Math 134 – Nonlinear ODE

University of California, Los Angeles

Duc Vu

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This is math 134 – Linear and Nonlinear System of Differential Equations taught by Professor Wink. The class lecture is prerecorded, and we have live session every Monday and Friday at 3:00 pm – 3:50 pm for Q & A. We use *Nonlinear Dynamics and Chaos* 2$^{\text{nd}}$ by *Steven Strogatz* as our main book for the class. Other course notes can be found through my github. Any error spotted in the notes is my responsibility, and please let me know through my email at ducvu2718@ucla.edu.

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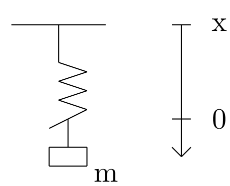
§1 Lec 1: Jan 4, 2021

§1.1 Intro to Dynamical Systems

There are two types of dynamical systems:

1. Discrete in time:
   - Difference equation
   - Iterated map: $a_{n+1} = f(a_n)$
2. Continuous in time: differential equation
   - Partial Differential Equation (PDE):
     e.g. heat equation
     $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$
     wave equation
     $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$
     where the derivatives w.r.t time and space.
   - Ordinary Differential Equation (ODE):
     i) Harmonic oscillator



m: mass
k: spring constant

$$m\dot{x} + kx = 0$$

If $\omega^2 = \frac{k}{m}$, then
$$x(t) = x_0 \cos(\omega t) + x_1 \sin(\omega t)$$

ii) Damped harmonic oscillator
$$m\ddot{x} + b\dot{x} + kx = 0, \qquad b\text{: damping constant}$$

iii) Forced, damped harmonic oscillator
$$m\ddot{x} + b\dot{x} + kx = F\cos(t), \qquad F\text{: force}$$

so derivatives w.r.t time only.

Definition 1.1 (Order of ODE) — Highest occurring derivative is defined as the order of the ODE.

Remark 1.2. We can always write an ODE of n^{th} order as a system of ODEs of 1^{st} order.

Trick: Consider the damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

Then,

$$\begin{aligned} \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x \\ &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{aligned}$$

i.e.,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{aligned}$$

General framework: $\dot{x} = f(t, x)$

$$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

i.e.,

$$\begin{aligned} \dot{x}_1 &= f_1(t_1, x_1, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, \dots, x_n) \end{aligned} \tag{1}$$

which is 1^{st} order n-dimensional ODE.

Definition 1.3 (Linear ODE) — The ODE (1) is called linear if $f(t, x) = A(t) \cdot x$ for a time dependent matrix $A(t)$, otherwise we call it non-linear.

Example 1.4

The damped harmonic oscillator is linear.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Question 1.1. Why are linear equations special?

They satisfy the principle of superposition. If ϕ, ψ solve $\dot{x} = A(t)x$, then $y(t) = c \cdot \phi(t) + \psi(t), c \in \mathbb{R}$ also solves $\dot{x} = A(t)x$. This is valid because $\dot{y} = c\dot{\phi} + \dot{\psi} = cA\phi + A\psi = A(c\phi + \psi) = Ay$. For non-linear ODEs, the principle of superposition fails.

Definition 1.5 (Autonomous ODE) — The ODE (1) is called <u>autonomous</u> if f does not depend on t, i.e., $f(t,x) = f(x)$.

Example 1.6

$$m\ddot{x} + b\dot{x} + kx = F\cos(t)$$

is non-autonomous.

However, we can always consider an autonomous system instead. Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= t \end{aligned}$$

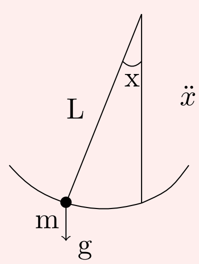
Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + F\cos(x_3) \\ \dot{x}_3 &= 1 \end{aligned}$$

We will primarily study autonomous 1st order system in 1 or 2 variables.

Example 1.7 (Swinging Pendulum)

Consider a swinging pendulum



$$\ddot{x} + \frac{g}{L}\sin(x) = 0$$

Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L}\sin(x_1) \end{aligned}$$

1st order, non-linear autonomous ODE in 2 variables.

Question 1.2. What can we say about the behavior of a solution $x_1(t), x_2(t)$ for larger time t? How does it depend on $\frac{g}{L}$?

<u>Idea</u>: Use geometric methods, without solving $\dot{x} = f(x)$ explicitly, to make qualitative statements about the long time behavior of the solution.

§2 Lec 2: Jan 6, 2021

§2.1 Phase Portraits

We want to study 1D autonomous dynamical systems

$$\dot{x} = f(x), \qquad f : \mathbb{R} \to \mathbb{R}$$

Remark 2.1. $x(t)$ is the solution to $\dot{x} = f(x)$ with $x(0) = x_0$. Find the solution $y(t)$ with $y(t_0) = x_0$.

Ans: $y(t) = x(t - t_0)$ because $y(t_0) = x(0) = x_0$ and $\dot{y}(t) = \dot{x}(t - t_0) = f(x(t - t_0)) = f(y(t))$.

Example 2.2

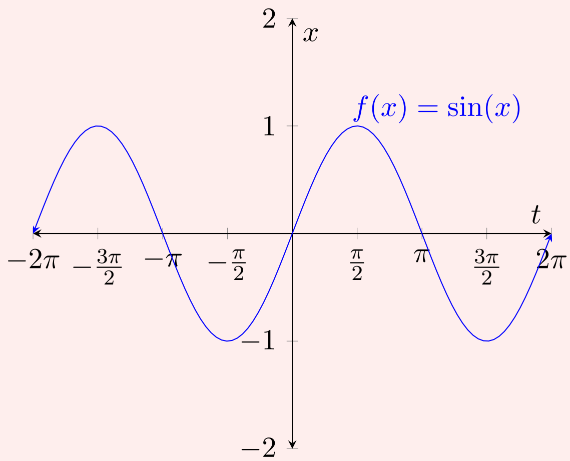
$x = \sin(x)$. Suppose $x_0 = \frac{\pi}{4}, x(t)$ solution with $x(0) = x_0$. Answer the followings

- Describe the long time behaviors of $x(t)$ as $t \to \infty$.
- How does the long time behavior depend on $x_0 \in \mathbb{R}$?

Attemp 1: Find explicit solution

$$\begin{aligned}\frac{dx}{dt} &= \sin(x)\\ dt &= \frac{dx}{\sin(x)}\\ t &= -\ln\left|\frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)}\right| + c\end{aligned}$$

We know $x(0) = x_0$, so $c = \ln\left|\frac{1+\cos(x_0)}{\sin(x_0)}\right|$. But what is $x(t) =$? This approach fails!
Attempt 2: Draw a phase portrait/diagram. We want to interpret the velocity $\dot{x} = f(x)$ as a vector field on the real line.

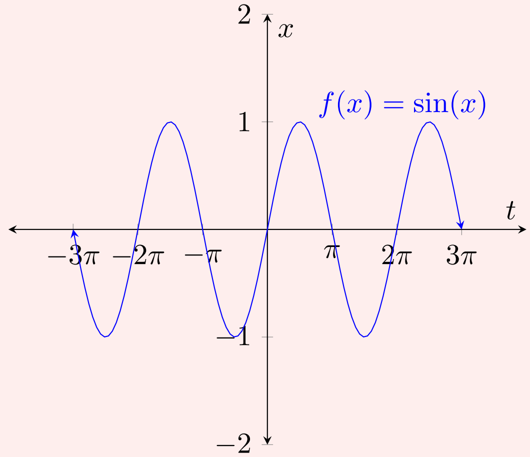


Idea:

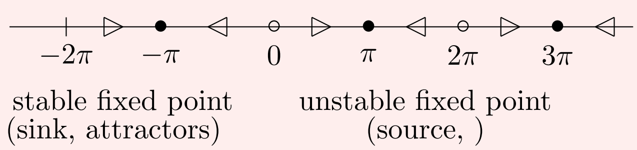
- If $f(x_0) > 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ increase near x_0.
- If $f(x_0) < 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ decrease near x_0.
- If $f(x_0) = 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ is $x(t) = x_0$ for all $t \in \mathbb{R}$, i.e., we have a fixed point/equilibrium point.

Example 2.3

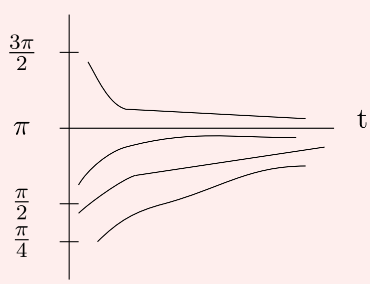
$\dot{x} = f(x) = \sin(x)$



Phase portrait:



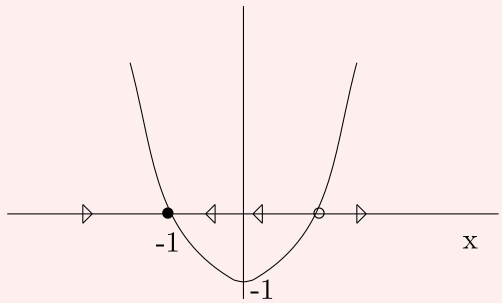
Qualitative plot of solution:



$x(t)$ is concave up
if $f(x(t))$ is increasing
$x(t)$ is concave down
if $f(x(t))$ is decreasing

Example 2.4

$\dot{x} = x^2 - 1$. Fixed points: $f(x) = x^2 - 1 = 0 \implies x = \pm 1$



<u>*Note*</u>: If $x_0 > 1$, then solution $x(t)$ with $x(0) = x_0 > 1$ is unbounded. In fact, $x(t) \to \infty$ in finite time.

§3 Lec 3: Jan 8, 2021

§3.1 Stability Types of Fixed Points

Definition 3.1 (Stability Types) — Consider the ODE $\dot{x} = f(x)$ and suppose that $f(x_*) = 0$. The fixed point x_* is called

1. Lyapunov stable if every solution $x(t)$ with $x(0) = x_0$ closed to x_* remain close to x_* for all $t \geq 0$, otherwise unstable.

2. Attracting if every solution $x(t)$ with $x(0) = x_0$ close to x_* satisfies $x(t) \to x_*$ as $t \to \infty$.

3. (asymptotically) stable if x_* is both Lyapunov stable and attracting.

