ME564 - Autumn 2022 # HW1

Henry Chang

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Exercise 1–1: Compute the derivative of the following functions

(a) $f(x) = cos(x^3)$

$$\begin{aligned} f'(x) &= -sin(x^3) \cdot 3x^2 \\ &= \boxed{-3x^2 sin(x^3)} \end{aligned}$$

(b) $f(x) = x^x$

$$\begin{aligned} y &= x^x \\ \ln y &= x \ln x \\ \frac{1}{y}\frac{dy}{dx} &= \ln x + 1 \\ \frac{dy}{dx} &= f'(x) = \boxed{x^x(\ln x + 1)} \end{aligned}$$

(c) $f(x) = e^{sin(2x)} cos(x)$

$$\begin{aligned} f'(x) &= -e^{sin(2x)} sin(x) + 2cos(2x)e^{sin(2x)} cos(x) \\ &= \boxed{e^{sin(2x)}[2cos(2x)cos(x) - sin(x)]} \end{aligned}$$

(d) Now, compute the derivative of $f(x, y) = cos(x^2 + y^2)$ with respect to t, assuming that x(t) and y(t) vary with time. You can write the solution in terms of dx/dt and dy/dt.

$$\begin{aligned} \frac{df(x,y)}{dt} &= -sin(x^2 + y^2)\frac{d(x^2 + y^2)}{dt} \\ &= \boxed{-sin(x^2 + y^2)(2x\frac{dx}{dt} + 2y\frac{dy}{dt})} \end{aligned}$$

Exercise 1–2: A given mass x of a radioactive element obeys the following differential equation in time:

$$\dot{x} = \lambda x$$

where λ is a constant describing the rate of decay.

(a) Write down the solution x(t) to the differential equation.

$$\begin{aligned} \frac{dx}{dt} &= \lambda x \\ dx &= \lambda x dt \\ \int \frac{dx}{x} &= \lambda \int dt \\ \ln x &= \lambda t + c \\ x(t) &= \boxed{e^{\lambda t} k} \end{aligned}$$

Note: We need initial condition to compute $k = x(0)$

(b) Plot the solution for an initial condition x(0) = 2 from time t = 0 to t = 5 for $\lambda = -5, -1, 0, 0.01, 0.1$. Please plot these all on the same figure using the hold on command in Matlab. Label your axes and include a legend.

