ME 564 Friday Worksheet

October 7, 2022

1. **Intuition for the simple harmonic oscillator**: Solve $\ddot{x} = -k^2 x$. What is the solution if k is a purely imaginary number? Show that if $x(0) = 1$, $\dot{x}(0) = 1$, the first solution (i.e. k a real number) is well-behaved but the second blows up to infinity. Can you find initial conditions such that the second equation produces a finite solution as $t \to \infty$?

2. **Solving variants of the simple harmonic oscillator symbolically in Python**: Work through the Jupyter notebook that was sent out and make sure (the math) makes sense. Can you confirm your solution and intuitions from Problem 1? Fill in the "TODO" parts that interest you.

3. **Solving challenging ODEs with the power series method**: In cylindrical coordinates, many partial differential equations can be reduced to an ordinary differential equation in the radial cylindrical coordinate x:

$$x^2 y'' + xy' + (x^2 - \alpha^2) y = 0. \tag{1}$$

Solve this equation for the case $\alpha = 0$ with the power series method. In particular, show that if

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then $a_n = 0$ if n is odd, and otherwise

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n}(n!)^2}.$$

Collect terms to show that one solution is proportional to the order-zero Bessel function of the 1st kind

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}. \tag{2}$$

The general solution to this problem for finite α are the Bessel functions of the 1st and 2nd kind, J_α and Y_α. Plot these functions for a few values of α, using scipy.special.jv and scipy.special.yv. What do you notice about the behavior of the Y_α near $x = 0$? Using python, verify that as $x \to \infty$,

$$J_\alpha(x) \to \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi + 2\alpha}{4}),$$

$$Y_\alpha(x) \to \sqrt{\frac{2}{\pi x}} \sin(x - \frac{\pi + 2\alpha}{4}),$$

by plotting these approximate formulas alongside the true functions. If you're more interested, code up a version of `https://scipython.com/book2/chapter-8-scipy/examples/drum-vibrations-with-bessel-functions/` to see how Bessel functions define the fundamental modes on a cylindrical drum.