Math 131AH – Honors Real Analysis I

University of California, Los Angeles

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This is math 131AH – Honors Real Analysis I taught by Professor Greene, and our TA is Haiyu Huang. We meet weekly on MWF from 1:00pm – 2:00pm for lectures. There are two textbooks used for the class, *Principles of Mathematical Analysis* by *Rudin* and *Metric Spaces* by *Copson*. You can find other lecture notes at my github site. Please let me know through my email if you spot any mathematical errors/typos.

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§1 Lec 1: Oct 2, 2020

Overview:

- Hmwrk: 30 %
- Midterm 1: 20 %
- Midterm 2: 20 %
- Final: 30 %

§1.1 Introduction

functions $\to 1, 2, 3, 4, 5, 6, 7 \ldots$
functions defined on $\mathbb{Q}$ with value in $\mathbb{Q}$

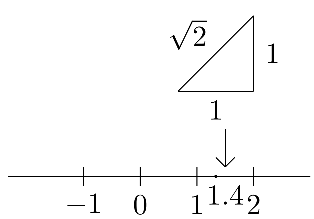
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

$a_i \in \mathbb{Q}$ $f(x) \in \mathbb{Q}$ if $x \in \mathbb{Q}$. Continuity makes sense.

$$x_0, x \quad x \text{close to } x_0 \implies f(x) \text{ close} f(x_0)$$

polynomials are continuous.
Somthing wrong: $\sqrt{2}$ is missing. What are these numbers that are not $\in \mathbb{Q}$? Choice:

1. Assume everything works and isolate what you need about "real numbers" (most of Rudin chap 1).
2. Construct the real numbers from rational numbers.



Classical argument:

$$x^2 \neq 2 \text{ if } x = \frac{p}{q} \in \mathbb{Q}$$

Proof. Suppose $\left(\frac{p}{q}\right)^2 = 2$

Note: wolog(without loss of generality)
can take $\frac{p}{q} > 0$ $p > 0$ $q > 0$

$$\left(\frac{p}{q}\right)^2 = 2$$
$$\frac{p^2}{q^2} = 2$$
$$p^2 = 2q^2$$

Now also wolog, can assume p and q are not both even numbers. But $p^2 = 2q^2$ means p has to be even (p^2 odd if p is odd).

$$p = 2n$$
$$p^2 = 2q^2$$
$$4n^2 = 2q^2$$

So $q^2 = 2n^2$, q is even. But it contradicts the initial assumption, p and q not both even $\square$

Related to: Why functions $\mathbb{Q}$ to $\mathbb{Q}$ not ideal for analysis?
– INFINITE DECIMAL

§2 Lec 2: Oct 5, 2020

§2.1 Mathematical Induction and More on Real Numbers

$P(n) \to 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$, where n is positive numbers.
Math induction: Proof by two steps:

1. Check $P(1)$ is true ✓
2. Assume $P(n)$ is true for all $n \le N$. Check that

$$P(N+1) \text{ is true}$$

Assume $1 + \ldots + N = \frac{N(N+1)}{2}$. Check

$$1 + \ldots + N + (N+1) = \frac{(N+1)(N+1+1)}{2}$$

Induction on k :

$$1^k + 2^k + \ldots + n^k$$

2nd illustration:

$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad r \neq 1$$

$r = 1 \implies 1 + r = \frac{1-r^2}{1-r}$

$$\begin{aligned} 1 + r + r^2 + \ldots + r^n + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \\ &= \frac{1 - r^{n+1} + r^{n+1} - r^{n+2}}{1 - r} \\ &= \frac{1 - r^{n+2}}{1 - r} \end{aligned}$$

$$(1 - r)(1 + r + \ldots + r^n) = 1 - r^{n+1} \qquad \text{Inspection}$$
$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}, \quad r \neq 1$$

$|r| < 1$ get inifite sum $\frac{1}{1-r}$

Example 2.1

Prime factors, prime = positive integers (> 1) with no factors except itself and 1, $p = ab$, $a > 1$, $b > 1$

$$2 \quad 3 \quad 5 \quad 7 \quad 11 \quad 13 \quad 17 \quad 19 \quad \ldots$$

Thin out as go along

Theorem 2.2 (Fundamental Theorem of Arithmetic)

Every positive integer > 1 is a product of primes.

Proof. Induction: $P(n)$ $n = 2, 3, \ldots$

$$P(2) = 2\checkmark$$

Assume P(n) $\ldots n \leq N$ $(N > 2)$. Every integer greater than 1 but smaller than or equal to N as a product of primes. We try to prove: $N + 1$ is a product of primes.

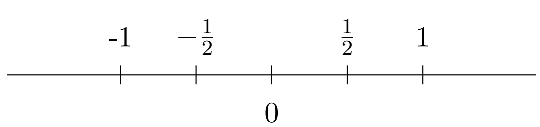
1. $N + 1$ is prime: Done $N + 1 = N + 1$
2. $N + 1$ is not a prime
$$N + 1 = a \cdot b \qquad a > 1 \;\; b > 1$$
Induction assumption ($a < N + 1$ since $b > 1$), a is a product of primes $a > 1 \implies b < N + 1$, b also a product of primes. So, $N + 1 = ab$ is a product of primes.

$N + 1 = ab$ is a product of prime. □

Why does induction work? If $P(n)$ not always true, $P(n)$ look at smallest n where $P(n)$ is false.
$n = 1$ not there $P(1)$ is supposed true (checked already). N_0 smallest one where $P(N_0)$ false $N_0 > 1$. Induction step says that $P(n)$ is true for all $n \leq \underbrace{N_0 - 1}_{>0} \implies P(N_0)$ true ($\times$
).
Let's go back to real numbers.

Last time: talked about $\sqrt{2}$ is irrational but $\sqrt{2}$ exists, so we need to enlarge our number system: $\mathbb{Q}$ rational numbers.

$$\frac{p}{q} > \frac{r}{s} \qquad ps > rq \qquad (p, q, r, s > 0)$$



x, y rational $x, y > 0$, $x + y > 0$, $xy > 0$

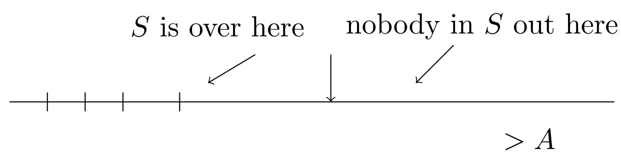
$x^2 = 2$ no answer in $\mathbb{Q}$. Enlarge number system, $\mathbb{Q} \subset \mathbb{R}$. What should $\mathbb{R}$ be like?

1. $\mathbb{R}$ ought ot have arithmetic like $\mathbb{Q}$

$$x + y \quad xy \quad \frac{x}{y} \quad 0 \quad 1$$

2. $\mathbb{Q} \subset \mathbb{R}$, arithmetic in $\mathbb{R}$ restricted to $\mathbb{Q}$, $\frac{1}{2} + \frac{1}{3}$ in $\mathbb{Q}$ ought to be $\frac{5}{6}$ in $\mathbb{R}$.
3. Order should positive in $\mathbb{Q} \implies$ in $\mathbb{R}$. $\mathbb{R}$ should have an order of its own too, $x \quad y$ positive then $x + y$ pos and xy pos.
4. want to fill in the holes in $\mathbb{Q}$. Want to have Least Upper Bound Property

   $S \subset \mathbb{R}$: An upper bound for S is a number A with property $A \geq x$ if $x \in S$



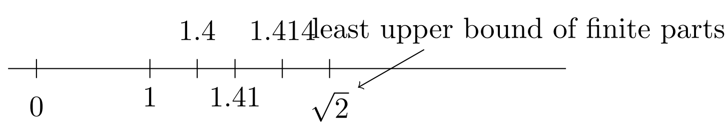
   $1, 2, 3, 4, \ldots$ have no upper bound.

   S is bounded above means that some upper bound A exists.

§2.2 Least Upper Bound Property

If S is bounded above ($S \neq \emptyset$) then it has a "least upper bound" where a number A_0 is called the least upper bound of S if A_0 is an upper bound for S & if A is an upper bound for S then $A_0 \leq A$.

Motivation: Think about $\sqrt{2}$



Denote: l.u.b(or supremum)(sequence) $= \sqrt{2}$
Means can define an infinite decimals: least upper bound of successive truncation.

$$0.99999\ldots \to 1.0$$

§3 Lec 3: Oct 7, 2020

§3.1 Cauchy Sequence

$\{x_n\}\ x_1, x_2, x_3, \ldots$ values $x_j \in \mathbb{Q} \quad x_j \in \mathbb{R}$
$S \quad x_1, x_i \ldots x_j \in S$

Definition 3.1 (Sequence) — A sequence with values in a set S is a function from positive integers $\{1, 2, 3 \ldots\}$ into S.

Definition 3.2 (Cauchy Sequence) — A Cauchy sequence is ($\mathbb{Q}$ valued or $\mathbb{R}$ valued) $\{x_i\}$ is sequence s.t. for every $\epsilon > 0$ there is a positive integer N_ϵ s.t.

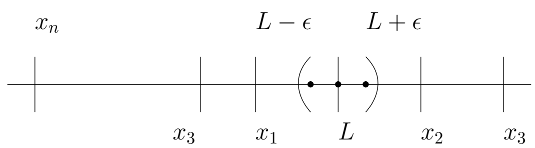
$$|x_i - x_j| < \epsilon \quad \text{if} \quad i, j > N_\epsilon$$

ϵ rational or real (same idea).

Lemma 3.3

If $\{x_j\}$ has a finite limit then it's a Cauchy sequence.

$\{x_i\}$ has L as a limit $\lim x_j = L$ means for every $\epsilon > 0$ then there is an N_ϵ such that $j \geq N_\epsilon$, $\quad |x_j - L| < \epsilon$

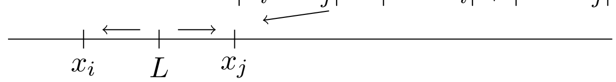


Everybody in $(L - \epsilon, L + \epsilon)$ except a finite number

Proof. Given $\epsilon > 0$, want to find N so that $i, j \geq N \implies |x_i - x_j| < \epsilon$
$|x_i - L|$ small, $|x_j - L|$ small and $\lim x_j = L$.

$$|x_i - x_j| \leq |x_i - L| + |x_j - L|$$

$$|x_i - x_j| = |L - x_i| + |L - x_j|$$



$i, j \geq N_{\frac{\epsilon}{2}}$:

$$|x_i - x_j| \leq \underbrace{|x_i - L|}_{<\frac{\epsilon}{2}} + \underbrace{|x_j - L|}_{<\frac{\epsilon}{2}}$$

Because $\lim x_n = L$, there is an $N_{\frac{\epsilon}{2}}$ s.t. $|L - x_n| < \frac{\epsilon}{2}$ if $n \geq N_{\frac{\epsilon}{2}}$
Get $|x_i - x_j| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ if $i, j \geq N$. Cauchy sequence: there exists number N s.t.

$$|x_i - x_j| < \epsilon \quad \text{if} \quad i, j \geq N$$ □

Cauchy sequence $\implies$ the existence of limit? Yes, for $\mathbb{R}$ valued sequences but NO for $\mathbb{Q}$ valued things.

$$\underbrace{\{x_n\}}_{\text{rational numbers}} \qquad \text{can be Cauchy seq without there being a rationl number } L \quad \text{such that } \lim x_j = L$$

But allow real L then $\exists L$ s.t. $\lim x_j = L$ if $\{x_j\}$ is Cauchy sequence(no rational limit – since $\sqrt{2}$ is irrational). Because $\mathbb{Q}$ has holes in it! (intuitive idea).

Example 3.4

$1, 1.4, 1.41, 1.414, 1.4142\ldots$ (decimal approx of $\sqrt{2}$) – Cauchy sequence. No – since $\sqrt{2}$ is irrational.

§3.2 Cauchy Completeness of $\mathbb{R}$

If $\{x_j\}, x_j \in \mathbb{R}$ is Cauchy sequence, then $\exists L \in \mathbb{R}$ s.t. $\lim x_j = L$.
"$\mathbb{Q}$ is not Cauchy complete" but $\mathbb{R}$ is. Why does this work?
Need: Least upper bound property. Assume L.U.B Property proof.

Proof. (Cauchy completeness from L.U.B Property)
Hypothesis: $\{x_i\}$ Cauchy seq

1. Prove that $\{x_i\}$ bounded $\Longleftarrow$ $\exists M > 0$ s.t. $|x_i| \leq M$ all i.

   Clear if take $\epsilon = 1$ in def. of Cauchy seq $\exists N$ s.t. $|x_i - x_j| < 1$ if $i, j \geq N \implies |x_N - x_j| < 1$ if $j \geq N \implies \quad |x_j| \leq |x_N| + 1 \quad j \geq N$

   So, $M = \max(|x_N| + 1, |x_1|, \ldots, |x_{N-1}|$ then $|x_j| \leq M$ all j !

   Next stage is to show that a bounded sequence always has a subsequence(tricky!) with a limit. Then if a Cauchy seq has a subseq with limit L, then L is limit of whole seq. (Bolzano – Weierstrass Theorem)

□

§4 Lec 4: Oct 9, 2020

§4.1 Bolzano – Weierstrass Theorem

– implied by Least Upper Bound Property

Theorem 4.1 (Bolzano – Weierstrass)

If $\{x_n\}$ sequence $(x_1, x_2, x_3 \ldots)$ that is bounded (means: $\exists M > 0 \ni |x_n| \leq M \forall\ n$), then $\exists L$ and a subsequence $\{x_{n_i}\}$ s.t. $\lim x_{n_i} = L$.

Slogan: Every bounded sequence has a convergent subsequence.

Example 4.2

$$1, 2, 1, 2, 1, 2, \ldots$$

The subsequence of the above sequence has either 1 or 2 as the limit.

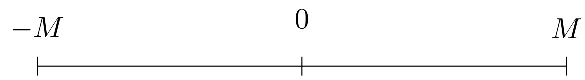
$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$$

Unbounded sequence – subsequence (limit 1, limit 2, limit 3...)

No claim of uniquneness of anything.

Proof – Summer 2008 Analysis Lec 4

Proof. So either $[-M, 0]$ or $[0, M]$ (maybe both) contains x_n for infinitely many n values. If each contained x_n for only finitely many n values X.

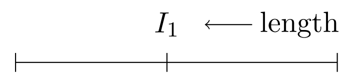


Every x_n is in $[-M, M]$ – $\{x_n\}$ is bounded

$$[-M, M] = [-M, 0] \cup [0, M]$$
$$I_1 = [-M, 0] \quad \text{or} \quad [0, M]$$

where chosen intervalhas x_n for infinitely many n values.
Do this again!

$$I_1 = [a_1, b_1] \qquad |b_1 - a_1| = M$$



left half of I_1, right half of I. Let I_2 = one of halves that contains x_n for infinitely many n values.

$$I_2 = [a_2, b_2] \qquad a_2 < b_2, \quad b_2 - a_2 = \frac{M}{2}$$

Continue

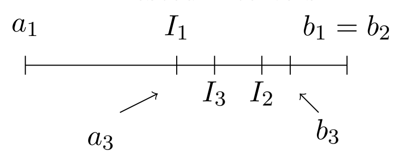
$$I_3 = [a_3, b_3] \qquad a_3 < b_3, \quad b_3 - a_3 = \frac{M}{4}$$

$$\vdots$$

$$I_k = [a_k, b_k] \qquad b_k - a_k = \frac{M}{2^{k-1}}$$

Each I_k contains x_n for infinitely many n values.

Nested Intervals



$$I_{k+1} \subset I_k \subset \ldots \subset I_1 \subset [-M, M]$$
$$a_{k+1} \geq a_k \ldots \qquad b_{k+1} \leq b_k \ldots$$

Claim $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$

Reason: $\sup a_k \in \bigcap_{k=1}^{\infty} I_k$ where sup = sup of left hand endpoint(=greatest lower bound of bs). l.u.b of a's $\leq b_k$, b_k bigger than or $\geq$ all a's.

$$\alpha = \text{lub a's}$$
$$\alpha \geq a_k \quad \forall k$$
$$\alpha \leq b_k \quad \forall k$$
$$\alpha \in [a_k, b_k]$$

Goal: $\alpha \in \bigcap_{k=1}^{\infty}$. Find a subsequence of $\{x_n\}$ converges to α.

Choose $x_k = x_n$ that belongs to I_k. Can also arrange successively:

$$n_1 < n_2 < n_3 < n_4$$

$x_{n_1} \in I_1 \quad x_{n_2} \in I_2$ can make $n_2 > n_1$ because infinitely possible $x_n's$ in I_2 n value. Continue to get subsequence, $\{x_{n_k}\}$ subsequence. Claim:

$$\lim_{k \to \infty} x_{n_k} = \infty$$

Reason:

$$\text{dis}(x_{n_k}, \alpha) \leq \text{length of } I_k \quad \alpha \in I_k, \quad x_{n_k} \in I_k$$

which is equivalent to

$$|x_{n_k} - \alpha| \leq \frac{M}{2^{k-1}} \quad \text{given } \epsilon > 0$$

When k is large,

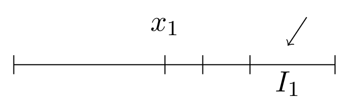
$$\frac{M}{2^{k-1}} < \epsilon$$

So $|x_{n_k} - \alpha| < \epsilon$ $\square$

This argument (or a variant) shows something else:
If $\{x_n\}$ sequence in $[0, 1]$ then there's an $\alpha \in [0, 1]$ with it never happening that

$$x_n = \alpha$$

"The real numbers in $[0, 1]$ are uncountable." (come from the least upper bound property)



I_1 one of $[0, \frac{1}{3}]$ $[\frac{1}{3}, \frac{2}{3}]$ $[\frac{2}{3}, 1]$ such that $x_1 \notin I_1$,

$$[0, \frac{1}{3}] \cap [\frac{1}{3}, \frac{2}{3}] \cap [\frac{2}{3}, 1] = \emptyset$$

$x_1 \notin I_2 \quad I_2 \subset I_1, \quad \& x_1 \notin I_1$. Continue. Get

$$I_1 \supset I_2 \supset I_3 \supset \dots$$

length $I_k = \frac{1}{3^k}$ and I_k is such that $x_1, x_2, x_3 \dots x_k$ are none of the ?n? in I_k. Same as before

$$\exists \alpha \in \bigcap_{\infty}^{k=1} I_k$$

$\alpha =$ sup of set of left hand endpoints of I_k. Claim α cannot be an x_N value. Clear: $x_N \notin I_N$ but $\alpha \in I_n \qquad \alpha \in \bigcap_{\infty}^{n=1} I_n$. But contrast:

There is a list of rational numbers in $[0, 1]$

$\frac{p}{q}$ $p < q$

	2	3	4	5	6	$\cdots$
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$			
2	-	$\frac{2}{3}$	$\frac{2}{4}$			
3	-	-	$\frac{3}{4}$			
$\vdots$	-	-	$\frac{\sqrt{2}}{2} \in [0,1] \to$ irrational - no exist			

$[0,1]$ not countable

Q is countable

§5 Lec 5: Oct 12, 2020

§5.1 Equivalence Relation

(p.10, Copson – Metric Space)
R set, relation of A and B $(A \times B)$ $\quad (a,b) \in R \quad aRb$
Functions: one b given a – exact one. $(A \to B)$

Example 5.1
$A = B = Q$
aRb or $(a,b) \in R$ if $a > b$
(mother,child)

- (Sara, Sebastian) $\in R$
- (Sara, Alita) $\in R$

Equivalence is a special kind of relation: (on a set $A; B \quad A = B$)
Properties:

1. $aRa \qquad A = Q$
2. $aRb \implies bRa$
3. aRb & bRc then aRc
   Example: $\mathbb{Z} \quad a \sim b$ means $a - b$ is divisible by 5
   $$1 \sim 6 \quad 0 \sim 5 \ldots$$
   $a \sim a \quad a - b$ div $\implies b - a$ div. by 5.

If $a - b$ div. by 5, and $b - c$ div by 5, then is $a - c$ div. by 5 true?
Sure, $a - b = 5k, \quad b - c = 5l \implies a - c = 5(k + l)$
"Equivalence classes": set $[a] = \{$ all b such that $aRb\}$

In the example above, $[a] = \{$ all b such that $a - b$ div. by 5$\}$

$$[2] = \{2, 7, -3, 12, -8, \ldots\}$$

$\mathbb{Z}_5$: integer mod 5.

1. $[a] \quad [p]$ either equal or have nothing in common.
2. $a \in [a]$ so is in some equivalence class.

A equivalence relation $\sim$ on $A \leftrightarrow$ a partition of A into subsets which are pairwise disjoint.
Q Cauchy seq. of rational numbers

$$\{x_n\} \sim \{y_n\}$$

means $\lim_{n\to\infty} |x_n - y_n| = 0$. Equivalence relation:

1. $\{x_n\} \sim \{x_n\}$ $(\lim(x_n - x_n) = 0)$
2. $\{x_n\} \sim \{y_n\} \implies \{y_n\} \sim \{x_n\}$
3. $\{x_n\} \sim \{y_n\} \,\&\, \{y_n\} \sim \{z_n\} \implies \{x_n\} \sim \{z_n\}$

Idea: Define a real number to be a (Cauchy seq. of rationals) equivalence class.
Homework: want to check that arithmetic extends to "real numbers"

$$[\{x_n\}] + [\{z_n\}] = [\{x_n + z_n\}]$$

Check that

1. $\{x_n + z_n\}$ is a Cauchy seq.
2. Only depends on equivalence classes.

Want

$$\{x_n\} \sim \{y_n\} \qquad \{z_n\} \sim \{w_n\}$$

then $\{x_n + z_n\} \sim \{y_n + w_n\}$. So,

$$[\{x_n + z_n\}] = [\{y_n + w_n\}]$$

Example 5.2

$$[2] + [11] = [2 + 11] = [13]$$

So, $[2+1] \sim [13]([11] = [1])$. Arithmetic (addition) in $\mathbb{Z}_5$ thus makes sense. How about multiplication? $\frac{[1]}{[a]} \leftarrow$ exists $[a] \neq 0$.

$$\frac{[1]}{[2]} = [3] \qquad [2][3] = [6] = [1]$$

Thus, $\mathbb{Z}_5$ is a field.

$\frac{p}{q} \sim \frac{r}{s}$, $\quad q, s \neq 0$ means $ps = rq$ (when talking about fractions – associate it with equivalence relation). Q = set of equivalences classes. $(\frac{p}{q})$: equivalence classes).
Last time, we proved that Cauchy seq. of real numbers have limits (lub property). Also, no sequence $\{x_n\}$ such that it hits all real numbers in $[0, 1]$ – this is important. Contrast with $Q \cap [0, 1]$, then there is a sequence that hits them all. Refer to the last figure in Lec 4 or math.ucla.edu/~greene – Summer 2008.

§6 Lec 6: Oct 14, 2020

Bolzano - Weierstrass:
Every bounded sequence has a convergent subsequence.
And we know about the Least Upper Bound Prop.

