

# Math 134 – Nonlinear ODE

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This is math 134 – Linear and Nonlinear System of Differential Equations taught by Professor Wink. The class lecture is prerecorded, and we have live session every Monday and Friday at 3:00 pm – 3:50 pm for Q & A. We use *Nonlinear Dynamics and Chaos* 2<sup>nd</sup> by *Steven Strogatz* as our main book for the class. Other course notes can be found through my [github](#). Any error spotted in the notes is my responsibility, and please let me know through my email at [ducvu2718@ucla.edu](mailto:ducvu2718@ucla.edu).

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# §1 | Lec 1: Jan 4, 2021

## §1.1 Intro to Dynamical Systems

There are two types of dynamical systems:

1. Discrete in time:

- Difference equation
- Iterated map:  $a_{n+1} = f(a_n)$

2. Continuous in time: differential equation

- Partial Differential Equation (PDE):  
e.g. heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

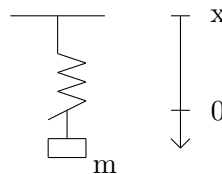
wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

where the derivatives w.r.t time and space.

- Ordinary Differential Equation (ODE):

i) Harmonic oscillator



m: mass  
k: spring constant

$$m\ddot{x} + kx = 0$$

If  $\omega^2 = \frac{k}{m}$ , then

$$x(t) = x_0 \cos(\omega t) + x_1 \sin(\omega t)$$

ii) Damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0, \quad b: \text{damping constant}$$

iii) Forced, damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = F \cos(t), \quad F: \text{force}$$

so derivatives w.r.t time only.

**Definition 1.1 (Order of ODE)** — Highest occurring derivative is defined as the order of the ODE.

**Remark 1.2.** We can always write an ODE of  $n^{\text{th}}$  order as a system of ODEs of  $1^{\text{st}}$  order.

Trick: Consider the damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

Set

$$\begin{aligned}x_1 &= x \\x_2 &= \dot{x}\end{aligned}$$

Then,

$$\begin{aligned}\dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x \\ &= -\frac{b}{m}x_2 - \frac{k}{m}x_1\end{aligned}$$

i.e.,

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1\end{aligned}$$

General framework:  $\dot{x} = f(t, x)$

$$f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

i.e.,

$$\begin{aligned}\dot{x}_1 &= f_1(t, x_1, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, \dots, x_n)\end{aligned}\tag{1}$$

which is  $1^{\text{st}}$  order  $n$ -dimensional ODE.

**Definition 1.3 (Linear ODE)** — The ODE (1) is called linear if  $f(t, x) = A(t) \cdot x$  for a time dependent matrix  $A(t)$ , otherwise we call it non-linear.

#### Example 1.4

The damped harmonic oscillator is linear.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

**Question 1.1.** Why are linear equations special?

They satisfy the principle of superposition. If  $\phi, \psi$  solve  $\dot{x} = A(t)x$ , then  $y(t) = c \cdot \phi(t) + \psi(t)$ ,  $c \in \mathbb{R}$  also solves  $\dot{x} = A(t)x$ . This is valid because  $\dot{y} = c\dot{\phi} + \dot{\psi} = cA\phi + A\psi = A(c\phi + \psi) = Ay$ . For non-linear ODEs, the principle of superposition fails.

**Definition 1.5 (Autonomous ODE)** — The ODE (1) is called autonomous if  $f$  does not depend on  $t$ , i.e.,  $f(t, x) = f(x)$ .

**Example 1.6**

$$m\ddot{x} + b\dot{x} + kx = F \cos(t)$$

is non-autonomous.

However, we can always consider an autonomous system instead. Set

$$x_1 = x$$

$$x_2 = \dot{x}$$

$$x_3 = t$$

Then

$$\dot{x}_1 = x_2$$

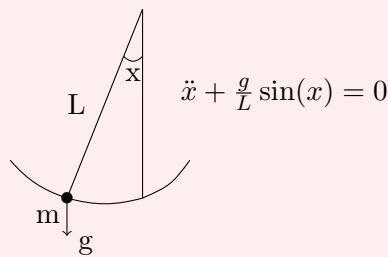
$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + F \cos(x_3)$$

$$\dot{x}_3 = 1$$

We will primarily study autonomous 1<sup>st</sup> order system in 1 or 2 variables.

**Example 1.7 (Swinging Pendulum)**

Consider a swinging pendulum



Set

$$x_1 = x$$

$$x_2 = \dot{x}$$

Then

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{L} \sin(x_1)$$

1<sup>st</sup> order, non-linear autonomous ODE in 2 variables.

**Question 1.2.** What can we say about the behavior of a solution  $x_1(t), x_2(t)$  for larger time  $t$ ? How does it depend on  $\frac{g}{L}$ ?

Idea: Use geometric methods, without solving  $\dot{x} = f(x)$  explicitly, to make qualitative statements about the long time behavior of the solution.

## §2 | Lec 2: Jan 6, 2021

### §2.1 Phase Portraits

We want to study 1D autonomous dynamical systems

$$\dot{x} = f(x), \quad f : \mathbb{R} \rightarrow \mathbb{R}$$

**Remark 2.1.**  $x(t)$  is the solution to  $\dot{x} = f(x)$  with  $x(0) = x_0$ . Find the solution  $y(t)$  with  $y(t_0) = x_0$ .

Ans:  $y(t) = x(t - t_0)$  because  $y(t_0) = x(0) = x_0$  and  $\dot{y}(t) = \dot{x}(t - t_0) = f(x(t - t_0)) = f(y(t))$ .

**Example 2.2**

$\dot{x} = \sin(x)$ . Suppose  $x_0 = \frac{\pi}{4}$ ,  $x(t)$  solution with  $x(0) = x_0$ . Answer the followings

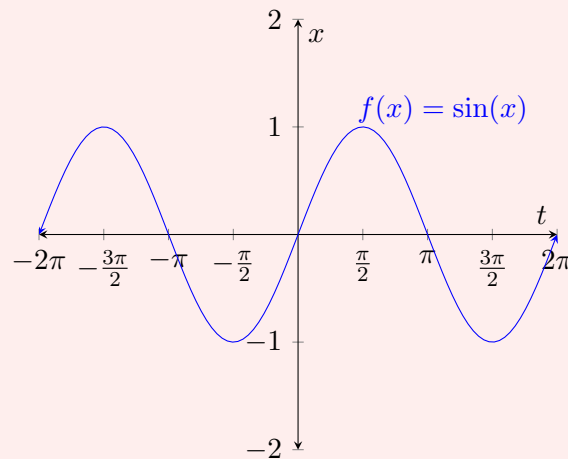
- Describe the long time behaviors of  $x(t)$  as  $t \rightarrow \infty$ .
- How does the long time behavior depend on  $x_0 \in \mathbb{R}$ ?