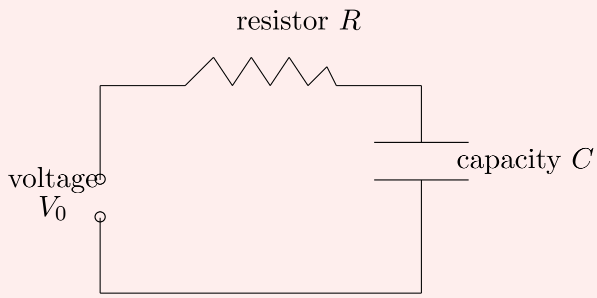
Example 3.2

Let $\alpha \in \mathbb{R}, \dot{x} = \alpha x$. General solution $x(t) = x_0 e^{\alpha t}$.

- $x_* = 0$ is always an equilibrium solution.
- $x_* = 0$ is
  1. attracting if $\alpha < 0$
  2. Lyapunov stable if $\alpha \leq 0$
  3. unstable if $\alpha > 0$

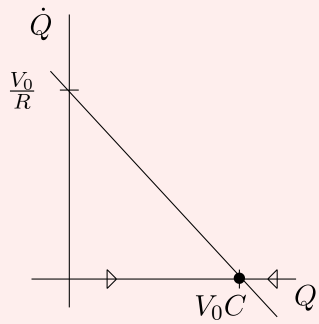
Example 3.3 (RC circuit)

We have the following circuit

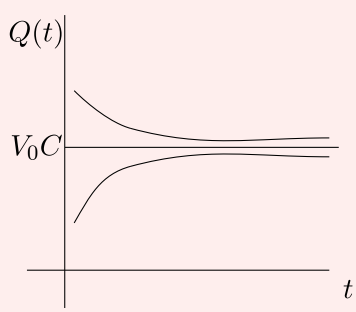


$$V_0 = RI + \frac{Q}{C}$$
$$I: \text{ current}, \ Q: \text{ change}$$
$$I = \dot{Q}$$
$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$$

Phase portrait



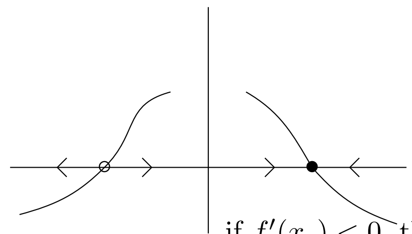
$Q_* = V_0C$ globally stable because every $Q(t)$ approaches Q_* as $t \to \infty$.



§3.2 Linear Stability Analysis

We have $\dot{x} = f(x), f(x_*) = 0$. Our task is to find an analytic criterion to decide if a fixed point x_* is stable/unstable.

Picture:



If $f'(x*) > 0$, then x_* is unstable. On the other hand, if $f'(x_*) < 0$, then x_* is a stable fixed point.
The linearization:
Consider: $\eta(t) = x(t) - x_*$ where $x(t)$ is the solution of $\dot{x} = f(x)$ with $x(0)$ close to x_*, $f(x_*) = 0$.
Note: $\dot{\eta}(t) = \dot{x}(t) = f(x(t)) = f(x(t) - x_* + x_*) = f(\eta(t) + x_*)$.
Taylor's Theorem:

$$f(x_* + \eta) = \underbrace{f(x_*)}_{=0} + f'(x_*)\eta + \underbrace{\mathcal{O}(\eta^2)}_{\text{error term and negligible if} f'(x_*)\neq 0 \text{ and } \eta \text{ is small}}$$

$\implies$ $\dot{\eta}(t) \approx f'(x_*)\eta(t)$ (as long as $\eta(t)$ is small) which is called the linearization of $\dot{x} = f(x)$ about x_*. The general solution is

$$\eta(t) = \eta_0 e^{f'(x_*)\cdot t}$$

In particular, η grows exponentially if $f'(x_*) > 0$ or decreases exponentially if $f'(x_*) < 0$.

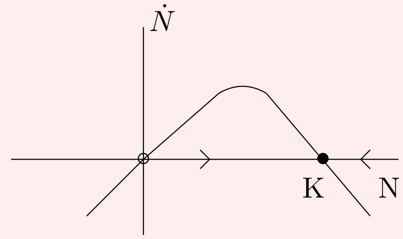
Definition 3.4 (Characteristics Time Scale) — $\frac{1}{|f'(x_*)|}$ is called the characteristics time scale.

Example 3.5 (Logstics Equation)

$N \geq 0$ population size, $r > 0$ growth rate, $K > 0$ carrying capacity

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

Fixed points: $\dot{N} = 0 \implies N_* = 0$ or $N_* = K$.
Let $f(N) = rN\left(1 - \frac{N}{K}\right) \implies f'(N) = r - 2\frac{r}{K}N$. In particular, $f'(0) = r > 0 \implies N_* = 0$ is an unstable fixed point and $f'(K) = r - 2r = -r < 0 \implies N_* = K$ is stable.
Phase portrait:



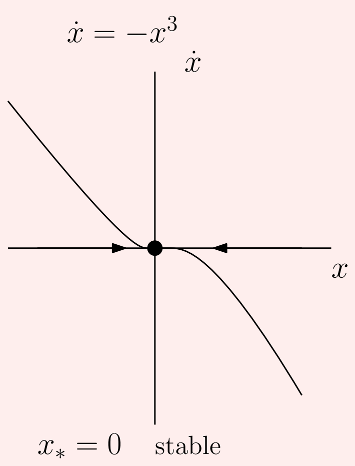
Thus, if $N(t)$ is the population with

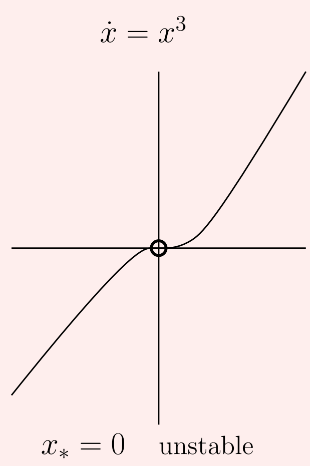
$$N(0) = N_0 > 0 \implies N(t) \to K \text{ as } t \to \infty$$
$$N(0) = 0 \to N(t) = 0 \quad \forall t \text{ (no spontaneous outbreak)}$$

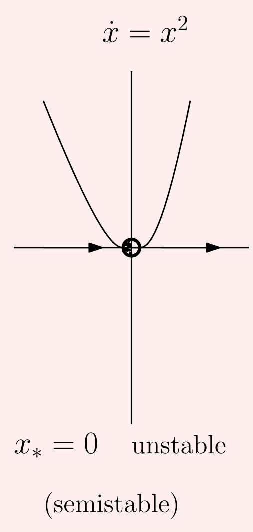
Characteristics time scale: $\frac{1}{|f'(N_*)|} = \frac{1}{r}$ for both $N_* = 0, K$.

Example 3.6

What if $f'(x_*) = 0$? Then we can't tell.





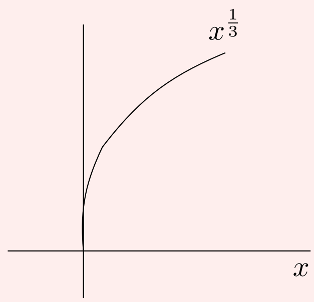


§4 Lec 4: Jan 11, 2021

§4.1 Existence and Uniqueness

Example 4.1 (Non-uniqueness)

$\dot{x} = x^{\frac{1}{3}} \implies x_1(t) \equiv 0$ (for all t) is a solution with $x_1(0) = 0$ but $x_2(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$ is also a solution with $x_2(0) = 0$



Is $x_0 = 0$ really a fixed point? No, it's unclear how it would behave (according to $x(t) = 0$ or $x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$).

Theorem 4.2 (Picard's)

Let $I = (a, b) \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ differentiable and f' continuous. Let $x_0 \in I$. Then there is $\tau > 0$ s.t. the initial value problem

$$\dot{x} = f(x), x(0) = x_0$$

has a unique solution $x : (-\tau, \tau) \to \mathbb{R}$.

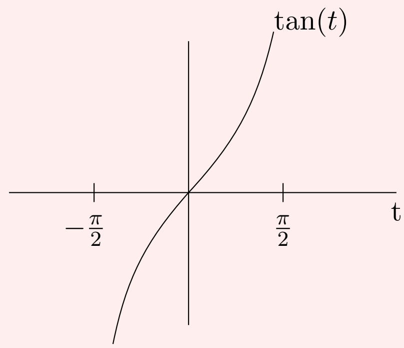
Example 4.3

(The solution might not exist for all times) Consider

$$\frac{dx}{dt} = \dot{x} = 1 + x^2, \quad x(0) = 0$$

So,

$$\begin{aligned} dt &= \frac{dx}{1+x^2} \\ t &= \int \frac{dx}{1+x^2} = \arctan x + C \\ 0 &= 0 + C \implies C = 0 \\ x(t) &= \tan(t) \end{aligned}$$



In particular,

$$x(t) \to +\infty \text{ as } t \to \frac{\pi}{2}$$
$$x(t) \to -\infty \text{ as } t \to \frac{-\pi}{2}$$

i.e., $x(t)$ reaches infinity in finite time, i.e., the solution $x(t)$ blows up in finite time.

Remark 4.4. (Hw 1) If $x_0 > 0$, then the solution to $\dot{x} = x^2, x(0) = x_0 > 0$ blows up in finite time. In fact, if $\alpha > 1$, then the solution to $\dot{x} = x^\alpha$, $x(0) = x_0 > 0$ blows up in finite time.

Theorem 4.5 (ODE Comparison)

If $x_1(t)$ solves $\dot{x} = f(x)$, $x_2(t)$ solves $\dot{x} = q(x)$ and $x_1(0) \leq x_2(0)$, $f(x) < q(x)$, then $x_1(t) \leq x_2(t)$ for all $t > 0$.
In particular, if $x_1(t) \to \infty$ in finite time, then $x_2(t) \to \infty$ in finite time.

Example 4.6

The solution to $\dot{x} = 1 + x^2 + x^3, x(0) = 0$ blows up in finite time.
Note: For $x \geq 0$:

$$1 + x^2 \leq 1 + x^2 + x^3$$

Recall: $\tan(t)$ solves $\dot{x} = 1 + x^2, x(0) = 0$. By comparison: the solution $x(t)$ to $\dot{x} = 1 + x^2 + x^3, x(0) = 0$ satisfies $x(t) \geq \tan(t)$. Thus, $x(t)$ blows up in finite time. We may indeed assume that $x(t) > 0$. Since $\dot{x}(0) = 1$, it follows that $x(t) > 0$ for $t > 0$ small. In fact, $\dot{x} = 1 + x^2 + x^3 > 0$ for $x(t)$ small, i.e., whenever $x(t)$ is close to zero, it must increase $\implies x(t) > 0$ for $t > 0$.