§6.1 Continuous Functions on Closed Interval

$f : S \to \mathbb{R}, \quad S \subset \mathbb{R}$

Example 6.1

$$S = [a, b]$$
$$S = \mathbb{R}$$

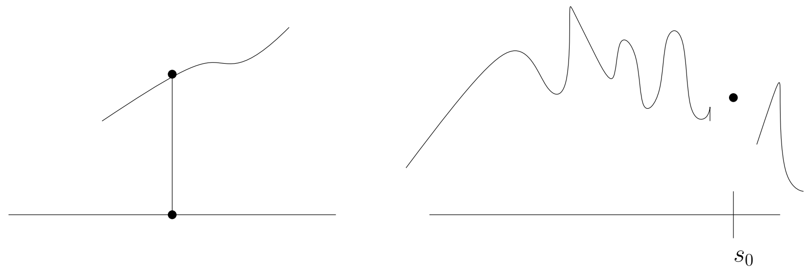
Definition 6.2 (Continuity) — $s_0 \in S, f$ is continuous at s_0 if given $\epsilon > 0, \exists \delta > 0$ s.t.

$$|s - s_0| < \delta_\epsilon \implies |f(s) - f(s_0)| < \epsilon$$

Three properties:

$$f : [a, b] \to \mathbb{R}$$
$$f \text{continuous}$$

1. f is bounded on $[a, b]$ means $\exists M$ s.t. for all $x \in [a, b], \quad |f(x)| \leq M$

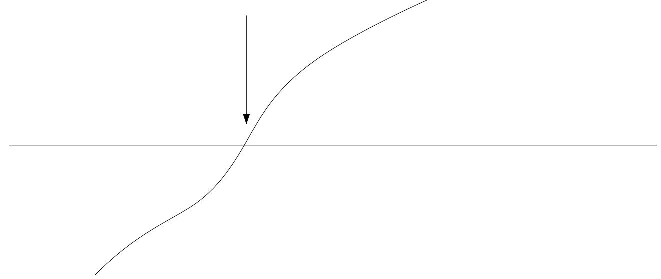


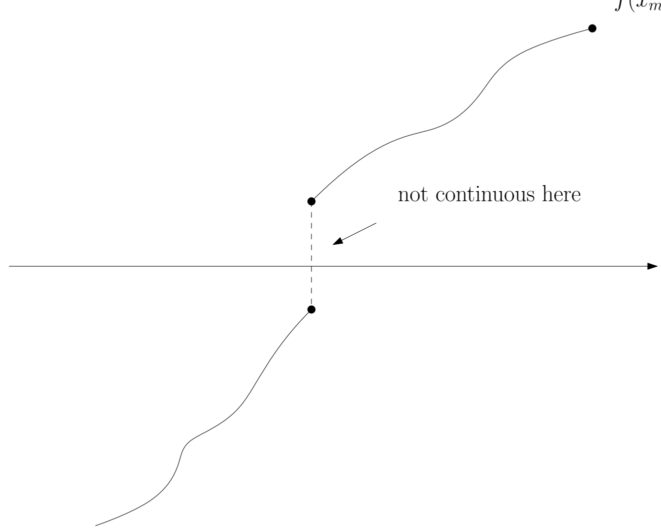
2. There exists $x_{\min}, x_{\max} \in [a,b]$ such that for all $x \in [a,b]$

$$f(x_{\min}) \le f(x) \le f(x_{\max})$$

Slogan: f attains its maximum and minimum.

3. If $\alpha, f(x_{\min}) < \alpha < f(x_{\max})$, then $\exists x \in S = [a,b]$ s.t. $f(X) = \alpha$.
"Intermediate Value Theorem" Need the least upper bound prop – "completeness of





real numbers"

Exercise: def of continuity $\{s_n\}$ converges to $s_0 \iff$ if $s_n \to s_0$, $s_n \in S, s_0 \in S$ then $\{f(s_n)\}$ converges to $f(s_0)$.

Example 6.3

For (3),

$$f(x) = x^2 - 2 \quad \text{on} \quad Q \cap [1,2]$$

Then $f(1) = -1$, $f(2) = 2$, but no rational $x \in [1,2]$ s.t. $f(x) = 0$.

Back to the properties:

1. f is bounded – Think about $|f| \leftarrow$ continuous if f is (exercise).

   $\exists M$ such $|f(x)| \leq M$ all $x \in [a,b]$. Suppose no such M exists.

   Try $M = 1,2,3,4,5,6,\ldots$. So $\exists x_1 \quad |f(x_1)| > 1$

   $$|f(x_2)| > 2$$
   $$\vdots$$
   $$|f(x_n)| > n$$

   But Bolzano – Weierstrass: subsequence $\left\{x_{n_j}\right\}$ that converges to x_0 say $|f(x_0)| \leftarrow$

   finite number. So $\exists N \ni \quad |f(x_0)| \leq N$.

   Now for j large enough

   $$\left|f(x_{n_j}) - f(x_0)\right| < 1$$

   x_{n_j} converges to x_0

   $$\left|f(x_{n_j})\right| < |f(x_0)| + \left|f(x_{n_j} - f(x_0)\right|$$

   So j is large enough that

   $$\underbrace{\left|f(x_{n_j})\right|}_{\geq |n_j| = n_j} \leq N + \text{ something less than 1 } \leq N$$

2. Attains max and min

   Similar: $\{f(x) : x \in [a,b]\}$ bounded set, has sup where

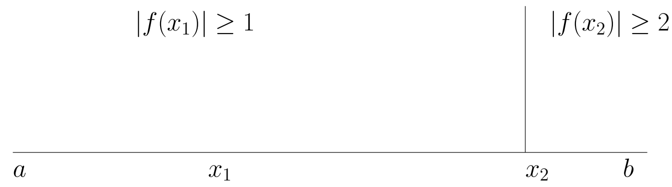
   $$\sup\{f(x) : x \in [a,b]\}$$

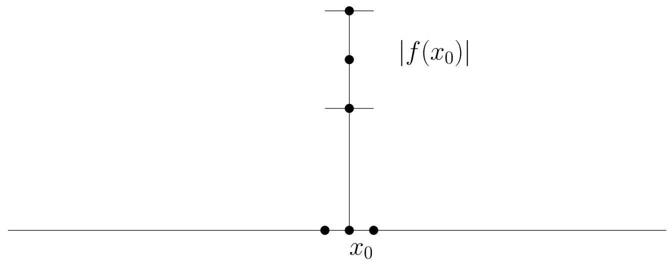
   either in the set of f-values (done if that's true), $\sup f = f(x_0)$.

   OR: $\sup f$ acutally not in the set $\{f(x) : x \in [a,b]\}$

   Now $\left\{x_{n_j}\right\}$ converges to $x_0 \in [a,b]$

   Claim 6.1. $f(x_0) = \sup\{f(x) : x \in [a,b]\}$

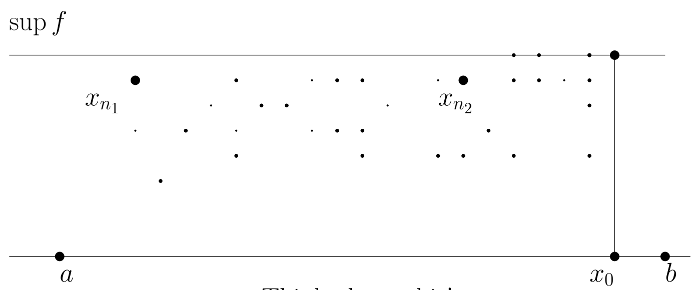




$$f(x_{n_j}) \leq \sup\{f(x) : x \in [a,b]\}$$

and $\lim f(x_{n_j}) = f(x_0) = f(\lim x_{n_j})$. So

$$f(x_0) = \sup f$$



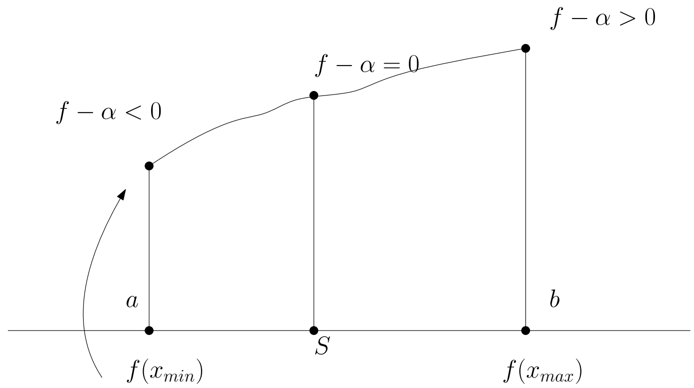
Think about this!

3. $\alpha \in [f(x_{\min}), f(x_{\max})]$ then x such that $f(x) = \alpha$.

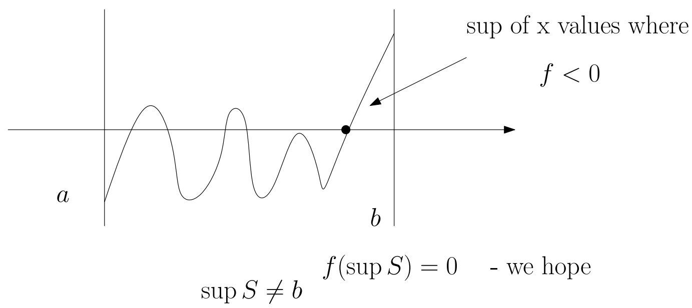
   Proof. Wolog:

   $$f(a) < 0 \quad \text{and} \quad f(b) > 0$$

   then $\exists x \in [a,b]$ with $f(x) = 0$.

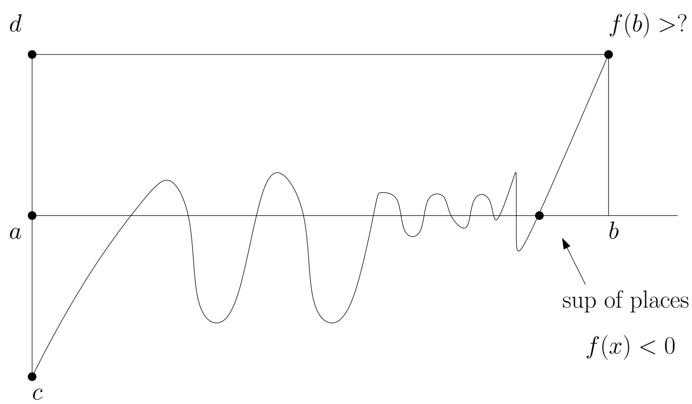


Use l.u.b: Look at S : $\{x : f(x) < 0\}$ and $S \neq \emptyset$ because $f(a) \in S$. Also, S is bounded above – $\exists$ l.u.b for S, $\sup S \in [a, b]$. Hope that $f(\sup S) = 0$.



$\sup S \neq b$ is clear because $f(b) > 0$ so $f(b - \epsilon) > 0$ for small ϵ.

So $\sup S = x_0, \quad a < x_0 < b$. What is $f(x_0)$? If it's negative, then there are slightly bigger $x \in [a_0, b] \ni f(x) < 0$ (continuity). In addition, x_0 cannot be a limit of x with $f(x) < 0$ – $x_0 = \sup$ places where $f < 0$. □



f continuous on $[a, b]$ if it is

1. bounded.
2. attains max and min.
3. attains every value between max value and min value.

$f([a, b]) = [c, d]$ where c is min of f and d is max of f.

§7 Lec 7: Oct 16, 2020

§7.1 Uniform Continuity

Definition 7.1 (Uniform Continuity) — $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$. f is uniformly continuous on S if given $\epsilon > 0$ there is a $\delta > 0$ s.t. $|f(x) - f(y)| < \epsilon$ if $x, y \in S$ and $|x - y| < \delta_\epsilon$

Example 7.2

$f : S \to \mathbb{R}$, $S = \mathbb{R}$, $f(x) = x^2$. Continuous on $\mathbb{R}$ but it is not uniformly continuous on $\mathbb{R}$.

Continuity: Given fixed x, and $\epsilon > 0$ want δ so that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

$|x^2 - y^2| = |x - y||x + y|$ and want it smaller than ϵ. Assume $\delta \le 1$.

$$\begin{gathered} |x + y| \le |x| + |y| \\ |y| < |x| + 1 \;\text{ if }\; |x - y| < \delta(\le 1) \end{gathered}$$

So, if $|x - y| < \delta(\le 1)$,

$$\begin{aligned} \left|x^2 - y^2\right| &= |x - y||x + y| \\ &\le |x - y|(2|x| + 1) \end{aligned}$$

Choose $\delta < \frac{\epsilon}{2|x|+1}$ (ok since x is fixed)

$$\begin{aligned} |x^2 - y^2| &< \frac{\epsilon}{2|x| + 1}(2|x| + 1) \\ &= \epsilon \;\text{ if }\; |x - y| < \min\left\{1, \frac{1}{2|x| + 1}\right\} \end{aligned}$$

Uniform continuity does not work on $\mathbb{R}$.

Claim 7.1. $\epsilon = 1 > 0$, there is no $\delta > 0$ s.t. $\left|x^2 - y^2\right| < 1 = \epsilon$ for all x, y with $|x - y| < \delta$.

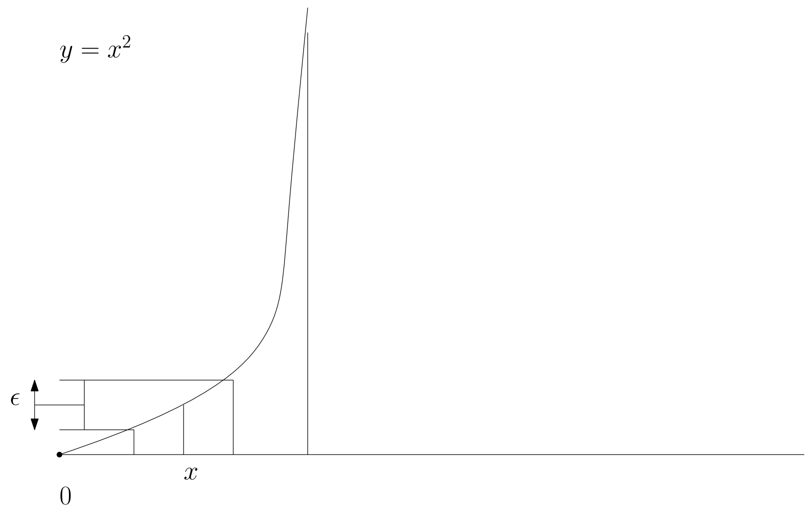
Why? Look at for $\delta > 0$, consider $y = \frac{1}{\delta} + \frac{\delta}{2}$, $\quad x = \frac{1}{\delta}$

$$|x - y| < \delta$$

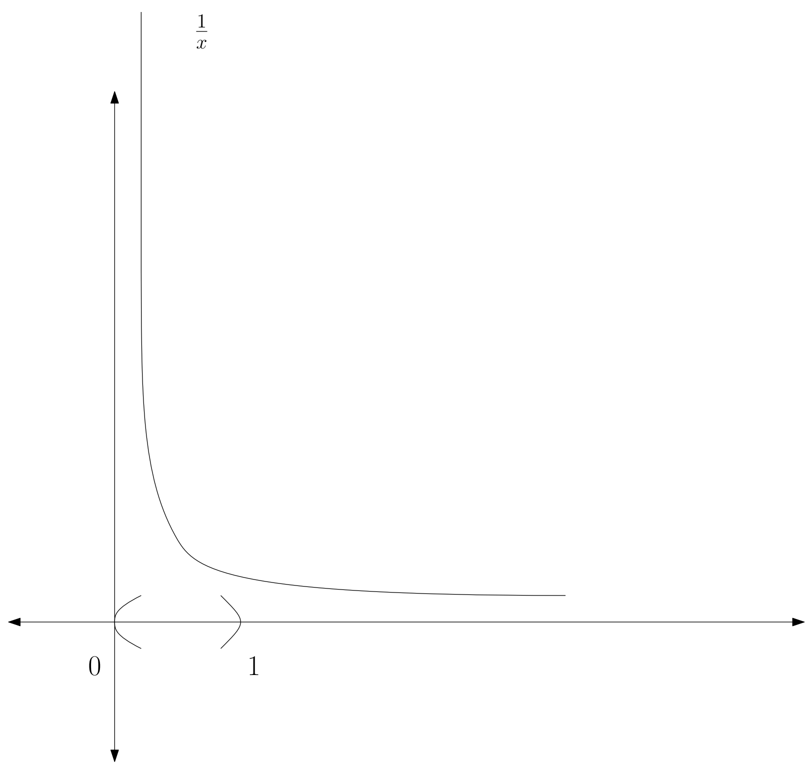
Also,

$$\begin{aligned} &\left|\left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \left(\frac{1}{\delta}\right)^2\right| \\ &= \left|\frac{1}{\delta^2} + 2\left(\frac{1}{\delta}\right)\left(\frac{\delta}{2}\right) + \left(\frac{\delta}{2}\right)^2 - \frac{1}{\delta^2}\right| \\ &= 1 + \left(\frac{\delta}{2}\right)^2 > 1 \end{aligned}$$

which is a contradiction.



Exercise 7.1. $\frac{1}{x}$ on $(0, 1)$ is continuous but <u>not</u> uniformly continuous. Sugges plausibly f



continuous on $[a, b]$ then it's uniformly continuous on $[a, b]$ where a, b are finite.

Theorem 7.3 (Heine – Cantor (Uniformly Continuous))
A continuous function f on a closed interval is uniformly continuous.

Proof. (By contradiction) Suppose not. Then $\epsilon > 0$ s.t. no δ "works". In particular, $\exists \epsilon > 0$

s.t. $\delta = 1$ fails, $\delta = \frac{1}{2}$ fails, etc. So $x, y \in [a, b]$ with $|f(x_1) - (fy_1)| \geq \epsilon$ but $|x_1 - y_1| < 1$. $x_n, y_n \in [a, b]$ with $|f(x_n) - f(y_n)| \geq \epsilon$ but $|x_n - y_n| < \frac{1}{n}$. Hope this is impossible. Bolzano - Weierstrass $\implies$ $\{n_j\}$ s.t. $\left\{x_{n_j}\right\}$ has a limit

$$x_0 = \lim, \qquad x_0 \in [a, b]$$

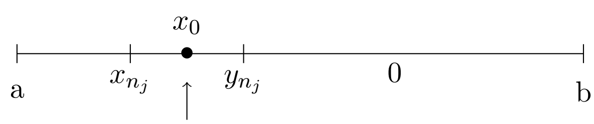
Now, claim $\left\{y_{n_j}\right\}$ also has limit x_0.

$$\left|x_{n_j} - y_{n_j}\right| < \frac{1}{n_j}$$

small when n_j large (j large).

$$\begin{aligned} \lim x_{n_j} &= x_0 \\ \lim y_{n_j} &= x_0 \\ \lim f(x_{n_j}) &= f(x_0) \\ \lim f(y_{n_j}) = f(x_0) \end{aligned}$$

So, $\lim f(x_{n_j}) - f(y_{n_j}) = 0$, but it contradicts $\left|f(x_{n_j} - f(y_{n_j})\right| \geq \epsilon$ for all j. $\square$



$$f(x_0) \leq \left|f(x_{n_j}) - f(x_0)\right| + \left|f(x_0) - f(y_{n_j})\right| \to 0$$

Ideas of continuity and uniform continuity and Bolzano - Weierstrass Theorem – all have reasons in metric spaces.

§8 Lec 8: Oct 19, 2020

§8.1 Convergence of Series

Series is "formal sum", an infinite sum

$$a_0 + a_1 + a_2 + \ldots = \sum_{j=1}^{\infty} a_j$$

A series $\iff$ sequence $a_1, a_2, a_3, \ldots$ add together. Associated to $a_1 + a_2 + a_3 + a_4 \ldots$ is a sequence of partial sum

$$S_N = \sum_{n=1}^{N} a_n, \qquad N = 1, 2, 3, 4, 5, \ldots$$

number valued sequence.

Definition 8.1 (Convergence of Series) — Series converges if sequence associated $\{S_N\}$ converges (has a limit).

Lots of things are defined by series such as ($x \in \mathbb{R}$),

$$e^x = \lim_{N\to\infty} \left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^N}{N!}\right)$$

Given series $a_0 + a_1 + a_2 + a_3 + \ldots$, when does it converge?

$$1 - 2 + 3 - 4 + 5 - 6 + 7 \ldots$$
$$S_1 = 1, \quad S_2 = -1, \quad S_3 = 2 \ldots$$

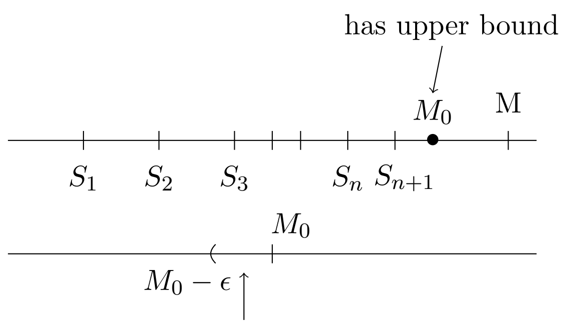
<u>NO LIMIT!</u> Series do not necessarily have to converge then it's okay to write

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

First thing to look at – Case where $a_j \geq 0$

$$S_N \leq S_{N+1}, \quad N = 1, 2, 3, \ldots$$

$S_{N+1} = S_N + a_{N+1}$ so $a_{N+1} \geq 0$ means $S_{N+1} \geq S_N$. Two cases:
Case 1: $\{S_n\}$ not bounded above.
$\lim S_N$ does not exist $\to$ Series diverges (sequences with limits are always bounded above and below).
Case 2: $\{S_n\}$ bounded above.
$\lim_{n\to\infty} S_n$ always exists. Namely, it is the least upper bound of set of values of S_n.



There is an S_{n_0} in this interval $(M_0 - \epsilon, M_0]$, M_0 is lub

From that n_0 on,

$$S_n \geq S_{n_0}, \quad S_n \leq M$$

S_n satisfies $|S_n - M_0| < \epsilon$ if $n \geq n_0$. So $\lim S_n = M_0$. This implies that S_n is a Cauchy sequence (it has a limit). Given $\epsilon > 0, \exists N_\epsilon$ s.t. $\left| \underbrace{\sum_1^{n_1} a_n}_{S_{n_1}} - \underbrace{\sum_1^{n_2} a_n}_{S_{n_2}} \right| < \epsilon$ if $n_1, n_2 \geq N_\epsilon$.

Suppose $n_1 > n_2 \geq N_\epsilon$

$$\sum_1^{n_1} a_n - \sum_1^{n_2} a_n = \sum_{n_2+1}^{n_1} a_n$$

Note: $S_7 - S_5 = a_6 + a_7$ which explains the above expression.

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} \dots$$

converges, but so does the following series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots < 2$$

This works for arbitrary choices of $+$ or $-$.

Theorem 8.2 (Absolute Convergence)
If $|b_1| + |b_2| + |b_3| + \dots$ converges, then

$$b_1 + b_2 + b_3 + \dots \quad \text{converges}$$

"Absolute convergence" $\implies$ convergence (but not necessarily the same limit).

Proof. Assume $\underbrace{\{S_n^A\}}_{A \text{ for absolute}}$ for absoluted series has limit. So

$$\sum_1^\infty |b_n| \quad \text{converges}$$

$\implies \{S_n^A\}$ Cauchy sequence.
We hope it $\implies \{S_n\} = \left\{\sum_{j=1}^n b_j\right\}$ is a Cauchy sequence.

$$S_{n_1}^A - S_{n_2}^A = |b_{n_2+1}| + |b_{n_2+2}| + \dots + |b_n|$$

But

$$|b_{n_2+1} + \dots + b_n| \le |b_{n_2+1}| + \dots + |b_n| (= S_{n_1}^A - S_{n_2}^A)$$

So,

$$|S_{n_1} - S_{n_2}| \le S_{n_1}^A - S_{n_2}^A < \epsilon \quad \text{for } n_1, n_2 \ge N_\epsilon$$

Then $|S_{n_1} - S_{n_2}| < \epsilon$ for $n_1, n_2 \ge N_\epsilon$. □

This is IMPORTANT – Better understand it thoroughly.

Corollary 8.3 (Root Test)
$|b_n| \le Cr^n, 0 < r < 1$, C, r fixed, then $\sum b_n$ converges.

Reason: $\sum_{n=0}^\infty Cr^n = C\frac{1}{1-r}$ (geometric series).

Exercise 8.1. $\sum_{n=0}^N Cr^n = C\frac{r^{N+1}-1}{r-1}, 0 < r < 1$ has limit $\frac{C}{1-r}$. Prove by induction.

Detail: Hypothesis:

$$|b_n| \leq Cr^n$$
$$\sum_1^\infty |b_n| \leq \sum_1^\infty Cr^n < \infty$$
$$\sum_b^N |b_n| \leq \sum_0^N Cr^n \leq M < \infty$$

So $\sum_0^N |b_n|$ converges and bounded by Cr, and $b_1 + b_2 + \dots$ converges absolutely.

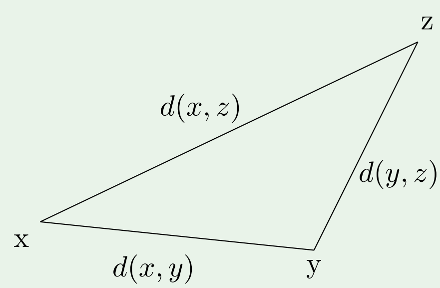
§9 Lec 9: Oct 21, 2020

§9.1 Metric Spaces

Definition 9.1 (Metric Spaces) — A set X, elements are "points", together with a function on $\underbrace{X \times X}_{\text{ordered pairs } (x,y)}$, $x \in X, y \in Y$, $\underbrace{d(x,y)}_{\text{distance}}$ with the following properties:

1. $d(x,y) \geq 0$ for all x, y.
   $d(x,y) = 0 \iff x = y$. Or $d(x,x) = 0$.
2. $d(x,y) = d(y,x)$.
3. $\triangle$ inequality:

$$d(x,y) + d(y,z) \geq d(x,z)$$
$$d(x,z) \leq d(x,y) + d(y,z)$$



Example 9.2 1. X set. Can you define a $d : X \times X \to \mathbb{R}$ to make (x, d) a metric space?

YES! Define given set X, $d(x_1, x_2) = 0$ if $x_1 = x_2$, or $d(x_1, x_2) = 1$ if $x_1 \neq x_2$. "discrete".

- $d(x, y) \geq 0$.
- $d(x, y) = d(y, x)$.
  $x = y$ both are 0.
  $x \neq y$ both are 1.
- $d(x, z) \leq d(x, y) + d(y, z)$
  $x = z \implies d = 0$.
  $x \neq z \implies d(x, z) = 1$.
  If $x = y$ then $y \neq z$ so $1 \leq 0 + 1$
  ...

2. (INTERESTING) $d(x, y) = |x - y|$ for $\mathbb{R}$.
   $d(\frac{p}{q}, \frac{r}{s}) = |\frac{p}{q} - \frac{r}{s}|$ for $\mathbb{Q}$.

   Note: X is a metric space $Y \subset X$ then $\left(Y, d\Big|_{Y \times Y}\right)$ is a metric space.

Motivation: Stuff about $\mathbb{R}$ involving e.g., continuity and limits can be transferred to metric space.

Example 9.3
$\{x_n\}$ is a sequence in a metric space (X, d) (or X) has limit $x_0 \in X$ if for every $\epsilon > 0$, there is an N_ϵ s.t. $d(x, x_0) < \epsilon$ if $n \geq N_\epsilon$. (If $X = \mathbb{R}$, $d(x, y) = |x - y|$ same as before)

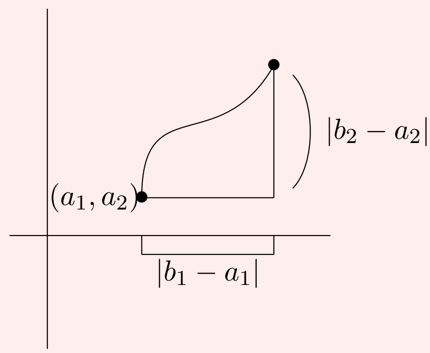
Example 9.4
Function: $f : (X, d_1) \to (Y, d_2)$. Continuity at $x_0 \in X$?
Real case: f cont at x_0 means given $\epsilon > 0$ $\exists \delta > 0$ s.t. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.
Metric space case: f cont at x_0 means given $\epsilon > 0 \exists \delta > 0$ s.t. $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \epsilon$.

More examples:

Example 9.5
$\mathbb{R}^2 = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$
$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}\}$
$\vdots$
$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}\}$

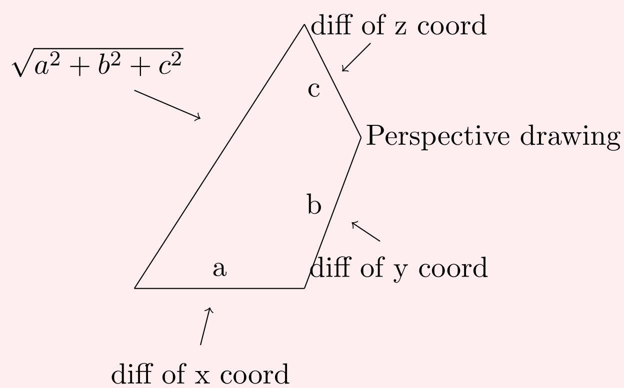
Interesting metric on $\mathbb{R}^2$ $d\left((a_1,a_2),(b_1,b_2)\right)$

$$d = \sqrt{(b_1-a_1)^2+(b_2-a_2)^2}$$



$\mathbb{R}^n(x_1,x_2,\dots,x_n),(y_1,\dots,y_n)$

$$d \coloneqq \sqrt{(y_1-x_1)^2+\ldots+(y_n-x_n)^2}$$



Is this function on $\mathbb{R}^n$ a metric?

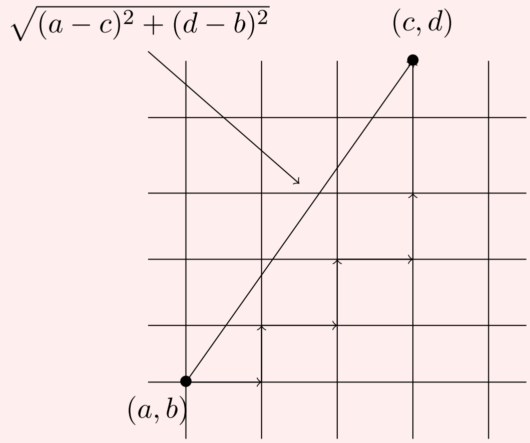
1. $d(x,y) \geq 0, = 0 \iff x = y$ where $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n)$ and
$$d(x,y) = \sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2}$$
2. $d(x,y) = d(y,x)$
3. BUT BUT BUT – $\triangle$ inequality is not so easy.
$$\begin{aligned}\sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2} \leq &\sqrt{(x_1-z_1)^2+\ldots+(x_n-z_n)^2}\\ &+\sqrt{(z_1-y_1)^2+\ldots+(z_n-y_n)^2}???\end{aligned}$$
Does $d(x,y) \leq d(x,z)+d(z,y)$ work?
YES but proof later :(

Realize that it's okay to assume $z = (0,0,\dots,0)$

Example 9.6

Try another metric

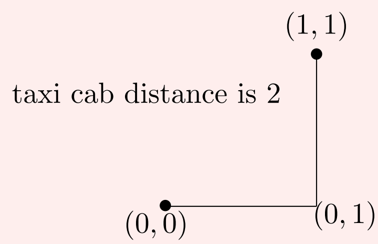
$\mathbb{R}^2$ – taxicab



$$|c-a|+|d-b| = d((a,b),(c,d))$$

min of length of taxi car

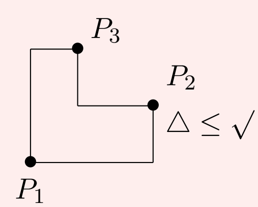
Easy to see that this d is really a metric. $\triangle$ inequality is easy!



Euclidean distance $=\sqrt{2}$

diff of x's $\leq$ Euc dis

diff of y's $\leq$ Euc dis

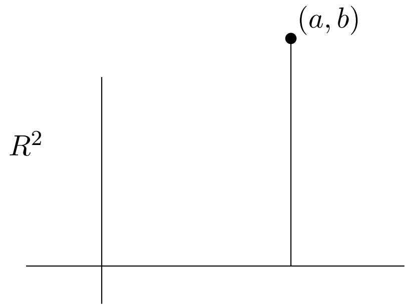


$$d(P_1,P_2)+d(P_2,P_3) \geq d(P_1,P_3)$$

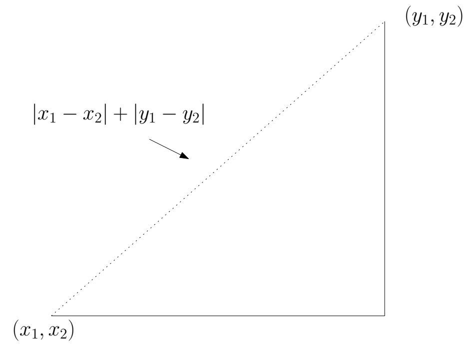
§10 Lec 10: Oct 23, 2020

§10.1 Metric on $\mathbb{R}^n$

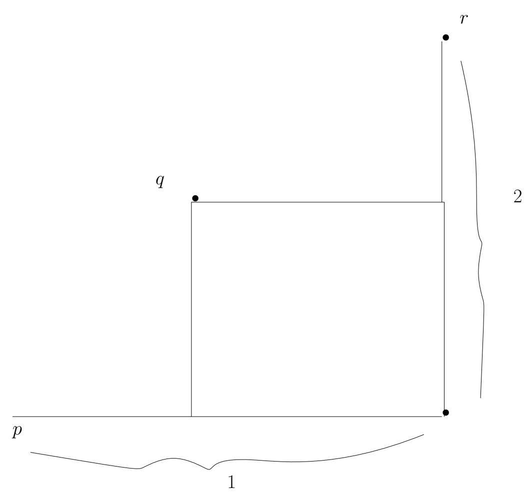
$\mathbb{R}^n : \{(x_1,\ldots,x_n) : x_j \in \mathbb{R}\}$



We want to make $\mathbb{R}^n$ a metric space. Last time, we defined "taxi cab metric", $d\left((x_1,\ldots,x_n),(y_1,\ldots,y_n)\right) = \sum_{i=1}^n |x_i - y_i|$ Verify $d(\vec{x},\vec{y}) \geq 0$ or $= 0$ if $\vec{x} = \vec{y}$ and $\triangle$ inequality,



$$d(p,q) + d(q,r) \geq d(p,r)$$



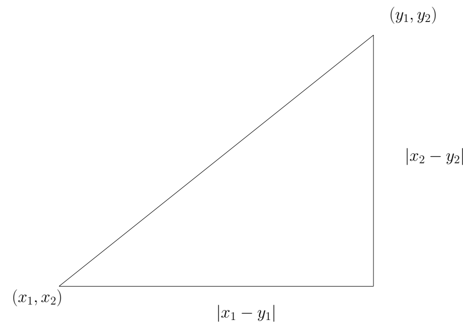
$d(p,r) = seg\,1 + seg\,2$

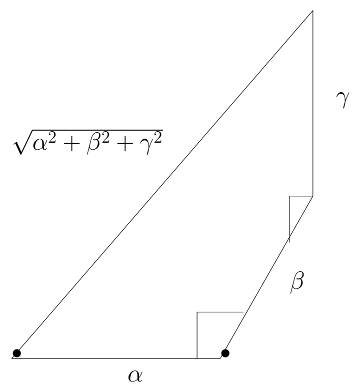
§10.2 Triangle Inequality in Euclidean Space

New idea: Euclidean distance (or Pythagorean distance)

$$d\left((x_1,x_2),(y_1,y_2)\right) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

For $\mathbb{R}^n : d\left((a_1,\ldots,a_n),(b_1,\ldots,b_n)\right) \coloneqq \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + \ldots + (a_n-b_n)^2}$.





We need to know:

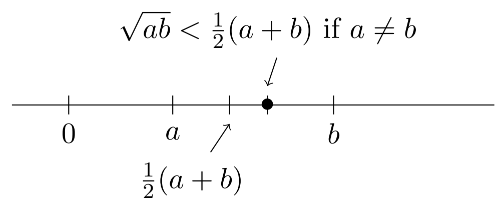
1. $d(\vec{a}, \vec{b}) \geq 0$
2. $d(\vec{a}, \vec{a}) = 0$ so $d(\vec{a}, \vec{b}) = 0 \implies \vec{a} = \vec{b}$
3. $d(\vec{a}, \vec{b}) = d(\vec{b}, \vec{a})$
4. ? $\triangle \leq 0$, $\vec{a}, \vec{b}, \vec{c}$
$$d(\vec{a}, \vec{c}) \leq d(\vec{a}, \vec{b}) + d(\vec{b}, \vec{c})$$
For $\mathbb{R}^n$,
$$\sqrt{(a_1-c_1)^2+\ldots+(a_n-c_n)^2} \leq \sqrt{(a_1-b_1)^2+\ldots+(a_n-b_n)^2}+\sqrt{(b_1-c_1)^2+\ldots+(b_n-c_n)^2}$$
We certainly need proof for $\triangle$ inequality: Copson($p > 1$) – for case $p = 2$

First step: $\alpha\beta \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2$ for all real α, β.
Reason:

$$\begin{aligned} 2\alpha\beta &\leq \alpha^2 + \beta^2 \\ \alpha^2 + \beta^2 - 2\alpha\beta &\geq 0 \\ (\alpha - \beta)^2 &\geq 0 \checkmark \end{aligned}$$

"Geometric mean $\leq$ Arithmetic mean"
Let $\alpha = \sqrt{a}, \beta = \sqrt{b}, a, b \geq 0$

$$\underbrace{\sqrt{ab}}_{\text{geometric mean of a,b}} \leq \frac{1}{2}(a) + \frac{1}{2}(b) = \underbrace{\frac{1}{2}(a+b)}_{\text{arithmetic mean}}$$



Second step:

$$\vec{a} = (a_1, \dots, a_n)$$
$$\vec{b} = (b_1, \dots, b_n)$$

and we know

$$a_i b_i \leq \frac{1}{2} a_i^2 + \frac{1}{2} b_i^2$$

Then,

$$\sum_{i=1}^n a_i b_i \leq \frac{1}{2} \sum_{i=1}^n a_i^2 + \frac{1}{2} \sum_{i=1}^n b_i^2$$

So, $\sum a_i^2 = 1$, $\sum b_i^2 = 1$, $\sum a_i b_i \leq 1$

Claim 10.1.

$$\sum a_i b_i \leq \left(\sum a_i^2\right)^{\frac{1}{2}} \left(\sum b_i^2\right)^{\frac{1}{2}}$$

But

$$\left|\vec{a} \cdot \vec{b}\right| \leq \|\vec{a}\| \|\vec{b}\|$$

So it's okay to define θ,

$$\cos\theta = \frac{\vec{a}\vec{b}}{\|\vec{a}\|\|\vec{b}\|} \in [-1, 1]$$

Verification of claim: $\vec{a}, \vec{b} \neq \vec{0}$

$$A_i = \frac{a_i}{\sqrt{\sum a_i^2}}, \quad B_i = \frac{b_i}{\sqrt{\sum b_i^2}}$$

And $\sum A_i^2 = 1$, $\sum B_i^2 = 1$ – also $\sum_{i=1}^n A_i B_i \leq 1$ which is equivalent to $\frac{\sum a_i b_i}{\sqrt{\sum a_i^2}\sqrt{\sum b_i^2}} \leq 1$.
So $|\sum a_i b_i| \leq \sqrt{\sum a_i^2}\sqrt{\sum b_i^2}$.
BIG DEAL: "Cauchy Schwarz inequality" What does this have to do with $\triangle$ inequality for Euclidean metric. Consider: $\vec{a}, \vec{b}$

$$\sum_{j=1}^n (a_j + b_j)^2 = \sum_{j=1}^n a_i(a_j + b_j) + \sum_{j=1}^n b_j(a_j + b_j)$$

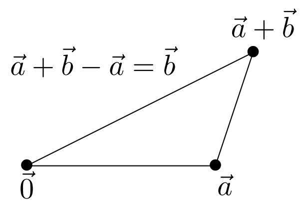
Now apply Cauchy – Schwarz

$$\sum_{j=1}^n (a_j + b_j)^2 \leq \left(\sum_{j=1}^n a_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^n (a_j + b_j)^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^n (b_j)^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^n (a_j + b_j)^2\right)^{\frac{1}{2}}$$

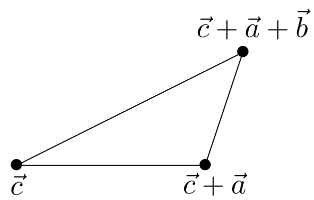
Divide through by $\left(\sum(a_j+b_j)^2\right)^{\frac{1}{2}}$

$$\left(\sum(a_j+b_j)^2\right)^{\frac{1}{2}} \le \left(\sum a_j^2\right)^{\frac{1}{2}} + \left(\sum b_j^2\right)^{\frac{1}{2}}$$

The above inequality is indeed the triangle inequality for $\vec{0}, \vec{a}, \vec{a}+\vec{b}$



But of course this gives you the triangle inequality in general.



△ inequality works in general

Last step: $\vec{p}, \vec{q}, \vec{r}$
Triangle inequality:

$$d(0, \vec{p}-\vec{r}) \le d(0, \vec{q}-\vec{p}) + d(\vec{q}-\vec{p}, \vec{r}-\vec{p})$$

Same as △ ineq for $0, \vec{q}-\vec{p}, (\vec{r}-\vec{q})+(\vec{q}-\vec{p})$ or
$0, \vec{a}, \vec{a}+\vec{b}$ if $\vec{a}=\vec{q}-\vec{p}, b=\vec{r}-\vec{q}$.

§11 Lec 11: Oct 26, 2020

§11.1 Metric Spaces Examples

Last time, we prove △ ineq. proof, taxi-cab metric, and sup norm metric. This gives rise to same "convergence idea". Namely $x_n \in X(X,d)$ converges to $L \in X$ means

$$\lim_{n\to\infty}(x_n - L) = 0$$

In all three metrics

$$\vec{x}_j \to L \qquad \lim \vec{x}_j = L$$

means (is same as) ith coordinate of $\vec{x}_j$ converges to ith coord of L for each $i = 1, 2, \ldots, n$.
$\{x_n\}$ Cauchy if given $\epsilon > 0 \exists N_\epsilon \ni n_1, n_2 \ge N_\epsilon$

$$d(x_{n_1}, x_{n_2}) < \epsilon$$

Exercise 11.1. $\{x_n\}$ Cauchy in $\mathbb{R}^n$ (any one of three metrics – Cauchy is the same idea in all three metrics) then $\{x_n\}$ has limit L, some L.

$$\sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2} \leq \sqrt{n}\max|x_j-y_j|, j=1,\ldots,n$$

which can be derived by the followings,

$$|x_j-y_j| \leq \max|x_j-y_j|$$
$$|x_j-y_j|^2 \leq \max{}^2|x_l-y_l|, l=1,\ldots,n$$
$$(x_1-y_1)^2+\ldots+(x_n-y_n)^2 \leq n\max{}^2|x_l-y_l|$$
$$\sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2} \leq \sqrt{n}\max|x_l-y_l|$$

$l_2 : \{x_j\}$ infinite sequences $j=1,2,3,\ldots$ where $\left\{\sum_{j=1}^{\infty} x_j^2 < \infty\right\}$ which means

$$\exists M \ni \sum_{j=1}^{M} x_j^2 \leq M$$

$$(1, \frac{1}{2}, \frac{1}{3}, \ldots) \in l_2$$
$$(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots) \notin l_2$$

because $1+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+\ldots \to \infty$ $(\frac{1}{n}) \to \infty$ as $n \to \infty$.
vector space:

$$c\{x_j\} = \{cx_j\}$$
$$\{x_j\} \in l_2 \implies \in l_2$$
$$\sum c^2 x_j^2 = c^2 \sum x_j^2$$

Also,

$$\{x_j\} + \{y_j\} = \{x_j+y_j\}$$
$$(x_j+y_j)^2 \leq 2(x_j^2+y_j^2)$$
$$x_j y_j \leq \frac{1}{2}(x_j^2+y_j^2)$$

$\{x_j\}, \{y_j\} \in l_2$ then

$$d(\{x_j\}, \{y_j\}) = \left[\sum (x_j-y_j)^2\right]^{\frac{1}{2}}$$

makes sense. (l_2, d) is a metric space obvious except $\triangle$ ineq. It's enough to check

$$d(0, \vec{x}) + d(\vec{x}, \vec{x}+\vec{y}) \geq d(0, \vec{x}+\vec{y})$$

which follows by taking limits of $\triangle$ ineq. for truncation up to level N.

$$d(\vec{0}, (x_1,\ldots,x_N)) + d((y_1,\ldots,y_N), (x+y) \text{ up to N}) \geq d(\vec{0}, (x+y)_N)$$

l_2 is metric space

l_2 is complete – Cauchy sequences have some limits.

Example 11.1

$C([0,1]) \coloneqq$ cont: $R --$ valued function $[0,1]$

$$\begin{aligned} d(f,g) &= \max|f(x)-g(x)| \\ &= \sup|f(x)-g(x)| \end{aligned}$$

"sup norm" All properties clear. "L^2 norm" – distance on $C[0,1]$:

$$d_2(f,g) = \left(\int_0^1 (f(x)-g(x))^2\right)^{\frac{1}{2}}$$

where $d_2 \geq 0,\ f,g,h \in C[0,1]$.
Imitate argument for $\triangle$ ineq. on $\mathbb{R}^n$: Cauchy Schwarz ineq.

$$\int_0^1 fg \leq \left(\int f^2\right)^{\frac{1}{2}} \left(\int g^2\right)^{\frac{1}{2}}$$

So,

$$\begin{aligned} f(x)g(x) &\leq \frac{1}{2}\left(f^2(x)+g^2(x)\right) \\ \int_0^1 f(x)g(x) &\leq \frac{1}{2}\int_0^1 f^2(x) + \frac{1}{2}\int_0^1 g^2(x) \end{aligned}$$

Apply these, $F = \frac{f(x)}{\sqrt{\int_0^1 f^2}}, G = \frac{g}{\sqrt{\int_0^1 g^2}}$, $\int F^2 = 1, \int G^2 = 1$. Also, we know $\int fg \leq 1$ if $\int f^2 = 1, \int g^2 = 1$.
Remainder argument for $\triangle$ ineq. is same as before

$$\int (f+g)^2 = \int f(f+g) + \int g(f+g)$$

Apply Cauchy – Schwartz,

$$\begin{aligned} \int (f+g)^2 &\leq \left(\int f^2\right)^{\frac{1}{2}} \left(\int (f+g)^2\right)^{\frac{1}{2}} + \left(\int g^2\right)^{\frac{1}{2}} \left(\int (f+g)^2\right)^{\frac{1}{2}} \\ \left(\int (f+g)^2\right)^{\frac{1}{2}} &\leq \left(\int f^2\right)^{\frac{1}{2}} + \left(\int g^2\right)^{\frac{1}{2}} \end{aligned}$$

§11.2 A Glance at Complex Number

Special case of $\mathbb{R}^n$, Euclidean norm

$$\begin{aligned} \mathbb{R}^2 : \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} &= d\left((x_1,x_2),(y_1,y_2)\right) \\ \mathbb{C} : \{(a+bi)\} - &\text{ Complex numbers} \end{aligned}$$

$(x_1, x_2) \leftrightarrow x_1 + ix_2$. Metric on $\mathbb{C}$, $z, w \in \mathbb{C}$

$$\begin{gathered} |z-w| = d(z,w) \quad \text{as pts in } \mathbb{R}^2 \\ z = a+bi \\ |z| = |a+bi| = \sqrt{a^2+b^2} \end{gathered}$$

We also define multiplication in $\mathbb{C}$ as follows

$$(a+bi)(c+di) \coloneqq (ac-bd) + (bc+ad)i$$

Example 11.2

$$\frac{1}{c+di} = \frac{c}{c^2+d^2} - \frac{d}{c^2+d^2}i$$

For $z = a+bi, w = c+di$ we define

$$\begin{aligned} |zw| &= |z||w| \\ &= \sqrt{a^2+b^2}\sqrt{c^2+d^2} \\ &= \sqrt{(ac-bd)^2+(bc+ad)^2} \end{aligned}$$

verify if the above step is actually equal

§12 Lec 12: Oct 28, 2020

§12.1 Midterm Announcement

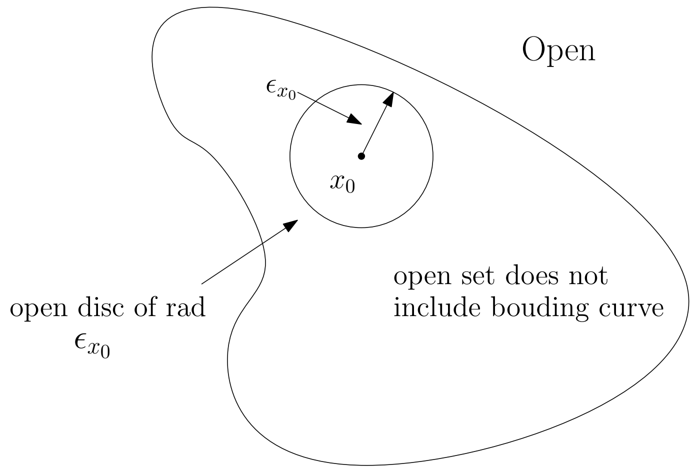
Midterm – Given out on Fri, Nov 6 at 3:00 pm. and due by Sat, Nov 7 at 11:00 pm.

§12.2 Open sets in Metric Space

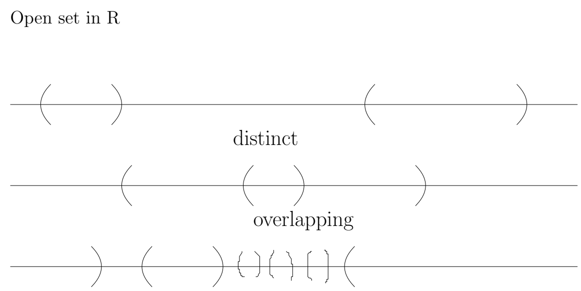
Beginning of "topology": (X, d) metric space

Definition 12.1 (Open sets) — $U \subset X$ open if for every $x_0 \in U$ there is an $\epsilon_{x_0} > 0$ s.t.

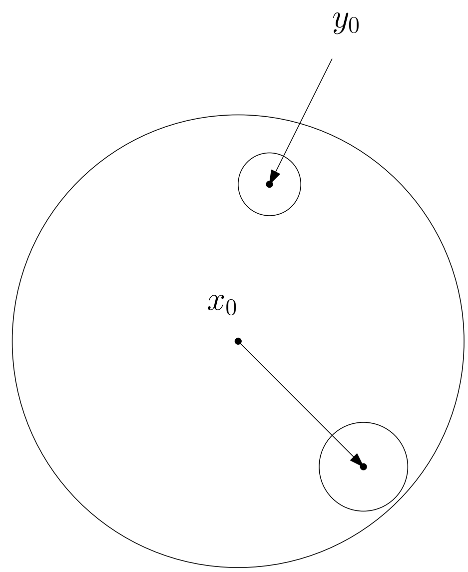
$$\underbrace{\{x \in X : d(x, x_0) < \epsilon_{x_0}\}}_{B(x_0, \epsilon_{x_0}) - \text{ open ball}} \subset U$$



open sets in R looks like unions of open intervals



> **Lemma 12.2**
> $B(x_0, \epsilon), \epsilon > 0$ open ball is open set.



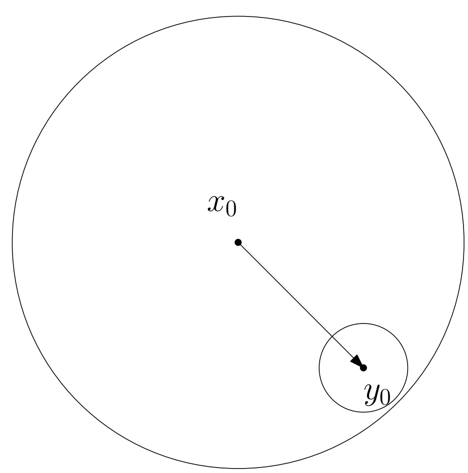
Proof. Need given $y \in B(x_0, \epsilon)$, $\lambda_y > 0$ s.t. $B(y, \lambda) \subset B(x_0, \epsilon)$.
Try $\lambda = \epsilon - d(x_0, y_0)$.
Suppose $y \in B(y_0, \epsilon) \iff d(y_0, y) < \epsilon - d(x_0, y_0)$

$$d(y_0, y) + d(x_0, y_0) < \epsilon$$

So,

$$d(x_0, y) \leq d(x_0, y_0) + d(y_0, y) < \epsilon$$

So $y \in B(x_0, \epsilon)$.



□

Reason why people care about open sets:
Remember: $f(X, d) \to (Y, d)$ continuous means given $\epsilon > 0, x_0 \in X$ there exists $\delta > 0$ s.t.

$$d_X(x_0, x) < \delta \implies d_Y\left(f(x_0), f(x)\right) < \epsilon$$

$\to$ Direct transcription of number – continuity of f can be described in terms of open sets in X and in Y. For this: $f : X \to Y$ and $V \subset Y$, then $f^{-1}(V) = \{x \in X : f(x) \in V\}$ f does not need to be invertible.

Example 12.3

$$f : \underbrace{X}_{\text{people}} \to \mathbb{Z}, \quad f(x) = \text{ integer age of x}$$

$$f^{-1}\left(\{20, 21, 22\}\right) = \text{everybody that's age 20,21, or 22}$$

Theorem 12.4 (Continuity – Open Sets)
$f : (X, d_x) \to (Y, d_y)$ is continuous if and only if (in δ, ϵ sense) $f^{-1}(V)$ is open in X for every V open in Y.
Slogan: continuity means inverses of open sets are open.

$f : X \to Y$, $g : Y \to Z \to g(f(x))$ compositions of f and g.

Claim 12.1. If f, g continuous then the composition is continuous

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Proof. (of Theorem) Suppose $f^{-1}(V)$ is open when V is open. Given $x_0 \in X, \epsilon > 0$ want $\delta > 0 \ni \underbrace{x \in B(x_0, \delta)}_{d(x,x_0)<\delta} \implies d\left(f(x), f(x_0)\right) < \epsilon$

$$\{y : d(y, f(x_0) < \epsilon\} = B(f(x_0), \epsilon)$$

Know that it's open by the above lemma. So,

$$f^{-1}(B(f(x_0), \epsilon)) \text{ open}$$

and $x_0 \in (B(f(x_0), \epsilon))$. So $f^{-1}(B(f(x_0), \epsilon))$ being open

$$\implies \delta > 0 \quad B(x_0, \delta) \subset f^{-1}\left(B(f(x_0), \epsilon)\right)$$

says $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \epsilon$ ✓
Took care of f^{-1}(open) is open $\implies$ continuity. Now,
Does continuity(ϵ, δ sense) $\implies f^{-1}$ (open) is open?
This also works: Suppose V open, and $x_0 \in f^{-1}(V)$. Need $\delta > 0$ s.t. $B(x_0, \delta) \subset f^{-1}(V)$. $f(x_0) \in V$ (meaning of $x_0 \in f^{-1}(V)$) $\exists \epsilon$ s.t. $B(f(x_0), \epsilon) < V$ (V is open). Then ϵ, δ defn of continuity $\exists \delta$ s.t. $f(B(x_0, \delta)) \subset B(f(x_0), \epsilon) \subset V$. So $B(x_0, \delta) \subset f^{-1}(V)$. ✓ □

Forward continuous images of open sets are not necessarily open.

Example 12.5
$f(x) = x^2, \quad f\left((-1, 1)\right) = [0, 1)$ which is not open.

<u>*Note*</u>: A notion to help understand the concept of open sets is thinking about how a map sends a point to a point but its inverse can send a point to a set.

§13 Lec 13: Oct 30, 2020

§13.1 Open Sets (Cont'd)

Recall: U open means $\forall x \in U$, $\exists \epsilon > 0$ s.t. $B(x, \epsilon) \subset U - \{y : d(x, y) < \epsilon\}$ (open ball)
$f : X \to Y$, $f^{-1}(V)$ open in X if V open in Y $\iff$ f continuous – δ, ϵ sense (p.91, Copson)
Properties of being open: (finiteness is important)

0. ϕ, X open sets – "trivial"

1. $U_\lambda, \lambda \in \Lambda$, open for each λ, $\bigcup_{\lambda\in\Lambda} U_\lambda$ is open.

2. $U_1, \ldots, U_n$ open then
$$\bigcap_{j=1}^{n} U_j \text{ open}$$
U open does not imply $X - U$ is open (not necessarily true).

3. $U_1, U_2, U_3, \ldots$ open
$$\bigcup_{j=1}^{\infty} U_j \text{ open}$$

Example 13.1
$U_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \subset \mathbb{R}$
$$\bigcap_{n=1}^{\infty} U_n = \{0\} \text{ one point}$$
which is not open.

$U_\lambda, \lambda \in \Lambda$ open (assume). We want $\bigcup U_\lambda$ is open.

Proof. Suppose $x \in \bigcup_{\lambda\in\Lambda} U_\lambda \implies x \in U_{\lambda_1}$ open. So $\exists \epsilon > 0 \ni B(x, \epsilon) \subset U_{\lambda_1}$
$$\implies B(x, \epsilon) \subset \bigcup_{\lambda\in\Lambda} U_\lambda$$
□

$u_1, \ldots, u_n$ open (finitely many $U's$). If $x \in \bigcap_{j=1}^n U_j, x \in U_j$ for each $j = 1, \ldots, n$. So for $\epsilon_j > 0$
$$B(x, \epsilon_j) \subset U_j \quad (U_j \text{ open})$$
Let $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n) > 0$. Then $B(x, \epsilon) \subset B(x, \epsilon_j) \subset U_j$. So $B(x, \epsilon) \subset U_j$ for all j. So $B(x, \epsilon) \subset \bigcap_{j=1}^n U_j$. Therefore, $\bigcap_{j=1}^n U_j$ is open. Contrast this with $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ example.

§13.2 Topological Space

Set S with some sets specified as open with

0. ϕ, X open.
1. $\cup$ open is open.
2. $\cap$ open is open.

This is a Topological Space.
We know (X, d) with our definition of $U \subset X$ open is a topological space.

§13.3 Closed Sets

Back to metric space (but also works in topological spaces)

Definition 13.2 (Closed Sets) — $C \subset X$ is closed if and only if $X - C$ is open.

Note: Being closed does not necessarily mean the opposite of open. For example, X is both closed and open – X open and $X - X = \emptyset$ open. Also, $\emptyset$ both closed and open – $\emptyset$ open & $X - \emptyset = X$ is open.
Closed sets:

0. ϕ, X closed (checked already)
1. $C_\lambda, \lambda \in \Lambda$ closed then $\bigcap_{\lambda\in\Lambda} C_\lambda$ is closed
2. $C_1, \ldots, C_n$ are closed then

$$\bigcup C_j = C_1 \cup \ldots \cup C_n \text{ is closed}$$

watch out for $\left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$ $X = \mathbb{R}$, $\mathbb{R} - [-1+\frac{1}{n}, 1-\frac{1}{n}]$ which is equivalent to $(-\infty, -1+\frac{1}{n}) \cup (1-\frac{1}{n}, +\infty)$. On the other hand,

$$\bigcup_{n=1}^{\infty} \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right] = (-1, 1) \text{ not closed}$$

Proof. (1) – $\bigcap_{\lambda\in\Lambda} C_\lambda$ is it closed? Closed means $X - \bigcap_{\lambda\in\Lambda}$ open – True? According to August de Morgan

$$X - \left(\bigcap_{\lambda\in\Lambda} C_\lambda\right) = \bigcup_{\lambda\in\Lambda} (X - C_\lambda)$$

□

A notion to understand this is people - (dog owners $\cap$ cat owners) = people who do not own both a dog and cat = (people who do not own a dog) $\cup$ (people who do not own a cat) = (people - dog owners) $\cup$ (people - cat owners).

Slogan: Complements of intersections is the union of complements. Or complements of unions is the intersection of complements – De Morgan's Laws.
Now, back to the closed sets, we have $X - \cap C_\lambda$ where C_λ closed then $= \cup(X - C_\lambda)$ open because C_λ are closed. So $\cup(X - C_\lambda)$ open (by prop(1) for open sets). So $\cap C_\lambda$ closed if each C_λ is closed.
Prop (2) for closed sets

$$C_1 \cup \ldots C_n$$

is closed if each $C_j j$ is closed. We need openness of $X-$ union:

$$X - (C_1 \cup \ldots \cup C_n) = \bigcap_{j=1}^{n} (X - C_j)$$

which is open by C_j being closed for each j and also is the finite intersection of open sets. So it's open by prop (2) of open sets. So $C_1 \cup \ldots \cup C_n$ closed (its complement is open).
Note: Continuity can be defined for functions from (S, Q_S) to (T, Q_T) : $f : S \to T$ continuous by definition if $f^{-1}(V) \forall V \subset T$ open is open in S.

§14 Lec 14: Nov 2, 2020

§14.1 Set, Tables, & Characteristics Functions

$A \subset X$, X_A is called characteristics function where

$$\begin{aligned} X_A &: X \to \{0,1\} \\ X_A(x) &= 1 \text{ if } x \in A \\ X_A(x) &= 0 \text{ if } x \notin A \\ A &= \{x : X_A(x) = 1\} \\ X_{X-A}(x) &= 1 - X_A(x) \end{aligned}$$

X_A	X_B	$X_{A\cup B}$	$X_{A\cap B}$	X_{X-A}	X_{X-B}
0	0	0	0	1	1
1	0	1	0	0	1
0	1	1	0	1	0
1	1	1	1	0	0

$X_{(X-A)\cap(X-B)}$		$X_{X-(A\cup B)}$
1		1
0	$\xleftrightarrow{\text{same}}$	0
0		0
0		0

De Morgan's Law:

$$X_{(X-A)\cap(X-B)} = X_{X-(A\cup B)}$$
$$\iff (X-A)\cap(X-B) = X-(A\cup B)$$

Exercise 14.1. $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

Reason: So, $(A\cap B)\cup(A\cap C) = A\cap(B\cup C)$.

A	B	C	$A\cap(B\cup C)$	$(A\cap B)\cup(A\cap C)$
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

$$\begin{array}{lll} A_1 & 0{,}1 & \overset{1}{x\in A_1}\ \overset{0}{x\notin A} \\ A_2 & 0{,}1 & \overset{1}{x\in A_2}\ \overset{0}{x\notin A_2} \\ A_3 & & \\ A_4 & & \end{array}$$

A_1	A_2	A_3	A_4
0,1	0,1	0,1	0,1

$X_{\liminf\{A_n\}} = 1$ if only 1's some n onward.
$X_{\limsup\{A_n\}} = 1$ for x such that table for x contains infinitely many 1s. – A way to do homework.

§14.2 Closed Sets in Metric Spaces

$C\subset X$, (X,d) metric space. It is closed if $X-C$ is open.
De Morgans' Laws: $\cap C_\lambda$ closed if C_λ are closed – $C_1\cup\ldots\cup C_n$ closed if $C_1,\ldots,C_n$ are closed.

Corollary 14.1

There is a minimal closed (closure) of a set containing a given A

$$A^- = \cap C$$

C closed, $A\subset C$ closed.

We can describe the closure of A in terms of limits of sequences. A point x is a limit point of A (Copson: adherent point) if

$$\exists\{a_n\}\in A \text{ s.t. } \{a_n\} \text{ converges and } \lim a_n \text{ is the point } x$$

If $x\in A$ then x is a limit point:
$x =$ limit of sequence, $a_n = x$, for all $n = 1,2,3,\ldots$.

- Set of limit points $\supset A$.
- Set of limit points is a closed set.

In order to understand that, we have to understand the characterization of a set being closed in terms of convergence of sequence:

A set A is closed $\iff$ every limit point of A is in A.

Proof. (of characterization) ($\rightarrow$) closed $\implies$ contains limit points
$\lim a_n = a_0$ want to know that a_0 must be in A. Suppose not: Then $X - A$ is open $\exists \epsilon > 0 B(a_0, \epsilon) \subset X - A$ which is impossible $\lim a_n = a_0$.
($\leftarrow$) A contains all limit points $\implies$ A closed.
Suppose $X - A$ is not open and $\exists$ some $a_0 \in X - A$ s.t. $B(a_0, \epsilon) \not\subset X - A$ for every $\epsilon > 0$. For $\epsilon = \frac{1}{n}, n = 1, 2, 3, \ldots$, $\exists x_n \in B(a_0, \frac{1}{n})$ with $x_n \in X - A$ so $x_n \in A$.

$$d(a_0, x_n) < \frac{1}{n}$$

$x_n \in A$, $\lim x_n = a_0$ where x_n is a sequence in A but $\lim \notin AX$. So $X - A$ is open. $\square$

think carefully through this proof

Back to set of limit points of A is always closed:

$$\lim x_n = x_0$$

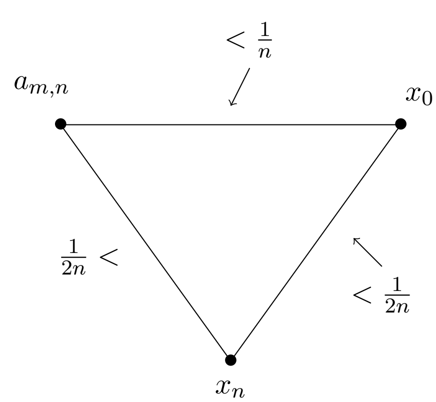
$\underbrace{\{x_n\}}_{\text{each is a limit point of A}}$. Hope x_0 is a limit point of A. To be a limit point

$$x_n = \lim_{n \to \infty} a_{m,n}$$

Passing to a subsequence, we can suppose for each n, choose $d(x_n, x_0) < \frac{1}{2n}$.
Watch out! To get from $d(x_n, x_0) \to 0$ that $d(x_n, x_0) < \frac{1}{2n}$, we need to pass to a subsequence!
For each n, there is an $x_{N(n)}$ with $d(x_N, x_0) < \frac{1}{2n}$. Relabel that as x_n, i.e., (new) $x_n =$ (old) $x_{N(n)}$. So x_0 an A-limit implies x_0 is a limit of sequence $\{x_n\}, x_n \in A$ with $d(x_n, x_0) < \frac{1}{2n}$.
Choose $a_{m,n}$ such that $d(x_n, a_{m,n}) < \frac{1}{2n}$. Consider the sequence $\{a_{m,n}\}, n = 1, 2, 3, \ldots$

$$\begin{aligned} d(x_0, a_{m,n}) &\leq d(x_0, x_n) + d(x_n, a_{m,n}) \\ &< \frac{1}{2n} + \frac{1}{2n} < \frac{1}{n} \end{aligned}$$

So x_0 is a limit of seq of points in a.



subsequence of orginal $\{x_n\}$
limit point of an A - sequence

A set of limit points is closed. C closed $\supset$ A. Then limit points of A in C. $C \supset$ set of limit points of A. So set points is a closed set $\supset A$ and every closed set that contains A contains set of limit points. So $A^- =$ set of limits of A.

Example 14.2

$\mathbb{Q}^- = \mathbb{R}$
$\sqrt{2}$ is a limit point of $\mathbb{Q}$. Every real number is a limit of sequence of rationals – " $\mathbb{Q}$ is dense in $\mathbb{R}$".

§15 Lec 15: Nov 4, 2020

§15.1 More on Open and Closed Sets

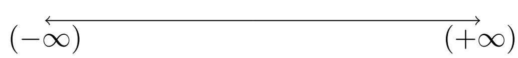
$x_n \to x_0$ where x_n is limit of a seq in A then $x_0 = A$ limit $\left[d(x_n, x_0) < \frac{1}{2n}\right]$.
Alternative view:
$x_n \to x_0$, $\lim d(x_n, x_0) = 0$. For each n, $\exists a_n \in A$ s.t. $d(a_n, x_n) < \frac{1}{n} \leftarrow$ because x_n is limit pt. So $\exists$ a seq in A converging to x_n. Then

$$\lim d(a_n, x_0) = 0$$

because $d(a_n, x_0) \le d(a_n, x_n) + d(x_n, x_0)$ as $n \to \infty$.
Closure of $A =$ set of sequence limits of sequences in A. Closed sets can be complicated. Open sets (at least in $\mathbb{R}$) seem simple. Open sets in $\mathbb{R}$:

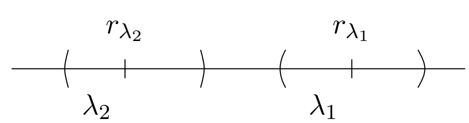
$$U \text{ open in } \mathbb{R} \iff U \text{ possibly infinite pairwise disjoint collection of } (a,b) \text{ or } (-\infty, a) \text{ or } (a, +\infty) \text{ or } \mathbb{R}$$

maximal open interval $\subset U$

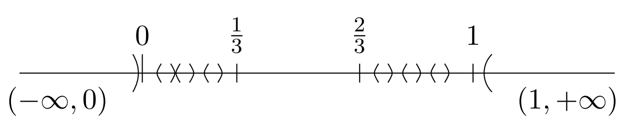


U = pairwise disjoint union of these

Number of intervals in U is countable (each contains a rational number). U_λ max open intervals $\subset U$ fixed. Pick λ_1 rational in $U_\lambda, U_{\lambda_1} \ne U_{\lambda_2}$ then $r_{\lambda_1} \ne r_{\lambda_2}$.



So rational numbers $\implies \{U_\lambda\}'s$ are countable (Λ is countable) if each max interval has one λ only.
Cantor set(closed set):



complement of $C = (+1, \infty), (-\infty, 0), (\frac{1}{3}, \frac{2}{3}), (\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9})$
At each stage, we remove open middle third of closed intervals that are left from previous stage. C is closed – complement is a union of open intervals hence open. C is not empty – $0 \in C, 1 \in C, \frac{1}{3} \in C, \frac{2}{3} \in C$. All the endpoint of $[0,1]$ and the removed open intervals $\subset C$.

C inifite: C contains some points that not endpoints.
Reason: set of endpoints is countable (can make a list of them) – (countable union of a finite sets). But C itself is uncountable. Why? We prove later a generalization of this!

$$x \in C \iff \text{a sequence } \{X_n\} \text{ in } x_n \in \{L, R\}$$

x_1 L if $x_1 \in$ down side of $[0,1) - \left(\frac{1}{3}, \frac{2}{3}\right)$, $x_1 \in$ upside of $[0,1) - \left(\frac{1}{3}, \frac{2}{3}\right)$
x_2 L if in downside – R if in upside.
Knowing x depends on a single LR valued sequence associated to unique $x \in C$ one and only $x \in C$ with that LR sequence being sequence for x. $LRL\ldots$ determined a sequence of closed intervals of successive length $\frac{1}{3^n}, n = 1, 2 \ldots$. Each is contained in previous ones "nested intervals".
Proofs earlier:

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \ldots$$

(nested intervals) of length $[a_n, b_n] \to 0$ as $n \to \infty$. $\exists$ one and one point in

$$\bigcap_{n=1}^{\infty} [a_n, b_n]$$

$$x \in C \to L, R \text{ seq}$$

L,R seq comes from exactly one $x \in C$. So to know C is uncountable just to have to know set of all L,R sequences is uncountable.

Proof. $\{L, R \text{ sequences}\}$ is countable.

1. L,R sequences no 1.
2. L,R seq no 2.

   $\vdots$

$\exists L, R$ sequences not in list: first element is L if first element here is R, if first element is L. 2nd: L if second element is R. R is second element of is L. New sequence is not in the list. Think about this – similar to subdivision argument to prove the accountability of $[0,1]$, powersets. □

Baire Category Theorem: later!
Sierpinski Carpet: (Check Wikipedia)

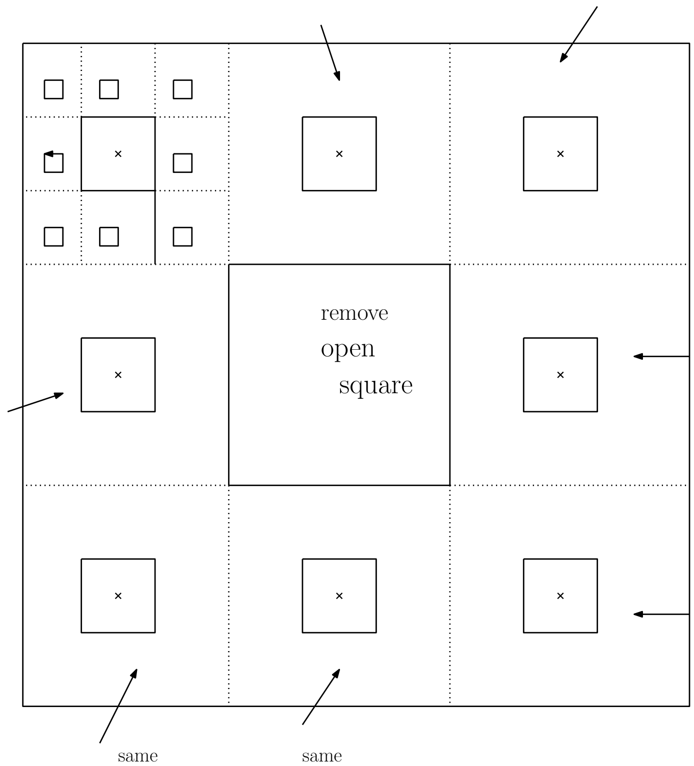


Figure 1: Sierpinski Carpet

interior (also interior of C) $= \emptyset$.
Closed uncountable set with interior $= \emptyset$.

§16 Midterm 1: Nov 6, 2020

Review for midterm 1, and it's distributed at 3:00 pm :D

§17 Lec 16: Nov 9, 2020

§17.1 Completeness (Cont'd)

Recall: (X, d) is complete means, by definition, Cauchy sequences have limits (in X).
Cauchy sequence – Given $\epsilon > 0, \exists N_\epsilon \ni n_1, n_2 \geq N_\epsilon \implies d(x_{n_1}, x_{n_2}) < \epsilon)$.
$\{x_n\} \to x_m$ means given $\epsilon > 0, \exists N_\epsilon \ni d(x_n, x_m) < \epsilon$ if $n \geq N_\epsilon$.
$\mathbb{R}$ is complete but $\mathbb{Q}$ is not.
Note: (X, d) is complete or not $\leftarrow$ depends on d as well as X. Any discrete metric space is, Cauchy $\iff$ eventually constant, complete. ✓

Example 17.1

$C([0,1])$ is complete in $d(f,g) = \sup(f(x), g(x))$ (sup norm) but not complete in $l^2, d(f,g) = \left(|f(x) - g(x)|^2\right)^{\frac{1}{2}}$. We will look at this later.

(X,d) metric space, $Y \subset X$, then $d\Big|_{Y\times Y}$ is a metric on Y – $(Y, d\Big|_{Y\times Y}$, Y is a subspace of X.

$$\underbrace{d_Y(y_1,y_2)}_{y_1,y_2\in Y} = \underbrace{d_X(y_1,y_2)}_{y_1,y_2\in X}$$

Lemma 17.2

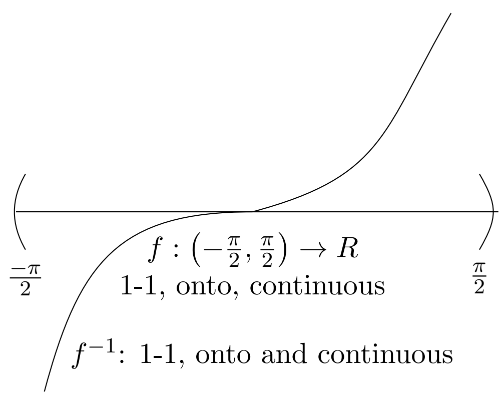
If (X,d) is complete, then $\left(Y, d\Big|_{Y\times Y}\right)$ is complete if and only if Y is closed in (X,d).

Proof. Left as exercise. $\square$

Example 17.3

$\mathbb{R}$ is complete but not $[0,1) \subset \mathbb{R}$.

Completeness is not a topological property not determined by knowing which sets are open.



f preserves open sets – homomorphism – 1-1 and onto mapping s.t. open sets are preserved. Define a new metric on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ has usual metric $d(\alpha,\beta) - |\alpha - \beta|$ in which it is not complete.

$$d(\alpha,\beta) = d\left(f(\alpha), f(\beta)\right)$$
$$(|\alpha - \beta - \text{old}|) \overset{\text{new}}{=} d(\alpha,\beta) = |\tan\beta - \tan\alpha|$$

This new metric on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is complete.
f "isometric" by definition, distances are preserved (by f). means $(()$, new metric) and $(\mathbb{R}$, usual metric), $()$ new metric has same open sets $()$, old metric not complete. $\mathbb{R}$ usual metric is complete.
(*) Completeness is a metric property, not topological property.

Example 17.4

(Complete)

1. Discrete metric space
2. $\mathbb{R}$
3. $\mathbb{R}^n$ in any of the three metric
   - $d(\vec{x},\vec{y}) = \sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2}$
   - sup metric $d(\vec{x},\vec{y}) = \max|x_i - y_1|$
   - taxicab metric $d(\vec{x},\vec{y}) = \sum_{i=1}^n |x_i - y_i|$

   All are complete metrics.

Proof. $d(\vec{x}_{n_1}, \vec{x}_{n_2} < \epsilon \implies$

$$|j \text{ th coord of } \vec{x}_{n_1} - j \text{ th coord of } \vec{x}_{n_2}| < \epsilon, j = 1,\ldots,n$$

Cauchy seq $\{\vec{x}_j\}$. Cauchy seq in $\mathbb{R}^n$, $l = 1,\ldots,n - l$ th coord of $\{\vec{x}_j\} \leftarrow$ numbers, is a Cauchy sequence. So it has a limit $\vec{L} \in \mathbb{R}^n, L = (L_1,\ldots,L_n)$ is the limit of $\{\vec{x}_n\}$ □

"Component-wise convergent (or Cauchyness) $\iff$ convergence or Cauchyness of vector sequences.
Issue: What l_2 infinite sequence? – Yes, Complete! Completeness of l_2 : next time.

§18 Lec 17: Nov 13, 2020

§18.1 Completeness of l_2 and $C([0,1])$ in sup norm

Recall $l_2 : \{x_n\} \implies \sum_{j=1}^\infty x_j^2 < \infty, \exists M$. This means, $\exists M$ s.t. $\sum_{j=1}^{n_0} x_j^2 \le M$ for $n_0 = 1,2,3,\ldots$.
Also, note that vector space $\{x_n + y_n\} \in l_2$ if $\{x_n\}, \{y_n\} \in l_2$

$$d(\{x_n\},\{y_n\}) := \left(\sum_{n=1}^\infty (x_n - y_n)^2\right)^{\frac{1}{2}}$$

$\triangle$ inequality : left as exercise.

Question 18.1. Is it complete?

Ans: Yes! Let us prove it

Proof. Cauchy sequence $\{x_n\}_N$, $N = 1,2,\ldots$ number of sequences. Note

$$x_{n,N} = \text{nth component of the Nth sequence}$$

$$\begin{array}{cccccc}
\text{1st} & x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} & \cdots \\
\text{2nd} & x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} & \cdots \\
\vdots & \downarrow & \downarrow & \downarrow & \downarrow & \\
\hline
\downarrow\ \vec{L} & L_1 & L_2 & L_3 & L_4 &
\end{array}$$

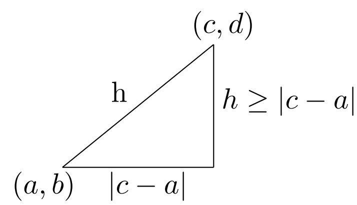
limit if $\{x_n\}_N$ is Cauchy

Find $\vec{L}$ given $\{x_n\}_N, N = 1, 2, 3, \ldots$ Cauchy sequence in l_2. Candidate for L : Try $L_n = \lim_{N\to\infty} x_{n,N}$. Does limit exist?

Yes: $x_{n,N}$ for $N = 1, 2, 3, \ldots$, n fixed is Cauchy sequence in $\mathbb{R}$. Cauchyness of $\{x_n\}_N, N = 1, 2, \ldots$ Given $\epsilon > 0, N_\epsilon \ni$

$$\begin{aligned}\left(\sum_{n=1}^{\infty} (x_{n,N_1} - x_{n,N_2})^2\right)^{\frac{1}{2}} &< \epsilon \\ &\geq |x_{n,N_1} - x_{n,N_2}| \checkmark\end{aligned}$$

for all $N_1, N_2 \geq N_\epsilon$.



$$h = \sqrt{(c-a)^2 + (d-b)^2} \geq \sqrt{(c-a)^2} = |c-a|$$

So $L_n = \lim_{N\to\infty}(x_{n,N})$ exists. $\{x_{n,N}\}, N = 1, 2, 3, \ldots$ n fixed Cauchy sequence. $\vec{L} =$ candidate for $\lim \{x_{n,N}\} = \{x_n\}_N, N = 1, 2, 3, \ldots$

Exercise 18.1. $\vec{L}$ has to be l_2-limit if there is a limit in l_2.

Component-wise convergence (for fixed n – number of component, $x_{n,N} \to L_n$) does not imply (in general) l_2 convergence $(1, 0, 0, \ldots), (0, 1, 0, 0, \ldots), (0, 0, 1, 0, \ldots) \ldots$ converges component-wise to $(0, 0, \ldots, 0)$ but does not l_2 converge to $(0, 0, \ldots, 0, \ldots)$ even though everything is in l_2. $\mathbb{R}^n, l_2 = \mathbb{R}^\infty$ – saved by the fact that $\{x_n\}_N$ is a Cauchy seq in l_2. Here is a basic observation: if $\|\{x_{n,N}\}\| \leq M$ and if $\vec{L} =$ component-wise limit of $\{x_{n,N}\}$ (includes limit L exists) then $\vec{L}$ is in l_2 and

$$\|\vec{L}\| \leq M$$

□

Proof. If $L_1^2 + \ldots + L_n^2 \leq M^2$ all n, then $\sum_{n=1}^{\infty} L_n^2 < \infty$ and indeed $\leq M^2$

$$L_1^2 + \ldots + L_n^2 = \lim \left(x_{1,N}^2 + \ldots + x_{n,N}^2\right)$$

where $\left(x_{1,N}^2 + \ldots + x_{n,N}^2\right) \leq \|(x_n)_N\|^2 \leq M^2$ and $\sum_{j=1}^{\infty} L_j^2 \leq M^2$. So, $\vec{L} \in l_2$ and $\|\vec{L}\| \leq M$ ✓

Second item: $\{x_{n,N}, N = 1, 2, 3, \ldots\}$ is a Cauchy sequence then $\exists \vec{L} = (L_1, L_2, \ldots)$ component-wise limit of $\{x_{n,N}\}$ ✓ □

Proof. Proper of completeness of l_2. Want: given $\epsilon > 0, \exists N_\epsilon$ s.t. $N_1 \geq N_\epsilon \implies \| \{x_n\}_{N_1} - \vec{L}\| < \epsilon$, $\vec{L} = l_2$ limit of $\{x_n\}_N$ (know $\vec{L}$ is in l_2).
Know: (from defn of Cauchy seq), $\exists N_0$ s.t. $N_1, N_2 \geq N_\epsilon$

$$\| \{x_n\}_{N_1} - \{x_n\}_{N_2} \| < \frac{\epsilon}{2}$$

Now fix $N_1 = A \geq N_\epsilon$

$$\| \{x_n\}_A - \{x_n\}_{N_2} \| < \frac{\epsilon}{2}$$

Now let $N_2 \to +\infty$ (A fixed), $\{x_n\}_A - \{x_n\}_{N_2}$ converges (as $N_2 \to \infty$) to $\{x_n\}_A - \vec{L}$ component-wise. So by lemma:

$$\| \{x_n\}_A - \vec{L}\| \leq \frac{\epsilon}{2} < \epsilon$$

True for all $A \geq N_0$. Done! $\{x_n\}_N$ converges to $\vec{L}$ in l_2 norm. $\square$

§19 Veterans Day: Nov 11, 2020

No class :D

§20 Lec 18: Nov 16, 2020

§20.1 Lec 17 (Cont'd)

$C([0,1])$ is complete. Uniform convergence: $f_n \to f_0$, $\{f_n\}$ converges uniformly to a function f_0 if given $\epsilon > 0, \exists N_\epsilon$ s.t.

$$n \geq N_\epsilon \implies |f_n(x) - f_0(x)| < \epsilon$$

for all $x \in X([0,1])$.
Uniform convergence associated to uniform Cauchyness: $\{f_n\}$ is Cauchy if given $\epsilon > 0, \exists N_\epsilon$ such that $n_1, n_2 \geq N_\epsilon \implies$

$$|f_{n_1}(x) - f_{n_2}(x)| < \epsilon$$

for all x.

Lemma 20.1

Unif Cauchy implies $\exists f_0$ such that $\{f_n\}$ converges uniformly to f_0. (not at first known to be $C([0,1])$ actually is)

Proof. Note that for each $x, \{f_n(x)\}$ is a Cauchy sequence. Candidate for $f_0, f_0(x) = \lim f_n(x)$. Need this things:

1. $f_n \to f_0$ uniformly

2. Need (for completeness) $f_0 \in C([0,1])$

For (1): Given $\epsilon > 0, \exists N_0 \ni n_1, n_2 \geq N_0 \implies$

$$|f_{n_1}(x) - f_{n_2}(x)| < \frac{\epsilon}{2}$$

for all x. Let $n_2 \to \infty$

$$|f_{n_1}(x) - f_{n_2}(x)| \to |f_{n_1}(x) - f_0(x)|$$

as $n_2 \to \infty$. So

$$|f_{n_1}(x) - f_0(x)| \leq \frac{\epsilon}{2} < \epsilon$$

for all x and all $n_1 \geq N_\epsilon$.
2) f_0 want f_0 to be continuous. Trick three term estimate:
Given $\epsilon > 0$, choose $N_\epsilon \ni |f_n(x) - f_0(x)| < \frac{\epsilon}{3}$ if $n > N_\epsilon$ (true for all x, one choice of N_ϵ)

$$|f_0(y) - f_0(x)| \leq |f_0(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f_0(x)|$$

We can choose $n \geq N_\epsilon$ fix n ($n = N_\epsilon$ ok)

$$|f_0(y) - f_n(y)| < \frac{\epsilon}{3}$$
$$|f_n(x) - f_0(x)| < \frac{\epsilon}{3}$$

Also, we can choose (u,x fixed) $\delta > 0$ s.t. $|f_n(y) - f_n(x)| < \frac{\epsilon}{3}$ if $|x - y| < \delta$
So $|f_0(y) - f_0(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ if $|y - x| < \delta$ So f_0 is continuous at x (all $x \in [0,1]$). So

$$f_0 \in C([0,1])$$

and $\{f_n\} \to f_0$ in $C([0,1])$ with sup norm metric. So $C([0,1])$ complete (in sup norm metric). □

Everything will work (no boundedness problem) if X is sequentially compact. $f : [0,1] \to \mathbb{R}$, f, g bounded because $[0,1]$ seq compact. $C([0,1])$ complete in any sup norm.
(X, d) sequentially compact if every sequence in X $\{x_n\}$ has a subsequence which converges to a point $x_0 \in X$. Special case: $[a,b] \subset \mathbb{R}$ usual metric, Bolzano-Weierstrass – $\exists$ a convergence subsequence in $[a,b]$, $[a,b]$ closed in $\mathbb{R}$.

Proof. C seq. compact. $\{x_n\} \in C$, BW $\implies x_{n_j} \to x_0 \in \mathbb{R}$. But $x_0 \in C$ because C is closed. Example: Cantor set is sequentially compact. Closed,bounded subset C of $\mathbb{R}^n$ (n finite) then C is sequentially compact. Bounded means, $x_0 \in C$

$$\underbrace{d(x_0, \vec{x})}_{\vec{x} \in C} \leq M \text{ some } M$$

$d(\vec{0}, \vec{x}) \leq M$ by triangle (exercise).
$\{x_n\}$ bdd $\exists$ subsequence s.t. 1st component converges. There exists subsequence of that subsequence such that 2nd component (and 1st one still) converges ... p times subs $\{\vec{x}_n\}$ where all p_n components converge. □

C seq. compact in $\mathbb{R}^p$ then C is closed and bounded. Boundedness: if it were unbounded, each n $\exists x_n \in C$ which

$$d\left(\vec{0}, \vec{x}_n\right) \geq n$$

where $\{x_n\}$ no convergent sub.
Closed: $x_n \in C$ with limit $= \vec{x}_0$ then seq. compact means $\exists \vec{y}_0$ subsequence limit in C. $\vec{y}_0 = \vec{x}_0$ but a sub limit = seq. limit if seq. limit exists.

Theorem 20.2 (Heine-Borel)
$C \subset \mathbb{R}^n$ is sequentially compact if and only if it is close in $\mathbb{R}^n$ and bounded.

Another kind of compact – covering compactness (which will be covered later). Continuous image of a compact space is compact.

$$f(X, d_X) \mapsto (Y, d_Y) \implies f(X) \subset Y \text{ is seq compact}$$

Proof. $y_j \in f(X)$ then $\exists x_j \ni f(x_j) = y_j$, for $n = 1, 2, \ldots$

$$x_j \to x_0 \in X \text{ subseq limit}$$

by cont.

$$f(x_0) = \lim f(x_{j_n})$$

$\{y_{j_n}\}$ converges to a pt in $f(X)$. □

§21 Lec 19: Nov 18, 2020

§21.1 Covering Compactness

Compactness(covering compactness). Note the difference with sequential compactness.

Definition 21.1 (Covering Compact) — (X, d) is (covering) compact if and only if for every collection $\{U_\lambda : \lambda \in \Lambda\}$, each $U_\lambda \subset X$ open, and with $\bigcup_{\lambda \in \Lambda} U_\lambda = X$, then $\exists \lambda_1, \ldots, \lambda_k$ finite numbers such that

$$\bigcup_{j=1}^{k} U_{\lambda_j} = X$$

Slogan: Every open cover of X has a finite subcover.

Fact 21.1. A metric space (X, d) is compact iff (X, d) is sequentially compact.

Proof. Later. □

Sequentially compact for $C \subset \mathbb{R}^n \iff C$ is closed in $\mathbb{R}^n$ and bounded in $\mathbb{R}^n$ (part of Heine-Borel Theorem).
Aside for subspaces: $Y \subset X, (X, d)$ is a metric space. Let $y_1, y_2 \in Y$, restrict the metric to $Y, d_{Y \times Y}$

$$d_{Y \times Y}(y_1, y_2) = d(y_1, y_2)$$

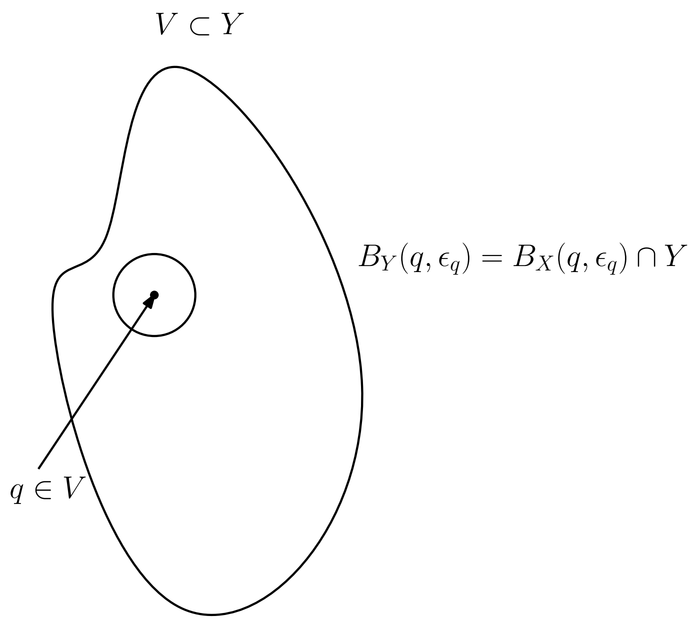
We have open set in Y defined as subset of $Y (\subset x)$ which is open in Y relative to $d_{Y \times Y}$.
Another candidate for being open in Y :Sets in Y of form $U \cap Y$ where U is open in X which is essentially the same idea.

Lemma 21.2

$Y \subset X, (X, d)$ a metric space. Then $V \subset Y$ is open in Y for $d_{Y\times Y}, d$ restricted to Y

$$\iff \exists U \subset X, U \text{ open with } U \cap Y = V$$

Proof. V open in Y iff $V = \bigcup_{q\in V} B_Y(q, \epsilon_q)$ for some suitable $\epsilon_q > 0$ for each q. An open set is an union of open balls and every union of open balls is open.



V open in Y means

$$V = (\bigcup_{q\in V} B_X(q, \epsilon_q)) \cap Y$$

Converse direction left as exercise. □

Y covering compact as a metric subspace (Y with $d_{Y\times Y}$ covering compact) $\iff$ For every U_λ, U_λ open in X

1. $\lambda \in \Lambda$ with $\bigcup_{\lambda\in\Lambda} U_\lambda \supset Y$ there exists a finite set $\lambda_1 \ldots, \lambda_k$ with

$$\bigcup_{j=1}^{k} U_{\lambda_j} \supset Y$$

$\{V_\lambda\}$ open in Y

$$\bigcup_{\lambda\in\Lambda} V_\lambda = Y$$

Each $V_\lambda = U_\lambda \cap Y$ for some U_λ open in X, so $\bigcup U_\lambda \supset Y$

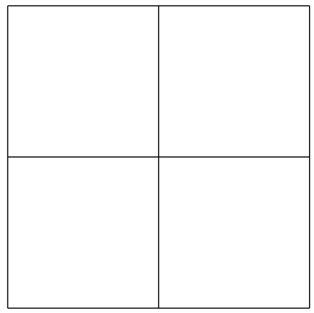
$$\bigcup_{j=1}^{h} U_{\lambda_k} \supset Y \implies \bigcup_{j=1}^{k} V_{\lambda_j} = Y$$

$C \subset \mathbb{R}^n$: What does it take to make C covering compact?

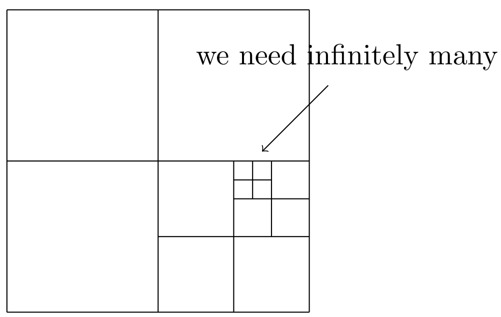
C closed and bounded $\iff$ C covering compact. (C is covering compact iff C is seq. compact).

Prove: If C is closed and bounded then C is (covering) compact.

Proof. Use technique of subdivision to prove first that a big cube $[-M, M] \times [-M, M] \times \ldots \times [-M, M] \subset \mathbb{R}^n$ is covering compact. e.g., we have a square in $\mathbb{R}^2$, a cube in $\mathbb{R}^3$,etc We will do the proof for $\mathbb{R}^2$ first.



Suppose U_λ open in $\mathbb{R}^2$ with $\bigcup_{\lambda \in \Lambda} \supset$ square. Suppose not finite collection of U_λ's covers square. This means we need infinitely many U_λ's to cover square. Use subdivision, we obtain 4 half-sized squares; one of these fours needs infinitely many U_λ's to cover it. Pick one that need infinitely many.



One of these need infinitely many. Pick one that does, and subdivide again. Get sequence of closed squares with side size going down by factor of $\frac{1}{2}$ each time, e.g., $2M \times 2M, M \times M, \frac{M}{2} \times \frac{M}{2} \ldots, S_1 \supset S_2 \supset S_3 \ldots$

$\exists x_0 \in U_{\lambda_0}, x_0 \in S_1$, so U_{λ_0} open in $\mathbb{R}^2$ for some λ_0. So, $B_{\mathbb{R}^2}(x_0, \epsilon) \subset U_{\lambda_0}$. For n large enough

$$x_0 \in S_n \implies S_n \subset B_{\mathbb{R}^2}(x_0, \epsilon)$$

In particular, n large S_n does not require infinitely many U_λ's. It only requires one of them. Note that C bounded and closed in $\mathbb{R}^2$ then C closed subset of $[-M, M] \times [-M, M]$ for some large M.

cool argument

If $\cup U_\lambda$ open $\supset C$ then $U_\lambda, \lambda \in \Lambda$ together with $\mathbb{R}^2 - C$ open in $\mathbb{R}^2$ cover C and $[-M, M]$. So, $U_{\lambda_1}, \ldots, U_{\lambda_k}$ and $\mathbb{R}^2 - C$ cover $[-M, M] \times [-M, M]$

$$\implies \bigcup_{j=1}^{k} U_{\lambda_j} \supset C$$

So C is covering compact. $\square$

$\mathbb{R}^n$ is similar: exercise.

§22 Lec 20: Nov 20, 2020

§22.1 Compactness (Cont'd)

$C \subset \mathbb{R}^n$, n finite. Sequential compactness is equivalent to C closed in $\mathbb{R}^n$ and bounded. Covering compact $C \iff$ closed bounded. So, we get: seq. compact $\iff$ compact. Note that we already done closed and bounded implies covering compactness (last time). Now, we need to prove compact $C \implies C$ is closed & bounded.
C compact $\implies C$ bounded. Look at $U_n = B(\vec{0}, n), n = 1, 2, 3, \ldots$, then $\bigcup_{n=1}^{\infty} U_n = \mathbb{R}^n$. Then,

$$\bigcup_{n=1}^{N} B(\vec{0}, n) = \bigcup_{n=1}^{N} U_n \supset C$$

and $d(\vec{0}, \vec{x}) \leq N$ for $\vec{x} \in C$, so C is bounded.
C compact in $\mathbb{R}^n \implies C$ closed in $\mathbb{R}^n$. Need to know: $\mathbb{R}^n - C$ is open. $x_0 \in \mathbb{R}^n - C$. Define $U_n = \mathbb{R}^n - \overline{B}(x_0, \frac{1}{n})$. U_n is open. If $x \in C \implies x = x_0$ because $x_0 \in \mathbb{R}^n - C$. Then, $d(x, x_0)$ is positive, set U_n for some large n

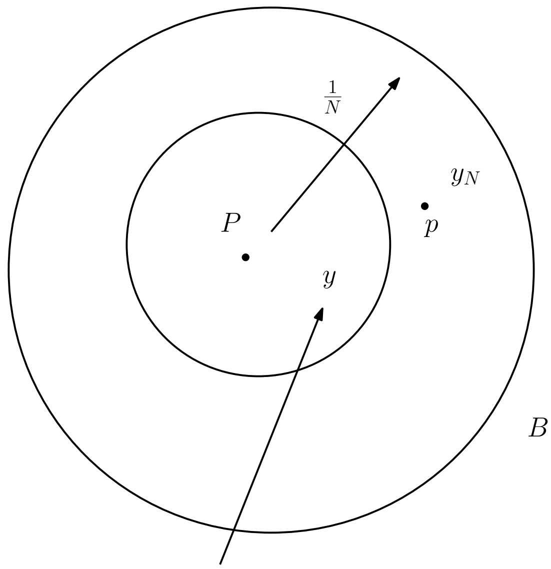
$$U_n = \left\{ x : d(x, x_0) > \frac{1}{n} \right\}$$

But true, given x, for some n (big enough). U_n are an open cover of C. Compactness of C gives $C \subset \bigcup_{n=1}^{N} U_n$ for some finite N. That means $x \in C$, get $d(x, x_0) > \frac{1}{N}$. So no sequence in C with a limit can have limit seq $= x_0$. So C is closed because given $x_0 \in \mathbb{R}^n - C, \exists \epsilon > 0$ s.t. $B(x_0, \epsilon) \subset \mathbb{R}^n - C \iff B(x_0, \epsilon) \cap C = \emptyset$. True because $\overline{B}(x_0, \epsilon) \cap C = \emptyset$.
Know that for subsets of $\mathbb{R}, C$ seq. compact $\iff$ C covering compact $\iff$ C closed and bounded.

Question 22.1. (X, d) seq. compact $\iff$ compact?

Ans: Yes, it's true.
Big deal theorem:
Covering compact $\implies$ sequence compact.

(Contradiction) Suppose we have $\{x_n\}$ with no convergent subsequence. Then, wolog, we can assume $\{x_n\}$ contains no infinite repetition: no $p \in X$ s.t. $x_n = p$ for infinitely many n. Passing for a subsequence $\left\{x_{n_j}\right\}$, no point occurs more than $x_{n_j} = p$ has points? at most once. Assume beyond $x_{n_2}, x_n : n > n_2$ different from x_1 and from x_{n_2}.
Passing to a subsequence all y_n's are distinct. Still no convergent subsequence. Now, for each $p \in X$, $\exists \epsilon_p > 0$ s.t. $B(p, \epsilon_p)$ contains only finitely many y.
Reason: If $B(p, \frac{1}{N})$ contains infinitely many y's for $N = 1, 2, 3, \ldots$ then we can find subsequence of $\{y_n\}$ converging to p.



Covering compactness of $X \implies X = B(p_1, \epsilon_{p_1} \cup \ldots \cup B(p_J, \epsilon_{p_J})$ for J finite which implies X contains only finitely many $y_n's$. Set $\{y_n : n = 1, 2, \ldots\}$ infinite set X, which is a contradiction.
Now, we need seq. compactness $\implies$ covering compactness (harder).

X covering compact:

Lemma 22.1

(X, d) compact and $U_\lambda, \lambda \in \Lambda$ open with $\cup U_\lambda = X \implies \exists \epsilon > 0$ s.t. for each $p \in X, B(p, \epsilon) \subset U_\lambda$ for some λ.

Proof. Suppose not. Then for each $n = 1, 2, 3 \ldots, \exists B(p_n, \frac{1}{n})$ that is not contained entirely in one U_λ. No U_λ contains $B(p_n, \frac{1}{n})$. Know (X seq. compact) $p_{n_j} \to p_0$, p_{n_j} conv. subsequence, $p_0 \in U_{\lambda_0}$ open since $\cup U_\lambda = X$, $\exists \delta > 0$ s.t.

$$B(p_0, \delta) \subset U_{\lambda_0} \text{ open}$$

Choose n s.t. $p_{n_j} \to p_0$, $d(p_0, p_n) < \frac{1}{2}\delta, \frac{1}{n} < \frac{1}{2}\delta$ and $B(p_n, \frac{1}{n})$. We claim $B(p_n, \frac{1}{n}) \subset B(p_0, \delta)$. Look at the $\triangle$ ineq.

$$d(p_0, x) < \frac{1}{2}\delta + \frac{1}{2}\delta = \delta$$

if $x \in B(p_n, \frac{1}{n})$. But, this is a contradiction since $B(p_n, \frac{1}{n})$ is not contained in any one U_λ (cannot be contained in U_{λ_0}. $\square$

Given $\cup U_\lambda$, we are done if we can show that $\exists$ a finite number of $p_j's$, $B(p_j, \epsilon), j = 1, \ldots, N$ s.t.

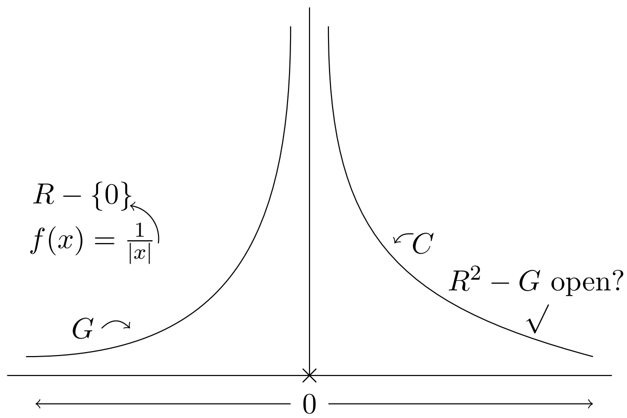
$$\cup B(p_j, \epsilon) = X$$

and $U_{\lambda_j} \supset B(p_j, \epsilon)$. Then $\cup U_{\lambda_j} = X$. We will do this for next lecture as we ran out of time.

§23 Lec 21: Nov 23, 2020

§23.1 Hw 6

Problem 3 – 4: $\mathbb{R} - \{0\}$, $\tan : \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$.
$f(x) = \frac{1}{|x|}$



$G = \left\{(x, y) : x \neq 0, y = \frac{1}{|x|}\right\} \subset \mathbb{R}^2$
$F : \mathbb{R} - \{0\} \to G$
$F(x) = \left(x, \frac{1}{|x|}\right)$, continuous, 1–1 and onto. Notice that F is homeomorphism which means F preserves convergent of sequences and F preserves closedness, that is F (closed set) is closed and F^{-1} (closed set) is closed because F is continuous. But, F may not preserve Cauchy sequences – Cauchy property maybe not preserved by homeomorphism.
$\mathbb{R} - \{0\}$ with usual metric, $d(x_1, x_2) = |x_1 - x_2|$, is not complete because

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \text{ is a Cauchy sequence}$$

but has no limit in $\mathbb{R} - \{0\}$. But $G = F(\mathbb{R} - \{0\})$ – the graph G is complete. It's because G is a closed set in $\mathbb{R}^2$ and $\mathbb{R}^2$ is complete (in taxicab metric). Closed subsets of complete metric spaces are complete in the induced metric.

$$F(x_1) = \left(x_1, \frac{1}{|x_1|}\right)$$
$$F(x_2) = \left(x_2, \frac{1}{|x_2|}\right)$$
$$d(F(x_1), F(x_2)) = |x_1 - x_2| + \left|\frac{1}{|x_1|} - \frac{1}{|x_2|}\right|$$

An alternative: $F(x) = \left(x, \frac{1}{x}\right)$. New idea – new metric on $\mathbb{R} - \{0\}$.

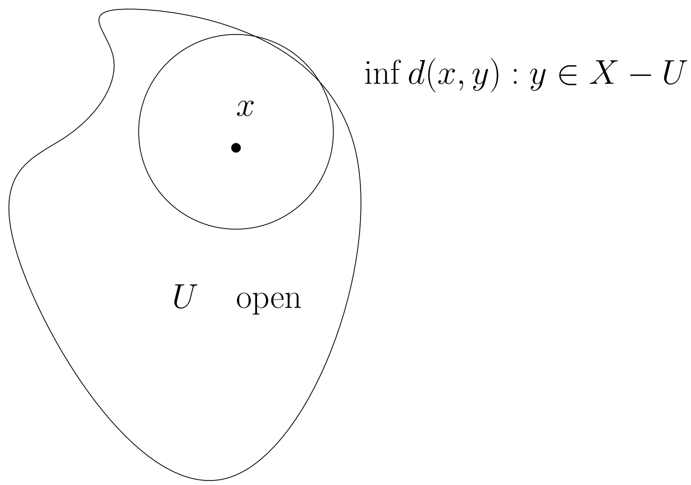
$$d(x_1, x_2) = |x_1 - x_2| + \left|\frac{1}{|x_1|} - \frac{1}{|x_2|}\right|$$

is a metric: induced metric on G. And d_G is complete.

Suppose $\{x_n\}$ is a sequence in $(\mathbb{R}-\{0\}, d_G)$. If it's Cauchy, then it has a limit in $\mathbb{R}-\{0\}$. Fix x_1,

$$d_G(x_1, x_n) \leq M$$

for some M. Cauchy seq. are bounded. So, $\frac{1}{|x_n|}$ is bounded. So, $|x_n| \geq \epsilon > 0$ for some ϵ. $U \subset X$ complete, $X - U \neq 0$, $|x| \leftrightarrow \operatorname{dis}(x, X-U) > 0$ try to understand this



§23.2 Compactness (Cont'd)

(X, d) seq. compact space. We want to prove compact (covering compact). Last time: If $\{U_\lambda \text{ open}\}, \lambda \in \Lambda$ open cover. Now, we want to prove the following claim $\exists p_1, p_2, \ldots, p_N$ finite set such that

$$\bigcup_{j=1}^{N} B(p_j, \epsilon) = X$$

Proof. Suppose not. $p_1 \in X$ (X is not empty). If $B(p_1, \epsilon) = X$, then we are done. If not, pick $p_2 \in X$ s.t. $d(p_1, p_2) \geq \epsilon$. Now suppose $B(p_1, \epsilon) \cup b(P_2, \epsilon) \neq X$ (if it equals, we are done). Pick p_3 s.t. $d(p_1, p_3) \geq \epsilon$ and $d(p_2, p_3) \geq \epsilon$. Continue: If

$$B(p_1, \epsilon) \cup B(p_2, \epsilon) \cup B(p_3, \epsilon) = X$$

then, we are done. If not, pick $p_4 \ni, \ldots$ Continue doing this. We get, if never happen $\bigcup_{j=1}^{N} B(p_j, \epsilon) = X$, $p_1, p_2, \ldots, \infty$ seq with $d(p_i, p_j) \geq \epsilon, i \neq j$, which is a contradiction since then $\{p_j\}$ has no convergent subsequence. We proved the claim. □

Now, we have $p_1, \ldots, p_N$

$$\bigcup_{j=1}^{N} B(p_j, \epsilon) = X$$

Now choose for each j

$$B(p_j, \epsilon) \subset U_{\lambda_j} \text{ for some } \lambda_j \in \Lambda$$

Then

$$\bigcup_{j=1}^{N} U_{\lambda_j} \supset \bigcup_{j=1}^{N} B(p_j, \epsilon) = X$$

$U_{\lambda_1}, \ldots, U_{\lambda_N}$ cover X. So, we are done with Heine Borel Theorem:
((X,d) is seq. compact $\iff$ (X,d) is compact). $C \subset \mathbb{R}^n$ then C closed and bounded $\iff$ C seq. compact $\iff$ C compact.
$l_2 : \{\vec{x} : \|x\| \le 1\}$ closed and bounded not seq. compact

$$(1,0,\ldots),(0,1,0,\ldots),(0,0,1,\ldots)$$

has no convergent subsequence.
(X,d) complete and "totally bounded" means For each $\epsilon > 0, \exists B(p_j,\epsilon), j = 1,\ldots,N_\epsilon$ s.t.

$$C \subset \cup B(p_j,\epsilon)$$

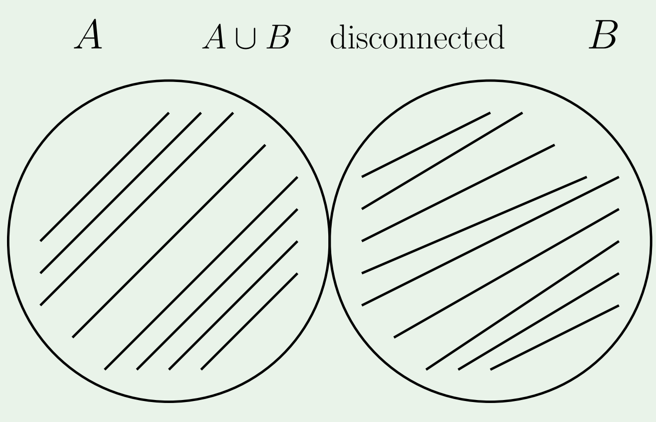
which implies C is seq. compact. l_2 is not totally bounded. Suppose $B(p_j,\frac{1}{2}), j = 1,\ldots,N$. These cannot cover $\{\vec{x} : \|x\| \le 1\}$.

§24 Lec 22: Nov 25, 2020

§24.1 Connectedness of Metric Spaces

Definition 24.1 (Disconnectedness) — (X,d) is disconnected if $\exists A,B \subset X, A \neq \emptyset, B \neq \emptyset$, $A \cup B = X, A \cap B = \emptyset$ and A,B are open.

Definition 24.2 (Connectedness) — (X,d) is connected if it is not disconnected.



A,B open discs

Definition 24.3 — $Y \subset (X,d)$, Y is connected if $(Y, d_X\big|_{Y\times Y})$ is connected.

Let $Y \subset (X,d)$. What does it mean for Y to be disconnected?
Let $A \neq \emptyset$ open in $Y \neq \emptyset$ and B open in Y and

$$A \cup B = Y$$
$$A \cap B = \emptyset$$

$\leftrightarrow \exists U_1 \neq \emptyset, U_2 \neq \emptyset$ open in X s.t.

$$\begin{aligned} A &= Y \cap U_1 \\ B &= Y \cap U_2 \\ U_1 \cup U_2 &\supset Y \end{aligned}$$

Question 24.1. Could we pick U_1, U_2 s.t. $U_1 \cap U_2 = \emptyset$ (Know that $U_1 \cap U_2 \cap Y = \emptyset$)?

$U_1 \cap U_2 \cap Y = \emptyset$ but maybe $U_1 \cap U_2 \neq \emptyset$

Fact 24.1. If Y is disconnected, and $A \neq \emptyset, B \neq \emptyset$ open in Y, $Y = A \cup B$, $A \cap B = \emptyset, A \cup B = Y$, then $\exists U_1, U_2$ open in X s.t.

$$U_1 \cap Y = A, \quad U_2 \cap Y = B, \quad U_1 \cap U_2 = \emptyset$$

Basic result: $[0,1]$ is connected.

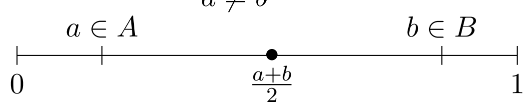
Proof. Suppose not connected. Then, $\exists A, B$ are open in $[0,1]$ s.t. $A \neq \emptyset, B \neq \emptyset$

$$A \cap B = \emptyset, \quad A \cup B = [0,1]$$

We want to show the above is impossible.

wolog $a < b$

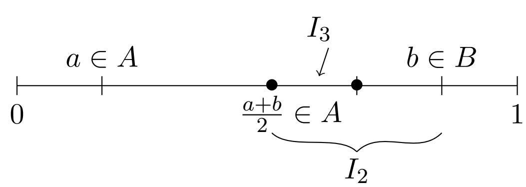
$a \neq b$



Either in A or in B

1. If $\frac{a+b}{2} \in A$, then look at $\frac{a+b}{2}, b$.
2. If $\frac{a+b}{2} \in B$, then look at $a, \frac{a+b}{2}$.

Now, we have new interval with length $= \frac{1}{2}|b-a|$ where one end is in A, other end is in B.

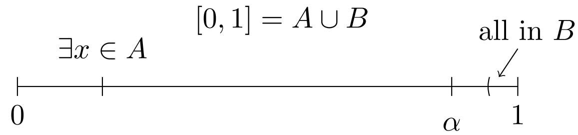


Continue to get I_3 with length $= \frac{1}{4}|b-a| = \frac{1}{2}\left(\frac{1}{2}|b-a|\right)$. Keep doing this, we have $I_1 \supset I_2 \supset I_3 \supset \ldots$ with length $\to 0$. Each I_n has one endpoint in A, one in B. We have $I_n = [\alpha_n, \beta_n]$ and α_n non decreasing and β_n non increasing and

$$\lim \alpha_n = \lim \beta_n = x_0$$

$x_0 \in A$ open or $x_0 \in B$ open. Open interval $I \subset A$ or open interval $\subset B$ n large: $x_0 \in \bigcap_{n=1}^{\infty} I_n$. $I_n \in I_\infty$, so ends of I_n are both in A or both in B, which is a contradiction. $\square$

Alternative view (exercise)



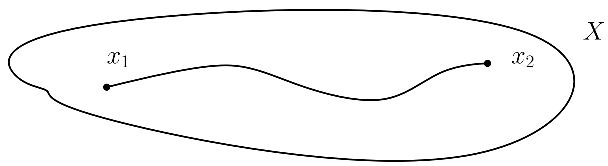
wolog, $1 \in B$. Now, we look at $\sup\{x \in [0,1] : x \in A\}$. Note that $\alpha < 1$ because B is open and $b \in B$. Where is α? Is $\alpha \in A$ or $\alpha \in B$ which is impossible (exercise to think about). So $[0,1]$ is connected and thus $[a,b]$ connected for $a, b \in \mathbb{R}, a \leq b$.

Definition 24.4 (Arcwise Connected) — A metric space (X,d) is arcwise connected if for each $x_1, x_2 \in X$ there is continuous function

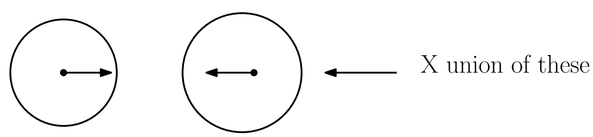
$$[0,1] \xrightarrow{\gamma} (X,d)$$

write

$$\gamma(0) = x_1 \text{ and } \gamma_1(1) = x_2$$



not arcwise connected



Theorem 24.5

If (X,d) is arcwise connected, then (X,d) is connected.

Proof. If (X,d) has a disconnection, $A \neq \emptyset, B \neq \emptyset$ open, $A \cup B = X, A \cap B = \emptyset$. Choose $a \in A, b \in B$

Claim 24.1. There is no arc γ from a to b.

