ME 564 hw3

Exercise: 3-1

$\ddot{x} + 5\dot{x} + 6x = f(t)$

(a) $f(t) = 0$, $\lambda = -2, -3$, $x(t) = C_1 e^{-2t} + C_2 e^{-3t}$

$\Rightarrow \begin{cases} C_1 + C_2 = \frac{1}{2} \\ -2C_1 - 3C_2 = -1 \end{cases} \Rightarrow -C_2 = 0,\ C_2 = 0 \Rightarrow C_1 = \frac{1}{2}$

$\Rightarrow \boxed{x(t) = \frac{1}{2} e^{-2t}}$ #

(b) assuming $x_p = ke^{-t}$, $\ddot{x} + 5\dot{x} + 6x = e^{-t}$

$\Rightarrow \dot{x}_p = -ke^{-t}$, $\ddot{x}_p = ke^{-t}$

$\Rightarrow ke^{-t} - 5ke^{-t} + 6ke^{-t} = e^{-t}$

$\Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$,

$\Rightarrow x(t) = C_1 e^{-2t} + C_2 e^{-3t} + \frac{1}{2} e^{-t}$, I.C. $\begin{cases} x(0) = \frac{1}{2} \\ \dot{x}(0) = -1 \end{cases}$

$\Rightarrow \begin{cases} x(0) = C_1 + C_2 + \frac{1}{2} = \frac{1}{2} \\ \dot{x}(0) = -2C_1 - 3C_2 - \frac{1}{2} = -1 \end{cases}$

$\Rightarrow C_1 = -C_2$, $2C_2 - 3C_2 = -\frac{1}{2}$, $C_2 = \frac{1}{2}$ $C_1 = -\frac{1}{2}$

$\Rightarrow \boxed{x(t) = -\frac{1}{2} e^{-2t} + \frac{1}{2} e^{-3t} + \frac{1}{2} e^{-t}}$ #

(c) assuming $x_p = A\cos t + B\sin t$, $\ddot{x} + 5\dot{x} + 6x = 50\cos t$

$\Rightarrow \dot{x}_p = -A\sin t + B\cos t$, $\ddot{x}_p = -A\cos t - B\sin t$

$\Rightarrow -A\cos t - B\sin t - 5A\sin t + 5B\cos t + 6A\cos t + 6B\sin t$

$= 50\cos t$

$\Rightarrow (5A+5B)\cos t + (5B-5A)\sin t = 50\cos t$

$\Rightarrow A+B = 10$, $B-A = 0$. $A=5$, $B=5$

$\Rightarrow x(t) = c_1 e^{-2t} + c_2 e^{-3t} + 5\cos + 5\sin$, I.C. $\begin{cases} x(0) = \frac{1}{2} \\ \dot{x}(0) = -1 \end{cases}$

$\Rightarrow c_1 + c_2 + 5 = \frac{1}{2}$, $-2c_1 - 3c_2 + 5 = -1$

$\Rightarrow -c_2 + 15 = 0$, $c_2 = 15$, $c_1 = -19.5$

$$\Rightarrow \boxed{x(t) = -19.5e^{-2t} + 15e^{-3t} + 5\cos t + 5\sin t}$$ #

Exercise 3-2 $\quad \ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = \gamma\cos(\omega t)$

(a)

for. $\beta = -1$, $\alpha = 1$, $\gamma = 0$

$$\Rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = -\delta v + x - x^3 \end{cases} \Rightarrow \dot{\underline{x}} = \underline{f}(\underline{x})$$

$$\Rightarrow \underline{f}(\underline{x}) = \begin{bmatrix} f_1(x, v) \\ f_2(x, v) \end{bmatrix} = \begin{bmatrix} v \\ -\delta v + x - x^3 \end{bmatrix}$$

$$\Rightarrow \boxed{\bar{x} = \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}}$$ #

