# ME 547: Linear Systems
# Observers and Observer State Feedback Control

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# Introduction

- full state feedback is usually not available
- the state estimation problem
  - deterministic case: observer design
  - stochastic case: the most frequent option is Kalman filter

# Outline

# Open-loop observer

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t),\; x(k+1) = Ax(k) + Bu(k)$$

- conceptually simplest scheme to estimate $x$:

$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t),\; \hat{x}(k+1) = A\hat{x}(k) + Bu(k)$$

  with a best guess of initial estimate $\hat{x}(0) \overset{e.g.}{=} 0$.

- error dynamics: $e = x - \hat{x}$:

$$\dot{e}(t) = Ae(t),\; e(k+1) = Ae(k),\; e(0) = x_0 - \hat{x}(0)$$

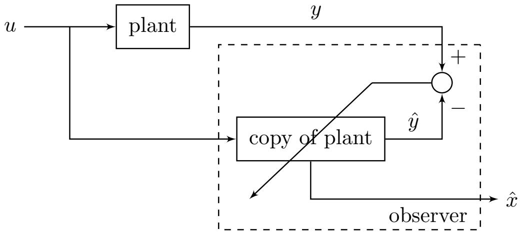
  - sensitive to input disturbances
  - if $A$ is not Hurwitz/Schur, the error diverges
- open-loop observers look simple but do not work in practice

# Luenberger (closed-loop) observer concept

- given system dynamics

$$\dot{x} = Ax + Bu,\ x(0) = x_0,\ A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}$$
$$y = Cx,\ y \in \mathbb{R}^{m\times n}$$

- in contrast to open-loop observers, the Luenberger observer adds correction based on output differences

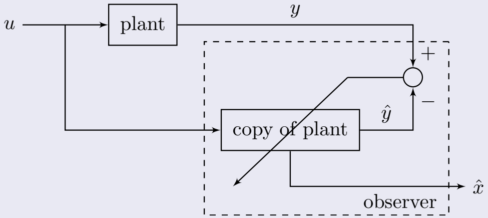


# Luenberger (closed-loop) observer algorithm

## plant:

$$\dot{x} = Ax + Bu, \;\; x(0) = x_0$$
$$y = Cx$$

## observer concept




- observer realization:

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L\left(y - \hat{y}\right) = A\hat{x} + Bu + L\left(y - C\hat{x}\right), \;\; \hat{x}(0) = 0 \\ &= \left(A - LC\right)\hat{x} + Ly + Bu\end{aligned}$$

# Luenberger (closed-loop) observer error dynamics

- system dynamics

$$\dot{x} = Ax + Bu,\ x(0) = x_0,\ A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times r}$$
$$y = Cx,\ y \in \mathbb{R}^{m \times n}$$

- Luenberger observer with correction:

$$\dot{\hat{x}} = A\hat{x} + Bu + L\left(y - \hat{y}\right) = A\hat{x} + Bu + L\left(y - C\hat{x}\right),\ \hat{x}(0) = 0$$
$$= \left(A - LC\right)\hat{x} + Ly + Bu$$

- error dynamics: $e = x - \hat{x}$:

$$\dot{e} = Ae - LCe = (A - LC)e,\ e(0) = x(0)$$

- if all eigenvalues of $A - LC$ are on the left half plane, then the error dynamics can be made asymptotically stable

# Luenberger (closed-loop) observer

**Theorem**

*If $(A, C)$ is an observable pair, then all the eigenvalues of $A - LC$ can be arbitrarily assigned, provided that they are symmetric with respct to the real axis of the complex plane.*

- we show the case when $A$ and $C$ are in observable canonical form (if not, a similarity transform can help out):

$$A = \begin{bmatrix} -\alpha_{n-1} & 1 & 0 & \dots \\ \vdots & 0 & \ddots & \ddots \\ -\alpha_1 & \vdots & \ddots & 1 \\ -\alpha_0 & 0 & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_{n-1} \\ \vdots \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \end{bmatrix}, \quad D = d$$

$$\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

# Observer eigenvalue placement: o.c.f.

- Luenberger observer with correction:

