ME565 hw3

1、 $u(x,y) = F(x)G(y)$

$$0 = \nabla^2 u = F_{xx}(x)G(y) + F(x)G_{yy}(y)$$

$$\Rightarrow \frac{F_{xx}}{F} = -\frac{G_{yy}}{G} = -\lambda^2 \quad \text{(Constant)}$$

①

$$\Rightarrow F_{xx} + \lambda^2 F = 0 \quad , \quad \text{eigenvalue} = \pm\lambda i$$

$$\Rightarrow F(x) = \alpha\cos(\lambda x) + \beta\sin(\lambda x)$$

$$\Rightarrow F'(x) = -\alpha\lambda\sin(\lambda x) + \lambda\beta\cos(\lambda x)$$

$$\Rightarrow u_x = F'(x)G(y)$$

to make the condition satisfiy $F_{xx} = -\lambda F$

$$\begin{cases} u_x(0,y) = \lambda\beta\cdot G(y) = 0 \quad , \quad \underline{\beta = 0} \\ u_x(L,y) = -\alpha\lambda\sin(\lambda L)\cdot G(y) = 0 \end{cases}$$

$$\Rightarrow \sin(\lambda L) = 0 \quad , \quad (\alpha \text{ shouldn't be zero})$$

$$\Rightarrow \underline{\lambda = \frac{n\pi}{L}} \quad , \quad n = 1, 2 \ldots$$

$$\therefore F(x) = \alpha_n\cos\left(\frac{n\pi}{L}x\right) \quad , \quad n = 1, 2 \ldots \quad (\text{choose } \alpha_n = 1)$$

②

$$\Rightarrow G_{yy} - \left(\frac{n\pi}{L}\right)^2 G = 0 \quad , \quad \text{eigenvalue} = \pm\frac{n\pi}{L}$$

$$\Rightarrow G(y) = c_n e^{\frac{n\pi}{L}y} + \gamma_n e^{-\frac{n\pi}{L}y}$$

$$\begin{cases} u(x, 0) = F(x)\cdot(C_n + \gamma_n) = 0, \quad \gamma_n = -C_n \\ \underline{u(x, H) = F(x)}\cdot C_n\left(e^{\frac{n\pi}{L}H} - e^{-\frac{n\pi}{L}H}\right) = f(x) \end{cases}$$

$$\Rightarrow G(y) = 2C_n \sinh\left(\frac{n\pi}{L}y\right) \quad n = 1, 2, \dots \quad (2C_n = C_n)$$

undefined.

$$\therefore f(x) = \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\cdot C_n \sinh\left(\frac{n\pi H}{L}\right)$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi H}{L}\right)\cos\left(\frac{n\pi}{L}x\right) \quad \#}$$

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx$$

$$= \int_0^L \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi H}{L}\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx$$

$$= \frac{C_m L}{2}\sinh\left(\frac{n\pi H}{L}\right)$$

$$\therefore \boxed{C_m = \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx \quad \#}$$

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} C_m \cos\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi}{L}y\right) \quad \#}$$

If $f(x) = x$

$$\boxed{C_m = \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L x\cos\left(\frac{m\pi}{L}x\right)dx \quad \#}$$

If left & right boundrics were fixed at zero, $u(x,y)$ <u>will consist of Sine function instead of Cosine function.</u>

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2. $u(x,t) = X(x)\,T(t)$, $u_t = u_{xx}$, $(t \geq 0)$

$\Rightarrow X(x)\,T'(t) = X''(x)\,T(t)$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -k^2 \quad (k>0)$$

①

$X'' + k^2 X = 0$ eig : $\underline{\pm k i}$

$X(x) = \alpha \operatorname{Cos}(kx) + \beta \operatorname{Sin}(kx)$ $\boxed{2\pi \text{ periodic}}$

from B.C. $X(0) = X(2\pi)$, $X'(0) = X'(2\pi)$

we can make sure we derived a reasonable form of $X(x)$!

