

1.8

Proposition 1. *Let $\mathbf{u} \in \mathbb{R}^n$. Prove that the normalization of \mathbf{u} has norm 1.*

Proof. Let $\mathbf{u} \in \mathbb{R}^n$. The normalization of a vector \mathbf{u} is the vector \mathbf{v} given by

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

The norm of a vector \mathbf{u} is the scalar $\|\mathbf{u}\|$ given by

$$\|\mathbf{u}\| = \sqrt{\sum_{k=1}^n u_k^2}, \text{ where } \mathbf{u} = [u_1 \ u_2 \ u_3 \ \dots \ u_n]^T.$$

In order to show the normalization of a vector \mathbf{u} has norm 1, we find the normalization of \mathbf{u} :

$$\frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

Now we must calculate the norm of the normalization of \mathbf{u} :

$$\begin{aligned} \|\text{Normalization of } \mathbf{u}\| &= \sqrt{\sum_{k=1}^n \left(\frac{u_k}{\|\mathbf{u}\|} \right)^2}, \text{ where normalization of } \mathbf{u} = \left[\frac{u_1}{\|\mathbf{u}\|} \ \frac{u_2}{\|\mathbf{u}\|} \ \frac{u_3}{\|\mathbf{u}\|} \ \dots \ \frac{u_n}{\|\mathbf{u}\|} \right]^T. \\ &= \sqrt{\sum_{k=1}^n \frac{u_k^2}{\|\mathbf{u}\|^2}} \\ &= \sqrt{\frac{1}{\|\mathbf{u}\|^2} \sum_{k=1}^n u_k^2} \\ &= \frac{1}{\|\mathbf{u}\|} \sqrt{\sum_{k=1}^n u_k^2} \\ &= \frac{1}{\|\mathbf{u}\|} \|\mathbf{u}\| \\ &= 1. \end{aligned}$$

So from the above steps we see that the normalization of a vector has norm 1. □

1.1

Let $\mathbf{v} = [1, 2, 3, \dots, n]^T$ and $\mathbf{w} = [1, 1, 1, \dots, 1]^T$, where $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

- (a) Compute $\mathbf{v} \cdot \mathbf{w}$.

Solution : The dot product between two vectors v and w is given by:

$$\sum_{i=1}^n v_i w_i.$$

So for this problem we have the following:

$$\mathbf{v} \cdot \mathbf{w} = 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + \dots + 1 \cdot n.$$

This can be written more succinctly in summation notation as

$$\sum_{i=1}^n i.$$

- (b) Compute $\|\mathbf{v}\|$.

Solution : We first remind ourselves that the definition of the norm of a vector \mathbf{u} is

$$\|\mathbf{v}\| = \sqrt{\sum_{k=1}^n v_k^2}.$$

Using this definition and substituting we get

$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + 3^2 + \dots + n^2}.$$

- (c) Compute $\|\mathbf{w}\|$.

Solution : Similarly using the definition from part (b) we can calculate the norm of the vector \mathbf{w} :

$$\begin{aligned}\|\mathbf{w}\| &= \sqrt{1^2 + 1^2 + 1^2 + \dots + 1^2} \\ &= \sqrt{1 + 1 + 1 + \dots + 1} \\ &= \sqrt{n}.\end{aligned}$$

1.3

Let c be a real number and let $\mathbf{v} \in \mathbb{R}^n$ with $v_k = c$ for $k = 1, 2, \dots, n$. Find $\|\mathbf{v}\|$.

Solution : The norm of a vector \mathbf{v} is the scalar $\|\mathbf{v}\|$ given by

$$\|\mathbf{v}\| = \sqrt{\sum_{k=1}^n v_k^2}, \text{ where } \mathbf{v} = [v_1 \ v_2 \ v_3 \ \dots \ v_k]^T.$$

So in our case we replace v_k with c then we have:

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{c^2 + c^2 + c^2 + \dots + c^2} \\ &= \sqrt{nc^2} \\ &= \sqrt{n} \cdot |c| \end{aligned}$$

So we have found that $\|\mathbf{v}\|$ is $\sqrt{n} \cdot |c|$.