\mathbb{Z}/12\mathbb{Z}$  a field?

- additive inverse ✓
- identity ✓
- comm ✓
- assoc ✓
- mult inverse ...  $\implies$  NO!

Or different argument:

$$2 \cdot 6 = 0^-$$

But  $2 \neq 0$  and  $6 \neq 0$ . This violates a property of fields:

$$\alpha\beta = 0 \implies \alpha = 0 \text{ or } \beta = 0$$

So  $\mathbb{Z}/12\mathbb{Z}$  can't be a field.

**Example 30.4**

$\mathbb{Z}/3\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}\}$

- additive id: 0
- mult id: 1



Mult inv:

$$1 \cdot 1 = 1$$

$$2 \cdot 2 = 1 \checkmark$$

Additive inverse:

$$0 + 0 = 0$$

$$1 + 2 = 0 \checkmark$$

$\mathbb{Z}/3\mathbb{Z}$  is a field!

When is  $\mathbb{Z}/n\mathbb{Z}$  is a field?

- $n = 2$ : yes
- $n = 3$ : yes
- $n = 4$ : no
- $n = 13$ : yes
- $\vdots$

Same sort of argument works whenever  $n$  is composite.

$\mathbb{Z}/p\mathbb{Z}$  is a field for  $p$  prime. Proof uses Bezat lemma (Euclidean algorithm)

### Example 30.5

$$\mathbb{Z}/7\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \dots, \bar{6}\}$$

Everything has a mult.inverse

## §31 | Dis 3: Oct 8, 2020

### §31.1 Characteristics of a Finite Field

Let  $F$  be a finite field. Then, there must be a repeat in the following list:

$$1, 1+1, 1+1+1, \dots$$

If there wasn't a repeat, clearly, this would be an infinite list of distinct elements in  $F$ . Then we have for some  $j < k$

$$\underbrace{1+1+1\dots+1}_{j \text{ times}} = \underbrace{1+1+\dots+1}_{k \text{ times}}$$

So,  $0 = \underbrace{1+1+\dots+1}_{k-j \text{ times}}$   $k-j > 0$ . Thus, in a finite field, adding 1 to itself repeatedly must at some point give 0. (need to add up 1 to itself at most  $|F|$  number of times)

**Claim 31.1.** There is no field with 10 elements and  $1 + 1 = 0$

*Proof.* Let  $F$  be a field of 10 elements with  $1 + 1 = 0$ . Let's list the elements

$$0, 1, \alpha (\alpha \neq 0, 1)$$

Is  $\alpha + 1$  already on my list?

$$\begin{aligned} \alpha + 1 = 0 &\implies \alpha + 1 + 1 = 0 + 1 = 1 \implies \alpha = 1 \\ \alpha + 1 = 1? &\implies \alpha = 0 \\ \alpha + 1 = \alpha? &\implies 1 = 0 \end{aligned}$$

None are possible so  $\alpha + 1$  is not on our list so far

$$0, 1, \alpha, \alpha + 1, \beta$$

Then,  $\beta + 1$  isn't on the list.

$$0, 1, \alpha, \alpha + 1, \beta, \beta + 1$$

Notice  $\alpha + \beta$  isn't on the list yet and so is  $\alpha + \beta + 1$ . There are 8 elements in  $F$ . Since  $|F| = 10$ , let  $\gamma \in F$  be something not on the list so far and  $\alpha + 1$  is not on the list so far, so it must be the last element of  $F$ .

$$0, 1, \alpha, \alpha + 1, \beta, \beta + 1, \alpha + \beta, \alpha + \beta + 1, \gamma, \gamma + 1$$

But then  $\gamma + \alpha$  is not on the list. This would give an 11<sup>th</sup> ... but  $|F| = 10$  contradiction  $\square$

*Note:* **Characteristics:** the number of times you add 1 to get 0 in a field. For the case of characteristics 2, EVERYTHING IS ITS OWN ADDITIVE INVERSE.

**Claim 31.2.** There is no field of 10 elements with  $1 + 1 \neq 0$  and  $1 + 1 + 1 = 0$

*Proof.* List the element:

$$0, 1, 2, \alpha, \alpha + 1, \alpha + 2, \beta, \beta + 1, \beta + 2, \gamma$$

But then  $\gamma + 1$  isn't on this list. – Contradiction.  $\square$

What if  $1 + 1 + 1 + 1 = 0$ ?

$$\begin{aligned} \underbrace{(1+1)}_x + \underbrace{(1+1)}_x &= 0 \\ x + x &= 0 \\ x(1+1) &= 0 \\ (1+1)(1+1) &= 0 \end{aligned}$$

So either  $(1+1) = 0$  or  $(1+1) = 0$ . We already ruled out  $1 + 1 = 0$ .

Can  $1 + 1 + 1 + 1 = 0$ ? List the element

$$0, 1, 2, 3, 4, \alpha, \alpha + 1, \alpha + 2, \alpha + 3, \alpha + 4$$

What is  $2\alpha$ ? Trick:  $2 \cdot 3 = (1+1)(1+1+1) = \underbrace{1+1+1+1+1}_0 + 1 = 1$

Can  $2\alpha = 0$ ?  $\implies \alpha = 0$  or  $2 = 0$ . Can  $2\alpha = 1$ ? Mult both sides by 3

$$3 \cdot 2\alpha = 3$$

$\implies \alpha = 3$  (nope!)

$$2\alpha = 2? \quad 2\alpha = 3? \quad 2\alpha = 4?$$

Proceed similarly and we can see that  $1 + 1 + 1 + 1 + 1 \neq 0$

$$1 + 1 + 1 + 1 + 1 + 1 = 0?$$

$$(1 + 1)(1 + 1 + 1) = 0$$

$1 + 1 = 0$  or  $1 + 1 + 1 = 0$  (but we already ruled out both cases). Now,

$$1 + 1 + 1 + 1 + 1 + 1 + 1 = 0?$$

List:

$$0, 1, 2, 3, 4, 5, 6, \alpha, \alpha + 1, \alpha + 2$$

$\alpha + 3$  is not on this list.

- $8 = 0$ ? We can have  $(1 + 1)(1 + 1)(1 + 1) = 0$  but  $(1 + 1) = 0$  also ruled out.
- $9 = 0 \implies (1 + 1 + 1) = 0$  also ruled out.
- $10 = 0 \implies (1 + 1) = 0$  or  $(1 + 1 + 1 + 1 + 1) = 0$  which is also ruled out above.

So there are no fields with 10 elements.  $\square$

## §32 | Dis 4: Oct 13, 2020

### §32.1 Vector Space and Subspace

**Definition 32.1 ((2.1))** — A vector space over a field  $F$  is a set  $V$  with some additional structure:

$(V, +)$  is an abelian group ( $V$  has addition which is assoc, comm, add. inv, add. iden)

Scalar mult

$$\cdot : F \times V \rightarrow V$$

with

$$1_F \cdot v = v \quad \forall v \in V$$

$$(\alpha\beta) \cdot v = \alpha(\beta v) \quad \forall \alpha, \beta \in F, v \in V$$

$$(\alpha + \beta)v = \alpha v + \beta v$$

$$\alpha(v + w) = \alpha v + \alpha w$$

We're overloading  $+$  and  $\cdot$ . In  $F$  :

$$\alpha + \beta, \quad \alpha \cdot \beta$$

In  $V$  :

$$v + w, \quad \alpha \cdot v$$

We say  $S \subseteq V$  is a subspace if

1. It is closed under addition.
2. It is closed under scalar multiplication.
3. It is not empty.

OR

4.  $0 \in S$

Then,  $S$  will automatically become a vector space over the same field(it inherits the nice properties from  $V$ ).

### Example 32.2 (Abstract Vector Space)

In general, vector spaces might not always have nice geometric descriptions like in  $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ , etc

1.  $\underbrace{\mathbb{R}[t]}_{\substack{\text{set of all polynomials in } t \text{ with real coeff} \\ n \in \mathbb{Z}, n \geq 0.}} = \{a_0 + a_1t + a_2t^2 + \dots + a_nt^n : a_i \in \mathbb{R} \forall i\}$  and
2.  $\mathbb{R}[t]_n$  : set of all polynomials in  $t$  with real coeff and degree  $\leq n$ .  
 $\mathbb{R}[t]_2 = \{a + bt + ct^2, \quad a, b, c \in \mathbb{R}\}$
3.  $M_n(\mathbb{R}), \mathbb{R}^{n \times n}$  : set of  $n$  by  $n$  matrices with real coeff.

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

4. Is  $GL_2(\mathbb{R})$  a subspace of  $M_2(\mathbb{R})$  (as an  $\mathbb{R}$ -vector space)?

$$GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and is invertible} \right\}$$

$GL_2(\mathbb{R}) \subseteq M_2(\mathbb{R})$ . No it is not a subspace.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin GL_2(\mathbb{R})$$

**Example 32.3** 1.  $C[a, b] = \{f : [a, b] \rightarrow \mathbb{R} : f \text{ is continuous}\}$  is a vector space over  $\mathbb{R}$  (with usual/natural choice of addition and scalar mult).

2.  $W = \{f \in C[a, b] : f(a) = 0\} \subseteq C[a, b]$

Is  $W$  is a subspace?

- $0 \in W$  since  $0(a) = 0$
- Let  $f, g \in W$ . Since  $f \in W, f(a) = 0, g \in W, g(a) = 0$ . Hence,  $(f + g)(a) = f(a) + g(a) = 0 + 0 = 0$ . Thus,  $f + g \in W$ .
- Let  $f \in W$  and  $\alpha \in \mathbb{R}$ . Then

$$\begin{aligned}\alpha(f)(a) &= \alpha(f(a)) \\ &= \alpha(0) \\ &= 0\end{aligned}$$

Thus,  $W \subseteq C[a, b]$  is a subspace.

$\mathbb{C}$  is a 2-dimensional vector space over  $\mathbb{R}$ .  $\{1, i\}$  will form a basis of  $\mathbb{C}$  as an  $\mathbb{R}$ - vector space.

$\left\{ \begin{array}{l} \text{Linearly Indep.} \\ \text{Span all of } \mathbb{C} \end{array} \right\} \implies$  Every complex number can be uniquely written as  $a + bi$  where  $a, b \in \mathbb{R}$

$\mathbb{C}$  is also a vector space over the field  $\mathbb{C}$ . It is a 1-dimensional vector space over  $\mathbb{C}$ .

Basis (every complex number can be written as  $z \cdot 1$  for some  $z \in \mathbb{C}$ ):  $\{1\}$  will work (don't need  $i$ , it is allowed to be a scalar).

$\mathbb{C}$  is also a vector space over  $\mathbb{Q}$ . As a  $\mathbb{Q}$ - vector space,  $\mathbb{C}$  is infinite dimensional!

*Proof.* (sketch)  $\mathbb{C}$  is uncountably infinite. There's too many of them for us to be able to write each as a  $\mathbb{Q}$ - linear combos of some finite list of vectors.

Alternative:  $1, \pi, \pi^2, \pi^3, \dots$  is an infinite list of vectors (elements of  $\mathbb{C}$ ) which are linearly indep. over  $\mathbb{Q}$ .  $\square$

### Example 32.4

Let  $V$  be a vector space over a field  $F$ . We define  $V^*$ , the dual vector space, as

$$V^* = \{T : V \rightarrow F \mid T \text{ is } F\text{-linear}\}$$

Then  $V^*$  is also a vector space over  $F$ .

