- Compute the gradient vector for a plane in 3D space (0.5 point)

$z = f(x, y) = ax + by + c$

$\nabla z = \langle f_x, f_y \rangle = \langle a, b \rangle$

- Compute the gradient vector for a hyperplane (0.5 point)
  - Note: derivative of the sum is the sum of the derivatives

$$z = f(\mathbf{x}) = f(x_1, x_2, \dots, x_N) = \sum^{N} a_i(x_i - b_i) + S = a_1x_1 + a_2x_2 + \cdots + a_Nx_N + d$$

$\nabla z = \langle f_{x_1}, f_{x_2}, \cdots, f_{x_N} \rangle = \langle a_1, a_2, \cdots, a_N \rangle$

- Compute the partial derivative of the paraboloid function (1.5 point)

$z = f(x, y) = A(x - x_o)^2 + B(y - y_o)^2 + C$

$$f_x(x, y) = \left(\frac{\partial f(x, y)}{\partial x}\right)_y = ?$$

$$f_y(x, y) = \left(\frac{\partial f(x, y)}{\partial y}\right)_x = ?$$

$z = A(x^2 - 2x_0x + x_0^2) + B(y^2 - 2y_0y + y_0^2) + C$

$f_x(x, y) = \left(\frac{\partial f(x, y)}{\partial x}\right) = 2Ax - 2Ax_0$

$f_y(x, y) = \left(\frac{\partial f(x, y)}{\partial y}\right) = 2By - 2By_0$

- Given the following matrices and vectors (1.5 point)

$$\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 2 & 5 & 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & 5 \\ 5 & 2 \\ 1 & 4 \end{pmatrix}$$

$[3 \times 1]$ $[1 \times 3]$ $[3 \times 3]$ $[3 \times 2]$

- compute the following quantities and specify the shape of the output, if an operation is not defined then just say "not defined"
  - (where dot specifies a dot product and x specifies a matrix product).

$\mathbf{x}^T$ $\mathbf{y}^T$ $\mathbf{B}^T$ $\mathbf{x} \cdot \mathbf{x}$ $\mathbf{x} \cdot \mathbf{y}^T$ $\mathbf{x} \times \mathbf{y}$ $\mathbf{y} \times \mathbf{x}$ $\mathbf{A} \times \mathbf{x}$ $\mathbf{A} \times \mathbf{B}$ **B.reshape(1,6)**

$x^T = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}^T = (3 \quad 1 \quad 4)$

$$y^T = (2 \ \ 5 \ \ 1)^T = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

$$x \cdot x = 3\times3 + 1\times1 + 4\times4 = 9+1+16 = 26$$

$$x \cdot y^T = 3\times2 + 1\times5 + 4\times1 = 6+5+4 = 15$$

$$x \times y = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \times (2 \ \ 5 \ \ 1) = \begin{pmatrix} 3\times2 & 3\times5 & 3\times1 \\ 1\times2 & 1\times5 & 1\times1 \\ 4\times2 & 4\times5 & 4\times1 \end{pmatrix} = \begin{pmatrix} 6 & 15 & 3 \\ 2 & 5 & 1 \\ 8 & 20 & 4 \end{pmatrix}$$

$$y \times x = (2 \ \ 5 \ \ 1) \times \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = (2\times3+5\times1+1\times4) = (15)$$

$$A \times x = \begin{pmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4\times3+5\times1+2\times4 \\ 3\times3+1\times1+5\times4 \\ 6\times3+4\times1+3\times4 \end{pmatrix} = \begin{pmatrix} 25 \\ 30 \\ 34 \end{pmatrix}$$

$$A \times B = \begin{pmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 5 \\ 5 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4\times3+5\times5+2\times1 & 4\times5+5\times2+2\times4 \\ 3\times3+1\times5+5\times1 & 3\times5+1\times2+5\times4 \\ 6\times3+4\times5+3\times1 & 6\times5+4\times2+3\times4 \end{pmatrix}$$

$$= \begin{pmatrix} 29 & 38 \\ 19 & 37 \\ 41 & 50 \end{pmatrix}$$

$$B.\text{reshape}(1, 6) = (3 \ \ 5 \ \ 5 \ \ 2 \ \ 1 \ \ 4)$$

- Use Calculus to analytically derive the expression for single variable linear regression fitting parameters using the sum of square error as the loss function (show your work)

Model: $y = M(x|\mathbf{p}) = mx + b$

$\mathbf{p} = (p_0, p_1) = (m, b)$

Loss surface: $L(\mathbf{p}) = L(m, b) = \sum_{i=1}^{N} \left(\hat{y}_i - M(\hat{x}_i, m, b)\right)^2$

solution: $m = \dfrac{cov(x,y)}{var(x)}$ $\qquad \bar{x} = \dfrac{1}{N}\sum_{i=1}^{N} x_i \qquad \text{var}(X) = \dfrac{1}{N}\sum_{i=1}^{n}(x_i - \bar{x})^2$

$b = \bar{y} - \dfrac{cov(x,y)}{var(x)}\bar{x}$ $\qquad \bar{y} = \dfrac{1}{N}\sum_{i=1}^{N} y_i \qquad \text{cov}(X,Y) = \dfrac{1}{N}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$

$$L(p) = \sum_{i=1}^{N} (\hat{y}_i - m x_i - b)^2.$$

$$\frac{d}{dm} = 2\sum_{i=1}^{N} (\hat{y}_i - m x_i - b)\cdot(-x_i)$$

Let $\dfrac{d}{dm} = 0$

$$\sum_{i=1}^{N} \hat{y}_i - \sum_{i=1}^{N} x_i \cdot m - Nb = 0$$

$$b = \bar{y} - \bar{x} \cdot m$$

$$\frac{d}{db} = -2\sum_{i=1}^{N} (\hat{y}_i - m x_i - b)$$

Let $\dfrac{d}{db} = 0 \Rightarrow \sum_{i=1}^{N} (\hat{y}_i - m x_i - \bar{y} + N\bar{x} \cdot m) = 0$

$$\sum_{i=1}^{N} x_i - \sum_{i=1}^{n} \bar{x} m = \sum_{i=1}^{n} \hat{y}_i - \sum_{i=1}^{n} \bar{y}$$

$$m = \sum_{i=1}^{n} \frac{\hat{y}_i - \bar{y}}{\hat{x}_i - \bar{x}}$$

$$= \frac{cov(x,y)}{var(x)}$$

$$b = \bar{y} - \frac{cov(x,y)}{var(x)} \cdot \bar{x}$$