Math 115AH – Honors Linear Algebra

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This is math 115AH – Honors Linear Algebra, a traditional first upper-division course that UCLA math students usually take. It's taught by Professor Elman, and our TA is Harris Khan. We meet weekly on MWF at 2:00pm – 2:50pm for lectures, and our discussion is on TR at 2:00pm – 2:50pm. With regard to book, we use *Linear Algebra* 2^{nd} by *Hoffman and Kunze* for the class. Note that some of the theorems' name are not necessarily the official name of the theorem; it's just a way to assign meaning to a theorem (easier for reference) instead of a tedious section number. Other course notes can be found through my github site. Please contact me at ducvu2718@ucla.edu if you find any concerning mathematical errors/typos.

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§1 Lec 1: Oct 2, 2020

Remark 1.1. To know a definition, theorem, lemma, proposition, corollary,etc., you must

1. Know its precise statement and what it means without any mistake
2. Know explicit example of the statement and specific examples that do not satisfy it
3. Know consequences of the statement
4. Know how to compute using the statement
5. At least have an idea why you need the hypotheses – e.g., know counter-examples,...
6. Know the proof of the statement
7. Know the important (key) steps of in the proof, separate from the formal part of the proof – i.e., the main idea(s) of the proof

THIS IS NOT EASY AND TAKES TIME – EVEN WHEN YOU THINK THAT YOU HAVE MASTERED THINGS.

§1.1 Field

What are the properties of the REAL NUMBERS?

$$\mathbb{R} \coloneqq \{x|x \text{ is a real no.}\}$$

– at least algebraically?
There are two FUNCTIONS (or MAPS)

- $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ called ADDITION write $a + b \coloneqq +(a, b)$
- $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ called MULTIPLICATION write $a \cdot b \coloneqq \cdot(a, b)$

that satisfy certain rule e.g., associativity, commutativity,...

Definition 1.2 (Field) — A set F is called a FIELD if there are two functions

- Addition: $+ : F \times F \to F$, write $a + b := +(a, b)$
- Multiplication: $\cdot : F \times F \to F$, write $a \cdot b := \cdot(a, b)$

satisfying the following AXIOMS(A: addition, M: multiplication, D: distributive)

A1 $(a + b) + c = a + (b + c)$ Associativity

A2 $\exists$ an element $0 \in F \ni a + 0 = a = 0 + a$ Existence of a Zero

A3 $\forall x \in F \exists y \in F \ni x + y = 0 = y + x$ Existence of an Additive Inverse

A4 $a + b = b + a$ Commutativity

M1 $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M2 (A2) holds and $\exists$ an element $\in F$ with $1 \neq 0 \ni a \cdot 1 = a = 1 \cdot a$ Existence of a One

M3 (M2) holds and $\forall 0 \neq x \in F \ \ \exists y \in F \ni xy = 1 = yx$ Existence of a Multiplicative Inverse

M4 $x \cdot y = y \cdot x$

D1 $a \cdot (b + c) = a \cdot b + a \cdot c$ Distributive Law

D2 $(a + b) \cdot c = a \cdot c + b \cdot c$

<u>Comments</u>: Let F be a field, $a, b \in F$. Then the following are true

1. $F \neq \emptyset$ (F at least has 2 elements)
2. 0 and 1 are unique
3. If $a + b = 0$, then b is unique write b as $-a$:
   if $a + b = a + c$, then

$$\begin{aligned} b &= b + 0 \\ &= b + (a + c) \\ &= (b + a) + c \\ &= (a + b) + c \\ &= 0 + c \\ &= c \end{aligned}$$

4. if $a + b = a + c$, then $b = c$
5. if $a \neq 0$ and $ab = 1 = ba$, then b is unique write a^{-1} for b.
6. $0 \cdot a = 0 \forall a \in F$

$$0 \cdot a + 0 \cdot a = (0 + 0) \cdot a = 0 \cdot a = 0 \cdot a + 0$$

so $0 \cdot a = 0$ by 3.

7. if $a \cdot b = 0$, then $a = 0$ or $b = 0$. If $a \neq 0$, then $0 = a^{-1}(ab) = (a^{-1}a)b = 1b = b$
8. if $a \cdot b = a \cdot c,\ a \neq 0$, then $b = c$
9. $(-a)(-b) = ab$
10. $-(-a) = a$
11. if $a \neq 0$, then $a^{-1} \neq 0$ and $(a^{-1})^{-1} = a$

Example 1.3

$$\begin{aligned}&\mathbb{Q} := \left\{\frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0\right\}\\ &\mathbb{R} := \text{set of real no.}\\ &\mathbb{C} := \{a + bi | a, b \in \mathbb{R}\} \text{ with}\end{aligned}$$

$$\begin{aligned}(a + b\sqrt{-1} + (c + d\sqrt{-1}) &= (a + c) + (b + d)\sqrt{-1}\\ (a + b\sqrt{-1}) \cdot (c + d\sqrt{-1}) &= (ac - bd) + (ad + bc)\sqrt{-1}\end{aligned}$$

$\forall a, b, c, d \in \mathbb{R}$
Under usual $+, \cdot$ of C

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

are all field and we say $\mathbb{Q}$ is a subfield of $\mathbb{R}$, $\mathbb{Q}, \mathbb{R}$ subfield of $\mathbb{C}$, i.e., they have the same $+, \cdot, 0, 1$.
$\mathbb{Z}$ is <u>not</u> a field as $\nexists n \in \mathbb{Z} \ni 2n = 1$, so $\mathbb{Z}$ do not satisfy (M3).

<u>*Note*</u>:To show something is FALSE, we need only one COUNTER-EXAMPLE. To show something is TRUE, one needs to show true for <u>all</u> elements – not just example.

§2 Lec 2: Oct 5, 2020

§2.1 Field(Cont'd)

<u>*Note*</u>: $\mathbb{Z}$ does satisfy the weaker properly if $a, b \in \mathbb{Z}$ then
(M3') if $ab = 0$ in $\mathbb{Z}$, then $a = 0$ or $b = 0$ and all other axioms except M3 hold

1. Let $F = \{0, 1\}$, $0 \neq 1$. Define $+, \cdot$ by following table Then F is a field.

Table 1: ADDITION

$+$	0	1
0	0	1
1	1	0

Table 2: MULTIPLICATION

$\cdot$	0	1
0	0	0
1	0	1

2. $\exists$ fields with n elements for

$$n = 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, \ldots$$

[conjecture?]

3. Let F be a field

$$F[t] := \{(\text{formal polynomial in one variable}\}$$

with t, given by

$$(a_0 + a_1t + a_2t^2 + \ldots) + (b_0 + b_1t + b_2t^2 + \ldots) := (a_0 + a_1) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \ldots$$
$$(a_0 + a_1t + a_2t^2 + \ldots) \cdot (b_0 + b_1t + b_2t^2 + \ldots) := a_0b_0 + (a_0b_1 + a_1b_0)t + \ldots$$

<u>*Note*</u>: $f, g \in F[t]$ are EQUAL iff they have the same COEFFICIENTS(coeffs) for each t^i (if t^i does not occur we assume its coeff is 0.) $F[t]$ is <u>not</u> a field but satisfy all axioms except (M3) but it does satisfy (M3') (compare $\mathbb{Z}$). Let

$$F(t) := \left\{\frac{f}{g} | f, g \in F[t], g \neq 0\right\} \quad \text{with}$$

- $\frac{f}{g} = \frac{h}{k}$ if $fk = gh$
- $\frac{f}{g} + \frac{h}{k} := \frac{fk+gh}{gk} \quad \forall f, g, h, k \in F[t]$
- $\frac{f}{g} \cdot \frac{h}{k} := \frac{fh}{gk} \quad g \neq 0, \quad k \neq 0$

is a field, the FIELD of RATIONAL POLYS over F.

<u>*Note*</u>:the 0 in $F[t]$ is $\frac{0}{f}$, $\quad f \neq 0$, and 1 in $F[t]$ is $\frac{f}{f}$, $f \neq 0$.

4. let F be a field.

$$M_nF := \{A | A \text{an} n \times n \text{matrix entries in} F\}$$

usual $+, \cdot$ of matrices, i.e. for $A, B \in M_nF$, let

$$A_{ij} := ij^{\text{th}} \text{ entry of A, etc}$$

Then

$$(A + B)_{ij} := A_{ij} + B_{ij}$$
$$(AB)_{ij} := C_{ij} := \sum_{k=1}^{n} A_{ik}B_{kj} \quad \forall i, j$$

<u>*Note*</u>: $A = B$ iff $A_{ij} = B_{ij} \ \ \forall i, j$.

If $n = 1$, then

F and M_1F and the "same" so M_1F is a field. If $n > 1$ then M_nF is not a field nor does it satisfy (M3), (M4), (M3'). It does satisfy other axioms with

$$I = I_n \coloneqq \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}, \quad 0 = 0_n \coloneqq \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

§2.2 Vector Space

$\mathbb{R}^2 \coloneqq \{(x,y)|x,y \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R}$ Vector in $\mathbb{R}^2$ are added as above and if $v \in \mathbb{R}^2$ is a vector,

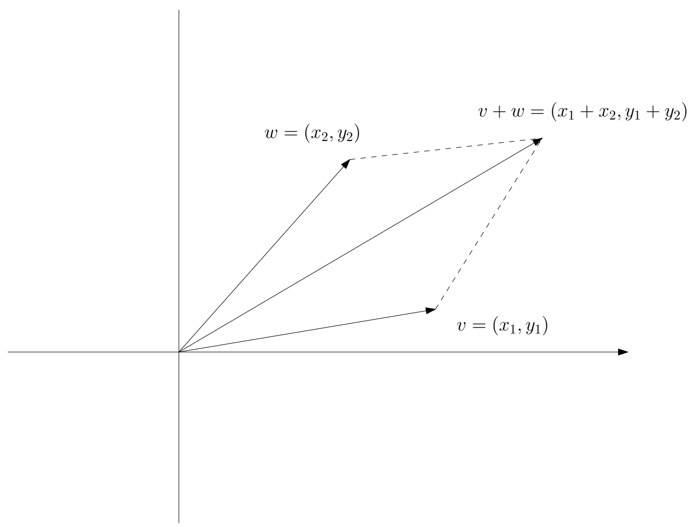


Figure 1: Geometry in $\mathbb{R}^2$

αv makes sense $\forall \alpha \in F$ by $\alpha(x,y) = (\alpha x, \alpha y)$ called SCALAR MULTIPLICATION. For $+$, scalar mult and $(0,0)$ is the ZERO VECTOR satisfying various axioms. e.g., assoc, comm, "distributive law…". To abstractify this

Definition 2.1 (Vector Space) — V is a vector space over F, via $+, \cdot$ or $(V, +, \cdot)$ is a vector space over F where

$$\begin{array}{cc} + : V \times V \to V & \cdot : F \times V \to V \\ \text{Addition} & \text{Scalar Multiplication} \\ \text{write:} v + w := +(v, w) & \text{write:} \alpha \cdot v := \cdot(\alpha, v) \ \text{ or } \ \alpha v \end{array}$$

if the following axioms are satisfied

$$\forall v, v_1, v_2, v_3 \in V, \quad \forall \alpha, \beta \in F$$

1. $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$
2. $\exists$ an element $0 \in V \ni \quad v + 0 = v = 0 + v$
3. (2) holds and the element $(-1)v$ in V satisfies

   $$v + (-1)v = 0 = (-1)v + v$$

   or (2) holds and $\forall v \in V \exists w \in V \ni \ v + w = 0 = w + v$
4. $v_1 + v_2 = v_2 + v_1$
5. $1 \cdot v = v$
6. $(\alpha \cdot \beta) \cdot v = \alpha(\beta \cdot v)$
7. $(\alpha + \beta)v = \alpha v + \beta v$
8. $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$

Elements of V are called vector, elements of F scalars .

Comments: V: a vector space over F

1. The zero of F is unique and is a scalar. The zero of V is unique and is a vector. They are different (unless $V = F$) even if we write 0 for both – should write $0_F, 0_V$ for the zero of F, V respectively.
2. if $v, w \in V, \alpha \in F$ then

   $$\begin{array}{rl} \alpha v + w & \text{makes sense} \\ v\alpha, vw & \text{do not make sense} \end{array}$$

3. We usually write

   vector using Roman letter

   scalar using Greek letter

   exception things like $(x_1, \dots, x_n) \in \mathbb{R}^n, x_i \in \mathbb{R} \forall i$
4. $+ : V \times V \to V$ says

   $$\text{if } \ v, w \in V, \quad \text{then } \ v + w \in V$$

write $v, w \in V \underbrace{\rightarrow}_{\text{implies}} v + w \in V$. We say V is CLOSED under +

5. $\cdot : F \times V \to V$ says $\alpha \in F, v \in V \to \alpha v \in V$. We say V is CLOSED under SCALAR MULTIPLICATION.

Example 2.2

F a field, e.g., $\mathbb{R}$ or $\mathbb{C}$

1. F is a vector space over F with $+, \cdot$ of a field, i.e., the field operation are the vector space operation with $0_F = 0_V$.

2. $F^n := \{\alpha_1, \ldots, \alpha_n\} \, | \alpha_i \in F \forall i$ is a vector space over F under COMPONENT-WISE OPERATION and
$$0_{F^n} := (0, \ldots, 0)$$
Even have
$$F^{\infty}_{\text{finite}} = \{(\alpha_1, \ldots, \alpha_n, \ldots\} \, |\alpha_i \in F \forall i \text{ with only FINITELY MANY } \alpha_i \neq 0$$

3. Let $\alpha < \beta$ in $\mathbb{R}$
$$I = [\alpha, \beta]\,, \quad (\alpha, \beta), \quad [\alpha, \beta), \quad (\alpha, \beta]$$
including ($\alpha = -\infty, \beta = \infty$). Let fxn $I := \{f : I \to \mathbb{R} | f \text{ a fxn}\}$ called the SET of REAL VALUE FXNS on I.

Define $+, \cdot$ as follows: $\forall f, g \in \text{Fxn } I$,
$$\begin{aligned} &f + g \quad \text{by } (f+g)(x) := f(x) + g(x) \\ &\alpha f \quad \text{by } (\alpha f)(x) := \alpha f(x) \quad \forall \alpha \in \mathbb{R} \end{aligned}$$
and 0 by $0(\alpha) = 0 \forall \alpha \in F$. Then Fxn I is a vector space over $\mathbb{R}$.

§3 Lec 3: Oct 7, 2020

§3.1 Vector Space(Cont'd)

Example 3.1

F is a field, e.g. $\mathbb{R}$ or $\mathbb{C}$

1. F is a vector space over F with $+, \cdot$ of a field, i.e. the field operation are the vector space operation with $0_F = 0_V$.

2. $F^n := \{(\alpha_1, \ldots, \alpha_n) | \alpha_i \in F \forall i\}$ is a vector space over F under COMPONENT-WISE OPERATIONS
$$(\alpha_1, \ldots, \alpha_n) + (\beta_1, \ldots, \beta_n) := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$$
$$\beta(\alpha_1, \ldots, \alpha_n) := (\beta\alpha_1, \ldots, \beta\alpha_n)$$

with $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in F$ and $0_{F^n} \coloneqq (0, \ldots, 0)$.

Even have:

$$F^\infty = F^\infty_{\text{this}} : \{(\alpha_1, \ldots, \alpha_n, \ldots) | \alpha_i \in F \forall i \text{ with only FINITELY MANY } \alpha_i \neq 0\}$$

3. Let $\alpha < \beta$ in $\mathbb{R}$
$$I = [\alpha, \beta], \quad (\alpha, \beta), \quad [\alpha, \beta), \quad (\alpha, \beta]$$
(including $\alpha = -\infty, \beta = \infty$. Let function $I \coloneqq \{f : I \to \mathbb{R} | f \text{ a function}\}$

   Define $+, \cdot$ as follows: $\forall f, g \in$ Fxn I,

$$\begin{aligned} f + g \quad &\text{by} \quad (f+g)(x) \coloneqq f(x) + g(x) \\ \alpha f \quad &\text{by} \quad (\alpha f)(x) \coloneqq \alpha f(x) \quad \forall \alpha \in \mathbb{R} \end{aligned}$$

   and 0 by $0(\alpha) = 0 \forall \alpha \in F$. Then Fxn I is a vector space over $\mathbb{R}$.

   Using this, we get subsets which are also vector space over $\mathbb{R}$ with same $+, \cdot, 0$.

   - $C(I) \coloneqq \{f \in \text{ fxn } I | f \text{ continuous on } I\}$
   - Diff $(I) \coloneqq \{f \in \text{ fxn } I | f \text{ differentiable on } I\}$
   - $C^n(I) \coloneqq \left\{f \in \text{ fxn } I | f(n) \text{ then}^{\text{th}} \text{ derivative of } f \text{ and f exists on I and is cont on I}\right\}$
   - $C^\infty(I) \coloneqq \{f \in \text{ fxn } I | f(n) \text{ exists} \forall n \geq 0 \text{ on I and is cont}\}$
   - $C^\omega(I) \coloneqq \{f \in \text{ fxn } I | \text{ f converges to its Taylor Series}\}$
     (in a neighborhood of every $x \in I$ – be careful at boundary points)
   - Int $(I) \coloneqq \{f \in \text{ fxn } I | f \text{ is integrable on } I\}$

4. $F[t]$ the set of polys, coeffs in F old $+, \cdot$ with scalar mult
$$\alpha(\alpha_0 + \alpha_1 t + \ldots + \alpha_n t^n) \coloneqq \alpha\alpha_0 + \alpha\alpha_1 t + \ldots + \alpha\alpha_n t^n$$

5. $\underbrace{F[t]_n}_{\text{truncating} F[t]} \coloneqq \{0 \in F[t]\} \cup \{f \in F[t] | \deg f \leq n\}$ (not closed under $\cdot$ of polys) where deg f = the highest power of t occurring non-trivially in f if $f \neq 0$ is a vector space over F with $+$, scalar mult,0.

Example 3.2 1. $F^{m \times n} \coloneqq$ set of $m \times n$ matrices entries in F where $A \in F^{m \times n}$, $A_{ij} = ij^{\text{th}}$ entry of A

$$\begin{aligned} (A+B)_{ij} &\coloneqq A_{ij} + B_{ij} \in F \qquad \forall A, B \in F^{m \times n} \\ (\alpha A)_{ij} &\coloneqq \alpha A_{ij} \in F \qquad \forall \alpha \in F \\ 0 &= \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix} (\text{ m rows and n columns}) \end{aligned}$$

COMPONENTWISE OPERATION! Then $F^{m \times n}$ is a vector space over F, e.g.

M_nF is a vector space over F.

Example to GENERALIZE

Let V be a vector space over F, $\emptyset \neq S$ a set. Set $W := \{f : S \to V | f \text{ a map}\}$. Define $+, \cdot$ on W by

$$\begin{aligned} f+g &\quad (f+g)(s) := f(s) + g(s) \in V \\ \alpha f &\quad (\alpha f)(s) := \alpha(f(s)) \in V \\ 0_W &\quad 0(s) = 0_V \quad \text{ZERO FUNCTION} \end{aligned}$$

$\forall f, g \in W; \alpha \in F; s \in S$. Then W is a vector space over F.(of componentwise operation)

2. Let $F \subset K$ be a fields under $+, \cdot$ on K. Same 0,1, i.e. F is a SUBFIELD of k e.g. $\mathbb{R} \subset \mathbb{C}$. Then K is a vector space over F by RESTRICTION of SCALARS.

   i.e., $+ = +$ on K. With scalar mult, $F \times K \to K$ by

$$\underbrace{\alpha v}_{\text{in K as a vector space over } F} = \underbrace{\alpha v}_{\text{in K as a field}} \quad \forall \alpha \in F \quad \forall v \in V$$

   e.g. $\mathbb{R}$ is a vector space over $\mathbb{Q}$ by $\frac{m}{n} r = \frac{mr}{n}, \quad m, n \in \mathbb{Z}, n \neq 0, r \in \mathbb{R}$. More generally, let V be a vector space over K, $F \subset K$ subfield, then it is a vector space over F by RESTRICTION of SCALARS.

$$\cdot|_{F \times V} : F \times V \to V$$

   e.g., K^n is a vector space over F (e.g. $\mathbb{C}^n$ is a vector space over $\mathbb{R}$).

Properties of Vector Space: Let V be a vector space over F. Then $\forall \alpha, \beta \in F, \quad \forall v, w \in V$, we have

1. The zero vector is unique write 0 or 0_V.
2. $(-1)v$ is the unique vector $w \ni w + v = 0 = v + w$ write $-v$.
3. $0 \cdot v = 0$
4. $\alpha \cdot 0 = 0$
5. $(-\alpha)v = -(\alpha v) = \alpha(-v)$
6. if $\alpha v = 0$, then either $\alpha = 0$ or $v = 0$
7. if $\alpha v = \alpha w$, $\alpha \neq 0$, then $v = w$
8. if $\alpha v = \beta v$, $v \neq 0$, then $\alpha = \beta$
9. $-(v + w) = (-v) + (-w) = -v - w$
10. can ignore parentheses in $+$

§3.2 Subspace

Definition 3.3 (Subspace) — Let V be a vector space over F, $W \subset V$ a subset. We say W is a subspace of V if W is a vector space over F with the operation $+, \cdot$ on V, i.e., $(V, +, \cdot)$ is a vector space over F, via $+ : V \times V \to V$ and $\cdot : F \times V \to V$ then W is a vector space over F via

- $+ = +/_{W \times W} : W \to W$: restrict the domain to $W \times W$
- $\cdot = \cdot|_{F \times W} : F \times W \to W$: restrict the domain to $F \times W$

  i.e. W is closed under $+, \cdot$ from V, $\forall^{w_1}_{w_2} \in W \quad \forall \alpha \in F, \quad w_1 + w_2 \in W$ and $\alpha w_1 \in W$ and $0_W = 0_V$.

Theorem 3.4 (Subspace)
Let V be a vector space over F, $\emptyset \neq W \subset V$ a subset. Then the following are equivalent:

1. W is a subspace for V
2. W is closed under $+$ and scalar mult from V
3. $\forall w_1, w_2 \in W$, $\forall \alpha \in F$, $\alpha w_1 + w_2 \in W$

Proof. Some of the implication are essentially ??
1) $\to$ 2) : by def. W is a subspace of V under $+, \cdot$ on V (and satisfies the axioms of a vector space over F) as $0_V = 0_W$.
2) $\to$ 1) claim: $0_V \in W$ and $0_W = 0_V$: As $\emptyset \neq W \exists w \in W$
By 2)$(-1)w \in W$, hence $0_V = w + (-w) \in W$. Since $0_V + w' = w' = w' + 0_V$ in V $\forall w' \in W$, the claim follows. The other axioms hold for elements of V hence for $W \subset V$.
2) $\to$ 3) : let $\alpha \in F$, $w_1, w_2 \in W$. As 2) holds, $\alpha w_1 \in W$ hence also $\alpha w_1 + w_2 \in W$
3) $\to$ 2) Let $\alpha \in F$, $w_1, w_2 \in W$. As above and 3)

$$0_V = w_1 + (-w_1) \in W \quad \text{and } 0_V = 0_W$$

Therefore,

$$w_1 + w_2 = 1 \cdot w_1 + w_2 \in W \quad \text{and } \alpha w_1 + \alpha w_1 + 0_V \in W$$

by 3). $\square$

Note:Usually 3) is the easiest condition to check. WARNING: must subsets of a vector space over F are NOT subspace.

Example 3.5
V a vector space over F.

1. $0 := \{0_V\}$ and V are subspace of V

2. Let $I \subset \mathbb{R}$ be an interval (not a point) then

$$\begin{gathered} C^\omega(I) < C^\infty(I) < \ldots < C^n(I) < \ldots < C'(I) \\ < \text{Diff I} < C(I) < \text{Int I} < \text{Fxn I} \end{gathered}$$

are subspaces of the vector space containing then... where we write

$$A < B \quad \text{if} \quad A \subset B \quad \text{and } A \neq B$$

3. Let F be afield, e.g $\mathbb{R}$. Then $F = F[t]_0 < F[t]_1 < \ldots < F[t_n] < \ldots < F[t]$ are vector space over F each a subspace of the vector space over F containing it.

4. If $W_1 \subset W_2 \subset V$, W_1, W_2 subspace of V,then $W_1 \subset W_2$ is a subspaces.

5. If $W_1 \subset W_2$ is a subspace and $W_2 \subset V$ is a subspace, then $W_1 \subset V$ is a subspace.

6. Let $W \coloneqq \{(0, \alpha_1, \ldots, \alpha_n | \alpha_i \in F, \quad 2 \leq i \leq n\} \subset F^n$ is a subspace, but $\{(1, \alpha_2, \ldots, \alpha_n | \alpha_i \in F, \quad 2 \leq i \leq n\}$ is not. Why?

7. Every line or plane through the origin in $\mathbb{R}^3$ is a subspace.

§4 Lec 4: Oct 9, 2020

§4.1 Span & Subspace

Definition 4.1 (Linear Combination) — Let V be a vector space over F, $v_1, \ldots, v_n \in V$ we say $v \in V$ is a LINEAR COMBINATION of $v_1, \ldots, v_n$ if $\exists \alpha_1, \ldots, \alpha_n \in F \ni v = \alpha v_1 + \ldots + \alpha_n v_n$.

Let

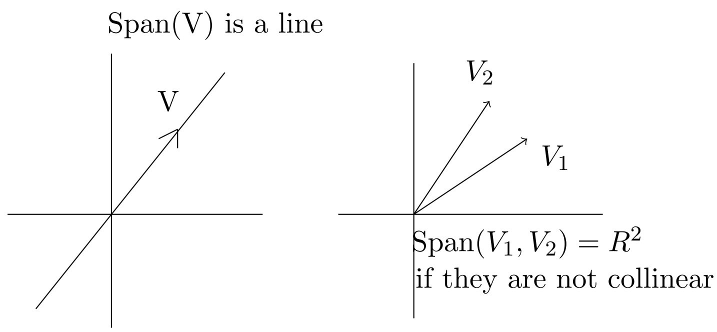
$$\text{Span}(v_1, \ldots, v_n) \coloneqq \{ \text{ all linear combos of} v_1, \ldots, v_n \}$$

Let $v_1, \ldots, v_n \in V$. Then

$$\text{Span}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^n \alpha_i v_i | \alpha_1, \ldots, \alpha_n \in F \right\}$$

is a subspace of V (by the Subspace Theorem) called the SPAN of $v_1, \ldots, v_n$. It is the (unique) smallest subsapce of V containing $v_1, \ldots, v_n$.

i.e., if $W \subset V$ is a subspace and $v_1, \ldots, v_n \in W$ then $\text{Span}(v_1, \ldots, v_n) \subset W$. We also let Span $\emptyset \coloneqq \{0_V\} = 0$, the smallest vector space containing no vectors.



Question: If we view $\mathbb{C}$ as a vector space over $\mathbb{R}$, then $\mathbb{R}$ is a subspace of $\mathbb{C}$, but if we view $\mathbb{C}$ is a vector space over $\mathbb{C}$, then $\mathbb{R}$ is not a subspace of $\mathbb{C}$ (why? What's going on?) – not closed under operation(s).

