Name: ______________________________ /66

120 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. Read problems very carefully. Don't do what you are not asked to and don't be wrong.

*Education is what remains after one has forgotten what one has learned in school.*

*A. Einstein*

**1.** *(6 pts)* Find the general solution to ODE $\frac{dP}{dt} = P^2 - P$.

$$\frac{dP}{P^2-P} = dt \; ; \; \int \frac{dP}{P^2-P} = \int \frac{dP}{P(P-1)} = \int \frac{P-(P-1)}{P(P-1)}\, dP$$

$$= \int \frac{dP}{P-1} - \int \frac{dP}{P} = \ln\left|\frac{P-1}{P}\right| + C$$

$$\ln\left|\frac{P-1}{P}\right| = x + C \; ; \; \frac{P-1}{P} = Ke^x \; ; \; 1 - \frac{1}{P} = ke^x \; ; \; 1 - ke^x = \frac{1}{P}$$

$$P = \frac{1}{1-ke^x}$$

**2.** *(6 pts)* Consider the IVP $2y'' + xy' = 0$, $y(0) = y'(0) = 1$. Find a power series solution to this DE. (You may leave your answer as a recurrence relation on coefficients together with initial conditions).

$$y(x) = \sum_{n\geq 0} c_n x^n \; ; \; y' = \sum_{n \geq 1} n c_n x^{n-1} \; ; \; xy' = \sum_{n\geq 1} n c_n x^n \; ;$$

$$2y'' = 2\sum_{n\geq 2} n(n-1) c_n x^{n-2} = \sum_{k\geq 0} 2(k+2)(k+1) c_{k+2} x^n$$

From IC, $\boxed{c_0 = c_1 = 1}$; Plug this all to the DE:

$$2\cdot 2\cdot 3 \cdot c_2 + \sum_{k\geq 1} 2(k+2)(k+1) c_{k+2} x^k + \sum_{k\geq 1} k c_k x^k = 0$$

$$\Downarrow$$

$$c_2 = 0$$

$$\Downarrow$$

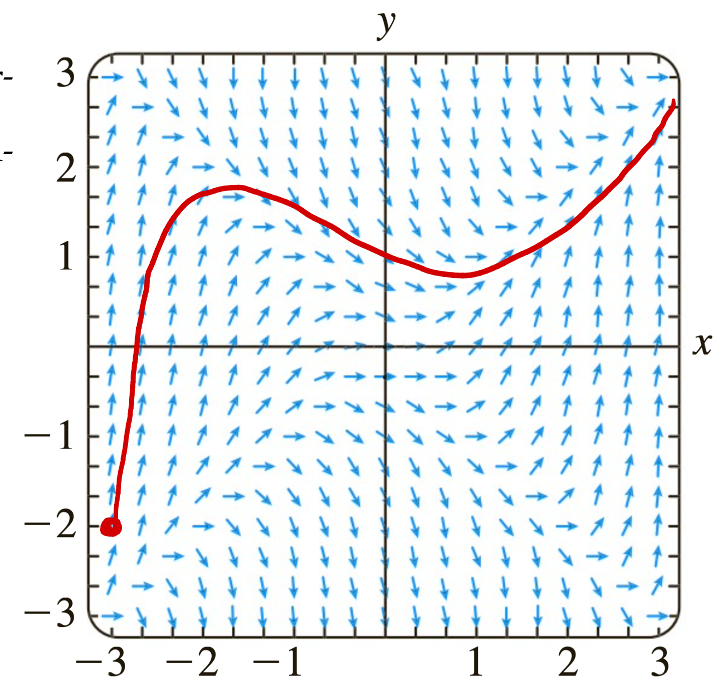
$$2(k+2)(k+\;\;) c_{k+2} + kc_k = 0$$

$$\boxed{c_0 = c_1 = 1, \qquad c_{k+2} = \frac{-kc_k}{2(k+2)(k+1)}}$$

**3.** Consider the DE $y' = x^2 - y^2$.

a) *(3 pts)* Its direction field is given. Plot a particular solution satisfying $y(-3) = -2$.

b) *(3 pts)* Use 3 steps of Euler's method to estimate $y(3)$.




$$y_{n+1} = y_n + 2f(x_n, y_n)$$
$$= y_n + 2x_n^2 - 2y_n^2$$

$x_0 = -3, \ y_0 = -2$

$x_1 = -1, \ y_1 = -2 + 18 - 8 = 8$

$x_2 = 1, \ y_2 = 8 + 2 - 128 = -118$

$x_3 = 3, \ y_3 = \underbrace{-118 + 2 - 2 \cdot 118^2}_{\text{this is fine}}$

$(= -27964)$

**4.** *(6 pts)* Show that the DE is exact and solve it: $\overbrace{(\sin y + y \sin x)}^{M}\, dx + (\underbrace{\cos x + x \cos y - y}_{N})\, dy = 0$.

