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Apollonian Tiling, the Lorentz Group and Regular Trees

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Abstract

The Apollonian tiling of the plane into circles is analyzed with respect to its group properties. The relevant group, which is non-compact and discrete, is found to be identical to the symmetry group of a particular geometric tree-graph in hyperbolic three-space. A linear recursive method to compute the radii is obtained. Certain modifications of the problem are investigated, and relations to other problems, such as the universal scaling of circle-maps, are pointed out.

1 Introduction

There have appeared a number of papers, see e.g. [1, 2, 3, 4], dealing with the fractal properties of various coverings of the plane by an infinite set of mutually tangent circles, such as the *Apollonian tiling* and *space filling bearings*. These can be seen as toy models of the fractal structures appearing in various systems, e.g. in foam or in turbulent flow, and they have a high degree of symmetry. This will be explored in this paper, where the Apollonian tiling and its complement $\mathcal{S}$ will be discussed from a group-theoretical point of view.

The fractal set $\mathcal{S}$ is defined in Section 2, and its basic properties are discussed in terms of mappings in the plane. In Section 3, a generic vector representation of spheres is defined, in terms of which basis sets of tangent spheres are discussed. The recursive structure of the Apollonian tiling is analyzed in Section 4, and based on this, a very fast linear recursive algorithm for computation of the radii of the circles is derived, and formulated in terms of matrix multiplication. The 4×4 matrices involved are shown in Section 5 to be equivalent to SO(3,1) matrices. A brief discussion of the curved versions of Apollonian tiling is also discussed there. A dual set of circles, orthogonal to the ones making up the Apollonian tiling, are defined in Section 6. In Section 7 some special scaling limits are analyzed, based on the previously defined formalism. A fully symmetrized version of $\mathcal{S}$ is defined in Section 8. The Apollonian tiling is related to a certain symmetric tree structure. A generic discussion of such structures in various dimensions is given in Section 9, and the symmetry group $\mathcal{G}$ of the Apollonian tiling, a certain non-compact discrete subgroup of the Lorentz group SO(3,1) = SL(2,$\mathbf{C}$)/$\mathbf{Z}_2$, is identified. Finally, various generalizations of the Apollonian tiling and their fractal properties are discussed in Section 10, where also certain relations to known problems (such as the Farey decomposition of the interval in one dimension) are pointed out, and the author's conclusions are presented.

2 The Apollonian set $\mathcal{S}$

The Apollonian tiling (from Apollonius of Perga, 200 BC) is a particular covering of the plane by an infinite set of open circular disks, where recursively new disks are inscribed in the enclosed space between triples of already defined, mutually tangent disks.

The complementary set $\mathcal{S}$ is a self-similar fractal set in the plane, defined e.g. by the following recursive process:

1. Pick three mutually tangent circles (the generators) A, B and C in the plane. They enclose an area in the form of a curvilinear triangle. Its closure is the lowest order approximation $\mathcal{S}^{(0)} = \mathcal{S}_{[]}$ to $\mathcal{S}$.