Attempt 1: Find explicit solution

$$\begin{aligned}\frac{dx}{dt} &= \sin(x) \\ dt &= \frac{dx}{\sin(x)} \\ t &= -\ln \left| \frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)} \right| + c\end{aligned}$$

We know  $x(0) = x_0$ , so  $c = \ln \left| \frac{1+\cos(x_0)}{\sin(x_0)} \right|$ . But what is  $x(t)$ ? This approach fails!

Attempt 2: Draw a phase portrait/diagram. We want to interpret the velocity  $\dot{x} = f(x)$  as a vector field on the real line.



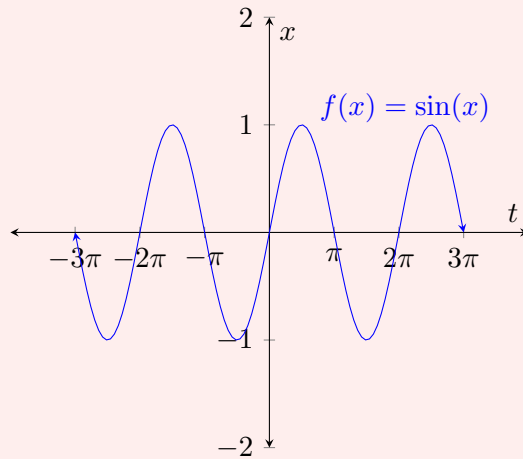
Idea:

- If  $f(x_0) > 0$ , then the solution to  $\dot{x} = f(x), x(0) = x_0$  increase near  $x_0$ .
- If  $f(x_0) < 0$ , then the solution to  $\dot{x} = f(x), x(0) = x_0$  decrease near  $x_0$ .
- If  $f(x_0) = 0$ , then the solution to  $\dot{x} = f(x), x(0) = x_0$  is  $x(t) = x_0$  for all  $t \in \mathbb{R}$ , i.e., we have a fixed point/equilibrium point.

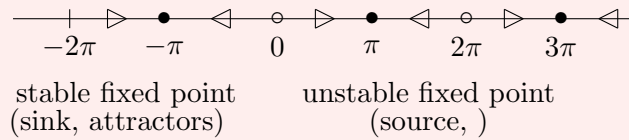


**Example 2.3**

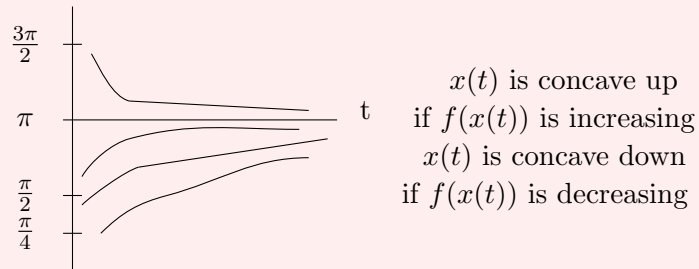
$$\dot{x} = f(x) = \sin(x)$$



Phase portrait:

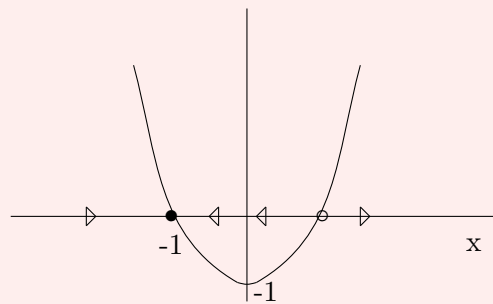


Qualitative plot of solution:



**Example 2.4**

$\dot{x} = x^2 - 1$ . Fixed points:  $f(x) = x^2 - 1 = 0 \implies x = \pm 1$



Note: If  $x_0 > 1$ , then solution  $x(t)$  with  $x(0) = x_0 > 1$  is unbounded. In fact,  $x(t) \rightarrow \infty$  in finite time.

## §3 | Lec 3: Jan 8, 2021

### §3.1 Stability Types of Fixed Points

**Definition 3.1 (Stability Types)** — Consider the ODE  $\dot{x} = f(x)$  and suppose that  $f(x_*) = 0$ . The fixed point  $x_*$  is called

1. Lyapunov stable if every solution  $x(t)$  with  $x(0) = x_0$  close to  $x_*$  remain close to  $x_*$  for all  $t \geq 0$ , otherwise unstable.
2. Attracting if every solution  $x(t)$  with  $x(0) = x_0$  close to  $x_*$  satisfies  $x(t) \rightarrow x_*$  as  $t \rightarrow \infty$ .
3. (asymptotically) stable if  $x_*$  is both Lyapunov stable and attracting.

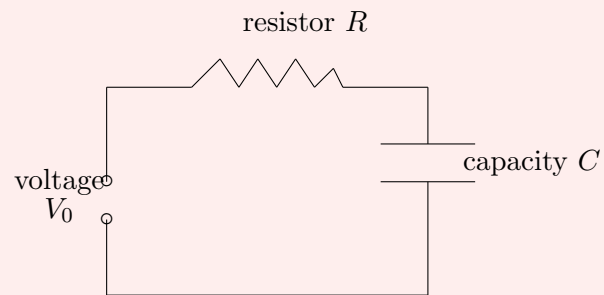
#### Example 3.2

Let  $\alpha \in \mathbb{R}$ ,  $\dot{x} = \alpha x$ . General solution  $x(t) = x_0 e^{\alpha t}$ .

- $x_* = 0$  is always an equilibrium solution.
- $x_* = 0$  is
  1. attracting if  $\alpha < 0$
  2. Lyapunov stable if  $\alpha \leq 0$
  3. unstable if  $\alpha > 0$

**Example 3.3** (RC circuit)

We have the following circuit



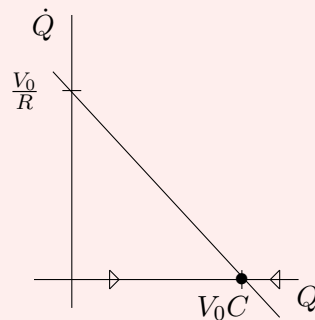
$$V_0 = RI + \frac{Q}{C}$$

$I$  : current,  $Q$  : charge

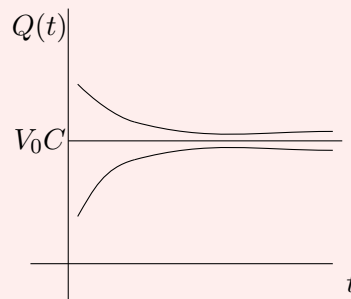
$$I = \dot{Q}$$

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$$

Phase portrait



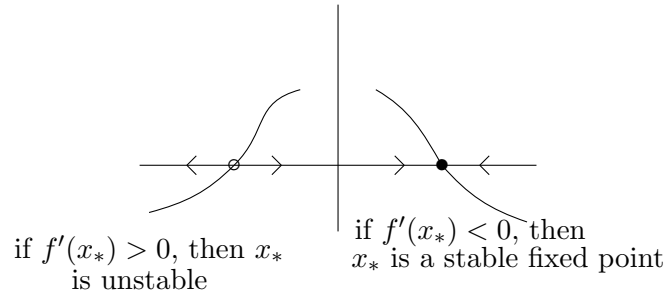
$Q_* = V_0C$  globally stable because every  $Q(t)$  approaches  $Q_*$  as  $t \rightarrow \infty$ .



### §3.2 Linear Stability Analysis

We have  $\dot{x} = f(x)$ ,  $f(x_*) = 0$ . Our task is to find an analytic criterion to decide if a fixed point  $x_*$  is stable/unstable.

Picture:



If  $f'(x_*) > 0$ , then  $x_*$  is unstable. On the other hand, if  $f'(x_*) < 0$ , then  $x_*$  is a stable fixed point.

The linearization:

Consider:  $\eta(t) = x(t) - x_*$  where  $x(t)$  is the solution of  $\dot{x} = f(x)$  with  $x(0)$  close to  $x_*$ ,  $f(x_*) = 0$ .

Note:  $\dot{\eta}(t) = \dot{x}(t) = f(x(t)) = f(x(t) - x_* + x_*) = f(\eta(t) + x_*)$ .

Taylor's Theorem:

$$f(x_* + \eta) = \underbrace{f(x_*)}_{=0} + f'(x_*)\eta + \underbrace{\mathcal{O}(\eta^2)}_{\text{error term and negligible if } f'(x_*) \neq 0 \text{ and } \eta \text{ is small}}$$

$\Rightarrow \dot{\eta}(t) \approx f'(x_*)\eta(t)$  (as long as  $\eta(t)$  is small) which is called the linearization of  $\dot{x} = f(x)$  about  $x_*$ . The general solution is

$$\eta(t) = \eta_0 e^{f'(x_*) \cdot t}$$

In particular,  $\eta$  grows exponentially if  $f'(x_*) > 0$  or decreases exponentially if  $f'(x_*) < 0$ .

**Definition 3.4** (Characteristics Time Scale) —  $\frac{1}{|f'(x_*)|}$  is called the characteristics time scale.

**Example 3.5** (Logistics Equation)

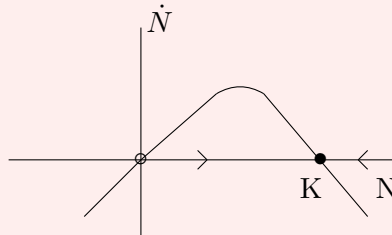
$N \geq 0$  population size,  $r > 0$  growth rate,  $K > 0$  carrying capacity

$$\dot{N} = rN \left(1 - \frac{N}{K}\right)$$

Fixed points:  $\dot{N} = 0 \implies N_* = 0$  or  $N_* = K$ .

Let  $f(N) = rN \left(1 - \frac{N}{K}\right) \implies f'(N) = r - 2\frac{r}{K}N$ . In particular,  $f'(0) = r > 0 \implies N_* = 0$  is an unstable fixed point and  $f'(K) = r - 2r = -r < 0 \implies N_* = K$  is stable.

Phase portrait:



Thus, if  $N(t)$  is the population with

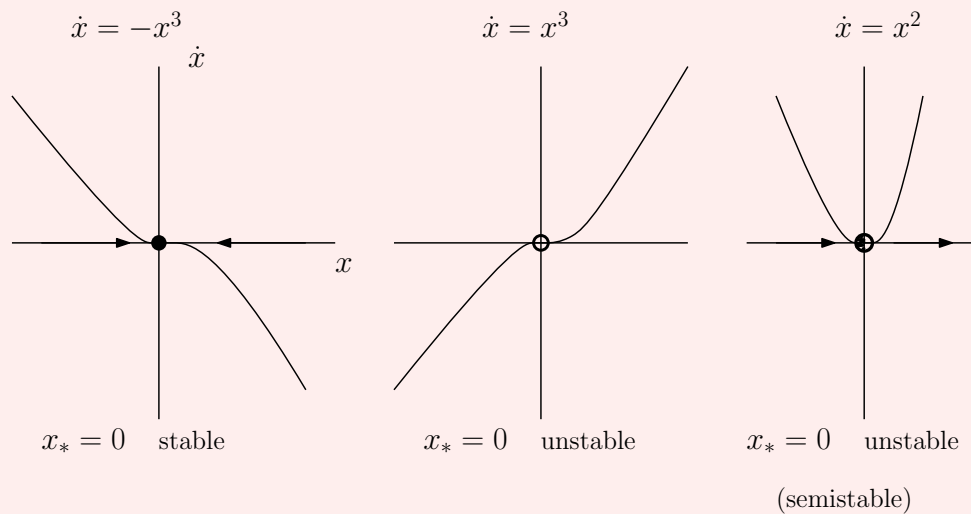
$$N(0) = N_0 > 0 \implies N(t) \rightarrow K \text{ as } t \rightarrow \infty$$

$$N(0) = 0 \rightarrow N(t) = 0 \quad \forall t \text{ (no spontaneous outbreak)}$$

Characteristics time scale:  $\frac{1}{|f'(N_*)|} = \frac{1}{r}$  for both  $N_* = 0, K$ .

**Example 3.6**

What if  $f'(x_*) = 0$ ? Then we can't tell.

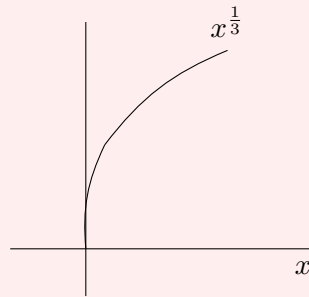


## §4 | Lec 4: Jan 11, 2021

### §4.1 Existence and Uniqueness

#### Example 4.1 (Non-uniqueness)

$\dot{x} = x^3 \implies x_1(t) \equiv 0$  (for all  $t$ ) is a solution with  $x_1(0) = 0$  but  $x_2(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$  is also a solution with  $x_2(0) = 0$



#### Theorem 4.2 (Picard's)

Let  $I = (a, b) \subseteq \mathbb{R}$  be an open interval,  $f : I \rightarrow \mathbb{R}$  differentiable and  $f'$  continuous. Let  $x_0 \in I$ . Then there is  $\tau > 0$  s.t. the initial value problem

$$\dot{x} = f(x), x(0) = x_0$$

has a unique solution  $x : (-\tau, \tau) \rightarrow \mathbb{R}$ .

**Example 4.3**

(The solution might not exist for all times) Consider

$$\frac{dx}{dt} = \dot{x} = 1 + x^2, \quad x(0) = 0$$

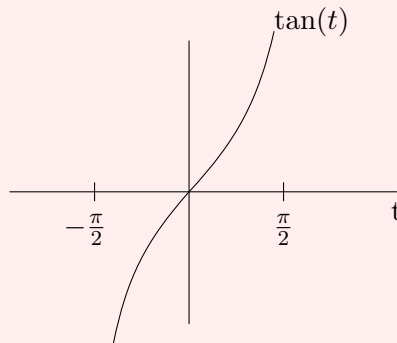
So,

$$dt = \frac{dx}{1 + x^2}$$

$$t = \int \frac{dx}{1 + x^2} = \arctan x + C$$

$$0 = 0 + C \implies C = 0$$

$$x(t) = \tan(t)$$



In particular,

$$x(t) \rightarrow +\infty \text{ as } t \rightarrow \frac{\pi}{2}$$

$$x(t) \rightarrow -\infty \text{ as } t \rightarrow \frac{-\pi}{2}$$

i.e.,  $x(t)$  reaches infinity in finite time, i.e., the solution  $x(t)$  blows up in finite time.

**Remark 4.4.** (Hw 1) If  $x_0 > 0$ , then the solution to  $\dot{x} = x^2, x(0) = x_0 > 0$  blows up in finite time. In fact, if  $\alpha > 1$ , then the solution to  $\dot{x} = x^\alpha, x(0) = x_0 > 0$  blows up in finite time.

**Theorem 4.5 (ODE Comparison)**

If  $x_1(t)$  solves  $\dot{x} = f(x)$ ,  $x_2(t)$  solves  $\dot{x} = q(x)$  and  $x_1(0) \leq x_2(0)$ ,  $f(x) < q(x)$ , then  $x_1(t) \leq x_2(t)$  for all  $t > 0$ .

In particular, if  $x_1(t) \rightarrow \infty$  in finite time, then  $x_2(t) \rightarrow \infty$  in finite time.



**Example 4.6**

The solution to  $\dot{x} = 1 + x^2 + x^3, x(0) = 0$  blows up in finite time.

Note: For  $x \geq 0$  :

$$1 + x^2 \leq 1 + x^2 + x^3$$

Recall:  $\tan(t)$  solves  $\dot{x} = 1 + x^2, x(0) = 0$ . By comparison: the solution  $x(t)$  to  $\dot{x} = 1 + x^2 + x^3, x(0) = 0$  satisfies  $x(t) \geq \tan(t)$ . Thus,  $x(t)$  blows up in finite time.

We may indeed assume that  $x(t) > 0$ . Since  $\dot{x}(0) = 1$ , it follows that  $x(t) > 0$  for  $t > 0$  small. In fact,  $\dot{x} = 1 + x^2 + x^3 > 0$  for  $x(t)$  small, i.e., whenever  $x(t)$  is close to zero, it must increase  $\implies x(t) > 0$  for  $t > 0$ .

**Example 4.7 (No Oscillating Solution in 1D)**

Let  $f \in C^1(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} | f \text{ differentiable, } f' \text{ continuous}\}$ . Suppose  $f(x_*) = 0, x(t)$  solution of  $\dot{x} = f(x)$ . If  $x(t_0) = x_*$  for some  $t_0$ . Then  $x(t) = x_*$  for all time  $t$ . Geometrically this says that a solution can never reach/cross a fixed point (unless it is a fixed point).

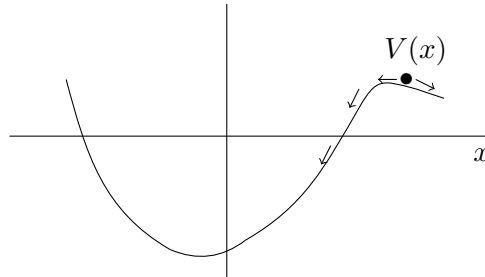
- $f(x(t)) > 0$  and  $\dot{x}(t) > 0$ , i.e.,  $x(t)$  increases.
- $f(x(t)) = 0$  and  $x(t) = \text{constant}$  for all  $t$ .
- $f(x(t)) < 0$  and  $\dot{x}(t) < 0$  i.e.,  $x(t)$  decreases.

In particular, there is no oscillating solution.

## §5 | Lec 5: Jan 13, 2021

### §5.1 Potential

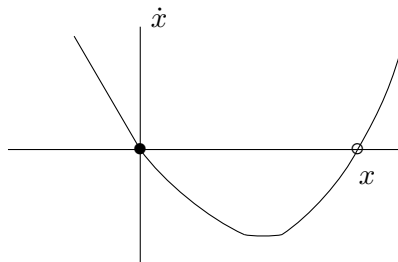
Consider the movement of a particle (with lots of friction) in a potential.



Notice:

- Particle approaches the local minimum of  $V(x)$  (minimum energy level) no fixed point.
- Local minima of  $V(x)$  are stable fixed points.
- Local maxima of  $V(x)$  are unstable fixed points.

$$\Rightarrow \dot{x} = f(x) = -\frac{dV}{dx} = -V'(x).$$



Expect  $t \rightarrow V(x(t))$  is non-increasing for a solution  $x(t)$  of  $\dot{x} = -V'(x)$ .

Indeed:

$$\begin{aligned} \frac{d}{dt}V(x(t)) &= V'(x(t)) \frac{d}{dt}x(t) \\ &= V'(x(t)) (-V'(x(t))) \\ &= -(V'(x(t)))^2 \leq 0 \end{aligned}$$

$\Rightarrow$  particle always moves towards a lower energy level.

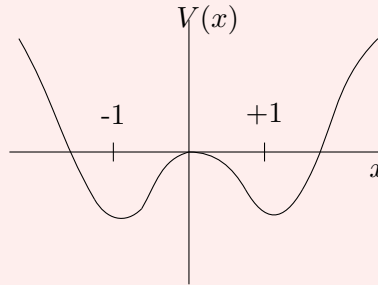
**Definition 5.1 (Potential)** — A function  $V(x)$  s.t.  $\dot{x} = f(x) = -\frac{dV}{dx}$  is called a potential.

**Example 5.2**

Graph potential for  $\dot{x} = x - x^3$ . Find/characterize equilibria (fixed points).

$$\dot{x} = f(x) = x - x^3 = -\frac{dV}{dx} \xRightarrow{f} V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$$

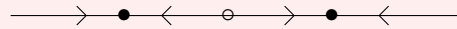
$\Rightarrow V$  is only defined up to a constant, we may choose any  $C \in \mathbb{R}$ , e.g., choose  $C = 0$ .



Local minima of  $V$  correspond to stable fixed points  $\Rightarrow 0 = -\frac{dV}{dx} = f(x) = x - x^3$ , i.e.,  $x = \pm 1$ .

Local maximum of  $V$  corresponds to an unstable fixed point at  $x = 0$ .

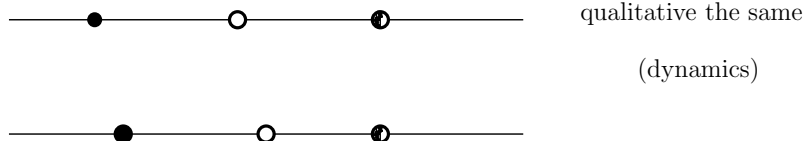
Phase portrait:



**Remark 5.3.** This system is often called bistable because it has two stable fixed points.

## §5.2 Bifurcations

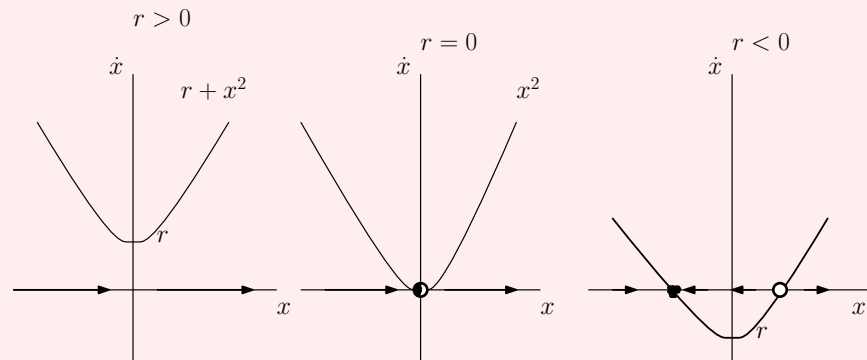
The qualitative behavior of 1D dynamical systems  $\dot{x} = f(x)$  is determined by fixed points.



If  $\dot{x} = f(r, x)$  depends on a parameter  $r$ , then the numbers of fixed points and their stability may change as  $r$  varies. This is called **bifurcation**.

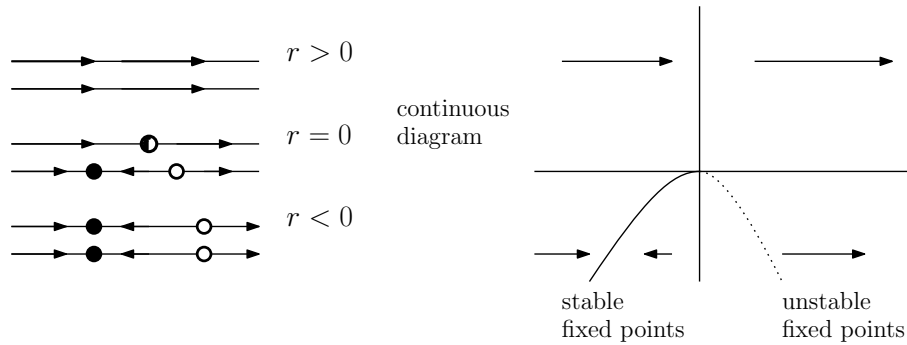
**Example 5.4** (Saddle-node, blue sky bifurcation)

$$\dot{x} = r + x^2, \quad r \in \mathbb{R}.$$

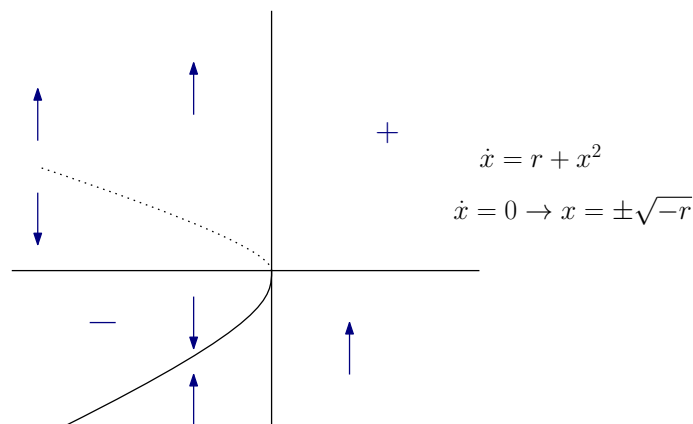


Hence, the qualitative behavior changes at  $r_* = 0$ , i.e.,  $r_* = 0$  is called a bifurcation point.

Ways to plot the dependence on the parameter:



Most common: bifurcation diagram



## §6 | Lec 6: Jan 15, 2021

### §6.1 Saddle-Node Example

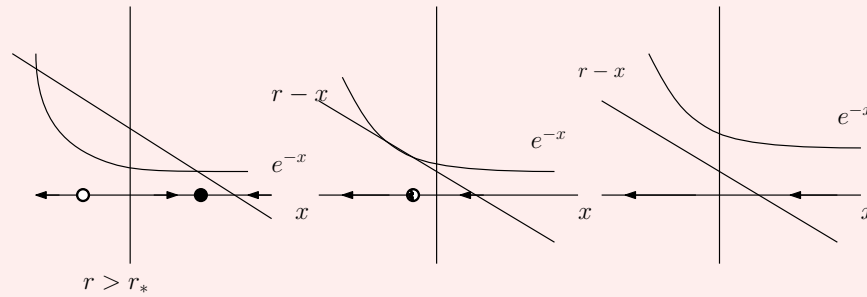
#### Example 6.1

Argue geometrically that the ODE

$$\dot{x} = r - x - e^{-x}$$

undergoes a saddle-node bifurcation. Furthermore, find the bifurcation point.

Note: Fixed points of  $\dot{x} = r - x - e^{-x}$  correspond to intersection points of the functions  $r - x, e^{-x}$  because  $r - x - e^{-x} = 0 \iff r - x = e^{-x}$ .



Indeed we have a saddle-node bifurcation.

Note: At  $r = r_*$ , the graph of  $r - x$  and  $e^{-x}$  intersect tangentially. Thus, for the bifurcation point we require:

$$\begin{aligned} 0 = \dot{x} = r - x - e^{-x} &\implies r - x = e^{-x} \\ 0 = \frac{d}{dx}(r - x - e^{-x}) &\implies \frac{d}{dx}(r - x) = \frac{d}{dx}e^{-x} \end{aligned}$$

So,

$$\begin{aligned} -1 &= -e^{-x} \\ e^{-x} &= 1 \\ x &= 0 \\ r_* &= x_* + e^{-x_*} = 0 + 1 = 1 \end{aligned}$$

Thus the bifurcation point is  $(r_*, x_*) = (1, 0)$ .

Note:

$$\begin{aligned} \dot{x} &= r - x - e^{-x} = r - x - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right) \\ &= r - 1 - \frac{1}{2}x^2 + \frac{x^3}{6} - \dots \\ &\approx (r - 1) - \frac{1}{2}x^2 \text{ for } x \text{ near } x_* = 0 \end{aligned}$$

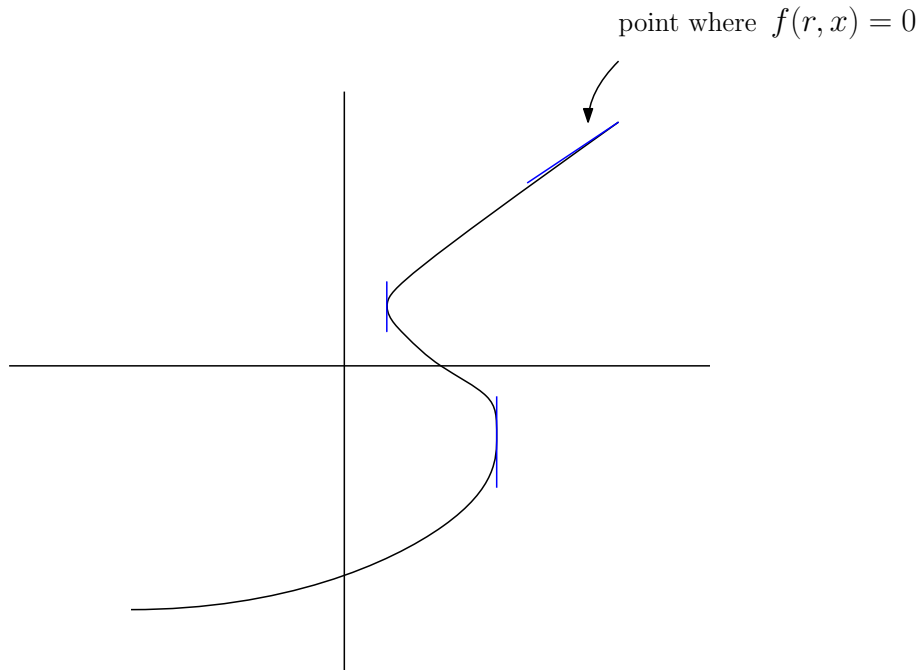
Set  $R = r - 1$ , then  $\dot{x} \approx R - \frac{1}{2}x^2$ .

Upshot: Up to appropriate rescalings/coordinate changes, every saddle-node bifurcation looks like its normal form

$$\dot{x} = r - x^2 \quad (\text{or } \dot{x} = r + x^2)$$

close to the bifurcation point  $(r_*, x_*) = (0, 0)$ .

## §6.2 Normal Forms



Recall:

- Normal vector:  $\begin{pmatrix} \partial_r f \\ \partial_x f \end{pmatrix}$
- Tangent vector:  $\begin{pmatrix} -\partial_x f \\ \partial_r f \end{pmatrix}$

Note: Bifurcation points have vertical tangent vectors, i.e.,  $\partial_x f = 0, \partial_r f \neq 0$ .

### Theorem 6.2 (Taylor's)

Suppose  $f(r_*, x_*) = 0$ .

$$\begin{aligned} f(r, x) = & f(r_*, x_*) + \underbrace{\frac{\partial f}{\partial r}(r_*, x_*)}_{p_1}(r - r_*) + \underbrace{\frac{\partial f}{\partial x}(r_*, x_*)}_{q_1}(x - x_*) \\ & + \frac{1}{2} \underbrace{\frac{\partial^2 f}{\partial r^2}(r_*, x_*)}_{p_2}(r - r_*)^2 + \underbrace{\frac{\partial^2 f}{\partial r \partial x}(r_*, x_*)}_{R}(r - r_*)(x - x_*) + \frac{1}{2} \underbrace{\frac{\partial^2 f}{\partial x^2}(r_*, x_*)}_{q_2}(x - x_*)^2 + \dots \end{aligned}$$

**Remark 6.3.** If  $q_1 \neq 0$ , then there is an bifurcation at  $(r_*, x_*)$ , linear stability (sign of  $q_1$ ) determines if  $(r_*, x_*)$  is (un)stable.

**Theorem 6.4**

Suppose that  $f(r_*, x_*) = 0, q_1 = 0, p_1 \neq 0, q_2 \neq 0$ , then  $\dot{x} = f(r, x)$  undergoes a saddle node bifurcation at  $(r_*, x_*)$  and

$$\dot{x} = \frac{\partial f}{\partial s}(r^*, x^*)(r - r^*) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

for  $|r - r_*| < \epsilon^2, \quad |x - x_*| < \epsilon.$

**Remark 6.5.** i) Note that TBA  $(r - r_*)(x - x_*)$  is  $\mathcal{O}(\epsilon^3)$

ii) With a coordinate change  $(t, x, \delta) \mapsto (s, y, R)$  we can arrange that ODE looks like

$$\frac{d}{ds}y = R + y^2$$

near  $(0, 0) = (R(r_*), y(x_*))$

**Example 6.6**

$\dot{x} = e^r - x - e^{-x}$  undergoes a saddle-node bifurcation near  $(r_*, x_*) = (0, 0)$ . Apply the theorem 6.4,

$$\begin{aligned} f(r, x) &= e^r - x - e^{-x} \\ f(0, 0) &= 1 - 0 - 1 = 0 \\ \frac{\partial f}{\partial x}(r, x) &= -1 + e^{-x} \implies \frac{\partial f}{\partial x}(0, 0) = 0 \\ \frac{\partial f}{\partial r}(r, x) &= e^r \implies \frac{\partial f}{\partial r}(0, 0) = 1 \neq 0 \\ \frac{\partial^2 f}{\partial x^2}(r, x) &= -e^{-x} \implies \frac{\partial^2 f}{\partial x^2}(0, 0) = -1 \neq 0 \end{aligned}$$

Therefore, by theorem 6.4,  $(r_*, x_*) = (0, 0)$  is a bifurcation point of a saddle-node bifurcation.