$M_y = \cos y + \sin x$

$N_x = -\sin x + \cos y$

— not exact

**5.** *(6 pts)* Write down a linear DE with constant coefficient which has solutions $e^{-2x}$ and $x \sin x$. For 3 extra credit points, use the DE of minimal possible order and explain why this is indeed the minimal order (don't write an essay, but use words).

$\chi(\lambda)$ must include $(\lambda+2)$ for $e^{-2x}$

and $(\lambda^2+1)$ of multiple. 2 to get $\sin x$.

minimal $\chi(\lambda) = (\lambda+2)(\lambda^2+1) = \lambda^5 + 2\lambda^4 + 2\lambda^3 + 4\lambda^2 + \lambda + 2$

DE: $y^{(5)} + 2y^{(4)} + 2y^{(3)} + 4y^{(2)} + y^{(1)} + 2y = 0.$

**6.** *(6 pts)* Consider the DE $(1-x^2)y'' + 2xy' = 0$. Verify that $y_1(x) = 1$ solves the DE and use the reduction of order to find the general solution.

$y_1' = y_1'' = 0 \Rightarrow y_1$ solves DE.

$(1-x^2)y'' + 2xy' = 0 \quad [u = y']$

$u' + \frac{2x}{1-x^2}u = 0$

$\frac{du}{dx} = -\frac{2x}{1-x^2}u$

$\frac{du}{u} = -\frac{2x}{1-x^2}dx$

$\ln|u| = -\int \frac{2x}{1-x^2}dx = -\int \frac{dx^2}{1-x^2} = \int \frac{dx^2}{x^2-1} = \ln|x^2-1| + c$

$u = x^2 - 1$

$y_2 = \frac{x^3}{3} - x \Rightarrow y(x) = c_1 + c_2\left(\frac{x^3}{3} - x\right)$

**7.** *(6 pts)* Use the definition of the Laplace transform to find $\mathcal{L}\{\cos 2t\}$.

$$\mathcal{L}\{\cos 2t\} = \int_0^{+\infty} e^{-st} \cos 2t \, dt = \int_0^{+\infty} e^{-st} \frac{e^{2it} + e^{-2it}}{2} dt$$

$$= \frac{1}{2} \int_0^{+\infty} e^{t(2i-s)} + e^{t(-2i-s)} dt$$

$$= \frac{1}{2} \left[ \frac{1}{2i-s} e^{(2i-s)t} - \frac{1}{2i+s} e^{t(-2i-s)} \right]_0^{+\infty}$$

$$= \frac{1}{2} \left[ -\frac{1}{2i-s} + \frac{1}{2i+s} \right] = \frac{s^2}{s^2+4}$$

**8.** *(6 pts)* Use the Laplace transform to solve $y' - y = te^t \sin t$, $y(0) = 0$.

$$sY - 0 - Y = \mathcal{L}\{te^t \sin t\}$$

$$\mathcal{L}\{t \sin t\} = \frac{2s}{(s^2+1)^2} \quad (\text{table})$$

$$\mathcal{L}\{e^t t \sin t\} = \frac{2(s-1)}{((s-1)^2+1)^2}$$

$$Y = \frac{1}{s-1} \cdot \frac{2(s-1)}{((s-1)^2+1)^2}$$

$$y = \mathcal{L}^{-1} \frac{2}{((s-1)^2+1)^2} = e^t(\sin x - x \cos x)$$

!! (will need to multiply by $e^t$)

$F(s-1)$

$F(s) = \frac{2}{(s^2+1)^2}$ ; $\mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)^2}\right\}$ = see page ~~8~~ 7 $= \sin \cancel{t} - \cancel{t} \cos \cancel{t}$

**9.** *(6 pts)* Consider the IVP $y'' - y = 0$, $y(0) = 1$, $y'(0) = -1$.

a) Write this IVP as a first order system $\mathbf{X}' = \mathbf{A}\mathbf{X}$;

$x_1 = y \qquad x_1' = x_2$

$x_2 = y' \qquad x_2' = x_1 \quad ; \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b) For the matrix $\mathbf{A}$ found in part a) compute and simplify $e^{\mathbf{A}t}$;

$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ (see P2 from HW for 8.4)

c) Use the result from part b) to solve the IVP.