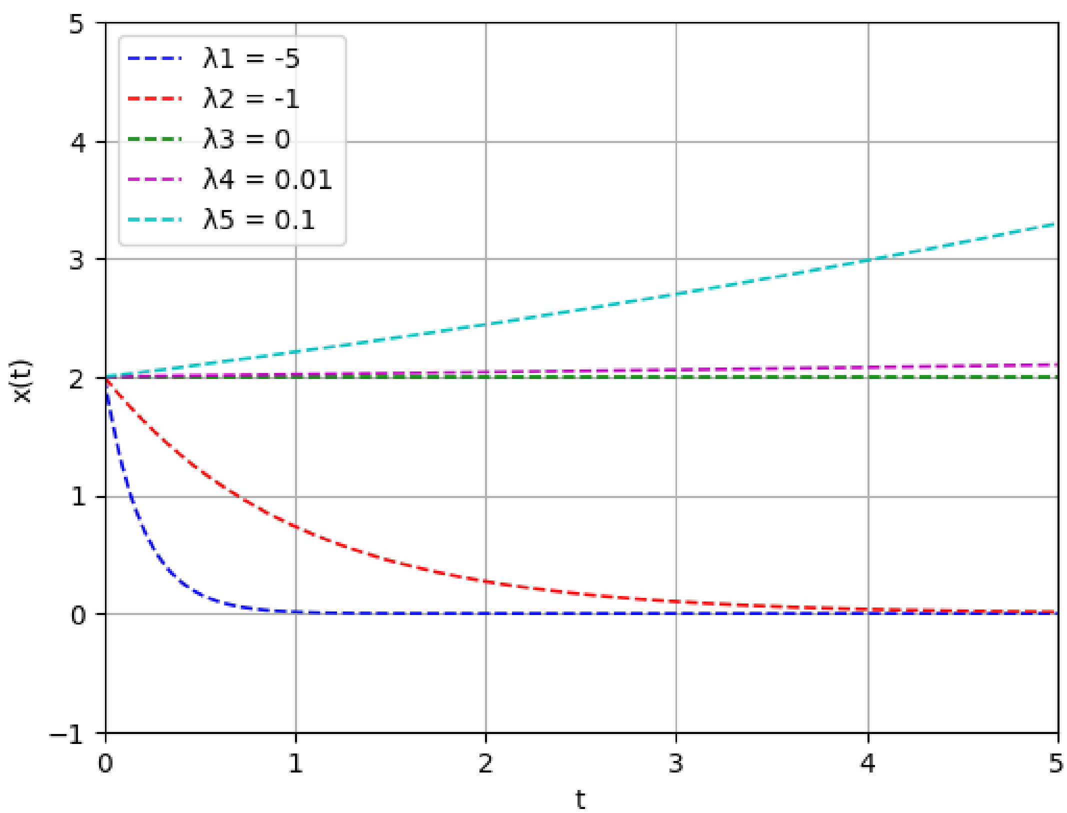
```
import numpy as np
from matplotlib import pyplot as plt

t = np.linspace(0, 5, 1000)

f1 = 2 * np.exp(-5 * t)
f2 = 2 * np.exp(-1 * t)
f3 = [2] * 1000
f4 = 2 * np.exp(0.01 * t)
f5 = 2 * np.exp(0.1 * t)

plt.xlim(0, 5)
plt.ylim(-1, 5)
plt.xlabel('t')
plt.ylabel('x(t)')
plt.grid(True)
plt.plot(t, f1, 'b--', linewidth=1.2)
plt.plot(t, f2, 'r--', linewidth=1.2)
plt.plot(t, f3, 'g--', linewidth=1.2)
plt.plot(t, f4, 'm--', linewidth=1.2)
plt.plot(t, f5, 'c--', linewidth=1.2)
plt.legend(['\lambda1 = -5', '\lambda2 = -1', '\lambda3 = 0', '\lambda4 =
0.01', '\lambda5 = 0.1'])
plt.show()

```



(c) The half-life T is defined as the time it takes for the material to be reduced to half of its mass through radioactive decay. The half-life of uranium-238 is 4.468 billion years. What is the corresponding value of λ?

$$\begin{aligned}
\frac{x(0)}{2} &= e^{\lambda \cdot 4.468 \cdot 10^9} x(0) \\
\frac{1}{2} &= e^{\lambda \cdot 4.468 \cdot 10^9} \\
\ln \frac{1}{2} &= \lambda \cdot 4.468 \cdot 10^9 \\
\lambda &= \boxed{-1.5514 \cdot 10^{-10} (\frac{1}{year})}
\end{aligned}$$

(d) If you start with 100kg of uranium-238, how long until you only have 5kg left?

$$\begin{aligned}
x(0) &= 100 \\
x(t) &= e^{\lambda t} \cdot 100 = 5 \\
e^{\lambda t} &= 0.05 \\
\lambda &= -1.5514 \cdot 10^{-10} \\
t &= \boxed{1.931 \cdot 10^{10}}
\end{aligned}$$

Note: $1.931 \cdot 10^{10}$ years = 19.31 billion years

Exercise 1–3: Compute the Taylor series expansion by hand for f(x). For each function, plot f(x) and the three-term expansion (i.e., the first three nonzero terms) from x = -5 to x = 5

(a) $f(x) = sin(x)/x$

Assume we know the Taylor series expansion of $sin(x)$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots,$