②

$T' + k^2 T = 0$, eig : $-k^2$

$T(t) = \gamma e^{-k^2 t}$

$\therefore u(x,t) = X(x) \cdot T(t)$

$= e^{-k^2 t}\left[\alpha \operatorname{Cos}(kx) + \beta \operatorname{Sin}(kx)\right]$

$\boxed{\text{fold } \gamma \text{ info } \alpha \,\&\, \beta}$

Let $k=3$, check if the solution reasonable

$$u = e^{-9t}[\alpha \cos(3x)+\beta \sin(3x)]$$

$$u_t = -9\,u(x,t)$$

$$u_x = 3e^{-9t}[-\alpha \sin(3x)+\beta \cos(3x)]$$

$$u_{xx} = -9e^{-9t}[\alpha \cos(3x)+\beta \sin(3x)]$$

$$\Rightarrow u_t = u_{xx} = -9\,u(x,t) \qquad k \in \mathbb{Z} \quad \checkmark$$

Linear, superposition holds.

$$\boxed{u(x,t) = \sum_{k=0}^{\infty} e^{-k^2 t}[\alpha_k \cos(kx)+\beta_k \sin(kx)]}$$

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③ Using I.C. $u(x,0)=f(x)$

$$u(x,0) = \sum_{k=0}^{\infty} \alpha_k \cos(kx)+\beta_k \sin(kx) = f(x)$$

$$\boxed{\Rightarrow \begin{cases} \alpha_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\,dx \\ \beta_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)\,dx \end{cases}}$$

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$$e^{-k^2 t} \longrightarrow 0 \quad \text{when } k>0,\ t\to\infty$$

$$e^{-kt} = 1 \quad \text{when } k=0$$

$$\therefore u(x,\infty) = 1\cdot[\alpha_0 \cos(0)+\beta_0 \sin(0)]$$

$$= \alpha_0 \cos(0) = \alpha_0$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \text{constant}$$

$\boxed{\text{Temperatme is uniformly distributed at steady state}}$ #

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3. equilibrium temperature $u_t = \nabla^2 u = 0$

Let $u(r,\theta) = F(r)\,G(\theta)$

$$\nabla^2 u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\Rightarrow \frac{1}{r}(u_r + r\,u_{rr}) + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$\Rightarrow \frac{1}{r}u_r + u_{rr} + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$\Rightarrow \frac{1}{r}F'(r)G(\theta) + F''(r)G(\theta) + \frac{1}{r^2}F(r)G''(\theta) = 0$$

$$\therefore \left[\frac{1}{r}F'(r) + F''(r)\right]G(\theta) = \left[-\frac{1}{r^2}G''(\theta)\right]F(r)$$

$$\Rightarrow \frac{\frac{1}{r}F'(r) + F''(r)}{F(r)} = \frac{-\frac{1}{r^2}G''(\theta)}{G(\theta)}$$

constant

$$\Rightarrow \frac{r^2F''(r) + rF'(r)}{F(r)} = \frac{-G''(\theta)}{G(\theta)} = k^2 \quad (k>0)$$

①

$$G''(\theta) + k^2G(\theta) = 0$$

$$\Rightarrow G(\theta) = \alpha \cos(k\theta) + \beta \sin(k\theta)$$

assume $F(r) = r^n$

$F'(r) = n r^{n-1}$

$F''(r) = n(n-1) r^{n-2}$

②

$$r^2 F''(r) + r F'(r) - k^2 F(r) = 0$$

$$\Rightarrow n(n-1)\cancel{r^n} + n\cancel{r^n} - k^2\cancel{r^n} = 0$$

$$\Rightarrow n^2 = k^2$$

$$\Rightarrow n = k$$

$$\Rightarrow F(r) = r^k$$

$$\therefore u(r,\theta) = F(r) G(\theta)$$

$$= \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} r^k \left[\alpha_n \cos(k\theta) + \beta_n \sin(k\theta)\right]$$

$$\Rightarrow u(1,\theta) = f(\theta)$$

$$= \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \left[\alpha_n \cos(k\theta) + \beta_n \sin(k\theta)\right]$$

where
$$\begin{cases} \alpha_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\, d\theta \\ \alpha_k = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(k\theta)\, d\theta \\ \beta_k = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(k\theta)\, d\theta \end{cases}$$

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$$\therefore \boxed{u(r,\theta) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} r^k \left[\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)\right]}$$

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