Example 4.7 (No Oscillating Solution in 1D)

Let $f \in C^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} | f \text{ differentiable}, f' \text{ continuous}\}$. Suppose $f(x_*) = 0, x(t)$ solution of $\dot{x} = f(x)$. If $x(t_0) = x_*$ for some t_0. Then $x(t) = x_*$ for all time t. Geometrically this says that a solution can never reach/cross a fixed point (unless it is a fixed point).

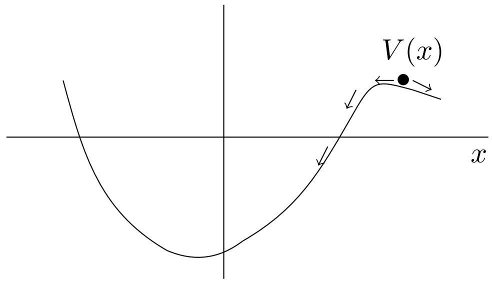
- $f(x(t)) > 0$ and $\dot{x}(t) > 0$, i.e., $x(t)$ increases.
- $f(x(t)) = 0$ and $x(t) = \text{constant}$ for all t.
- $f(x(t)) < 0$ and $\dot{x}(t) < 0$ i.e., $x(t)$ decreases.

In particular, there is no oscillating solution.

§5 Lec 5: Jan 13, 2021

§5.1 Potential

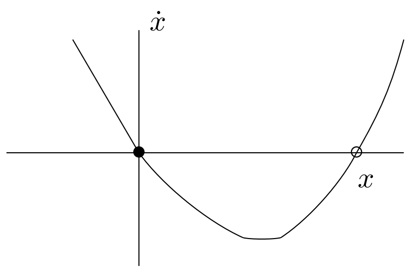
Consider the movement of a particle (with lots of friction) in a potential.



Notice:

- Particle approaches the local minimum of $V(x)$ (minimum energy level) no fixed point.
- Local minima of $V(x)$ are stable fixed points.
- Local maxima of $V(x)$ are unstable fixed points.

$\implies$ $\dot{x} = f(x) = -\frac{dV}{dx} = -V'(x)$.



Expect $t \to V(x(t))$ is non-increasing for a solution $x(t)$ of $\dot{x} = -V'(x)$.
Indeed:

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= V'(x(t))\frac{d}{dt}x(t) \\ &= V'(x(t))\left(-V'(x(t))\right) \\ &= -\left(V'(x(t))\right)^2 \leq 0\end{aligned}$$

$\implies$ particle always moves towards a lower energy level.

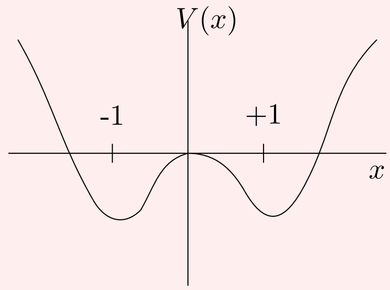
Definition 5.1 (Potential) — A function $V(x)$ s.t. $\dot{x} = f(x) = -\frac{dV}{dx}$ is called a potential.

Example 5.2

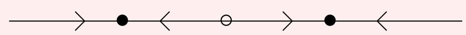
Graph potential for $\dot{x} = x - x^3$. Find/characterize equilibria (fixed points).

$$\dot{x} = f(x) = x - x^3 = -\frac{dV}{dx} \overset{\int}{\Longrightarrow} V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$$

$\Longrightarrow$ V is only defined up to a constant, we may choose any $C \in \mathbb{R}$, e.g., choose $C = 0$.



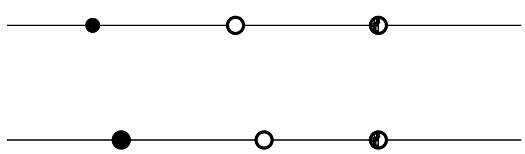
Local minima of V correspond to stable fixed points $\Longrightarrow$ $0 = -\frac{dV}{dx} = f(x) = x - x^3$, i.e., $x = \pm 1$.
Local maximum of V corresponds to an unstable fixed point at $x = 0$.
Phase portrait:



Remark 5.3. This system is often called <u>bistable</u> because it has two stable fixed points.

§5.2 Bifurcations

The qualitative behavior of 1D dynamical systems $\dot{x} = f(x)$ is determined by fixed points.

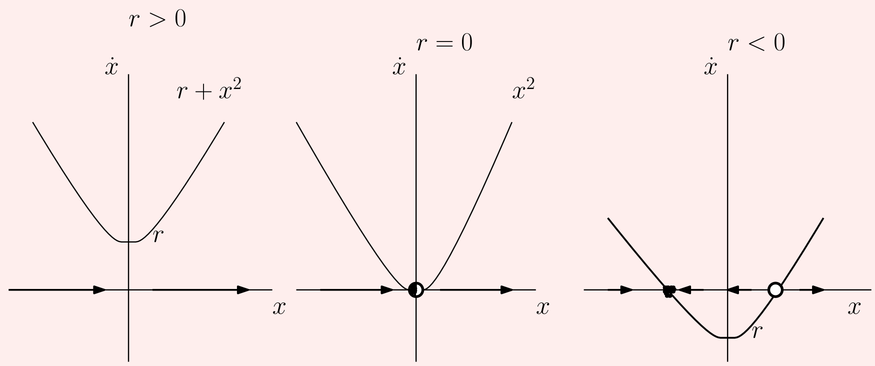


qualitative the same

(dynamics)

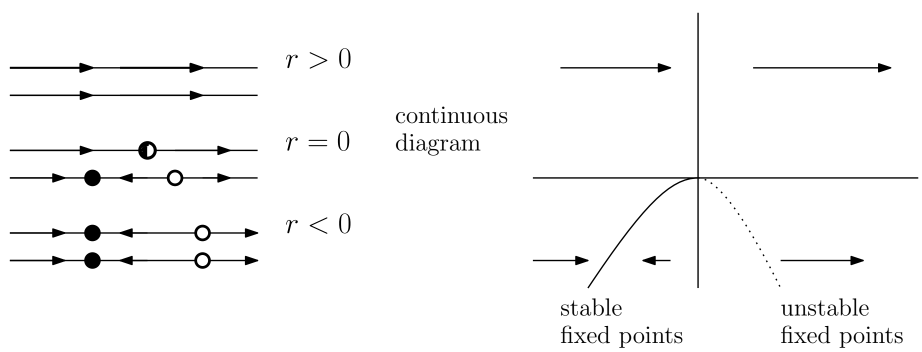
If $\dot{x} = f(r, x)$ depends on a parameter r, then the numbers of fixed points and their stability may change as r varies. This is called bifurcation.

Example 5.4 (Saddle-node, blue sky bifurcation)
$\dot{x} = r + x^2, \quad r \in \mathbb{R}.$

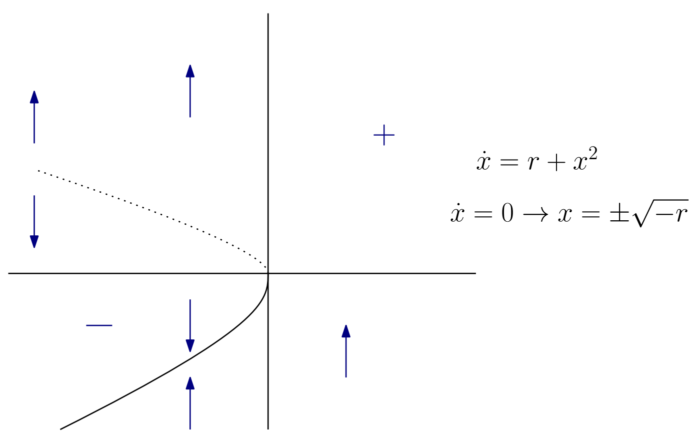


Hence, the qualitative behavior changes at $r_* = 0$, i.e., $r_* = 0$ is called a bifurcation point.

Ways to plot the dependence on the parameter:



Most common: bifurcation diagram



§6 Lec 6: Jan 15, 2021

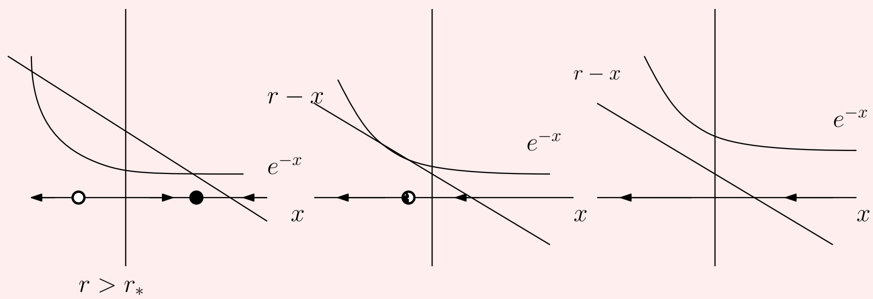
§6.1 Saddle-Node Example

Example 6.1

Argue geometrically that the ODE

$$\dot{x} = r - x - e^{-x}$$

undergoes a saddle-node bifurcation. Furthermore, find the bifurcation point.
Note: Fixed points of $\dot{x} = r - x - e^{-x}$ correspond to intersection points of the functions $r - x, e^{-x}$ because $r - x - e^{-x} = 0 \iff r - x = e^{-x}$.



Indeed we have a saddle-node bifurcation.
Note: At $r = r_*$, the graph of $r - x$ and e^{-x} intersect tangentially. Thus, for the bifurcation point we require:

$$0 = \dot{x} = r - x - e^{-x} \implies r - x = e^{-x}$$
$$0 = \frac{d}{dx}(r - x - e^{-x}) \implies \frac{d}{dx}(r - x) = \frac{d}{dx}e^{-x}$$

So,

$$\begin{aligned} -1 &= -e^{-x} \\ e^{-x} &= 1 \\ x &= 0 \\ r_* &= x_* + e^{-x_*} = 0 + 1 = 1 \end{aligned}$$

Thus the bifurcation point is $(r_*, x_*) = (1, 0)$.