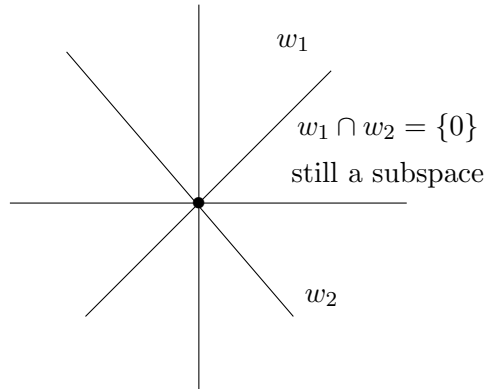
Relatedly, if  $V, W$  are  $F$ - vector spaces, then

$$L(V, W) = \{T : V \rightarrow W \mid T \text{ is } F\text{-linear}\}$$

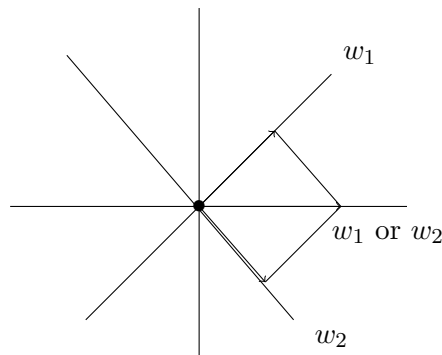
is also a vector space.

Note: Let  $V$  be a vector space over  $F$ . The intersection of subspaces of  $V$  will still be a subspace of  $V$

Example:



Let  $V$  be a vector space over  $F$ . The union of subspaces of  $V$  might not be a subspace.



not closed under addition in general

Note:

$$A \cup B = \{v \in V : v \in A \text{ or } v \in B\}$$

$$A + B = \{v \in V : \text{there exists } a \in A, b \in B \ni a + b = v\}$$

## §33 | Dis 5: Oct 15, 2020

### §33.1 Linear Independence, Span, & Subspaces

**Problem 33.1.** Let  $\mathbb{F}$  be a field on  $V$  be a vector space over  $\mathbb{F}$ . Let  $\alpha \in \mathbb{F}$  be arbitrary. Show  $\alpha \cdot \vec{0} = \vec{0}$ .

*Proof.* Let  $\alpha \in \mathbb{F}$  be arbitrary. Note:

$$\alpha \cdot \vec{0} = \alpha(\vec{0} + \vec{0}) = \alpha \cdot \vec{0} + \alpha \cdot \vec{0}$$

Adding the additive inverse of  $\alpha \cdot \vec{0}$  to both sides we see

$$\begin{aligned} -(\alpha \cdot \vec{0}) + \alpha \cdot \vec{0} &= \left( -(\alpha \cdot \vec{0}) + \alpha \cdot \vec{0} \right) + \alpha \cdot \vec{0} \\ \vec{0} &= \vec{0} + \alpha \cdot \vec{0} \end{aligned}$$

Thus,  $\vec{0} = \alpha \cdot \vec{0}$  as desired. □

**Problem 33.2.** If  $\alpha \in \mathbb{F}$  and  $\vec{v} \in V$  satisfy  $\alpha \vec{v} = \vec{0}$ , then either  $\alpha = 0$  or  $\vec{v} = \vec{0}$ .

*Proof.* If  $\alpha = 0$ , we're done. If  $\alpha \neq 0$ , it has a mult. inverse,  $\alpha^{-1} \in F^x$ . Then

$$\begin{aligned}\alpha^{-1}(\alpha \vec{v}) &= (\alpha^{-1}\alpha) \cdot \vec{v} \\ &= 1_{\mathbb{F}} \cdot \vec{v} = \vec{v}\end{aligned}$$

On the other hand, since  $\alpha \vec{v} = \vec{0}$ , we have

$$\alpha^{-1}(\alpha \vec{v}) = \alpha^{-1}(\vec{0}) = \vec{0}$$

So we see that if  $\alpha \neq 0$  then  $\vec{v} = \vec{0}$ . This completes the proof.  $\square$

**Problem 33.3.** a) Find a nonempty subset  $U \subseteq \mathbb{R}^2$  which is closed under scalar mult but is not a subspace of  $\mathbb{R}^2$  (over  $\mathbb{R}$ ).

b) Find a nonempty subset  $U \subseteq \mathbb{R}^2$  which is closed under addition but is not a subspace.

a) Take  $U = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \cup \text{span} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ . One can show  $U$  is closed under scalar mult but not under addition.

b) Left as exercise.

Let  $V$  be a vector space over  $\mathbb{F}$ . For  $u, w \subseteq V$  subspaces, define

$$u + w = \{ \vec{u} + \vec{w} : \vec{u} \in U, \vec{w} \in W \}$$

**Claim 33.1.**  $u + w \subseteq V$  is also a subspace.

*Proof.* (Sketch)

- Show  $0 \in u + w$ .
- Show  $u + w$  is closed under  $+$ .
- Show  $u + w$  is closed under scalar mult.

$\square$

Let  $v_1, \dots, v_k \in V$ . Define

$$\text{span}(\{v_1, \dots, v_k\}) = \left\{ \sum_{i=1}^k \alpha_i v_i : \alpha_i \in \mathbb{F} \right\} \subseteq V$$

(it's the set of all  $\mathbb{F}$ -linear combinations of  $v_1, \dots, v_k$ ).

**Claim 33.2.**  $\text{Span}(\{v_1, \dots, v_k\}) \subseteq V$  is a subspace of  $V$ .

If  $S \subseteq V$  is an infinite subset of  $V$ , we can still define  $\text{span}(S)$  as the set of all finite linear combos.

$$\sum_{i=1}^k \alpha_i v_i \quad \text{where } \alpha_i \in \mathbb{F}, v_i \in S$$

$\text{Span}(S) \subseteq V$  is also a subspace.

**Example 33.1** $\mathbb{F} = \mathbb{R}, V = \mathbb{R}^3$ .

$$U = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} = x\text{-axis}$$

$$W = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = xy\text{-plane}$$

What is  $u + w$ ? $u + w = xy\text{-plane}$ .

Claim:  $u + w = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

*Proof.* Let  $\vec{x} \in u + w$ . Then there exists  $\vec{u} \in U$  and  $\vec{w} \in W$  s.t.  $\vec{x} = \vec{u} + \vec{w}$ . Since

$$\vec{u} \in U = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \text{ there exists an } \alpha \in \mathbb{R} \text{ with } \vec{u} = \alpha \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \text{ Similarly, since } \vec{w} \in$$

$$W = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \text{ there exists } \beta, \gamma \in \mathbb{R} \text{ with } \vec{w} = \beta \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta \\ \beta + \gamma \\ 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned} \vec{x} = \vec{u} + \vec{w} &= \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \beta \\ \beta + \gamma \\ 0 \end{pmatrix} \\ &= (\alpha + \beta) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (\beta + \gamma) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

Thus,  $\vec{x} \in \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ . This shows  $U + W \subseteq \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ .

Conversely, suppose  $\vec{x} \in \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ . Then, there exists real number  $a, b \in \mathbb{R}$  s.t.

$$\vec{x} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



Note:  $a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ . Meanwhile,

$$b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Thus, there exists vectors  $\vec{u} \in U$  and  $\vec{w} \in W$  namely  $\vec{u} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  and  $\vec{w} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  and

$\vec{w} = b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$  s.t.  $\vec{x} = \vec{u} + \vec{w}$ . Thus,  $\vec{x} \in u + w$ . This shows

$$\text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subseteq U + W$$

We showed earlier that

$$U + W \subseteq \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

We conclude  $U + W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ . □

More generally, one can show  $\text{span}(S) + \text{span}(T) = \text{span}(S \cup T)$ .

Let  $V$  be a vector space over  $\mathbb{F}$ . Let  $S \subseteq V$  be any subset. Properties of  $\text{span}(S)$ :

- $\text{span}(S) \subseteq V$  is a subspace.
- $S \subseteq \text{span}(S)$ .
- If  $W$  is a subspace of  $V$  and  $S \subseteq W$ , then  $\text{span}(S) \subseteq W$ .

Suppose  $v_1, \dots, v_n$  span  $V$ . (i.e.,  $\text{span}(v_1, \dots, v_n) = V$ . Show  $v_1, v_2 - v_1, v_3 - v_2, \dots, v_n - v_{n-1} = V$ ).

Notation: Let  $w_i = v_i - v_{i-1}$  if  $i > 1$  and  $w_1 = v_1$ .

*Proof.* We'll show  $\forall i \text{span}(w_1, \dots, w_n)$  for  $i = 1, \dots, n$ . Use induction

- For  $i = 1$ ,  $v_1 = w_1 \in \text{span}(w_1, \dots, w_n)$ .
- Suppose  $v_k \in \text{span}(w_1, \dots, w_n)$  where  $k < n$ .
- 

$$\begin{aligned} v_{k+1} &= v_{k+1} - v_k + v_k \\ &= w_{k+1} + v_k \end{aligned}$$

Since  $w_{k+1} \in \text{span}(w_1, \dots, w_n)$  and  $v_k \in \text{span}(w_1, \dots, w_n)$  and since  $\text{span}(w_1, \dots, w_n) \subseteq V$  is a subspace, we have  $v_{k+1} = w_{k+1} + v_k \in \text{span}(w_1, \dots, w_n)$ . By induction, we get

$$v_1, \dots, v_n \in \text{span}(v_1, \dots, v_n) \subseteq \text{span}(w_1, \dots, w_n)$$

But  $\text{span}(v_1, \dots, v_n) = V$ . So, we have

$$V \subseteq \text{span}(w_1, \dots, w_n) \subseteq V$$

So  $\text{span}(w_1, \dots, w_n) = V$ . □

## §34 | Dis 6: Oct 20, 2020

### §34.1 Review of Linear Independence & Dependence

Let  $V$  be a vector space over  $\mathbb{F}$ . Recall  $v_1, \dots, v_k \in V$  are linearly indep. if for any  $\alpha_1, \dots, \alpha_k \in \mathbb{F}$  with  $\sum_{i=1}^k \alpha_i v_i = 0$ , we have  $\alpha_i = 0$  for  $i = 1, \dots, k$ .

To show  $v_1, \dots, v_k \in V$  are linearly indep:

Suppose we have  $\alpha_1, \dots, \alpha_m \in \mathbb{F}$  with

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0$$

Try to show  $\alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_k = 0$ .

#### Example 34.1

$$V = \mathbb{F}^3. \text{ Let } v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

**Claim 34.1.**  $v_1, v_2, v_3$  are linearly indep.

*Proof.* Let  $\alpha, \beta, \gamma \in \mathbb{F}$  s.t.

$$\alpha v_1 + \beta v_2 + \gamma v_3 = \vec{0}$$

Then, we have

$$\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$$\begin{pmatrix} \alpha + \beta + \gamma \\ \beta + \gamma \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From this we see:

$$\alpha + \beta + \gamma = 0$$

$$\beta + \gamma = 0$$

$$\gamma = 0$$

Solve the above system, we have  $\alpha = 0, \beta = 0, \gamma = 0$ . Thus  $\{v_1, v_2, v_3\}$  are linearly indep.  $\square$

### Example 34.2

Let  $V$  be a vector space over  $\mathbb{F}$ . Let  $\vec{v}, \vec{w} \in V$ . Set  $\vec{x} = \vec{v} + \vec{w}$ ,  $\vec{y} = \vec{v} - \vec{w}$ , and  $\vec{z} = 4\vec{v} + 2\vec{w}$ .

**Claim 34.2.**  $\{\vec{x}, \vec{y}, \vec{z}\}$  are linearly dep.

*Proof.* Observe  $3\vec{x} + \vec{y} = \vec{z}$ , so  $3\vec{x} - \vec{y} - \vec{z} = \vec{0}$ . This is a nontrivial linear combos. So,  $\{\vec{x}, \vec{y}, \vec{z}\}$  are linearly dep.  $\square$

**Remark 34.3.** More generally, any 3 vectors in  $\text{span}(\vec{v}, \vec{w})$  are necessarily going to be linearly dep. But this requires proof!

$V$  is a vector space over  $\mathbb{F}$ .

**Claim 34.3.**  $\vec{v}, \vec{w} \in V$  are linearly dep  $\iff \vec{v}$  is a multiple of  $\vec{w}$  or  $\vec{w}$  is a multiple of  $\vec{v}$ .

*Proof.* Suppose  $\vec{v}, \vec{w} \in V$  are linearly dep. Then,  $\exists \alpha, \beta \in \mathbb{F}$  not both 0 with  $\alpha\vec{v} + \beta\vec{w} = \vec{0}$ .