2. There is a unique circle $o_{[]}$ that can be inscribed in $\mathcal{S}^{(0)}$ such, that it touches the

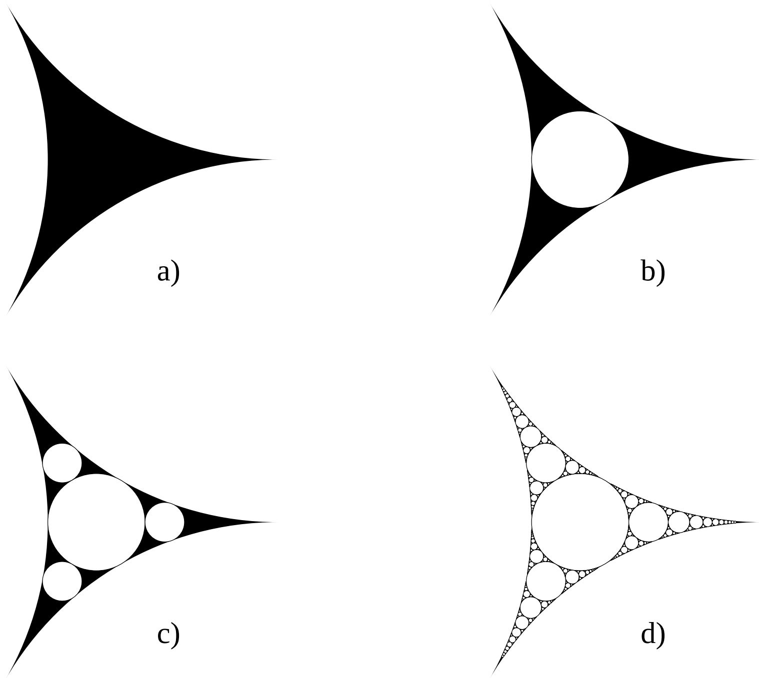

Figure 1: Some low order approximations to the Apollonian set $\mathcal{S}$: a) $\mathcal{S}^{(0)}$, b) $\mathcal{S}^{(1)}$, c) $\mathcal{S}^{(2)}$, and d) $\mathcal{S}$ itself.

three circular parts of the boundary. Subtracting the corresponding open disk from $\mathcal{S}_{[]}$, we are left with three (almost) disjoint parts, to be labelled $\mathcal{S}_{[a]}$ (tangent to B and C), $\mathcal{S}_{[b]}$ and $\mathcal{S}_{[c]}$. Each of them is a conformal "copy" of $\mathcal{S}^{(0)}$. Their union defines the first order set $\mathcal{S}^{(1)} = \mathcal{S}_{[a]} \cup \mathcal{S}_{[b]} \cup \mathcal{S}_{[c]} \subset \mathcal{S}^{(0)}$.

3. Repeat the procedure for each of the three subsets of $\mathcal{S}^{(1)}$, removing from $\mathcal{S}_{[a]}$ the open disk inside the inscribed circle $o_{[a]}$ to define $\mathcal{S}_{[aa]}$, $\mathcal{S}_{[ab]}$ and $\mathcal{S}_{[ac]}$, etc. In this way $\mathcal{S}^{(2)} \subset \mathcal{S}^{(1)}$ is defined, consisting of nine almost disjoint subsets.
Treating $\mathcal{S}^{(2)}$ the same way gives $\mathcal{S}^{(3)} \subset \mathcal{S}^{(2)}$, etc.

The N'th order set $\mathcal{S}^{(N)}$ will consist of 3^N almost disjoint subsets, labeled by *words* of the type $[abc\ldots]$ of length N constructed from the three-letter alphabet $\{abc\}$.

In the limit as $N \to \infty$ we obtain the Cantor set $\mathcal{S}$, cf. Fig. 1.

SL(2,C) approach

If the plane is regarded as the complex plane, there are three fractional linear mappings (FLM's) F_a, F_b, F_c, that generate the set $\mathcal{S}$ from $\mathcal{S}_{[]}$.

An FLM is a mapping of the type

$$z \to \frac{az+b}{cz+d},\ a, b, c, d\,\text{complex},\ ad - bc = 1 \tag{1}$$

FLM's always map circles onto circles (if straight lines are regarded as circles). They are conveniently represented in an obvious way by complex two-by-two matrices with determinant one, defining the group SL(2,**C**). Functional composition then corresponds to the group operation, i.e. matrix multiplication.

Now, F_a is defined by demanding that it map A, B and C onto $o_{[]}$, B and C, respectively. It obviously also maps $\mathcal{S}_{[]}$ onto $\mathcal{S}_{[a]}$, and $\mathcal{S}_{[b]}$ onto $\mathcal{S}_{[ab]}$, etc. F_b and F_c are defined in a similar way. In effect, F_a (F_b, F_c) adds an 'a' ('b', 'c') to the *head* of the word.