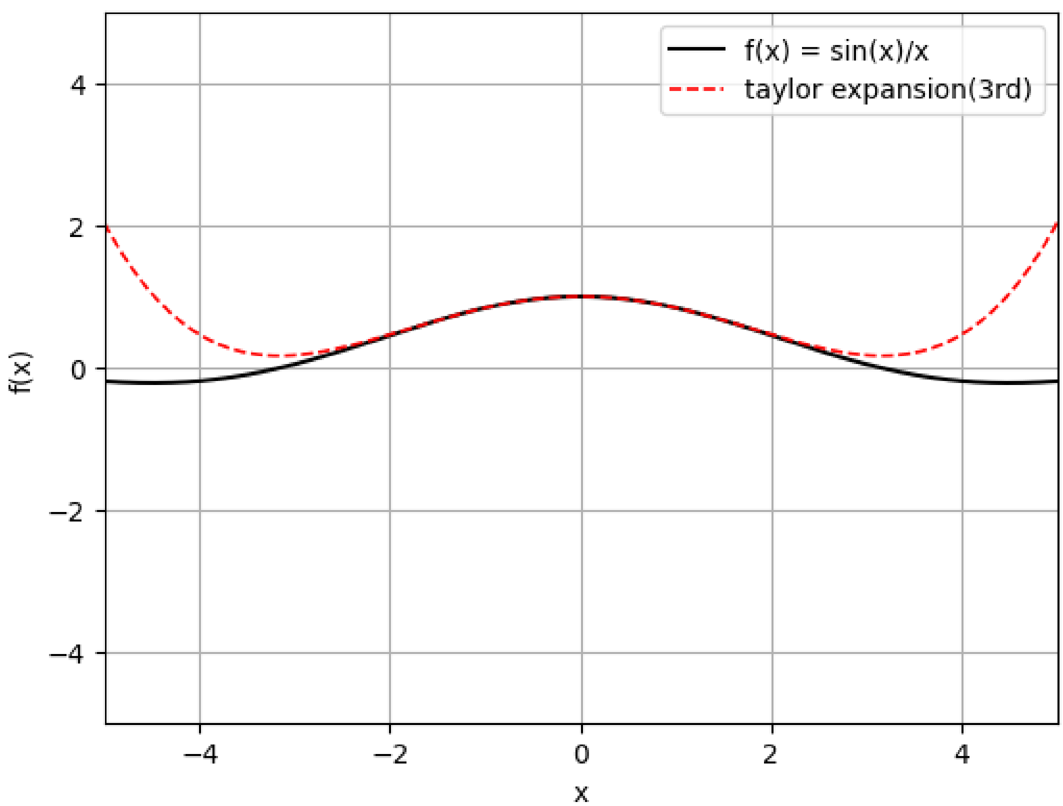
$$f(x) = [x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots]/x$$
$$f(x) = \boxed{1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots}$$

Plot $f(x)$,

```
x = np.linspace(-5, 5, 2000)
fx1 = np.sin(x) / x
fx1_taylor = 1 - x**2/factorial(3) + x**4/factorial(5)

plt.xlim(-5, 5)
plt.ylim(-5, 5)
plt.xlabel('x')
plt.ylabel('f(x)')
plt.grid(True)
plt.plot(x, fx1, 'k', linewidth=1.5)
plt.plot(x, fx1_taylor, 'r--', linewidth=1.2)
plt.legend(['f(x) = sin(x)/x', 'taylor expansion(3rd)'])
plt.show()