**Case 1:**  $\alpha \neq 0$

Then we have  $\alpha\vec{v} = -\beta\vec{w}$ . Since  $\alpha \neq 0$ , we have  $\alpha^{-1} \in \mathbb{F}^\times$ . Then  $\alpha^{-1}\alpha\vec{v} = \alpha^{-1}(-\beta\vec{w})$ . So

$$\vec{v} = \underbrace{-\alpha^{-1}\beta}_{\in \mathbb{F}} \vec{w}$$

Thus, in this case,  $\vec{v}$  is a multiple of  $\vec{w}$ .

**Case 2:**  $\alpha = 0$

In this case,  $\beta \neq 0$  (since  $\alpha, \beta$  are not both 0). Then  $0\vec{v} + \beta\vec{w} = \vec{0}$ . So,  $\vec{0} + \beta\vec{w} = \vec{0}$  and thus  $\beta\vec{w} = \vec{0}$ . Then either  $\beta = 0$  or  $\vec{w} = \vec{0}$ . But  $\beta \neq 0$ , so  $\vec{w} = \vec{0} = 0 \cdot \vec{v}$ . Hence, in this case,  $\vec{w}$  is a multiple of  $\vec{v}$ .

Conversely, suppose  $\vec{v}$  is a multiple of  $\vec{w}$ . Then there is an  $\alpha \in \mathbb{F}$  with  $\vec{v} = \alpha\vec{w}$ . So,

$$1\vec{v} - \alpha\vec{w} = \vec{0}$$

The coefficient of  $\vec{v}$  is nonzero. Thus, this is a nontrivial linear combos of  $\vec{v}$  and  $\vec{w}$  which gives  $\vec{0}$ . Hence  $\vec{v}$  and  $\vec{w}$  are linearly dep. A similar argument works in the case where  $\vec{w}$  is a multiple of  $\vec{v}$ .  $\square$

### Example 34.4

$$f : \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = \sin x$$

$$g : \mathbb{R} \rightarrow \mathbb{R} \quad g(x) = \cos x$$

$f, g \in C(\mathbb{R})$  where  $C(\mathbb{R}) = \{h : \mathbb{R} \rightarrow \mathbb{R} | h \text{ is continuous}\}$ ,  $\mathbb{R}$ -vector space. Show  $f, g$  are linearly indep.

*Proof.* Suppose we have  $\alpha, \beta \in \mathbb{R}$  with  $\alpha f + \beta g = \underbrace{0}_{\text{zero function}} \in C(\mathbb{R})$ . Then for any

$x \in \mathbb{R}$ ,

$$(\alpha f + \beta g)(x) = 0(x) = 0$$

So,

$$\alpha \sin(x) + \beta \cos(x) = 0$$

Plugging in  $x = 0$ , we see

$$\alpha \sin(0) + \beta \cos(0) = 0$$

so,  $\beta = 0$ . Plugging in  $x = \frac{\pi}{2}$ , we see

$$\alpha \sin\left(\frac{\pi}{2}\right) + \beta \cos\left(\frac{\pi}{2}\right) = 0$$

So,  $\alpha = 0$ . We have  $\alpha = 0, \beta = 0$ . Thus,  $f$  and  $g$  are linearly indep in  $C(\mathbb{R})$ , as desired.  $\square$

**Claim 34.4.**  $e^x, e^{4x} \in C(\mathbb{R})$  are linearly indep.

*Proof.* Suppose we have  $\alpha, \beta \in \mathbb{R}$  with  $\alpha e^x + \beta e^{4x} = 0 \in C(\mathbb{R})$ . That is, for any  $x \in \mathbb{R}$ ,

$$\alpha \cdot e^x + \beta \cdot e^{4x} = 0$$

Then for any real number  $x$ , we have

$$\beta = -\alpha e^x \cdot e^{-4x} = -\alpha e^{-3x}$$

For any  $x \in \mathbb{R}$ , we have  $\beta = -\alpha e^{-3x}$  (but  $\beta$  is a constant).

- At  $x = 1$ ,  $\beta = \alpha e^{-3}$ .
- At  $x = 0$ ,  $\beta = -\alpha$ .

So  $\alpha e^{-3} = -\alpha$ , which gives

$$\alpha(1 - e^{-3}) = 0$$

Thus either  $\alpha = 0$  or  $e^{-3} = 1$ . So  $\alpha = 0$ . Then  $\beta = -\alpha = -0 = 0$ . So  $\alpha = 0, \beta = 0$ , as desired.  $\square$

$V$  is a vector space over  $\mathbb{F}$ . True or false:

1. If  $v_1, \dots, v_k \in V$  are linearly indep., and  $\alpha \neq 0 \in \mathbb{F}$ , then  $\alpha v_1, \dots, \alpha v_k$  are also linearly indep. – **TRUE**
2. If  $v_1, \dots, v_k \in V$  are linearly dep., and  $\alpha \neq 0 \in \mathbb{F}$ , then  $\alpha v_1, \dots, \alpha v_k$  are also linearly indep. – **TRUE**
3.  $v_1, \dots, v_k \in V$  are linearly indep., and  $w \in V$ , then  $v_1 + w, v_2 + w, \dots, v_k + w$  are linearly indep. as well. – **FALSE**
4.  $v_1, \dots, v_k \in V$  are linearly dep., and  $w \in V$ , then  $v_1 + w, v_2 + w, \dots, v_k + w$  are linearly indep. as well. – **FALSE**

**Claim 34.5.**  $v_1, \dots, v_k \in V$  are linearly indep.,  $\alpha \neq 0 \in \mathbb{F}$ . Then  $\alpha v_1, \dots, \alpha v_k \in V$  are also linearly indep.

*Proof.* Suppose we have  $\beta_1, \dots, \beta_k \in \mathbb{F}$  with  $\beta_1(\alpha v_1) + \beta_2(\alpha v_2) + \dots + \beta_k(\alpha v_k) = \vec{0}$ . Then

$$(\beta_1 \alpha) v_1 + \dots + (\beta_k \alpha) v_k = 0$$

Since  $v_1, \dots, v_k$  are linearly indep., this gives

$$\beta_i \alpha = 0 \quad \text{for all } i = 1, \dots, k$$

So either  $\beta_i = 0$  or  $\alpha = 0$  for each  $i = 1, \dots, k$ . But,  $\alpha \neq 0$ . So  $\beta_i = 0$  for each  $i = 1, \dots, k$ . Thus,  $\alpha v_1, \dots, \alpha v_k$  are linearly indep.  $\square$

**Claim 34.6.** Let  $v_1, \dots, v_k \in V$  be linearly dep,  $\alpha \neq 0 \in \mathbb{F}$ . Then  $\alpha v_1, \dots, \alpha v_k \in V$  are also linearly dep.

*Proof.* Suppose  $\alpha v_1, \dots, \alpha v_k$  are actually linearly dep. Taking  $\alpha^{-1} \neq 0 \in \mathbb{F}$  by previous claim, multiplying through by  $\alpha^{-1}$  keeps the list linearly indep. Then,  $v_1, \dots, v_k \in V$  are linearly indep. Contradiction! Thus, by contradiction,  $\alpha v_1, \dots, \alpha v_k$  are also linearly dep.  $\square$

**Claim 34.7.**  $v_1, \dots, v_k \in V$  and  $w \in V$ . Then  $v_1 + w, \dots, v_k + w$  maybe linearly dep. or linearly indep. depending on the choice of  $w$ .

*Proof.* Take  $\vec{w} = -\vec{v}_1$ . Then

$$\vec{v}_1 + \vec{w} = \vec{0}$$

Then

$$1(\vec{v}_1 + \vec{w}) + 0(\vec{v}_2 + \vec{w}) + \dots + 0(\vec{v}_k + \vec{w}) = \vec{0}$$

This gives a nontrivial linear combos. So not linearly indep.  $\square$

**Claim 34.8.**  $v_1, \dots, v_k \in V$  linearly dep.,  $\vec{w} \in V$ . Then  $\vec{v}_1 + \vec{w}, \dots, \vec{v}_k + \vec{w}$  maybe linearly dep. or indep. depending on the choice of  $\vec{w} \in V$ .

*Proof.* Left as exercise. (Actually for the choice of  $\vec{w} \in \text{Span}(v_1, \dots, v_k)$  the proof is right below.  $\square$

## §35 | Dis 7: Oct 22, 2020

### §35.1 Review of Linear Independence and Linear Transformation

*Proof.* Since  $v_1 + w, \dots, v_k + w \in V$  are linearly dep., there exists scalars  $a_1, \dots, a_k \in \mathbb{F}$ , not all 0, s.t.

$$a_1(v_1 + w) + \dots + a_k(v_k + w) = 0$$

Then we have

$$\sum_{i=1}^k a_i v_i + \sum_{i=1}^k a_i w = 0$$

So,

$$-\left(\sum_{i=1}^k a_i\right)w = \sum_{i=1}^k a_i v_i$$

Let  $\beta = -\sum_{i=1}^k a_i \in \mathbb{F}$ . Then  $\beta \vec{w} = \sum_{i=1}^k a_i \vec{v}_i$ . Suppose  $\beta = 0$ . Then this would read

$$0 = \sum_{i=1}^k a_i \vec{v}_i$$

Since  $v_1, \dots, v_k$  are linear indep., this gives  $\alpha_i = 0$  for all  $i = 1, \dots, k$ . Yet, we choose  $\alpha_1, \dots, \alpha_k$  not all 0. Contradiction!

So  $\beta \neq 0$ . Hence,

$$\vec{w} = \beta^{-1} \sum_{i=1}^k a_i v_i = \sum_{i=1}^k (\beta^{-1} a_i) v_i$$

which in the span of  $v_1, \dots, v_k$ , as desired.  $\square$

### Corollary 35.1

$v_1, \dots, v_k \in V$  are linearly indep.,  $w \in V$ , and  $w \notin \text{span}(v_1, \dots, v_k)$ . Then  $v_1 + w, \dots, v_k + w$  are also linearly indep.

**Claim 35.1.**  $v_1, \dots, v_k \in V$  are linearly indep.,  $w \in V$  and  $w \notin \text{span}(v_1, \dots, v_k)$ . Then  $v_1, \dots, v_k, w$  are linearly indep.

*Proof.* Suppose we have  $a_1, \dots, a_{k+1} \in \mathbb{F}$  with

$$a_1 v_1 + \dots + a_k v_k + a_{k+1} w = 0$$

**Case 1:**  $a_{k+1} = 0$ .

Then, we have

$$a_1 v_1 + \dots + a_k v_k = 0$$

Since  $v_1, \dots, v_k$  are linearly indep., we have  $a_1 = 0, \dots, a_k = 0$ . Hence in this case, all coeffs are 0, as desired.

**Case 2:**  $a_{k+1} \neq 0$

Then we have

$$-a_{k+1} w = a_1 v_1 + \dots + a_k v_k$$

Since  $-a_{k+1} \neq 0$ , it has a mult. inverse  $\beta \in \mathbb{F}$ . Then we have

$$\vec{w} = \beta a_1 v_1 + \dots + \beta a_k v_k \in \text{span}(v_1, \dots, v_k)$$

But  $\vec{w} \notin \text{span}(v_1, \dots, v_k)$ . So we have a contradiction. So  $a_{k+1}$  must be 0. By the previous case,  $a_1 = 0, \dots, a_{k+1} = 0$ . Thus,  $v_1, \dots, v_k, w$  are linearly indep., as desired.  $\square$

**Definition 35.2 ((6.1))** —  $(v_1, \dots, v_m)$  is a basis for  $V$  if

1.  $v_1, \dots, v_m \in V$  are linearly indep.
2.  $\text{span}(v_1, \dots, v_m) = V$ .

**Theorem 35.3**

Every vector space has a basis.(potentially an infinite list of vectors)

**Theorem 35.4**

If  $(v_1, \dots, v_m)$  and  $(w_1, \dots, w_l)$  are both bases for  $V$ , then  $l = m$ .