$\underline{X} = e^{At}\underline{X}_0 = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \cos t - \sin t \\ \text{something} \end{bmatrix}$

$y = x_1 = \cos t - \sin t$

**10.** *(6 pts)* Initially 50 pounds of salt is dissolved in a large tank holding 300 gallons of water. A brine solution of salt begins to flow at a constant rate of 3 gallons/min. The concentration of salt in the brine entering the tank is 2 pounds per gallon. The solution inside the tank is kept well stirred and is flowing out of the tank at the same rate.

a) Set up an IVP which will determine the amount $A(t)$ of salt in the tank at time t.

$$A' = 2\cdot 3 - \frac{A}{300}\cdot 3 \iff A' = 6 - \frac{A}{100}$$

$$A(0) = 50$$

b) Find the amount of salt in the tank after a long time.

$A' + \frac{A}{100} = 6$; $A = e^{-t/100} + 600$ solves DE; from Picard-Lindelöf indeed the solution.

$$A(+\infty) = \lim_{t\to+\infty}\left(e^{-t/100} + 600\right) = 600$$

**11.** *(6 pts)* Solve $y'' - 3y' + 2y = e^{t}$ by variation of parameters (no credit for solving it by a different method).

$$y_c = c_1 e^{t} + c_2 e^{2t}$$

$$y_p = u_1 e^{t} + u_2 e^{2t}$$

$$W = \begin{vmatrix} e^{t} & e^{2t} \\ e^{t} & 2e^{2t} \end{vmatrix} = e^{3t}$$

$$W_1 = \begin{vmatrix} 0 & e^{2t} \\ e^{t} & 2e^{2t} \end{vmatrix} = -e^{3t}$$

$$W_2 = \begin{vmatrix} e^{t} & 0 \\ e^{t} & e^{t} \end{vmatrix} = e^{2t}$$

$$u_1 = \int \frac{-e^{3t}}{e^{3t}}\, dt = -t$$

$$u_2 = \int \frac{e^{2t}}{e^{3t}}\, dt = -e^{-t}$$

$$y_p = -te^{t} - e^{-t}e^{2t}$$

$$y = c_1 e^{t} + c_2 e^{2t} - te^{t}$$

EXTRA SPACE

8 cont.

Need to compute $\mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)^2}\right\}$

Looks like $\mathcal{L}\{t \sin t\}$ (see table)

But need one $s$ on top. If $g' = t\sin t$, $g(0)=0$

then $\mathcal{L}g' = s\mathcal{L}g$

$$\frac{2s}{(s^2+1)^2} = s\mathcal{L}g \implies \frac{2}{(s^2+1)^2} = \mathcal{L}g$$

So $\mathcal{L}^{-1}\left(\frac{2}{(s^2+1)^2}\right) = \int_0^t x\sin x \, dx = $ (IBP)

$= \sin t - t\cos t$

## Brief Table of Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\quad n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int u\,dv = uv - \int v\,du$$

$$\int e^x\,dx = e^x + c$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = \ln|\sec x| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x + \sqrt{x^2-a^2}\right| + c$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x + \sqrt{x^2+a^2}\right| + c$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2+a^2}\right| + c$$

## Some Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots$$

## Table of Laplace Transforms

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$\mathcal{L}\left\{t^{1/2}\right\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$$

$$\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$$

$$\mathcal{L}\left\{te^{at}\right\} = \frac{1}{(s-a)^2}$$

$$\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\left\{e^{at}\sin(kt)\right\} = \frac{k}{(s-a)^2+k^2}$$

$$\mathcal{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}$$

$$\mathcal{L}\{t\sin(kt)\} = \frac{2ks}{(s^2+k^2)^2}$$

$$\mathcal{L}\{t\cos(kt)\} = \frac{s^2-k^2}{(s^2+k^2)^2}$$

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$$

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$\mathcal{L}\{f * g\} = F(s)G(s)$$