```



(b) $f(x) = 3^x$

Taylor expansion at $x = 0$,

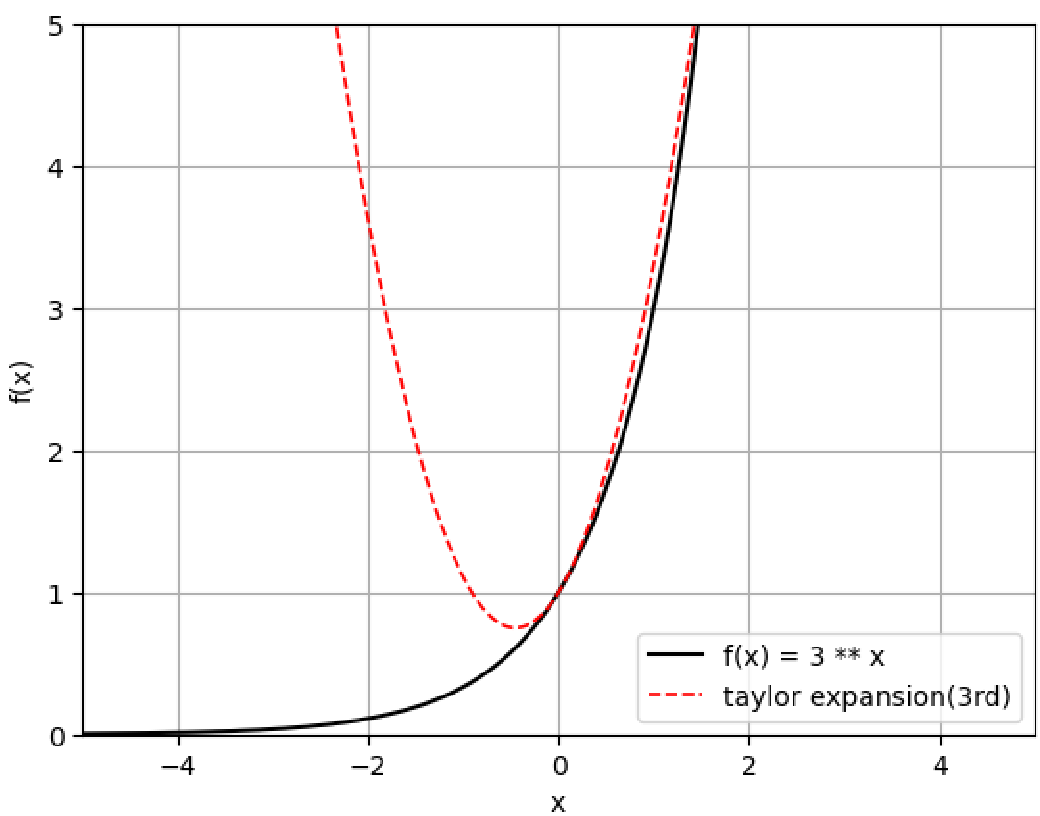
$$f(x) = f(0) + f^{'}(0)(x) + \frac{f^{''}(0)}{2!}(x-a)^2 + \ldots$$
$$f^{'}(0) = \ln 3 \cdot 3^x|_{x=0} = \ln 3$$
$$f^{''}(0) = (\ln 3)^2 \cdot 3^x|_{x=0} = \ln 3)^2$$
$$f(x) = \boxed{1 + \ln 3 \cdot x + \frac{(\ln 3)^2}{2!}x^2 + \ldots}$$

Plot $f(x)$,

```
x = np.linspace(-5, 5, 2000)
fx2 = 3 ** x
fx2_taylor = 1 + np.log(3) * x + (np.log(3) * x) ** 2

plt.xlim(-5, 5)
plt.ylim(-0, 5)
plt.xlabel('x')
plt.ylabel('f(x)')
plt.grid(True)
plt.plot(x, fx2, 'k', linewidth=1.5)
plt.plot(x, fx2_taylor, 'r--', linewidth=1.2)
plt.show()

```



Exercise 1–4: Please compute an analytic expression, by hand, for the real and imaginary parts of the following complex functions. Please also plot these in Matlab for $t = 0 : .01 : 10$.

