ME565 HW1

5-1 Show the following:

(a) $\nabla \times (\nabla \phi) = 0$ for any potential field.

$$\Rightarrow \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]^T \times \left[\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right]^T$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix} = \vec{i}\left(\cancel{\frac{\partial^2\phi}{\partial y \partial z} - \frac{\partial^2\phi}{\partial y \partial z}}\right) - \vec{j}\left(\cancel{\frac{\partial^2\phi}{\partial x \partial z} - \frac{\partial^2\phi}{\partial x \partial z}}\right)$$

$$+ \vec{k}\left(\cancel{\frac{\partial^2\phi}{\partial x \partial y} - \frac{\partial^2\phi}{\partial x \partial y}}\right) = \boxed{0} \#$$

(b) $\nabla^2 \phi = \nabla \cdot (\nabla \phi)$ for any potential field.

$$\text{LHS: } \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\Rightarrow \quad \nabla^2 \phi = \underline{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}} \quad ①$$

$$\text{RHS: } \nabla \cdot (\nabla \phi) = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{bmatrix} = \underline{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}} \quad ②$$

$$\boxed{① = ②, \quad \nabla^2 \phi = \nabla \cdot (\nabla \phi)} \#$$

(c) $\nabla \cdot (\nabla\phi \times \nabla\psi) = 0$ for all potential flow ϕ & ψ

$$\Rightarrow \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \left(\begin{bmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial z \end{bmatrix} \times \begin{bmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \\ \partial\psi/\partial z \end{bmatrix} \right)$$

$$\Rightarrow \quad " \quad \cdot \begin{vmatrix} i & j & k \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \\ \partial\psi/\partial x & \partial\psi/\partial y & \partial\psi/\partial z \end{vmatrix}$$

$$\Rightarrow \quad " \quad \cdot \begin{bmatrix} \frac{\partial\phi\partial\psi}{\partial x\partial y} - \frac{\partial\phi\partial\psi}{\partial x\partial y} \\ -\frac{\partial\phi\partial\psi}{\partial x\partial z} + \frac{\partial\phi\partial\psi}{\partial x\partial z} \\ \frac{\partial\phi\partial\psi}{\partial x\partial y} - \frac{\partial\phi\partial\psi}{\partial x\partial y} \end{bmatrix} \Rightarrow \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot 0$$

$$\Rightarrow \boxed{0} \#$$

5-2

(a) $$(\vec{a}\times\vec{b}) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 \quad a_2b_1 \end{bmatrix}$$

$$- (\vec{a}\times\vec{b})\cdot\vec{a} = a_1(a_2b_3 - a_3b_2) + a_2(-a_1b_3 + a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$= a_1a_2b_3 - a_1a_3b_2 - a_1a_2b_3 + a_2a_3b_1 + a_1a_3b_2 - a_2a_3b_1 = 0$$

$$- (\vec{a}\times\vec{b})\cdot\vec{b} = b_1(a_2b_3 - a_3b_2) + b_2(-a_1b_3 + a_3b_1)$$

$$+\ b_3(a_1b_2 - a_2b_1)$$

$$= a_2b_1b_3 - a_3b_1b_2 - a_1b_2b_3 + a_3b_1b_2$$

$$+\ a_1b_2b_3 - a_2b_1b_3 = 0$$

$$\boxed{\begin{array}{l} \because\ \vec{a}\cdot(\vec{a}\times\vec{b}) = 0,\quad \vec{b}\cdot(\vec{a}\times\vec{b}) = 0, \\ \therefore\ \vec{a}\times\vec{b} \text{ is orthogonal to } \vec{a}\ \&\ \vec{b} \end{array}}$$ #

(b) $\nabla\cdot(\nabla\times\vec{f}) = 0$ for any vector field. $\vec{f}$

$$\Rightarrow \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} \Rightarrow \cdots \begin{bmatrix} \partial f_3/\partial y - \partial f_2/\partial z \\ -\partial f_3/\partial x + \partial f_1/\partial z \\ \partial f_2/\partial x - \partial f_1/\partial y \end{bmatrix}$$

$$\Rightarrow \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} - \frac{\partial^2 f_3}{\partial x\partial y} + \frac{\partial^2 f_1}{\partial y\partial z} + \frac{\partial^2 f_2}{\partial x\partial z} - \frac{\partial^2 f_1}{\partial y\partial z} = \boxed{0}$$ #

5-3

(a) $$D_v f = \nabla f\cdot\vec{V} = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x\cdot y^2 \\ 2\sin x\, y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \boxed{\cos x\cdot y^2 + 4\sin x\cdot y}$$ #