(b)

$$\dot{x} = f(x) = f(\bar{x}+\Delta x) \doteq f(\bar{x}) + \frac{Df}{Dx}(\bar{x})(x-\bar{x})$$

$$\Rightarrow \frac{d}{dt}\Delta x = \frac{Df}{Dx}(\bar{x})\Delta x$$

$$\frac{Df}{Dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x} , & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 , & 1 \\ 1-3x^2 , & -\delta \end{bmatrix}$$

① $\Rightarrow \left.\frac{Df}{Dx}\right|_{x=\begin{bmatrix}1\\0\end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -2 & -\delta \end{bmatrix} \Rightarrow \boxed{\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -\delta \end{bmatrix}\underline{x}}$ #

② $\Rightarrow \left.\frac{Df}{Dx}\right|_{x=\begin{bmatrix}0\\0\end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \Rightarrow \boxed{\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix}\underline{x}}$ #

③ $\Rightarrow \left.\frac{Df}{Dx}\right|_{x=\begin{bmatrix}-1\\0\end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -2 & -\delta \end{bmatrix} \Rightarrow \boxed{\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -\delta \end{bmatrix}\underline{x}}$ #

(c) calculate $\det(A-\lambda I)=0$

① $\Rightarrow \lambda^2+2=0 , \lambda = \pm\sqrt{2}i \rightarrow$ $\boxed{\text{stable center}}$ #

② $\Rightarrow \lambda^2-1=0 , \lambda=\pm 1 \rightarrow$ $\boxed{\text{unstable saddle}}$ #

③ $\Rightarrow \lambda^2+2=0 . \lambda=\pm\sqrt{2}i \rightarrow$ $\boxed{\text{stable center}}$ #

(d)

①③ $\Rightarrow -\lambda(-\delta-\lambda)+2=0 \Rightarrow \lambda^2+\delta\lambda+2=0$

$\Rightarrow \lambda = \frac{-\delta\pm\sqrt{\delta^2-8}}{2} = -\frac{\delta}{2} \pm \frac{\sqrt{\delta^2-8}}{2}$ (if δ is small positive assuming $\delta^2<8$)

for case ①③, both λ has negative real part, (with complex conjugate) it will become an stable spiral sink #

② $\Rightarrow -\lambda(-\delta-\lambda)-1=0 \Rightarrow \lambda^2+\delta\lambda-1=0$

$\Rightarrow \lambda = \frac{-\delta \pm \sqrt{\delta^2+4}}{2} = \frac{-\delta}{2} \pm \frac{\sqrt{\delta^2+4}}{2}$, (if δ is small positive $\frac{\delta}{2} < \frac{\sqrt{\delta^2+4}}{2}$)

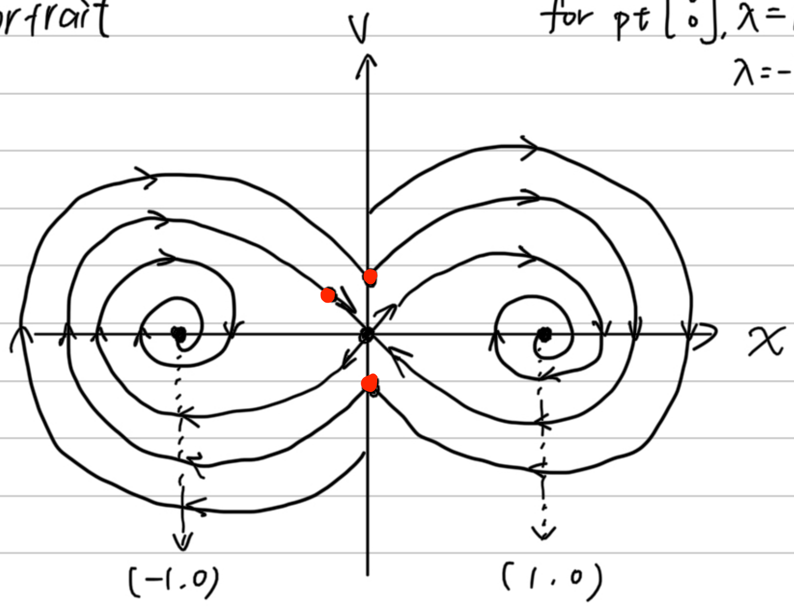
for case ②, λ will be one negative real, one positive real number, This is a unstable saddle #