Define  $\dim V = \#$  of elements in any basis of  $V$ .

**Proposition 35.5**

Let  $V$  be a vector space with  $\dim V = n$ . Suppose  $v_1, \dots, v_n$  are linearly indep. Then,  $\text{span}(v_1, \dots, v_n) = V$ . So,  $v_1, \dots, v_n$  is a basis of  $V$ .

**Proposition 35.6**

Let  $V$  be a vector space with  $\dim V = n$ . Suppose  $\text{span}(v_1, \dots, v_n) = V$ . Then,  $v_1, \dots, v_n$  are linearly indep. So  $(v_1, \dots, v_n)$  is a basis of  $V$ .

Recall  $T$  is injective  $\iff$  whenever  $T(x) = T(y)$  we have  $x = y$ .  $T$  is surjective  $\iff$  for all  $\vec{w} \in W$ , there is a  $\vec{v} \in V$  s.t.  $T(\vec{v}) = \vec{w}$ .

From the perspective of list of vectors,

- $v_1, \dots, v_k \in V$  are linearly indep. in  $V$ , then  $T(v_1), \dots, T(v_k)$  are also linearly indep in  $W \iff T$  is injective.
- $T$  is surjective  $\iff$  if  $v_1, \dots, v_k \in V$  have  $\text{span}(v_1, \dots, v_k) = V$ , then  $T(v_1), \dots, T(v_k) \in W$  satisfy  $\text{span}(T(v_1), \dots, T(v_k)) = W$ .
- $T$  is bijective  $\iff T$  is injective and surjective  $\iff$  if  $v_1, \dots, v_k \in V$  are a basis of  $V$ , then  $T(v_1), \dots, T(v_k) \in W$  are a basis of  $W$ .

From the perspective of subspace,

$$\ker(T) \subseteq V = \{v \in V : T(v) = 0\}$$

$$\text{im}(T) \subseteq W = \{T(\vec{v}) : \vec{v} \in V\}$$

- $T$  is injective  $\iff \ker T = \{0\}$ .
- $T$  is surjective  $\iff \text{im}(T) = W$ .
- $T$  is bijective  $\iff \ker T = \{0\}$  and  $\text{im } T = W$ .

**§35.2 Hw2 # 1**

Let  $V \neq 0$  be a vector space over  $\mathbb{F}$ . Suppose  $V$  can be spanned by one vector, say  $V = \text{span}(\vec{x})$  for some  $\vec{x} \in V$ . Let  $W \subseteq V$  be a subspace. Then, either  $W = \{0\}$  or  $W = V$ .

*Proof.* Let  $W \subseteq V$  be a subspace of  $V$ , and suppose  $W \neq \{0\}$ . We want to show  $W = V$ . Let  $\vec{w} \in W \setminus \{0\}$ , i.e.  $\vec{w} \in W, \vec{w} \neq 0$ . Since  $\vec{w} \in W$ , we have  $\text{span}(\vec{w}) \subseteq W$ . Meanwhile,  $\vec{w} \in W \subseteq V = \text{span}(\vec{x})$ . So there is an  $\alpha \in \mathbb{F}$  with  $\vec{w} = \alpha\vec{x}$ . If  $\alpha = 0$ , this would give  $\vec{w} = \vec{0}$  but  $\vec{w} \neq \vec{0}$ . So we have  $\alpha \neq 0$ . But then  $\alpha \in \mathbb{F}^\times$  has a mult. inv  $\alpha^{-1} \in \mathbb{F}^\times$ , so  $\vec{x} = \alpha^{-1}\vec{w} \in \text{span}(\vec{w})$ .

Thus,  $\text{span}(\vec{x}) \subseteq \text{span}(\vec{w})$ . So we have

$$V = \text{span}(\vec{x}) \subseteq \text{span}(\vec{w}) \subseteq W \subseteq V$$

So all containments must be equality. Thus,  $W = V$ .

Part 2: Suppose  $V \neq 0$  has  $V = \text{span}(\vec{v}, \vec{w})$  where  $\vec{v}, \vec{w} \in V$  are linearly indep.

**Claim 35.2.** If  $W \subseteq V$ , then either  $W = 0$ ,  $W = V$ , or  $W = \text{span}(\vec{w})$  for some  $\vec{w} \in V \setminus \{0\}$ .

Setup: Let  $W \subseteq V$  be a subspace with  $W \neq 0$  and  $W \neq V$ . Consider nonzero vector  $\vec{w}_1 \in W \setminus \{0\}$ . Suppose  $\text{span}(\vec{w}_1) \neq W$ . Then, find some  $\vec{w}_2 \in W \setminus \text{span}(\vec{w}_1)$ . Argue

- $\vec{w}_1, \vec{w}_2$  are linearly indep.
- In fact, they span all of  $V$ .  
Assume they don't, find  $\vec{w}_3 \in V \setminus \text{span}(\vec{w}_1, \vec{w}_2)$
- $\vec{w}_1, \vec{w}_2, \vec{w}_3$  are linearly indep.
- Ch2, Thm 4:  $V$  cannot have a list of linearly indep. vectors with more than 2 elements.  $\square$

## §36 | Dis 8: Oct 27, 2020

### §36.1 A Note on Object Types

Types of objects:

- Field:  $(\mathbb{F}, +, \cdot, 0, 1)$
- Vector space over a field  $\mathbb{F}$ :  $(V, +, \vec{0})$  with scalar mult.  $(\cdot : \mathbb{F} \times V \rightarrow V)$
- Subspace of a vector space  $V$ :  $S \subseteq V$  with  $\vec{0} \in S$  and  $S$  is closed under linear combo.  
AND,  $(S, +, \vec{0})$  and  $\mathbb{F} \times S \rightarrow S$  will make  $S$  into a vector space over  $\mathbb{F}$ .
- Vector in  $V$ : an element  $\vec{v} \in V$
- A list of vectors:  $v_1, \dots, v_k \in V$ . We might write it as a k-tuple,

$$(v_1, \dots, v_k)$$

to emphasize the order of the vectors.

For an infinite list of vectors, we might write it as a set of vectors

$$\{v_1, v_2, \dots\} \subseteq V$$

(more generally, a subset  $A \subseteq V$ )

1. The span of a list of vectors in  $V$  is a subspace of  $V$ .



**Example 36.1**

$V = \mathbb{R}^2$ ,  $W = x\text{-axis} \subseteq V$

$$\text{span} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right) = \underbrace{W}_{\text{subspace of } V}$$

More generally, span of a list of vectors in  $V$  is a vector space. This list of vectors spans this subspace (or a vector space).

2. Linearly Independent: – list of vectors.
3. A basis of a vector space is a list of vectors. That list of vectors is linearly indep. The span of that list of vectors gives us the vector space.
4. The dimension of a vector space is just a non-negative integer.  
 “Dimension of this basis – nonsense”  
 “number of vectors in this basis” ✓

**§36.2 Review**

1. Suppose  $\text{span}(v_1, \dots, v_k) = V$  and  $w_1, \dots, w_m$  are linearly indep. Then,  $m \leq k$  –  
 Proof idea: Get a system eqns with more vars than eqns. RREF. Get a nontrivial solution. – Thm 4, Ch. 2.  
 Meaning: suppose the  $\text{span}(v_1, \dots, v_k) = V$ . Then, there is no linearly indep. list of vectors whose length exceeds  $k$ .
2. If  $V$  has one basis which is finite, then every basis must be finite.  
 Proof: the first list spans  $V$ , so the second list of linearly indep. vectors must be shorter (or equal) by 1).
3. In fact, if you have two different finite bases of  $V$ , they have the same number of vectors.  
 Proof idea: length of the first list is longer or equal to second list. Length of second list is longer or equal to length of first.
4. Dimension of  $V$  is well-defined in this case. That is, any two different finite bases have same number of vectors.
5. Suppose  $\dim(V)$  is finite. Any linearly indep. list of vectors in  $V$  can be extended to a basis.  
 Proof:  $v_1, \dots, v_k$  linearly indep. Either  $\text{span}(v_1, \dots, v_k) = V$  and we’re done, or, pick  $v_{k+1} \notin \text{span}(v_1, \dots, v_k)$ . Then,  $v_1, \dots, v_k, v_{k+1}$  is still linearly indep. But we have an upper bound on how long linearly indep. list of vectors can be, since  $V$  is finite dimensional. This process must end.
6. Any finite spanning set of  $V$  can be shortened to get a basis of  $V$ .

*Proof.*  $\text{Span}(v_1, \dots, v_k) = V$ . If  $v_1, \dots, v_k$  linearly indep., done. If they're not, we can solve for one of them as a linear combo of the rest.

$$v_i = \sum_{j \neq i} a_j v_j \quad a_j \in \mathbb{F}$$

Then,  $\text{span}(v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_k)$  is still  $V$  ( $v_i$  is a redundant vector). We started with a finite list, so the process must terminate.  $\square$

7.  $\dim V$  is finite  $\iff V$  is the span of some finite list of vectors.

8. Let  $v_1, \dots, v_k \in V$  be linearly indep. Then  $\dim \text{span}(v_1, \dots, v_k) = k$ .

*Proof.*  $W = \text{span}(v_1, v_2, \dots, v_k) \subseteq V$   $v_1, \dots, v_k$  are linearly indep. in  $W$ . Clearly,  $\text{span}(v_1, \dots, v_k) = W$  so they form a basis of  $W$ . So,  $\dim W = \#$  of vectors in the basis  $v_1, \dots, v_k = k$ .  $\square$

9. Let  $V$  be finite dimensional.

$\dim V =$  length of the shortest list of vectors whose span is all of  $V$

$\dim V =$  length of the longest list of vectors that are linearly indep.

Linearly indep. lists can be expanded to get a basis.

10.  $\dim W \leq k$  where  $W = \text{span}(v_1, \dots, v_k) \subseteq V$ .

11.  $V = \text{span}(v_1, \dots, v_k)$ . If  $\dim V = k$ , then  $v_1, \dots, v_k$  are linearly indep.

*Proof.* By 10),  $\dim V \leq k$

$\dim V =$  length of shortest list of whose span is all of  $V$

We can't get rid of any of  $v_1, \dots, v_k$  in toss-out thm. If they had been dependent, we could toss something out, so they must be linearly indep.  $\square$

## §37 | Dis 9: Oct 29, 2020

### §37.1 Dis 8 (Cont'd)

$V$  finite dimensional,  $\dim V = n$ ,  $v_1, \dots, v_k \in V$  are linearly indep. Then, we can extend to a basis of all of  $V$ . That is,  $\exists v_{k+1}, \dots, v_n \in V$  s.t.  $v_1, \dots, v_k, v_{k+1}, \dots, v_n \in V$  form a basis of  $V$  (Extension Theorem). – useful fact for proofs.

Suppose  $v_1, \dots, v_n \in V$

1. If  $v_1, \dots, v_n \in V$  are linearly indep., then  $\text{span}(v_1, \dots, v_n) = V$ . So  $v_1, \dots, v_n$  form a basis of  $V$ .
2. If  $v_1, \dots, v_n \in V$  have  $\text{span}(v_1, \dots, v_n) = V$ , then  $v_1, \dots, v_n$  are linearly indep. So  $v_1, \dots, v_n \in V$  form a basis of  $V$ .

**Claim 37.1.** Let  $W \subseteq V$  be a subspace of  $V$  where  $V$  is a vector space over  $\mathbb{F}$ . Suppose  $V$  is finite dimensional. Then,  $W$  is also finite dimensional, and  $\dim W \leq \dim V$

*Proof.* If  $V$  is finite dimensional, say  $\dim V = n$ , then no list of linearly indep. vectors in  $W$  can exceed length  $n$ . Any basis of  $W$  will therefore have  $\leq n$  elements, since a basis of  $W$  is, in particular, a list linearly indep. So  $\dim W \leq n$ .  $\square$

**Remark 37.1.** On homework 3 # 2,  $\dim W = \dim V \implies W = V$ .

### Theorem 37.2

$V$  finite dimensional vector space over  $\mathbb{F}$ ,  $W_1, W_2 \subseteq V$  subspaces.