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L\left(y - \hat{y}\right) = A\hat{x} + Bu + L\left(y - C\hat{x}\right), \ \hat{x}(0) = 0 \\ &= \left(A - LC\right)\hat{x} + Ly + Bu\end{aligned}$$

- Goal: place eigenvalues of the observer at locations $\bar{p}_1, \cdots, \bar{p}_n$:

$$\begin{aligned}\det\left(sI - (A - LC)\right) &= (s - \overline{p}_1)(s - \overline{p}_2)\cdots(s - \overline{p}_n) \\ &= s^n + \overline{\gamma}_{n-1}s^{n-1} + \cdots + \overline{\gamma}_1 s + \overline{\gamma}_0\end{aligned}$$

# Observer eigenvalue placement: o.c.f.

- Goal: place eigenvalues of the observer at locations $\bar{p}_1, \cdots, \bar{p}_n$:

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Let $L = [l_0, l_1, \ldots, l_{n-1}]^T$. The unique structures of $A$ and $C$ give

$$LC = \begin{bmatrix} l_0 \\ \vdots \\ l_{n-2} \\ l_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} = \begin{bmatrix} l_0 & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & \vdots \\ l_{n-2} & \ddots & \ddots & 0 \\ l_{n-1} & 0 & \ldots & 0 \end{bmatrix}$$

$$A - LC = \begin{bmatrix} -\alpha_{n-1} - l_0 & 1 & 0 & \ldots & 0 \\ -\alpha_{n-2} - l_1 & 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ -\alpha_1 - l_{n-2} & \vdots & \ddots & 0 & 1 \\ -\alpha_0 - l_{n-1} & 0 & \ldots & 0 & 0 \end{bmatrix}$$

# Observer eigenvalue placement: o.c.f.

- $A$ and $A - LC$ have the same structure:

$$A = \begin{bmatrix} -\alpha_{n-1} & 1 & 0 & \dots \\ \vdots & 0 & \ddots & \ddots \\ -\alpha_1 & \vdots & \ddots & 1 \\ -\alpha_0 & 0 & \dots & 0 \end{bmatrix}, \ A - LC = \begin{bmatrix} -\alpha_{n-1} - l_0 & 1 & 0 & \dots \\ \vdots & 0 & \ddots & \ddots \\ -\alpha_1 - l_{n-2} & \vdots & \ddots & 1 \\ -\alpha_0 - l_{n-1} & 0 & \dots & 0 \end{bmatrix}$$

- Recall: $\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_1 s + \alpha_0$.
- Thus

$$\det(sI - (A - LC)) = s^n + \underbrace{(\alpha_{n-1} + l_0)}_{\text{target: } \overline{\gamma}_{n-1}} s^{n-1} + \dots + \underbrace{(\alpha_0 + l_{n-1})}_{\text{target: } \overline{\gamma}_0}$$

- Hence

$$l_0 = \overline{\gamma}_{n-1} - \alpha_{n-1}$$
$$\vdots$$
$$l_{n-1} = \overline{\gamma}_0 - \alpha_0$$

# General observer eigenvalue placement

- What if $(A, B, C, D)$ is not in the observable canonical form?
- We can transform it to o.c.f. via a similarity transform:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \overset{x=R^{-1}x_{ob}}{\Longrightarrow} \quad \begin{cases} \dot{x}_{ob} & = \underbrace{RAR^{-1}}_{A_o} x_{ob} + \underbrace{RB}_{B_o} u \\ y & = C_o x_{ob} = CR^{-1} x_{ob} \end{cases}$$

