# Exercise 5-1

(a)

$$\frac{1}{4-3i} = \frac{4+3i}{(4-3i)(4+3i)} = \frac{4+3i}{25} = \boxed{\frac{4}{25} + \frac{3}{25} i} \ \#$$

$$= \frac{1}{5} e^{i\theta}, \left(\theta = \cos^{-1}\frac{4}{5} \fallingdotseq 0.64\right) = \boxed{\frac{1}{5} e^{i \cdot 0.64}} \ \#$$

(b)

$$\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)^4 = \left(\frac{3}{4} - \frac{1}{4} - \frac{\sqrt{3}}{2} i\right)^2$$

$$= \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^2 = \left(\frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2} i\right)$$

$$= \boxed{-\frac{1}{2} - \frac{\sqrt{3}}{2} i} \ \# \quad = 1 \cdot e^{i\theta}, \left(\theta = \frac{4}{3}\pi\right)$$

$$= \boxed{e^{i \frac{4}{3}\pi}} \ \#$$

(c)

$$i^2, i^3, i^4, i^5 \cdots = -1, -i, 1, i \cdots$$

$$i = Re^{i\theta} = e^{i\frac{\pi}{2}}, \left(R = 1, \ \theta = \frac{\pi}{2}\right)$$

$$\boxed{i^n = e^{i\frac{n\pi}{2}}, \ n = 2, 3, 4, 5 \ldots} \ \#$$

$$i = \cos\theta + i\sin\theta \quad , \quad \theta = \frac{n\pi}{2}$$

$$\boxed{i^n = \cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right), \quad n = 2, 3, 4 \dots}$$ #

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## Exercise 5-2

(a) $e^z = i = e^{i\frac{\pi}{2}} \Rightarrow \boxed{z = \left(\frac{\pi}{2} + 2n\pi\right)i, \text{ for } n = 0, 1, 2 \dots}$ #

(b) $e^z = -1 = e^{i\pi} \Rightarrow \boxed{z = (\pi + 2n\pi)i, \text{ for } n = 0, 1, 2 \dots}$ #

## Exercise 5-3

(a) $z^4 = 1 \Rightarrow R^4 e^{i4\theta} = e^{i\cdot 0} \quad , \quad R = 1$

$4\theta = 0, 2\pi, 4\pi \dots \quad , \quad \theta = \frac{n\pi}{2} \quad , \quad n = 0, 1, 2, \dots$

$$\boxed{z = e^{i\frac{n\pi}{2}}, \quad n = 0, 1, 2, 3 \dots}$$ #

$$\boxed{n = 0,\ z = 1,\ n = 1,\ z = i,\ n = 2,\ z = -1,\ n = 3,\ z = -i}$$ #

(b)

$$z^2 = 4i \Rightarrow R^2 e^{i2\theta} = 4e^{i\cdot\frac{\pi}{2}} , \quad R = 2$$

$$2\theta = \frac{\pi}{2}, \ \frac{\pi}{2}+2\pi, \ \frac{\pi}{2}+4\pi \ \cdots$$

$$\theta = \frac{\pi}{4}, \ \frac{\pi}{4}+\pi, \ \frac{\pi}{4}+2\pi \ \cdots$$

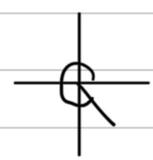
$$\boxed{z = 2e^{i\left(\frac{\pi}{4}+n\pi\right)}, \quad n = 0, 1, 2, 3 \cdots \ \#}$$

$$\boxed{\begin{aligned} n=0, \ z &= 2e^{i\frac{\pi}{4}} = \sqrt{2}+\sqrt{2}i \\ n=1, \ z &= 2e^{i\frac{5}{4}\pi} = -\sqrt{2}-\sqrt{2}i \ \# \end{aligned}}$$

(c)

$$z^2 = 1-i \Rightarrow R^2 e^{i2\theta} = \sqrt{2}e^{i\frac{7}{4}\pi}, \quad R = 2^{\frac{1}{4}}$$

$$2\theta = \frac{7}{4}\pi, \ \frac{7}{4}\pi+2\pi, \ \frac{7}{4}\pi+4\pi \ \cdots$$



$$\theta = \frac{7}{8}\pi, \ \frac{7}{8}\pi+\pi, \ \frac{7}{8}\pi+2\pi \ \cdots$$

$$\boxed{z = 2^{\frac{1}{4}} e^{i\left(\frac{7}{8}\pi+n\pi\right)}, \quad n = 0, 1, 2, 3 \cdots \ \#}$$

$$\boxed{n=0, \ z = 2^{\frac{1}{4}} e^{i\frac{7}{8}\pi}, \quad n=1, \ z = 2^{\frac{1}{4}} e^{i\frac{15}{8}\pi} \ \#}$$

# Exercise 5-4

(a) $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$

$= e^x(\cos y + i\sin y) = e^x \cos y + i e^x \sin y$

$= u(x.y) + i V(x.y)$

by Cauchy - Riemann Condition

$$\begin{cases} \dfrac{\partial u}{\partial x} = e^x \cos y, & \dfrac{\partial V}{\partial y} = e^x \cos y, \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial V}{\partial y} \\ \dfrac{\partial u}{\partial y} = -e^x \sin y, & \dfrac{\partial V}{\partial x} = e^x \sin y \Rightarrow \dfrac{\partial u}{\partial y} = -\dfrac{\partial V}{\partial x} \end{cases}$$

therefore, $\boxed{f(z) = e^z \text{ is analytic}}$ #

(b) $f(z) = \cos z = \cos(x+iy)$

$= \cos x \cos(iy) - \sin x \sin(iy)$

$= \cos x \cosh y - i(\sin x \sinh y)$

$= u(x;y) + i V(x.y)$

by Cauchy - Riemann Condition

$$\begin{cases} \dfrac{\partial u}{\partial x} = -\sin x \cosh y , & \dfrac{\partial v}{\partial y} = -\sin x \cosh y \\[2em] \dfrac{\partial u}{\partial y} = \cos x \sinh y , & \dfrac{\partial v}{\partial x} = -\cos x \sinh y \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[1em] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

$\boxed{\text{therefore, } f(z) = \cos z \text{ is analytic}}$ #

## Exercise 5-5

$f = u + iv$ with $u(x, y) = 2xy$

by Cauchy - Riemann Condition

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = 2y , & u = y^2 + C_1 \\[1em] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = 2x , & u = -x^2 + C_2 \end{cases}$$

$$u = -x^2 + y^2 + C$$

$$\therefore \boxed{f(z) = 2xy + i(-x^2 + y^2 + C)}$$ #