Definition 4.2 (Span) — Let V be a vector space over $F, \emptyset \neq S \subset V$ a subset. Then, Span S := the set of all FINITE linear combos of vectors in S. i.e., if $V \in$ Span S, then

$$\exists v_1, \ldots, v_n \in S, \quad \alpha_1, \ldots, \alpha_n \in F \ni v = \alpha_1 v_1 + \ldots + \alpha_n v_n$$

Span $S \subset V$ is a subspace. What is Span V?

Example 4.3

1. Let $V = \mathbb{R}^3$.

$$\text{Span}(i+j, i-j, k) = \text{Span}V = \text{Span}(i, j, i+j, k) = \text{Span}(i+j, i-j, k+i)$$

2. Define
$$\text{Symm}_n F := \left\{ A \in M_n F | A = A^\top \right\}$$
Recall: $A^\top$ is the transpose of A, i.e.,
$$(A^\top)_{ij} := A_{ji} \quad \forall i, j$$
is a subspace of $M_n F$

3.
$$V = \left\{ \begin{pmatrix} a & c+di \\ c-di & b \end{pmatrix} | a, b, c, d \in \mathbb{R} \right\} \subset M_2 C$$
is NOT a subspace as a vector space over $\mathbb{C}$,eg,
$$i \begin{pmatrix} a & c+di \\ c-di & b \end{pmatrix} = \begin{pmatrix} ai & -d+ci \\ d+ci & bi \end{pmatrix}$$
does not lie in V if either $a \neq 0$ or $b \neq 0$ (cannot be imaginary). Also V is not a subspace of $M_2\mathbb{R}$ as a vector space over $\mathbb{R}$ as $V \not\subset M_2\mathbb{R}$. $V \subset M_2\mathbb{C}$ is a subspace as a vector space over $\mathbb{R}$.

4. (Important computational example) Fix $A \in F^{m\times n}$. Let

$$\ker A := \left\{ x \in F^{n\times 1} | Ax = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \text{ in } F^{m\times 1} \right\}$$

called the KERNEL or NULL SPACE of A. Ker $A \subset F^{n\times 1}$ is a subspace and it is the SOLUTION SPACE of the system of m linear equations in n unknowns. – which we can compute by Gaussian elimination.

5. Let $W_i \subset V_i, i \in \underbrace{I}_{\text{indexing set}}$ be subspaces. Then $\bigcap_I W = \bigcap_{i\in I} W_i := \{x \in V | x \in W_i \quad \forall i \in I\}$ is a subspaces of V (why?)

6. In general, if $W_1, W_2 \subset V$ are subspaces, $W_1 \cup W_2$ is NOT a subspace.

   e.g., Span(i) $\cup$ Span(j) $= \{(x,0)|x \in \mathbb{R}\} \cup \{(0,y)|y \in \mathbb{R}\}$ is not a subspace

$$(x,y) = (x,0) + (0,y) \notin \quad \text{Span(i)} \cup \text{Span(j)}$$

   if $x \neq 0$ and $y \neq 0$

Definition 4.4 (Subspace & Span) — Let $W_1, W_2 \subset V$ be subspaces. Define

$$\begin{aligned} W_1 + W_2 &:= \{w_1 + w_2 | w_1 \in W_1, w_2 \in W_2\} \\ &= \text{Span}(W_1 \cup W_2) \end{aligned}$$

So $w_1 + w_2 \subset V$ is a subspace and the smallest subsapce of V containing W_1 and W_2.

More generally, if $W_i \in V$ is a subspace $\forall i \in I$ let

$$\sum_I W_i = \sum_{i\in I} W_i := +W_i := \text{Span}(\bigcup_I W_i)$$

the smallest subspace of V containing $W_i \forall i \in I$. What do elements in $\sum_I W_i$ look like? Determine the span of vector $v_1, \ldots, v_n$ in $\mathbb{R}^n$

Suppose $v_i = (a_{i_1}, \ldots, a_{ni}, i = 1, \ldots, n$. To determine when $w \in \mathbb{R}^n$ lies in Span$(u_1, \ldots, u_n$) i.e., if $w = (b_1, \ldots, b_n) \in \mathbb{R}^n$ when does

$$w = \alpha_1 v_1 + \ldots + \alpha_n v_n, \qquad \alpha_1, \ldots, \alpha_n \in \mathbb{R}$$

What v_i is an $n \times 1$ column matrix $\begin{pmatrix} \alpha_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$

$$A = (a_{ij}), \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

view w as $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. To solve

$$Ax = B, \qquad X = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

is equivalent to finding all the $n \times 1$ matrices B (actually $B^\top$) s.t.

$$Ax = B$$

when the columns of A are the $v_i(v_i^\top)$.
Note: If $m = n$ an A is invertible then all B work.

§4.2 Linear Independence

We know that $\mathbb{R}^n$ is an n-dimensional vector space over $\mathbb{R}$. Since we need n coordinates (axes) to describe all vector in $\mathbb{R}^n$ but no fewer will do.
We want something like the following:

Let V be a vector space over F with $V \neq \emptyset$. Can we find distinct vectors $v_1 \ldots, v_n \in V$, some n with following properties

1. $V = \text{Span}(v_1, \ldots, v_n\ )$
2. No v_i is a linear combos of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ (i.e. we need them all)

Then we want to call V an n-DIMENSIONAL VECTOR SPACE OVER F.

Lemma 4.5

Let V be a vector space over F, $n > 1$. Suppose $v_1, \ldots, v_n$ are distinct. Then (2) is equivalent to

$$\text{If } \alpha_1 v_1 + \ldots + \alpha_n v_n = \beta_1 v_1 + \ldots + \beta_n v_n, \quad \alpha_i, \beta_i \in F \forall i, j$$

i.e. the "coordinates" are unique.

Proof. $(- >)$ If not, relabelling the $v_i's$, we may assume that $\alpha_1 \neq \beta_2$ in(*), then

$$(\alpha_1 - \beta_1)v_1 = \sum_{i=2}^{n} (\beta_i - \alpha_i)v_i$$

As $\alpha_1 - \beta_1 \neq 0$ in F, a field, $(\alpha_1 - \beta_1)^{-1}$ exists, so

$$v_1 = \sum_{i=2}^{n} (\alpha_1 - \beta_1)^{-1}(\beta_i - \alpha_i)v_i \in \ \text{Span}(v_1, \ldots, v_n)$$

a contradiction.
$(< -)$ Relabelling, we may assume that

$$v_1 = \alpha_2 v_2 + \ldots + \alpha_n v_n, \quad \text{some } \alpha_i \in F$$

Then,

$$1 \cdot v_1 + 0v_2 + \ldots + 0v_n = v_1 = 0 \cdot v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

so $1 = 0$, a contradiction. □

Remark 4.6. The case $n = 1$ is special because there are two possibilities
Case 1: $v \neq 0$: then $\alpha v = \beta v \to \alpha = \beta$
Case 2: $v = 0$: then $\alpha v = \beta v \forall \alpha, \beta \in F$

So the only time the above lemma is false is when $n = 1$ and $v = 0$. We do not want to say this, so we use another definition.

§5 Lec 5: Oct 12, 2020

§5.1 Linear Independence(Cont'd)

Definition 5.1 (Linear Independence & Dependence) — Let V be a vector space over F, $v_1, \ldots, v_n$ in V all distinct. We say $\{v_1, \ldots, v_n\}$ is LINEARLY DEPENDENT if $\exists \alpha_1, \ldots, \alpha_n \in F$ not all zero $\ni$

$$\alpha_1 v_1 + \ldots + \alpha_n v_n = 0$$

and $\{v_1, \ldots, v_n\}$ is LINEARLY INDEPENDENT if it is NOT linearly dependent, i.e., if for any eqn

$$0 = \alpha v_1 + \ldots + \alpha_n v_n, \quad \alpha_1, \ldots, \alpha_n \in F,$$

then $\alpha_i = 0 \forall i$, i.e., the only linear comb of $v_1, \ldots, v_n$ – the zero vector is the TRIVIAL linear combo (we shall also say that distinct $v_1, \ldots, v_n$ are linearly independent if $\{v_1, \ldots, v_n\}$ is. More generally, a set $\emptyset \neq S \subset V$ is called LINEARLY DEPENDENT if for some FINITE subset (of distinct elements of S) of S is linearly dependent and it is called LINEARLY INDEPENDENT if every FINITE subset of S (of distinct elements) is linearly independent.
We say $v_i, i \in F$, all distinct are LINEARLY INDEPENDENT if $\{v_i\}_{i \in I}$ is linearly independent and $v_i \neq v_j \forall i, j \in I, i \neq j$.

Remark 5.2. Let V be a vector space over F, $\emptyset \neq S \subset V$ a subset

1. If $0 \in S$, then S is linearly dependent as $l \cdot 0 = 0$
2. distinct: $v_1, \ldots, v_n$ in V are linearly independent iff
   - no $v_i = 0$
   - $\alpha_1 v_1 + \ldots + \alpha_n v_n = \beta_1 v_1 + \ldots + \beta_n v_n, \quad \alpha_i, \beta_i \in F$ implies $\alpha_i = \beta_i \forall i$

   Note: v, v are linearly dependent if we allow repetitions – and $\{v, v\} = \{v\}$.

For homework, make sure to show this:

Suppose $v_1, \ldots, v_n$ are distinct, $n > 2$, no $v_i = 0$. Suppose no v_i is a scalar multiple of another v_j, $j \neq i$. It does not follow that $v_1, \ldots, v_n$ are linearly independent (in general).

Example 5.3 (counter-example)

$$(1,0),(0,1),(1,1) \text{ in } \quad V = \mathbb{R}^2$$

$(1,0),(0,1)$ are linearly indep. but not $(1,0),(0,1)$, and $(1,1)$.

Remark 5.4. Let $\emptyset \neq T \subset S$ be a subset. If T is linearly dependent, so is S. Then the contraposition is also true: if S is linearly indep., so is T.

More remarks:

1. Let $0 \neq v \in V$. Then $\{v\}$ is linearly independent and
$$Fv \coloneqq \operatorname{Span}(v)$$
is called a LINE in V:
$$\alpha v = 0 \to \alpha = 0$$

2. $u, v, w \in V \setminus \{0\}$ and $v \notin \operatorname{Span}(w\ )$ (equivalently, $w \notin \operatorname{Span}(v\ )$, then $\{v, w\}$ is linearly indep. and span(v, w) is called a PLANE in V.

3. $(1,1),(-2,-2)$ are linearly dep. in $\mathbb{R}^2$.

4. $(1,1),(2,-2)$ are linearly indep. in $\mathbb{R}^2$ (show coefficients are equal to each other and to 0).

5. More generally,
$$v_i = (a_{i_1}, \ldots, a_{i_n}) \text{ in } \mathbb{R}^n, \quad i = 1, \ldots, m \text{ (distinct)}$$
Then
$$\exists \alpha_1, \ldots, \alpha_m \in \mathbb{R} \text{ not all } 0 \ \ni \quad \alpha_1 v_1 + \ldots + \alpha_m v_m = 0$$
iff $v_1, \ldots, v_m$ are linearly dep – iff $\exists \alpha_1, \ldots, \alpha_m \in \mathbb{R}$ not all 0 s.t.
$$\alpha_1(a_{11}, \ldots, a_{1m}) + \ldots + \alpha_m(a_{m1}, \ldots, a_{mn}) = 0$$
iff the matrix
$$A = \begin{pmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & & \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$$
with rows v_i row reduced to echelon form with a zero row. Also,
$$B = A^\top = \begin{pmatrix} a_{11} & \ldots & a_{m1} \\ \vdots & & \\ a_{1m} & & a_{mn} \end{pmatrix}$$
i.e., write the vectors v_i as columns then
$$\underbrace{B}_{n \times m} \underbrace{X}_{m \times 1} = 0$$

has a NON-TRIVIAL solution, i.e.,

$$\ker B \neq 0$$

where

$$\ker B := \left\{X \in F^{m\times 1} | BX = 0\right\}$$

the kernel of B.

6. Let $f_1, \ldots, f_n \in C^{n-1}(I), \quad I = (\alpha, \beta), \alpha < \beta$ in $\mathbb{R}$ and

$$\alpha_1 f_1 + \ldots + \alpha_n f_n = \underbrace{0}_{\text{the zero func}}$$

i.e., $(\alpha_1 f_1 + \ldots + \alpha_n f_n)(x) = 0 \ \ \forall x \in (\alpha, \beta)$. Taking the derivatives $(n-1)$ times and put them in matrix form, we have

$$\begin{pmatrix} f_1 & \ldots & f_n \\ f_1' & \ldots & f_n' \\ \vdots & \ldots & \vdots \\ f_1^{n-1} & \ldots & f_n^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

In particular, the Wronskian of $f_1, \ldots, f_n$ is not the zero func, i.e., $\exists x \in (\alpha, \beta) \ni W(f_1, \ldots, f_n)(x) \neq 0$. This means that the matrix above is invertible for some $x \in (\alpha, \beta)$. Then, $\alpha_1 = 0, \ldots, \alpha_n = 0$ by Cramer's rule – only the trivial soln.

Conclusion: $W(f_1, \ldots, f_n) \neq 0 \to \{f_1, \ldots, f_n\}$ is linearly indep.

WARNING: the converse is false.

Example 5.5 (of the conclusion)
Let $\alpha < \beta$ in $\mathbb{R}$.

1. $\sin x, \cos x$ are linearly indep. on (α, β).
2. We need some (sub) defns for this example.

For $x \in \mathbb{R}$, define the map

$$e_x : \mathbb{R}[t] \to \mathbb{R} \text{ by}$$

$g = \sum a_i t^i \mapsto g(x) := \sum a_i x^i$ called EVALUATION at x.

We call a map $f : \mathbb{R} \to \mathbb{R}$ (or some $f : I \to \mathbb{R}(I \subset \mathbb{R})$) a POLYNOMIAL FUNCTION if

$$\exists P_f = \sum_{i=1}^{n} a_i t^i \in \mathbb{R}[t]$$

and

$$f(x) = e_x P_f = P_f(x) = \sum_{i=1}^{n} a_i x^i \quad \forall x \in \mathbb{R}$$

i.e., the function arising from a (formal) polynomial by evaluation at each x. We let

$$\mathbb{R}[x] \coloneqq \{f : \mathbb{R} \to \mathbb{R} | f \text{ a poly fcn }\}$$

Note:Polynomial fcns are defined on all of $\mathbb{R}$. $\mathbb{R}[x]$ is a vector space over $\mathbb{R}$.

Warning: if we replace $\mathbb{R}$ by F, $F[t]$ may be "very different" from $F[x]$, e.g., let $F = \{0, 1\}$. Then

$$t, t^2 \in F[t], \quad t \neq t^2 \text{ but } P_t = P_{t^2}$$

Now we can give our example using Wronskians

$$\{1, x, \ldots, x^n\}$$

is linearly indep. on (α, β) assuming $\alpha < \beta$.
HOMEWORK: Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ be distinct, then

$$e^{\alpha_1 t}, \ldots, e^{\alpha_n t}$$

are linearly indep. on (α, β). THINK OVER IT!

Theorem 5.6 (Toss In)
Let V be a vector space over F, $\emptyset \neq S \subset V$ a linearly indep. subset. Suppose that $v \in V\setminus$ Span S. Then $S \cup \{v\}$ is linearly indep.

Proof. Suppose this is false which is $S \cup \{v\}$ is linearly dep. Then $\exists v_1, \ldots, v_n \in S$ and $\alpha, \alpha_1, \ldots, \alpha_n \in F$ some n not all zero s.t.

$$\alpha v + \alpha_1 v_1 + \ldots + \alpha_n v_n = 0$$

Case 1: $\alpha = 0$
Then $\alpha_1 v_1 + \ldots + \alpha_n v_n = 0$ not all $\alpha_1, \ldots, \alpha_n$ zero so $\{v_1, \ldots, v_n\}$ is linearly dep., a contradiction.
Case 2: $\alpha \neq 0$
Then α^{-1} exists.

$$v = -\alpha^{-1}\alpha_1 v_1 - \ldots - \alpha^{-1}\alpha_n v_n$$

is a linear combo of $v_1, \ldots, v_n$, i.e., $v \in$ Span $(v_1, \ldots, v_n)$ – a contradiction. Therefore, $S \cup \{v\}$ is linearly indep. □

Corollary 5.7
Let V be a vector space over F and $v_1, \ldots, v_n \in V$ linearly indep. if

$$\text{Span}(v_1, \ldots, v_n) < V$$

then $\exists v_{n+1} \in V \ni v_1, \ldots, v_n, v_{n+1}$ are linearly indep. and

$$\text{Span}(v_1, \ldots, v_n) < \text{Span}(v_1, \ldots, v_{n+1}) \subset V$$

Question 5.1. Why can't we get a linearly indep. set spanning any vector space over F using this theorem?

Ans: Certainly we may not get a finite set. We shall only be interested in the case, much of the time, when such a finite linearly indep. set spans our vector space over F.

Example 5.8

$(1,3,1) \in \mathbb{R}^3$ is linearly indep. but Span $(1,3,1) < \mathbb{R}^3$.
$(1,1,0) \notin$ Span $(1,3,1)$ so $(1,3,1),(1,1,0)$ are linearly indep. Similarly for $(0,0,1)$.
$\mathbb{R}^3 = \text{Span}((1,3,1),(1,1,0),(0,0,1))$

§6 Lec 6: Oct 14, 2020

§6.1 Bases

Definition 6.1 (Basis) — Let $\emptyset \neq V$ be a vector space over F. A BASIS B for V is a linearly indep. set in V and spans V. i.e.,

1. $V = \text{Span } B$.
2. B is linearly indep.

We say V is a FINITE DIMENSIONAL VECTOR SPACE OVER F if there exists B for V with finitely many elements, i.e., $|B| < \infty$.

Notation: If $V = 0$, we say V is a finite dimensional vector sapce over F of DIMENSION ZERO.
Goal: To show if V is finite dimensional vector space over F with bases B and b then $|B| = |b| < \infty$. This common integer is called the DIMENSION of V.

Example 6.2

Let V be a vector space over F, $S \subset V$ a linearly indep. set. Then S is a basis for Span S.
Warning: S is not a subspace just a subset.

Definition 6.3 (Ordered Basis) — If V is a finite dimensional vector space over F with a basis $B = \{v_1, \ldots, v_n\}$ we called it an ORDERED BASIS if the given order of $v_1, \ldots, v_n$ is to be used, i.e., the i^{th} vector in B is the i^{th} in the written list, e.g., $\{v_1, v_2, v_4, v_3, \ldots\}$ then v_4 is the 3^{rd} element in the ordered list if we want B to be ordered in this way.

Theorem 6.4 (Coordinate)
Let V be a finite dimensional vector space over F with basis $B = \{v_1, \ldots, v_n\}$ and $v \in V$. Then $\exists! \alpha_1, \ldots, \alpha_n \in F \ni v = \alpha_1 v_1 + \ldots + \alpha_n v_n$. We call $\alpha_1, \ldots, \alpha_n$ the COORDINATE of v relative to the basis B and call α_i the i^{th} coordinate relative to B.

Proof. Existence: By defn, $V = \text{Span } B$, so if $v \in V$

$$\exists \alpha_1, \ldots, \alpha_n \in F \ni v = \alpha_1 v_1 + \ldots + \alpha_n v_n$$

Uniqueness: Let $v \in V$ and suppose that $\alpha_1 v_1 + \ldots + \alpha_n v_n = \beta_1 v_1 + \ldots + \beta_n v_n$, for some $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in F$. Then

$$(\alpha_1 - \beta_1) v_1 + \ldots + (\alpha_n - \beta_n) v_n = 0$$

Since B is linearly indep,

$$\alpha_i = \beta_i = 0 \quad \text{for } i = 1, \ldots, n$$

□

Question 6.1. Does the above theorem hold if the basis B is not necessarily finite? If so prove it!

Exercise 6.1. Let V be a vector space over F, $v_1, \ldots, v_n \in V$ then

$$\text{Span}(v_1, \ldots, v_n) = \text{Span}(v_2, \ldots, v_n) \iff v_1 \in \text{Span}(v_2, \ldots, v_n)$$

Make sure to PROVE THIS

Note:For induction, you CAN'T assume n in the induction hypothesis is special in any way except it is greater than 1. Also, you can start induction at $n = 0$,i.e., show $P(0)$ true (or at any $n \in \mathbb{Z}$).

Theorem 6.5 (Toss Out)
Let V be a vector space over F. If V can be spanned by finitely many vector then V is a finite dimensional vector space over F. More precisely, if

$$V = \text{Span}(v_1, \ldots, v_n)$$

then a subset of $\{v_1, \ldots, v_n\}$ is a basis for V.

Proof. If $V = 0$, there is nothing to prove. So we may assume that $V \neq 0$. Suppose that $V = \text{Span}(v_1, \ldots, v_n)$. We can use induction on n and show a subset of $\{v_1, \ldots, v_n\}$ is a basis.

- $n = 1$: $V = \text{Span}(v_1) \neq 0$ as $V \neq 0$, so $v_1 \neq 0$. Hence $\{v_1\}$ is linearly indep and it is the basis.

- Assume $V = \text{Span}(w_1, \ldots, w_n\ )$ – the induction hypothesis – to be true. Then a subset of $w_1, \ldots, w_n$ is a basis for V. Now suppose that $v = \text{Span}(v_1, \ldots, v_{n+1})$. To show a subset of $\{v_1, \ldots, v_{n+1}\}$ is a basis for V, we need to show if $\{v_1, \ldots, v_{n+1}\}$ is linearly indep., then it is a basis for V and it spans V and we are done. So let us assume that $\{v_1, \ldots, v_{n+1}\}$ is linearly dep. Hence,

$$\exists \alpha_1, \ldots, \alpha_{n+1} \in F \text{ not all zero } \ni$$

$$\alpha_1 v_1 + \ldots + \alpha_{n+1} v_{n+1} = 0$$

Assume $\alpha_{n+1} \neq 0$, then

$$v_{n+1} = -\alpha_{n+1}^{-1} \alpha_1 v_1 - \ldots - \alpha_{n+1}^{-1} \alpha_n v_n$$

lies in $\text{Span}(v_1, \ldots, v_n)$. By the Exercise above,

$$V = \text{Span}(v_1, \ldots, v_{n+1}) = \text{Span}(v_1, \ldots, v_n)$$

By the induction hypo, a subset of $\{v_1, \ldots, v_n\}$ is a basis for V.

$\square$

Example 6.6 1. Let $e_i = \{(0, \ldots, 0, 1, 0, \ldots)\} \in F^n$

$$s = s_n \coloneqq \{e_1, \ldots, e_n\} \subset F^n$$

If $v \in F^n$, then

$$v = (\alpha_1, \ldots, \alpha_n) = \alpha_1 e_1 + \ldots + \alpha_n e_n$$

since $\alpha_i \in F$, so $F^n = \text{Span } s$. If $0 = \alpha_1 e_1 + \ldots + \alpha_n e_n = (\alpha_1, \ldots, \alpha_n) = (0, \ldots, 0)$, then $\alpha_i = 0 \forall i$. So s is linearly indep. Hence s is a basis for F^n called the standard basis. More generally, let

$$e_{ij} \in F^{m \times n} \text{ be the } m \times n \text{ matrix with all entries 0 except in the ith place.}$$

Then $s_{mn} \coloneqq \{e_{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for $F^{m \times n}$ called the STANDARD BASIS for $F^{m \times n}$ – same proof – everything is done componentwise.

2. $V = F[t] \coloneqq \{$ polys in t, coeffs in F.$\}$ $(F = \mathbb{R}\ )$. Let $f \in V$. Then, there exists $n \geq 0$ in $\mathbb{Z}$ and $\alpha_0, \ldots, \alpha_n$ in F s.t.

$$f = \alpha_0 + \alpha_1 t + \ldots + \alpha_n t^n$$

So $B = \{t^n | n \geq 0\} = \left\{1, t, t^2, \ldots\right\}$ spans V and by defn if

$$\alpha_0 + \alpha_1 t + \ldots + \alpha_n t^n = \underbrace{0}_{\text{zero poly}}$$

then $\alpha_i = 0$ for all i so B is linearly indep. Hence B is a basis for $F[t]$. B is not a finite set. We shall see that $F[t]$ is not a finite dimensional vector space over F.

How?

3. $F[t]_n := \{f \in F[t] | f = 0 \text{or} \deg f \le n\} \subset F[t]$ is spanned by $\{1, t, t^2, \dots, t^n\}$. It is a subset of linearly indep. set. $\{1, t, t^2, \dots\} = \{t^n | n \ge 0\}$ so also linearly indep. and therefore a basis.

4. $\{1, \sqrt{-1}\}$ is a basis for $\mathbb{C}$ as a vector space over $\mathbb{R}$. $\{1\}$ is a basis for C as a vector space over $\mathbb{C}$(indeed, if F is a field, F is a vector space over F and if $0 \neq \alpha \in F$, then α^{-1} exists and $x = x\alpha^{-1}\alpha \in \text{Span } F$ so $\{\alpha\}$ is a basis. e.g., $\{\pi\}$ is a basis for $\mathbb{R}$ as a vector space over $\mathbb{R}$).

5. $\{e^{-x}, e^{3x}\}$ is a basis for
$$V := \{f \in \mathbb{C}^2(-\infty, \infty) | f'' - 2f' - 3f = 0\}$$
a vector space over $\mathbb{R}$.

6. Given $v_1, \dots, v_n \in F^n$, you know how to find $W = \text{Span}(v_1, \dots, v_n)$. *Note*:If $m > n$ then rows reducing $A^\top$ must lead to a zero row so $v_1, \dots, v_m$ cannot be linearly indep. If $m = n$ we can see if
$$\det A^\top = 0 \quad (\text{or } \det \text{A} = 0)$$
then linearly dep. And if
$$\det A^\top \neq 0 \quad (\text{or } \det \text{A } \neq 0)$$
then linearly indep.

§7 Lec 7: Oct 16, 2020

§7.1 Replacement Theorem

Theorem 7.1 (Replacement)
Let V be a vector space over F, $\{v_1, \dots, v_n\}$ a basis for V. Suppose that $v \in V$ satisfies
$$v = \alpha_1 v_1 + \ldots + \alpha_n v_n, \qquad \alpha_1, \dots, \alpha_n \in F, \alpha_i \neq 0$$
Then
$$\{v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_n\}$$
is also a basis for V.