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

*Proof.* Note that  $\dim(W_1 \cap W_2) = k$

Let  $w_1, \dots, w_k$  be a basis of  $W_1 \cap W_2$ . Then  $w_1, \dots, w_k \in W_1$  are linearly indep. So, they can be extended to a basis. That is,  $\exists x_1, \dots, x_{m-k} \in W_1$  s.t.  $w_1, \dots, w_k, x_1, \dots, x_{m-k} \in W_1$  form a basis of  $W_1$  where  $\dim W_1 = m$ . Similarly,  $\exists y_1, \dots, y_{n-k} \in W_2$  s.t.  $w_1, \dots, w_k, y_1, \dots, y_{n-k} \in W_2$  form a basis of  $W_2$  where  $\dim W_2 = n$ .

**Claim 37.2.**  $w_1, \dots, w_k, x_1, \dots, x_{m-k}, y_1, \dots, y_{n-k} \in W_1 + W_2$  form a basis of  $W_1 + W_2$ .

Check

1. Every vector  $z \in W_1 + W_2$  can be written as a linear comb of the vectors on our list.
2. The list of vectors is linearly indep.

Suppose

$$a_1 w_1 + \dots + a_k w_k + b_1 x_1 + \dots + b_{n-k} x_{n-k} + c_1 y_1 + \dots + c_{n-k} y_{n-k} = 0$$

Then note

$$\sum_{i=1}^{n-k} c_i y_i \in W_2$$

But also,

$$\sum_{i=1}^{n-k} c_i y_i = -\sum_{i=1}^k a_i w_i - \sum_{i=1}^{m-k} b_i x_i \in W_1$$

So  $\sum_{i=1}^{n-k} c_i y_i \in W_1 \cap W_2$ . But  $w_1, \dots, w_k$  form a basis of  $W_1 \cap W_2$ , so  $\exists d_1, \dots, d_k \in \mathbb{F}$  with

$$\sum_{i=1}^{n-k} c_i y_i = \sum_{i=1}^k d_i w_i$$

Similarly,  $\exists \alpha_1, \dots, \alpha_k \in \mathbb{F}$

$$\sum_{i=1}^{n-k} b_i x_i = \sum_{i=1}^k \alpha_i w_i$$

So, we get

$$\sum_{i=1}^k (a_i + d_i + \alpha_i) w_i = 0$$

So each  $\alpha_i + d_i \alpha_i = 0$  – try to show  $a_i = 0$ , etc □

## §37.2 Homework 3 Problem 7

Try  $\dim V = 2$ .

## §38 | Dis 10: Nov 3, 2020

### §38.1 Dimension

$T : V \rightarrow W$  linear. Say  $\dim V = n, \dim W = m$ .

$T$  is injective  $\iff \dim \ker T = 0$ . Similarly,  $T$  is surjective  $\iff \dim \operatorname{im} T = m \iff \operatorname{rank}(T) = m$

Important Result:

Rank – Nullity:  $\dim(\ker(T)) + \dim(\operatorname{im}(T)) = \dim V$ . This says the subspaces, in general, are not related but their dimensions add to  $\dim(V)$ .

#### Corollary 38.1

$T : V \rightarrow V$  linear,  $\dim V = n$ . Then  $T$  is injective  $\iff T$  is surjective  $\iff T$  is bijective.

*Proof.* By rank – nullity,

$$\dim(\ker(T)) + \operatorname{rank}(T) = n = \dim V$$

Hence,

$$\dim(\ker(T)) = n - \operatorname{rank}(T)$$

So,  $T$  is injective  $\iff \dim(\ker(T)) = 0 \iff \operatorname{rank}(T) = n = \dim V \iff T : V \rightarrow V$  is surjective.

Finally,  $T$  is bijective  $\implies T$  is injective and  $T$  is injective  $\implies T$  is surjective and thus  $T$  is bijective. So  $T$  is bijective  $\iff T$  is injective  $\iff T$  is surjective. □

**Remark 38.2.** Need  $T : V \rightarrow V$  and  $\dim V$  finite.

In the infinite dimensional case, this fails.

**Exercise 38.1.** Find an example.

#### Corollary 38.3

$T : V \rightarrow W$ ,  $\dim V = n, \dim W = m$ . Then  $\operatorname{rank}(T) \leq n = \dim V$  and  $\operatorname{rank}(T) \leq m = \dim W$ .

*Proof.* Note by rank – nullity,  $\text{rank}(T) = n - \dim(\ker(T)) \leq n$ . Meanwhile,  $\text{im}(T) \subseteq W$ , so  $\text{rank}(T) = \dim(\text{im}(T)) \leq \dim W = m$ .  $\square$

#### Corollary 38.4

$T : V \rightarrow W, \dim V = n, \dim W = m$ . Then  $\dim \ker(T) \geq n - m$ .

*Proof.*  $\dim(\ker(T)) = n - \text{rank}(T) \geq n - m$ .  $\square$

#### Example 38.5

$T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  any  $\mathbb{R}$  linear map.

$$\dim(\ker T) \geq 3 - 2 = 1$$

So there is no injective linear map from  $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ .

#### Example 38.6

$T : \mathbb{R}^5 \rightarrow \mathbb{R}^6$

$$\dim(\ker(T)) \geq 5 - 6 = -1$$

This result does not tell us anything.

#### Example 38.7

$T : \mathbb{R}^3 \rightarrow \mathbb{R}^2, T(x, y, z) = (x, y)$

$$\dim \ker(T) = 1 = 3 - 2$$

$S : \mathbb{R}^3 \rightarrow \mathbb{R}^2, S(x, y, z) = (y, 0)$

$$\dim \ker(T) = 2 > 3 - 2$$

$T : V \rightarrow W, \dim V = n, \dim W = m$  and  $\dim(\ker T) \geq n - m$ . When do we get equality?

$$\dim(\ker T) = n - m$$

$$\iff \text{rank}(T) = m$$

$$\iff T \text{ is surjective}$$

$T : V \rightarrow W$  linear and  $T$  bijective. Then we have a function,  $T^{-1} : W \rightarrow V$ , the inverse function of  $T$ .

**Claim 38.1.**  $T^{-1}$  is also linear.

*Proof.* Let  $w_1, w_2 \in W$  and  $\alpha \in \mathbb{F}$  be arbitrary. We want to show

$$T^{-1}(\alpha w_1 + w_2) = \alpha T^{-1}(w_1) + T^{-1}(w_2)$$

Let  $T^{-1}(w_1) = v_1 \in V$  and  $T^{-1}(w_2) = v_2 \in V$ . Then  $T(\alpha v_1 + v_2)$

$$\begin{aligned} &= \alpha T(v_1) + T(v_2) \\ &= \alpha w_1 + w_2 \end{aligned}$$

Then,

$$\begin{aligned} T(\alpha v_1 + v_2) &= \alpha w_1 + w_2 \\ T^{-1}(\alpha w_1 + w_2) &= \alpha v_1 + v_2 \end{aligned}$$

So,  $T^{-1} : W \rightarrow V$  is linear. □

### §38.2 Homework 4 # 4

$$\begin{array}{ccc} U & \xrightarrow{S} & V \xrightarrow{T} W \\ U & \xrightarrow{S} & \text{im}(S) \xrightarrow{T|_{\text{im}(S)}} W \\ T : V \rightarrow W & \text{im}(S) \subseteq V & \text{subspace} \\ T|_{\text{im}(S)} : \text{im}(S) & \rightarrow & W \end{array}$$

Note

$$(T|_{\text{im}(S)})(a) = T(a), \quad \forall a \in \text{im}(S)$$

$$\text{im}(T \circ S) = \text{im}(T|_{\text{im}(S)}) \subseteq W$$

*Proof.* Let  $y \in \text{im}(T \circ S)$ . Then there is an  $x \in U$  s.t.  $(T \circ S)(x) = y$ . So,  $T(S(x)) = y$ . Note  $a := S(x) \in \text{im}(S)$ . Moreover,  $T|_{\text{im}(S)}(a) = T(a) = T(S(x)) = y$ . So,  $y \in \text{im}(T|_{\text{im}(S)})$

Conversely, let  $y \in \text{im}(T|_{\text{im}(S)})$

$$T|_{\text{im}(S)} : \text{im}(S) \rightarrow W$$

Then,  $\exists z \in \text{im}(S)$  with

$$T|_{\text{im}(S)}(z) = y$$

Since  $z \in \text{im}(S)$ , there exists an  $x \in U$  with  $S(x) = z$ . Then

$$(T \circ S)(x) = T(S(x)) = T(z) = T|_{\text{im}(S)}(z) = y$$

So,  $y \in \text{im}(T \circ S)$ . By double containment, we get

$$\text{im}(T \circ S) = \text{im}(T|_{\text{im}(S)})$$

□

**Claim 38.2.**  $\ker(T|_{\text{im}(S)}) = \ker(T) \cap \text{im}(S)$ .

$$\begin{array}{ccc} U & \xrightarrow{S} & V \xrightarrow{T} W \\ & \searrow \subset & \nearrow \subset \\ & \text{im}(S) & \ker(T) \\ & \xrightarrow{T|_{\text{im}(S)}} & W \\ \cup & & \\ \ker(T|_{\text{im}(S)}) & & \end{array}$$

*Proof.* Let  $x \in \ker(T|_{\text{im}(S)})$ . Then  $x \in \text{im}(S)$  and

$$T(x) = T|_{\text{im}(S)}(x) = 0$$

So,  $x \in \text{im}(S)$  and  $x \in \ker(T)$ . So  $x \in \text{im}(S) \cap \ker(T)$ .

Conversely, suppose  $x \in \text{im}(S) \cap \ker(T)$ . Then  $x \in \text{im}(S)$ , so it is a valid input to  $T|_{\text{im}(S)}$ . Moreover,

$$T|_{\text{im}(S)}(x) = T(x) = 0$$

where we see  $T(x) = 0$ , since  $x \in \ker(T)$ . So,  $x \in \ker(T|_{\text{im}(S)})$ . By double containment,

$$\ker(T|_{\text{im}(S)}) = \ker(T) \cap \text{im}(S)$$

□

Rank – Nullity gives

$$\begin{aligned} \dim(\ker T \cap \text{im } S) &= \dim(\ker T|_{\text{im}(S)}) \\ &= \dim(\text{im}(S)) - \dim(\text{im}(T|_{\text{im}(S)})) \\ &= \dim(\text{im}(S)) - \dim(\text{im}(T \circ S)) \end{aligned}$$

What can we say about  $\ker(T \circ S)$ ?

$$\begin{array}{ccccc} U & \xrightarrow{S} & V & \xrightarrow{T} & W \\ & \searrow & & \nearrow & \\ & & T \circ S & & \end{array}$$

**Claim 38.3.**  $u \in \ker(T \circ S) \iff u \in \ker(S) \text{ or } u \notin \ker(S) \text{ and } S(u) \in \ker(T)$ .

*Proof.* Let  $u \in \ker(T \circ S)$ . Then  $T(S(u)) = 0$ . So  $S(u) \in \ker(T)$ . So either  $S(u) = 0$  or  $S(u) \neq 0$  and  $S(u) \in \ker(T)$ . So either  $u \in \ker(S)$  or  $u \notin \ker(S), S(u) \in \ker(T)$ .

Conversely, suppose  $u \in \ker(S)$ . Then  $S(u) = 0$ , so  $T(S(u)) = T(0) = 0$ . Similarly, if  $u \notin \ker(S)$  but  $S(u) \in \ker(T)$ , then  $T(S(u)) = 0$ . In both cases,  $u \in \ker(T \circ S)$ . □

## §39 | Dis 11: Nov 5, 2020

Discussion on homework # 4 and # 5.

### §39.1 Coordinates

Two types of questions:

- Coordinates of a vector in a vector space given a basis.
- Matrix of a linear transformation between two vector spaces each with a basis.