(a) $f(t) = e^{it}$

Using power series expansion,

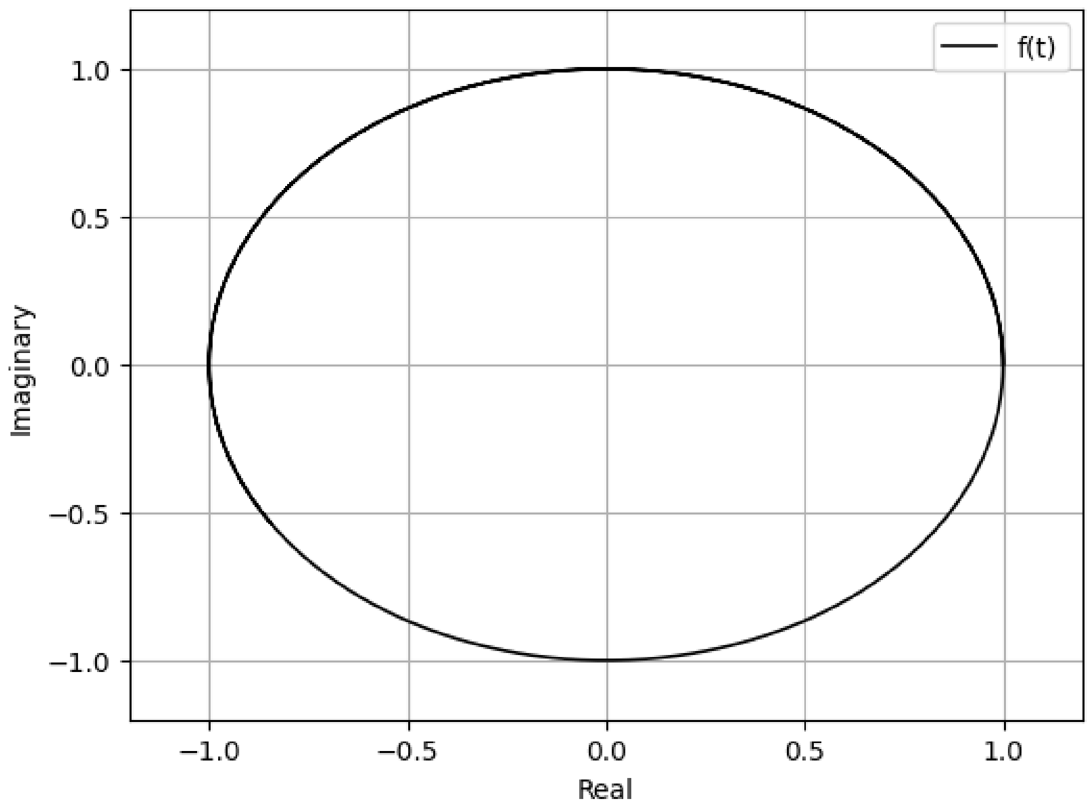
$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + ... = 1 + it - \frac{t^2}{2!} - \frac{it^3}{3!} + \frac{t^4}{4!} + \frac{it^5}{5!} - \frac{t^6}{6!} - ...$$
$$= (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}) + i(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}) + ...$$
$$e^{it} = \boxed{cos(t) + isin(t)}$$

Thus, we derived Euler's formula. Plot $f(t) = cos(t) + isin(t)$,

```
t = np.arange(0, 10, 0.01, dtype=float)

ft1_real = np.cos(t)
ft1_imag = np.sin(t)
plt.xlim(-1.2, 1.2)
plt.ylim(-1.2, 1.2)
plt.xlabel('Real')
plt.ylabel('Imaginary')
plt.grid(True)
plt.plot(ft1_real, ft1_imag, 'k', linewidth=1.2)
plt.legend(['f(t)'])
plt.show()