- use previous formulas to design $\tilde{L}$ in:

$$\dot{\hat{x}}_{ob} = \left(A_o - \tilde{L}C_o\right)\hat{x}_{ob} + \tilde{L}y + B_o u \qquad \text{(analysis form)}$$

correspondingly in the original state space (via $\hat{x}_{ob} = R\hat{x}$):

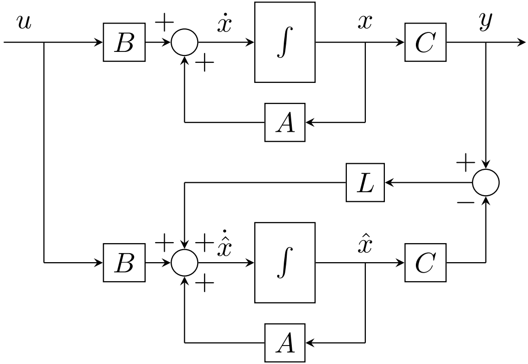
$$R\dot{\hat{x}} = \left(RAR^{-1} - \tilde{L}CR^{-1}\right)R\hat{x} + \tilde{L}y + RBu$$

$$\Rightarrow \dot{\hat{x}} = (A - \overbrace{R^{-1}\tilde{L}}^{L}C)\hat{x} + Ly + Bu \qquad \text{(implementation form)}$$

- **Powerful fact**: if system $\Sigma = (A, B, C, D)$ is observable, then we can arbitrarily place the observer eigenvalues.

# Luenberger observer summary

- observer dynamics: $\dot{\hat{x}} = A\hat{x} + Bu + L\left(y - C\hat{x}\right), \ \hat{x}(0) = 0$
- block diagram



# Luenberger observer summary

- system dynamics

$$\dot{x} = Ax + Bu,\ x(0) = x_0,\ A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}$$
$$y = Cx,\ y \in \mathbb{R}^{m\times 1}$$

- observer dynamics

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L\left(y - C\hat{x}\right),\ \hat{x}(0) = 0\\ &= \left(A - LC\right)\hat{x} + LCx + Bu\end{aligned}$$

- augmented system

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u$$

# Luenberger observer summary

- augmented system

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u$$
$$y = Cx$$

- to see the distribution of eigenvalues, note the error dynamics $\dot{e} = (A - LC)e \Rightarrow$

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$\Rightarrow$eigenvalues are separated into: $\lambda(A)$ and observer eigenvalues

- underlying similarity transform: $\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$

# Discrete-time observers: Introduction

- full state feedback is usually not available
- often observers are implemented in the discrete-time domain
- the discrete-time observer design
  - basic form: analogous to the continuous-time Luenberger observer
  - predict and correct form:
    - direct DT design
    - leverages discrete-time signal properties

# Discrete-time full state observer

- standard discrete-time observer:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k)) \\
y(k) &= Cx(k)
\end{aligned}
$$

- error dynamics: $e(k) = x(k) - \hat{x}(k)$,
  $e(k+1) = Ae(k) - LCe(k)$
- overall dynamics

$$
\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k)
$$

$$
y(k+1) = [C,\ 0] \begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix}
$$

- **Powerful fact**: the error dynamics can be arbitrarily assigned if the system is observable.

# DT full state observer with predictor

- motivation: $\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k))$ doesn't use most recent measurement $y(k+1) = Cx(k+1)$
- discrete-time observer **with predictor**:

predictor: $\hat{x}(k+1|k) = A\hat{x}(k|k) + Bu(k)$

corrector: $\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + L(y(k+1) - C\hat{x}(k+1|k))$

  - $\hat{x}(k|k)$: estimate of $x(k)$ based on measurements up to time $k$
  - $\hat{x}(k|k-1)$: estimate based on measurements up to time $k-1$
  - $e(k) \triangleq x(k) - \hat{x}(k|k)$: estimation error
- error dynamics

$$\begin{aligned}
\hat{x}(k+1|k+1) &= (I - LC)\hat{x}(k+1|k) + Ly(k+1) \\
&= (I - LC)A\hat{x}(k|k) + (I - LC)Bu(k) + Ly(k+1) \\
\Rightarrow e(k+1) &= x(k+1) - Ly(k+1) - (I - LC)A\hat{x}(k|k) - (I - LC)Bu(k) \\
&= (A - LCA)e(k)
\end{aligned}$$