For the set $\mathcal{S}$, we now have the self-similarity identity

$$\mathcal{S} = F_a(\mathcal{S}) \cup F_b(\mathcal{S}) \cup F_c(\mathcal{S}), \tag{2}$$

which can be used to generate $\mathcal{S}$ recursively, starting from $\mathcal{S}_{[]}$.

The effect of choosing a different set of generating circles A, B, C merely corresponds to an FLM transformation of $\mathcal{S}$; the corresponding F_a, F_b and F_c will undergo a similarity transform. The fractal properties of $\mathcal{S}$ are of course invariant under such transformations, as long as the image remains compact.

3 Spheres as vectors

A sphere in $\mathbf{R}^D$ is defined by its center $\vec{a}$ and its radius R. It is the union of all solutions $\vec{x}$ to the equation

$$\vec{x}^2 - 2\vec{a}\cdot\vec{x} + (\vec{a}^2 - R^2) = 0. \tag{3}$$

Out of linear combinations of the $D+2$ coefficients of the terms in the left member, divided by R, we can define a vector in $\mathbf{R}^{D+2}$ as follows:

$$V = (V_0, \ldots, V_{D+1}) = (\frac{1+a^2-R^2}{2R}, \frac{1-a^2+R^2}{2R}, \frac{a_1}{R}, \ldots, \frac{a_D}{R}). \tag{4}$$

This is a space-like, normalized vector in the Minkowski space $\mathbf{M}_{D+2}$:

$$V \cdot V \equiv V_0^2 - V_1^2 \ldots - V_{D+1}^2 = -1. \tag{5}$$

The vector V constitutes a particularly useful representation of oriented spheres. A conformal transformation of the sphere is represented by a SO(1,D+1) transformation of V. SO(1,D+1) is isomorphic to the conformal group in $\mathbf{R}^D$.

The curvature $q = 1/R$ (defined positive for orientation outwards) is obtained from V as the scalar product with the light-like vector $\lambda = (1, -1, 0, \ldots, 0)$. The cutting angle α between two different spheres, i.e. the angle between the normals at the contact points, is obtained from the scalar product between their respective vectors V_1 and V_2:

$$V_1 \cdot V_2 = -\cos\alpha. \tag{6}$$

In particular, $\cos\alpha = -1$ corresponds to oppositely oriented tangent spheres, whereas if $|V_1 \cdot V_2| > 1$ the spheres do not touch.

Tangent spheres

For a $(D+2)$-tuple of mutually tangent spheres, represented by $V_i, i = 1, \ldots D+2$, we have

$$V_i \cdot V_j = 1 - 2\delta_{ij} \equiv g_{ij}, \tag{7}$$

defining the "metric" g. They make up a complete basis in $\mathbf{M}_{D+2}$. The completeness is expressed by

$$\sum g_{ij}^{-1}(V_i \cdot x)(V_j \cdot y) = x \cdot y,\ x, y \in \mathbf{M}_{D+2}, \tag{8}$$

with g^{-1} the matrix inverse of the metric g:

$$g_{ij}^{-1} = (1 - D\delta_{ij})/2D. \tag{9}$$

In particular, if we set $x = y = \lambda$ in eq. (8), we obtain a quadratic identity for the curvatures q_i:

$$(\sum_i q_i)^2 = D \sum_i q_i^2. \tag{10}$$

The dual set

Of interest is also the dual basis $\{U_i\}$,

$$U_i = \sum_j g_{ij}^{-1} V_j \;\Rightarrow\; U_i \cdot V_j = \delta_{ij}. \tag{11}$$

Properly normalized, the U_i represent a set of spheres, each orthogonal to all but one of the V_i. Their mutual cutting angle β is given by $\cos\beta = -\frac{1}{D-1}$. Thus, only for $D = 2$ do we have the self-dual case that also the U_i are mutually tangent.

Suppose the V_i cut each other at an arbitrary angle α, instead of being tangent. Then,

$$\cos\beta = -\frac{\cos\alpha}{D\cos\alpha + 1}. \tag{12}$$

If $\cos\alpha = -\frac{2}{D}$, self-duality is reinforced: $\cos\beta = \cos\alpha$. Conversely, if $\cos\alpha = -\frac{1}{D-1}$, then $\cos\beta = -1$, and the dual spheres are tangent.