```



(b) $f(t) = e^{(-1-i)t}$

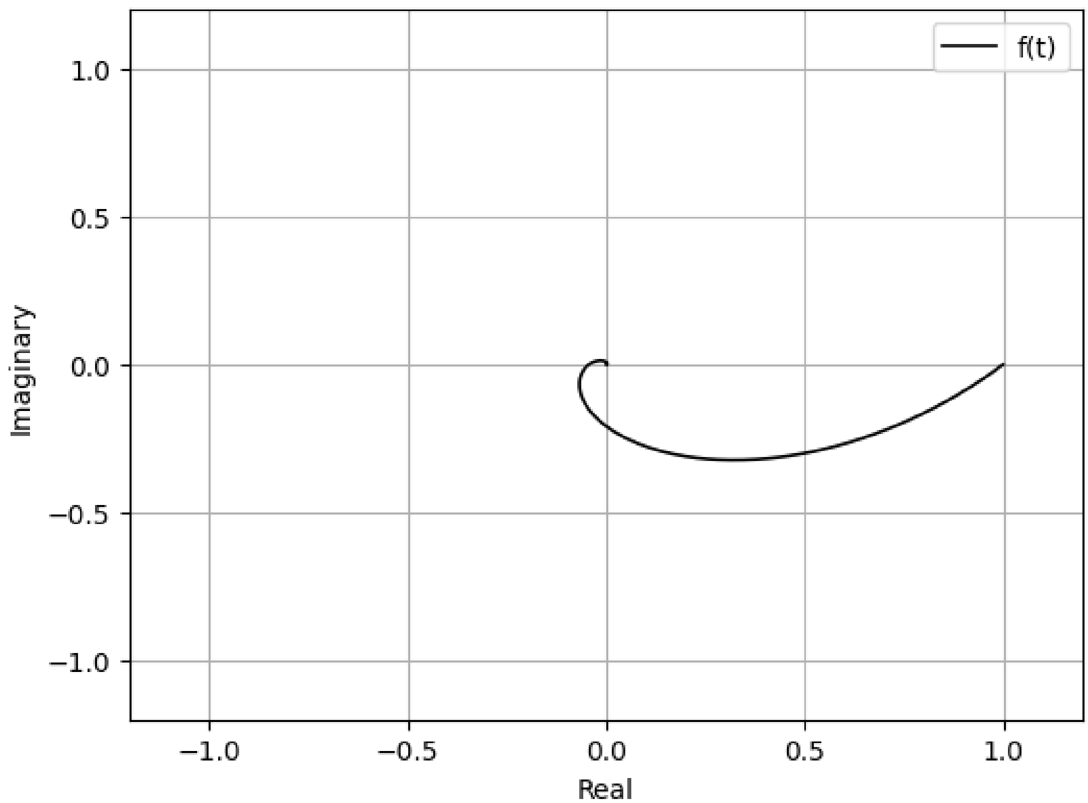
Using Euler's formula,

$$\begin{aligned} e^{(-1-i)t} &= e^{-t}e^{i(-t)} \\ &= e^{-t}(cos(-t) + isin(-t)) \\ e^{(-1-i)t} &= \boxed{e^{-t}(cos(t) - isin(t))} \end{aligned}$$

Plot $f(t) = e^{-t}(cos(t) - isin(t))$,

```
t = np.arange(0, 10, 0.01, dtype=float)

ft2_real = np.exp(-t) * np.cos(t)
ft2_imag = -np.exp(-t) * np.sin(t)
plt.xlim(-1.2, 1.2)
plt.ylim(-1.2, 1.2)
plt.xlabel('Real')
plt.ylabel('Imaginary')
plt.grid(True)
plt.plot(ft2_real, ft2_imag, 'k', linewidth=1.2)
plt.legend(['f(t)'])
plt.show()
```