$\dim(V) = n$ ,  $\mathcal{B} = (v_1, \dots, v_n)$  is an ordered basis of  $V$ ,  $v \in V$ . Since  $v_1, \dots, v_n$  is a basis of  $V$ , and  $v \in V$  there exists a unique  $a_1, \dots, a_n \in \mathbb{F}$  with  $v = a_1v_1 + a_2v_2 + \dots + a_nv_n$ .

$$[v]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^{n \times 1}$$

### Example 39.1

$\mathbb{F} = \mathbb{R}$ ,  $V = \mathbb{R}^2$ ,  $\mathcal{B} = \left\{ \underbrace{(1, 1)}_{v_1}, \underbrace{(2, 1)}_{v_2} \right\}$ . Say we have  $v_0 \in V$  has  $[v_0]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . What is  $v_0$ ? (which element of  $\mathbb{R}^2$  is it?)

$$\begin{aligned} v_0 &= 1v_1 + 2v_2 \\ &= (1, 1) + (4, 2) \\ &= (5, 3) \in \mathbb{R}^2 \end{aligned}$$

Conversely, we have  $w_0 = (4, 3) \in \mathbb{R}^2$ . What are its  $\mathcal{B}$ -coordinates? Or  $[w_0]_{\mathcal{B}}$ ?

$$\begin{aligned} w_0 &= c_1v_1 + c_2v_2 \\ 4 &= c_1 + 2c_2 \\ 3 &= c_1 + c_2 \end{aligned}$$

$\implies c_2 = 1$  and  $c_1 = 2$  and we have

$$[w_0]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

## §40 | Dis 12: Nov 10, 2020

### §40.1 Dis 11 (Cont'd)

Coordinates:  $\dim V = n$ ,  $\beta = \{v_1, \dots, v_n\}$  an ordered basis of  $V$ ,  $v \in V$ ,  $[v]_{\beta} \in F^{n \times 1}$ . There exists unique  $\alpha_1, \dots, \alpha_n \in \mathbb{F}$  with  $v = \alpha_1v_1 + \dots + \alpha_nv_n$ . Set

$$[v]_{\beta} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$T : V \rightarrow F^{n \times 1}$  given by  $T(v) = [v]_{\beta}$ .  $T$  is linear. It's an isomorphism of vector space.

### Example 40.1



$\mathbb{F} = \mathbb{R}, V = \mathbb{R}[t]_2, \beta = \{1, t, t^2\}$  basis of  $V$ .

$$p = 2 + 3t + 4t^2 \in V$$

$$[p]_{\beta} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$\gamma = \{1 + t, t^2 - 1, t\}$  basis of  $V$  – needs proof.

$$p = 2 + 3t + 4t^2$$

$$p = 6(1 + t) + 4(t^2 - 1) - 3(t)$$

$$[p]_{\gamma} = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$$

Above is question type 1:  $\beta$  basis of  $V$ ,  $v \in V, [v]_{\beta}$

Question type 2:  $T : V \rightarrow W$  linear,  $\dim V = n, \dim W = m$   $\beta$  ordered basis of  $V$  and  $\gamma$  ordered basis of  $W$ ,  $[T]_{\beta, \gamma} \in M_{m \times n}(\mathbb{F})$ . We want a matrix  $A \in M_{m \times n}(\mathbb{F})$  such that

$$[T(v)]_{\gamma} = A[v]_{\beta}$$

such a matrix  $A$  exists and is unique.

$$[T]_{\beta, \gamma} := A$$

So,

$$[T(v)]_{\gamma} = [T]_{\beta, \gamma} [v]_{\beta}$$

for all  $v \in V$ . “Matrix associated to  $T : V \rightarrow W$  does not make sense on its own – it gives  $\beta$  an ordered basis of  $V, \gamma$  an ordered basis of  $W$ ”.

Nice Trick:

$T : V \rightarrow W, \beta = \{v_1, \dots, v_n\}$  basis of  $V, \gamma = \{w_1, \dots, w_m\}$  basis of  $W$ .  $[T]_{\beta, \gamma}$  can be figured out column by column. First column of the matrix will just be  $[T(v_1)]_{\gamma}, \dots, i^{\text{th}}$  column will just be  $[T(v_i)]_{\gamma}$  which is Type 1 question.

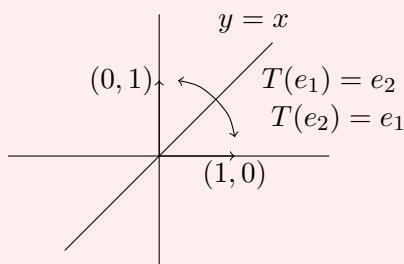
### Example 40.2

$\mathbb{F} = \mathbb{R}, V = W = \mathbb{R}^2$

$\beta = \{e_1, e_2\}$  basis of  $V$  but also of  $W$ .

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  reflection about the line  $y = x$ .  $[T]_{\beta, \beta} = ?$

First column is  $[T(e_1)]_{\beta}$ , second column is  $[T(e_2)]_{\beta}$ .



$$[T(e_1)]_\beta = [e_2]_\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T(e_2)]_\beta = [e_1]_\beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

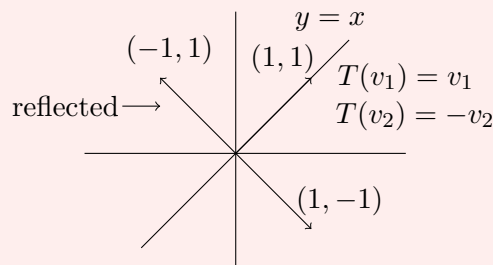
$$[T]_{\beta,\beta} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

**Example 40.3**

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  reflection about  $y = x$

$$\gamma = \{e_1 + e_2, e_1 - e_2\}$$

First column:  $[T(v_1)]_\gamma$ , second column:  $[T(v_2)]_\gamma$



$$[T]_{\gamma,\gamma} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**Example 40.4**

$V = \mathbb{R}[t]_3 = W$ ,  $\beta = \{1, t, t^2, t^3\}$  basis of  $V$  and basis of  $W$ ,  $D : \mathbb{R}[t]_3 \rightarrow \mathbb{R}[t]_3 -$

$D(f) = f'$ .  $D$  is linear (check this).  $[D]_{\beta,\beta} = ?$

$$D(1) = 0 \qquad [D(1)]_{\beta} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(t) = 1 \qquad [D(t)]_{\beta} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(t^2) = 2t \qquad [D(t^2)]_{\beta} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$D(t^3) = 3t^2 \qquad [D(t^3)]_{\beta} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

So,

$$[D]_{\beta,\beta} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

### Example 40.5

$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\gamma = \left\{ \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \left( -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right) \right\}$$

$\gamma$  is a basis of  $\mathbb{R}^3$ . In fact, it's an ON basis and  $v_1 \times v_2 = v_3$ .  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  rotation by  $\frac{\pi}{2}$  counterclockwise about  $v_3$ .  $[T]_{\gamma,\gamma}$ ?

$$T(v_1) = v_2$$

$$T(v_2) = -v_1$$

$$T(v_3) = v_3$$

So,

$$\begin{aligned} [T(v_1)]_\gamma &= [v_2]_\gamma = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ [T(v_2)]_\gamma &= [-v_1]_\gamma = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \\ [T(v_3)]_\gamma &= [v_3]_\gamma = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ [T]_{\gamma,\gamma} &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

for  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  rotation by  $\theta$  counterclockwise about  $v_3$

$$[S]_{\gamma,\gamma} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(try to show this)

## §41 | Dis 13: Nov 12, 2020

### §41.1 Dis 12 (Cont'd)

#### Example 41.1

$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  rotation by  $\theta$  counterclockwise about  $v_3 = v_1 \times v_2$

$\gamma = \{e_1, e_2, e_3\}$  standard basis.  $[T]_\gamma$ ?

Method 1:  $\begin{bmatrix} [T(e_1)]_\gamma & [T(e_2)]_\gamma & [T(e_3)]_\gamma \end{bmatrix}$

Method 2: Can we convert between  $\beta$  coordinates and  $\gamma$  coordinates?

Trick:

$$\begin{aligned} \text{Id} : \mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ [x]_\gamma &= [\text{Id}]_{\beta,\gamma} [x]_\beta \\ [x]_\beta &= [\text{Id}]_{\gamma,\beta} [x]_\gamma \end{aligned}$$

Let  $S = [\text{Id}]_{\beta,\gamma}$ , so  $[\text{Id}]_{\gamma,\beta} = S^{-1}$

$$\begin{aligned} [T(x)]_\gamma &= S [T(x)]_\beta \\ &= SB[x]_\beta \\ &= SBS^{-1}[x]_\gamma \end{aligned}$$

$[T(x)]_\gamma = SBS^{-1}[x]_\gamma$  for all  $x \in \mathbb{R}^3$ . So

$$\begin{aligned} [T]_\gamma &= SBS^{-1} \\ B = [T]_\beta &= \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ S = [\text{Id}]_{\beta, \gamma} &= \begin{bmatrix} [v_1]_\gamma & [v_2]_\gamma & [v_3]_\gamma \end{bmatrix} \end{aligned}$$

### Example 41.2

$$\beta = \left\{ \frac{(1, 1, 1)}{\sqrt{3}}, \frac{(-1, 1, 0)}{\sqrt{2}}, \frac{(-1, -1, 2)}{\sqrt{6}} \right\}$$

$$B = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$[T]_\gamma = SBS^{-1}$$

$\dim V = n$

**Claim 41.1.**  $T : V \rightarrow V$ ,  $\beta = \{v_1, \dots, v_n\}$  basis of  $V$ . Then,  $[T]_\beta$  is diagonal iff  $\beta$  is an eigenbasis ( $v_i$  is an eigenvector of  $T$  for each  $i = 1, \dots, n$ )

*Proof.* ( $\rightarrow$ ) Suppose  $[T]_\beta$  is diagonal. The  $i$ th column of this matrix is  $[T(v_i)]_\beta$ . On the

other hand, since  $[T]_\beta$  is diagonal, the  $i$ th column is  $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_i \\ 0 \\ \vdots \end{bmatrix}$ , i.e., the  $i$ th column is  $a_i e_i \in \mathbb{F}^{n \times 1}$

where  $a_i \in \mathbb{F}$ . So,

$$[T(v_i)]_\beta = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_i \\ 0 \\ \vdots \end{bmatrix}$$

$$T(v_i) = a_i v_i$$

Hence,  $v_i$  is an eigenvector. This holds for any  $i$ . So  $\beta$  is a basis consisting entirely of eigenvectors. We conclude  $\beta$  is an eigenbasis.

( $\leftarrow$ ) Suppose  $\beta$  is an eigenbasis. Then, for each  $i$ ,  $T(v_i) = \lambda_i v_i$  for some  $\lambda_i \in \mathbb{F}$ . So

$$[T(v_i)]_\beta = \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ 0 \\ \vdots \end{bmatrix} = \lambda_i e_i$$

So,

$$[T]_\beta = \begin{bmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

is diagonal. □

**Claim 41.2.**  $T : V \rightarrow W$

$$\dim V = \dim W = n$$

Then,  $T$  is an isomorphism if and only if there exists a basis  $\beta$  of  $V$  and a basis  $\gamma$  of  $W$  s.t.

$$[T]_{\beta, \gamma} = I_n \in M_n(\mathbb{F})$$

which is similar to # 8. Hint on # 8: The matrix can be turned to look like a matrix with 1 and 0 as its entries.

Another hint: Revisit the proof of rank – nullity. That will help with basis to choose.

*Proof.* ( $\rightarrow$ ) Suppose  $T : V \rightarrow W$  is an isomorphism. Let  $\beta = \{v_1, \dots, v_n\}$  be a basis of  $V$ . We claim  $\gamma = \{Tv_1, \dots, Tv_n\}$  is a basis of  $W$ .