(e) phase portrait

for pt $\begin{bmatrix}0\\0\end{bmatrix}$, $\lambda=1$, $\xi_1=\begin{bmatrix}1\\1\end{bmatrix}$

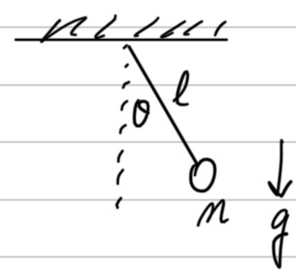
$\lambda=-1$, $\xi_2=\begin{bmatrix}1\\-1\end{bmatrix}$

Those red dots I.C. will end up at the saddle (0,0) #



Exercise 3-3

$$\begin{cases} T = \frac{1}{2} m l^2 \dot{\theta}^2 \\ V = -mgl\cos\theta \end{cases}$$



$$\begin{cases} \mathcal{L} = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta \quad ① \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad ② \end{cases}$$

$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta}$, $\frac{\partial \mathcal{L}}{\partial \theta} = -mgl\sin\theta$

$\Rightarrow$ plug into ② $\quad ml^2\ddot{\theta} + mgl\sin\theta = 0$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0, \quad \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta = f(\theta, \dot{\theta})} \ \#$$

$$\Rightarrow \dot{\theta} = \omega, \quad \dot{\omega} = -\frac{g}{l}\sin\theta = -\sin\theta \quad (l = g = 1)$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}\omega \\ -\sin\theta\end{bmatrix} = \begin{bmatrix}f_1(\theta, \omega) \\ f_2(\theta, \omega)\end{bmatrix}$$

(a)

$$\Rightarrow \begin{cases} f_1(\bar{x}) = \underline{\omega = 0} \\ f_2(\bar{x}) = -\sin\theta = 0, \ \theta = \underline{0, \pi, 2\pi} \end{cases}$$

$$\boxed{\bar{x}_1 = \begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}, \ \bar{x}_2 = \begin{bmatrix}\pi \\ 0\end{bmatrix}, \ \bar{x}_3 = \begin{bmatrix}2\pi \\ 0\end{bmatrix}} \ \#$$

(b)

$$\frac{Df}{Dx} = \begin{bmatrix}\frac{df_1}{d\theta} & \frac{df_1}{d\omega} \\ \frac{df_2}{d\theta} & \frac{df_2}{d\omega}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\cos\theta & 0\end{bmatrix}$$

$$\lambda^2 + 1 = 0$$

$$\left.\frac{Df}{Dx}\right|_{\bar{x} = \begin{bmatrix}0 \\ 0\end{bmatrix}} = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix} \Rightarrow \boxed{\begin{cases}\dot{\theta} = \omega, & \lambda = \pm i \\ \dot{\omega} = -\theta, & \text{stable center}\end{cases}} \ \#$$

$$\lambda^2 - 1 = 0$$

$$\left.\frac{Df}{Dx}\right|_{\bar{x} = \begin{bmatrix}\pi \\ 0\end{bmatrix}} = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} \Rightarrow \boxed{\begin{cases}\dot{\theta} = \omega, & \lambda = \pm 1 \\ \dot{\omega} = \theta, & \text{unstable saddle}\end{cases}} \ \#$$

$$\left.\frac{Df}{Dx}\right|_{\bar{x} = \begin{bmatrix}2\pi \\ 0\end{bmatrix}} = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix} \Rightarrow \boxed{\begin{cases}\dot{\theta} = \omega, & \lambda = \pm i \\ \dot{\omega} = -\theta, & \text{stable center}\end{cases}} \ \#$$