(b) take gradient to find the slope of f

$$\nabla f = \begin{bmatrix} -4x^3 \\ -2y \\ 0 \end{bmatrix}, \quad \nabla f(1, 2, 15) = \begin{bmatrix} -4 \\ -4 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{\text{Direction} = \left[-\tfrac{4}{4\sqrt{2}}, -\tfrac{4}{4\sqrt{2}}, 0\right]^T = \left[-\tfrac{2}{\sqrt{2}}, -\tfrac{2}{\sqrt{2}}, 0\right]^T}$$ #

(c) take gradient to find the slope of T

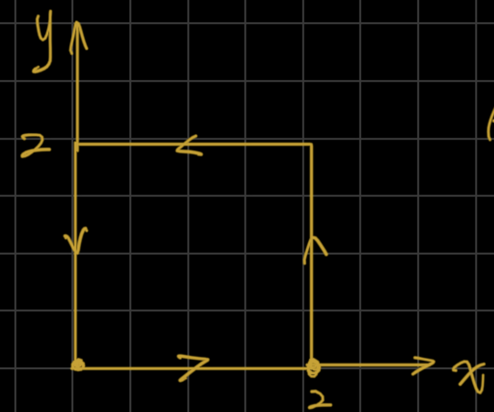
$$\nabla T = \begin{bmatrix} -\sin x \cos y \cdot \sin z \\ -\cos x \sin y \cdot \sin z \\ \cos x \cos y \cos z \end{bmatrix}$$ (x, y, z in rad)

$$, \nabla T(2, 1, 1) = \begin{bmatrix} -\sin(2)\cos(1)\sin(1) \\ -\cos(2)\sin(1)\sin(1) \\ \cos(2)\cos(1)\cos(1) \end{bmatrix}$$

$$\therefore \boxed{\text{Direction} = \begin{bmatrix} -0.413 \\ 0.294 \\ -0.121 \end{bmatrix}}$$ #

5-4

(a)



$$\text{Area} = \frac{1}{2} \int_{\partial S} x\,dy - y\,dx$$

$$\Rightarrow A = \frac{1}{2}\left[\int_{y=0}^{2} 2\,dy - \int_{x=2}^{0} 2\,dx + \int_{y=2}^{0} 0\,dy - \int_{x=0}^{2} 0\,dx\right]$$

$$= \frac{1}{2}\left[\,2y\,\Big|_0^2 - 2x\,\Big|_2^0\,\right] = \frac{1}{2}\left[4-(-4)\right]$$

$$= \boxed{4} \#$$

(b)

$$f(x,y) = x^2+y^2 = 1, \quad x=\cos\theta, \quad y = \sin\theta$$

$$\frac{dx}{d\theta} = -\sin\theta, \quad \frac{dy}{d\theta} = \cos\theta$$

$$\Rightarrow A = \frac{1}{2}\left[\int_{\theta=0}^{2\pi} \cos\theta\,(\cos\theta\,d\theta) - \int_{\theta=0}^{2\pi} \sin\theta\,(-\sin\theta\,d\theta)\right]$$

$$= \frac{1}{2}\left(\int_0^{2\pi} \cos^2\theta\,d\theta + \int_0^{2\pi} \sin^2\theta\,d\theta\right)$$

$$= \frac{1}{2}\left(\int_0^{2\pi} 1\,d\theta\right) = \boxed{\pi} \#$$

(c)

$$f(x,y) = x^{2/3}+y^{2/3} = 1, \quad x=\cos^3\theta, \quad y=\sin^3\theta$$

$$\frac{dx}{d\theta} = 3\cos^2(\theta)(-\sin\theta)$$

$$\frac{dy}{d\theta} = 3\sin^2(\theta)(\cos\theta)$$

$$\Rightarrow A = \frac{1}{2}\int_{\partial S} x\,dy - y\,dx$$

$$= \frac{1}{2}\left[\int_0^{2\pi} \cos^4\theta\cdot 3\sin^2\theta + \sin^4\theta\cdot 3\cos^2\theta\,d\theta\right]$$

$$= \frac{3}{2}\int_0^{2\pi} (\sin^2\theta \cos^2\theta)(\cos^2\theta + \sin^2\theta)\, d\theta$$

$$= \frac{3}{2}\int_0^{2\pi} \sin^2\theta \cos^2\theta \, d\theta$$

trig identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\Rightarrow \frac{1}{4}\sin^2 2\theta = \sin^2\theta\cos^2\theta$$

$$= \frac{3}{8}\int_0^{2\pi} \sin^2(2\theta)\, d\theta$$

trig identity: $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\Rightarrow \sin^2(2\theta) = \frac{1-\cos 4\theta}{2}$$

$$= \frac{3}{8}\int_0^{2\pi} \frac{1}{2}\, d\theta - \cancelto{0}{\frac{3}{8}\int_0^{2\pi} \frac{1}{2}\cos 4\theta\, d\theta}$$

$$= \frac{3}{8}\left(\frac{\theta}{2}\Big|_0^{2\pi} \right) = \boxed{\frac{3}{8}\pi}$$ #