Homework 8.4

due 11:59pm Monday 8 December, by Gradescope as usual

The matrix exponential. For a square matrix $\mathbf{A}$, the matrix exponential function $e^{\mathbf{A}t}$ is defined as

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \dots + \mathbf{A}^k\frac{t^k}{k!} + \dots = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!}$$

Notice that if $t = 0$ then the matrix exponential is just the identity matrix:

$$e^{\mathbf{A}\,0} = \mathbf{I}$$

The formula for $e^{\mathbf{A}t}$ includes the ordinary (scalar) exponential as a special case: if $\mathbf{A} = (\, a \,)$ is a 1×1 matrix with entry a then $e^{\mathbf{A}t} = e^{at} = 1 + at + (at)^2/2 + (at)^3/3! + \dots$

One can only compute $e^{\mathbf{A}t}$ by hand in easy cases. First of all you need to be able to compute powers of $\mathbf{A}$; you have to know how to do matrix-matrix multiplication. Then one "easy case" is when $\mathbf{A}$ is a diagonal matrix. Another is when $\mathbf{A}$ is a nonzero matrix for which there is a power $\mathbf{A}^k$ which is the zero matrix, because then the infinite series becomes a finite sum.

The key fact about the matrix exponential, which makes it useful for differential equations, is the usual derivative rule for exponentials:

$$\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A} e^{\mathbf{A}t}$$

The matrix exponential allows us to solve any 1st-order, linear, constant-coefficient system of differential equations with ease. For homogeneous systems

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

the general solution is

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$$

Here $\mathbf{X}(t)$ is a column vector of the solution components and $\mathbf{C}$ is a column vector of constants:

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Notice that $\mathbf{X}(0) = \mathbf{C}$ in the general solution formula (because $e^{\mathbf{A}0} = \mathbf{I}$). If $\mathbf{X}_0$ is a vector then the solution of the initial value problem

$$\mathbf{X}' = \mathbf{A}\mathbf{X}, \qquad \mathbf{X}(0) = \mathbf{X}_0$$

is

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{X}_0$$

In other words, $\mathbf{C}$ is the vector of initial values.

As usual, a computer can do the job! In Matlab or Octave, $e^{\mathbf{A}t}$ = `expm(A * t)`. (*The command* `exp()` *gives the wrong answer here. It exponentiates entrywise.*) So, if you have entered a square matrix `A` and a vector `C`, then the solution $X(t)$ at a particular time `t` is

```
>> X = expm(A * t) * C
```

In Problems 1 and 2 use the definition to compute $e^{\mathbf{A}t}$ by hand, and simplify.

Problem 1. $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}$

Problem 2. $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Problems 3 and 4 require technology, presumably Matlab. Compute the matrix $e^{\mathbf{A}t}$ and the (particular) vector $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$.

Problem 3. $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $t = 1$

Problem 4. $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $t = 0.5$

Problems 5 and 6 relate the above calculations to familiar stuff. Use the **auxiliary equation method** (§4.3) to solve the initial value problem by hand. You will get the same numbers as in Problems 3 and 4. Explain this by **writing the differential equation as a 1st-order system** and **stating explicitly** what $y(t)$ corresponds to in Problems 3 and 4.

Problem 5. Solve the initial value problem and compute $y(t)$ at $t = 0.5$:

$$y'' - y' - 6y = 0, \qquad y(0) = 1,\ y'(0) = 0$$

Problem 6. Solve the initial value problem and compute $y(t)$ at $t = 1$:

$$y'' + y = 0, \qquad y(0) = 0,\ y'(0) = 2$$

Use the definition to compute $e^{\mathbf{A}t}$ by hand, and simplify.

Problem 7. $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix}$

Use technology to compute the matrix $e^{\mathbf{A}t}$ and the vector $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$.

Problem 8. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \\ -2 & -3 & -4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -1 \\ 9 \\ 1 \end{pmatrix}$, $t = \pi$