# DT full state observer with predictor

$$e(k+1) = \left(A - L\underbrace{CA}_{\tilde{C}}\right) e(k), \quad e(0) = (I - LC)\, x_0$$

- the error dynamics can be arbitrarily assigned if the pair $\left(A,\ \tilde{C}\right) = (A,\ CA)$ is observable
- observability matrix

$$\tilde{Q}_d = \begin{bmatrix} \tilde{C} \\ \tilde{C}A \\ \vdots \\ \tilde{C}A^{n-1} \end{bmatrix} = \overbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}^{Q_d} A$$

- if $A$ is invertible, then $\tilde{Q}_d$ has the same rank as $Q_d$
- $\left(A,\ \tilde{C}\right)$ is observable if $(A,\ C)$ is observable and $A$ is nonsingular (guaranteed if discretized from a CT system)

# Example

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u(k),$$

$y(k) = x_1(k)$. Place all eigenvalues of an **observer with predictor** at the origin.

$$\begin{aligned} A - LCA &= \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} -a_2 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (l_1 - 1) a_2 & 1 - l_1 & 0 \\ l_2 a_2 - a_1 & -l_2 & 1 \\ l_3 a_2 - a_0 & -l_3 & 0 \end{bmatrix} \end{aligned}$$

$\det(A - LCA - \lambda I) = ((l_1 - 1) a_2 - \lambda)(l_2 + \lambda)\lambda +$
$(1 - l_1)(l_3 a_a - a_0) + l_3((l_1 - 1) a_2 - \lambda) + \lambda(1 - l_1)(l_2 a_2 - a_1)$
roots must be all 0 $\Rightarrow l_1 = 1,\ l_2 = l_3 = 0$.

# Observer state feedback

given system dynamics:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

- state feedback control: arbitrary eigenvalue assignment if system controllable
- observer design: arbitrary observer eigenvalue assignment for state estimation if system observerable
- when full states are not available, what's the performance if we combine both?

$$u = -K\hat{x} + v$$

# Closed-loop dynamics

- full closed-loop system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L\left(y - C\hat{x}\right)$$
$$u = -K\hat{x} + v$$

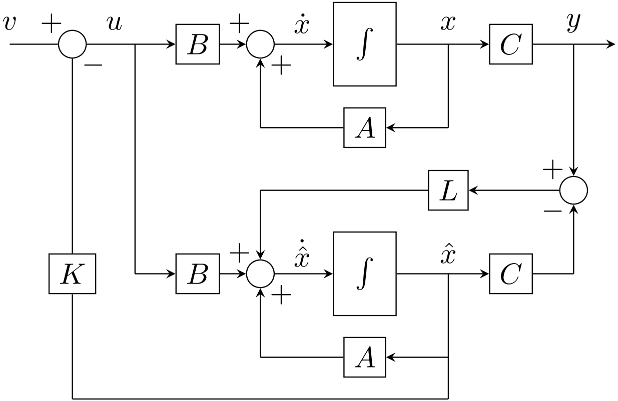
$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v$$

- using again similarity transform $\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$ gives

$$\frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v$$

# Block diagram

- $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$, $u = -K\hat{x} + v$



# The separation theorem

- closed-loop dynamics

$$\frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v$$

- **powerful result: separation theorem:** closed-loop eigenvalues consist of
  - eigenvalues of $A - BK$ from the state feedback control design
  - eigenvalues of $A - LC$ from the observer design
- can design $K$ and $L$ separately based on discussed tools
- if system is controllable and observable, we can arbitrarily assign the closed-loop eigenvalues