Note:

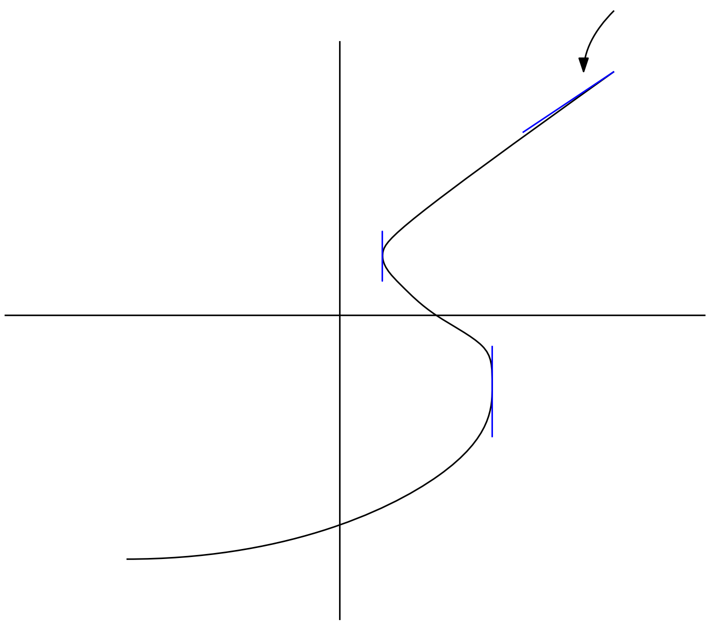
$$\begin{aligned} \dot{x} = r - x - e^{-x} &= r - x - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots\right) \\ &= r - 1 - \frac{1}{2}x^2 + \frac{x^3}{6} - \ldots \\ &\approx (r-1) - \frac{1}{2}x^2 \text{ for x near } x_* = 0 \end{aligned}$$

Set $R = r - 1$, then $\dot{x} \approx R - \frac{1}{2}x^2$.
Upshot: Up to appropriate rescalings/coordinate changes, every saddle-node bifurcation looks like its normal form

$$\dot{x} = r - x^2 \qquad (\text{or } \dot{x} = r + x^2)$$

close to the bifurcation point $(r_*, x_*) = (0, 0)$.

§6.2 Normal Forms



Recall:

- Normal vector: $\begin{pmatrix} \partial_r f \\ \partial_x f \end{pmatrix}$
- Tangent vector: $\begin{pmatrix} -\partial_x f \\ \partial_r f \end{pmatrix}$

<u>*Note*</u>: Bifurcation points have vertical tangent vectors, i.e., $\partial_x f = 0, \partial_r f \neq 0$.

Theorem 6.2 (Taylor's)
Suppose $f(r_*, x_*) = 0$.

$$f(r,x) = f(r_*, x_*) + \underbrace{\frac{\partial f}{\partial r}(r_*, x_*)}_{p_1}(r - r_*) + \underbrace{\frac{\partial f}{\partial x}(r_*, x_*)}_{q_1}(x - x_*)$$
$$+\frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial r^2}(r_*, x_*)}_{p_2}(r-r_*)^2 + \underbrace{\frac{\partial^2 f}{\partial r \partial x}(r_*, x_*)}_{R}(r-r_*)(x-x_*) + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial x^2}(r_*, x_*)}_{q_2}(x-x_*)^2 + \dots$$

Remark 6.3. If $q_1 \neq 0$, then there is no bifurcation at (r_*, x_*), linear stability (sign of q_1) determines if (r_*, x_*) is (un)stable.

Theorem 6.4

Suppose that $f(r_*, x_*) = 0, q_1 = 0, p_1 \neq 0, q_2 \neq 0$, then $\dot{x} = f(r, x)$ undergoes a saddle node bifurcation at (r_*, x_*) and

$$\dot{x} = \frac{\partial f}{\partial r}(r^*, x^*)(r - r^*) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

for $|r - r_*| < \epsilon^2, \quad |x - x_*| < \epsilon$.

Remark 6.5.

i) Note that the constant $(r - r_*)(x - x_*)$ is $\mathcal{O}(\epsilon^3)$

ii) With a coordinate change $(t, x, r) \mapsto (s, y, R)$ we can arrange that ODE looks like

$$\frac{d}{ds}y = R + y^2$$

near $(0, 0) = (R(r_*), y(x_*))$

Example 6.6

$\dot{x} = e^r - x - e^{-x}$ undergoes a saddle-node bifurcation near $(r_*, x_*) = (0,0)$. Apply the theorem 6.4,

$$\begin{aligned}
f(r,x) &= e^r - x - e^{-x} \\
f(0,0) &= 1 - 0 - 1 = 0 \\
\frac{\partial f}{\partial x}(r,x) &= -1 + e^{-x} \implies \frac{\partial f}{\partial x}(0,0) = 0 \\
\frac{\partial f}{\partial r}(r,x) &= e^r \implies \frac{\partial f}{\partial r}(0,0) = 1 \neq 0 \\
\frac{\partial^2 f}{\partial x^2}(r,x) &= -e^{-x} \implies \frac{\partial^2 f}{\partial x^2}(0,0) = -1 \neq 0
\end{aligned}$$

Therefore, by theorem 6.4, $(r_*, x_*) = (0,0)$ is a bifurcation point of a saddle-node bifurcation.
Normal form near $(r_*, x_*) = (0,0)$:

$$\begin{aligned}
\dot{x} &= e^r - x - e^{-x} \\
&= 1 + r + \frac{r^2}{2} + \mathcal{O}(r^3) - x - \left(1 - x + \frac{x^2}{2} + \mathcal{O}(x^3)\right) \\
&= r + \underbrace{\frac{r^2}{2}}_{\mathcal{O}(\epsilon^4)} - \frac{x^2}{2} + \mathcal{O}(r^3) + \mathcal{O}(x^3) \\
&= \underbrace{r - \frac{x^2}{2}}_{\mathcal{O}(\epsilon^2)} + \mathcal{O}(\epsilon^3) \text{ if } |r - r_*| = |r| < \epsilon^2 \\
&\qquad\qquad\qquad \text{if } |x - x_*| = |x| < \epsilon
\end{aligned}$$

Set $y = \frac{x}{2}$, then

$$\dot{y} = \frac{1}{2}\dot{x} = \frac{r}{2} - \frac{x^2}{4} + \mathcal{O}(\epsilon^3) = \frac{r}{2} - y^2 + \mathcal{O}(\epsilon^3)$$

Set $s = -t$, then

$$\frac{d}{ds}y = -\frac{d}{dt}y = -\frac{r}{2} + y^2 + \mathcal{O}(\epsilon^3)$$

Set $R = -\frac{r}{2}$, then

$$\underbrace{\frac{d}{ds}y = R + y^2}_{\text{normal form of a saddle-node bifurcation}} \qquad + \mathcal{O}(\epsilon^3)$$

§7 Lec 7: Jan 20, 2021

§7.1 Classification of Bifurcations

Let's rewrite $\dot{x}$ in theorem 6.4 as

$$\dot{x} = p(r - r_*) + \frac{c}{2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

if $|r - r_*| < \epsilon^2, |x - x_*| < \epsilon$. After a coordinate change $(t, x, r) \mapsto (s, y, R)$ such that

$$\begin{aligned} s &= t \\ y &= \frac{c}{2}(x - x_*) \\ R &= p\frac{c}{2}(r - r_*) \end{aligned}$$

the ODE is represented by the normal form.

$$\frac{d}{ds}y = \dot{y} = R + y^2 + \mathcal{O}(\epsilon^3)$$

for $|R| < \epsilon^2, |y| < \epsilon$.
If $f(x_*, r_*) = 0$, and also $\frac{\partial f}{\partial x}(x_*, r_*) = 0 = \frac{\partial f}{\partial r}(x_*, r_*)$, then the second derivatives determines the bifurcation type.

$$\text{Hessian} \quad \text{Hessf} = \begin{pmatrix} \frac{\partial^2 f}{\partial r^2} & \frac{\partial^2 f}{\partial r \partial x} \\ \frac{\partial^2 f}{\partial r \partial x} & \frac{\partial^2 f}{\partial x^2} \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Second test: if $AC - B^2 > 0$, (r_*, x_*) is a local maximum/minimum. In particular, (r_*, x_*) is an isolated fixed point. (irrelevant case)
Practically relevant case: If $AC - B^2 < 0$: (r_*, x_*) is a saddle. If also $C \neq 0$: transcritical bifurcation.

$$\dot{y} = Ry - y^2 + \mathcal{O}(\epsilon^2)$$

for $|R| < \epsilon, |y| < \epsilon$ (after an appropriate coordinate change)

$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

If also $C = 0$: Pitchfork bifurcation

- Supercritical Pitchfork bifurcation:

$$y' = Ry - y^3 + \mathcal{O}(\epsilon^3)$$

- Subcritical Pitchfork bifurcation

$$y' = Ry + y^3 + \mathcal{O}(\epsilon^3)$$

  for $|R| < \epsilon^2, |y| < \epsilon$

Again,

$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

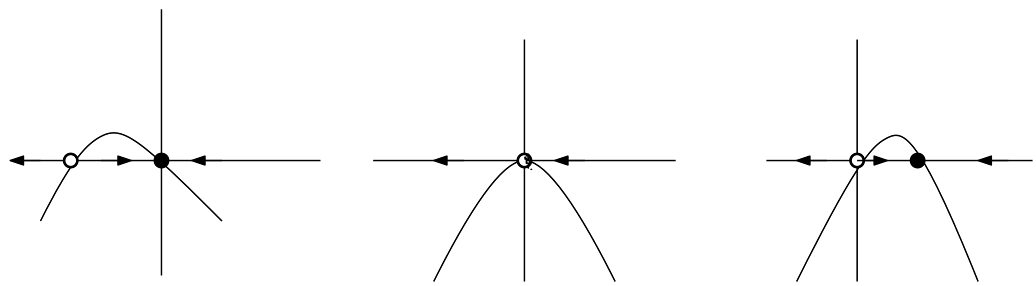
§7.2 Transcritical Bifurcation

Normal form:

$$\dot{x} = rx - x^2 = x(r - x)$$

In particular, $x_* = 0$ is always a fixed point but it changes stability.