4 Curvature and recurrence

When it comes to analysing the fractal properties of the Apollonian set $\mathcal{S}$, a size measure is needed on the subsets $\mathcal{S}_{[w]}$ (w word). The obvious choice is the radius $r_{[w]}$ of the inscribed circle $o_{[w]}$.

The V vectors, as defined in the previous section for $D = 2$, can be used to represent the inscribed circles. The FLM's that generate $\mathcal{S}$ recursively act as SO(1,3) transformations on V. These can in principle be used to compute the radii recursively, but in a way that depends on the initial choice of generating circles A, B and C. An equivalent, but manifestly universal, method will be derived below.

Parents and daughters

The circle $o_{[]}$ touches A, B and C. Similarly, every circle $o_{[w]}$ touches three mutually tangent lower order circles, defining its *parents.*

The three parent words are given by truncating the tail of the daughter word just before the last 'a', 'b' or 'c', thus defining the a-, b- or c-parent, respectively. If no 'a' is found, the a-parent is A, etc. Thus, for $o_{[acc]}$, e.g., the a-, b- and c-parents are $o_{[]}$, B and $o_{[ac]}$.

The parent closest in order to the daughter is called the mother. Its label is obviously obtained by removing the last letter in the word, and it is thus of order one less than the daughter. The remaining two parents are inherited from the mother.

Conversely, every circle is the mother of three daughters, obtained by adding an 'a', 'b' or 'c' to the tail of the mother word.

Recurrence

In the Apollonian tiling problem, we have $D = 2$ and $\cos\alpha = -1$, and the identity (10) becomes the well-known relation between the curvatures $q_i = 1/r_i$ of four mutually tangent circles in the plane:

$$(q_1 + q_2 + q_3 + q_4)^2 = 2(q_1^2 + q_2^2 + q_3^2 + q_4^2) \tag{13}$$

Thus, the curvature q of a circle is given in terms of the ones of its parents, q_A, q_B, q_C, by Soddy's formula [1]:

$$q = q_A + q_B + q_C + 2\sqrt{q_A q_B + q_A q_C + q_B q_C}. \tag{14}$$

This constitutes a (four term) non-linear recurrence relation for the radii, and was used in Ref. [2] to compute the fractal dimension of $\mathcal{S}$, $d_F = 1.305684 \pm 0.000010$.

The recurrence relations can be even more simplified, however.

Linear recurrence

As previously stated, the two remaining parents are inherited from mother to daughter. Given a circle and its three parents, denote their curvatures q, q_A, q_B and q_C, and denote by q_a the curvature of the a-daughter. We then have

$$q_a = q + q_B + q_C + 2\sqrt{q q_B + q q_C + q_B q_C}. \tag{15}$$

Replacing q_a with q_A will not change the quadratic relation leading to this equation, so the only change must be in the sign in front of the squareroot:

$$q_A = q + q_B + q_C - 2\sqrt{q q_B + q q_C + q_B q_C}. \tag{16}$$

This gives a five term **linear** recurrence relation,

$$q_a = 2(q + q_B + q_C) - q_A, \tag{17}$$

that provides an even faster recursive method to compute the radii of the Apollonian tiling. Analogous linear relations exist for other problems of this type, also in higher dimensions.