1.  $\gamma$  is a linearly indep. set as follows: Suppose  $a_1, \dots, a_n \in \mathbb{F}$  have

$$a_1 T(v_1) + \dots + a_n T(v_n) = 0$$

Then  $T(a_1 v_1 + \dots + a_n v_n)$ , by linearity

$$= \sum_{i=1}^n a_i T(v_i) = 0$$

Hence,  $\sum_{i=1}^n a_i v_i \in \ker(T)$ . Since  $T$  is an isomorphism, it is injective. So,  $\ker T = 0$ . Hence,

$$\sum_{i=1}^n a_i v_i = 0$$

Since  $v_1, \dots, v_n$  forms a basis of  $V$ , they are linearly indep. We conclude  $a_j = 0 \forall j = 1, \dots, n$ . Hence, we conclude  $T(v_1), \dots, T(v_n)$  are linearly indep.

2. Next, we claim  $\text{span}(T(v_1), \dots, T(v_n)) = W$ . To see this, note since each  $T(v_i) \in W$ , certainly  $\text{span}(T(v_1), \dots, T(v_n)) \subset W$ . Next, let  $w \in W$ . Since  $T$  is an isomorphism, it is surjective. Hence,  $\text{im}(T) = W$ . Since  $w \in W = \text{im}(T)$ , we have  $w \in \text{im}(T)$ . Thus, there exists  $v \in V$  with  $w = T(v)$ . Since  $v_1, \dots, v_n$  are a basis of  $V$ , then  $v \in V = \text{span}(v_1, \dots, v_n)$ . Hence, there exists  $a_1, \dots, a_n \in \mathbb{F}$  s.t.  $v = a_1v_1 + \dots + a_nv_n$ . Now,

$$\begin{aligned} w = T(v) &= T\left(\sum_{i=1}^n a_i v_i\right) \\ &= \sum_{i=1}^n a_i T(v_i) \end{aligned}$$

So,  $w \in \text{span}(T(v_1), \dots, T(v_n))$ . So, we conclude

$$W \subset \text{span}(T(v_1), \dots, T(v_n))$$

By double containment,  $\text{span}(T(v_1), \dots, T(v_n)) = W$ .

By 1) and 2),  $\gamma = \{T(v_1), \dots, T(v_n)\}$  form a basis of  $W$ . Finally,  $[T]_{\beta, \gamma}$  has  $i$ th column

$$[T(v_i)]_{\gamma} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = e_i \in \mathbb{F}^{n \times 1}$$

for all  $i = 1, \dots, n$ . So  $[T]_{\beta, \gamma} = I_n \in M_n(\mathbb{F})$ .

( $\leftarrow$ ) Suppose there exists a basis  $\beta$  of  $V$  and a basis  $\gamma$  of  $W$  s.t.

$$[T]_{\beta, \gamma} = I_n \in M_n(\mathbb{F})$$

Write  $\beta = \{v_1, \dots, v_n\}$ ,  $\gamma = \{w_1, \dots, w_n\}$ . Argument outline:

- Observe  $w_i = T(v_i)$  for all  $i = 1, \dots, n$ .
- $T$  is surjective, which means every  $w \in W$  is a linear combo of  $w_1, \dots, w_n \in \text{im}(T)$ .
- $T$  is injective? Use rank – nullity

$$\begin{aligned} \dim(\ker T) &= \dim V - \dim \text{im } T \\ &= \dim V - \dim W = 0 \end{aligned}$$

Thus,  $T$  is an isomorphism. □

## §42 | Dis 14: Nov 17, 2020

Hw 7 – “Take home midterm” which is due on Dec 10th.

work on it  
asap

## §42.1 Midterm Solutions

1.  $V$  a vector space over  $\mathbb{F}$ ,  $S \subseteq V$  nonempty subset.

a)  $v \in \text{Span}(S) \iff \exists k \in \mathbb{N}, v_1, \dots, v_k \in S, a_1, \dots, a_k \in \mathbb{F}$  with  $v = a_1 v_1 + \dots + a_k v_k$ .

b)  $S \subseteq V$  is a linearly dependent subset of  $V \iff \exists k \in \mathbb{N}$  with  $v_1, \dots, v_k \in S, a_1, \dots, a_k \in \mathbb{F}^x$  such that

$$a_1 v_1 + \dots + a_k v_k = 0$$

or phrase as:  $a_1, \dots, a_k \in \mathbb{F}$  with  $a_1 v_1 + \dots + a_k v_k = 0$  and not all  $a_i = 0$ .

c)  $S \subseteq V$  is a linearly independent subset of  $V \iff S$  is not a linearly dependent subset of  $V$ .

d)  $W_i \subset V$  subspace for each  $i \in I$  such that  $V = \sum_{i \in I} W_i \iff V \subseteq \sum_{i \in I} W_i \iff v \in V$ , there exists  $k \in \mathbb{N}, i_1, \dots, i_k \in I, v_j \in W_{i_j} (j = 1, \dots, k)$  s.t.  $v = \sum_{j=1}^k v_j$

2. a) # of subspaces of dim 1 in  $M_2(\mathbb{F}_2)$ ,  $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$  field of 2 elements

$$V = M_2(\mathbb{F}_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_2 \right\}$$

$|V| = 2^4 = 16$  and  $\dim_{\mathbb{F}_2} V = 4$  with the usual basis.  $U \subseteq V$  with  $\dim_{\mathbb{F}_2} U = 1 \iff U = \text{span}\{A\}$  for  $A \in M_2(\mathbb{F}_2) = \{0, A\}$  for  $A \neq 0$ . So, there are 15 of them.

- 0 dim subspaces  $\{0\}$
- 1 dim subspaces  $\{0, A\}, A \neq 0$
- 2 dim subspaces  $\{0, A, B, A + B\}, A \neq 0, B \neq 0, A \neq B$
- 3 dim subspaces  $\{0, A, B, A + B, C, A + C, B + C, A + B + C\}$  for  $A, B, C \neq 0$  pairwise disjoint and TBA

b)  $V = M_2(\mathbb{F}), u = \{A \in M_2(\mathbb{F}) | A^\top = -A\}$  Similar to a homework problem.

i)  $B = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$

ii)  $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

d)  $T : \mathbb{R}[t] \rightarrow \mathbb{R}[t]$ . An example could be  $T(f) = f'''$ .

**Claim 42.1.**  $\ker(T) = \text{span}\{1, t, t^2\}$

Over a field of characteristics  $p$ ,

$$\begin{aligned} t^p &\mapsto 0 \\ t^{p+1} &\mapsto 0 \\ t^{p+2} &\mapsto 0 \end{aligned}$$

We have more in  $\ker(T)$ .

e)  $V = W = \mathbb{R}[t]$ .  $T : \mathbb{R}[t] \rightarrow \mathbb{R}[t]$  where  $T(f) = t \cdot f$ .  $S : \mathbb{R}[t] \rightarrow \mathbb{R}[t]$  where  $S(f) = f'$ . In general, think about a basis of  $V$  and a basis of  $W$  (need  $V, W$  infinite dimensional).



3. a)  $\implies$  b) Suppose we have  $w_1 \in W_1, w_2 \in W_2, w_3 \in W_3$  with  $w_1 + w_2 + w_3 = 0$ . We want to show  $w_1 = w_2 = w_3 = 0$ . Observe that  $0 \in V$  can be written as  $0 = 0 + 0 + 0 = x_1 + x_2 + x_3$  where  $x_1 = 0 \in W_1, x_2 = 0 \in W_2, x_3 = 0 \in W_3$ . So,

$$0 = x_1 + x_2 + x_3 = w_1 + w_2 + w_3$$

By a), such a decomposition must be unique. So, we must have

$$w_1 = x_1 = 0$$

$$w_2 = x_2 = 0$$

$$w_3 = x_3 = 0$$

- b)  $\implies$  c) Let  $i, j, k \in \{1, 2, 3\}$  be distinct. We want to show  $W_i \cap (W_j + W_k) = \{0\}$ . Let  $x \in W_i \cap (W_j + W_k)$ . We WTS  $x = 0$ . Since  $x \in W_i \cap (W_j + W_k)$ , we have  $x \in W_i$  and  $x \in W_j + W_k$ . So there exists  $v_j \in W_j$  and  $v_k \in W_k$  with  $x = v_j + v_k$ . Now,

$$x - v_j - v_k = 0$$

By part b),  $x = 0, -v_j = 0, -v_k = 0$ . In particular,  $x = 0$ , as desired.

- c)  $\implies$  a) Let  $v \in V$  be arbitrary. Since  $V = W_1 + W_2 + W_3$ , there exists  $x_1 \in W_1, x_2 \in W_2, x_3 \in W_3$  s.t.  $v = x_1 + x_2 + x_3$ . Suppose we also have  $y_1 \in W_1, y_2 \in W_2, y_3 \in W_3$  with  $v = y_1 + y_2 + y_3$ . Then,  $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$ , so  $x_1 - y_1 = (y_2 - x_2) + (y_3 - x_3)$ .

$$x_1 - y_1 \in W_1 \cap (W_2 + W_3) = \{0\}$$

by c). So  $x_1 = y_1$ . Similarly,  $x_2 = y_2, x_3 = y_3$ . We conclude  $v = x_1 + x_2 + x_3$  for unique  $x_1 \in W_1, x_2 \in W_2, x_3 \in W_3$ . Since  $v \in V$  was arbitrary, this holds for all  $v \in V$ .

**Remark 42.1.** In this setting, we say  $v = w_1 \oplus w_2 \oplus w_3$ .

## §43 | Dis 15: Nov 19, 2020

### §43.1 Some Notes about Take-home Midterm

Open book – cite your sources. Prove everything not done in lecture or previous homework. No collaborations allowed.

### §43.2 Dual Space

$V$  finite dimensional vector space. Recall  $V^* = \mathcal{L}(V, \mathbb{F})$  then  $\dim V^* = \dim V$ . Given a basis  $e_1, \dots, e_n$  of  $V$ , we have a dual basis  $f_1, \dots, f_n$  of  $V^*$  with  $f_i \in V^*$  is given by

$$f_i : V \rightarrow \mathbb{F} \text{ with } f_i \left( \sum_{j=1}^n a_j e_j \right) = a_i. \text{ That is, } f_i(e_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

**Example 43.1**

$V = \mathbb{R}^2, e_1 = (1, 0), e_2 = (0, 1). V^*, f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}, f_1(x, y) = x, f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}, f_2(x, y) = y.$   
TBA

Let  $S \subset V$ , the annihilator  $S$  is the subset  $S^\circ \subset V^*$ ,

$$S^\circ := \{f \in V^* : f(x) = 0, \forall x \in S\}$$

**Example 43.2**

$S = \{0\} \subset V - S^\circ = V^*$   
 $S = V \subset V, S^\circ = \{0\} \subset V^*$

**Claim 43.1.**  $W \subset V$  subspace,  $V$  finite dimensional, then  $\dim W^\circ = \dim V - \dim W$ .

*Proof.* Pick  $w_1, \dots, w_k$  a basis of  $W$ . Extend to a basis of  $V : w_1, \dots, w_k, v_{k+1}, \dots, v_n$ ,  
 $T : V \rightarrow V^*$

$$w_1, \dots, w_k, v_{k+1}, \dots, v_n \mapsto 0, \dots, 0, f_{k+1}, \dots, f_n$$

- $\ker T = W$
- $\text{im}(T) = W^\circ$

Use rank – nullity. □

Analogy:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r & s \end{bmatrix}$$

$$\begin{bmatrix} z & w \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} t & u \end{bmatrix}$$

$T : V \rightarrow W$ . Define  $T^t : W^* \rightarrow V^*$ . Given  $g \in W^*, g : W \rightarrow \mathbb{F}$ . We want

$$T^t g \in V^*$$

$$T^t g : V \rightarrow \mathbb{F}$$

$$(T^t g)(v) = g(T(v)) \text{ for all } v \in V$$

That is,  $T^t TBA$

**Theorem 43.3**

$V, W$  finite dimensional vector space.  $T : V \rightarrow W, T^t : W^* \rightarrow V^*$ . Then,

$$\ker T^t = \text{im}(T)^\circ$$

$$\text{im}(T^t) = \ker(T)^\circ$$

$$\text{rank}(T) = \text{rank}(T^t)$$

## §44 | Dis 16: Nov 24, 2020

Takehome Midterm: Prove everything you use not done in lecture or HW. Cite anything you reference.