Normal form near  $(r_*, x_*) = (0, 0)$  :

$$\begin{aligned} \dot{x} &= e^r - x - e^{-x} \\ &= 1 + r + \frac{r^2}{2} + \mathcal{O}(r^3) - x - \left(1 - x + \frac{x^2}{2} + \mathcal{O}(x^3)\right) \\ &= r + \underbrace{\frac{r^2}{2}}_{\mathcal{O}(\epsilon^4)} - \frac{x^2}{2} + \mathcal{O}(r^3) + \mathcal{O}(x^3) \\ &= \underbrace{r - \frac{x^2}{2}}_{\mathcal{O}(\epsilon^2)} + \mathcal{O}(\epsilon^3) \text{ if } |r - r_*| = |r| < \epsilon^2 \\ &\quad \text{if } |x - x_*| = |x| < \epsilon \end{aligned}$$

Set  $y = \frac{x}{2}$ , then

$$\dot{y} = \frac{1}{2}\dot{x} = \frac{r}{2} - \frac{x^2}{4} + \mathcal{O}(\epsilon^3) = \frac{r}{2} - y^2 + \mathcal{O}(\epsilon^3)$$

Set  $s = -t$ , then

$$\frac{d}{ds}y = -\frac{d}{dt}y = -\frac{r}{2} + y^2 + \mathcal{O}(\epsilon^3)$$

Set  $R = -\frac{r}{2}$ , then

$$\underbrace{\frac{d}{ds}y = R + y^2}_{\text{normal form of a saddle-node bifurcation}} + \mathcal{O}(\epsilon^3)$$



## § 7 | Dis 1: Jan 7, 2021

### § 7.1 Fixed points and Stability

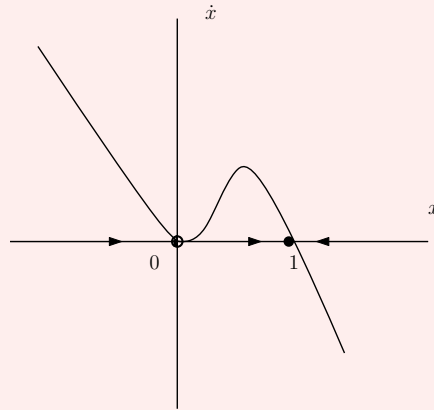
$$\dot{x} = f(x)$$

**Example 7.1**

$$\dot{x} = -x^3 + x^2$$

a) Sketch the vector field, classify the fixed points.

“vector fields” = x-axis with arrows



so:

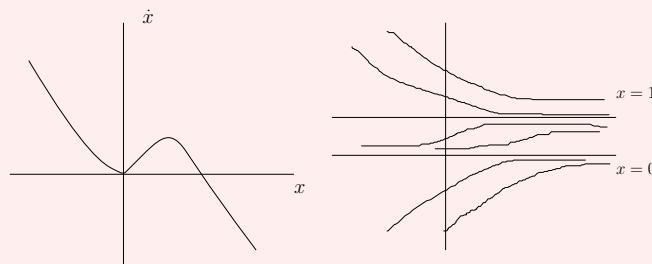
- $\dot{x} > 0 \implies x(t)$  increasing
- $\dot{x} < 0 \implies x(t)$  decreasing

“Fixed point”  $\iff x_*$  s.t.  $f(x_*) = 0 \iff x_*$  s.t. the constant function  $x(t) = x_*$  is a solution.

We have 2 fixed points:

- $x_* = 0$  is semi-stable.
- $x_* = 1$  is stable.

b) Sketch various solutions of  $x(t)$ .

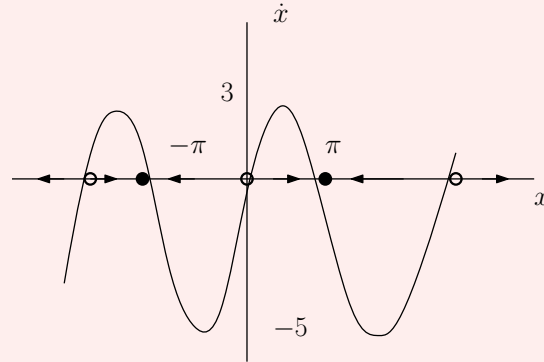


- $\dot{x} = 0$  for  $x = 0, 1 \implies x(t) = 0, 1$  are solutions.
- $\dot{x} > 0$  for  $x < 0 \implies x(t)$  increasing.
- $\dot{x} > 0$  for  $0 < x < 1 \implies x(t)$  increasing.
- $\dot{x} < 0$  for  $x > 1 \implies x(t)$  decreasing.

**Example 7.2**

$$\dot{x} = -1 + 4 \sin x$$

a) Sketch vector field, classify fixed points.

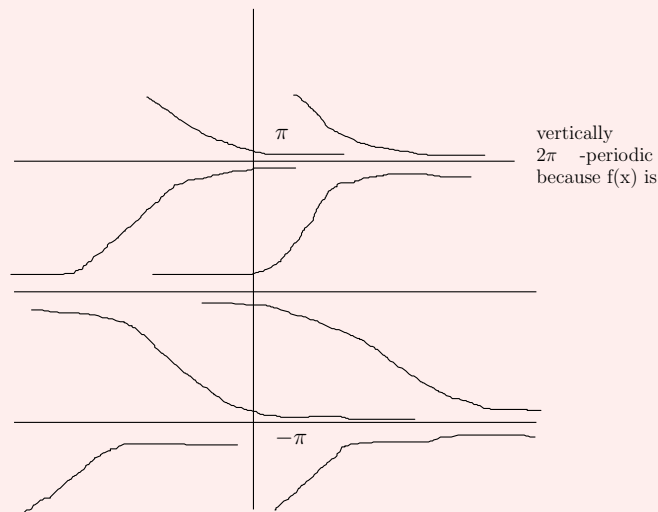


Fixed points:

$$\sin(x_*) = \frac{1}{4}$$

- $x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$  for  $n = 0, \pm 1, \dots$  are unstable.
- $x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$  for  $n = 0, \pm 1, \dots$  are stable.

b) Sketch various solutions  $x(t)$ .



## §7.2 First Order Autonomous System

$\vec{x} = \vec{f}(\vec{x})$  – first order and autonomous.