Proof. Changing notation, we may assume $\alpha_1 \neq 0$. To show $\{v_1, v_2, \dots, v_n\}$ is a basis for V, we have to show $\{v, v_2, \dots, v_n\}$ spans V. Since
$$v = \alpha_1 v_1 + \ldots + \alpha_n v_n, \quad \alpha_1 \neq 0$$
α_1^{-1} exists, so
$$v_1 = \alpha_1^{-1} v - \alpha_1^{-1}\alpha_2 v_2 - \ldots - \alpha_1^{-1}\alpha_n v_n$$

lies in $\text{Span}(v, v_2, \dots, v_n\ )$. By Exercise ...,

$$V = \text{Span}(v, v_1, \dots, v_n) = \text{Span}(v, v_2, \dots, v_n)$$

So $\{v, v_2, \dots, v_n\}$ spans V. Thus, $\{v, v_2, \dots, v_n\}$ is linearly indep.
Suppose $\exists \beta_1, \beta_2, \dots, \beta_n \in F$ not all $0 \ni$

$$\beta v + \beta_2 v_2 + \ldots + \beta_n v_n = 0$$

Case 1: $\beta = 0$
Then $\beta_2 v_2 + \ldots + \beta_n v_n = 0$ not all $\beta_i = 0$. So $\{v_2, \dots, v_n\}$ is linearly dep., a contradiction.
Case 2: $\beta \neq 0$, so β^{-1} exists.
Then using (*), we see

$$v = 0 \cdot v_1 - \beta^{-1}\beta_2 v_2 - \ldots - \beta^{-1}\beta_n v_n = \alpha_1 v_1 + \ldots + \alpha_n v_n$$

As $\{v_2, \dots, v_n\}$ is a basis, by the Coordinate Theorem, we have

$$\alpha_1 = 0 \quad \text{and } \alpha_1 = \beta^{-1}\beta_i$$

a contradiction. □

Question 7.1. In the Replacement Theorem, do we need the basis to be finite?

Ans: I think it can be infinite ...

§7.2 Main Theorem

Theorem 7.2 (Main)
Suppose V is a vector space over F with $V = \text{Span}(v_1, \dots, v_n)$. Then any linearly indep. subset of V has at most n elements.

Proof. We know that a subset of $B = \{v_1, \dots, v_n\}$ is a basis for V by Toss Out Theorem. So we may assume B is a basis for V. It suffices to show any linearly indep. set in V has at most $|B| = n$ elements where B is a basis. Let $\{w_1, \dots, w_m\} \subset V$ be linearly indep. where no $w_i = 0$. To show $m \leq n$, the idea is to use Toss In and Toss out in conjunction with the Replacement Theorem.

Claim 7.1. After changing notation, if necessary, for each $k \leq n$

$$\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$$

is a basis for V.

Suppose we have shown the above claim for $k = n$. Apply the claim to $k = n$ if $m > k$, then $\{w_1, \dots, w_{n+1}\}$ is linearly dep., a contradiction as $\{w_1, \dots, w_n\}$ is a basis. Thus, we prove the claim for $m \leq n$ as needed. We prove it by induction on k. BY the argument above, we may assume $k \leq n$.

- $k = 1$: As $w_1 \in$ Span $B =$ Span $(v_1, \ldots, v_n)$ and $w_1 \neq 0$, $\exists \alpha_1, \ldots, \alpha_n \in F$ not all 0 $\ni$
$$w_1 = \alpha_1 v_1 + \ldots + \alpha_n v_n$$
Changing notation, we may assume $\alpha_1 \neq 0$. By the Replacement Theorem,
$$\{w_1, v_2, \ldots, v_n\} \text{ is a basis for } V$$
- Assume the claim hold for $k (k < n)$.
- We must show the claim holds for $k + 1$,
$$\{w_1, \ldots, w_k, v_{k+1}, \ldots, v_n\} \text{ is a basis for } V$$
We can write
$$0 \neq w_{k+1} = \beta_1 w_1 + \ldots + \beta_k w_k + \alpha_{k+1} v_{k+1} + \ldots + \alpha_n v_n$$
for some (new) $\beta_1, \ldots, \beta_k, \alpha_{k+1}, \ldots, \alpha_n \in F$ not all 0
Case 1: $\alpha_{k+1} = \alpha_{k+2} = \ldots = \alpha_n = 0$
Then $w_{k+1} \in \text{Span}(w_1, \ldots, w_k)$, hence $\{w_1, \ldots, w_{k+1}\}$ is linearly dep., a contradiction.
Case 2: $\exists i \ni \alpha_i \neq 0$:
Changing notation, we may assume $\alpha_{k+1} \neq 0$. By the Replacement Theorem
$$\{w_1, \ldots, w_{k+1}, v_{k+2}, \ldots, v_n\}$$
is a basis for V. This completes the induction step thus prove the claim and establish the theorem. □

§7.3 A Glance at Dimension

Corollary 7.3
Let V be a finite dimensional vector space over F, B_1, B_2 two bases for V. Then $|B_1| = |B_2| < \infty$. We call $|B_1|$ the dimension of V, <u>write</u> $\dim V = \dim_F V = |B_1|$ (dropping F if F is clear).

Proof. By defn of finite dimensional vector space over F, $\exists$ a basis b for V with $|b| < \infty$. By the Main Theorem, $|B| \leq |b|$, if B is a basis for V, so B is finite. Again by the Main Theorem, $|b| \leq |B|$ if B is a basis for V, so $|b| = |B|$ for any basis B of V. □

The corollary above says $\dim V$ is well-defined for all finite dimensional vector space over F, i.e., "dim" : {finite dimensional vector space over $F \to \mathbb{Z}^+ \cup \{0\}$} is a function.
<u>Warning</u>: F makes a difference.

Example 7.4

$$\begin{aligned} \dim_{\mathbb{C}} \mathbb{C} &= 1 \qquad \text{basis } \{1\} \\ \dim_{\mathbb{R}} \mathbb{C} &= 2 \qquad \text{basis } \{1, \sqrt{-1}\} \\ \dim_{\mathbb{Q}} \mathbb{C} &= ? \end{aligned}$$

Corollary 7.5

$\dim_F F^n = n$.

Corollary 7.6

$\dim_F F^{m\times n} = mn$.

Corollary 7.7

$\dim_F F[t]_n = 1 + n$.

Note: If V is a finite dimensional vector space over F with bases B, then the Replacement Theorem allows us to find many other bases.

Corollary 7.8

Let V be a finite dimensional vector space over F, $n = \dim V$, $\emptyset \neq S \subset V$ a subset. Then

- If $|S| > n$, then S is linearly dep.
- If $|S| < n$, then Span $S < V$.

Proof.

- First bullet point: The Main Theorem says:

  A maximal linearly indep. set in V is a basis and can have at most n elements by Toss In Theorem.

- Second bullet point: By Toss Out Theorem, we can assume that S is linearly indep., so it cannot be a basis by Corollary ?.

□

Question 7.2. What is $\dim_{\mathbb{R}} M_n(\mathbb{C})$?

§8 Lec 8: Oct 19, 2020

§8.1 Extension and Counting Theorem

Theorem 8.1 (Extension)
Let V be a finite dimensional vector space over F, $W \subset V$ a subspace. Then every linearly independent subset S in W is finite and part of a basis for W which is a finite dimensional vector space over F.

Proof. Any linearly indep. set in W is linearly indep. subset S in V so $|S| \leq \dim V < \infty$ by the Main Theorem. In particular,

$$\dim \operatorname{Span} S \leq \dim V$$

if $W = \operatorname{Span}\, S$, we are done.
If not, $\exists w_1 \in W \setminus \operatorname{Span}\, S$, and hence $S_1 = S \cup \{w_1\}$ is linearly indep. by Toss In Theorem and

$$|S_1| = |S \cup \{w_1\}| = |S| + 1 \leq \dim V$$

if $\operatorname{Span}\, S_1 < W$, then $\exists w_2 \in W \setminus \operatorname{Span}\, S_1$, so $S_2 = S \cup \{w_1, w_2\} \subset W$ is linearly indep., hence

$$|S_2| = |S| + 2 \leq \dim V$$

Continuing in this manner, we must stop when $n \leq \dim V - \dim \operatorname{Span}\, S$ as $\dim V < \infty$. So S is a part of a basis for W and W is a finite dimensional vector space over F. □

Think about the proof for this

Corollary 8.2
Let V be a finite dimensional vector space over F. Then any linearly indep. set in V can be EXTENDED to a basis for V, i.e., is part of a basis for V. We often call this special case the Extension Theorem.

Corollary 8.3
Let V be a finite dimensional vector space over F, $W \subset V$ a subspace. Then W is a finite dimensional vector space over F and $\dim W \leq \dim V$ with equality iff $W = V$.

Proof. Left as exercise. □

Theorem 8.4 (Counting)
Let V be a finite dimensional vector space over F, $W_1, W_2 \subset V$ subspaces. Suppose that both W_1 and W_2 are finite dimensional vector space over F. Then

1. $W_1 \cap W_2$ is a finite dimensional vector space over F.
2. $W_1 + W_2$ is a finite dimensional vector space over F.
3. $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$.

Proof.

1. $W_1 \cap W_2 \subset W_i, i = 1, 2$, so it is a finite dimensional vector space over F by corollary 8.2.
2. Let B_i be a basis for W_i, $i = 1, 2, \ldots$. Then $W_1 + W_2 = \text{Span}\,(B_1 \cup B_2)$ and
$$|B_1 \cup B_2| \leq |B_1| + |B_2| < \infty$$
So $W_1 + W_2$ is a finite dimensional vector space over F by Toss Out.
3. Let $B = \{v_1, \ldots, v_n\}$ be a basis for $W_1 \cap W_2$. Extend B to a basis
$$b_1 = \{v_1, \ldots, v_n, y_1, \ldots, y_r\} \text{ for } W_1$$
$$b_2 = \{v_1, \ldots, v_n, z_1, \ldots, z_s\} \text{ for } W_2$$
using the Extension Theorem.

Claim 8.1. $b_1 \cup b_2 = \{v_1, \ldots, v_n, y_1, \ldots, y_r, z_1, \ldots, z_s\}$ is a basis for $W_1 + W_2$ and has $n + r + s$ elements. So if we show the claim, the result will follow.

Certainly,
$$\text{Span}(b_1 \cup b_2) = \text{Span } b_1 + \text{ Span } b_2 = W_1 + W_2$$
So we need only to show $b_1 \cup b_2$ is linearly indep. Suppose this is false. Then
$$0 = \alpha_1 v_1 + \ldots + \alpha_n v_n + \beta_1 y_1 + \ldots + \beta_r y_r + \gamma_1 z_1 + \ldots + \gamma_s z_s \qquad (*)$$
for some $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_s$ in F not all zero.

Case 1: All the $\gamma_i = 0$. Since b_1 is linearly indep., this is a contradiction.

Case 2: Some $\gamma_i \neq 0$.

Changing notation, we may assume $\gamma_1 \neq 0$. Since b_2 is a basis, (*) leads to an equation
$$0 \neq z = \gamma_1 z_1 + \ldots + \gamma_s z_s = -\alpha_1 v_1 - \ldots - \alpha_n v_n - \beta_1 y_1 - \ldots - \beta_r y_r$$
Therefore, $0 \neq z$ lies in Span $b_2 \cap$ Span $b_1 = W_2 \cap W_1$. So we can write $zi \in W_1 \cap W_2$ using basis B as
$$0 \neq z = \delta_1 v_1 + \ldots + \delta_n v_n \quad \text{some } \delta_1, \ldots, \delta_n \in F$$
Thus $W_2 = \text{Span } b_2$, we have
$$\delta_1 v_1 + \ldots + \delta_n v_n - 0z_1 + \ldots + 0z_s = z = 0v_1 + \ldots + 0v_n + \gamma_1 z_1 + \ldots + \gamma_s z_s$$
By the Coordinate Theorem, $\gamma_1 = 0$, a contradiction.

□

Corollary 8.5

Let V be a vector space over F, $W_1, W_2 \subset V$ finite dimensional subspaces of V. Then
$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2$$
iff
$$W_1 \cap W_2 = \emptyset$$
In this case, we write $W_1 + W_2 = W_1 \oplus W_2$ called the DIRECT SUM.

§8.2 Linear Transformation

In mathematics, whenever you have a collection of objects, one studies maps between them that preserves any special properties of the objects in the collection and tries to see what information can be gained from such maps.

Definition 8.6 (Linear Transformation) — Let V, W be a vector space over F. A map $T : V \to W$ is called a Linear Transformation, write $T : V \to W$ is linear if $\forall v_1, v_2 \in V, \forall \alpha \in F$

- $T(v_1 + v_2) = T(v_1) + T(v_2)$.
- $T(\alpha v_1) = \alpha T(v_1)$.
- $T(0_V) = 0_W$.

Notation: We write Tv for $T(v)$.

Remark 8.7. Let V, W be a vector space over F, $T : V \to W$ a map.

1. If T satisfies 1) and 2), then it satisfies 3):
$$0_W + T(0_V) = T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$$
so $0_W = T(0_V)$.
2. T is linear iff $T(\alpha v_1 + v_2) = \alpha T v_1 + T v_2 \quad \forall v_1, v_2 \in V, \forall \alpha \in F$.
3. If T is linear, $\alpha_1, \ldots, \alpha_n \in F, v_1, \ldots, v_n \in V$, then
$$T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i T v_i$$
We leave a proof of 2) and 3) as exercises.

Example 8.8

Let V, W be a vector space over F. The followings are linear transformations

1. $0_{V,W} : V \to W$ by $v \mapsto 0_W$.
2. $V = W$, $1_V : V \to V$ by $v \mapsto v$.
   A linear transformation $T : V \to V$ is called a Linear Operator.
3. If $\emptyset \neq Z \subset W$ is a subset, then we have a map
$$\text{inc} : Z \to W$$
given by $z \mapsto z$ called the Inclusion Map. Then, Z is a subspace of V iff inc: $Z \hookrightarrow W$ is linear.

Note: inc = $\underbrace{1_W\Big|_Z}_{\text{Restriction map}}$.

This is the Subspace Theorem.

4. $T : F^n \to F^{n-1}$ by $(\alpha_1, \dots, \alpha_n) \mapsto (\alpha_1, \dots, \overbrace{i}^{\text{omit}}, \dots, \alpha_n$ for a fixed i.
5. $T : F^n \to F$ by $(\alpha_1, \dots, \alpha_n) \mapsto \alpha_i$ for a fixed i.
6. $T : \mathbb{R}^{n-1} \to \mathbb{R}^n$ by $(\alpha_1, \dots, \alpha_{n-1} \mapsto (\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_i, \dots, \alpha_n)$ for fixed i.
7. $T : \mathbb{R} \to \mathbb{R}^n$ by $\alpha \mapsto (0, 0, \dots, \alpha, 0, \dots, 0)$ for fixed i.
8. If $\alpha < \beta$ in $\mathbb{R}$, $D : C'(\alpha, \beta) \to C(\alpha, \beta)$ by $f \mapsto f'$.
9. If $\alpha < \beta$ in $\mathbb{R}$, Int: $C(\alpha, \beta) \to C'(\alpha, \beta)$ by $f \mapsto \int f$ where $\int f$ is the antiderivative – constant of integration 0.
10. Fix $\alpha \in F$, then $\lambda\alpha : V \to V$ by $v \mapsto \alpha v$. Left translation by α.
11. Let $A \in F^{m\times n}$. Define

$$T : F^{n\times 1} \to F^{m\times 1} \quad \text{by } T \cdot X = A \cdot X$$
$$\text{i.e. } \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \mapsto A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Matrices can be viewed as linear transformation. We should see the converse is true IF V is a finite dimensional vector space over F. It is not true in general.

§9 Lec 9: Oct 21, 2020

§9.1 Kernel, Image, and Dimension Theorem

Definition 9.1 (Kernel(Nullspace)) — Let V, W be a vector space over F, $T : V \to W$ linear set

$$N(T) = \ker T := \{v \in V | Tv = 0_W\}$$

called the nullspace or kernel of T.

Definition 9.2 (Range(Image)) — Let V, W be a vector space over F, $T : V \to W$ linear set

$$\begin{aligned} \text{im } T = T(V) &:= \{w \in W | \exists v \in V \ni Tv = w\} \\ &= \{Tv | v \in V\} \end{aligned}$$

called the range or image of T.

Proposition 9.3

Let $T : V \to W$ be linear. Then

1. $\ker T \subset V$ is a subspace.
2. $imT \subset W$ is a subspace.

Proof. Left as exercise. □

Theorem 9.4 (Dimension)

Let $T : V \to W$ be linear with V is a finite dimensional vector space over F. Then

1. $im\ T$ and $\ker T$ are finite dimensional vector space over F.
2. $\dim V = \dim \ker T + \dim imT$.

<u>*Note*</u>: $\dim \ker T$ is also called the NULLITY of T and $\dim imT$ is also called the RANK of T.

Proof. Let $n = \dim V$.

$\ker T \subset V$ is a subspace, V is a finite dimensional vector space over F so $\ker T$ is a finite dimensional vector space over F and $\dim \ker T \leq \dim V = n$. Say $m = \dim \ker T$. Let $\mathscr{B}_0 = \{v_1, \ldots, v_m\}$ be a basis for $\ker T$. By the Extension Theorem $\exists \mathscr{B} = \{v_1, \ldots, v_m, \ldots, v_n\}$ a basis for V.

Claim 9.1. $Tv_{m+1}, \ldots, Tv_n$ are linearly indep. (in particular, distinct) and

$$\mathscr{C} = \{Tv_{m+1}, \ldots, Tv_n\}$$

is a basis for imT.

If we prove the claim above, then imT is a finite dimensional vector space over F of dimension $n - m$ and we are done.

<u>**Step 1**</u>: $\mathscr{C}$ spans imT:

Let $w \in imT$. By definition, $\exists v \in V \ni Tv = w$. As $\mathscr{B}$ is a basis for V $\exists \alpha_1, \ldots, \alpha_n \in F \ni$

$$v = \alpha_1 v_1 + \ldots + \alpha_n v_n$$

Hence

$$\begin{aligned} w = T(v) = T(\alpha_1 v_1 + \ldots + \alpha_n v_n) &= \alpha_1 T v_1 + \ldots + \alpha_n T v_n \\ &= \alpha_1 0_W + \ldots + \alpha_m 0_W + \alpha_{m+1} T v_{m+1} + \ldots + \alpha_n T v_n \end{aligned}$$

lies w Span($\mathscr{C}$) (as $v_1, \ldots, v_m \in \ker T$). need recheck

Case 2: $\mathscr{C}$ is linearly indep.
Suppose $\alpha_{m+1}, \ldots, \alpha_n \in F$ and

$$\alpha_{m+1} T v_{m+1} + \ldots + \alpha_n T v_n = 0_W$$

Then

$$0_W = T(\alpha_{m+1} v_{m+1} + \ldots + \alpha_n v_n$$

So $\alpha_{m+1} v_{m+1} + \ldots + \alpha_n v_n \in \ker T$. By defn, $\mathscr{B}_0$ is a basis for $\ker T$. So $\exists \beta_1, \ldots, \beta_m \in F \ni$

$$\alpha_{m+1} v_{m+1} + \ldots + \alpha_n v_n = \beta_1 v_1 + \ldots + \beta_m v_n$$

Hence

$$0 = -\beta_1 v_1 - \ldots - \beta_m v_m + \alpha_{m+1} v_{m+1} + \ldots + \alpha_n v_n$$

As $\mathscr{B}$ is a basis for V, it is linearly indep, so $\beta_1 = 0, \ldots, \beta_m = 0, \alpha_{m+1} = 0, \ldots, \alpha_n = 0$ (Coordinate Theorem) and the claim follows. ☐

Note: Let V be a finite dimensional vector space over F, $W \subset V$ a subspace, V/W the quotient space, then $- : V \to V/W$, $v \mapsto \overline{v} = v + W$ and $\dim V/W = \dim V - \dim W$.

§9.2 Algebra of Linear Transformation

We want to study the set of all linear transformation from a vector space over F V to a vector space over F W. Let V, W be a vector space over F. Set

$$L(V, W) \coloneqq \{T : V \to W | T \text{ is linear}\}$$

Check: if $T, S \in L(V, W), \alpha \in F$, then $\alpha T + S \in L(V, W)$. Since we know $\mathscr{F}(V, W) = \{f : V \to W | f \text{ a map}\}$ is a vector space over F, by the Subspace Theorem, $L(V, W) \subset \mathscr{F}(V, W)$ is a subspace.

Proposition 9.5
Let V, W be a vector space over F, then $L(V, W) \subset \mathscr{F}(V, W)$ is a subspace.

Now we know if we have maps

$$f : X \to Y \quad \text{and} \quad g : y \to Z,$$

we have the COMPOSITE MAP

$$g \circ f : X \to Z \quad \text{by } (g \circ f)(x) = g(f(x)) \, \forall x \in X$$

where $\circ$ is called the COMPOSITION (and often omitted when clear). Then we have

Proposition 9.6

Let V, W, X, U be vector space over F, $T, T' : V \to W, \quad S, S' : W \to X, \quad R : X \to U$ all be linear. Then,

1. $S \circ T : V \to W$ is linear.(the composition of linear transformations is linear).
2. $R \circ (S \circ T) = (R \circ S) \circ T$ and linear.
3. $S \circ (T + T') = S \circ T + S \circ T'$ and linear.
4. $(S + S') \circ T = S \circ T + S' \circ T$ and linear.

Proof.

$$\begin{aligned}(S \circ T)(\alpha v_1 + v_2) &= S\left(T(\alpha v_1 + v_2)\right) = S(\alpha T v_1 + T v_2) \\ &= \alpha S \circ T(v_1) + S \circ T(v_2)\end{aligned}$$

$\forall v_1, v_2 \in V, \alpha \in F$.
The rest are left as exercises. ☐

Definition 9.7 (Linear Operator) — Let V be a vector space over F, $T : V \to V$ linear, so a linear operator is defined as

$$\begin{aligned}&T^n := \underbrace{T \circ \ldots \circ T}_{n} \quad \text{if } n \in \mathbb{Z}^+ \\ &T^0 = 1_V\end{aligned}$$

Proposition 9.8

Let V be a vector space over F. Then $L(V, V)$ under $+$ and $\circ$ of functions $V \to V$ satisfies all the axioms of a field except possibly (M3) and (M4) with

$$\begin{aligned}\text{one} = 1_V : V \to V \quad &\text{by } v \mapsto v \\ \text{zero} = 0_V : v \to v \quad &\text{by } v \mapsto 0\end{aligned}$$

We say $L(V, V)$ is a (non-commutative) ring of $M_n F$.

§9.3 Linear Transformation Theorems

Definition 9.9 (Properties/Consequences of Linear Transformation) — Let $T : V \to W$ be linear. We say that T is

1. a MONOMORPHISM (write mono or monic) or NONSINGULAR if T is $1-1$. (i.e., injective).
2. an EPIMORPHISM (write epi or epic) if T is onto (i.e., surjective).
3. an ISOMORPHISM (write iso) or INVERTIBLE if T is bijective and $T^{-1} : W \to V$ is linear. We say V, W vector spaces over F are ISOMORPHIC (write $V \cong W$ if $\exists$ an isomorphism $S : V \to W$, we also write an isomorphism $S : V \to W$ as $S : V \xrightarrow{\sim} W$

Remark 9.10. $V \cong W$ vector space over F means that we cannot take V and W apart algebraically.

Example 9.11

$F^{n+1} \cong F[t]_n$ as $F^{n+1} \to F[t]_n$ by $(\alpha_0, \dots, \alpha_n) \mapsto \alpha_0 + \alpha_1 t_1 + \dots + \alpha_n t^n$ is an isomorphism with inverse $F[t]_n \to F^{n+1}$ by $\alpha_0 + \alpha_1 t + \dots + \alpha_n t^n \mapsto (\alpha_0, \dots, \alpha_n)$

$$\begin{aligned} T^{-1}(\alpha w_1 + w_2) &= T^{-1}(\alpha T v_1 + T v_2) = T^{-1}\left(T(\alpha v_1 + v_2)\right) \\ &= T^{-1}T(\alpha v_1 + v_2) \\ &= \alpha v_1 + v_2 \\ &= \alpha T^{-1} w_1 + T^{-1} w_2 \quad \square \end{aligned}$$

Corollary 9.12

Let $T : V \to W$ be a monomorphism. Then $V \cong imT$ via T.

Remark 9.13. If V, W, X are vector space over F, then

1. $V \cong V$
2. $V \cong W \to W \cong V$
3. $V \cong W$ and $W \cong X$ then $V \cong X$

In algebra, isomorphisms are usually easier to check than are one might assume, because the following result is often true.

Proposition 9.14

Let $T : V \to W$ be linear. Then T is an isomorphism iff T is bijective.

Proof. $(\to)$ immediate.

($\leftarrow$) Let $T^{-1} : W \to V$ be the set inverse of $T : V \to W$, so

$$T \circ T^{-1} = 1_W \quad \text{and } T^{-1} \circ T = 1_V$$

In particular, if $v \in V$ and $w \in W$,

$$w = Tv \quad \text{iff} \quad T^{-1}w = v$$

Let $w_1, w_2 \in W$, $\alpha \in F$. To show

$$T^{-1}(\alpha w_1 + w_2) = \alpha T^{-1} w_1 + T^{-1} w_2$$

T is onto so

$$\exists v_i \in V \ni Tv_i = w_i, i = 1, \ldots$$

Hence, we have

$$\begin{aligned} T^{-1}(\alpha w_1 + w_2) &= T^{-1}(\alpha T v_1 + T v_2) = T^{-1}(T(\alpha v_1 + v_2)) \\ &= T^{-1}T(\alpha v_1 + v_2) = \alpha v_1 + v_2 \\ &= \alpha T^{-1} w_1 + T^{-1} w_2 \end{aligned}$$

□

§10 Lec 10: Oct 23, 2020

§10.1 Monomorphism, Epimorphism, and Isomorphism

Corollary 10.1

Let $T : V \to W$ be a monomorphism. Then $V \cong \operatorname{im} T$ via T.

Definition 10.2 (Linear Map) — Let $T : V \to W$ be linear. We say T takes linearly independent sets to linearly independent sets if $v_i, i \in I$ are linearly independent in V (in particular, distinct). Then, $Tv_1, i \in I$ are linearly indep. in W. ($Tv_i \neq Tv_j$ if $i \neq j$ in I)

Theorem 10.3 (Monomorphism)

Let $T : V \to W$ be linear. Then the followings are true

1. T is $1-1$, so it's monomorphism.
2. T takes linearly indep. sets in V to linearly indep. sets in W.
3. $\ker T = 0 := \{0_V\}$.
4. $\dim \ker T = 0$.

Proof.

- 3) iff 4) is the defn of the 0-space.

- 1) $\to$ 2) It suffices to show that T takes finite linearly indep. sets in V to linearly indep. sets in W.

  Suppose that $v_1, \ldots, v_n \in V$ are linearly indep. and $\alpha_1, \ldots, \alpha_n \in F$ satisfy
  $$0_W = \alpha_1 T v_1 + \ldots + \alpha_n T v_n$$
  Then
  $$T(0_V) = 0_W = T(\alpha_1 v_1 + \ldots + \alpha_n v_n)$$
  As T is $1-1$
  $$0_V = \alpha_1 v_1 + \ldots + \alpha_n v_n$$
  Since $v_1, \ldots, v_n$ are linearly indep. $\alpha_i = 0, i = 1, \ldots, n$ as needed.

- 2) $\to$ 3) Let $v \in \ker T$. Then $Tv = 0_W$. If $v \neq 0$, then $\{v\}$ is linearly indep. By 2) $Tv \neq 0_W$ as then $\{Tv\}$ is linearly indep. So $v \neq 0$.

- 3) $\to$ 1) If $Tv_1 = Tv_2$, $v_1, v_2 \in V$, then
  $$0_W = Tv_1 - Tv_2 = T(v_1 - v_2)$$
  So $v_1 - v_2 = 0_V$ by 3), i.e., $v_1 = v_2$ □

Remark 10.4. The Monomorphism Theorem says $\ker T$ measures the deviation of T from being $1-1$.