If $\gamma : [0,1] \to X$ continuous and has $\gamma(0) = x_1 \in A$ and $\gamma(1) = x_2 \in B$ then

$$\gamma^{-1}(A), \gamma^{-1}(B)$$

is a disconnection of $[0,1]$ which does not exist. $\gamma^{-1}(A)$ open since Y continuous and A open. $\gamma^{-1}(B)$ open.

$$\gamma^{-1}(A \cap B) = \gamma^{-1}(\emptyset) = \emptyset$$
$$\gamma^{-1}(A) \cap \gamma^{-1}(B) = \gamma^{-1}(\emptyset) = \emptyset$$
$$\gamma(A \cup B) = [0,1]$$
$$x_1 \in \gamma^{-1}(A) \neq \emptyset, x_2 \in \gamma^{-1}(B) \neq \emptyset$$

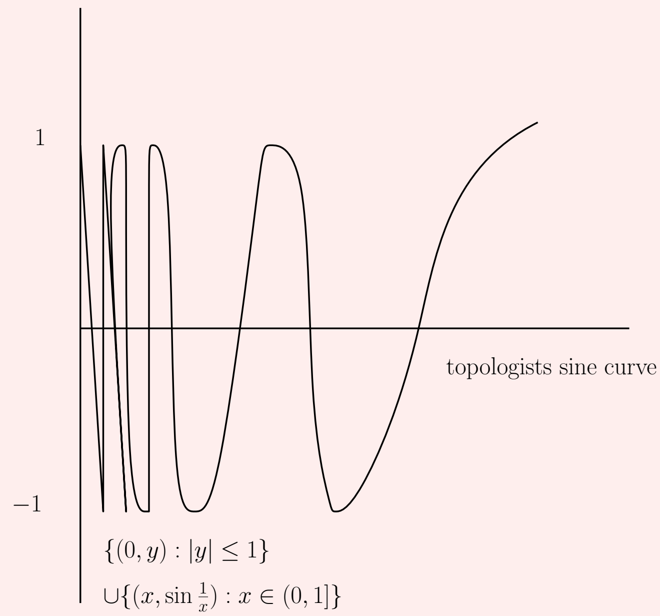
So $\gamma^{-1}(A), \gamma^{-1}(B)$ is a disconnection of $[0,1]$. There is no disconnection of $[0,1]$. So there is no disconnection of X. So X is connected. $\square$

Question 24.2. Does the converse work?

Ans: No, not in general.

Example 24.6

Consider the following graph of $\sin(\frac{1}{x})$



A, B disconnection, A open in $\mathbb{R}^2$, and $A = U \cap$ top sin curve.
$U \cap \left\{ \left(x, \sin\frac{1}{x}\right) : x \in (0,1] \right\}$ nonempty. Induces a disconnection of $\left\{ \left(x, \sin\frac{1}{x}\right) : x \in (0,1] \right\}$ which is impossible since it is arcwise connected.
It's easy to see that there is no continuous arc from $(1, \sin\frac{1}{1})$ to $(0,0)$.

Note: $U \subset \mathbb{R}^2$ open then U connected $\implies$ U arcwise connected.

§25 Dis 1: Oct 1, 2020

Notation:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$$
$$\mathbb{Q} = \left\{ \frac{p}{q} | p, q \in \mathbb{Z}, q \neq 0 \right\}$$
$$\mathbb{R} = \text{real numbers}$$
$$\mathbb{C} = \{a + bi, \quad a, b \in \mathbb{R}\}$$

Set theory:

- $A \subset B$ (or $A \subseteq B$) means $x \in A \implies x \in B$
- $x \in A \cap B$ means $x \in A$ and $x \in B$
- $x \in A \cup B$ means $x \in A$ or $x \in B$
- $x \in A \setminus B \iff x \in A$ and $x \notin B$
- $A = B \iff A \subset B$ and $B \subset A$

§25.1 Induction

Given a sequence of mathematical statement $P(n)$ indexed by $\mathbb{N}$. If $P(1)$ is true and $P(k) \implies P(k+1)$ is true $\forall k \in \mathbb{N}$, then $P(n)$ is true $\forall n \in \mathbb{N}$.

Example 25.1

Prove $\sum_{k=1}^{n}(2k-1) = n^2$ (*) using induction.
Base case $n = 1 : 1 = 1^2$ ✓
Induction step: assume as induction hypothesis that (*) holds

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1) &= \sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2\end{aligned}$$

Or we can prove it the following way

$$\begin{aligned}S &= 1 + 3 + 5 + \ldots + (2n-1) \\ S &= (2n-1) + (2n-3) + \ldots + 3 + 1 \\ 2S &= 2n \cdot n \\ S &= n^2\end{aligned}$$

Example 25.2

$a_{n+1} = \sqrt{2 + a_n}, \quad a_1 = 1$. Prove $a_n > 0$ and a_n increasing.
$a_1 > 0$ assume $a_n > 0$, $a_{n+1} = \sqrt{2 + a_n} > 0$

$$a_2 = \sqrt{3} \approx 1.732 > 1 = a_1$$

Assume $a_n \leq a_{n+1}$, want to show $a_{n+1} \leq a_{n+2} \impliedby \sqrt{a_n + 2} \leq \sqrt{a_{n+1} + 2} \impliedby a_n \leq a_{n+1}$

Example 25.3

$(1+x)^n \geq 1 + nx$: Bernoulli Inequality
$\quad x \geq -1, \quad n \geq 0$
base case $1 \geq 1$

Assume $(1+x)^n \geq 1 + nx$

$$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1 + (n+1)x + nx^2$$
$$= 1 + (n+1)x$$

Strong Induction:

If $P(1)$ true and $P(1), P(2), \dots P(k) \implies P(k+1)$ true $\forall k \in \mathbb{N}$ then $P(n)$ holds for all $n \in \mathbb{N}$

Remark 25.4. Induction $\iff$ strong induction

Example 25.5

Every integer greater than 1 is a product of primes.
Assume $2, 3, \dots, n$ is a product of primes. n+1 is either a prime or a composite, in which case $n+1 = ab, \quad 1 < a, b < n+1$.
By strong induction hypothesis, both a and b are product of primes, hence so is $n+1 = ab$.

Exercise 25.1. Every integer greater than 1 has a prime divisior.

Proof of infinitude of primes by Euclid:

Proof. Assume on the contrary there are finitely many primes $\{p_1, p_2, \dots, p_k\}$. Define $N = p_1 \dots p_k + 1 > 1$ and (by above exercise) let p be a prime divisior of N but $p \neq p_j$ for any $1 \leq j \leq k$ otherwise if $p = p_j$ then $p|p_2 \dots p_k$ also $p|N \implies p|N - p_1 \dots p_k \implies p|1$,a contradiction. (no primes divide 1) □

§26 Dis 2: Oct 8, 2020

§26.1 Number System

- $(\mathbb{N}, +, \cdot, <) : + : \mathbb{N} \times \mathbb{N} = \mathbb{N}^2 \to \mathbb{N}$ satisfies commutativity and associativity. Note that 0 is the identity with respect to addition, but $\mathbb{N}$ has no additive inverse.

- $(\mathbb{Z}, +, \cdot, <) : (\mathbb{Z}, +)$ is a commutative group (associativity, identity, inverse). $(\mathbb{Z}, \cdot)$ satisfies commutativity, associativity with 1 as mult identity but 2 has no mult inverse.

- $(\mathbb{Q}, +, \cdot, <) : (\mathbb{Q}, +)$ and $(\mathbb{Q}, \cdot)$ are commutative group(i). $+$ and $\cdot$ are compatible with distributive law: $a(b+c) = ab + ac$ (ii). Both (i) and (ii) mean $(\mathbb{Q}, +, \cdot)$ is a FIELD. $(Q, <)$ is an ordered set with $<$ satisfying trichotomy and transitivity. $+, \cdot$ are compatible : $y < z \implies x + y < x + z \forall x$, $x > 0, y > 0 \implies xy > 0$. With the above compatibility, $(\mathbb{Q}, +, \cdot, <)$ is an ordered field. Even though $\mathbb{Q}$ is additivity adn multiplicatively complete, $\mathbb{Q}$ is not satisfying in that

  1. $\mathbb{Q}$ is not algebraically closed, $x^2 - 2$ is a polynomial with no root in $\mathbb{Q}$.

2. $\mathbb{Q}$ is not complete in a metric space: there exists subsets of $\mathbb{Q}$ bounded above but with no least upper bound (supremum), e.g. $A := \left\{p \in \mathbb{Q} : p < 0 \text{ or } p^2 < 2\right\}$ and $B = \mathbb{Q} \setminus A$. A contains no largest number and B contains no smallest.

$$\forall p \in A \exists q \in A \quad q > p$$

Let $p \in A$. Define $q := p - \frac{p^2-2}{p+2} > p$

$$q^2 - 2 = \left(\frac{2p+2}{p+2}\right)^2 - 2 = \frac{2(p^2-2)}{(p+2)^2} < 0 \implies q^2 < 2$$

If A has an upper bound α, $\alpha \notin A$: then $\alpha \in B$. It follows that B is the set of all upper bounds for A. Since B contains no smallest number, A has no least upper bound in $\mathbb{Q}$.

Definition 26.1 (Least Upper Bound Property) — S has the least-upper-bound property if $\forall E \subset S$ nonempty, bounded above $\sup E \in S$.

Remark 26.2. $\mathbb{Q}$ does not satisfy the least-upper-bound property.

$(\mathbb{R}, +, \cdot, <)$ there exists an ordered field with the l.u.b property that contains an isomorphic copy of $\mathbb{Q}$.

§26.2 Equivalence Relation

An equivalence relation given $\sim$ on $A \times A$ satisfies

- $x \sim x$ reflexity
- $x \sim y \iff y \sim x$ symmetry
- $x \sim y \cdot y \sim z \implies x \sim z$ transitivy

Example 26.3

$\mathbb{Q}$ Define $\sim$ on $\{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$ by $(a, b) \sim (c, d)$ if $ad = bc$

$$A = \mathbb{Z}^2 \setminus \{(a, 0) : a \in \mathbb{Z}\}$$

$$\begin{aligned} \mathbb{Q} &= \text{ the set of all equivalence classes of A write } \sim \\ &= A/\sim = \{[x] : x \in A\} \end{aligned}$$

In this construction, $\quad \mathbb{Z} \to \mathbb{Q}, \quad n \to [(n, 1)]$
$+ \setminus \cdot : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$: note that $+$ and $\cdot$ need to be well-defined on $\mathbb{Q}^2$. (need to show $\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}$ if $\frac{a}{b} \sim \frac{a'}{b'}$ and $\frac{c}{d} \sim \frac{c'}{d'}$.

Example 26.4

$S' = [0,1]/0_m$

Definition 26.5 (Convergent Sequences) — $\{a_n\}_{n\geq 1} \subseteq \mathbb{R}$ is said to be convergent to l if $\forall \epsilon > 0 \quad \exists N(\epsilon) > 0$ s.t. $\forall n \geq N, \quad |a_n - l| < \epsilon$

§27 Dis 3: Oct 13, 2020

§27.1 Equivalence Relation (Cont'd)

Example 27.1

Define $\sim p$ on $\mathbb{Z}$ by $a \sim pb$ if $a - b \in p\mathbb{Z}(p|a-b)$.

$\forall a \exists! b \in \mathbb{Z}, \quad 0 \leq r < p$ s.t. $a = bp + r$.

$$F_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/\sim p = \{[0]_p, [1]_p, [2]_p, \ldots, [p-1]_p\}$$

$[a]_p + [b]_p = [a+b]_p \quad \& \quad [a]_p[b]_p = [ab]_p$

Remark 27.2. $(F_p, +, \cdot)$ is a finite field. F_p cannot be ordered: $1 > 0, 1+1 > 0, \ldots, p-1 > 0$ but $p - 1 = -1$

Example 27.3

$T = \mathbb{R}/\mathbb{Z} \quad a \sim b$ if $ab \in \mathbb{Z}$

$[0,1]/0 \sim 1$

$\forall a \in \mathbb{R}, \quad \exists b = \underbrace{\{a\}}_{\text{fractional part of } a} \in [0,1)$ s.t. $a \sim b$

§27.2 Construction of $\mathbb{R}$ via Cauchy Sequences(Cantor)

S = set of rational Cauchy sequences.

$\sim$ on $S: \{x_n\} - \{y_n\}$ if $\lim(x_n - y_n) = 0$ (Q3 – Homework 2)

$Q = S/\sim = \{[\{x_n\}] : \{x_n\} \in S\}$. First we need to define arithmetic on Q.

$$[\{p_n\}] + [\{q_n\}] = [\{p_n + q_n\}]$$
$$[\{p_n\}] - [\{q_n\}] = [\{p_n - q_n\}]$$
$$[\{p_n\}] \cdot [\{q_n\}] = [\{p_n q_n\}]$$
$$[\{p_n\}] / [\{p_n/q_n\}] = [\{p_n/q_n\}], \quad [\{q_n\}] \neq 0, = [\{0,0,0,\ldots\}]$$

$+ : Q \times Q \to Q$. Check well-defined

- $\{x_n\} \cdot \{y_n\}$ cauchy then so is $\{x_n + y_n\}$(Q4)

- $\{x_n\} \sim \{y_n\} \quad \& \quad \{z_n\} \sim \{w_n\}$ then $\{x_n + z_n\} \sim \{y_n + w_n\}$ (Q5)
  Commutativity, assoc, identity, $(0 = [\{0, 0, 0, \ldots\}]$, inverse.
- Well-defined: $\{x_n\}, \{y_n\}$ so is $\{x_n y_n\}$ (Q4).
- $\{x_n\} \sim \{y_n\}$ & $\{z_n\} \sim \{w_n\}$ (Q6, Q7)
  comm, assoc, iden, $(1 = [\{1, 1, \ldots, 1\}]$
  mult. inverse (Q9,Q10).
  $<$: trichotomy (Q11), transitivity
  various compatibility (distributivity, etc)
  l.u.b property (Q12)

Note:All the Q used above is assumed to be Q^{hat}

Remark 27.4.

$$Q \to Q^{\text{hat}}$$
$$q \mapsto [q^*]$$
$$p < q \iff [p^*] < [q^*]$$

Sequences:

- Cauchy seq. are bounded.
- Convergent seq. is Cauchy.
  Theorem: in $\mathbb{R}$, every Cauchy seq. is convergent.

Example 27.5
$a_n = \frac{1}{n}$
$\forall \epsilon > 0 \exists N$ s.t. $\epsilon N > 1$.
$\forall n \geq N \quad \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$. $\square$

§28 Dis 4: Oct 20, 2020

§28.1 Least Upper Bound and Its Applications

Remark 28.1 (ϵ – Principle)**.** $a, b \in \mathbb{R}, \forall \epsilon > 0,\ a \leq b + \epsilon \implies a \leq b$.

- $x, y \in \mathbb{R} \quad \forall \epsilon > 0,\ |x - y| \leq \epsilon \implies x = y$.

Supremum: $E \subset S$ bounded above. Suppose $\sup E \in S$

- $e \leq \sup E \forall e \in E$.
- $\forall \beta < \sup E, \quad \exists e \in E$ s.t. $\beta < e < \sup E$
  <u>OR</u>
  $\forall \epsilon > 0, \exists e \in E$ s.t. $\sup E - \epsilon < e \leq \sup E$.

Example 28.2

$\sup\left\{\frac{1}{n}\right\}_{n\geq 1} = 1, \quad \inf\left\{\frac{1}{n}\right\} = 0.$

- $0 \leq \frac{1}{n} \forall n \in \mathbb{N}$.
- $\forall \epsilon > 0, \exists n \in \mathbb{N}$ s.t. $0 \leq \frac{1}{n} < \epsilon$ by Archimedean Prop.

Theorem 28.3 (Nested Interval)

$\{I_n = [a_n, b_n]\}_{n\geq 1} \subset \mathbb{R}$, $I_n \supset I_{n+1} \implies \bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Moreover, if $|I_n| \to 0$, then $\bigcap I_n$ is a singleton (a set with exactly one element).

Proof. $\sup a_n \in \bigcap I_n$. □

Theorem 28.4 ((4.1))

(Bolzano – Weierstrass): Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. $I_0 = [-M, M] \supset I_1 \supset I_2 \supset \ldots$.

$$|I_n| = (2M) \cdot 2^{-n} \to 0 \quad \text{as } n \to \infty$$

From Nested Interval Thm, $\bigcap_{n=0}^{\infty} I_n = \{x\}$. Choose $x_{n_k} \in I_k, x_{n_k} \to x$. □

Remark 28.5. l.u.b property of $\mathbb{R} \implies$ Nested Interval $\implies$ Bolzano – Weierstrass $\overset{(*)}{\implies}$ Cauchy Completeness.

(*) Exercise: $\{x_n\}$ Cauchy. $x_{n_k} \to x \implies x_n \to x$.

Remark 28.6. In $\mathbb{R}$, to check convergence, it suffices to check Cauchyness. Useful especially when you don't have a candidate for the limit. Cauchy criterion for series $\sum_{n=1}^{\infty} a_n$ converges $(\lim_{n\to\infty} \sum_{k=0}^{n} a_k)$ exists. $\iff \sum a_n$ Cauchy $(\forall \epsilon > 0 \exists N \, |\sum_{k=n}^{m} a_k| < \epsilon \quad \forall m \geq n \geq N)$.

Corollary 28.7

Absolute convergence $\implies$ convergence. $(\sum |a_n|$ converges $\implies \sum a_n$ converges$)$.

Monotone convergence theorem, $\{a_n\}$ monotone. Then $\{a_n\}$ bounded $\iff$ $\{a_n\}$ convergent. (HW 3 – Q1).

Definition 28.8 (Monotone Sequence) — $\{a_n\}$ monotone if $a_n \leq a_{n+1} \forall n$ or $a_n \geq a_{n+1} \forall n$.

Corollary 28.9
$\sum |a_n| < \infty \iff \sum |a_n|$ converges.

§28.2 Continuity

Definition 28.10 ((6.2)) — $f : X \to \mathbb{R}$ is continuous at x (local prop) if

1. ($\epsilon - \delta$ def) $\forall \epsilon > 0, \exists \delta(\epsilon, x) > 0$ s.t. $\forall y \in X$, $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.
2. (Sequential def) $\forall \{x_n\} \subset X, x_n \to x \implies f(x_n) \to f(x)$ (f preserves sequential convergence).
3. $\lim_{y \to x} f(y) = f(x)$

$f : X \to \mathbb{R}$ is continuous if f is continuous at all $x \in X$.

Definition 28.11 ((7.1)) — f is uniformly continuous on X (global prop) if

1. ($\epsilon - \delta$) $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ s.t. $\forall x, y \in X$ $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.
2. (Sequential) $\forall \{x_n\} \subset X$, $\{x_n\}_{n \geq 1}$ Cauchy $\implies \{f(x_n)\}_{n \geq 1}$ Cauchy. (f preserves Cauchy seq).

Remark 28.12. Uniform continuity $\implies$ continuity.

Example 28.13
$f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is continuous.

$$\left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x - y|}{xy} < \frac{|x - y|}{x \cdot \frac{x}{2}} = |x - y| \cdot 2x^{-2} < \epsilon$$

$\delta = \min \left\{ \frac{x}{2}, \frac{\epsilon x^2}{2} \right\}$.

Remark 28.14. $x \mapsto \frac{1}{x}$ is uniformly continuous on $(a, \infty) \forall a > 0$.
$x \mapsto \frac{1}{x}$ is NOT uniformly continuous on $(0, \infty)$.

- $x_n = \frac{1}{n}, y_n = \frac{1}{n+1}$ $\quad |x_n - y_n| \to 0$ but $|\frac{1}{x_n} - \frac{1}{y_n} = 1 \forall n$.
- $\{\frac{1}{n}\}_{n \geq 1}$ Cauchy but $\{n\}$ is not.

§29 Dis 5: Oct 27, 2020

§29.1 Metric Spaces

Definition 29.1 ((9.1)) — A metric on a set X is a function $d : X \times X \to [0, \infty]$ s.t.

- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \le d(x, y) + d(y, z) \quad \forall x, y, z \in X$

Thus (X, d) is called a metric space.

Example 29.2

- $(X, d), A \subset X$. $d\Big|_{A \times A}$ is a metric on A.
- (Discrete metric) Given any set X, define

$$d(x, y) = \begin{cases} 1, x \neq y \\ 0, x = y \end{cases}$$

Check d is a metric on X.

Remark 29.3 (norm)**.** Given a vector space X. A norm on X is a function $\| \cdot \| : x \to [0, \infty)$ s.t.

- $\|x\| = 0 \iff x = 0$
- $\|\alpha x\| = |\alpha| \|x\|$
- $\|x + y\| \le \|x\| + \|y\|$

Then $d(x, y) = \|x - y\|$ is a metric on X.

Example 29.4

- $\mathbb{R}^d, | \cdot | = \| \cdot \|_2$ where $|x| = \|x\|_2 = \sqrt{\sum_{i=1}^d |x_i|^2}$
- On $\mathbb{R}^d$, define $\|x\|_p = \left(\sum_{i=1}^d \|x_i\|^p\right)^{\frac{1}{p}}, 1 \le p < \infty$

Inequalities:

- Young's Inequality:
$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, a, b \ge 0, \frac{1}{p} + \frac{1}{q} = 1$$
- Holden's Inequality:
$$\|xy\|_1 \le \|x\|_p \|y\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty$$
- Minkowski's Inequality (triangle inequality for $\| \cdot \|_p$)
$$\|x + y\|_p \le \|x\|_p + \|y\|_p$$

Define $\|x\|_\infty = \max_{i=1}^{d} |x_j|$. Then

$$\|xy\|_1 \le \|x\|_1 \|y\|_\infty$$
$$\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$$

Hence $(\mathbb{R}^d, \|\cdot\|_p)$ is a metric space $\forall 1 \le p \le \infty$. *Note*:

- $p = 1:$ taxicab / Manhattan metric
- $p = 2:$ Euclidean metric
- $p = \infty:$ sup metric

Notation: $\mathbb{R}^N = \{(x_i)_{i\ge 1} : x_i \in \mathbb{R}\} = \{f : \mathbb{N} \to \mathbb{R}\}$

Definition 29.5 — Given $x \in \mathbb{R}^N$, $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}, 1 \le p < \infty$. $\|x\|_\infty = \sup |x_i|$

Example 29.6

$l^p(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{R}, \|f\|_p < \infty\}$, $1 \le p \le \infty$. So $(l^p, \|\cdot\|_p)$ is a metric space and a vector space.

Definition 29.7 (Completeness of Metric Space) — A metric space (X, d) is complete if every Cauchy sequence with respect to d is convergent with respect to d.

Example 29.8

- $(\mathbb{Q}, |\cdot|)$ is not complete; $(\mathbb{R}, |\cdot|)$ is complete.
- $(\mathbb{R}^d, \|\cdot\|_p)$ is complete.
- $(l^p(\mathbb{N}), \|\cdot\|_p)$ is complete $(1 \le p \le \infty)$.
- $([0,1], \mathbb{R}) = \{f : [0,1] \to \mathbb{R}\}$ continuous

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)| \to \|f-g\| = \sup_{x\in[0,1]} |f(x) - g(x)|$$

$(C([0,1]), \|\cdot\|_\infty)$ is a complete metric space.

Special structure when $p = 2$

Inner product space:

Given vector space $X/\mathbb{R}$ a real inner product on X is $\langle\cdot,\cdot\rangle : x \succ x \to [0,\infty]$ s.t.

- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle, \forall a, b \in \mathbb{R}, x, y, z \in X$.
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle x, x\rangle \in (0, \infty)$ and is 0 $\iff x = 0$.

With the inner product: $\|x\| = \sqrt{(x,x)}$ is a norm, then $(X, \|\cdot\|)$ is a metric space.

Example 29.9

$\mathbb{R}^d : \langle x, y\rangle = x \cdot y = \sum x_i y_i$

also, $\|x\|_2 = \sqrt{\sum x_i^2} = \sqrt{\langle x, x\rangle}$

Example 29.10

$l^2 : \langle f, g\rangle = \sum_{i=1}^{\infty} f(i)g(i)$ and $\|f\|_2 = \sqrt{\langle f, f\rangle} = \sqrt{\sum_{i=1}^{\infty} |f(i)|^2}$

Definition 29.11 (Orthogonality) — $x \perp y \iff \langle x, y\rangle = 0$

Theorem 29.12 (Cauchy – Schwarz)

$|\langle x, y\rangle \leq \|x\| \cdot \|y\||$ and equality holds $\iff$ x, y are linearly dependent.

$\forall x, y \in X, \alpha \in \mathbb{R}$

$$\langle x - \alpha y \cdot x - \alpha y\rangle = \|x - \alpha y\|^2 \geq 0$$

Goal: find α that minimize $\|x - \alpha y\|$

The intuition here is $\|x - \alpha y\|$ is shortest when $x - \alpha y \perp y$.

$$\langle x - \alpha y \cdot x - \alpha y\rangle = \|x\|^2 + \alpha^2\|y\|^2 - 2\alpha\langle x, y\rangle$$

is minimal when $\alpha = \frac{\langle x,y\rangle}{\|y\|^2}$. Let us set α to such value, so

$$\begin{aligned} &= \|x\|^2 + \frac{|\langle x, y\rangle|^2}{\|y\|^2} - \frac{2\,|\langle x, y\rangle|^2}{\|y\|^2} \\ &= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2} \geq 0 \end{aligned}$$

§30 Dis 6: Nov 3, 2020

§30.1 Basic Topology – Metric Space

(X, d) metric space. If $x \in X$, the (open) ball of radius r about x is denoted $B_r(x) = B(r, x) = \{y \in X : d(x, y) < r\}$ where r is radius and x is the center.

Definition 30.1 (Open/Closed Sets) — $E \subset X$ open if $\forall x \in E \exists r > 0$ s.t. $B(r, x) \subset E$. E is closed if $E^c = X \setminus E$ is open.

Example 30.2

$B(r,x)$ is open: $\forall y \in B(r,x), B(r-d(x,y),y) \subset B(r,x)$

Example 30.3

$X, \emptyset$ is both open and closed, also known as clopen.

Example 30.4

Subsets of $\mathbb{R}$

	open	closed
$[0,1]$	$\times$	$\checkmark$
$(0,1)$	$\checkmark$	$\times$
$(0,1]$	$\times$	$\times$
$\mathbb{Z}$	$\times$	$\checkmark$
$\left\{\frac{1}{n}\right\}_{n\geq 1}$	$\times$	$\times$

We can observe for the last case, $\left\{\frac{1}{n}\right\}_{n\geq 1}$ is not closed since any neighborhood around 0 intersects $\left\{\frac{1}{n}\right\}_{n\geq 1} \implies \left\{\frac{1}{n}\right\}_{n\geq 1}^c$ is not open.

Example 30.5

Subset of $\mathbb{R}^2$

	open	closed
$\left\{x^2+y^2<1\right\} = B(1,0)$	$\checkmark$	$\times$
$\left\{x^2+y^2\leq 1\right\}$	$\times$	$\checkmark$
A where $\lvert A\rvert < \infty$	$\times$	$\checkmark$
$\{(x,y) : x = 1\}$	$\times$	$\checkmark$
$(0,1) = \{(x,0) : x \in (0,1)\}$	$\times$	$\times$

Remark 30.6. Open/Closed is relative: $(0,1)$ open in $\mathbb{R}$ but not open in $\mathbb{R}^2$.

- $\{V_\alpha\}_{\alpha\in A}$ open $\implies \bigcup_{\alpha\in A} V_\alpha$ is open
  $\{F_\alpha\}_{\alpha\in A}$ closed $\implies \bigcap_{\alpha\in A} F_\alpha$ is closed.
- $V_1,\dots,V_n$ open $\implies \bigcap_{i=1}^n V_i$ is open
  $F_1,\dots,F_n$ closed $\implies \bigcup_{j=1}^m F_j$ is closed.
- Infinite intersection (union) of open (closed) sets need <u>not</u> be open (closed, respectively).

$$\bigcap_{n\geq 1}\left(-\frac{1}{n},\frac{1}{n}\right) = \{0\} \qquad \bigcup_{n\geq 1}\left[\frac{1}{n}, 1-\frac{1}{n}\right] = (0,1)$$

Theorem 30.7

f is continuous $(X_1, d_1) \to (X_2, d_2) \iff f^{-1}(U)$ is open in $X_1 \forall U$ open in X_2.

Remember to prove this

Definition 30.8 (Boundedness) — Diameter of E : diam $E = \sup\{d(x,y) : x, y \in E\}$.
E is bounded if diam $E < \infty$.
An alternative definition: E is bounded if $\exists x \in E, R > 0$ s.t. $E \subset B_R(x)$

Definition 30.9 (Closure) — $E \subset X$. The closure of E in X is denoted $\overline{E} = \bigcap^F_{E \subset F, F \text{closed}}$. Note $\overline{E}$ is closed.

The interior of E in X is denoted

$$\mathring{E} = \bigcup_{E \supset G, G \text{ open}} G \qquad \mathring{E} \text{ is open}$$

Remark 30.10. E closed $\iff E = \overline{E}$. E open $\iff E = \mathring{E}$.

Theorem 30.11

The followings are equivalent

1. $x \in \overline{E}$
2. $\forall r > 0, B_r(x) \cap E \neq \emptyset$
3. $\exists \{x_n\}_{n \geq 1} \subset E$ s.t. $x_n \to x$

Proof. (1) $\iff$ (2) $\equiv x \notin \overline{E} \iff r > 0 B_r(x) \cap E = \emptyset \iff \exists r > 0, B_r(x) \subset E^c$. So this implies $x \in \left(\overline{E}\right)^c$. $\exists r > 0, B_r(x) \subset \left(\overline{E}\right)^c \subset E^c$

$$\impliedby \exists r > 0, B_r(x) \subset E^c \iff E \subset B_r(x)^c \implies \overline{E} \subset B_r(x)^c \implies x \notin \overline{E}$$

Note: above argument shows $\left(\overline{E}\right)^c = \left(\mathring{E^c}\right)$
(2) $\iff$ (3) – obvious. ☐

Definition 30.12 (Limit Point) —

$$\begin{aligned} E' &= \{x \in X : \exists r > 0 \, (B(r,x) \setminus \{x\}) \cap E \neq \emptyset\} \\ &:= \{x \in X : \exists \{x_n\} \subset E \setminus \{x\} \ni x_n \to x\} \end{aligned}$$

Example 30.13

$$
\begin{aligned}
E &= \left\{\frac{1}{n}\right\}_{n\geq 1} \\
E' &= \{0\} \\
\overline{E} &= \left\{\frac{1}{n}\right\}_{n\geq 1} \cup \{0\} \\
\left(\overline{E}\right)' &= Ex \\
\left(E'\right)' &= Ex
\end{aligned}
$$

Remark 30.14. $\overline{E} = E \cup E'$.

Theorem 30.15

The followings are equivalent

1. E closed (E^c is open).
2. $\overline{E} \subset E(\iff E = \overline{E})$
3. $E' \subset E$ – Rudin Definition
4. $\underbrace{\forall \{x_n\} \subset E \text{ if } x_n \to x}_{x\in\overline{E}}$ then $x \in E$.

§31 Dis 7: Nov 10, 2020

§31.1 Some Nice Theorems

Theorem 31.1 (Extreme Value)

$f : [a, b] \to \mathbb{R}$ continuous $\implies \exists x_n, x_m \in [a, b]$ s.t. $f(x_n) \leq f(x) \leq f(x_m) \forall x \in [a, b]$.

Remark 31.2.
1. $f : X \to \mathbb{R}$ continuous and if X is sequentially compact then f attains its extrema in X.

   Proof. Suppose $f(x_n) \to \sup_{x\in X} f(x)$ (allowing infinity), then by sequential compactness of X, $\exists x_{n_k} \to x \in X$. By continuity, $f(x_{n_k}) \to f(x)$ but $f(x_{n_k}) \to \sup_{x\in X} f(x)$ as well. By uniqueness of limit, $f(x) = \sup_{y\in Y} f(y) < \infty$. $\square$

2. $[a, b]$ is sequentially compact (HW)

3. Sequential compactness $\implies$ closed and bounded (HW).

   In $\mathbb{R}^n$, the converse is true by (high-dimensional) Bolzano-Weierstrass. So, in $\mathbb{R}^n$, sequential compactness $\iff$ closed and bounded.

Theorem 31.3 (Intermediate Value)

$f : [a, b] \to \mathbb{R}$ continuous. For every λ between $f(a), f(b)$, then $\exists c \in (a, b)$ s.t. $f(c) = \lambda$.

Remark 31.4. Image of connected set under continuous mapping is connected (later).

Example 31.5

- $\exists \alpha \in \mathbb{R} \ni \alpha^2 = 2$.
- Every odd polynomial $p(x)$ has a root in $\mathbb{R}$. Note: all polynomials are continuous.
- $f : [0, 1] \to [0, 1]$ continuous has a fixed point x s.t. $f(x) = x$. Show $g(x) = f(x) - x$ has a root. Note that g is also continuous $g(0) = f(0) = 0 : g(1) = f(1) - 1 \le 0$.

  If $f(0) = 0$ or $f(1) = 1$, we have the fixed point; if not, $g(0) > 0, g(1) < 0$ so IVT $\implies \exists c \in (0, 1)$ s.t. $g(1) = f(1) - c = 0$.

Theorem 31.6 (Heine – Cantor)

$f : [a, b] \to \mathbb{R}$ continuous $\implies$ f is uniformly continuous.

Remark 31.7. This also generalizes to any sequentially compact space.

Example 31.8

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), x \neq 0 \\ c, x = 0 \end{cases}$$

f is not continuous: $x_n = \frac{1}{\frac{\pi}{2} + n\pi}$ so that $\sin\left(\frac{1}{x_n}\right) = (-1)^n$. $x_n \to 0$ but $\sin\left(\frac{1}{x_n}\right) = (-1)^n$ does not converge. For any $c \in \mathbb{R}$, f is not continuous. So there exists no continuous extension of $\sin\left(\frac{1}{x}\right)$ to the origin.

Remark 31.9. $f : [a, b] \to \mathbb{R}$ uniformly continuous. Then $\exists! \mathscr{F} : [a, b] \to \mathbb{R}$ continuous s.t. $\mathscr{F}\big|_{(a,b)} = f$. Exercise!

§31.2 Completeness

- $A \subset B \implies \overline{A} \subset \overline{B} : A \subset B \subset \overline{B} \implies \overline{A} \subset \overline{B}$
- $A \subset B \implies A^\circ \subset B^\circ$
- $\overline{A \cup B} = \overline{A} \cup \overline{B} \iff (A \cap B)^\circ = A^\circ \cap B^\circ$ $(\overline{X}^c = \overset{\circ}{X^c})$
  $\supset$: $\overline{A} \subset \overline{A \cup B}, \overline{B} \subset \overline{A \cup B} \implies \overline{A} \cup \overline{B} \subset \overline{A \cup B}$
  $\subset$: $A \cup B \subset \overline{A} \cup \overline{B} \implies \overline{A \cup B} \subset \overline{A} \cup \overline{B}$.
- $\bigcup_{k=1}^\infty \overline{A_k} \subset \overline{\bigcap_{n=1}^\infty A_k}$, however, let $A_k = \{q_k\}$ where $Q = \{q_k\}_{k \geq 1}$ is enumeration of Q.
  $\bigcup_{k=1}^\infty \overline{A_k} = \bigcup_{k=1}^\infty \{q_k\} = Q \subsetneq \overline{\bigcup_{k=1}^\infty A_k} = \overline{Q} = \mathbb{R}$
  Similarly, $\left(\bigcap_{k=1}^\infty A_k\right)^\circ \subset \bigcap_{k=1}^\infty A_k^\circ$ but in general not equal.
- $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$; however,

$$\emptyset = \overline{\emptyset} = \overline{(-1, 0) \cap (0, 1)} \subsetneq \overline{(-1, 0)} \cap \overline{(0, 1)} = [-1, 0] \cap [0, 1] = \{0\}$$

  Similarly, $A^\circ \cup B^\circ \subset (A \cup B)^\circ$ but in general not equal.

Definition 31.10 (Dense Set) — $A \subset X$ is dense if $\overline{A} = X$ $(x \in X = \overline{A} \implies \forall r > 0, B_r(x) \cap A \neq \emptyset)$.

Example 31.11
$\overline{\mathbb{Q}} = \mathbb{R}$.

§32 Dis 8: Nov 17, 2020

§32.1 Complete Metric Spaces

- Closed subset of a complete metric space is complete.
- Complete subset of an arbitrary metric space is closed.

Definition 32.1 — A topological property/invariant is a property of a space invariant under homeomorphism.

Example 32.2

Sequential compactness, connectedness(later), (covering) compactness.

Remark 32.3. Completeness is NOT a topological property.

Example 32.4

$\phi : (\underbrace{\mathbb{R}, |\cdot|}_{\text{complete}}) \xrightarrow{\sim} \underbrace{((-1,1), |\cdot|)}_{\text{not complete}}, \phi(x) = \frac{x}{1+|x|}$. Define $\psi(x) = \phi^{-1}(x) = \frac{x}{1-|x|}$ and the pushforward metric $d \coloneqq \phi\,|\cdot|$ on $(-1,1)$ by $d(x,y) = |\psi(x) - \psi(y)|$. $((-1,1), d)$ is complete.

Note also that boundedness is NOT a topological property.

Theorem 32.5 (Completion)

Every metric space can be completed uniquely.

Proof. Confer Rudin Exercise 3.24. □

- l_2 is complete
- Define $B([0,1]) \coloneqq \{f : [0,1] \to \mathbb{R}, \|f\|_\infty < \infty\}$. $(B([0,1]).\|\cdot\|_\infty)$ is complete.
- $C([0,1]) \subset B([0,1])$ is closed subspace and hence complete. This fact is referred to as "uniform limit theorem".
- $((C[0,1]), \|\cdot\|_2)$ is not complete:

  Define $f_n : [0,\infty) \to \mathbb{R}$ by $f_n(x) = \max(1 - nx, 0)$ figure here f_n is Cauchy in $L_2, \|f_n - f_m\|_2 < \frac{1}{m}$ if $n \geq m$. The point wise limit

$$f(x) = \begin{cases} 1, x = 0 \\ 0, x \neq 0 \end{cases} \quad \text{is not continuous}$$

§32.2 Equivalent Metric

Definition 32.6 (Equivalent Metric) — Two metric d_1, d_2 on X are equivalent if $\exists C, C' > 0$ s.t. $Cd_1 \leq d_2 \leq C'd_1$.

Remark 32.7. Equivalent metrics define the same open/closed sets, same convergent and Cauchy sequences, and the same continuous and uniformly continuous mappings.

Observe that $B^{d_1}_{\frac{r}{C'}}(x) \subset B^{d_2}_r(x)$, $B^{d_2}_{Cr}(x) \subset B^{d_1}_r(x)$. This also implies the identity map $(X, d_1) \to (X, d_2)$ is homeomorphism.

Example 32.8 (Product Metric)
$(X, d), (X', d')$ define d_∞ on $X \times X'$ by $d_\infty((X, X'), (Y, Y')) = \max(d(x, y), d'(x', y'))$. We can also define $d_1((x, x'), (y, y')) = d(x, y) + d'(x', y')$ or $d_2((x, x'), (y, y')) = \left(d(x, y)^2 + d'(x', y')^2\right)^{\frac{1}{2}}$. d_1, d_2, d_∞ are all metrics on $X \times X'$ (check!) and either one of them is called a product metric on $X \times X'$ because d_1, d_2, d_∞ are equivalent:

$$d_\infty \le d_2 \le d_1 \le 2d_\infty$$

Corollary 32.9
A sequence in $\mathbb{R}^d$ converges $\iff$ each of its component converges.

Corollary 32.10
$\left(\mathbb{R}^d, \|\cdot\|_p\right)$ is complete $1 \le p \le \infty$.

Definition 32.11 (Topological Equivalent) — Two metrics d_1, d_2 on X are topological equivalent if they generate the same topology on X. ,i.e., either of the following holds

- $A \subset X$ is d_1-open $\iff$ d_2-open.
- $\forall x \in X, r > 0, \exists r_1, r_2$ s.t. $B^{d_1}_{r_1}(x) \subset B^{d_2}_r(x)$ and $B^{d_2}_{r_2}(x) \subset B^{d_1}_r(x)$.
- Identity map $(X, d_1) \to (X, d_2)$ is homeomorphic.

Remark 32.12. Equivalence $\implies$ topological equivalence.

Topological equivalence defines the same open sets, same convergent sequences, same continuous mappings.

Example 32.13
Identity map $((-1, 1), |\cdot|) \to ((-1, 1), d)$ is a homeomorphism. (Prove this)
$\{1 - 2^{-n}\}_{n \ge 1}$ is $|\cdot|$-Cauchy, but not d-Cauchy:

$$d(1 - 2^{-n}, 1 - 2^{-m}) = |2^n - 2^m|$$

and

$$\begin{aligned}\psi(x) &= \frac{x}{1-|x|}\\ \psi(1-2^{-n}) &= \frac{1-2^{-n}}{2^{-n}}\\ &= 2^n - 1\end{aligned}$$

Note also $\{1-2^{-n}\}$ is not $|\cdot|$ convergent, $1 \notin (-1,1)$.

- Bounded metric: (X,d). Define ρ by

$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

  ρ is a bounded metric which generates the same topology and the same Cauchy sequences.

Note $\beta(x,y) = \min(d(x,y),1)$ does the same trick.

Example 32.14 (Inifinite Product Metric)
Given (X_i,d_i), define $d : \prod_{i=1}^{\infty} X_i \times \prod_{i=1}^{\infty} X_i \to [0,\infty)$ by $d(x,y) = \sum_{j=1}^{\infty} 2^{-j} \frac{d(x,y)}{1+d_i(x,y)}$, d is a metric on $X = \prod X_i$. $d'(x,y) = \sum 2^{-j}\min(d_i(x,y),1)$ does the same trick.

§33 Dis 9: Nov 24, 2020

§33.1 Hw 6 – Ex. 4,5

$\phi \neq E \subset X$, define

$$d_E(x) = d(x,E) = \inf_{y\in E} d(x,y)$$

Prove these two things:

- $d_E(x) = 0 \iff x \in \overline{E}$.
- d_E is (uniformly) continuous on X: $|d_E(x) - d_E(y)| \leq d(x,y)$.

§33.2 Hw 7

Definition 33.1 (Cardinality) — X is finite if $\exists n \in \mathbb{N}$ s.t. $\exists f : X \to \{1,\ldots,n\}$ bijection. We write $|X| = n$.

Definition 33.2 (Countability) — X is countable if $\exists f : X \hookrightarrow \mathbb{N}$ injection. X is countably infinite if $\exists f : X \to \mathbb{N}$ bijection.

Example 33.3
Set of even number and set of squares.

The below facts are freely quoted for hw 7.

Fact 33.1.
1. Countable $\iff$ finite or countably infinite.
2. X, Y countable $\implies X \times Y$ countable(enough to show $\mathbb{N}^2$ is countable)
3. A is countable and X_α is countable $\forall \alpha \in A$, then $\bigcup_{\alpha \in A} X_\alpha$ is countable.
4. $\mathbb{Z}, \mathbb{Q}$ countable ($\exists f : \mathbb{Z} \to \mathbb{Q}, f(m,n) = \frac{m}{n}$ if $n \neq 0$ or $= 0$ if $n = 0$.
5. $\mathbb{R}$ is uncountable.

§33.3 Compactness

Definition 33.4 (Compactness) — $E \subset X$ compact if $\forall \{V_\alpha\}_{\alpha \in A}, E \subset \bigcup_{\alpha \in A} U_\alpha$ $\exists F \subset A, |F| < \infty$ s.t. $E \subset \bigcup_{\alpha \in F} V_\alpha$.

Example 33.5
- $[0,1)$ – NOT compact: $\bigcup_{n \geq 1} \left(-1, 1 - \frac{1}{n}\right) \supset [0,1)$ but no finite subcover. Suppose $[0,1) \subset \bigcup_{k=1}^m \left(-1, 1 - \frac{1}{nk}\right) \subset \left(-1, 1 - \frac{1}{\max_{k=1}(nk)}\right)$.
- $\mathbb{Z}$ – NOT compact: $\mathbb{Z} \subset \bigcup_{n \geq 1} (-n, n)$ again no finite subcover.
- $\left\{\frac{1}{n}\right\} \cup \{0\} \subset \mathbb{R}$ – compact

$$\left(\{U_\alpha\}_{\alpha \in A} \supset \left\{\frac{1}{n}\right\}_{n \geq 1} \cup \{0\} \right)$$

Theorem 33.6
$(X, d), E \subset X$. The followings are equivalent
- E is complete & totally bounded (implies boundedness)
- E is sequentially compact.
- E is compact.

Corollary 33.7
Subsets of $\mathbb{R}^n$ is compact $\iff$ closed and bounded.

$\implies$ true in any metric space because complete $\implies$ closed and totally bounded $\implies$ bounded.
$\impliedby$ in $\mathbb{R}^n$: closed $\implies$ complete as $\mathbb{R}^n$ is complete. Bounded $\implies$ totally bounded.

Remark 33.8. Unit ball in l_2 is not totally bounded, so not compact.

Theorem 33.9

$f : X \to Y$ continuous, X is compact. Then f is uniformly continuous.

Proof. Rudin 4.19. □

Theorem 33.10 ((30.1))

$f : X \to Y$ continuous, X is compact. Then $f(X)$ is compact.

Proof. Rudin. □

§33.4 Separability

Definition 33.11 (Separability) — X is separable if $\exists$ a countable dense subset.

Example 33.12

$\mathbb{R}$ is separable ($\overline{\mathbb{R}} = \mathbb{R}$). $\mathbb{R}^d$ is separable ($\overline{\mathbb{Q}^d} = \mathbb{R}^d$.
Compact metric space is separable (Hw 7 Ex 1).
Subspaces of separable space.

Example 33.13

$l^\infty = \{f : \mathbb{N} \to \mathbb{R} : \sup_{n \in \mathbb{N}} |f(n)| < \infty\}$ is not separable.

Proof. $A = \{f \in l^\infty : f(n) \in \{0, 1\} \, \forall n \in \mathbb{N}\}$. A is uncountable. $\forall f \neq g \in A, \|f - g\|_\infty = 1$. $\left\{B_{\frac{1}{2}}(f)\right\}_{f \in A}$ is an uncountable collection of distinct open balls in l^∞. Let C be a dense set. Each $B_{\frac{1}{2}}(f)$ contains at least one element in C which implies $\exists$ an injection $A \hookrightarrow C \implies C$ is uncountable. Hence, l^∞ is NOT separable. □

Example 33.14

l^p is separable $\forall 1 \leq p \leq \infty$: the set of finite rational sequences are dense in l^p.