(c) $f(t) = e^{1-it}$

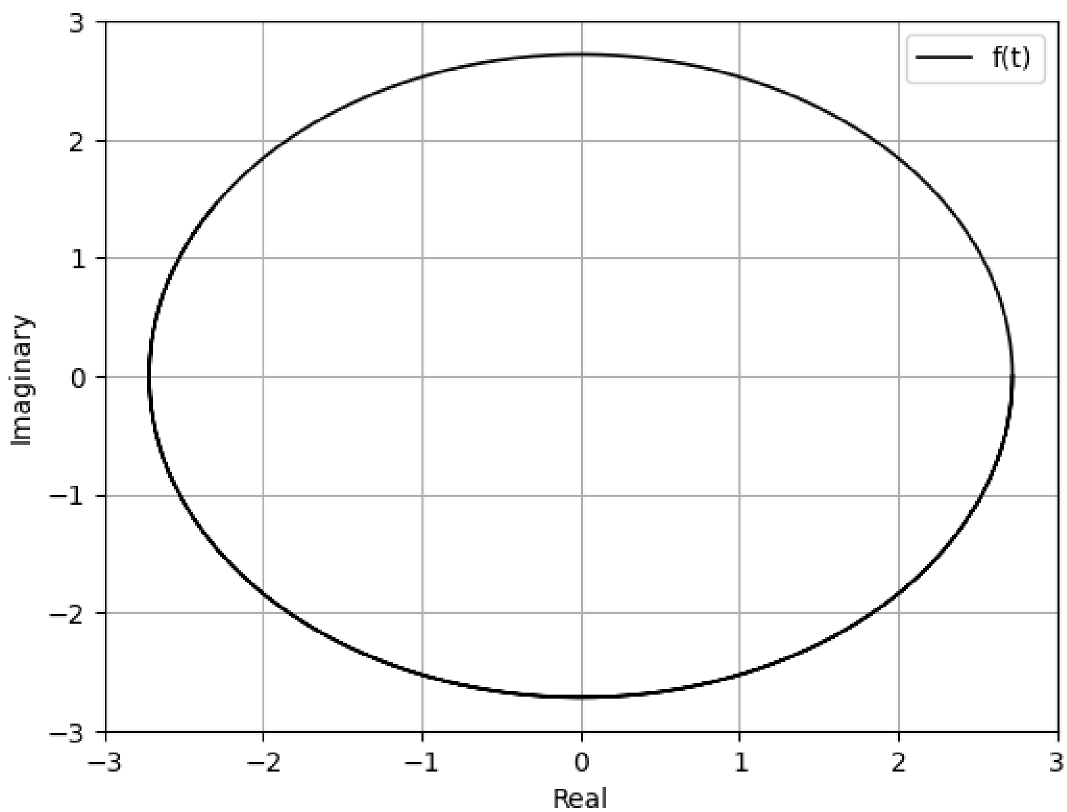
Using Euler's formula,

$$\begin{aligned} e^{1-it} &= e \cdot e^{i(-t)} \\ &= e(cos(-t) + isin(-t)) \\ e^{(1-i)t} &= \boxed{e(cos(t) - isin(t))} \end{aligned}$$

Plot $f(t) = e(cos(t) - isin(t))$,

```
t = np.arange(0, 10, 0.01, dtype=float)

ft3_real = np.exp(1) * np.cos(t)
ft3_imag = -np.exp(1) * np.sin(t)
plt.xlim(-3, 3)
plt.ylim(-3, 3)
plt.xlabel('Real')
plt.ylabel('Imaginary')
plt.grid(True)
plt.plot(ft3_real, ft3_imag, 'k', linewidth=1.2)
plt.legend(['f(t)'])
plt.show()
```



(d) $f(t) = e^{(-.2+3\pi i)t}$

Using Euler's formula,

$$
\begin{aligned}
e^{(-.2+3\pi i)t} &= e^{-0.2t} e^{i(3\pi t)} \\
&= e^{-0.2t}(cos(-3\pi t) + isin(3\pi t)) \\
e^{(-.2+3\pi i)t} &= \boxed{e^{-0.2t}(cos(3\pi t) + isin(3\pi t))}
\end{aligned}
$$

Plot $f(t) = e^{-0.2t}(cos(3\pi t) - isin(3\pi t))$,

```python
    t = np.arange(0, 10, 0.01, dtype=float)

    ft4_real = np.exp(-0.2*t) * np.cos(3*np.pi*t)
    ft4_imag = np.exp(-0.2*t) * np.sin(3*np.pi*t)
    plt.xlim(-1.2, 1.2)
    plt.ylim(-1.2, 1.2)
    plt.xlabel('Real')
    plt.ylabel('Imaginary')
    plt.grid(True)
    plt.plot(ft4_real, ft4_imag, 'k', linewidth=1.2)
    plt.legend(['f(t)'])
    plt.show()

```