Note: In the Monomorphism Theorem, we do not assume that V or W is a finite dimensional vector space over F.

Theorem 10.5 (Isomorphism)

Suppose $T : V \to W$ is linear with $\dim V = \dim W < \infty$,i.e., V, W are finite dimensional vector space over F of the same dimension. Then the followings are true

1. T is an isomorphism.
2. T is a monomorphism.
3. T is an epimorphism.
4. If $\mathscr{B} = \{v_1, \ldots, v_n\}$ is a basis for V, then $\{Tv_1, \ldots, Tv_n\}$ is a basis for W (so $Tv_1, \ldots, Tv_n$ are distinct), i.e., T takes basis of V to basis of W.
5. There exists a basis $\mathscr{B}$ of V that maps to a basis of W.

Remark 10.6.

1. The condition that $\dim V = \dim W < \infty$ is crucial

   Come up with a counter example

2. Let $V \cong W$ with V, W be finite dimensional vector space over F. So $\dim V = \dim W$. Let $S : V \to W$ be linear. Then S may or may not be an isomorphism, e.g., if S is the zero map then it is not an isomorphism unless $V = 0$. The theorem only says that $\exists$ an isomorphism and any such satisfies the theorem.

3. Let $f : A \to B$ be a map of finite sets with $|A| = |B|$. Then f is a bijection iff f is an injection iff f is a surjection.

Proof. (of Theorem)

- 1) $\to$ 2) follows by defn.
- 2) $\to$ 3) By the Dimension Theorem

  $$\dim W = \dim V = \dim \ker T + \dim \ \operatorname{im} T$$

  Thus, T is onto iff im $T = W$ iff $\dim W = \dim$ im T (by the Corollary to the Existence Theorem) iff $\dim \ker T = 0$ iff T is $1 - 1$.
- 3) $\to$ 1) as 3) $\to$ 2) and 1) $=$ 2) $+$ 3) by the Proposition ?
- 2) $\to$ 4) Let $\{v_1, \ldots, v_n\}$ be a basis for V. By the Monomorphism Theorem, $Tv_1, \ldots, Tv_n$ are linearly indep. in W, so

  $$n \leq \dim W = \dim V = n$$

  Hence $\{Tv_1, \ldots, Tv_n\}$ also spans as $\dim W = \dim V$.
- 4) $\to$ 5) $\to$ 3) are clear.

$\square$

§10.2 Existence of Linear Transformation

The next result is really the defining property of finite dimensional vector space and linear transformation.

Theorem 10.7 (Existence of Linear Transformation (UPVS))

– (Universal Property of Vector Space) Let V be a finite dimensional vector space over F, $\mathscr{B} = \{v_1, \dots, v_n\}$ a basis for V and W an arbitrary vector space over F. Let $w_1, \dots, w_n \in W$, not necessarily distinct. Then

$$\exists! \quad T : V \to W \text{ linear } \ni Tv_i = w_i \forall i$$

We can write this in an other way as follows:
Let $B \hookrightarrow V$ be a basis for V, V a finite dimensional vector space over F and W a vector space over F. Given a diagram,

$$\begin{tikzcd} B \arrow[r, hook] \arrow[dr, "f"'] & V \\ & W \end{tikzcd} \quad \begin{array}{l}\text{of sets and}\\ \text{set maps}\end{array}$$

then $\exists! T : V \to W$ linear $\ni$

$$\begin{tikzcd} B \arrow[r, hook, "\text{inc}"'] \arrow[dr, "f"'] & V \arrow[d, "T"] \\ & W \end{tikzcd}$$

commutes , i.e., $T \circ \text{inc} = f$.

Proof. Define $T : V \to W$ as follows: let $V \in V$. The $\exists! \alpha_1, \dots, \alpha_n \in F \ni v = \alpha_1 v_1 + \dots + \alpha_n v_n$ by the Coordinate Theorem. Define

$$Tv = T(\alpha_1 v_1 + \dots + \alpha_n v_n) \coloneqq \alpha_1 w_1 + \dots + \alpha_n w_n$$

Since the α_i ARE UNIQUE, this defines a map – we say $T : V \to W$ is WELL – DEFINED. Certainly, $Tv_i = w_i, i = 1, \dots, n$. To show T is linear, let $v = \sum_{i=1}^n \alpha_i v_i, v' = \sum_{i=1}^n \beta_i v_i$, $\alpha, \alpha_i, \beta_j \in F \forall i, j$. Then

$$\begin{aligned} T(\alpha v + v') &= T\left(\alpha \sum_{i=1}^n \alpha_i v_i + \sum_{i=1}^n \beta_i v_i\right) \\ &= T\left(\sum_{i=1}^n (\alpha\alpha_i + \beta_i) v_i\right) = \sum_{i=1}^n (\alpha\alpha_i + \beta_i) w_i \\ &= \alpha \sum_{i=1}^n \alpha_i w_i + \sum_{i=1}^n \beta_i w_i = \alpha Tv + Tv' \end{aligned}$$

as needed. This shows existence.
Uniqueness: Let $T : V \to W$ by (*) and $S : V \to W$ linear s.t. $Sv_i = w_i \forall i$. To show $S = T$, let $v = \sum_{i=1}^n \alpha_i v_i, \alpha_i \in F$ unique, $i = 1, \dots, n$. Then $Tv = \sum_{i=1}^n \alpha_i Tv_i = \sum_{i=1}^n \alpha_i w_i$ which is equivalent to

$$= \sum_{i=1}^n \alpha_i S v_i = S\left(\sum_{i=1}^n \alpha_i v_i\right) = Sv$$

So S is T and we have proven uniqueness. □

Remark 10.8. The theorem says a linear transformation from a finite dimensional vector space over F is completely determined by what it does to a fixed basis. i.e., as there are no non − trivial RELATIONS on linear combos of elements in $\mathscr{B}$, the only relation in $\operatorname{im} T$ will arise from the kernel of T.

§11 Lec 11: Oct 26, 2020

§11.1 Lec 10 (Cont'd)

Remark 11.1.
1. In the above, given $fv_i = w_i \forall i$, we say that $T : V \to W$ by $\sum \alpha_i v_i \mapsto \alpha_i w_i$ EXTENDS f linearly.
2. Let V be any vector space over F (not necessarily finite dimensional). Suppose V has a basis $\mathscr{B}$, then every $v \in V$ is a finite linear combo elements in $\mathscr{B}$. Using the same proof of UPVS, shows

   if W is a vector space over F, then given a diagram

$$\begin{tikzcd} B \arrow[r, hook] \arrow[dr, "f"'] & V \\ & W \end{tikzcd} \quad \text{of sets and set maps}$$

   of set and set maps. $\exists! T : V \to W$ linear s.t.

$$\begin{tikzcd} B \arrow[r, hook, "\text{inc}"'] \arrow[dr, "f"'] & V \arrow[d, "T"] \\ & W \end{tikzcd}$$

   commutes. I.E., if $\mathscr{B} = \{v_i\}_I$ is a basis for V, $w_i \in W$, $i \in I$ (not necessarily distinct), $f : V \to W$ by $v_i \mapsto w_i \forall i \in I$. Then $\exists! T : V \to W$ linear s.t. $Tv_i = w_i \forall i \in I$. So any linear transformation from a vector space over F V having a basis is completely determined by what it does to that basis.
3. <u>Axiom</u>: Every vector space over F has a basis. This is equivalent to the Axiom of Choice.

Theorem 11.2 (Classification of Finite Dimensional Vector Space)
Let V, W be finite dimensional vector space over F. Then

$$V \cong W \iff \dim V = \dim W$$

Proof. ($\to$)Let $T : V \to W$ be an isomorphism, $\mathscr{B} = \{v_1, \dots, v_n\}$ a basis for V (so $\dim V = n$). By the Monomorphism Theorem,

$$\mathscr{C} = \{Tv_1, \dots, Tv_n\}$$

is linearly indep. in W. Since $|\mathscr{C}| = n$ and $\operatorname{span}(\mathscr{C}) = w$ (as T is onto), $\mathscr{C}$ is a basis for W and $\dim W = \dim V$.
($\leftarrow$) Suppose $n = \dim V = \dim W$. Let $\mathscr{B} = \{v_1, \dots, v_n\}$ be a basis for V, $\mathscr{C} = \{w_1, \dots, w_n\}$ a basis for W. By the UPVS, $\exists! T : V \to W$ linear $v_i \mapsto w_i \forall i$, i.e., T takes the basis $\mathscr{B}$ of V to the basis $\mathscr{C}$ of W. By the Isomorphism Theorem, T is an isomorphism. $\square$

Example 11.3
1. $F^{n\times m} \cong F^{m\times n} \cong F^{mn}$
2. $M_n F \cong F^{n^2}$
3. $F[t]_n \cong F^{n+1}$

Let $T : V \to W$ be linear with V, W arbitrary. Since T only tells us about im T, we replace the target W by im $T = T(V)$, i.e., view $T : V \to W$ surjective linear. Let $\mathscr{B}_0$ be a basis for $\ker T \subset V$ subspace. Then Extension. Theorem holds even when V is not finite dimensional. Extend $\mathscr{B}_0$ to a basis $\mathscr{B} = \mathscr{B}_0 \cup \mathscr{C}$ so $\mathscr{C} \cap \mathscr{B}_0 = \emptyset$ and $V = \text{span } \mathscr{B}$. By the argument proving the Dimension Theorem,

$$T(\mathscr{C}) = \{T(y) | y \in \mathscr{C}\}$$

is linearly indep. and since T is onto $T(\mathscr{C})$ is a basis for W. The new relation in $W = $ im T comes from

$$Tx = 0, x \in \mathscr{B}_0$$

In the extra section (3), we showed

$$V/\ker T = \{\overline{v} | v \in V\}$$

where

$$\overline{v} = v + \ker T = \{v + z | z \in \ker T\}$$

is a vector space over F. In fact, $\{\overline{y} | y \in \mathscr{C}\}$ is a basis for $V/\ker T$. By the UPVS, $\exists!$ linear transformation

$$\overline{T} : V/\ker T \to W$$

given by $\overline{0} = \overline{x} \mapsto 0, x \in \mathscr{B}_0, \overline{y} \mapsto Ty, y \in \mathscr{C}$. $\overline{T}$ is clearly onto and $\overline{T}$ is $1-1$,

$$\overline{T}(\overline{v}) = T(v) \quad \forall v \in V$$

So

$$\overline{T} : V/\ker T \to W = \text{ im } T$$

is an isomorphism.
As $- : V \to V/\ker T$ by $v \mapsto \overline{v}$ is a surjective linear transformation, by definition,

$$\overline{\alpha v + v'} = \alpha \overline{v} + \overline{v'}$$

<u>*Note*</u>: $\ker - = \ker T$.
We have a commutative diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "-"'] & \text{im } T \\ V/\ker T \arrow[ur, "\overline{T}"'] & \end{tikzcd} \quad \text{commutes}$$

with - an epimorphism
$\overline{T}$ an isomorphism

Notiece if $W \neq$ im $T, \overline{T}$ is only a monomorphism.
We shall show that all of this is true without using bases (or the Extension Theorem in the Extra Lecture). In particular,

$$V/\ker T \cong \text{ im } T$$

§11.2 Matrices and Linear Transformations

Goal: Let V, W be finite dimensional vector spaces over F. Reduce the study of linear transformations $T : V \to W$ to matrix theory, hence often to computation (Deabstractify).

> **Remark 11.4.** In this section, all bases are ORDERED.

Set up and Notation: Let V, W be finite dimensional vector space over F. $\mathscr{B} = \{v_1, \ldots, v_n\}$ an ordered basis for V, so $\dim V = n$. $\mathscr{C} = \{w_1, \ldots, w_m\}$ an ordered basis for W, so $\dim W = m$.
Step 1: If $v \in V$, write

$$v = \alpha_1 v_1 + \ldots + \alpha_n v_n$$

i.e., $\alpha_1, \ldots, \alpha_n$ are the unique coordinate of v relative to $\mathscr{B}$. Then let

$$[v]_{\mathscr{B}} \coloneqq \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in F^{n\times 1}$$

the coordinate matrix of v relative to the ordered basis $\mathscr{B}$. E.g.,

$$[v_i]_{\mathscr{B}} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{pmatrix} i^{\text{th}}$$

and set

$$v_{\mathscr{B}} \coloneqq \{[v]_{\mathscr{B}} | v \in V\} = F^{n\times 1}$$

Then

$$v \to v_{\mathscr{B}} \quad \text{by } v \mapsto [v]_{\mathscr{B}} \quad \text{isomorphism}$$

as

$$v_i \mapsto e_i \coloneqq \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} i^{\text{th}}, f_{n,1} = \{e_1, \ldots, e_n\}$$

the standard basis for $F^{n\times 1}$. **Step 2**: Let $T : V \to W$ be linear, then

$$Tv_i \in W = \text{Span } \mathscr{C} = \text{Span}(w_1, \ldots, w_m)$$

as $\mathscr{C}$ is a basis for W. Therefore,

$$\exists! \alpha_{ij} \in F, 1 \le i \le m, 1 \le j \le n \ni$$
$$Tv_j = \sum_{i=1}^{m} \alpha_{ij} w_i, \quad j = 1, \ldots, n$$

Let $A = (\alpha_{ij} \in F^{m\times n})$, i.e., $A_{ij} = \alpha_{ij} \forall i, j$. Then the j^{th} COLUMN of A is

$$\begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{pmatrix} = [Tv_j]_{\mathscr{C}} \in W_{\mathscr{C}} = F^{m\times 1}$$

<u>**Step 3**</u>: Let

$$A : V_{\mathscr{B}} \to W_{\mathscr{C}} \text{ by } A\left([v]_{\mathscr{B}}\right) = A \cdot [v]_{\mathscr{B}}$$

This is a linear transformation.

$$A : F^{n\times 1} \to F^{m\times 1}$$

Since

$$A\left([v_j]_{\mathscr{B}}\right) = [Tv_j]_{\mathscr{C}}, j = 1, \dots, n$$

A is the unique linear transformation s.t.

$$A[v_j]_{\mathscr{B}} = [Tv_j]_{\mathscr{C}}$$

So by UPVS,

$$A[v]_{\mathscr{B}} = [Tv]_{\mathscr{C}} \quad \forall v \in V \tag{*}$$

Definition 11.5 (Matrix Representation) — The unique matrix $A \in F^{m\times n}$ in (*) is called the matrix representation of T relative to the ordered bases, $\mathscr{B}, \mathscr{C}$. We denote A by $[T]_{\mathscr{B},\mathscr{C}}$.

<u>Notation</u>: if $V = W$, $\mathscr{B} = \mathscr{C}$, we usually write $[T]_{\mathscr{B}}$ for $[T]_{\mathscr{B},\mathscr{B}}$.

§12 Lec 12: Oct 28, 2020

§12.1 Lec 11 (Cont'd)

<u>Summary</u>: Let $T : V \to W$ be linear with V, W finite dimensional vector space over F

$$\begin{aligned} \mathscr{B} &= \{v_1, \dots, v_n\} \text{ an ordered basis for } V, \dim V = n \\ \mathscr{C} &= \{w_1, \dots, w_n\} \text{ an ordered basis for } W, \dim W = m \end{aligned}$$

Then $\exists!\ A = [T]_{\mathscr{B},\mathscr{C}} \in F^{m\times n}$ satisfying

$$A[v]_{\mathscr{B}} = [T]_{\mathscr{B},\mathscr{C}}[v]_{\mathscr{B}} = [Tv]_{\mathscr{C}} \forall v \in V$$

Moreover, if

$$Tv_j = \sum_{i=1}^{m} \alpha_{ij} w_i, \quad j = 1, \dots, n$$

then the j^{th} column of $A = [T]_{\mathscr{B},\mathscr{C}}$ is precisely

$$[Tv_j]_{\mathscr{C}} = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{pmatrix} \in F^{m\times 1}$$

i.e.,

$$[T]_{\mathscr{B},\mathscr{C}} = \left(\underbrace{[Tv_1]_{\mathscr{C}} \ldots [Tv_n]_{\mathscr{C}}}_{\text{columns}} \right)$$

Warning: If $\mathscr{B}', \mathscr{C}'$ are two other ordered bases for V, W respectively (even the same vectors in $\mathscr{B}, \mathscr{C}$ written in a different order), then in general

$$[T]_{\mathscr{B},\mathscr{C}} \neq [T]_{\mathscr{B}',\mathscr{C}'}$$

Example 12.1

1. Let $\mathscr{B} = \{v_1, \ldots, v_n\}, \mathscr{C} = \{w_1, \ldots, w_n\}$ be two ordered bases for V. Let

$$T : V \to V \text{ linear by } v_i \mapsto w_i, i = 1, \ldots, n$$

Then $[T]_{\mathscr{B},\mathscr{C}} = I$, the identity matrix. Moreover, if

$$Tv_j = w_j = \sum_{i=1}^{n} \alpha_{ij} v_i$$

then

$$[T]_{\mathscr{B}} = [T]_{\mathscr{B},\mathscr{B}} = (\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & & \alpha_{nn} \end{pmatrix}$$

2. $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $(\alpha, \beta) \mapsto (\beta, \alpha)$, $\mathscr{S} = \mathscr{S}_2 = \{e_1, e_2\}$, the standard ordered basis for $\mathbb{R}^2$. Then

$$[T]_{\mathscr{S}} = ([Te_1]_{\mathscr{S}}, [Te_2]_{\mathscr{S}}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and if $\mathscr{B}$ is the ordered bases $\mathscr{B} = \{e_2, e_1\}$ then

$$[T]_{\mathscr{S},\mathscr{B}} = ([Te_1]_{\mathscr{B}}, [Te_2]_{\mathscr{B}}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3. Let $\mathscr{B} = \left\{1, x, x^2, x^3\right\}$ be a basis for $\mathbb{R}[x]_3$, the polynomial functions of degree ≤ 3 (and 0), and

$$D : \mathbb{R}[x]_3 \to \mathbb{R}[x]_3 \text{ differentiation}$$

Find $[D]_{\mathscr{B}}$

$$D \cdot 1 = 0 \text{ so } [D \cdot 1]_{\mathscr{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Dx = 1 \text{ so } [Dx]_{\mathscr{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Dx^2 = 2x \text{ so } [Dx^2]_{\mathscr{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$Dx^3 = 3x^2 \text{ so } [Dx^3]_{\mathscr{B}} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

Hence,

$$[D]_{\mathscr{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Some more examples

Example 12.2 1. Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation by an $\angle\theta$

$$\begin{aligned} T_\theta e_1 &= \cos\theta e_1 + \sin\theta e_2 \\ T_\theta e_2 &= (-\sin\theta) e_1 + \cos\theta e_2 \end{aligned}$$

So

$$[T_\theta]_{\mathscr{S}} = ([T_\theta e_1]_{\mathscr{S}} [T_\theta e_2]_{\mathscr{S}}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

2. Let $\mathscr{B} = \{v_1, v_2\}$ be an ordered basis for V and $\mathscr{C} = \{w_1, w_2, w_3\}$ an ordered basis for W. Suppose

$$T : V \to W \text{ by } \begin{cases} Tv_1 = 3w_1 + w_3 \\ Tv_2 = w_1 + 6w_2 + w_3 \end{cases}$$

then $[T]_{\mathscr{B},\mathscr{C}} = \begin{pmatrix} 3 & 1 \\ 0 & 6 \\ 1 & 1 \end{pmatrix}$

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection about the e_1, e_2 plane. What is $[T]_{\mathscr{S}}$?

$$\begin{aligned} e_1 &\mapsto e_1 \\ e_2 &\mapsto e_2 \\ e_3 &\mapsto -e_3 \end{aligned}$$

So $[T]_{\mathscr{S}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Theorem 12.3 (Matrix Theory)

(MTT) Let V, W be finite dimensional vector space F, $\dim V = n$, $\dim W = m$, and $\mathscr{B}, \mathscr{C}$ ordered bases for V, W. Then the map

$$\phi : L(V, W) \to F^{m \times n} \text{ by } T \mapsto [T]_{\mathscr{B},\mathscr{C}}$$

is an isomorphism. In particular

$$\dim L(V, W) = mn$$

Proof. Left as exercise (Homework). □

Using the fact that $W \to W_{\mathscr{C}}$ is an isomorphism if $w \mapsto [w]_{\mathscr{C}}$ show that

1. ϕ is linear
2. ϕ is onto
3. ϕ is $1 - 1$
4. $\dim L(V, W) = mn$

Theorem 12.4

Let V, W, U be finite dimensional vector space over F with ordered bases $\mathscr{B}, \mathscr{C}, \mathscr{D}$ respectively, $T : V \to W$, $S : W \to U$ linear. Then

$$[S \circ T]_{\mathscr{B},\mathscr{D}} = [S]_{\mathscr{C},\mathscr{D}} \cdot [T]_{\mathscr{B},\mathscr{C}}$$

Proof.

$$\begin{aligned} [S]_{\mathscr{C},\mathscr{D}}[T]_{\mathscr{B},\mathscr{C}}[v]_{\mathscr{B}} &= [S]_{\mathscr{C},\mathscr{D}}[Tv]_{\mathscr{C}} \\ &= [S(Tv)]_{\mathscr{D}} \\ &= [(S \circ T)(v)]_{\mathscr{D}} \\ &= [S \circ T]_{\mathscr{B},\mathscr{D}} [v]_{\mathscr{B}} \end{aligned}$$

□

Exercise: Let V, W be finite dimensional vector space over F with $\dim V = \dim W$, $\mathscr{B}, \mathscr{C}$ ordered bases of V, W respectively, $T : V \to W$ linear. Then, T is an isomorphism iff $[T]_{\mathscr{B},\mathscr{C}}$ is invertible.
Let V be a finite dimensional vector space over F, $\dim V = n$, $\mathscr{B}$ an ordered basis for V. Then

$$\phi : L(V, V) \to M_n F \text{ by } T \mapsto [T]_{\mathscr{B}}$$

satisfies all of the following: $\forall T, S \in L(V, V)$

(i) $\phi(T + S) = \phi(T) + \phi(S)$

(ii) $\phi(T \circ S) = \phi(T)\phi(S)$

(iii) $\phi(0_V) = 0_{F^{n\times 1}}$

(iv) $\phi(1_V) = 1_{F^{n\times 1}}$

By the exercise, ϕ is bijection linear transformation. Both $L(V, V)$ and $M_n F$ satisfy all the axioms of a field except (M3) and (M4). We call them (NON COMMUTATIVE) rings and since ϕ preserves all the structure i) – iv) as does its inverse(?), we say ϕ is an ISOMORPHISM of rings

Definition 12.5 (Change of Basis Matrix) — Let V be a finite dimensional vector space over F with ordered bases $\mathscr{B}, \mathscr{C}$. Then the invertible matrix $[1_V]_{\mathscr{B},\mathscr{C}}$ is called a CHANGE OF BASIS MATRIX.

Example 12.6

1. $\mathscr{S} = \{e_1, e_2\}, \mathscr{B} = \{(1, 1), (2, 1)\}, \mathscr{C} = \{(3, 4), (6, 1)\}$ ordered bases for $\mathbb{R}^2$.

$$[1_{\mathbb{R}^2}]_{\mathscr{B},\mathscr{S}} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad [1_{\mathbb{R}^2}]_{\mathscr{S}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$[1_{\mathbb{R}^2}]_{\mathscr{C},\mathscr{S}} = \begin{pmatrix} 3 & 6 \\ 4 & 1 \end{pmatrix}, \quad [1_{\mathbb{R}^2}]_{\mathscr{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. $\mathscr{B}$ an ordered basis for V, a finite dimensional vector space over F, $\dim V = n$, then $[1_V]_{\mathscr{B}} = I \in M_n F$

3. V a finite dimensional vector space over F, $\mathscr{B}, \mathscr{C}$ ordered bases for V, then $[1_V]_{\mathscr{B},\mathscr{C}}$ is invertible and

$$\begin{aligned} [1_V]^{-1}_{\mathscr{B},\mathscr{C}} &= [1_V]_{\mathscr{C},\mathscr{B}} \\ [1_V]_{\mathscr{B},\mathscr{C}}[1_V]_{\mathscr{C},\mathscr{B}} &= [1_V]_{\mathscr{C}} \\ &= I \\ &= [1_V]_{\mathscr{C},\mathscr{B}}[1_V]_{\mathscr{B},\mathscr{C}} \end{aligned}$$

4. Apply 3) to 1)

$$\begin{aligned}
[1_V]_{\mathscr{S},\mathscr{C}} &= [1_V]^{-1}_{\mathscr{C},\mathscr{S}} = \begin{pmatrix} 3 & 6 \\ 4 & 1 \end{pmatrix}^{-1} = -\frac{1}{21}\begin{pmatrix} 1 & -6 \\ -4 & 3 \end{pmatrix} \\
[1_V]_{\mathscr{B},\mathscr{C}} &= [1_V]_{\mathscr{S},\mathscr{C}}[1]_{\mathscr{B},\mathscr{S}} \\
&= -\frac{1}{21}\begin{pmatrix} 1 & -6 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \\
&= -\frac{1}{21}\begin{pmatrix} -5 & -4 \\ -1 & -5 \end{pmatrix}
\end{aligned}$$

Some more examples

Example 12.7 1. Any invertible matrix $A \in M_n F$ is a change of basis matrix for some ordered bases $\mathscr{B}, \mathscr{C}$ for F^n : if $A = (\alpha_{ij})$ is invertible, define

$$v_j = \sum_{i=1}^n \alpha_{ij} e_i, \quad \mathscr{B} = \{v_1, \ldots, v_n\}$$

Then $A = [A]_{\mathscr{B},\mathscr{S}}$ since A is invertible, so $\mathscr{B}$ is linearly indep., hence a basis by counting and $A = [\mathscr{F}_v]_{\mathscr{B},\mathscr{S}}$.

2. The j^{th} column of $[1_v]_{\mathscr{B},\mathscr{C}}$, V a finite dimensional vector space over F is the j^{th} vector of $\mathscr{B}$ expressed as a linear combo of vectors in $\mathscr{C}$.

3. Generalizing (1), (3) from above example, we get the following crucial computational device: if $V = F^n, \mathscr{B}, \mathscr{C}$ ordered bases for V, then

$$[1_v]_{\mathscr{B},\mathscr{C}} = [1_v]_{\mathscr{S},\mathscr{C}}[1_v]_{\mathscr{B},\mathscr{S}} = [1_v]^{-1}_{\mathscr{C},\mathscr{S}}[1_v]_{\mathscr{B},\mathscr{S}}$$

if we only have $V \cong F^n$, then we have to use an isomorphism $V \to F^n$ – how?

Since $[1_v]_{\mathscr{B},\mathscr{S}}$ and $[1_v]_{\mathscr{C},\mathscr{S}}$ are usually (often?) easy to write down, this is quite useful. What if $V = F^{m\times n}$?

Theorem 12.8 (Change of Basis)
Let V, W be finite dimensional vector space over F with ordered bases $\mathscr{B}, \mathscr{B}'$ for V and $\mathscr{C}, \mathscr{C}'$ for W. Let $T : V \to W$ be linear. Then

$$\begin{aligned}
[T]_{\mathscr{B},\mathscr{C}} &= [1_W]_{\mathscr{C}',\mathscr{C}}[T]_{\mathscr{B}',\mathscr{C}'}[1_V]_{\mathscr{B},\mathscr{B}'} \\
&= [1_W]^{-1}_{\mathscr{C},\mathscr{C}'}[T]_{\mathscr{B}',\mathscr{C}'}[1_V]_{\mathscr{B},\mathscr{B}'} \\
&= [1_W]_{\mathscr{C}',\mathscr{C}}[T]_{\mathscr{B}',\mathscr{C}'}[1_V]^{-1}_{\mathscr{B}',\mathscr{B}}
\end{aligned}$$

Proof. We have

$$[1_W]^{-1}_{\mathscr{C},\mathscr{C}'} = [1_W]_{\mathscr{C}',\mathscr{C}} \text{ and } [1_V]_{\mathscr{B},\mathscr{B}'} = [1_V]^{-1}_{\mathscr{B}',\mathscr{B}}$$

Since

$$\begin{aligned}[1_W]_{\mathscr{C}',\mathscr{C}}[T]_{\mathscr{B}',\mathscr{C}'}[1_V]_{\mathscr{B},\mathscr{B}'} &= [1_W \circ T]_{\mathscr{B}',\mathscr{C}}[1_V]_{\mathscr{B},\mathscr{B}'} \\ &= [1_W \circ T \circ 1_V]_{\mathscr{B},\mathscr{C}} \\ &= [T]_{\mathscr{B},\mathscr{C}}\end{aligned}$$

the result follows. $\square$

To use (and remember) this, do it as follows – to let the notation help you:

$$T : V \to W$$

$$\begin{tikzcd}
V_B \arrow[r, "{[T]_{B,C}}"] \arrow[d, "{[1_V]_{B,C}}"] \arrow[dr, "{[T]_{B,C'}}" near start] & W_C \arrow[d, "{[1_W]_{C,C'}}"] \\
V_{B'} \arrow[r, "{[T]_{B',C'}}"'] \arrow[ur, "{[T]_{B',C}}"' near start] \arrow[u, bend left=60, "{[1_V]_{B',B} \,\|\, [1_V]^{-1}_{B,B'}}"] & W_{C'} \arrow[u, bend right=60, "{[1_W]^{-1}_{C',C}}"']
\end{tikzcd}$$

COMMUTES, i.e., can compose along any allowable arrows in the correct direction if we arrive at the same place in different way starting at the same place we get the same answer. <u>Warning</u>: You can only reverse direction if the arrow is an isomorphism and then you can take the inverse. To remember the theorem, we write

$$T : V \to W$$

$$\begin{tikzcd}
V_B \arrow[r, "{[T]_{B,C}}"] \arrow[d, "{[1_V]_{B,B'}}"'] & W_C \arrow[d, "{[1_W]_{C,C'}}"] \\
V_{B'} \arrow[r] & W_{C'}
\end{tikzcd}$$

and fill in arrows you can find in the diagram before.

§13 Lec 13: Oct 30, 2020

§13.1 Some Examples of Change of Basis

If V, W are finite dimensional vector space over F with ordered bases $\mathscr{B}, \mathscr{C}$ respectively and if $T : V \to W$ is linear

$$[Tv]_{\mathscr{C}} = [T]_{\mathscr{B},\mathscr{C}}[v]_{\mathscr{B}} \forall v \in V$$

<u>Note</u>: There is nothing about the bases in which v was written.

1. $V = \mathbb{R}^2$, $\mathscr{S} = \{e_1, e_2\}$, $\mathscr{B} = \{v_1 = (1,1), v_2 = (2,1)\}$ ordered bases. Find $[T]_{\mathscr{S}}$ in the following (equivalently, $[T]_{\mathscr{S}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{\mathscr{S}} \leftrightarrow T(\alpha, \beta)$)

   (i) $T(1,1) = (2,1)$ and $T(2,1) = (1,1)$

$$\begin{tikzcd} V_B \arrow[r, "{[T]_B}"] \arrow[d, "{[1_V]_{B,S}}"'] & V_B \arrow[d, "{[1_V]_{B,S} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}}"] \\ V_S \arrow[r, "{[T]_S}"'] & V_S \end{tikzcd}$$

So

$$\begin{aligned} [T]_{\mathscr{S}} &= [1_V]_{\mathscr{B},\mathscr{S}}[T]_{\mathscr{B}}[1_V]^{-1}_{\mathscr{B},\mathscr{S}} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

So $T(\alpha, \beta) = (-\alpha + 3\beta, \beta)$

(ii) $T(1,1) = 6(1,1) + (2,1)$ and $T(2,1) = -2(1,1) + (2,1)$

$$\begin{tikzcd} V_B \arrow[r, "{[T]_B}"] \arrow[d, "{[1_V]_{B,S}}"'] & V_B \arrow[d, "{[1_V]_{B,S} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}}"] \\ V_S \arrow[r, "{[T]_S}"'] & V_S \end{tikzcd}$$

So

$$[T]_{\mathscr{S}} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -8 & 16 \\ -8 & 15 \end{pmatrix}$$

(iii) $T(1,1) = (3,1)$ and $T(2,1) = (5,1)$

$$\begin{tikzcd} V_B \arrow[r] \arrow[d] \arrow[dr, "{[T]_{B,S}}"] & V_B \arrow[d, "{[1]_{B,S}}"] \\ V_S \arrow[r] & V_S \end{tikzcd}$$

$$[T]_{\mathscr{B},\mathscr{S}} = ([T(1,1)]_{\mathscr{S}}[T(2,1)]_{\mathscr{S}}) = ([(3,1)][(5,1)]_{\mathscr{S}})$$

So $[T]_{\mathscr{S}} = [T]_{\mathscr{B},\mathscr{S}}[1_V]^{-1}_{\mathscr{B},\mathscr{S}}$ which is equal to $\begin{pmatrix} 3 & 5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}$

2. Let T be a rotation about the axis $(1,1,1) \in V = \mathbb{R}^3$ of an $\angle\theta$ in the counter-clockwise direction with $(1,1,1)$ up. We will use stuff from 33A – dot product. Normalize $(1,1,1)$ to

$$v_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{(1,1,1)}{\|(1,1,1)\|}$$

a unit vector in the DIRECTION of v_1. Find a vector $\perp$ to v_1, say

$$v_2' = (0, 1, -1)$$

and normalize it to

$$v_2 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

Let $v_3 = v_1 \times v_2$ the cross product of v_1, v_2. It is orthogonal to v_1 and v_2 and by the right hand rule in the correct orientation

$$v_3 = \begin{pmatrix} i & j & k \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

a unit vector (or use Gram – Schmidt and check you have $v_3 = v_1 \times v_2$ and not $-(v_1 \times v_2)$

§13.2 Orthonormal Basis

Definition 13.1 (Orthonormal Basis) — Let $\mathscr{B} = \{v_1, v_2, v_3\}$ an ordered bases of vectors of length 1 and each $\perp$ to the others, called an ORTHONORMAL BASIS.

$$\begin{aligned} Tv_1 &= v_1 \\ Tv_2 &= \cos\theta v_2 + \sin\theta v_3 \\ Tv_3 &= -\sin\theta v_2 + \cos\theta v_3 \\ [T]_{\mathscr{B}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \\ [1_V]_{\mathscr{B},\mathscr{S}} &= \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \end{aligned}$$

$$\begin{tikzcd} V_B \arrow[r, "{[T]_B}"] \arrow[d, "{[1_V]_{B,S}}"'] & V_B \arrow[d, "{[1_V]_{B,S}}"] \\ V_S \arrow[r, "{[T]_S}"'] & V_S \end{tikzcd}$$

$$[T]_{\mathscr{S}} = [1_V]_{\mathscr{B},\mathscr{S}}[T]_{\mathscr{B}}[1_V]_{\mathscr{B},\mathscr{S}}^{-1} = [1_V]_{\mathscr{B},\mathscr{S}}[T]_{\mathscr{B}}[1_V]_{\mathscr{S},\mathscr{B}}$$

Since both $\mathscr{S}$ and $\mathscr{B}$ are orthonormal bases and $F = \mathbb{R}$, it turns out that

$$[1_V]_{\mathscr{B},\mathscr{S}}^{-1} = [1_V]_{\mathscr{B},\mathscr{S}}^{\top}$$

This is, however, not true in general.

3. $V = \mathbb{R}^3$, $T : V \to V$ as in 2) and $S : V \to V$ a reflection about the plane $\perp (1,2,3)$. Find $[S]_{\mathscr{S}}$ and $[S \circ T]_{\mathscr{S}}$.

   Find an orthonormal basis with $(1,2,3)$ direction of the first vector

$$(1,2,3), (0,3,-2), (-13,2,3)$$

then normalize as follows:

$$\begin{aligned}
w_1 &= \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) \\
w_2 &= \left(0, \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}}\right) \\
w_3 &= \left(\frac{-13}{\sqrt{182}}, \frac{2}{\sqrt{182}}, \frac{3}{\sqrt{182}}\right)
\end{aligned}$$

So $\mathscr{C} = \{w_1, w_2, w_3\}$ is an orthonormal basis and

$$[S]_{\mathscr{C}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{tikzcd}
V_B \arrow[r, "{[T]_B}"] \arrow[d, "{[1_V]_{B,S}}"'] & V_B \arrow[d] & \\
V_S \arrow[r, "{[T]_S}"'] & V_S \arrow[r, "{[S]_S}"] & V_S \\
& V_C \arrow[u] \arrow[r, "{[S]_C}"'] & V_C \arrow[u, "{[1_V]_{C,S}}"']
\end{tikzcd}$$

$$[1_V]_{\mathscr{C},\mathscr{S}} = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & \frac{13}{\sqrt{182}} \\ \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{182}} \\ \frac{3}{\sqrt{14}} & -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{182}} \end{pmatrix}$$

$$\begin{aligned}
[S]_{\mathscr{S}} &= [1_V]_{\mathscr{C},\mathscr{S}}[S]_{\mathscr{C}}[1_V]^{-1}_{\mathscr{C},\mathscr{S}} \\
[S \circ T]_{\mathscr{S}} &= [1_V]_{\mathscr{C},\mathscr{S}}[S]_{\mathscr{C}}[1_V]_{\mathscr{B},\mathscr{S}}[T]_{\mathscr{B}}[1_V]^{-1}_{\mathscr{B},\mathscr{S}}
\end{aligned}$$

The only reason to normalize $\mathscr{C}$ to an orthonormal basis is

$$[1_V]))\mathscr{C},\mathscr{S}^{-1} = [1_V]^{\top}_{\mathscr{C},\mathscr{S}}$$

§13.3 Similarity

Definition 13.2 (Similar Matrices) — Let $A, B \in M_nF$. We say A is SIMILAR to B write $A \sim B$ if $\exists C \in M_nF$ invertible $\ni$

$$A = C^{-1}BC$$

Remark 13.3. $A, B \in M_n F$:

1. $A \sim B \to B \sim A$:
   $A = C^{-1}BC, C$ invertible $\to B = (C^{-1})^{-1}AC^{-1}$ as $CC^{-1} = I = C^{-1}C$
2. If $A \sim B$, then $\det A = \det B$. If $A = C^{-1}BC$, invertible, then

$$\begin{aligned} \det A &= \det\left(C^{-1}BC\right) = \det(C^{-1}) \det B \det C \\ &= (\det C)^{-1} \det B \det C = \det B \end{aligned}$$

3. $\sim$ is an equivalence relation.

Theorem 13.4 (Similar Matrices)
Let $A, B \in M_n F$. Then $A \sim B$ iff $\exists V$ a vector space over F, $\dim V = n$, $T : V \to V$ linear and ordered bases $\mathscr{B}, \mathscr{C}$ for V s.t

$$A = [T]_{\mathscr{B}} \quad \text{and} \quad B = [T]_{\mathscr{C}}$$

i.e., $A \sim B$ iff they represent the same linear transformation relative to (possibly) different ordered bases.

§14 Lec 14: Nov 2, 2020

§14.1 Lec 13 (Cont'd)

Proof. (Of Similar Matrices Theorem) ($\leftarrow$) If $A = [T]_{\mathscr{B}}, B = [T]_{\mathscr{C}}$, then $C = [1_V]_{\mathscr{B},\mathscr{C}} \in M_n F$ is invertible with $A = C^{-1}BC$ by the Change of Basis Theorem.
($\to$) Suppose $C \in M_n F$ is invertible, $A = C^{-1}BC$. Define $V = F^n$, $T : V \to V$ by

$$T_{ij} = \sum_{i=1}^{n} A_{ij} e_i$$

with $\mathscr{S} = \{e_1, \dots, e_n\}$ the standard basis

$$[T]_{\mathscr{S}} = A = C^{-1}BC$$

Let $w_j := \sum_{i=1}^{n} (C^{-1})_{ij} e_i$, i.e., $(C^{-1})_{ij}$ is the ij^{th} entry of C^{-1}. As C is invertible, C^{-1} exists and is invertible. Then

$$\mathscr{B} = \{w_1, \dots, w_n\}$$

is a basis for V and $[1_V]_{\mathscr{B},\mathscr{S}} = C^{-1}$ figure here so $A = C^{-1}[T]_{\mathscr{B}}C$ and $B = [T]_{\mathscr{B}}$ works. $\square$

§14.2 Eigenvalues and Eigenvectors

Definition 14.1 (Eigenvalues, Eigenvectors & Eigenspace) — Let $0 \neq V$ be a vector space over F, $T : V \to V$ a linear operator and $\lambda \in F$. Set

$$S_\lambda := T - \lambda 1_V : V \to V,$$

a linear operator, so

$$S_\lambda(v) = Tv - \lambda v \forall v \in V$$

We say λ is an EIGENVALUE of T if S_λ is not $1-1$, i.e., $\ker S_\lambda \neq 0$. Let

$$\begin{aligned} E_T(\lambda) &:= \ker S_\lambda = \{v \in V | Tv - \lambda v = 0\} \\ &= \{v \in V | Tv = \lambda v\} \end{aligned}$$

if $E_T(\lambda) \neq 0$, we call $E_T(\lambda)$ an EIGENSPACE of V relative T, λ and any $v \in E_T(\lambda)$ an EIGENVECTOR of T relative to λ. So if $T : V \to V$ is linear, $\lambda \in F$ is an eigenvalue of T iff

$$\exists 0 \neq v \in V \ni Tv = \lambda v$$

Remark 14.2. Let $0 \neq V$ be a vector space over F and $T : V \to V$ linear

1. Eigenvalues occur as measured quantities in science and engineering, e.g., resonance, quantum number – measurable values.
2. If $\lambda \in F$ is an eigenvalue of T, then

$$0 \neq E_T(\lambda) \subset V \text{ is a subspace}$$

3. If $\lambda \in F$ an eigenvalue, any $v \in E_T(\lambda)$ is an eigenvector. In particular, any basis for $E_T(\lambda)$ consists of eigenvectors of T relative to λ. Hence

$$T\Big|_{E_T(\lambda)} = \lambda 1_{E_T(\lambda)}$$

(the notation above means we restrict the domain to $E_T(\lambda)$. In particular, if $V = E_T(\lambda)$, then $T = \lambda 1_V$.
4. If $T = 0$, then $V = E_T(\lambda)$ with eigenvalue $\lambda = 0 (\lambda = 1)$.

Example 14.3 5. Let $V = \mathbb{R}^3$, $T : V \to V$ a counterclockwise rotation by an $\angle\theta, 0 < \theta < 2\pi$ around the axis determined by $0 \neq v \in V$. Then

$$T(\alpha v) = \alpha T v = \alpha v \forall \alpha \in F$$

So $\text{Span}(v) \subset E_T(1)$. Note if $0 \neq v$ is an eigenvector with eigenvalue μ of linear $S : V \to V$, then

$$Sv \in \text{Span}(v) = Fv \text{ so } \text{Span}(v) \subset E_S(\mu)$$

Do there exist other eigenvalues of T? Ever? So the only other possibilities would be

$$\theta = \pi, \lambda = -1$$

In that case

$$E_T(-1) = \operatorname{Span}(w_1, w_2)$$

where w_1, w_2 are linearly indep. with $w_i \perp v, i = 1, 2$. (of course, if one allows $\theta = 0, T = 1_V$.)

6. Let $0 \neq v \in V$. Suppose that

$$\mu v = Tv = \lambda v, \quad \lambda, \mu \in F$$

Then $\mu = \lambda$ so $0 \neq v \in V$ is an eigenvector of at most one eigenvalue of T – usually none. In particular,

$$E_T(\lambda) \cap E_T(\mu) = 0 \text{ if } \lambda \neq \mu$$

and we write

$$E_T(\lambda) \oplus E_T(\mu) = E_T(\lambda) + E_T(\mu)$$

and call it the DIRECT SUM of the subspace $E_T(\lambda)$ and $E_T(\mu)$.

What do you think is $W_1 \bigoplus W_2 \bigoplus W_3$?

7. Suppose $\dim V = n$, $\mathscr{B} = \{v_1, \dots, v_n\}$ is an ordered basis for V. Suppose that that

$$Tv_i = \alpha_i v_i, \qquad i = 0, \dots, n$$

$\lambda_1, \dots, \lambda_n \in F$ not necessarily distinct. Then

$$[T]_{\mathscr{B}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix}$$

is a DIAGONAL MATRIX, i.e., all non-diagonal entries 0. We say T is DIAGONALIZABLE if $\exists$ an ordered bases $\mathscr{C}$ for $V \ni [T]_{\mathscr{C}}$ is diagonal.

8. Suppose $\dim V = n(< \infty)$ and T is diagonalizable, i.e., $\exists$ an ordered basis $\mathscr{C} = \{w_1, \dots, w_n\}$ for V s.t.

$$[T]_{\mathscr{C}} = \begin{pmatrix} \mu_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mu_n \end{pmatrix}$$

Then $Tw_i = \mu_i w_i, i = 1, \dots, n$ and $\mathscr{C}$ is an ordered basis for V consisting of eigengenvalues for T.

<u>Conclusion</u>: Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Then T is diagonalizable iff $\exists$ a basis for V consisting of eigenvectors of T.
<u>*Note*</u>: If T is diagonalizable, $T : V \to V$ linear, V a finite dimensional vector space over F, ordered basis $\mathscr{B}$ for V. Then $\exists C \in M_n F$, invertible, $n = \dim V \ni C^{-1}[T]_{\mathscr{B}} C$ is diagonal by the Change of Basis Theorem.

Example 14.4 9. Let V be a finite dimensional vector space over F, $n = \dim V$, $\mathscr{B}$ an ordered basis for V, $S : V \to V$ linear. Then by the Isomorphism Theorem, S is 1-1 iff S is onto. Apply this to

$$S_\lambda = T - \lambda 1_V : V \to V$$

to conclude:

λ is an eigenvalue of T iff $S_\lambda = T - \lambda 1_V$ is singular (i.e., S_λ is not 1-1)

iff

$$[S_\lambda]_{\mathscr{B}} = [T - \lambda 1_V]_{\mathscr{B}} \text{ is not invertible}$$

iff

$$\det[T - \lambda 1_V]_{\mathscr{B}} = 0 \text{ (by properties of det)}$$

iff

$$\det\left([T]_{\mathscr{B}} - \lambda[1_V]_{\mathscr{B}}\right) = 0$$

iff

$$\det\left([T]_{\mathscr{B}} - \lambda I\right) = 0$$

iff

$$\det\left(\lambda I - [T]_{\mathscr{B}}\right) = 0$$

Summary: Let V be a finite dimensional vector space over F, $\dim V = n$, $T : V \to V$ linear, $\mathscr{B}$ an ordered basis for V, $\lambda \in F$. Then, λ is an eigenvalue of T iff $\det(\lambda I - [T]_{\mathscr{B}}) = 0$.

Definition 14.5 (Characteristics Polynomial) — Let $A \in M_n F$. Define

$$f_A := \det(tI - A) \in F[t]$$

called the Characteristics Polynomial of A.

The properties of the determinant on $F[t]$ is the same as on F except that $A \in M_n F[t]$ is invertible iff $\det A \in F \setminus \{0\}$ and we assume these properties.

Proposition 14.6

If $A, B \in M_n F$ are similar, then $f_A = f_B$

Proof. If $A = C^{-1}BC$, $C \in M_n F$ in

$$\begin{aligned} f_A &= \det(C^{-1}(tI - B)C) = \det C^{-1} \det(tI - B) \det C \\ &= \det(tI - B) = f_B \end{aligned}$$

□

Warning: Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then, A and B are not similar, but $f_A = f_B$, i.e., the converse is false.

Corollary 14.7

Let V be a finite dimensional vector space over F, $T : V \to V$ linear, $\mathscr{B}, \mathscr{C}$ ordered bases for V. Then

$$f_{[T]_{\mathscr{B}}} = f_{[T]_{\mathscr{C}}}$$

Proof. Change of Basis Theorem. □

Definition 14.8 (Characteristics Polynomial) — Let V be a finite dimensional vector space over F, $T : V \to V$ linear, $\mathscr{B}$ ordered basis for V. We call $f[t]_{\mathscr{B}}$ the characteristics polynomial of T. By the corollary, it is independent of $\mathscr{B}$, so we denote it by $f_T (= f_{[T]_{\mathscr{B}}})$ and write $f_T = \det(t1_V - T) := \det(tI - [T]_{\mathscr{B}})$

Theorem 14.9 (Eigenvalue – Root of f_T)

Let V be a finite dimensional vector space over F, $T : V \to V$ linear. Then, the eigenvalues of T are precisely, the roots of f_T, i.e., those $\alpha \in F \ni f_T(\alpha) = 0$.

Proof. det $\lambda \in F, \mathscr{B}$ an ordered basis for V. Set $A = [T]_{\mathscr{B}}$, so $f_T = \det(tI - A)$. Then λ is a root of f_T iff evaluating f_T at λ, i.e., $f_T(\lambda)$, we have

$$f_T(\lambda) = \det(tI - A)\Big|_{t=\lambda} = 0 \iff \lambda \text{ is an eigenvalue of } T$$

i.e., expanding the polynomial $\det(tI - A)$ and plugging λ for t gives 0. □

We cannot use the following theorem if we fully prove it.

Theorem 14.10 (Cayley – Hamilton)

Let $A \in M_n F$. Then

$$f_A(A) = 0$$

plugging A into the expansion of the determinant f_A, you get 0.

Remark 14.11. By HW, we have $\left\{I, A, A^2, \dots, A^{n^2}\right\} \subset M_n F$ is linearly dep., i.e., $\left\{I, A, \dots, A^N\right\}$ is linearly dep. for some $N > 0$. This means $\exists 0 \neq g \in F[t]$ with $\deg g \leq N$ and $g(A) = 0$ – why?

So Cayley – Hamilton's Theorem says $\{I, A, \dots, A^n\}$ in $M_n F$ is always linearly dep. in $M_n F$ with $f_A(A)$ giving a dependence relation.
Note: If you know Cramer's Rule in determinant theory, one can prove Cayley – Hamilton follows from it. In fact, it is essentially Cramer's Rule.

Remark 14.12. Let V be a finite dimensional vector space over F, $T : V \to V$ linear. You will show in your Take home Exam. There exists a polynomial $q \in F[t]$ satisfying

1. $q \neq 0$
2. $q(A) = 0$
3. $\deg q$ is the minimal degree for a poly $g \neq 0$ in $F[t]$ to satisfy $g(A) = 0$
4. q is MONIC, i.e., leading coeff is 1.

Moreover, q is unique and called the MINIMAL POLYNOMIAL of A and denoted q_T. Using it we shows a stronger form of the Cayley – Hamilton Theorem.

§15 Lec 15: Nov 4, 2020

§15.1 Lec 14 (Cont'd)

Cayley – Hamilton (Stronger Form): Let V be a finite dimensional vector space over F, $T : V \to V$ linear, then

$$q_T | f_T \text{ in } F[t]$$

(where $q_T = q[T]_{\mathscr{B}}$, $\mathscr{B}$ an ordered basis and q_T is indep. of $\mathscr{B}$). Why does this show the other form?
Computation: Let V be a finite dimensional vector space over F, $T : V \to V$ linear. To find eigenvalues and eigenvectors of T, you must solve

$$Tv = \alpha v$$

By Matrix Theory Theorem, this is equivalent to

$$[T]_{\mathscr{B}}[v]_{\mathscr{B}} = \lambda[v]_{\mathscr{B}} \tag{*}$$

$\mathscr{B}$ an ordered basis for V. To find eigenvalues, we find the roots of f_T. To find the eigenvectors, we solve (*).

Theorem 15.1
Let $T : V \to V$ be linear and $\lambda_1, \ldots, \lambda_n$ in F distinct eigenvalues of $T, 0 \neq v_i \in E_T(\lambda_i), i = 1, \ldots, n$. Then $\{v_1, \ldots, v_n\}$ is linearly indep.

Proof. We induct on n.

- $n = 1 : v_1 \neq 0$ so $\{v\}$ is linearly indep.
- $n > 1$ – Induction Hypothesis (IH) : If $\lambda_1, \ldots, \lambda_{n-1}$ are distinct eigenvalues of $T, 0 \neq v_i \in E_T(\lambda_i), i = 1, \ldots, n-1$ then $\{v_1, \ldots, v_{n-1}\}$ is linearly indep. Suppose that
$$0 = \alpha_1 v_1 + \ldots + \alpha_n v_n, \alpha_1, \ldots, \alpha_n \in F \tag{*}$$
Apply the linear operator $S_{\lambda_n} = T - \lambda_n 1_V$ to (*). As
$$S_{\lambda_n}(v_i) = Tv_i - \lambda_n v_i = \lambda_i v_i - \lambda_n v_i = (\lambda_i - \lambda_n)v_i$$

We get

$$\begin{aligned}S_{\lambda_n}(\alpha_1 v_1 + \ldots + \lambda_n v_n) &= \alpha_1 S_{\lambda_{v_n}} v_1 + \ldots + \alpha_n S_{\lambda_{v_n}} v_n \\ 0 &= \alpha_1(\alpha_1 - \alpha_n)v_1 + \ldots + \alpha_{n-1}(\lambda_{n-1} - \lambda_n)v_{n-1}\end{aligned}$$

By the IH, $\alpha_i(\lambda_i - \lambda_n) = 0, i = 1, \ldots, n-1$

As $\lambda_i - \lambda_n \neq 0, i = 1, \ldots, n-1, \alpha_i = 0, i = 1, \ldots, n-1$. So $0 = \alpha_n v_n$. As $v_n \neq 0$, $\alpha_n = 0$ also.

□

Proof. (Alternative) Take T of (*) to get an eqn 1). Multiply (*) by λ_n to get an eqn 2). Subtract eqn 2) from eqn 1). The proof that if $\alpha_1, \ldots, \alpha_n$ are distinct then $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ are linearly indep. □

Corollary 15.2

Let V be a finite dimensional vector space over F, $\dim V = n$ if $T : V \to V$ linear has n distinct eigenvalues, then T is diagonalizable. The converse is false, e.g., $T = 1_V$.

Corollary 15.3

If V is a finite dimensional space over F, $\dim V = n$, $T : V \to V$ linear, then T has at most n distinct eigenvalues. This also follows as any $0 \neq f \in F[t]$ has at most $\deg f$ roots.

Corollary 15.4

Let V be a vector space over F, $T : V \to V$ linear, $\lambda_1, \ldots, \lambda_n$ distinct eigenvalues of T. Set

$$w = E_T(\lambda_1) + \ldots + E_T(\lambda_n)$$

if $v_i \in E_T(\lambda_i), i = 1, \ldots, n$ satisfy

$$v_1 + \ldots + v_n = 0$$

then $v_i = 0, i = 1, \ldots n$. We write this as

$$W = E_T(\lambda_1) \oplus \ldots \oplus E_T(\lambda_n)$$

Exercise 15.1. Let V be a vector space over F, $W_1, \ldots, W_n \subset V$ subspaces. Let $W = W_1 + \ldots + W_n$. Then the followings are equivalent

1. If $w_i \in W_i, i = 1, \ldots, n$ satisfy $w_1 + \ldots + w_n = 0$ then $w_i = 0 \forall i$. We say W_i are indep.
2. If $v \in W \exists ! w_i \in W_i \ni v = w_1 + \ldots + w_n$
3. $W_i \cap \sum_{j \neq i, j=1}^{n} W_j = 0 \forall i = 1, \ldots, n$

4. If $\mathscr{B}_i$ is a basis for $W_i, i = 1, \ldots, n$ then $\mathscr{B} = \mathscr{B}_1 \cup \ldots \cup \mathscr{B}_n$ is a basis for W.

If these hold for W, we say W is an (internal) direct sum of the W_i and write

$$W = W_1 \oplus \ldots \oplus W_n$$

> **Remark 15.5.** This generalizes to $W = \oplus W_i$, general I – How. What is the proof?

Exercise 15.2. Let V be a vector space over F, $W_1, \ldots, W_n \subset V$ subspaces $\ni V = W_1 + \ldots + W_n$. Let

$$W = W_1 \times \ldots \times W_n = \{(W_1, \ldots, W_n) | w_i \subset W_i \forall i\}$$

a vector space over F via component wise operations. Show

$$v = W_1 \oplus \ldots \oplus W_n \iff T : W_1 \times \ldots \times W_n \to V$$

by $(w_1, \ldots, w_n) \mapsto w_1 + \ldots w_n$ is an isomorphism. We call W the external direct sum of the W_i.

Consequences: Let V be a finite dimensional vector space over F, $\lambda_1, \ldots, \lambda_n$ distinct eigenvalues of $T : V \to V$ linear, $?_i = \dim E_T(\lambda_i), \mathscr{B}_i$ ordered basis for $E_T(\lambda_i), i = 1, \ldots, n$ if

$$V = E_T(\lambda_1) + \ldots + E_T(\lambda_n)$$

then

$$V = E_T(\lambda_1) \oplus \ldots \oplus E_T(\lambda_n)$$

and $\mathscr{B} = \mathscr{B}_1 \cup \ldots \cup \mathscr{B}_n$ is an ordered basis for V and

$$[T]_{\mathscr{B}} = \begin{pmatrix} \left[\lambda_1 1_{E_T(\lambda_1)}\right]_{\mathscr{B}_1} & & \\ & \ddots & \\ & & \left[\lambda_n 1_{E_T(\lambda_n)}\right]_{\mathscr{B}_n} \end{pmatrix}$$

(Block form) is a diagonal matrix. In particular,

$$f_T = \det(T 1_V - T) = (t - \lambda_1)^{r_1} \ldots (t - \lambda_n)^{r_n}$$

By determinant theory,

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det A \det B$$

A, B square matrices and T is diagonalizable.

> **Remark 15.6.** $T : V \to V$ linear may or may not have eigenvalues
>
> 1. $V = \mathbb{R}^2, f_T = t^2 + 1$, then T has not eigenvalues.
> 2. If V is a finite dimensional vector space over $\mathbb{C}$, then T has an eigenvalue as f_T has a root by the FUNDAMENTAL THEOREM OF ALGEBRA (which we shall always assume to be true).

§15.2 Inner Product Space

We know that the dot product of vectors in $\mathbb{R}^3$ allows us to define $\perp$, $\angle$, distance, etc. We want to generalize this to "inner product spaces". When we talk about inner product spaces, we shall always assume that OUR FIELD F LIES in $\mathbb{C}$ (e.g., $\mathbb{Q}, \mathbb{R}, \mathbb{C}$) as a subfield. Let $- : \mathbb{C} \to \mathbb{C}$ by $\alpha + \beta\sqrt{-1} \mapsto \alpha - \beta\sqrt{-1} \forall \alpha, \beta \in \mathbb{R}$ denoted complex conjugation.
Note:Let $a = \alpha + \beta\sqrt{-1}$ in $\mathbb{C}$, $\alpha, \beta \in \mathbb{R}$. Then

1. $a = \overline{a}$ iff $a \in \mathbb{R}$
2. $\overline{\overline{a}}$
3. $|a|^2 \coloneqq a\overline{a} \geq 0$ in $\mathbb{R}$ as $a\overline{a} = \alpha^2 + \beta^2$ and $= 0$ iff a= 0.

   As we shall assume $F \subset \mathbb{C}$, we define:

   $$\overline{F} \coloneqq \{\overline{z} \in \mathbb{C} | z \in F\}$$

   and we shall also assume that

   $$F = \overline{F}$$

   This is true if $F \subset \mathbb{R}$ or $F = \mathbb{C}$, but does not always hold UNLESS we only consider those F that do which we will.

Definition 15.7 (Inner Product Space) — Let $F \subset \mathbb{C}$ be a subfield satisfying $F = \overline{F}, V$ a vector space over F. We call V an inner product space over F, write V is an ips / F, under the map

$$\langle , \rangle \coloneqq \langle , \rangle_V : V \times V \to F$$

Write: $\langle v, w \rangle$ for $\langle , \rangle(v, w)$ if $\langle , \rangle$ satisfies $\forall v_1, v_2, v_3, v \in V, \forall \alpha \in F$

1. $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle$
2. $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$
3. $\langle \alpha v_1, v_2 \rangle = \alpha \langle v_1, v_2 \rangle = \langle v_1, \overline{\alpha} v_2 \rangle$
4. $\langle v, v \rangle \in \mathbb{R}$ and $\langle v, v \rangle \geq 0$ with $\langle v, v \rangle = 0$ iff $v = 0$.

If V is an inner product space over F (under $\langle , \rangle$, the LENGTH (or NORM or MAGNITUDE) of $v \in V$ is given by

$$\|v\| \coloneqq \sqrt{\langle v, v \rangle} \geq 0 \in \mathbb{R}$$

Note: If $F < \mathbb{C}$, $\|v\|^2 \in F$, but it is possible that $\|v\| \notin F$, e.g., if $V = \mathbb{Q}^2$ a vector space over $\mathbb{Q}$ and an inner product space over $\mathbb{Q}$ under the dot product $\|(1, 1)\| = \sqrt{2} \notin \mathbb{Q}$. This is a reason to work only with $F = \mathbb{R}$ or $\mathbb{C}$.

§16 Extra Lec: Nov 2, 2020

§16.1 Dual Bases – Dual Spaces

Let $0 \neq V$ be a vector space over F with basis $\mathscr{B}$. For each $v_0 \in \mathscr{B}$, we define a map

$$f_{v_0} : V \to F \text{ linear}$$

as follows: by the UPVS (which also holds if the basis is infinite, let fv_0 be the unique linear transformation) s.t.

$$\begin{aligned} v_0 &\mapsto 1 \\ v &\mapsto 0 \quad \forall v_0 \neq v \in \mathscr{B} \end{aligned}$$

We have

$$0 < \text{ im } fv_0 \subset F \text{ a subspace}$$

(im $fv_0 \neq 0$ as $v_0 \neq 0$). As $\dim_F F = 1$, we must have $\dim fv_0 = 1$, so $fv_0 : V \to F$ is an epimorphism and

$$\begin{aligned} \ker fv_0 &= \{w \in V | w \text{ has } v_0 \text{ coordinate} = 0\} \\ &= \text{Span}(\mathscr{B} \setminus \{v_0\}) \end{aligned}$$

So if $w \in V$, $w = \sum \alpha_v v, \alpha_v \in F$ almost all 0 with α_v unique.

$$fv_0(w) = \alpha_{v_0}$$

the coordinate of w on v_0. We can do this for each $v \in \mathscr{B}$. If $v' \in \mathscr{B}, f_V : V \to F$ is the linear transformation determined by

$$f_{v'}(v) = \delta_{vv'} = \begin{cases} i, \text{ if } v = v' \\ 0, \text{ if } v \neq v', v \in \mathscr{B} \end{cases} \quad \text{, the Kronecker } \delta$$

Set

$$\mathscr{B}^* := \{fv | v \in \mathscr{B}\} \; f_v \text{ is the coordinate function} f_v \text{ on } v$$

The vector space

$$V^* := L(V, F)$$

is called the DUAL SPACE of V. So by the above if $w \in V$

$$w = \sum_{v \in \mathscr{B}} \alpha_v v, \alpha_v \in F \text{ almost all } 0$$

then

$$\alpha_v = f_v(w) \text{ the coordinate } w, v \in \mathscr{B}$$

so

$$w = \sum_{\mathscr{B}} \alpha_v v = \sum_{\mathscr{B}} f_v(w) v$$

Now by the UPVS, we have a unique linear transformation

$$D_{\mathscr{B}} : V \to V^{\times}$$

determined by $v \in \mathscr{B} \mapsto f_v$. So $\sum_{\mathscr{B}} \alpha_v v \mapsto \sum_{\mathscr{B}} \alpha_v f_v$ almost all $\alpha_v = 0$

Claim 16.1. $D_{\mathscr{B}}$ is 1-1.

Suppose $w = \sum_{\mathscr{B}} \alpha_v v \mapsto 0$ almost all $\alpha_v = 0$ i.e., $\sum_{\mathscr{B}} \alpha_v f_v = 0 \leftarrow$ in v^*
Let $v_0 \in \mathscr{B}$, then

$$0 = \left(\sum_{\mathscr{B}} \alpha_v f_v\right)(v_0) = \sum_{\mathscr{B}} \alpha_v f_v(v_0) = \sum_{\mathscr{B}} \alpha_v S_{vv_0} = \alpha v_0$$

Hence $\sum \alpha_v f_v = 0 \to \alpha_v = 0 \forall v \in \mathscr{B}$, so $w = 0$. $D_{\mathscr{B}}$ is therefore 1-1 as claimed. $\square$
Warning: If V is not finite dimensional, then $D_{\mathscr{B}}$ is not onto, i.e., $\mathscr{B}^*$ does not span V^*. $\left(|V^*| = |F|^{|\mathscr{B}|}\right.$ and $|F| = |V|$ by UPVS if F is infinite$\left.\right)$
Note: $D_{\mathscr{B}} : V \to V^*$ depends on the choice of basis $\mathscr{B}$.

Definition 16.1 (Linear Functionals) — If V is a vector space over F, eigenvalues in $V^* = L(V, F)$ are called LINEAR FUNCTIONALS.

Fact 16.1. If S is a linearly indep. set in a vector space over F (even infinite) then S is part of a basis for V, i.e., the Extension Theorem holds (This needs the Axiom of Choice).

Example 16.2

V a vector space over F. Then followings are linear functionals

1. If $0 \neq v \in V$, then $\{v\}$ extend to a basis $\mathscr{B}$ for V and $\mathscr{B}^*$ satisfies $\mathscr{B}^*$ is linearly indep.
$$f_v(x) = S_{vx} \forall x \in \mathscr{B}$$
Let $w = \sum_{x \in \mathscr{B}} \alpha_x x, \alpha_x = 0$ almost all $x \in \mathscr{B}$. Then $f_x(w) = \alpha_x \in F \forall x \in \mathscr{B}$, $w = \sum f_x(w) x$
2. $\pi_i : F^n \to F$ by $(\alpha_1, \dots, \alpha_n) \mapsto \alpha_i \forall i$
3. Let $\text{Int} : C[\alpha, \beta] \to \mathbb{R}, \alpha < \beta$ be given by
$$\text{Int } f \mapsto \int_\alpha^\beta f$$
4. trace: $M_n F \to F$ by
$$A \mapsto \sum_{i=1}^n A_{ii}$$
The sum of the diagonal entries of A called the TRACE of A.

We can iterate our constructions as follows:

Let $\mathscr{C}$ be a basis for $V^* = L(V, F)$ a vector space over F, where V is a vector space over F. Then
$$D_{\mathscr{C}} : V^* \to (V^*)^* := V^{**}$$

V^{**} is called the DOUBLE DUAL of V, is induced by

$$f_0 \in \mathscr{C} \mapsto G_{f_0} \in \mathscr{C}^*$$

the coordinate function on f_0, i.e.,

$$\sum_{\mathscr{C}} \alpha_f f \mapsto \sum_{\mathscr{C}^*} \alpha_f G_f$$

with

$$G_{f_0}(f) = \delta_{tf_0} = \begin{cases} 1 \text{ if } f = f_0 \forall f, f_0 \in \mathscr{C} \\ 0 \text{ if } f \neq f_0 \end{cases}$$

So we have

$$V \overset{\mathscr{D}_{\mathscr{B}}}{\to} V^* \overset{\mathscr{D}_{\mathscr{C}}}{\to} V^{**}$$

and the composition is a monomorphism.

<u>Wonderful Result</u>: $\exists$ a monomorphism

$$L : V \to V^{**}$$

INDEPENDENT OF CHOICE OF BASES. We know want to show this:

For each $v \in V$ define the following linear functionals on V^*

$$L_v : V^* \to F \text{ by } L_v(f) \coloneqq f(v)$$

EVALUATION at v.

<u>Check</u>. $L_v : V^* \to F$ is linear, i.e., $L_v \in V^{**} = (V^*)^*$:

$$\begin{aligned} L_v(\alpha f + g) &= (\alpha f + g)(v) = \alpha f(v) + g(v) \\ &= \alpha L_v f + L_v g \end{aligned}$$

$\forall t, g \in V^* \forall \alpha \in F$ as needed. Now define

$$L : V \to V^{**} \text{ by } v \mapsto L_v$$

i.e., $L(v) = L_v$

Claim 16.2. L is linear.

$\forall f \in V^*$, $v, v' \in V, \alpha \in F$, we have

$$\begin{aligned} L(\alpha v + v')(f) &= L_{\alpha v + v'}(f) = f(\alpha v + v') \\ &= \alpha f(v) + f(v') = \alpha L_v f + L_{v'} f \\ &= (\alpha L_v + L_{v'})(f) \end{aligned}$$

as needed.

Claim 16.3. $L : V \to V^{**}$ is monic.

Suppose $v \neq 0$. By Example TBA, $\exists f \in V^* \ni L_v(f) = f(v) \neq 0$. As L is linear, L is a monomorphism. Hence

$$L : V \to V^{**}$$

is a NATURAL or CANONICAL MONOMORPHISM, i.e., no basis is needed to define it. We now assume that V is a finite dimensional vector space over F, let

$$\mathscr{B} = \{v_1, \ldots, v_n\} \text{ be a basis for } V$$
$$\mathscr{B}^* = \{f_1, \ldots, f_n\} \subset V^* \text{ defined by } f_i(v_j) = \delta_{ij} \forall i, j$$

i.e., the f_i are the coordinate functions relative to $\mathscr{B}$. Then, as before, we have a monomorphism

$$D_{\mathscr{B}} : V \to V^* \text{ induced by } v_i \mapsto f_i$$

But we also have

$$\dim V^* = \dim L(V, F) = \dim V \dim F = \dim V$$

by the Matrix Theory Theorem, so $D_{\mathscr{B}}$ is an isomorphism by the Isomorphism Theorem with $\mathscr{B}^*$ a basis for V^* called the DUAL BASIS of $\mathscr{B}$. We also have

$$V \cong V^* \cong V^{**}, \text{ so } V \cong V^{**}$$

and

$$\mathscr{B}^{**} := \{L_{v_1}, \ldots, L_{v_n}\}$$

with

$$L_{v_i} := L_{f_i}, f_i \in \mathscr{B}^*$$
$$L_{f_i}(f_j) = L_{v_i}(f_j) = f_j(v_i) = \delta_{ij}$$

So $\mathscr{B}^{**}$ is the DUAL BASIS of $\mathscr{B}^*$. We also now $L : V \to V^{**}$ is now a natural isomorphism by the Isomorphism Theorem and even better that

$$f(v) = L_v(f) \quad \forall v \in V \quad \forall f \in V^*$$

EVALUATION at v. So when V is a finite dimensional vector space over F, we can and do identify L_v and v $\forall v \in V$.
Any $v \in V$ is determined by the $t \in V^*$ and every $f \in V^*$ is determined by the $L_v \in V^{\times\times}$ and

$$f(v) = L_v(f)$$

So now we have: if V is a finite dimensional vector space over F

$$\mathscr{B} = \{v_1, \ldots, v_n\} \text{ a basis for } V$$
$$\mathscr{B}^* = \{f_1, \ldots, f_n\} : \{f_{v_1}, \ldots, f_{v_n}\} \text{ the dual basis of } \mathscr{B}$$
$$\mathscr{B}^{**} = \left\{L_{f_{v_1}}, \ldots, L_{f_{v_n}}\right\} = \{Lv_1, \ldots, Lv_n\} \text{ the dual basis of } \mathscr{B}^*$$

i.e.,

$$f_i = f_{v_i}$$
$$L_{f_{v_i}} = L_{v_i}$$

and these satisfy

$$f_|(v_i) = tv_j(v_i) = \delta_{ij} = L_{f_{v_i}}(v_j) = L_{v_i}(f_|)$$

If $v \in V$, then

$$\begin{aligned} v &= \alpha_1 v_1 + \ldots + \alpha_n v_n \text{ unique } \alpha_1, \ldots, \alpha_n \in F \\ f_j(v) &= f_j(\alpha_1 v_1 + \ldots + \alpha_n v_n) \\ &= \alpha_j \end{aligned}$$

So

$$\boxed{v = \sum_{i=1}^{n} f_i(v) v_i}$$

where $f_i(v)$ is the coordinate function relative to $\mathscr{B}$ and if $f \in V^*$, then

$$f = \beta_1 f_1 + \ldots + \beta_n f_n \text{ unique } \beta_1, \ldots, \beta_n \in F$$

As

$$\begin{aligned} L_{v_1}(f) &= (\beta_1 f_1 + \ldots + \beta_n f_n)(v_j) \\ &= \beta_1 f_1(v_1) + \ldots + \beta_n f_n(v_j) = \beta_| \end{aligned}$$

And

$$\begin{aligned} f &= \beta_1 f_1 + \ldots + \beta_n f_n \\ &= L_{v_1}(f) f_1 + \ldots + L_{v_n}(f) f_n \\ &= f(v_1) f_1 + \ldots + f(v_n) f_n \end{aligned}$$

So,

$$\boxed{f = \sum f(v_i) f_i}$$

where $f(v_i)$ is the coordinate function.

§17 Dis 1: Oct 1, 2020

Overview of the class:

- HW – 20%
- Takehome Midterm – 20(25)%
- Midterm – 20(0)%
- Final – 40(55)%

Note: For starred homework problems, we can resubmit these problems (if we did not get full credit for it).

Plan:

1. Proofs
2. Sets
3. Functions

§17.1 Sets

- $\mathbb{N}$ = set of natural numbers = $\{1, 2, 3, 4, \ldots\}$
- $\mathbb{Z}$ = set of integers = $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Q}$ = set of rational numbers = $\left\{\frac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0\right\}$
- $\mathbb{R}$ = set of real numbers(number line)
- $\mathbb{C}$ = set of complex numbers = $\{a + bi | a, b \in \mathbb{R}\}$
- $\mathbb{R}^2$ = (xy)-plane = $\{(a, b) : a, b \in \mathbb{R}\}$

Notation: subset – $\subseteq$, proper subset – $\subsetneq$ (subset and not equal), empty subset – $\emptyset$.

§17.2 Functions

What is a set?
– A collection of elements

Example 17.1
- $A = \{\text{cat, dog}\}$
- $B = \{1, 2, 3\}$
- $C = \mathbb{R}^2$

So what is a function?

$$f : \underbrace{A}_{\text{set called the domain of f}} \mapsto \underbrace{B}_{\text{this set is called the codomain of f}}$$

In general, range and codomain are two different thing.
Given any element $a \in A$, it gives an element $f(a) \in B$.

Example 17.2
- $f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = x^2$ for any $x \in \mathbb{R}$
- $g : \mathbb{R} \mapsto \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ given by $g(\theta) = \tan(\theta)$
- Is $h(x) = \frac{1}{x}$ a function? No – Poorly defined. If $\mathbb{R} \mapsto \mathbb{R}$ is included, still not defined because of $h(0)$

  $h : \mathbb{R} \setminus \{0\} \mapsto \mathbb{R}$ is a function
- $k : (0, 1) \mapsto \mathbb{R}$ given by $k(x) = x^2$. Still a function but it's differemt from $f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = x^2$

  Note: Domain and codomain are part of the function
- $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by $T(a, b) = (a + b, a - b)$. Yes, this is a function
- $S : \mathbb{R}^3 \mapsto \mathbb{R}^2$ given by

$$S(x, y, z) = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This is also a function. S and T are linear transformations (functions from one vector space to another)

Definition 17.3 (Injection & Surjection) — A function $f : A \mapsto B$ is <u>injective</u> (one-to-one) if for any $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$.

A function $f : A \mapsto B$ is <u>surjective</u> (onto) if for all $b \in B$, there is an $a \in A$ such that $f(a) = b$.

Example 17.4

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be given by $T(a, b) = (a + b, a - b)$. Show T is injective. Show T is surjective.

Suppose $T(x_1, y_1) = T(x_2, y_2)$, then $(x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2)$. So,

$$\begin{aligned} x_1 + y_1 &= x_2 + y_2 \\ x_1 - y_1 &= x_2 - y_2 \end{aligned}$$

Solve the above system of linear equations, we obtain $(x_1, y_1) = (x_2, y_2)$ We conclude T is injective. $\square$

T is surjective?

Let $(c, d) \in \mathbb{R}^2$ be arbitrary. We want to show there exists an $(a, b) \in \mathbb{R}^2$ with $T(a, b) = (c, d)$

$$\begin{aligned} a + b &= c \\ a - b &= d \\ a &= \frac{c + d}{2} \\ b &= \frac{c - d}{2} \end{aligned}$$

Note: $(a, b) \in \mathbb{R}^2$ is a valid input.

Take $a = \frac{c+d}{2}$ and $b = \frac{c-d}{2}$. Then,

$$\begin{aligned} T(a, b) &= \left(\frac{c + d}{2} + \frac{c - d}{2}, \frac{c + d}{2} - \frac{c - d}{2} \right) \\ &= \left(\frac{2c}{2}, \frac{2d}{2} \right) \\ &= (c, d) \end{aligned}$$

Since $(c, d) \in \mathbb{R}^2$ was arbitrary, we conclude T is surjective $\square$

§18 Dis 2: Oct 6, 2020

§18.1 Field

Definition 18.1 ((1.2)) — A field consists of a set F with two elements $0, 1 \in F$ ($0 \neq 1$) and two operations, multiplication ($\cdot$) and addition ($+$)
$(F, +)$

- $+$ is associative
- $+$ is commutative
- has an additive identity (0)
- has an additive inverse

"abelian group"
$(F^*, \cdot)$ (everything except 0) – $F \setminus \{0\} = F^*$

- assoc
- comm
- has an identity (1)
- has mult inverse

"abelian group"
Finally, distributive prop also holds

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

Linear Algebra works over any field! (Not just $\mathbb{R}$ like we did in lower div Lin Alg class).

Claim 18.1. Let F be a field. Let $\alpha \in F$ be an arbitrary element of the field. Then $0\alpha = 0$

Proof. Note since $0 + 0 = 0$

$$0\alpha = (0 + 0)\alpha$$

However, by the dist. prop,

$$(0 + 0)\alpha = 0\alpha + 0\alpha$$

Then $0\alpha = 0\alpha + 0\alpha$. Substract 0α from both sides (i.e. add its additive inverse to both sides)

$$-(0\alpha) + (0\alpha) = -(0\alpha) + 0\alpha + 0\alpha$$

So,

$$0 = 0 + 0\alpha = 0\alpha$$

So, $0\alpha = 0$ □

Claim 18.2. Let F be a field, and let $\alpha, \beta \in F$ s.t $\alpha\beta = 0$. Then either $\alpha = 0$ or $\beta = 0$.

Proof. If $\alpha = 0$, there is nothing to show. Suppose $\alpha \neq 0$. We want to show $\beta = 0$. Since $\alpha \neq 0$, $\alpha \in F^*$ has a multiplicative inverse $\alpha^{-1} \in F^*$.

Since $\alpha\beta = 0$, we can mult both sides by α^{-1} on the left to get $\alpha^{-1}(\alpha\beta) = \alpha^{-1}(0) = 0$. Moreover, by associativity,

$$\alpha^{-1}(\alpha\beta) = (\alpha^{-1}\alpha)\beta = 1\beta = \beta$$

Hence, $\beta = 0$. So, $\beta = 0$ as desired. $\square$

Example 18.2

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields. $\mathbb{Z}$ is not a field.

Example 18.3

$$\underbrace{\mathbb{Z}/12\mathbb{Z}}_{\text{integers mod 12}} = \{0, 1, 2, 3, \dots 11\}$$

Clock arithmetic. Addition is clock addition:

$$2 + 11 = 1$$

Multiplication is "clock mult"

$$2 \cdot 11 = 10$$

Multiply and add like normal but then substract nultiples of 12 until you get an element of the set.

- Additive identity: 0
- Multiplicative identity: 1

Is $\mathbb{Z}/12\mathbb{Z}$ a field?

- additive inverse ✓
- identity ✓
- comm ✓
- assoc ✓
- mult inverse ... $\implies$ NO!

Or different argument:

$$2 \cdot 6 = 0^-$$

But $2 \neq 0$ and $6 \neq 0$. This violates a property of fields:

$$\alpha\beta = 0 \quad \implies \quad \alpha = 0 \text{ or } \beta = 0$$

So $\mathbb{Z}/12\mathbb{Z}$ can't be a field.

Example 18.4

$\mathbb{Z}/3\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}\}$

- additive id: 0
- mult id: 1

Mult inv:

$$\begin{aligned} 1 \cdot 1 &= 1 \\ 2 \cdot 2 &= 1 \checkmark \end{aligned}$$

Additive inverse:

$$\begin{aligned} 0 + 0 &= 0 \\ 1 + 2 &= 0 \checkmark \end{aligned}$$

$\mathbb{Z}/3\mathbb{Z}$ is a field!

When is $\mathbb{Z}/n\mathbb{Z}$ is a field?

- $n = 2$: yes
- $n = 3$: yes
- $n = 4$: no
- $n = 13$: yes

  $\vdots$

Same sort of argument works whenever n is composite.
$\mathbb{Z}/p\mathbb{Z}$ <u>is</u> a field for p prime. Proof uses Bezat lemma (Eucledian algorithm)

Example 18.5

$$\mathbb{Z}/7\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{7}\}$$

Everything has a mult.inverse

§19 Dis 3: Oct 8, 2020

§19.1 Characteristics of a Finite Field

Let F be a finite field. Then, there must be a repeat in the following list:

$$1, 1+1, 1+1+1, \ldots$$

If there wasn't a repeat, clearly, this would be an infinite list of distinct elements in F. Then we have for some $j < k$

$$\underbrace{1+1+1\ldots+1}_{\text{j times}} = \underbrace{1+1+\ldots+1}_{\text{k times}}$$

So, $0 = \underbrace{1+1+\ldots+1}_{k-j\text{ times}} \quad k-j > 0$. Thus, in a finite field, adding 1 to itself repeatedly must at same point give 0.(need to add up 1 to itself at most $|F|$ number of times)

Claim 19.1. There is no field with 10 elements and $1+1=0$

Proof. Let F be a field of 10 elements with $1+1=0$. Let's list the elements

$$0, 1, \alpha (\alpha \neq 0, 1)$$

Is $\alpha + 1$ already on my list?

$$\begin{aligned}
\alpha + 1 = 0 \quad &\implies \alpha + 1 + 1 = 0 + 1 = 1 \implies \alpha = 1 \\
\alpha + 1 = 1? \quad &\implies \alpha = 0 \\
\alpha + 1 = \alpha? \quad &\implies 1 = 0
\end{aligned}$$

None are possible so $\alpha + 1$ is not on our list so far

$$0, 1, \alpha, \alpha + 1, \beta$$

Then, $\beta + 1$ isn't on the list.

$$0, 1, \alpha, \alpha + 1, \beta, \beta + 1$$

Notice $\alpha + \beta$ isn't on the list yet and so is $\alpha + \beta + 1$. There are 8 elements in F. Since $|F| = 10$, let $\gamma \in F$ be something not on the list so far and $\alpha + 1$ is not on the list so far, so it must be the last element of F.

$$0, 1, \alpha, \alpha + 1, \beta, \beta + 1, \alpha + \beta, \alpha + \beta + 1, \gamma, \gamma + 1$$

But then $\gamma + \alpha$ is not on the list. This would give an 11^{th} ... but $|F| = 10$ contradiction □

Note: Characteristics: the number of times you add 1 to get 0 in a field. For the case of characteristics 2, EVERYTHING IS ITS OWN ADDITIVE INVERSE.

Claim 19.2. There is no field of 10 elements with $1+1 \neq 0$ and $1+1+1=0$

Proof. List the element:

$$0, 1, 2, \alpha, \alpha + 1, \alpha + 2, \beta, \beta + 1, \beta + 2, \gamma$$

But then $\gamma + 1$ isn't on this list. – Contradiction. □

What if $1+1+1+1=0$?

$$\begin{aligned}
\underbrace{(1+1)}_{x} + \underbrace{(1+1)}_{x} &= 0 \\
x + x &= 0 \\
x(1+1) &= 0 \\
(1+1)(1+1) &= 0
\end{aligned}$$

So either $(1+1)=0$ or $(1+1)=0$. We already ruled out $1+1=0$.
Can $1+1+1+1+1=0$? List the element

$$0,1,2,3,4,\alpha,\alpha+1,\alpha+2,\alpha+3,\alpha+4$$

What is 2α? Trick: $2\cdot 3=(1+1)(1+1+1)=\underbrace{1+1+1+1+1}_{0}+1=1$

Can $2\alpha=0$? $\implies$ $\alpha=0$ or $2=0$. Can $2\alpha=1$? Mult both sides by 3

$$3\cdot 2\alpha=3$$

$\implies$ $\alpha=3$ (nope!)

$$2\alpha=2? \quad 2\alpha=3? \quad 2\alpha=4?$$

Proceed similarly and we can see that $1+1+1+1+1\neq 0$

$$1+1+1+1+1+1=0?$$
$$(1+1)(1+1+1)=0$$

$1+1=0$ or $1+1+1=0$ (but we already ruled out both cases). Now,

$$1+1+1+1+1+1+1+1=0?$$

List:

$$0,1,2,3,4,5,6,\alpha,\alpha+1,\alpha+2$$

$\alpha+3$ is not on this list.

- $8=0$? We can have $(1+1)(1+1)(1+1)=0$ but $(1+1)=0$ also ruled out.
- $9=0$ $\implies$ $(1+1+1)=0$ also ruled out.
- $10=0$ $\implies$ $(1+1)=0$ or $(1+1+1+1+1)=0$ which is also ruled out above.

So there are no fields with 10 elements. □

§20 Dis 4: Oct 13, 2020

§20.1 Vector Space and Subspace

Definition 20.1 ((2.1)) — A vector space over a field F is a set V with some additional structure:

$(V,+)$ is an abelian group (V has addition which is assoc, comm, add. inv, add. iden)

Scalar mult

$$\cdot : F\times V\to V$$

with

$$1_F\cdot v=v \quad \forall v\in V$$
$$(\alpha\beta)\cdot v=\alpha(\beta v) \quad \forall \alpha,\beta\in F, v\in V$$
$$(\alpha+\beta)v=\alpha v+\beta v$$
$$\alpha(v+w)=\alpha v+\alpha w$$

We're overloading $+$ and $\cdot$. In F :

$$\alpha+\beta, \quad \alpha\cdot\beta$$

In V :

$$v+w, \quad \alpha\cdot v$$

We say $S \subseteq V$ is a subspace if

1. It is closed under addition.
2. It is closed under scalar multiplication.
3. It is not empty.

   OR

4. $0 \in S$

Then, S will automatically become a vector space over the same field(it inherits the nice properties from V).

Example 20.2 (Abstract Vector Space)

In general, vector spaces might not always have nice geometric descriptions like in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$, etc

1. $\underbrace{\mathbb{R}[t]}_{\text{set of all polynomials in t with real coeff}} = \left\{a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n : a_i \in \mathbb{R} \forall i\right\}$ and $n \in \mathbb{Z}, n \geq 0$.

2. $\mathbb{R}[t]_n$: set of all polynomials in t with real coeff and degree $\leq n$.

   $\mathbb{R}[t]_2 = \left\{a + bt + ct^2, \quad a, b, c \in \mathbb{R}\right\}$

3. $M_n(\mathbb{R})$, $\mathbb{R}^{n\times n}$: set of n by n matrices with real coeff.

$$M_2(\mathbb{R}) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} : a, b, c, d \in \mathbb{R}\right\}$$

4. Is $GL_2(\mathbb{R})$ a subspace of $M_2(\mathbb{R})$ (as an R-vector space)?

$$GL_2(\mathbb{R}) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and is invertible }\right\}$$

   $GL_2(\mathbb{R}) \subseteq M_2(\mathbb{R})$. No it is not a subspace.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix} \notin GL_2(\mathbb{R})$$

Example 20.3 1. $C[a, b] = \{f : [a, b] \to \mathbb{R} : f \text{ is continuous}\}$ is a vector space over $\mathbb{R}$ (with usual/natural choice of addition and scalar mult).

2. $W = \{f \in C[a, b] : f(a) = 0\} \subseteq C[a, b]$

Is W is a subspace?

- $0 \in W$ since $0(a) = 0$
- Let $f, g \in W$. Since $f \in W, f(a) = 0, g \in W g(a) = 0$. Hence, $(f+g)(a) = f(a) + g(a) = 0 + 0 = 0$. Thus, $f + g \in W$.
- Let $f \in W$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\alpha(f)(a) &= \alpha(f(a)) \\ &= \alpha(0) \\ &= 0\end{aligned}$$

Thus, $W \subseteq C[a, b]$ is a subspace.

$\mathbb{C}$ is a 2-dimensional vector space over $\mathbb{R}$. $\{1, i\}$ will form a <u>basis</u> of $\mathbb{C}$ as an $\mathbb{R}$- vector space.

$$\begin{cases}\text{Linearly Indep.} \\ \text{Span all of } \mathbb{C}\end{cases} \implies \text{Every complex number can be uniquely written as } a + bi \text{ where } a, b \in \mathbb{R}$$

$\mathbb{C}$ is also a vector space over the field $\mathbb{C}$. It is a 1-dimensional vector space over $\mathbb{C}$.
Basis(every complex number can be written as $z \cdot 1$ for some $z \in \mathbb{C}$): $\{1\}$ will work (dont need i, it is allowed to be a scalar).
$\mathbb{C}$ is also a vector space over $\mathbb{Q}$. As a $\mathbb{Q}$- vector space, $\mathbb{C}$ is infinite dimensional!

Proof. (sketch) $\mathbb{C}$ is uncountably infinite. There's too many of them for us to be able to write each as a $\mathbb{Q}-$ linear combos of some finite list of vectors.
Alternative: $1, \pi, \pi^2, \pi^3, \ldots$ is an infinite list of vectors (elements of $\mathbb{C}$) which are linearly indep. over $\mathbb{Q}$. □

Example 20.4

Let V be a vector space over a field F. We define V^*, the dual vector space, as

$$V^* = \{T : V \to F | \text{T is F-linear}\}$$

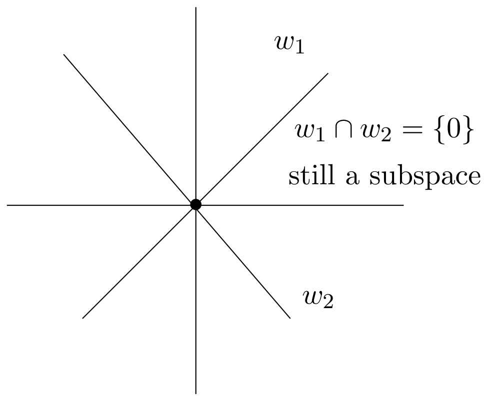
Then V^* is also a vector space over F.
Relatedly, if V, W are $F-$ vector spaces, then

$$L(V, W) = \{T : V \to W | T \text{is F-linear}\}$$

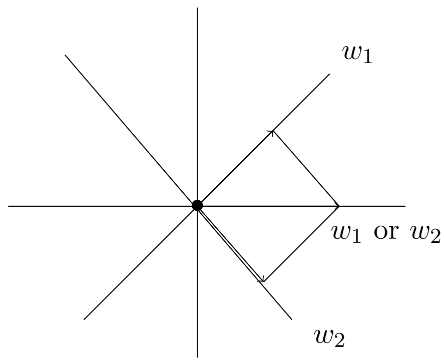
is also a vector space.

<u>*Note*</u>:Let V be a vector space over F. The intersection of subspaces of V will still be a subspace of V

Example:



Let V be a vector space over F. The union of subspaces of V might not be a subspace.



not closed under addition in general

<u>Note</u>:

$$A \cup B = \{v \in V : v \in A \text{ or } v \in B\}$$
$$A + B = \{v \in V : \text{there exists } a \in A, b \in B \ni a + b = v\}$$

§21 Dis 5: Oct 15, 2020

§21.1 Linear Independence, Span, & Subspaces

Problem 21.1. Let $\mathbb{F}$ be a field on V be a vector space over $\mathbb{F}$. Let $\alpha \in \mathbb{F}$ be arbitrary. Show $\alpha \cdot \vec{0} = \vec{0}$.

Proof. Let $\alpha \in \mathbb{F}$ be arbitrary. Note:

$$\alpha \cdot \vec{0} = \alpha(\vec{0} + \vec{0}) = \alpha \cdot \vec{0} + \alpha \cdot \vec{0}$$

Adding the additive inverse of $\alpha \cdot \vec{0}$ to both sides we see

$$-(\alpha \cdot \vec{0}) + \alpha \cdot \vec{0} = \left(-(\alpha \cdot \vec{0}) + \alpha \cdot \vec{0}\right) + \alpha \cdot \vec{0}$$
$$\vec{0} = \vec{0} + \alpha \cdot \vec{0}$$

Thus, $\vec{0} = \alpha \cdot \vec{0}$ as desired. □

Problem 21.2. If $\alpha \in \mathbb{F}$ and $\vec{v} \in V$ satisfy $\alpha \vec{v} = \vec{0}$, then either $\alpha = 0$ or $\vec{v} = \vec{0}$.

Proof. If $\alpha = 0$, we're done. If $\alpha \neq 0$, it has a mult. inverse, $\alpha^{-1} \in F^x$. Then

$$\begin{aligned} \alpha^{-1}(\alpha\vec{v}) &= (\alpha^{-1}\alpha)\cdot\vec{v} \\ &= 1_{\mathbb{F}}\cdot\vec{v} = \vec{v} \end{aligned}$$

On the other hand, since $\alpha\vec{v} = \vec{0}$, we have

$$\alpha^{-1}(\alpha\vec{v}) = \alpha^{-1}(\vec{0}) = \vec{0}$$

So we see that if $\alpha \neq 0$ then $\vec{v} = \vec{0}$. This completes the proof. $\square$

Problem 21.3. a) Find a nonempty subset $U \subseteq \mathbb{R}^2$ which is closed under scalar mult but is not a subspace of $\mathbb{R}^2$ (over $\mathbb{R}$).
b) Find a nonempty subset $U \subseteq \mathbb{R}^2$ which is closed under addition but is not a subspace.

a) Take $U = \operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} \cup \operatorname{span}\left\{\begin{pmatrix}-1\\1\end{pmatrix}\right\}$. One can show U is closed under scalar mult but not under addition.
b) Left as exercise.
Let V be a vector space over $\mathbb{F}$. For $u, w \subseteq V$ subspaces, define

$$u + w = \{\vec{u} + \vec{w} : \vec{u} \in U, \vec{w} \in W\}$$

Claim 21.1. $u + w \subseteq V$ is also a subspace.

Proof. (Sketch)

- Show $0 \in u + w$.
- Show $u + w$ is closed under $+$.
- Show $u + w$ is closed under scalar mult.

$\square$

Let $v_1, \ldots, v_k \in V$. Define

$$\operatorname{span}(\{v_1, \ldots, v_k\}) = \left\{\sum_{i=1}^{k} \alpha_i v_i : \quad \alpha_i \in \mathbb{F}\right\} \subseteq V$$

(it's the set of all $\mathbb{F}$–linear combinations of $v_1, \ldots, v_k$).

Claim 21.2. $\operatorname{Span}(\{v_1, \ldots, v_k\}) \subseteq V$ is a subspace of V.

If $S \subseteq V$ is an infinite subset of V, we can still define span(S) as the set of all <u>finite</u> linear combos.

$$\sum_{i=1}^{k} \alpha_i v_i \quad \text{where } \alpha_i \in \mathbb{F}, v_i \in S$$

$\operatorname{Span}(S) \subseteq V$ is also a <u>subspace</u>.

Example 21.1

$\mathbb{F} = \mathbb{R}, V = \mathbb{R}^3$.

$$U = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\} = x - \text{axis}$$

$$W = \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} = xy - \text{plane}$$

What is $u + w$?

$u + w = xy-$ plane.

<u>Claim</u>: $u + w = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

Proof. Let $\vec{x} \in u + w$. Then there exists $\vec{u} \in U$ and $\vec{w} \in W$ s.t. $\vec{x} = \vec{u} + \vec{w}$. Since $\vec{u} \in U = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$, there exsits an $\alpha \in \mathbb{R}$ with $\vec{u} = \alpha \cdot \begin{pmatrix}1\\0\\0\end{pmatrix}$. Similary, since $\vec{w} \in W = \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$, there exists $\beta, \gamma \in \mathbb{R}$ with $\vec{w} = \beta \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} + \gamma \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}\beta\\\beta+\gamma\\0\end{pmatrix}$. Hence,

$$\begin{aligned}\vec{x} = \vec{u} + \vec{w} &= \begin{pmatrix}\alpha\\0\\0\end{pmatrix} + \begin{pmatrix}\beta\\\beta+\gamma\\0\end{pmatrix}\\ &= (\alpha+\beta)\begin{pmatrix}1\\0\\0\end{pmatrix} + (\beta+\gamma)\begin{pmatrix}0\\1\\0\end{pmatrix}\end{aligned}$$

Thus, $\vec{x} \in \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$. This shows $U + W \subseteq \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$.

Conversely, suppose $\vec{x} \in \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$. Then, there exists real number $a, b \in \mathbb{R}$ s.t.

$$\vec{x} = a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix}$$

<u>*Note*</u>: $a\begin{pmatrix}1\\0\\0\end{pmatrix} \in \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$. Meanwhile,

$$b\begin{pmatrix}0\\1\\0\end{pmatrix} = 0\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} \in \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

Thus, there exists vectors $\vec{u} \in U$ and $\vec{w} \in W$ namely $\vec{u} = a\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\vec{u} = a\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\vec{w} = b\begin{pmatrix}0\\1\\0\end{pmatrix}$ s.t. $\vec{x} = \vec{u} + \vec{w}$. Thus, $\vec{x} \in u + w$. This shows

$$\text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} \subseteq U + W$$

We shoed earlier that

$$U + W \subseteq \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

We conclude $U + W = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$. □

More generally, one can show $\text{span}(S) + \text{span}(T) = \text{span}(S \cup T)$.
Let V be a vector space over $\mathbb{F}$. Let $S \subseteq V$ be any subset. Properties of $\text{span}(S\ )$:

- $\text{span}(S) \subseteq V$ is a subspace.
- $S \subseteq$ span(S).
- If W is a subspace of V and $S \subseteq W$, then $\text{span}(S\ ) \subseteq W$.

Suppose $v_1, \ldots, v_n$ span V. (i.e., $\text{span}(v_1, \ldots, v_n) = V$. Show $v_1, v_2 - v_1, v_3 - v_2, \ldots, v_n - v_{n-1} = V\ )$.
Notation: Let $w_i = v_i - v_{i-1}$ if $i > 1$ and $w_1 = v_1$.

Proof. We'll show $\forall i \text{span}(w_1, \ldots, w_n)$ for $i = 1, \ldots, n$. Use induction

- For $i = 1$, $v_1 = w_1 \in \text{span}(w_1, \ldots, w_n)$.
- Suppose $v_k \in \text{span}(w_1, \ldots, w_n)$ where $k < n$.
- $$\begin{aligned} v_{k+1} &= v_{k+1} - v_k + v_k \\ &= w_{k+1} + v_k \end{aligned}$$

Since $w_{k+1} \in \text{span}(w_1, \ldots, w_n)$ and $v_k \in \text{span}(w_1, \ldots, w_n)$ and since $\text{span}(w_1, \ldots, w_n) \subseteq V$ is a subspace, we have $v_{k+1} = w_{k+1} + v_k \in \text{span}(w_1, \ldots, w_n\ )$. By induction, we get

$$v_1, \ldots, v_n \in \text{span}(v_1, \ldots, v_n) \subseteq \text{span}(w_1, \ldots, w_n)$$

But $\text{span}(v_1 \ldots, v_n) = V$. So, we have

$$V \subseteq \text{span}(w_1, \ldots, w_n) \subseteq V$$

So $\text{span}(w_1, \ldots, w_n) = V$. □

§22 Dis 6: Oct 20, 2020

§22.1 Review of Linear Independence & Dependence

Let V be a vector space over $\mathbb{F}$. Recall $v_1, \ldots, v_k \in V$ are linearly indep. if for any $\alpha_1, \ldots, \alpha_k \in F$ with $\sum_{i=1}^{k} \alpha_i v_i = 0$, we have $\alpha_i = 0$ for $i = 1, \ldots, k$.
To show $v_1, \ldots, v_k \in V$ are linearly indep:
Suppose we have $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ with

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = 0$$

Try to show $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_k = 0$.

Example 22.1

$V = \mathbb{F}^3$. Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Claim 22.1. v_1, v_2, v_3 are linearly indep.

Proof. Let $\alpha, \beta, \gamma \in \mathbb{F}$ s.t.

$$\alpha v_1 + \beta v_2 + \gamma v_3 = \vec{0}$$

Then, we have

$$\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$$\begin{pmatrix} \alpha + \beta + \gamma \\ \beta + \gamma \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From this we see:

$$\begin{aligned} \alpha + \beta + \gamma &= 0 \\ \beta + \gamma &= 0 \\ \gamma &= 0 \end{aligned}$$

Solve the above system, we have $\alpha = 0, \beta = 0, \gamma = 0$. Thus $\{v_1, v_2, v_3\}$ are linearly indep. $\square$

Example 22.2

Let V be a vector space over $\mathbb{F}$. Let $\vec{v}, \vec{w} \in V$. Set $\vec{x} = \vec{v} + \vec{w}$, $\vec{y} = \vec{v} - \vec{w}$, and $\vec{z} = 4\vec{v} + 2\vec{w}$.

Claim 22.2. $\{\vec{x}, \vec{y}, \vec{z}\}$ are linearly dep.

Proof. Observe $3\vec{x} + \vec{y} = \vec{z}$, so $3\vec{x} - \vec{y} - \vec{z} = \vec{0}$. This is a nontrivial linear combos. So, $\{\vec{x}, \vec{y}, \vec{z}\}$ are linearly dep. $\square$

Remark 22.3. More generally, any 3 vectors in $\operatorname{span}(\vec{v}, \vec{w})$ are necessarily going to be linearly dep. But this requires proof!

V is a vector space over $\mathbb{F}$.

Claim 22.3. $\vec{v}, \vec{w} \in V$ are linearly dep $\iff$ $\vec{v}$ is a multiple of $\vec{w}$ or $\vec{w}$ is a multiple of $\vec{v}$.

Proof. Suppose $\vec{v}, \vec{w} \in V$ are linearly dep. Then, $\exists \alpha, \beta \in \mathbb{F}$ not both 0 with $\alpha\vec{v} + \beta\vec{w} = \vec{0}$.
Case 1: $\alpha \neq 0$

Then we have $\alpha\vec{v} = -\beta\vec{w}$. Since $\alpha \neq 0$, we have $\alpha^{-1} \in \mathbb{F}^x$. Then $\alpha^{-1}\alpha\vec{v} = \alpha^{-1}(-\beta\vec{w})$. So

$$\vec{v} = \underbrace{-\alpha^{-1}\beta}_{\in\mathbb{F}}\vec{w}$$

Thus, in this case, $\vec{v}$ is a multiple of $\vec{w}$.
Case 2: $\alpha = 0$

In this case, $\beta \neq 0$ (since α, β are not both 0). Then $0\vec{v} + \beta\vec{w} = \vec{0}$. So, $\vec{0} + \beta\vec{w} = \vec{0}$ and thus $\beta\vec{w} = \vec{0}$. Then either $\beta = 0$ or $\vec{w} = \vec{0}$. But $\beta \neq 0$, so $\vec{w} = \vec{0} = 0 \cdot \vec{v}$. Hence, in this case, $\vec{w}$ is a multiple of $\vec{v}$.
Conversely, suppose $\vec{v}$ is a multiple of $\vec{w}$. Then there is an $\alpha \in \mathbb{F}$ with $\vec{v} = \alpha\vec{w}$. So,

$$1\vec{v} - \alpha\vec{w} = \vec{0}$$

The coefficient of $\vec{v}$ is nonzero. Thus, this is a nontrivial linear combos of $\vec{v}$ and $\vec{w}$ which gives $\vec{0}$. Hence $\vec{v}$ and $\vec{w}$ are linearly dep. A similar argument works in the case where $\vec{w}$ is a multiple of $\vec{v}$. $\square$

Example 22.4

$f : \mathbb{R} \to \mathbb{R} \quad f(x) = \sin x$
$g : \mathbb{R} \to \mathbb{R} \quad g(x) = \cos x$
$f, g \in C(\mathbb{R})$ where $C(\mathbb{R}) = \{h : \mathbb{R} \to \mathbb{R} | h \text{ is continuous}\}$, $\mathbb{R}-$ vector space. Show f, g are linearly indep.

Proof. Suppose we have $\alpha, \beta \in \mathbb{R}$ with $\alpha f + \beta g = \underbrace{0}_{\text{zero function}} \in C(\mathbb{R})$. Then for any

$x \in \mathbb{R}$,

$$(\alpha f + \beta g)(x) = 0(x) = 0$$

So,

$$\alpha \sin(x) + \beta \cos(x) = 0$$

Plugging in $x = 0$, we see

$$\alpha \sin(0) + \beta \cos(0) = 0$$

so, $\beta = 0$. Plugging in $x = \frac{\pi}{2}$, we see

$$\alpha \sin\left(\frac{\pi}{2}\right) + \beta \cos\left(\frac{\pi}{2}\right) = 0$$

So, $\alpha = 0$. We have $\alpha = 0, \beta = 0$. Thus, f and g are linearly indep in $C(\mathbb{R})$, as desired. $\square$

Claim 22.4. $e^x, e^{4x} \in C(\mathbb{R})$ are linearly indep.

Proof. Suppose we have $\alpha, \beta \in \mathbb{R}$ with $\alpha e^x + \beta e^{4x} = 0 \in C(\mathbb{R})$. That is, for any $x \in \mathbb{R}$,

$$\alpha \cdot e^x + \beta \cdot e^{4x} = 0$$

Then for any real number x, we have

$$\beta = -\alpha e^x \cdot e^{-4x} = -\alpha e^{-3x}$$

For any $x \in \mathbb{R}$, we have $\beta = -\alpha e^{-3x}$ (but β is a constant).

- At $x = 1$, $\beta = \alpha e^{-3}$.
- At $x = 0$, $\beta = -\alpha$.

So $\alpha e^{-3} = -\alpha$, which gives

$$\alpha(1 - e^{-3}) = 0$$

Thus either $\alpha = 0$ or $e^{-3} = 1$. So $\alpha = 0$. Then $\beta = -\alpha = -0 = 0$. So $\alpha = 0, \beta = 0$, as desired. $\square$

V is a vector space over $\mathbb{F}$. True or false:

1. If $v_1, \ldots, v_k \in V$ are linearly indep., and $\alpha \neq 0 \in \mathbb{F}$, then $\alpha v_1, \ldots, \alpha v_k$ are also linearly indep. – TRUE

2. If $v_1, \ldots, v_k \in V$ are linearly dep., and $\alpha \neq 0 \in \mathbb{F}$, then $\alpha v_1, \ldots, \alpha v_k$ are also linearly indep. – TRUE

3. $v_1, \ldots, v_k \in V$ are linearly indep., and $w \in V$, then $v_1 + w, v_2 + w, \ldots, v_k + w$ are linearly indep. as well. – FALSE

4. $v_1, \ldots, v_k \in V$ are linearly dep., and $w \in V$, then $v_1 + w, v_2 + w, \ldots, v_k + w$ are linearly indep. as well. – FALSE

Claim 22.5. $v_1, \ldots, v_k \in V$ are linearly indep., $\alpha \neq 0 \in \mathbb{F}$. Then $\alpha v_1, \ldots, \alpha v_k \in V$ are also linearly indep.

Proof. Suppose we have $\beta_1, \ldots, \beta_k \in \mathbb{F}$ with $\beta_1(\alpha v_1) + \beta_2(\alpha v_2) + \ldots + \beta_k(\alpha v_k) = \vec{0}$. Then

$$(\beta_1 \alpha)v_1 + \ldots + (\beta_k \alpha)v_k = 0$$

Since $v_1, \ldots, v_k$ are linearly indep., this gives

$$\beta_i \alpha = 0 \quad \text{for all } i = 1, \ldots, k$$

So either $\beta_i = 0$ or $\alpha = 0$ for each $i = 1, \ldots, k$. But, $\alpha \neq 0$. So $\beta_i = 0$ for each $i = 1, \ldots, k$. Thus, $\alpha v_1, \ldots, \alpha v_k$ are linearly indep. □

Claim 22.6. Let $v_1, \ldots, v_k \in V$ be linearly dep, $\alpha \neq 0 \in \mathbb{F}$. Then $\alpha v_1, \ldots, \alpha v_k \in V$ are also linearly dep.