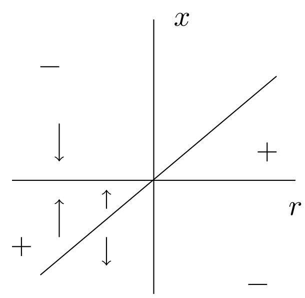
$r < 0$ $\quad$ $r = 0$ $\quad$ $r > 0$



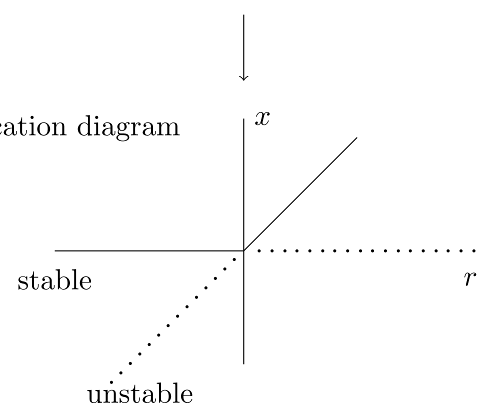
Bifurcation diagram: $\dot{x} = x(r - x) = rx - x^2 = f(x)$. Fixed points:

$$x_* = 0, \quad x_* = r \quad r \in \mathbb{R}$$

intermediate step:
draw fixed points
(without stability)



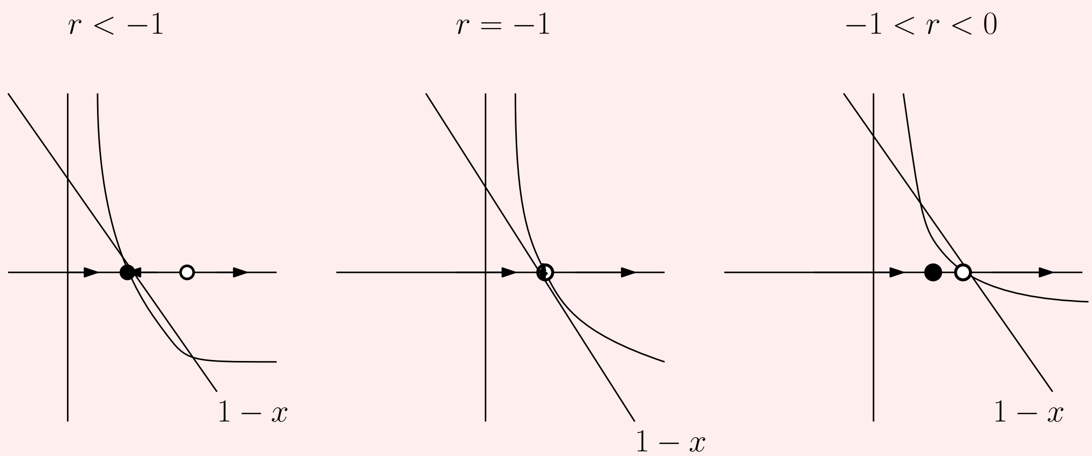
bifurcation diagram



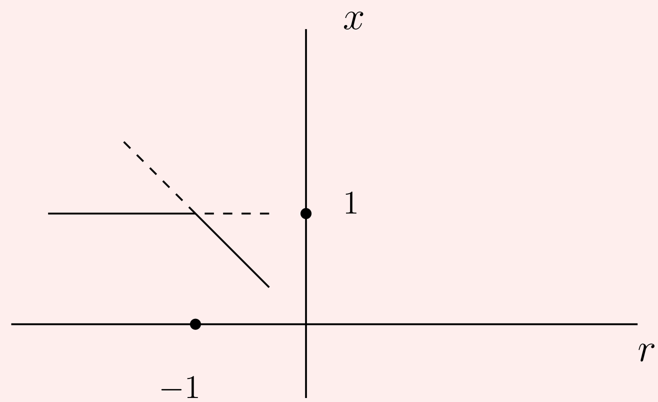
Example 7.1

$\dot{x} = r\ln(x) + x - 1$ has a transcritical bifurcation at $(r_*, x_*) = (-1, 1)$.
Geometric approach:

$$\dot{x} = 0 \iff r\ln(x) = 1 - x$$



Bifurcation near $(r_*, x_*) = (-1, 1)$



Normal form: $\dot{x} = r\ln(x) + x - 1$.

Remark 7.2. $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \quad |x| < 1$

So,

$$\begin{aligned}
\dot{x} &= r\ln(x) + x - 1 \\
&= r(x - 1 - \frac{1}{2}(x-1)^2 + \mathcal{O}((x-1)^3) + x - 1 \\
&= (r+1)(x-1) - \frac{1}{2}\left((r+1) - 1\right)(x-1)^2 + \mathcal{O}\left(r(x-1)^3\right) \\
&= (r+1)(x-1) + \frac{1}{2}(x-1)^2 + \mathcal{O}(\epsilon^3)
\end{aligned}$$

if $|r-(-1)| < \epsilon$ and $|x - 1 < \epsilon|$.
Now, set $R = r+1, y = c\cdot(x-1)$. Then,

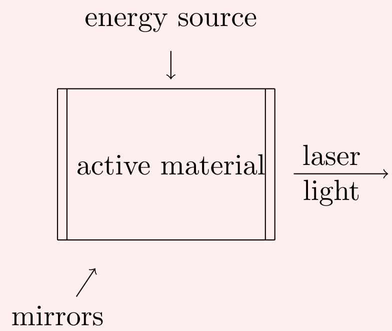
$$\begin{aligned}\dot{y} &= c\dot{x}\\ &= (r+1)c(x-1) + \frac{1}{2}c(x-1)^2 + \mathcal{O}(\epsilon^3)\\ &= Ry + \frac{1}{2c}\left(c(x-1)\right)^2 + \mathcal{O}(\epsilon^3)\\ &= Ry + \underbrace{\frac{1}{2c}}_{=1} y^2 = Ry + y^2\end{aligned}$$

for $c = \frac{1}{2}$.

§7.3 Application of Transcritical Bifurcations

Example 7.3 (Laser Threshold)

Consider



Simple model:

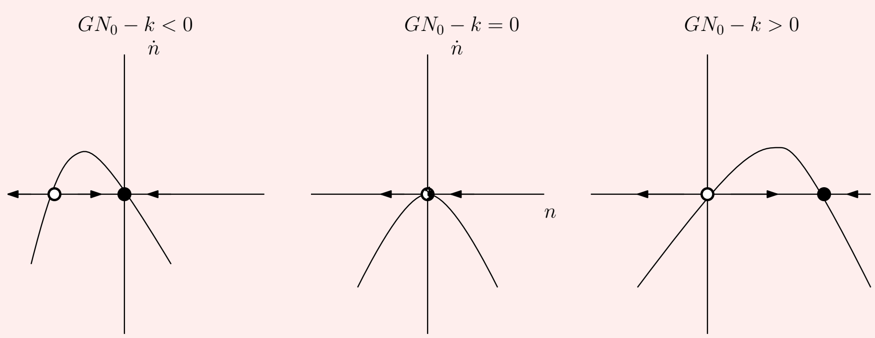
$$n = n(t) = \#\text{ photons in the laser}$$

Then

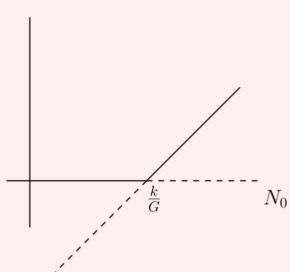
$$\begin{aligned}\dot{n} &= G \cdot \underbrace{N}_{\#\text{ excited atoms}} \cdot n - kn \\ &= N_0 - \alpha \cdot n \\ &= G(N_0 - \alpha n)n - kn \\ &= (GN_0 - k)n - \alpha G n^2\end{aligned}$$

where $G, k, \alpha > 0$. Fixed points:

$$\dot{n} = 0 \iff n = 0 \text{ or } n = \frac{GN_0 - k}{\alpha G}$$



Bifurcation diagram



transcritical bifurcation at

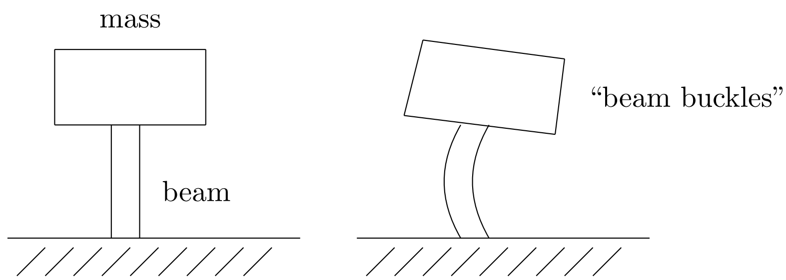
$(N, n) = (\frac{k}{G}, 0)$

$\frac{k}{G} =$ laser threshold

§8 Lec 8: Jan 22, 2021

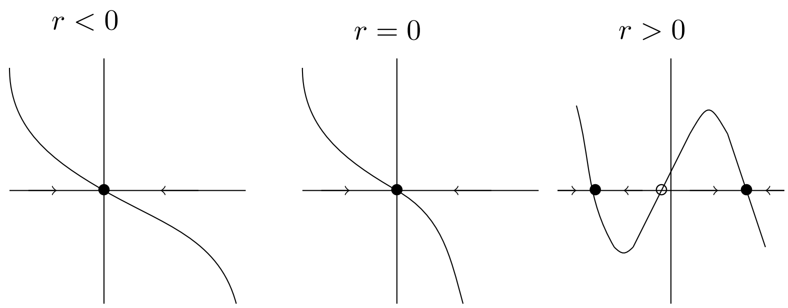
§8.1 Supercritical Pitchfork Bifurcation

Fixed points appear/disappear in symmetric pairs



Supercritical Pitchfork Bifurcation:

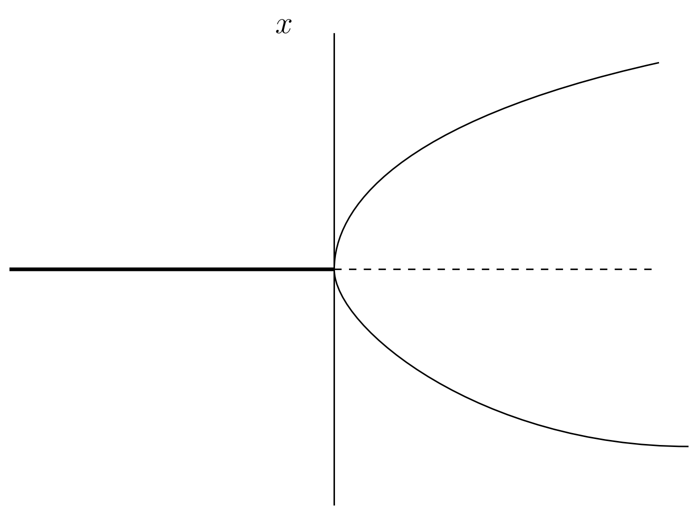
$$\dot{x} = rx - x^3$$



> **Remark 8.1.** Decay towards $x_* = 0$ is not exponential in time for $r = 0$.