**Example 7.3**

A unit mass with displacement  $x(t)$  attached to a spring with spring constant 6 obeys:

$$\ddot{x} = -6x - b(t)\dot{x}$$

where  $b(t) \geq 0$  is the friction coefficient.

a) Show that this can be expressed as a first order autonomous system

$$\begin{aligned} x_1 &= x, & x_2 &= \dot{x} \\ \dot{x}_2 &= \ddot{x} = -6x_1 - b(t)x_2 \\ \vec{x} &:= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \vec{\dot{x}} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -6x_1 - b(x_3)x_2 \end{pmatrix} \end{aligned}$$

where  $x_3 = t \implies \dot{x}_3 = 1$ .

b) In the case  $b(t) = 5$ , find the explicit solution for  $x(0) = x_0, \dot{x}(0) = v_0$ .

$$\ddot{x} = -6x - 5\dot{x} \implies \ddot{x} + 5\dot{x} + 6x = 0$$

Try  $x(t) = e^{kt}$  :

$$\begin{aligned} 0 &= \ddot{x} + 5\dot{x} + 6x = k^2 e^{kt} + 5k e^{kt} + 6e^{kt} \\ &= e^{kt}(k^2 + 5k + 6) \implies k = -3, -2 \end{aligned}$$

Now,  $x(t) = c_1 e^{-3t} + c_2 e^{-2t}, c_1, c_2 \in \mathbb{R}$ . Using the initial conditions, we obtain

$$x(t) = (-2x_0 - v_0)e^{-3t} + (3x_0 + v_0)e^{-2t}$$

## §8 | Dis 2: Jan 14, 2021

### §8.1 Linearization and Potentials

#### Example 8.1

$$\dot{x} = -x^3 + x^2$$

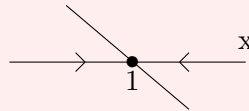
- a) Use linear stability analysis to classify the fixed points. If it fails, use a graphical argument.

Idea: For  $x$  near a fixed point  $x_*$ ,  $\dot{x} = f(x) \approx f(x_*) (= 0) + f'(x_*)(x - x_*) = \dots$

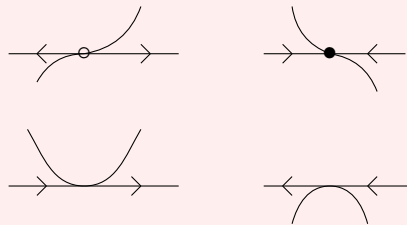
$$f(x) = -x^3 + x^2, \quad f'(x) = -3x^2 + 2x$$

$$0 = f(x_*) = -x_*^2(x_* - 1) \implies x_* = 0, 1$$

- $x_* = 1 : f'(1) = -1 < 0 \implies$  stable



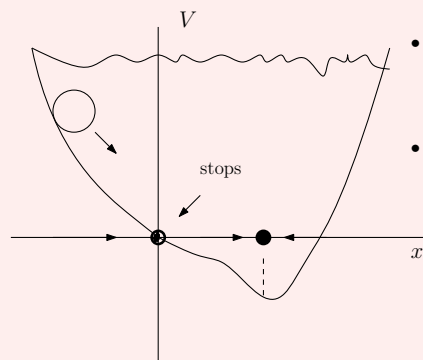
- $x_* = 0 : f'(0) = 0 : \text{inconclusive}$



- b) Find and plot a potential function.

“Potential function”  $\iff V(x)$  s.t.  $\dot{x} = -\frac{dV}{dx}$

$$\dot{x} = -x^3 + x^2 = -V'(x) \implies V(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 + C \text{ (choose 0)}$$



- fixed points
  - $\iff f(x_*) = 0$
  - $\iff V'(x_*) = 0$
  - $\iff$  critical point

- Trajectories move toward decreasing  $V$ , like a ball rolling down the graph of  $V$

**Example 8.2**

$$\dot{x} = 4 \sin x - 1.$$

- a) Use linear stability analysis to classify the fixed points.

$$f(x) = 4 \sin x - 1, \quad f'(x) = 4 \cos x$$

Last time: Fixed points are

$$\bullet x_* = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \dots$$

$$f'(x_*) = 4 \cos\left(\sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) > 0$$

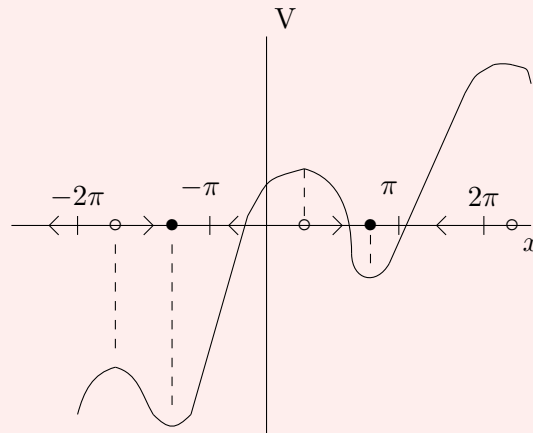
$\implies$  unstable

$$\bullet x_* = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, n = 0, \pm 1, \dots$$

$$f'(x_*) = 4 \cos\left(\pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n\right) < 0$$

$\implies$  Stable.

- b) Plot potential  $-1 + 4 \sin x = -V'(x) \implies V(x) = x + 4 \cos x$



## §8.2 Existence of Solutions

**Example 8.3** a) Let  $a > 0$  be a constant. Show that the solution of

$$\begin{cases} \dot{x} = ax^2 \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

$$\begin{aligned} \frac{dx}{dt} = ax^2 &\implies \int \frac{dx}{x^2} = \int a \, dt \\ &\implies -\frac{1}{x} = at + c \\ &\implies x(t) = \frac{1}{c - at} \quad \forall c \in \mathbb{R} \\ x(0) > 0 &\implies c > 0 \implies \lim_{t \rightarrow T} x(t) = +\infty \end{aligned}$$

for some  $T > 0$ . In fact,  $c = \frac{1}{x_0}$  and so  $T = \frac{c}{a} = \frac{1}{ax_0}$ .

b) Let  $0 < \epsilon < 1$  be a constant. Show that the solution of

$$\begin{cases} \dot{x} = x^2 (1 + \epsilon \sin x) \\ x(0) = x_0 > 0 \end{cases}$$

blows up in finite time.

Idea:  $\dot{x} \geq ax^2$  for some  $a > 0$ , so our solution grows faster than a function which blows up, and must blow up too.

$$\begin{aligned} |\sin x| \leq 1 &\implies 1 + \epsilon \sin x \geq 1 - \epsilon \\ \implies \dot{x} = x^2 (1 + \epsilon \sin x) &\geq \underbrace{1 - \epsilon}_{>0} x^2 \end{aligned}$$

Let  $x(t)$  be the solution to

$$\begin{cases} \dot{x} = x^2 (1 + \epsilon \sin x) \\ x(0) = x_0 \end{cases}$$

Let  $y(t)$  be the solution to

$$\begin{cases} \dot{x} = (1 - \epsilon)x^2 \\ x(0) = x_0 \end{cases}$$

By part a),  $y(t)$  blows up at some time  $T > 0$ . Since  $x(0) = y(0)$  and  $\dot{x} \geq \dot{y}$ , then  $x(t) \geq y(t)$  for all  $t \geq 0$  (ODE Comparison Lec 4). Therefore,  $x(t)$  must blow up in finite time. In fact, blow up time must be  $\leq T = \frac{1}{(1-\epsilon)x_0}$ .