Proof. Suppose $\alpha v_1, \ldots, \alpha v_k$ are actually linearly dep. Taking $\alpha^{-1} \neq 0 \in \mathbb{F}$ by previous claim, multiplying through by α^{-1} keeps the list linearly indep. Then, $v_1, \ldots, v_k \in V$ are linearly indep. Contradiction! Thus, by contradiction, $\alpha v_1, \ldots, \alpha v_k$ are also linearly dep. □

Claim 22.7. $v_1, \ldots, v_k \in V$ and $w \in V$. Then $v_1 + w, \ldots, v_k + w$ maybe linearly dep. or linearly indep. depending on the choice of $\vec{w}$.

Proof. Take $\vec{w} = -\vec{v_1}$. Then

$$\vec{v_1} + \vec{w} = \vec{0}$$

Then

$$1(\vec{v_1} + \vec{w}) + 0(\vec{v_2} + \vec{w}) + \ldots + 0(\vec{v_k} + \vec{w}) = \vec{0}$$

This gives a nontrivial linear combos. So not linearly indep. □

Claim 22.8. $v_1 \ldots, v_k \in V$ linearly dep., $\vec{w} \in V$. Then $\vec{v_1} + \vec{w}, \ldots, \vec{v_k} + \vec{w}$ maybe linearly dep. or indep. depending on the choice of $\vec{w} \in V$.

Proof. Left as exercise. (Actually for the choice of $\vec{w} \in \text{Span}(v_1, \ldots, v_k)$ the proof is right below. □

§23 Dis 7: Oct 22, 2020

§23.1 Review of Linear Independence and Linear Transformation

Proof. Since $v_1 + w, \ldots, v_k + w \in V$ are linearly dep., there exists scalars $a_1, \ldots, a_k \in \mathbb{F}$, not all 0, s.t.

$$a_1(v_1 + w) + \ldots + a_k(v_k + w) = 0$$

Then we have

$$\sum_{i=1}^{k} a_i v_i + \sum_{i=1}^{k} a_i w = 0$$

So,

$$-\left(\sum_{i=1}^{k} a_i\right) w = \sum_{i=1}^{k} a_i v_i$$

Let $\beta = -\sum_{i=1}^{k} a_i \in \mathbb{F}$. Then $\beta \vec{w} = \sum_{i=1}^{k} a_i \vec{v_i}$. Suppose $\beta = 0$. Then this would read

$$0 = \sum_{i=1}^{k} a_i \vec{v_i}$$

Since $v_1, \ldots, v_k$ are linear indep., this gives $\alpha_i = 0$ for all $i = 1, \ldots, k$. Yet, we choose $\alpha_1, \ldots, \alpha_k$ not all 0. Contradiction!
So $\beta \neq 0$. Hence,

$$\vec{w} = \beta^{-1} \sum_{i=1}^{k} a_i v_i = \sum_{i=1}^{k} (\beta^{-1} a_i) v_i$$

which in the span of $v_1, \ldots, v_k$, as desired. □

Corollary 23.1

$v_1, \ldots, v_k \in V$ are linearly indep., $w \in V$, and $w \notin \text{span}(v_1, \ldots, v_k)$. Then $v_1 + w, \ldots, v_k + w$ are also linearly indep.

Claim 23.1. $v_1, \ldots, v_k \in V$ are linearly indep., $w \in V$ and $w \notin \text{span}(v_1, \ldots, v_k)$. Then $v_1, \ldots, v_k, w$ are linearly indep.

Proof. Suppose we have $a_1, \ldots, a_{k+1} \in \mathbb{F}$ with

$$a_1 v_1 + \ldots + a_k v_k + a_{k+1} w = 0$$

Case 1: $a_{k+1} = 0$.
Then, we have

$$a_1 v_1 + \ldots + a_k v_k = 0$$

Since $v_1, \ldots, v_k$ are linearly indep., we have $a_1 = 0, \ldots, a_k = 0$. Hence in this case, all coeffs are 0, as desired.
Case 2: $a_{k+1} \neq 0$
Then we have

$$-a_{k+1} w = a_1 v_1 + \ldots + a_k v_k$$

Since $-a_{k+1} \neq 0$, it has a mult. inverse $\beta \in \mathbb{F}$. Then we have

$$\vec{w} = \beta a_1 v_1 + \ldots + \beta a_k v_k \in \text{ span } (v_1, \ldots, v_k)$$

But $\vec{w} \notin \text{ span}(v_1, \ldots, v_k)$. So we have a contradiction. So a_{k+1} must be 0. By the previous case, $a_1 = 0, \ldots, a_{k+1} = 0$. Thus, $v_1, \ldots, v_k, w$ are linearly indep., as desired. □

Definition 23.2 ((6.1)) — $(v_1, \ldots, v_m)$ is a basis for V if

1. $v_1, \ldots, v_m \in V$ are linearly indep.
2. span $(v_1, \ldots, v_m) = V$.

Theorem 23.3
Every vector space has a basis.(potentially an infinite list of vectors)

Theorem 23.4
If $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_l)$ are both bases for V, then $l = m$.

Define $\dim V = \#$ of elements in any basis of V.

Proposition 23.5
Let V be a vector space with $\dim V = n$. Suppose $v_1, \ldots, v_n$ are linearly indep. Then, $\operatorname{span}(v_1, \ldots, v_n) = V$. So, $v_1, \ldots, v_n$ is a basis of V.

Proposition 23.6
Let V be a vector space with $\dim V = n$. Suppose $\operatorname{span}(v_1, \ldots, v_n) = V$. Then, $v_1, \ldots, v_n$ are linearly indep. So $(v_1, \ldots, v_n)$ is a basis of V.

Recall T is injective $\iff$ whenever $T(x) = T(y)$ we have $x = y$. T is surjective $\iff$ for all $\vec{w} \in W$, there is a $\vec{v} \in V$ s.t. $T(\vec{v}) = \vec{w}$.
From the perspective of list of vectors,

- $v_1, \ldots, v_k \in V$ are linearly indep. in V, then $T(v_1), \ldots, T(v_k)$ are also linearly indep in W $\iff$ T is injective.
- T is surjective $\iff$ if $v_1, \ldots, v_k \in V$ have $\operatorname{span}(v_1, \ldots, v_k) = V$, then $T(v_1), \ldots, T(v_k) \in W$ satisfy $\operatorname{span}(T(v_1), \ldots, T(v_k)) = W$.
- T is bijective $\iff$ T is injective and surjective $\iff$ if $v_1, \ldots, v_k \in V$ are a basis of V, then $T(v_1), \ldots, T(v_k) \in W$ are a basis of W.

From the perspective of subspace,

$$\ker(T) \subseteq V = \{v \in V : T(v) = 0\}$$
$$\operatorname{im}(T) \subseteq W = \{T(\vec{v}) : \vec{v} \in V\}$$

- T is injective $\iff$ $\ker T = \{0\}$.
- T is surjective $\iff$ $\operatorname{im}(T) = W$.
- T is bijective $\iff$ $\ker T = \{0\}$ and $\operatorname{im} T = W$.

§23.2 Hw2 # 1

Let $V \neq 0$ be a vector space over $\mathbb{F}$. Suppose V can be spanned by one vector, say $V = \operatorname{span}(\vec{x}\ )$ for some $\vec{x} \in V$. Let $W \subseteq V$ be a subspace. Then, either $W = \{0\}$ or $W = V$.

Proof. Let $W \subseteq V$ be a subspace of V, and suppose $W \neq \{0\}$. We want to show $W = V$. Let $\vec{w} \in W \setminus \{0\}$, i.e. $\vec{w} \in W, \vec{w} \neq 0$. Since $\vec{w} \in W$, we have $\text{span}(\vec{w}\ ) \subseteq W$. Meanwhile, $\vec{w} \in W \subseteq V = \text{span}(\vec{x})$. So there is an $\alpha \in \mathbb{F}$ with $\vec{w} = \alpha\vec{x}$. If $\alpha = 0$, this would give $\vec{w} = \vec{0}$ but $\vec{w} \neq \vec{0}$. So we have $\alpha \neq 0$. But then $\alpha \in \mathbb{F}^x$ has a mult. inv $\alpha^{-1} \in \mathbb{F}^x$, so $\vec{x} = \alpha^{-1}\vec{w} \in \text{span}(\vec{w})$.
Thus, $\text{span}(\vec{x}\ ) \subseteq \text{span}(\vec{w})$. So we have

$$V = \text{span}(\vec{x}) \subseteq \text{span}(\vec{w}) \subseteq W \subseteq V$$

So all containments must be equality. Thus, $W = V$.
Part 2: Suppose $V \neq 0$ has $V = \text{span}(\vec{v}, \vec{w})$ where $\vec{v}, \vec{w} \in V$ are linearly indep.

Claim 23.2. If $W \subseteq V$, then either $W = 0, W = V$, or $W = \text{span}(\vec{w})$ for some $\vec{w} \in V \setminus \{0\}$.

Setup: Let $W \subseteq V$ be a subspace with $W \neq 0$ and $W \neq V$. Consider nonzero vector $\vec{w_1} \in W \setminus \{0\}$. Suppose $\text{span}(\vec{w_1}) \neq W$. Then, find some $\vec{w_2} \in W \setminus \text{span}(\vec{w_1})$. Argue

- $\vec{w_1}, \vec{w_2}$ are linearly indep.
- In fact, they span all of V.
  Assume they don't, find $\vec{w_3} \in V \setminus \text{span}(\vec{w_1}, \vec{w_2})$
- $\vec{w_1}, \vec{w_2}, \vec{w_3}$ are linearly indep.
- Ch2, Thm 4: V cannot have a list of linearly indep. vectors with more than 2 elements. $\square$

§24 Dis 8: Oct 27, 2020

§24.1 A Note on Object Types

Types of objects:

- Field: $(\mathbb{F}, +, \cdot, 0, 1)$
- Vector space over a field $\mathbb{F}$: $(V, +, \vec{0})$ with scalar mult. $(\cdot : \mathbb{F} \times V \to V)$
- Subspace of a vector space V: $S \subseteq V$ with $\vec{0} \in S$ and S is closed under linear combo.
  AND, $(S, +, \vec{0})$ and $\mathbb{F} \times S \to S$ will make S into a vector space over $\mathbb{F}$.
- Vector in V: an element $\vec{v} \in V$
- A list of vectors: $v_1, \ldots, v_k \in V$. We might write it as a k-tuple,

  $$(v_1, \ldots, v_k)$$

  to emphasize the order of the vectors.
  For an infinite list of vectors, we might write it as a set of vectors

  $$\{v_1, v_2, \ldots\} \subseteq V$$

  (more generally, a subset $A \subseteq V$)

1. The span of a list of vectors in V is a subspace of V.

Example 24.1

$V = \mathbb{R}^2, W = x-$axis $\subseteq V$

$$\operatorname{span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}\right) = \underbrace{W}_{\text{subspace of } V}$$

More generally, span of a list of vectors in V is a vector space. This list of vectors spans this subspace (or a vector space).

2. Linearly Independent: – list of vectors.

3. A basis of a vector space is a list of vectors. That list of vectors is linearly indep. The span of that list of vectors gives us the vector space.

4. The dimension of a vector space is just a non-negative integer.

   "Dimension of this basis – nonsense"

   "number of vectors in this basis" ✓

§24.2 Review

1. Suppose $\operatorname{span}(v_1, \ldots, v_k) = V$ and $w_1, \ldots, w_m$ are linearly indep. Then, $m \le k$ – Proof idea: Get a system eqns with more vars then eqns. RREF. Get a nontrivial solution. – Thm 4, Ch. 2.

   Meaning: suppose the $\operatorname{span}(v_1, \ldots, v_k) = V$. Then, there is no linearly indep. list of vectors whose length exceeds k.

2. If V has one basis which is finite, then every basis must be finite.

   Proof: the first list spans V, so the second list of linearly indep. vectors must be shorter (or equal) by 1).

3. In fact, if you have two different finite bases of V, they have the same number of vectors.

   Proof idea: length of the first list is longer or equal to second list. Length of second list is longer or equal to length of first.

4. Dimension of V is well-defined in this case. That is, any two different finite bases have same number of vectors.

5. Suppose $\dim(V)$ is finite. Any linearly indep. list of vectors in V can be extended to a basis.

   Proof: $v_1, \ldots, v_k$ linearly indep. Either $\operatorname{span}(v_1, \ldots, v_k) = V$ and we're done, or, pick $v_{k+1} \notin \operatorname{span}(v_1, \ldots, v_k)$. Then, $v_1, \ldots, v_k, v_{k+1}$ is still linearly indep. But we have an upper bound on how long linearly indep. list of vectors can be, since V is finite dimensional. This process must end.

6. Any finite spanning set of V can be shortened to get a basis of V.

Proof. $\text{Span}(v_1, \dots, v_k) = V$. If $v_1, \dots, v_k$ linearly indep., done. If they're not, we can solve for one of them as a linear combo of the rest.

$$v_i = \sum_{j \neq i} a_j v_j \qquad a_j \in \mathbb{F}$$

Then, $\text{span}(v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_k)$ is still V (v_i is a redundant vector). We started with a finite list, so the process must terminate. □

7. $\dim V$ is finite $\iff$ V is the span of some finite list of vectors.

8. Let $v_1, \dots, v_k \in V$ be linearly indep. Then $\dim\ \text{span}(v_1, \dots, v_k) = k$.

   Proof. $W =\ \text{span}(v_1, v_2, \dots v_k) \subseteq V$ $v_1, \dots, v_k$ are linearly indep. in W. Clearly, $\text{span}(v_1, \dots, v_k) = W$ so they form a basis of W. So, $\dim W = \#$ of vectors in the basis $v_1, \dots, v_k = k$. □

9. Let V be finite dimensional.

   $$\dim V = \text{ length of the shortest list of vectors whose span is all of } V$$

   $$\dim V = \text{ length of the longest list of vectors that are linearly indep.}$$

   Linearly indep. lists can be expanded to get a basis.

10. $\dim W \leq k$ where $W = \text{span}(v_1, \dots, v_k) \subseteq V$.

11. $V = \text{span}(v_1, \dots, v_k)$. If $\dim V = k$, then $v_1, \dots, v_k$ are linearly indep.

    Proof. By 10), $\dim V \leq k$

    $$\dim V = \text{ length of shortest list of whose span is all of V}$$

    We can't get rid of any of $v_1, \dots, v_k$ in toss-out thm. If they had been dependent, we could toss something out, so they must be linearly indep. □

§25 Dis 9: Oct 29, 2020

§25.1 Dis 8 (Cont'd)

V finite dimensional, $\dim V = n$, $v_1, \dots, v_k \in V$ are linearly indep. Then, we can extend to a basis of all of V. That is, $\exists v_{k+1}, \dots, v_n \in V$ s.t. $v_1, \dots, v_k, v_{k+1}, \dots, v_n \in V$ form a basis of V (Extension Theorem). – useful fact for proofs.
Suppose $v_1, \dots, v_n \in V$

1. If $v_1, \dots, v_n \in V$ are linearly indep., then $\text{span}(v_1, \dots, v_n) = V$. So $v_1, \dots, v_n$ form a basis of V.

2. If $v_1, \dots, v_n \in V$ have $\text{span}(v_1, \dots, v_n) = V$, then $v_1, \dots, v_n$ are linearly indep. So $v_1, \dots, v_n \in V$ form a basis of V.

Claim 25.1. Let $W \subseteq V$ be a subspace of V where V is a vector space over $\mathbb{F}$. Suppose V is finite dimensional. Then, W is also finite dimensional, and $\dim W \leq \dim V$

Proof. If V is finite dimensional, say $\dim V = n$, then no list of linearly indep. vectors in W can exceed length n. Any basis of W will therefore have $\leq n$ elements, since a basis of W is, in particular, a list linearly indep. So $\dim W \leq n$. $\square$

> **Remark 25.1.** On homework 3 # 2, $\dim W = \dim V \implies W = V$.

Theorem 25.2

V finite dimensional vector space over $\mathbb{F}$, $W_1, W_2 \subseteq V$ subspaces.

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Proof. Note that $\dim(W_1 \cap W_2) = k$
Let $w_1, \ldots, w_k$ be a basis of $W_1 \cap W_2$. Then $w_1, \ldots, w_k \in W_1$ are linearly indep. So, they can be extended to a basis. That is, $\exists x_1, \ldots, x_{m-k} \in W_1$ s.t. $w_1, \ldots, w_k, x_1, \ldots, x_{m-k} \in W_1$ form a basis of W_1 where $\dim W_1 = m$. Similarly, $\exists y_1, \ldots, y_{n-k} \in W_2$ s.t. $w_1, \ldots, w_k, y_1, \ldots, y_{n-k} \in W_2$ form a basis of W_2 where $\dim W_2 = n$.

Claim 25.2. $w_1, \ldots, w_k, x_1, \ldots, x_{m-k}, y_1, \ldots, y_{n-k} \in W_1 + W_2$ form a basis of $W_1 + W_2$.

Check

1. Every vector $z \in W_1 + W_2$ can be written as a linear comb of the vectors on our list.
2. The list of vectors is linearly indep.

   Suppose

   $$a_1 w_1 + \ldots + a_k w_k + b_1 x_1 + \ldots + b_{n-k} x_{n-k} + c_1 y_1 + \ldots + c_{n-k} y_{n-k} = 0$$

   Then note

   $$\sum_{i=1}^{n-k} c_i y_i \in W_2$$

   But also,

   $$\sum_{i=1}^{n-k} c_i y_i = -\sum_{i=1}^{k} a_i w_i - \sum_{i=1}^{m-k} b_i x_i \in W_1$$

   So $\sum_{i=1}^{n-k} c_i y_i \in W_1 \cap W_2$. But $w_1, \ldots, w_k$ form a basis of $W_1 \cap W_2$, so $\exists d_1, \ldots, d_k \in \mathbb{F}$ with

   $$\sum_{i=1}^{n-k} c_i y_i = \sum_{i=1}^{k} d_i w_i$$

   Similarly, $\exists \alpha_1, \ldots, \alpha_k \in \mathbb{F}$

   $$\sum_{i=1}^{n-k} b_i x_i = \sum_{i=1}^{k} \alpha_i w_i$$

So, we get

$$\sum_{i=1}^{k}(a_i + d_i + \alpha_i)w_i = 0$$

So each $\alpha_i + d_i\alpha_i = 0$ – try to show $a_i = 0$, etc ☐

§25.2 Homework 3 Problem 7

Try $\dim V = 2$.

§26 Dis 10: Nov 3, 2020

§26.1 Dimension

$T : V \to W$ linear. Say $\dim V = n, \dim W = m$.
T is injective $\iff$ $\dim \ker T = 0$. Similarly, T is surjective $\iff$ dim im $T = m$ $\iff$ $\text{rank}(T) = m$
Important Result:
Rank – Nullity: $\dim(\ker(T)) + \dim(\text{ im}(T)) = \dim V$. This says the subspaces, in general, are not related but their dimensions add to $\dim(V)$.

Corollary 26.1
$T : V \to V$ linear, $\dim V = n$. Then T is injective $\iff$ T is surjective $\iff$ T bijective.

Proof. By rank – nullity,

$$\dim(\ker(T)) + \text{rank}(T) = n = \dim V$$

Hence,

$$\dim(\ker(T)) = n - \text{ rank}(T)$$

So, T is injective $\iff$ $\dim(\ker(T)) = 0$ $\iff$ $\text{rank}(T) = n = \dim V$ $\iff$ $T : V \to V$ is surjective.
Finally, T is bijective $\implies$ T is injective and T is injective $\implies$ T is surjective and thus T is bijective. So T is bijective $\iff$ T is injective $\iff$ T surjective. ☐

Remark 26.2. Need $T : V \to V$ and $\dim V$ finite.

In the infinite dimensional case, this fails.

Exercise 26.1. Find an example.

Corollary 26.3
$T : V \to W$, $\dim V = n, \dim W = m$. Then rank(T) $\leq n = \dim V$ and rank(T) $\leq m = \dim W$.

Proof. Note by rank – nullity, $\text{rank}(T) = n - \dim(\ker(T)) \leq n$. Meanwhile, im(T)$\subseteq W$, so rank(T) $= \dim(\text{im}(T)) \leq \dim W = m$. ☐

Corollary 26.4

$T : V \to W, \dim V = n, \dim W = m$. Then $\dim \ker(T) \geq n - m$.

Proof. $\dim(\ker(T)) = n -$ $\text{rank}(T) \geq n - m$. ☐

Example 26.5

$T : \mathbb{R}^3 \to \mathbb{R}^2$ any $\mathbb{R}$ linear map.

$$\dim(\ker T) \geq 3 - 2 = 1$$

So there is no injective linear map from $\mathbb{R}^3 \to \mathbb{R}^2$.

Example 26.6

$T : \mathbb{R}^5 \to \mathbb{R}^6$

$$\dim(\ker(T)) \geq 5 - 6 = -1$$

This result does not tell us anything.

Example 26.7

$T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (x, y)$

$$\dim \ker(T) = 1 = 3 - 2$$

$S : \mathbb{R}^3 \to \mathbb{R}^2$, $S(x, y, z) = (y, 0)$

$$\dim \ker(T) = 2 > 3 - 2$$

$T : V \to W, \dim V = n, \dim W = m$ and $\dim(\ker T) \geq n - m$. When do we get equality?

$$\begin{aligned} &\dim(\ker T) = n - m \\ \iff &\text{rank}(T) = m \\ \iff &\text{ T is surjective} \end{aligned}$$

$T : V \to W$ linear and T bijective. Then we have a function, $T^{-1} : W \to V$, the inverse function of T.

Claim 26.1. T^{-1} is also linear.

Proof. Let $w_1, w_2 \in W$ and $\alpha \in \mathbb{F}$ be arbitrary. We want to show

$$T^{-1}(\alpha w_1 + w_2) = \alpha T^{-1}(w_1) + T^{-1}(w_2)$$

Let $T^{-1}(w_1) = v_1 \in V$ and $T^{-1}(w_2) = v_2 \in V$. Then $T(\alpha v_1 + v_2)$

$$\begin{aligned} &= \alpha T(v_1) + T(v_2) \\ &= \alpha w_1 + w_2 \end{aligned}$$

Then,

$$\begin{aligned} T(\alpha v_1 + v_2) &= \alpha w_1 + w_2 \\ T^{-1}(\alpha w_1 + w_2) &= \alpha v_1 + v_2 \end{aligned}$$

So, $T^{-1} : W \to V$ is linear. □

§26.2 Homework 4 # 4

$$\text{U} \xrightarrow{\text{S}} \text{V} \xrightarrow{\text{T}} \text{W}$$

$$\text{U} \xrightarrow{\text{S}} \text{im(S)} \xrightarrow{T|_{im(S)}} \text{W}$$

$$T : V \to W \quad \text{im(S)} \subseteq V \text{ subspace}$$

$$T|_{im(S)} : im(S) \to W$$

Note

$$(T|_{\text{im}(S)})(a) = T(a), \quad \forall a \in \text{ im}(S)$$

$\text{im}(T \circ S) = \text{ im}(T|_{\text{im}(S)}) \subseteq W$

Proof. Let $y \in \text{im}(T \circ S)$. Then there is an $x \in U$ s.t. $(T \circ S)(x) = y$. So, $T(S(x)) = y$. Note $a := S(x) \in \text{ im}(S)$. Moreover, $T|_{\text{im}(S)}(a) = T(a) = T\left(S(x)\right) = y$. So, $y \in \text{im}(T|_{\text{im}(S)}$
Conversely, let $y \in \text{ im}\left(T|_{\text{im}(S)}\right)$

$$T|_{\text{im}(S)} : \text{ im }(S) \to W$$

Then, $\exists z \in \text{ im}(S)$ with

$$T|_{\text{im}(S)}(z) = y$$

Since $z \in \text{ im}(S)$, there exists an $x \in U$ with $S(x) = z$. Then

$$(T \circ S)(x) = T\left(S(x)\right) = T(z) = T|_{\text{im}(s)}(z) = y$$

So, $y \in \text{ im}(T \circ S)$. By double containment, we get

$$\text{im}(T \circ S) = \text{ im}(T|_{\text{im}(S)})$$

□

Claim 26.2. $\ker\left(T|_{\text{im}(S)}\right) = \ker(T) \cap \text{ im}(S)$.

$$\begin{tikzcd} \text{U} \arrow[r, "\text{S}"] & \text{V} \arrow[r, "\text{T}"] & \text{W} \\ \text{im(S)} \arrow[ur, "\subset"] & \ker(T) \arrow[u, "\subset"'] & \end{tikzcd}$$

$$\begin{tikzcd} \text{im(S)} \arrow[r, "T|_{im(S)}"] & \text{W} \\ \ker\left(T|_{im(S)}\right) \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

Proof. Let $x \in \ker\left(T\big|_{\text{im}(S)}\right)$. Then $x \in \text{ im}(S)$ and

$$T(x) = T\big|_{\text{im}(S)}(x) = 0$$

So, $x \in \text{ im}(S)$ and $x \in \ker(T)$. So $x \in \text{ im}(S) \cap \ker(T)$.
Conversely, suppose $x \in \text{ im}(S) \cap \ker(T)$. Then $x \in \text{ im}(S)$, so it is a valid input to $T\big|_{\text{im}(S)}$. Moreover,

$$T\big|_{\text{im}(S)}(x) = T(x) = 0$$

where we see $T(x) = 0$, since $x \in \ker(T)$. So, $x \in \ker\left(T\big|_{\text{im}(S)}\right)$. By double containment,

$$\ker\left(T\big|_{\text{im}(S)}\right) = \ker(T) \cap \text{ im}(S)$$

□

Rank – Nullity gives

$$\begin{aligned}
\dim(\ker T \cap \text{ im } S) &= \dim\left(\ker T\big|_{\text{im}(S)}\right) \\
&= \dim\left(\text{im}(S)\right) - \dim\left(\text{ im } \left(T\big|_{\text{im}(S)}\right)\right) \\
&= \dim\left(\text{ im}(S)\right) - \dim\left(\text{im}(T \circ S)\right)
\end{aligned}$$

What can we say about $\ker(T \circ S)$?

$$\begin{tikzcd}
U \arrow[r, "S"] \arrow[rr, bend right, "T \circ S"'] & V \arrow[r, "T"] & W
\end{tikzcd}$$

Claim 26.3. $u \in \ker(T \circ S) \iff u \in \ker(S)$ or $u \notin \ker(S)$ and $S(u) \in \ker(T)$.

Proof. Let $u \in \ker(T \circ S)$. Then $T(S(u)) = 0$. So $S(u) \in \ker(T)$. So either $S(u) = 0$ or $S(u) \neq 0$ and $S(u) \in \ker(T)$. So either $u \in \ker(S)$ or $u \notin \ker(S), S(u) \in \ker(T)$.
Conversely, suppose $u \in \ker(S)$. Then $S(u) = 0$, so $T(S(u)) = T(0) = 0$. Similarly, if $u \notin \ker(S)$ but $S(u) \in \ker(T)$, then $T(S(u)) = 0$. In both cases, $u \in \ker(T \circ S)$. □