Bifurcation diagram:

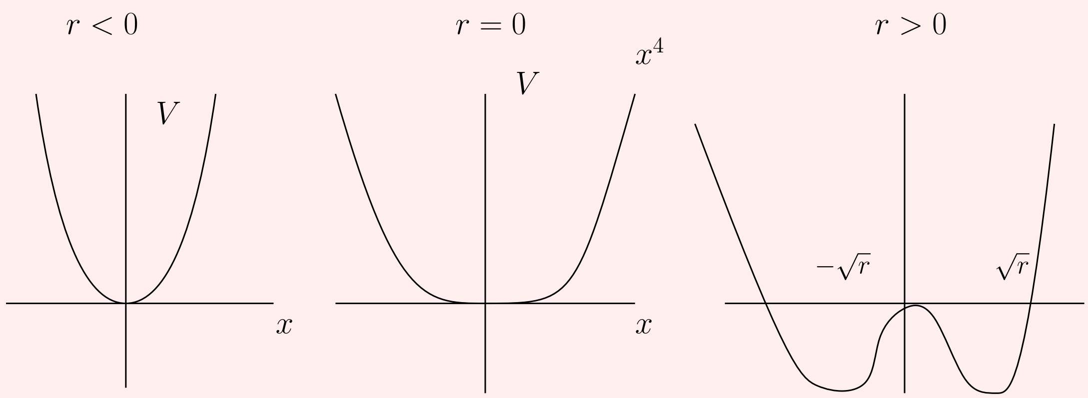
$$\dot{x} = rx - x^3 = 0$$
$$\implies x = 0, \qquad x = \pm\sqrt{r}, \quad r > 0$$



Example 8.2

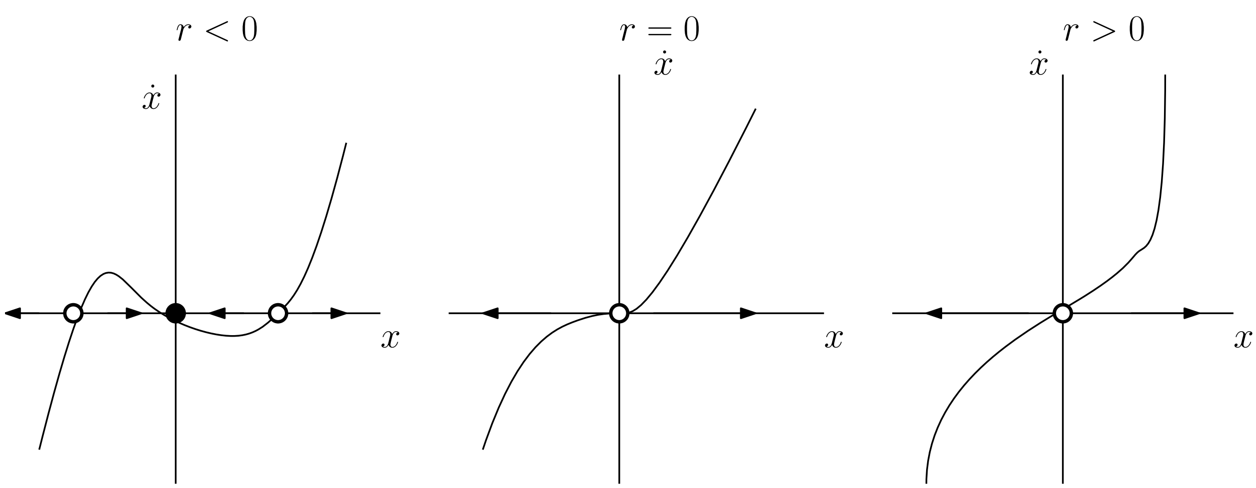
Potential for $\dot{x} = rx - x^3 = -\frac{dV}{dx}$

$$\implies V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4 + \underbrace{C}_{=0}$$



§8.2 Subcritical Pitchfork Bifurcation

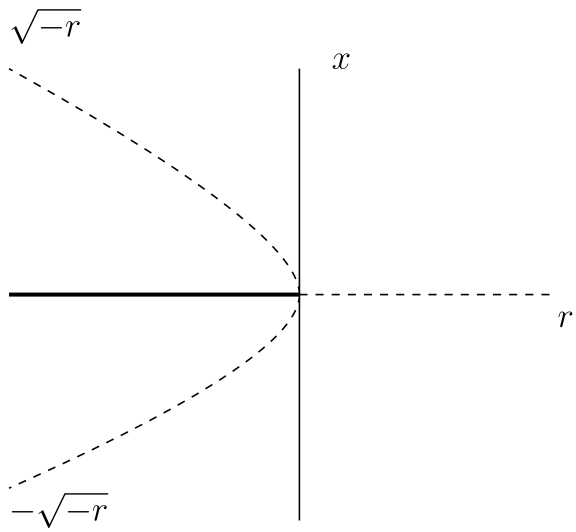
$$\dot{x} = rx + x^3$$



Fixed points:

$$\dot{x} = rx + x^3 = 0$$
$$\implies x = 0, \qquad x = \pm\sqrt{-r}, \quad r < 0$$

<u>Bifurcation Diagram</u>:



> **Remark 8.3.** If $r > 0, x_0 > 0$, then the solution $x(t)$ with $x(0) = x_0 > 0$ blows up in finite time (cf. homework). Interpretation: $+x^3$ is destabilizing.

Physically more realistic scenario:

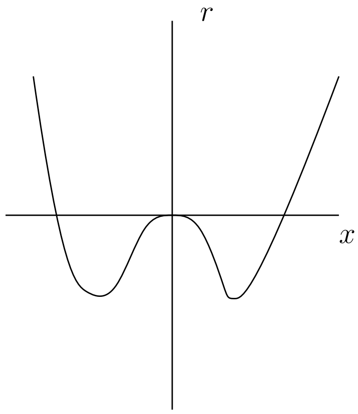
$$\dot{x} = rx + x^3 - x^5$$

where x^5 is the stabilizing higher order term.
Fixed points:

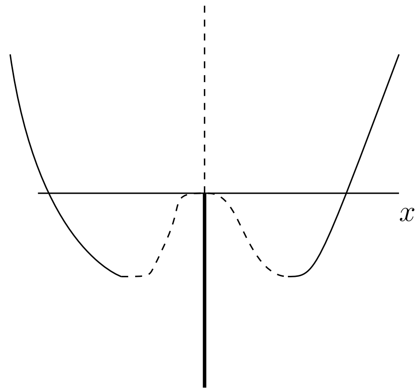
$$\dot{x} = 0 \iff x = 0, \quad r = -x^2 + x^4$$

Bifurcation diagram:

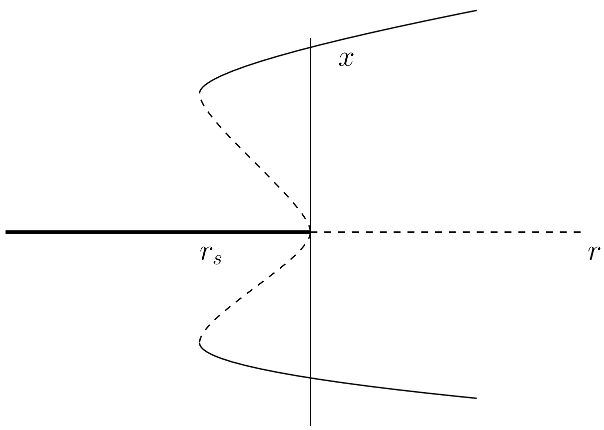
1. Intermediate step



2. Stability Types

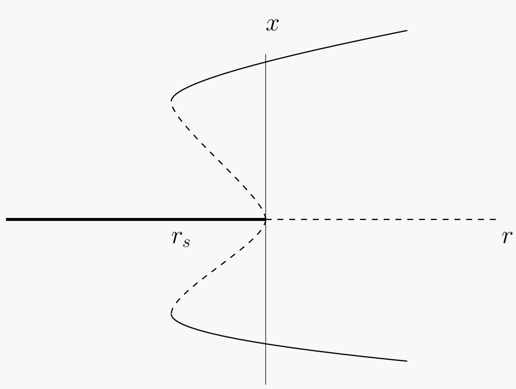


3. Change axes: bifurcation diagram



$$\dot{x} = rx + x^3 - x^5$$

Remark 8.4. i) Subcritical pitchfork bifurcation at $(r_*, x_*) = (0, 0)$ and saddle node bifurcation at $(r_s, x_*) = \left(-\frac{1}{4}, \pm\sqrt{2}\right)$.



$$\dot{x} = r(t)x + x^3 - x^5$$

ii) jump at $r_* = 0$: A small perturbation of a stable fixed point at $(0, r)$ with $r < 0$ jumps to the stable large amplitude branch as r becomes positive, but does not jump back until $r < r_s$.

This non-reversibility is called hysteresis.

§9 Dis 1: Jan 7, 2021

§9.1 Fixed points and Stability

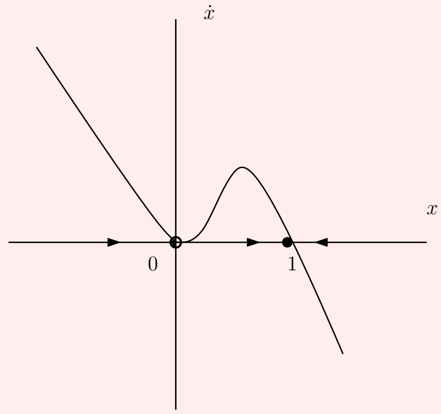
$\dot{x} = f(x)$

Example 9.1

$\dot{x} = -x^3 + x^2$

a) Sketch the vector field, classify the fixed points.

"vector fields" = x-axis with arrows



so:

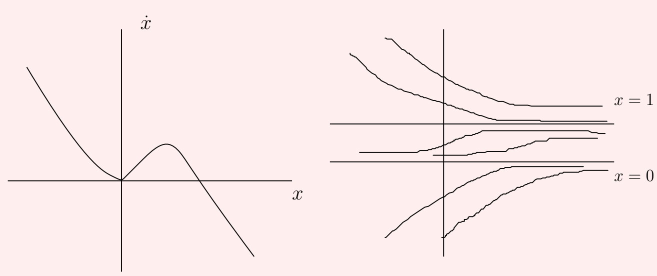
- $\dot{x} > 0 \implies x(t)$ increasing
- $\dot{x} < 0 \implies x(t)$ decreasing

"Fixed point" $\iff$ x_* s.t. $f(x_*) = 0$ $\iff$ x_* s.t. the constant function $x(t) = x_*$ is a solution.

We have 2 fixed points:

- $x_* = 0$ is semi-stable.
- $x_* = 1$ is stable.

b) Sketch various solutions of $x(t)$.

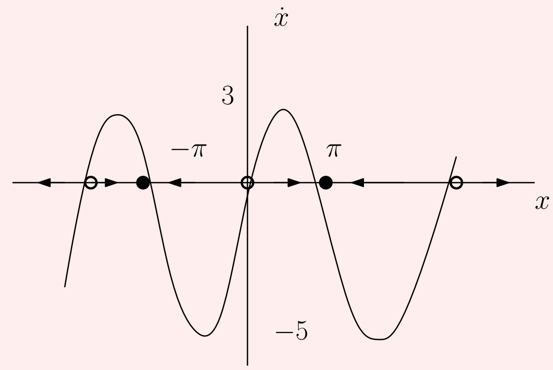


- $\dot{x} = 0$ for $x = 0, 1 \implies x(t) = 0, 1$ are solutions.
- $\dot{x} > 0$ for $x < 0 \implies x(t)$ increasing.
- $\dot{x} > 0$ for $0 < x < 1 \implies x(t)$ increasing.
- $\dot{x} < 0$ for $x > 1 \implies x(t)$ decreasing.

Example 9.2

$\dot{x} = -1 + 4\sin x$

a) Sketch vector field, classify fixed points.

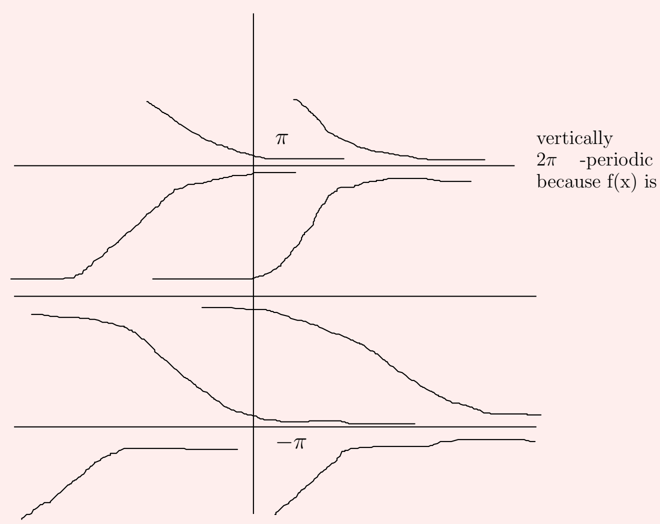


Fixed points:

$$\sin(x_*) = \frac{1}{4}$$

- $x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$ for $n = 0, \pm 1, \dots$ are unstable.
- $x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$ for $n = 0, \pm 1, \dots$ are stable.

b) Sketch various solutions $x(t)$.



§9.2 First Order Autonomous System

$\dot{\vec{x}} = \vec{f}(\vec{x})$ – first order and autonomous.

Example 9.3

A unit mass with displacement $x(t)$ attached to a spring with spring constant 6 obeys:

$$\ddot{x} = -6x - b(t)\dot{x}$$

where $b(t) \geq 0$ is the friction coefficient.

a) Show that this can expressed as a first order autonomous system

$$\begin{aligned}
x_1 &= x, \quad x_2 = \dot{x} \\
\dot{x}_2 &= \ddot{x} = -6x_1 - b(t)x_2 \\
\vec{x} &\coloneqq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
\dot{\vec{x}} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -6x_1 - b(x_3)x_2 \end{pmatrix}
\end{aligned}$$

where $x_3 = t \implies \dot{x}_3 = 1$.

b) In the case $b(t) = 5$, find the explicit solution for $x(0) = x_0, \dot{x}(0) = v_0$.

$$\ddot{x} = -6x - 5\dot{x} \implies \ddot{x} + 5\dot{x} + 6x = 0$$

Try $x(t) = e^{kt}$:

$$\begin{aligned}
0 &= \ddot{x} + 5\dot{x} + 6x = k^2 e^{kt} + 5ke^{kt} + 6e^{kt} \\
&= e^{kt}(k^2 + 5k + 6) \implies k = -3, -2
\end{aligned}$$

Now, $x(t) = c_1 e^{-3t} + c_2 e^{-2t}, c_1, c_2 \in \mathbb{R}$. Using the initial conditions, we obtain

$$x(t) = (-2x_0 - v_0)e^{-3t} + (3x_0 + v_0)e^{-2t}$$

§10 Dis 2: Jan 14, 2021

§10.1 Linearization and Potentials

Example 10.1

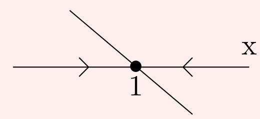
$\dot{x} = -x^3 + x^2$

a) Use linear stability analysis to classify the fixed points. If it fails, use a graphical argument.

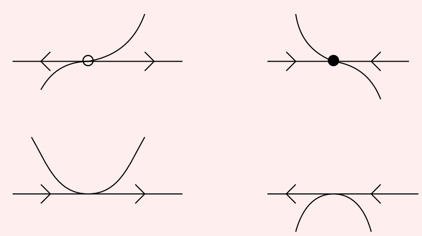
Idea: For x near a fixed point x_*, $\dot{x} = f(x) \approx f(x_*)(= 0) + f'(x_*)(x - x_*) = \dots$

$$f(x) = -x^3 + x^2, \quad f'(x) = -3x^2 + 2x$$
$$0 = f(x_*) = -x_*^2(x_* - 1) \implies x_* = 0, 1$$

- $x_* = 1 : f'(1) = -1 < 0 \implies$ stable



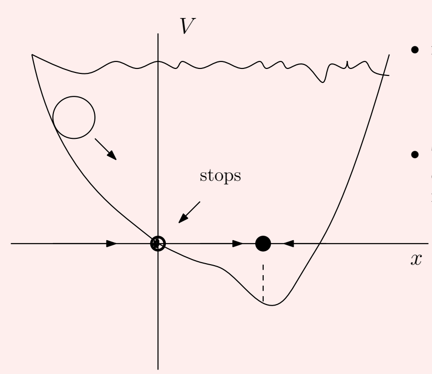
- $x_* = 0 : f'(0) = 0 :$ inconclusive



b) Find and plot a potential function.

"Potential function" $\iff V(x)$ s.t. $\dot{x} = -\frac{dV}{dx}$

$$\dot{x} = -x^3 + x^2 = -V'(x) \implies V(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 + C \text{ (choose 0)}$$



- fixed points $\iff f(x_*) = 0 \iff V'(x_*) = 0 \iff$ critical point
- Trajectories move toward decreasing V, like a ball rolling down the graph of V

Example 10.2

$\dot{x} = 4\sin x - 1$.

a) Use linear stability analysis to classify the fixed points.

$$f(x) = 4\sin x - 1, \quad f'(x) = 4\cos x$$

Last time: Fixed points are

- $x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \dots$

$$f'(x_*) = 4\cos\left(\sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) > 0$$

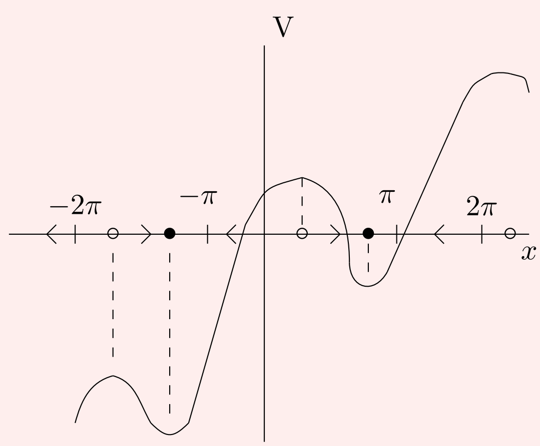
$\implies$ unstable

- $x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \dots$

$$f'(x_*) = 4\cos\left(\pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) < 0$$

$\implies$ Stable.

b) Plot potential $-1 + 4\sin x = -V'(x) \implies V(x) = x + 4\cos x$



§10.2 Existence of Solutions

Example 10.3 a) Let $a > 0$ be a constant. Show that the solution of

$$\begin{cases} \dot{x} = ax^2 \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

$$\frac{dx}{dt} = ax^2 \implies \int \frac{dx}{x^2} = \int a\, dt$$
$$\implies -\frac{1}{x} = at + c$$
$$\implies x(t) = \frac{1}{c - at} \forall c \in \mathbb{R}$$
$$x(0) > 0 \implies c > 0 \implies \lim_{t \to T} x(t) = +\infty$$

for some $T > 0$. In fact, $c = \frac{1}{x_0}$ and so $T = \frac{c}{a} = \frac{1}{ax_0}$.

b) Let $0 < \epsilon < 1$ be a constant. Show that the solution of

$$\begin{cases} \dot{x} = x^2 \left(1 + \epsilon \sin x\right) \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

Idea: $\dot{x} \geq ax^2$ for some $a > 0$, so our solution grows faster than a function which blows up, and must blow up too.

$$|\sin x| \leq 1 \implies 1 + \epsilon \sin x \geq 1 - \epsilon$$
$$\implies \dot{x} = x^2 \left(1 + \epsilon \sin x\right) \geq \underbrace{1 - \epsilon}_{>0} x^2$$

Let $x(t)$ be the solution to

$$\begin{cases} \dot{x} = x^2 \left(1 + \epsilon \sin x\right) \\ x(0) = x_0 \end{cases}$$

Let $y(t)$ be the solution to

$$\begin{cases} \dot{x} = (1 - \epsilon) x^2 \\ x(0) = x_0 \end{cases}$$

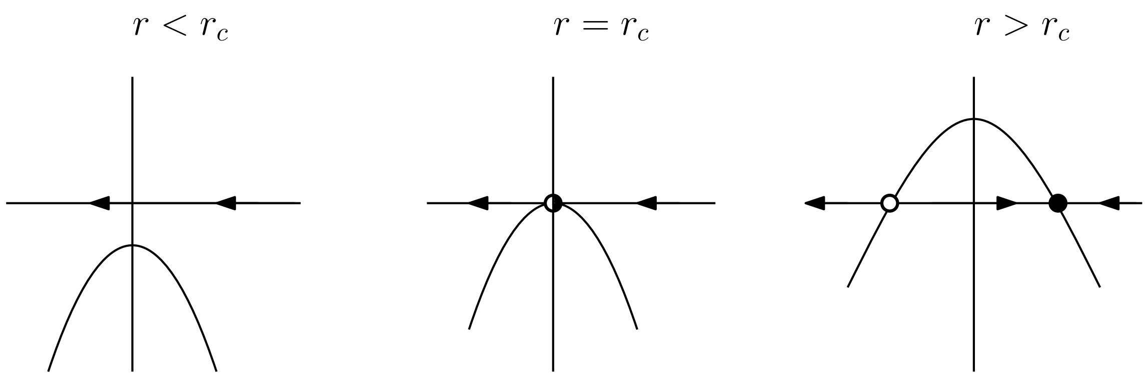
By part a), $y(t)$ blows up at some time $T > 0$. Since $x(0) = y(0)$ and $\dot{x} \geq \dot{y}$, then $x(t) \geq y(t)$ for all $t \geq 0$ (ODE Comparison Lec 4). Therefore, $x(t)$ must blow up in finite time. In fact, blow up time must be $\leq T = \frac{1}{(1-\epsilon)x_0}$.