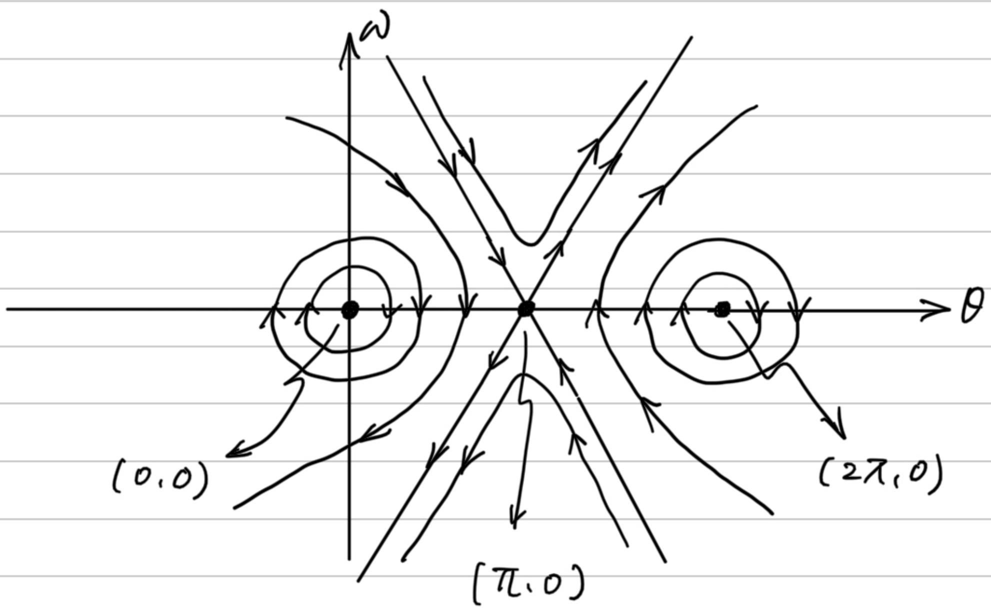
(c)

$\underline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,
$$\boxed{\begin{array}{l} \lambda = -i, \quad \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} \theta \\ w \end{bmatrix} = \begin{cases} i\theta + w = 0 \\ -\theta + iw = 0 \end{cases} \Rightarrow \xi_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \\ \lambda = i, \quad \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} \theta \\ w \end{bmatrix} = \begin{cases} -i\theta + w = 0 \\ -\theta - iw = 0 \end{cases} \Rightarrow \xi_2 = \begin{bmatrix} i \\ -1 \end{bmatrix} \end{array}}$$ #

$\underline{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$,
$$\boxed{\begin{array}{l} \lambda = 1, \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \theta \\ w \end{bmatrix} \Rightarrow \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda = -1, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \theta \\ w \end{bmatrix} \Rightarrow \xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}}$$ #

$\underline{x} = \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$,
$$\boxed{\begin{array}{l} \lambda = -i \Rightarrow \xi_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \\ \lambda = i \Rightarrow \xi_2 = \begin{bmatrix} i \\ -1 \end{bmatrix} \end{array}}$$ #

(d) phase portrait



(e)

The phase portrait matches my physical intuition about the fixed point. At $\bar{x} = (\pi, 0)$ it is an unstable inverted pedulum. When $\theta = 0, 2\pi$, these are stable configuration. The system peforms harmonic motion or stay still. #

Exercise 3-4

- All stable centers will become stable spiral sinks. Unstable saddle points remain the same status #

- Five fixed points.

Without friction, 2 unstabe saddles, 3 stable centers

with friction, 2 unstable saddes, 3 stable spiral sinks #

Exercise 3-4

- All stable centers will become stable spiral sinks. Unstable saddle points remain the same status #