### §44.1 Inner Product Spaces

Let  $V$  be a vector space over  $\mathbb{F}$ . ( $\mathbb{F}$  is either  $\mathbb{R}$  or  $\mathbb{C}$ ). An inner product on  $V$  is a map

$$\langle, \rangle : V \times V \rightarrow \mathbb{F}$$

(e.g., dot product in  $\mathbb{R}^n$ ) with (for every  $v \in V, v_1, v_2, v_3 \in V, a \in \mathbb{F}$ )

- $\langle v, v \rangle \geq 0$  with equality  $\iff v = 0$ .
- $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$
- $\langle av_1, v_2 \rangle = a\langle v_1, v_2 \rangle$
- $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$

$(V, \langle, \rangle)$  is called an inner product space.

#### Proposition 44.1

Let  $(V, \langle, \rangle)$  be an inner product space. Then, for all  $v, w, x \in V$  and  $a \in \mathbb{F}$ , we have

$$\langle v, aw + x \rangle = \bar{a}\langle v, w \rangle + \langle v, x \rangle$$

That is,

1.  $\langle v, aw \rangle = \bar{a}\langle v, w \rangle$
2.  $\langle v, w + x \rangle = \langle v, w \rangle + \langle v, x \rangle$

*Proof.* 1.

$$\begin{aligned} \langle v, aw \rangle &= \overline{\langle aw, v \rangle} \\ &= \overline{a\langle w, v \rangle} \\ &= \bar{a}\overline{\langle w, v \rangle} \\ &= \bar{a}\langle v, w \rangle \end{aligned}$$

2.

$$\begin{aligned} \langle v, w + x \rangle &= \overline{\langle w + x, v \rangle} \\ &= \overline{\langle w, v \rangle + \langle x, v \rangle} \\ &= \overline{\langle w, v \rangle} + \overline{\langle x, v \rangle} \\ &= \langle v, w \rangle + \langle v, x \rangle \end{aligned}$$

□

**Proposition 44.2**

For all  $v \in V$ ,  $\langle v, 0 \rangle = 0 = \langle 0, v \rangle$ .

*Proof.* Let  $v \in V$ . Then

$$\begin{aligned}\langle v, 0 \rangle &= \langle v, 0 + 0 \rangle \\ &= \langle v, 0 \rangle + \langle v, 0 \rangle\end{aligned}$$

So,  $0 = \langle v, 0 \rangle$ . Finally  $\langle 0, v \rangle = \overline{\langle v, 0 \rangle} = \overline{0} = 0$ .  $\square$

$(V, \langle, \rangle)$  inner product space. For any  $v \in V$ , define  $\|v\| = \sqrt{\langle v, v \rangle}$ . Note  $\langle v, v \rangle \geq 0$  so  $\|v\| = \sqrt{\langle v, v \rangle}$  is defined. Moreover,

$$\|v\| = 0 \iff v = 0$$

**Remark 44.3.** Let  $V = \mathbb{R}^n$ ,  $\mathbb{F} = \mathbb{R}$ . Define inner product on  $V$  via

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = x_1 y_1 + \dots + x_n y_n$$

and

$$\|(x_1, \dots, x_n)\| = \sqrt{\langle (x_1, \dots, x_n), (x_1, \dots, x_n) \rangle} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

Inner product gives a notion of angles. Norm gives a notion of length. Metric gives a notion of distance.

**Theorem 44.4 (Pythagorean)**

$(V, \langle, \rangle)$  inner product space. Let  $u, v \in V$ . If  $\langle u, v \rangle = 0$ , then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

*Proof.*

$$\begin{aligned}\|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \langle u, u + v \rangle + \langle v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + \|v\|^2\end{aligned}$$

$\square$

**Theorem 44.5 (Cauchy – Schwarz)**

$(V, \langle, \rangle)$  an inner product space. Let  $u, v \in V$ . Then

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

*Proof.* If  $u \neq 0$ ,

$$v_{\parallel} := \frac{\langle u, v \rangle}{\|u\|^2} u$$

$$v_{\perp} := v - v_{\parallel}$$

Then,

$$\begin{aligned} \langle u, v_{\perp} \rangle &= \langle u, v \rangle - \langle u, v_{\parallel} \rangle \\ &= \langle u, v \rangle - \langle u, \frac{\langle v, u \rangle}{\|u\|^2} u \rangle \\ &= \langle u, v \rangle - \frac{\langle u, v \rangle}{\|u\|^2} \langle u, u \rangle \\ &= 0 \end{aligned}$$

So,  $\langle u, v \rangle = \langle u, v_{\parallel} \rangle$ . Also check  $\langle v_{\parallel}, v_{\perp} \rangle = 0$  where  $v_{\parallel} = au$  for some scalar  $a \in \mathbb{F}$ .

$$\begin{aligned} \|v\|^2 &= \|v_{\parallel} + v_{\perp}\|^2 \\ &= \|v_{\parallel}\|^2 + \|v_{\perp}\|^2 \\ &\geq \|v_{\parallel}\|^2 = \frac{|\langle v, u \rangle|^2}{\|u\|^2} \end{aligned}$$

This will give

$$\|u\| \cdot \|v\| \geq |\langle v, u \rangle|$$

If  $u = 0$ ,  $\|u\| = 0$  and  $\langle v, u \rangle = 0$  so  $|\langle v, u \rangle| \leq \|u\| \cdot \|v\|$  because both sides are 0.  $\square$

#### Example 44.6

$\mathbb{F} = \mathbb{R}, V = \mathbb{R}^n, \langle, \rangle$  dot product.

$\mathbb{F} = \mathbb{C}, V = \mathbb{C}^n$

$$\langle (z_1, \dots, z_n), (w_1, \dots, w_n) \rangle = z_1 \overline{w_1} + \dots + z_n \overline{w_n}$$

Note that

$$\begin{aligned} \langle (z_1, \dots, z_n), (z_1, \dots, z_n) \rangle &= z_1 \overline{z_1} + \dots + z_n \overline{z_n} \\ &= |z_1|^2 + \dots + |z_n|^2 \end{aligned}$$

#### Example 44.7

$n = 1, \mathbb{F} = \mathbb{C}, V = \mathbb{C}$

$$\begin{aligned} \langle a + bi, c + di \rangle &= (a + bi)(c - di) \\ \langle a + bi, a + bi \rangle &= (a + bi)(a - bi) = a^2 + b^2 = |a + bi|^2 \end{aligned}$$

**Example 44.8**

$\mathbb{F} = \mathbb{R}, V = \mathbb{R}[t]$ . Define  $\langle p, q \rangle = \int_0^1 p(t)q(t)dt$

$$\begin{aligned}\langle t, t^2 + 2t \rangle &= \int_0^1 t^3 + 2t^2 dt \\ &= \frac{11}{12}\end{aligned}$$

**Example 44.9**

$\mathbb{F} = \mathbb{C}, V = \mathbb{C}[t]$ . Define

$$\langle p, q \rangle = \int_0^1 p(t)\overline{q(t)}dt$$

Note

$$\langle (1+i)t+1, (1+i)t^2+it \rangle = \int_0^1 ((1+i)t+1)((1-i)t^2-it)dt$$

## §45 | Dis 17: Dec 1, 2020

### §45.1 Orthogonal Projection

Let  $V$  be a finite dimensional inner product space over  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ .  $W \subset V$  subspace. We need to find the orthogonal projection of  $\vec{v}$  onto  $W$ .

Useful idea:

1. Find a basis of  $W$ .
2. GS to get orthonormal basis of  $W$ .
3. Extend to an orthonormal basis of  $V$ .

Pick  $u_1, \dots, u_m$  to be orthonormal basis of  $W$  and extend to  $u_1, \dots, u_m, u_{m+1}, \dots, u_n$  which is an orthonormal basis of  $V$ .

Let  $\vec{v} \in V$ . We have

$$\vec{v} = a_1u_1 + a_2u_2 + \dots + a_nu_n$$

where  $a_i \in \mathbb{F}$ . In fact,  $a_i = \langle \vec{v}, u_i \rangle$  for each  $i$ .

$$\text{proj}_W(\vec{v}) = a_1u_1 + \dots + a_mu_m$$

$\text{proj}_W : V \rightarrow V$  and  $\text{proj}_W(\vec{w}) = \vec{w}$  for  $\vec{w} \in W$ .

$$\begin{aligned}\text{im}(\text{proj}_W) &= W \\ \text{ker}(\text{proj}_W) &= W^\perp\end{aligned}$$

**Fact 45.1.** •  $\langle \text{proj}_W(\vec{v}), \vec{v} - \text{proj}_W(\vec{v}) \rangle = 0$ . As a result,

$$\begin{aligned}\|\vec{v}\|^2 &= \|\vec{v} - \text{proj}_W(\vec{v}) + \text{proj}_W(\vec{v})\|^2 \\ &= \|\vec{v} - \text{proj}_W(\vec{v})\|^2 + \|\text{proj}_W(\vec{v})\|^2\end{aligned}$$

In particular,

$$\|\text{proj}_W(\vec{v})\| \leq \|\vec{v}\|$$

## §45.2 Hw 8

1.  $V = C[1, 3]$  over  $\mathbb{R}$

$$\langle f, g \rangle = \int_1^3 fg$$

Find the constant function  $g$  “closest” to a given function  $f \in V$ .

Idea:  $W = \{g \in C[1, 3] : g \text{ is constant}\}$ . Find  $\text{proj}_W(f)$ . This will be the constant function closest to  $f$ .

Let  $V$  be an inner product space. Let  $W \subseteq V$  be a finite dimensional subspace. Pick  $u_1, \dots, u_m$  an orthonormal basis of  $W$ .

$$\text{proj}_W(\vec{v}) = \langle \vec{v}, u_1 \rangle u_1 + \dots + \langle \vec{v}, u_m \rangle u_m$$

**Claim 45.1.** To minimize  $\|\vec{v} - \vec{w}\|$  for  $\vec{w} \in W$ , pick  $\vec{w} = \text{proj}_W(\vec{v})$

*Proof.*

$$\begin{aligned}\|\vec{v} - \vec{w}\| &= \|\vec{v} - \text{proj}_W(\vec{v}) + \text{proj}_W(\vec{v}) - \vec{w}\|^2 \\ &= \|\vec{v} - \text{proj}_W(\vec{v})\|^2 + \|\text{proj}_W(\vec{v}) - \vec{w}\|^2 \\ &\geq \|\vec{v} - \text{proj}_W(\vec{v})\|^2\end{aligned}$$

with equality if and only if

$$\|\text{proj}_W(\vec{v}) - \vec{w}\| = 0 \iff \vec{w} = \text{proj}_W(\vec{v})$$

□

### Example 45.1

$V = \mathbb{R}^3$  standard inner product (dot product). Let  $W = \{(t, -t + s, -s) : t, s \in \mathbb{R}\}$ . Find

$$1. \text{proj}_W \left( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right).$$

2. An orthonormal basis for  $W^\perp$ .

General method:

1. Find a basis for  $W$

2. Turn it into an orthonormal basis
3. Extend to an orthonormal basis of  $\mathbb{R}^3$
4. Compute

$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$  form a basis of  $W$ . Extend to a basis of  $\mathbb{R}^3$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ basis of } \mathbb{R}^3$$

Now, we apply GS to obtain

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$u_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Now,

$$\begin{aligned} \text{proj}_W \left( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right) &= \langle \vec{v}, u_1 \rangle u_1 + \langle \vec{v}, u_2 \rangle u_2 \\ &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \in W \end{aligned}$$

Other sanity check:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \in W^\perp$$

Orthonormal basis of  $W^\perp$  :  $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .