Christopher David Miller
FSMAT 201
Problem 1.8
Version 3

1.8

Proposition 1. *Let* $\mathbf{u} \in \mathbb{R}^n$. *Prove that the normalization of* $\boldsymbol{u}$ *has norm 1.*

Proof. Let $\mathbf{u} \in \mathbb{R}^n$. The normalization of a vector $\mathbf{u}$ is the vector $\mathbf{v}$ given by

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

The norm of a vector $\mathbf{u}$ is the scalar $\|\mathbf{u}\|$ given by

$$\|\mathbf{u}\| = \sqrt{\sum_{k=1}^{n} u_k^2} \text{ ,where } \mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 \ldots u_n \end{bmatrix}^T.$$

In order to show the normalization of a vector $\mathbf{u}$ has norm 1, we find the normalization of $\mathbf{u}$:

$$\frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

Now we must calculate the norm of the normalization of $\mathbf{u}$:

$$\begin{aligned} \frac{\mathbf{u}}{\|\mathbf{u}\|} &= \sqrt{\sum_{k=1}^{n} (v_k)^2} \text{ ,where } v_k = \frac{u_k}{\|\mathbf{u}\|} \\ &= \sqrt{\sum_{k=1}^{n} \frac{u_k^2}{\|\mathbf{u}\|^2}} \\ &= \sqrt{\frac{1}{\|\mathbf{u}\|^2} \sum_{k=1}^{n} u_k^2} \\ &= \frac{1}{\|\mathbf{u}\|} \sqrt{\sum_{k=1}^{n} u_k^2} \\ &= \frac{1}{\|\mathbf{u}\|} \|\mathbf{u}\| \\ &= 1. \end{aligned}$$

So from the above steps we see that the normalization of a vector has norm 1. □

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Problem 1.1
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1.1

Let $\mathbf{v} = [1, 2, 3, \ldots, n]^T$ and $\mathbf{w} = [1, 1, 1 \ldots, 1]^T$, where $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

(a) Compute $\mathbf{v} \cdot \mathbf{w}$.

Solution :

Suppose there are two vectors $\mathbf{v} = [v_1, v_2, v_3 \ldots, v_i]$ and $\mathbf{w} = [w_1, w_2, w_3 \ldots, w_i]$. The dot product between two vectors $\mathbf{v}$ and $\mathbf{w}$ is given by:

$$\sum_{i=1}^{n} v_i w_i.$$

So for this problem we have the following:

$$\mathbf{v} \cdot \mathbf{w} = 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + \cdots + 1 \cdot n.$$

This can be written more succinctly in summation notation as

$$\sum_{i=1}^{n} i.$$

(b) Compute $\|\mathbf{v}\|$.

Solution : We first remind ourselves that the definition of the norm of a vector $\mathbf{u}$, where $\mathbf{u} = [u_1, u_2, u_3 \ldots, u_i]$ is

$$\|\mathbf{u}\| = \sqrt{\sum_{k=1}^{n} u_i^2}.$$

Using this definition and substituting we get

$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + 3^2 + \cdots + n^2}.$$

(c) Compute $\|\mathbf{w}\|$.

Solution : Similarily using the definition from part (b) we can calculate the norm of the vector $\mathbf{w}$:

$$\begin{aligned}
\|\mathbf{w}\| &= \sqrt{1^2 + 1^2 + 1^2 + \cdots + 1^2} \\
&= \sqrt{1 + 1 + 1 + \cdots + 1} \\
&= \sqrt{n}.
\end{aligned}$$

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Problem 1.3
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1.3

Let c be a real number and let $\mathbf{v} \in \mathbb{R}^n$ with $v_k = c$ for $k = 1, 2, \ldots, n$. Find $\|\mathbf{v}\|$.

Solution : The norm of a vector $\mathbf{v}$ is the scalar $\|\mathbf{v}\|$ given by

$$\|\mathbf{v}\| = \sqrt{\sum_{k=1}^{n} v_k^2} \text{ ,where } \mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \ldots v_n \end{bmatrix}^T.$$

So in our case we replace ,v_k with c and then we have:

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{c^2 + c^2 + c^2 + \cdots + c^2} \\ &= \sqrt{nc^2} \\ &= \sqrt{n} \cdot |c| \end{aligned}$$

So we have found that $\|\mathbf{v}\|$ is $\sqrt{n} \cdot |c|$.