Matrix formulation

Any multi-term scalar recurrence relation such as eq. (17) can be replaced by a two term vector relation. Define the vector Q for a circle to have the components (q, q_A, q_B, q_C). Thus, e.g.

$$Q_{[abcb]} = (q_{[abcb]}, q_{[\,]}, q_{[abc]}, q_{[ab]}). \tag{18}$$

Then, the corresponding vector for the a-, b- or c-daughter, is obtained simply by multiplying Q by one of the integer matrices

$$A^\top = \begin{pmatrix} 2 & -1 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; B^\top = \begin{pmatrix} 2 & 2 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; C^\top = \begin{pmatrix} 2 & 2 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

These matrices are of a degenerate type, in that the spectral equation has the single solution 1. They all satisfy the equation

$$(M-1)^3 = 0, \; M = A, B, C. \tag{20}$$

We now obtain Q for a given word by multiplying $Q_{[\,]}$ by the product of matrices defined by the letters of the word, but in opposite order. Thus, e.g.,

$$Q_{[abbc]} = C^\top B^\top B^\top A^\top Q_{[\,]} = T^\top_{[abbc]} Q_{[\,]}, \tag{21}$$

and the curvature q is simply the first component of Q, or, formally

$$q_{[abbc]} = X^\top_{[\,]} Q_{[abbc]}, \tag{22}$$

using the coefficient vector $X_{[\,]} = (1, 0, 0, 0)$.

Transposing to get things in the right order, we obtain

$$q_{[abbc]} = Q^\top_{[\,]} ABBCX_{[\,]} = Q^\top_{[ab]} X_{[bc]} = Q^\top_{[\,]} X_{[abbc]} = Q^\top_{[\,]} T_{[abbc]} X_{[\,]}. \tag{23}$$

with obvious notation.

The universality modulo the choice of generating circles A, B, C is obvious – the dependence sits entirely in $Q_{[\,]}$.

5 Minkowski version: SO(1,3)

Obviously, the quadratic identity satisfied by Q is preserved under the above transformations. The identity can be written as

$$Q^\top G Q = 0, \text{ with } G = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}. \tag{24}$$

The invariance of the metric G under the transformations can be expressed as

$$AGA^\top = BGB^\top = CGC^\top = G. \tag{25}$$

G is equivalent to the Minkowski metric $H = \mathrm{diag}(1,-1,-1,-1)$, by the orthogonal similarity transformation

$$G = U^\top H U, \text{ with } U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{26}$$

Thus, Q is related to a light-like Minkowski vector $\tilde{Q}$, and the daughter- generating transformations A, B and C are in fact disguised SO(1,3) (i.e. Lorentz) transformations. The Minkowski versions are also integer:

$$\tilde{A} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \tilde{B} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \tilde{C} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}. \tag{27}$$

They are the exponentials of degenerate SO(1,3) generators, e.g.

$$\tilde{A} = \exp\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \tag{28}$$

with 'electric' and 'magnetic' parts $\vec{K} = (0,1,1)$, $\vec{J} = (0,1,-1)$, leading to the vanishing of the two invariants $2\vec{K}\cdot\vec{J} = \vec{K}^2 - \vec{J}^2 = 0$. The expression for q is of course still of the same form:

$$q_{[abbc]} = \tilde{Q}_{[]}^\top \tilde{A}\tilde{B}\tilde{B}\tilde{C}\tilde{X}_{[]} = \tilde{Q}_{[]}^\top \tilde{T}_{[abbc]}\tilde{X}_{[]} = \tilde{Q}_{[]}^\top \tilde{X}_{[abbc]} = \tilde{Q}_{[abbc]}^\top \tilde{X}_{[]} \tag{29}$$

with $\tilde{Q}_{[]}$ light-like and $\tilde{X}_{[]}$ given by the space-like vector

$$\tilde{X}_{[]} = \frac{1}{2}(1, 1, 1, 1). \tag{30}$$

Obviously, $\tilde{Q_{[]}}$ and $\tilde{X}_{[w]}$ are related to the Lorentz vectors λ and V, introduced in section 3, by a transformation carrying the dependence on the initial choice of generating circles A, B, C.

Apollonian tiling on the sphere

If the set $\mathcal{S}$ is conformally mapped onto the unit sphere, the radii will be modified, but the basic geometrical properties will remain unchanged. The relation (13) between the curvatures, leading to the linear recurrence (17), will undergo a simple modification.

