Christopher David Miller
FSMAT 201
Problem 3.1
Version 1

3.1

Let $z = a + bi$ and $w = c + di$. Prove that $\overline{zw} = \overline{z} \cdot \overline{w}$.

Now we have enough definitions to finish our proof.

Proof. To prove this statement we need to introduce the definition of the conjugate of a complex number. Let q be a complex number, in other words $q = c + di$ where c and d are real numbers and where $i = \sqrt{-1}$. The conjugate of q, denoted by $\overline{q}$, is defined by

$$\overline{q} = c - di.$$

Let $z = a + bi$ and $w = c + di$. In order to show that $\overline{zw} = \overline{z} \cdot \overline{w}$. We need to calculate $\overline{zw}$ and $\overline{z} \cdot \overline{w}$.

$$\begin{aligned} \overline{zw} &= \overline{(a + bi)(c + di)} \\ &= \overline{ac + adi + bci - bd} \\ &= \overline{(ac - bd) + i(ad + bc)} \\ &= (ac - bd) - i(ad + bc) \end{aligned}$$

$$\begin{aligned} \overline{z} \cdot \overline{w} &= \overline{(a + bi)} \cdot \overline{(c + di)} \\ &= (a - bi) \cdot (c - di) \\ &= ac - adi - bci - bd \\ &= (ac - bd) - i(ad + bc) \end{aligned}$$

We have shown that $\overline{zw} = \overline{z} \cdot \overline{w}$

$\square$

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Problem 3.2
Version 1

3.2
Let $z = a + bi$ and $w = c + di$. Prove that $|zw| = |z| \cdot |w|$.

Proof. To prove this statement we need to introduce the definition of the modulus of a complex number. Let q be a complex number, in other words $q = c + di$ where c and d are real numbers. The modulus of q, denoted by $|q|$, is defined by

$$|q| = \sqrt{c^2 + d^2}$$

Let $z = a + bi$ and $w = c + di$. In order to show that $|zw| = |z| \cdot |w|$. We need to calculate $|zw|$ and $|z| \cdot |w|$.

$$\begin{aligned} |z| \cdot |w| &= \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2} \end{aligned}$$

$$\begin{aligned} |zw| &= \overline{(a + bi)} \cdot \overline{(c + di)} \\ &= (a - bi) \cdot (c - di) \\ &= ac - adi - bci - bd \\ &= (ac - bd) - i(ad + bc) \end{aligned}$$

We have shown that $\overline{zw} = \overline{z} \cdot \overline{w}$

$\square$

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Problem 3.3
Version 1

3.3
Let $z = a + bi$. Find the real and imaginary parts of $\frac{1}{z}$.

Solution: In order to find the real and imaginary parts of a complex number we must express $\frac{1}{z}$ of the form $c + di$ where c and d are real numbers.First we have

$$\frac{1}{z} = \frac{1}{a+bi},$$

then we multiply by 1 where $1 = \frac{a-bi}{a-bi}$,

$$\begin{aligned} &= \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} \\ &= \frac{a-bi}{a^2+b^2} \\ &= \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2} \\ &= \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} \end{aligned}$$

. Now our complex number is of the proper form. So our real part is $\frac{a}{a^2+b^2}$ and the imaginary part is $-\frac{b}{a^2+b^2}$.

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Problem 3.4
Version 1

3.4

Let $z = a + bi$. Prove that z is a real number if and only if $z = \overline{z}$

Proof. To prove this statement we need to introduce the definition of the conjugate of a complex number. Let q be a complex number, in other words $q = c + di$ where c,the real part, and d, the imaginary part, are real numbers and where $i = \sqrt{-1}$. The conjugate of q, denoted by $\overline{q}$, is defined by

$$\overline{q} = c - di.$$

Let $z = a + bi$. Assume z is a real number.Thus z does not have an imaginary part or in other words $z = a + 0i$. Notice from the definition of conjugate that $\overline{z} = a - 0i$. Thus $z = \overline{z}$. Now assume that $z = \overline{z}$. We will show that z is a real number. For z to be a real number then the imaginary part of z must be 0.Notice since $z = \overline{z}$. then $a + bi = a - bi$ the only real number b that this is true for is 0. Thus z is a real number. $\square$

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FSMAT 201
Problem 3.5
Version 1

3.5
Prove DeMoivre's Theorem, namely, that for a nonnegative integer n,

$$(\cos\omega + i\sin\omega)^n = \cos n\omega + i\sin n\omega$$

Proof. To prove this statement we need to introduce the definition of the exponetial function. Let q be a complex number, in other words $q = c + di$ where c,the real part, and d, the imaginary part, are real numbers and where $i = \sqrt{-1}$, then we define the exponetial function e^q by

$$e^q = e^c(\cos d + i\sin d).$$

Notice if the real part of $q = 0$ then $e^q = (\cos d + i\sin d)$

Assume $(\cos\omega + i\sin\omega)^n$, then following from above we have

$$(\cos\omega + i\sin\omega)^n = (e^{i\omega})^n$$

Following from laws of exponets we have

$$(e^{i\omega})^n = (e^{i\omega n}).$$

Then using our defintion from earlier we have,

$$(e^{i\omega n}) = \cos n\omega + i\sin n\omega.$$

The proof for the other way is similar. $\square$