- Five fixed points,

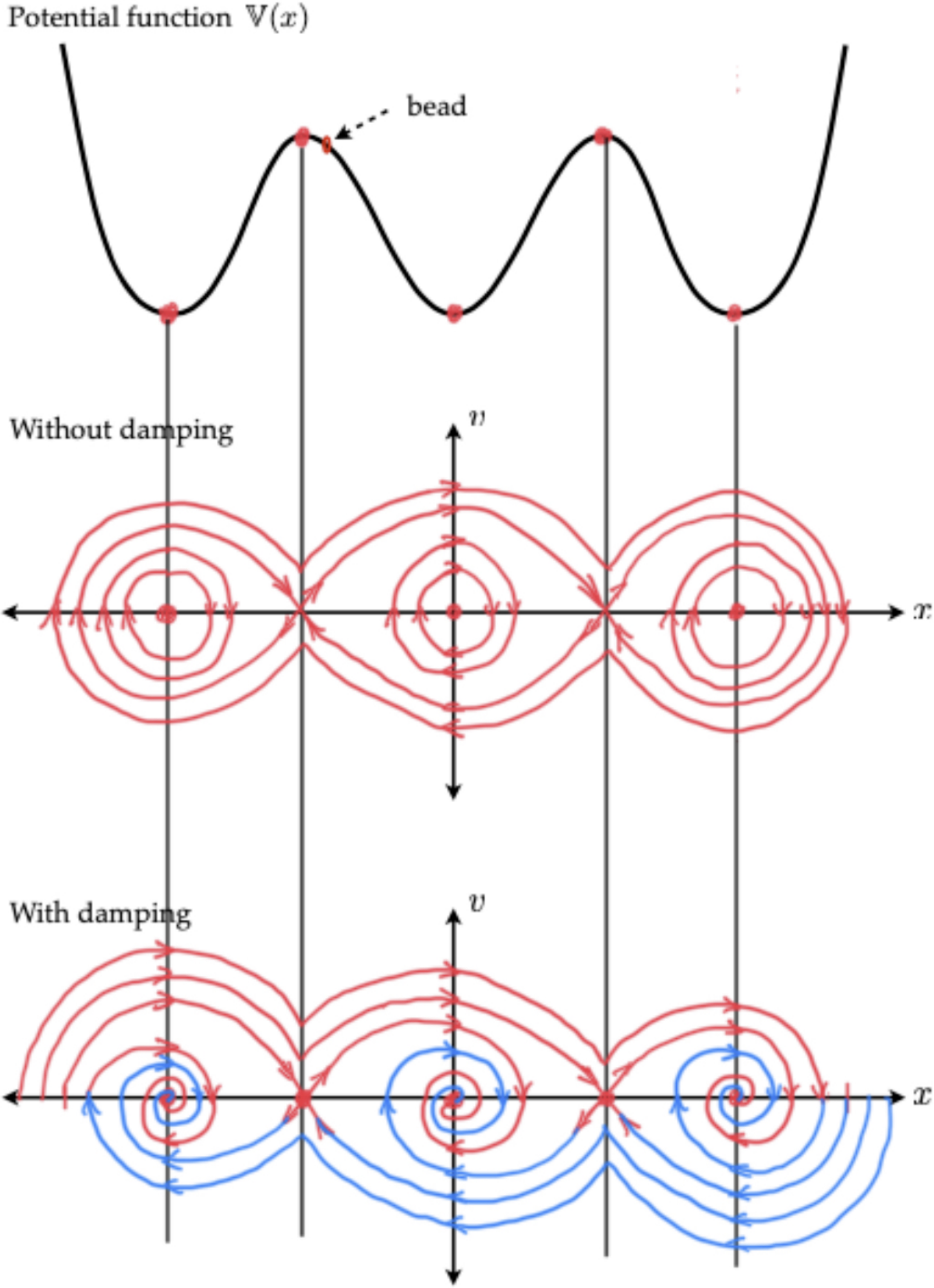
Without friction, 2 unstabe saddles,
3 stable centers

with friction, 2 unstable saddes
3 stable spiral sinks #

- for saddle points, $\lambda \rightarrow$ one positive real
one negative real

- for spiral sinks, $\lambda \rightarrow$ negative real, with complex conjugate

- for stable center, $\lambda \rightarrow$ a pair of complex conjugate, #


Potential function $\mathbb{V}(x)$
bead
Without damping
v
x
With damping
v
x