With $r_i,\ i = 1, \ldots, 4$ denoting the radii (arc-lengths) of four mutually tangent circles on the sphere, the curvatures are given by $q_i = \cot(r_i)$. The relation between the curvatures then reads

$$(q_1 + q_2 + q_3 + q_4)^2 = 2(q_1^2 + q_2^2 + q_3^2 + q_4^2) + 4. \tag{31}$$

as can be shown by considering five spheres in three-dimensional Euclidean space. Thus, the curvature q of a circle is given in terms of the ones, q_A, q_B, q_C, of its parents as

$$q = q_A + q_B + q_C + 2\sqrt{q_A q_B + q_A q_C + q_B q_C - 1}. \tag{32}$$

But this does not change the linear recurrence relations, thus e.g. for the a-daughter, we still have

$$q_a = 2(q + q_B + q_C) - q_A. \tag{33}$$

Hence, we can use the same disguised SO(1,3) matrices T_a, T_b and T_c (and the same $X_{[]}$).

The only difference is that Q corresponds to a time-like rather than a light-like Minkowski vector, due to the inhomogeneity in the relation (31). Similarly, on a manifold of negative curvature, the Q-vectors will be space-like. Hence, the formalism is universal also with respect to the curvature of the underlying space – all dependence on the particular representation of $\mathcal{S}$ resides in $Q_{[]}$ alone.

6 Dual circles

All geometric information about a certain subset $\mathcal{S}_{[w]}$ can be extracted from the corresponding Q vector. In the above we have concentrated on the curvature q of the inscribed circle, but that gives only partial information on the local geometry.

Another measure with a simple interpretation is the curvature p of the **dual** circle $O_{[w]}$, which circumscribes the subset and cuts the parents of the inscribed circle at right angles at their three points of contact, i.e. at the 'corners' of the subset. It is simply given by

$$p = \sqrt{q_A q_B + q_B q_C + q_B q_C} = \frac{1}{2}(q - q_A - q_B - q_C). \tag{34}$$

It is obtained simply by replacing the coefficient vector $X_{[]}$ by

$$Y_{[]} = \frac{1}{2}(1, -1, -1, -1) = -GX_{[]}. \tag{35}$$

The Minkowski version is

$$\tilde{Y}_{[]} = \frac{1}{2}(-1, 1, 1, 1), \tag{36}$$

which is a space-like vector, just like $\tilde{X}_{[]}$, from which it can be obtained by a 'time' inversion.

Obviously, the dual curvatures p have their own linear recurrence relations, obtained by inserting $GG = 1$ between all the factors in the expression (34) for p. We obtain

$$p \;=\; Y_{[]}^\top T^\top Q_{[]} \;=\; Y_{[]}^\top G\, GT^\top G\, GQ_{[]} \;=\; X_{[]}^\top T^{-1}\, P_{[]} \tag{37}$$

where in the last expression we have introduced the P-vector

$$P = -GQ = (p, -p_a, -p_b, -p_c), \tag{38}$$

with p_a referring to p for the a-daughter, etc., and where we also have used the identity

$$GT^\top G = T^{-1}. \tag{39}$$

Thus, the formalism for P is the same as for Q, provided the transformation matrix $T^\top$ is replaced by T^{-1}.

Note that the six points of contact among a dual circle $O_{[w]}$ and its daughters are identical to the ones among the circle $o_{[w]}$ and its parents, and that the former cut the latter at

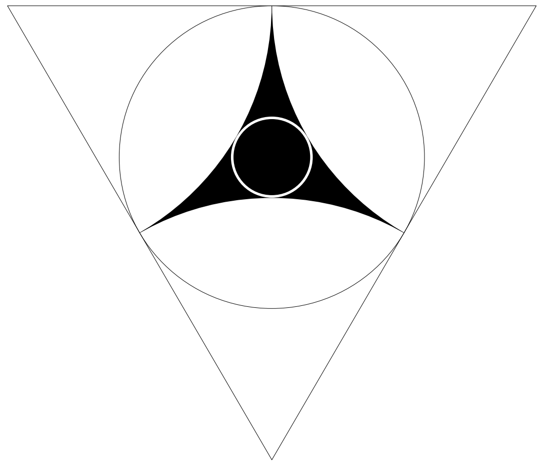

Figure 2: The inscribed circle, the subset, the dual circle and the triangle corresponding to one and the same word.

these points at right angles. The curvatures of the circumscribed circles $O_{[w]}$ transform with the inverse of the transposed transfer matrix, and they make a four-tuple of tangent circles with the daughters rather than with the parents, which further underlines their dual relationship to the inscribed ones.