§11 Dis 3: Jan 21, 2021

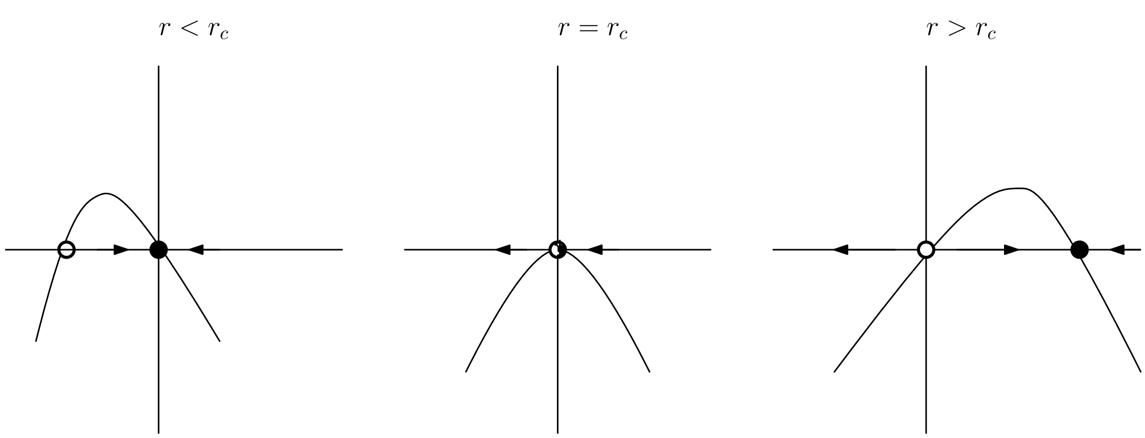
§11.1 Bifurcations

$\dot{x} = f(x, r)$, r parameter
"Bifurcation" $\iff$ change in the number or stability of fixed points.
There are two types:

- Saddle-node: $0 \to 1 \to 2$ fixed points



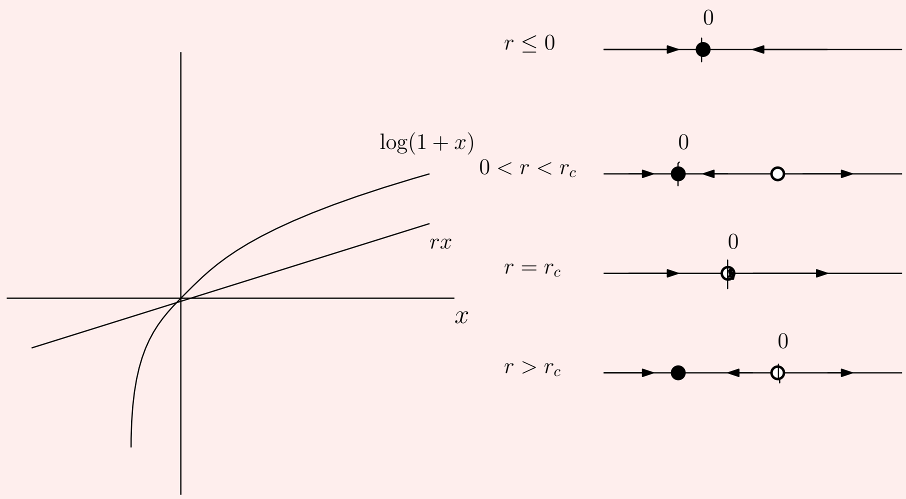
- Transcritical: $2 \to 1 \to 2$ fixed points



Example 11.1

$\dot{x} = rx - \log(1+x)$

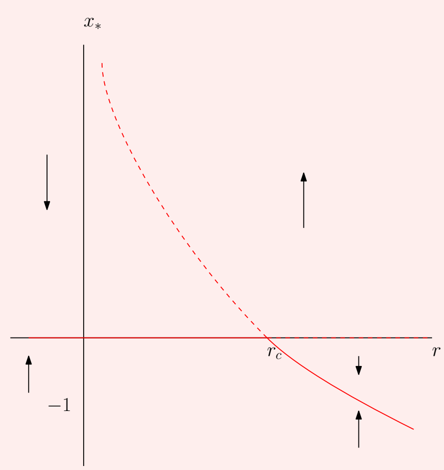
(a) Sketch all qualitatively different vector fields, sketch bifurcation diagram, find and classify bifurcations.



Transcritical bifurcation at $r = r_c$

"Bifurcation diagram" $\iff$ Plot of the fixed points x_* as a function of r.

$$0 = rx_* - \log(1+x_*)$$
$$r = \frac{1}{x_*}\log(1+x_*) \quad \text{or} \quad x_* = 0$$



- Vertical slices of constant r are vector fields
- Whole regions have same arrow direction

Bifurcation point: $(r_c, x_c) = (1, 0)$

Example 11.2 (Cont'd of example 11.1)

From the above example,

(b) Show that there is a transcritical bifurcation at $(r_c, x_c) = (1, 0)$ using normal forms. Taylor expand about $r = 1, x = 0$

$$\begin{aligned}\dot{x} &= rx - \log(1+x) \\ &= (r-1)x + x - (x - \frac{1}{2}x^2 + \mathcal{O}(x^3)) \\ &= (r-1)x + \frac{1}{2}x^2 + \mathcal{O}(x^3)\end{aligned}$$

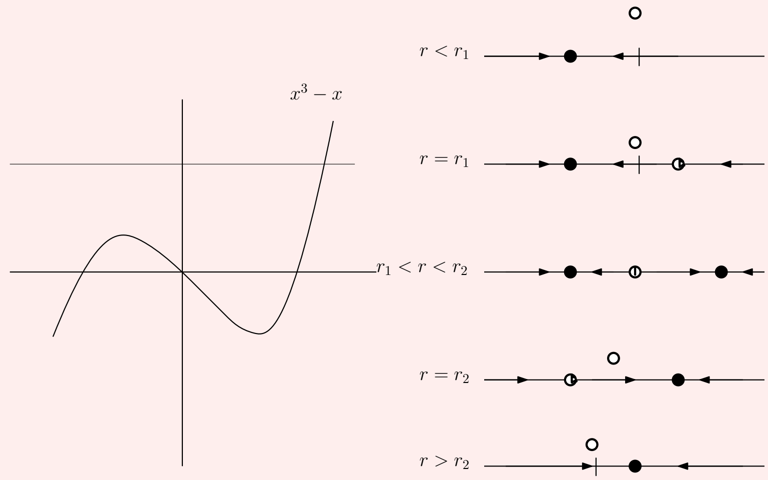
$\dot{x} = (r-1)x + \frac{1}{2}x^2 + \mathcal{O}(\epsilon^3)$ for $|x| < \epsilon, |r-1| < \epsilon$

This is normal form for transcritical bifurcation.

Example 11.3

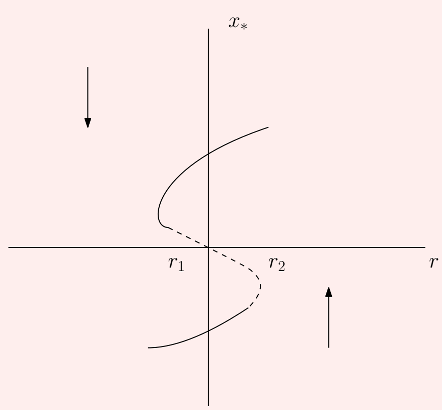
$\dot{x} = r + x - x^3$

(a) Sketch all vector field, sketch bifurcation diagram, find and classify bifurcations.



2 saddle node bifurcations at $r = r_1, r_2$

$$0 = r + x_* - x_*^3 \implies r = x_*^3 - x_*$$



Bifurcation point (x_c, r_c) satisfies:

- Fixed point: $0 = f(x_c, r_c) = r_c + x_c - x_c^3$
- $0 = \frac{\partial f}{\partial x}(x_c, r_c) = 1 - 3x_c^2$

$$0 = 1 - 3x_c^2 \implies x_c = \pm\frac{1}{\sqrt{3}}$$

$$0 = r_c + x_c - x_c^3 \implies r_1 = -\frac{2}{3\sqrt{3}}, r_2 = \pm\frac{2}{3\sqrt{3}}$$

$$(r_c, x_c) = \left(\frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Example 11.4 (Cont'd of example 11.3) (b) Show that there is a saddle-node bifurcation at $(r_c, x_c) = \left(\frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ using normal forms.

Taylor expand about $r = \frac{2}{3\sqrt{3}}, x = -\frac{1}{\sqrt{3}}$.

$$\dot{x} = r + x - x^3, \quad r = \frac{2}{3\sqrt{3}} + (r - \frac{2}{3\sqrt{3}})$$

$$x - x^3 = -\frac{2}{3\sqrt{3}} + 0(x + \frac{1}{\sqrt{3}}) + \frac{1}{2} \cdot \frac{6}{\sqrt{3}}(x + \frac{1}{\sqrt{3}})^2 + \mathcal{O}\left(\left(x + \frac{1}{\sqrt{3}}\right)^3\right)$$

Plug these in, get

$$\dot{x} = \left(r - \frac{2}{3\sqrt{3}}\right) + \sqrt{3}\left(x + \frac{1}{\sqrt{3}}\right)^2 + \mathcal{O}(\epsilon^3)$$

for $\left|r - \frac{2}{3\sqrt{3}}\right| < \epsilon^2, \left|x + \frac{1}{\sqrt{3}}\right| < \epsilon$. This is normal form for a saddle-node bifurcation ($\dot{y} = R + y^2$).