The dual circle *circumscribing* a subset $\mathcal{S}_{[w]}$ is in turn *inscribed* in a triangle $\Delta_{[w]}$ defined by joining the centers of the inscribed circles of the parents by straight lines. We thus have (int = interior)

$$\mathrm{int}(o_{[w]}) \subset \mathcal{S}_{[w]} \subset \mathrm{int}(O_{[w]}) \subset \Delta_{[w]}, \tag{40}$$

cf. Fig. 2. The actual area of such a triangle is $p/q_A q_B q_C$. The set of all such triangles at a given level (= word length) constitute a partitioning of the initial triangle $\Delta_{[]}$, independently of the level.

7 Some special scaling limits

We can now easily compute the scaling properties at different corners of $\mathcal{S}$. We consider some simple periodic words.

1. $[a^N]$:
   The dynamics is governed by the limiting behaviour of high powers of $T_{[a]}$. Using

$(T-1)^3 = 0$, we easily obtain

$$T_{[a]}^N = \begin{pmatrix} 1+N & N & 0 & 0 \\ -N & 1-N & 0 & 0 \\ N(N+1) & N(N-1) & 1 & 0 \\ N(N+1) & N(N-1) & 0 & 1 \end{pmatrix}, \tag{41}$$

with a leading N^2 behaviour. Acting on $X_{[]}$ or $Y_{[]}$, it gives

$$X_{[aaa...]} = (1+N, -N, N(N+1), N(N+1)), \tag{42}$$

$$Y_{[aaa...]} = \frac{1}{2}(1, -1, 2N-1, 2N-1), \tag{43}$$

with different power behaviours, since $Y_{[]}$ kills the leading terms. For q and p, we get an N^2 and N behaviour, respectively. The subsets in this limit become very long and narrow, with width $\propto 1/N^2$ and length $\propto 1/N$.

2. $[(ab)^{\frac{N}{2}}]$:
   We get the leading behaviour by finding the solutions to the characteristic equation for $T_{[ab]}$, which are in this case also the eigenvalues. These are $1, 1, \gamma^4, 1/\gamma^4$, where γ is the golden mean (≈ 1.618). The dominant eigenvalue is γ^4, and r and R will both go like γ^{-2N} ($\approx 1/2.618^N$).

3. $[(abc)^{\frac{N}{3}}]$:
   Doing the same thing for $T_{[abc]}$, we get the eigenvalues $(\gamma \pm \sqrt{\gamma})^3, (-1/\gamma \pm i\sqrt{1/\gamma})^3$. The dominant eigenvalue is here $(\gamma + \sqrt{\gamma})^3$, and r and R go like $(\gamma - \sqrt{\gamma})^N$ ($\approx 1/2.890^N$). This is the case with the fastest decrease of size, and it is the analogy for this system of the golden mean for the rationals.

For longer periodic words, the scaling properties can be computed in a similar way – by making a spectral analysis of corresponding transfer matrix.

8 Extension of $\mathcal{S}$

The Apollonian set $\mathcal{S}$ is not in itself invariant under the FLM's F_a, F_b and F_c, but it can easily be extended to a larger set $\mathcal{S}^*$, that is. It is defined by adding to $\mathcal{S}$ three copies of it: $\mathcal{S}_A$, $\mathcal{S}_B$ and $\mathcal{S}_C$, where

$$\mathcal{S}_A = F_b^{-1}\mathcal{S}_{[a]} = F_c^{-1}\mathcal{S}_{[a]}, \tag{44}$$

etc.

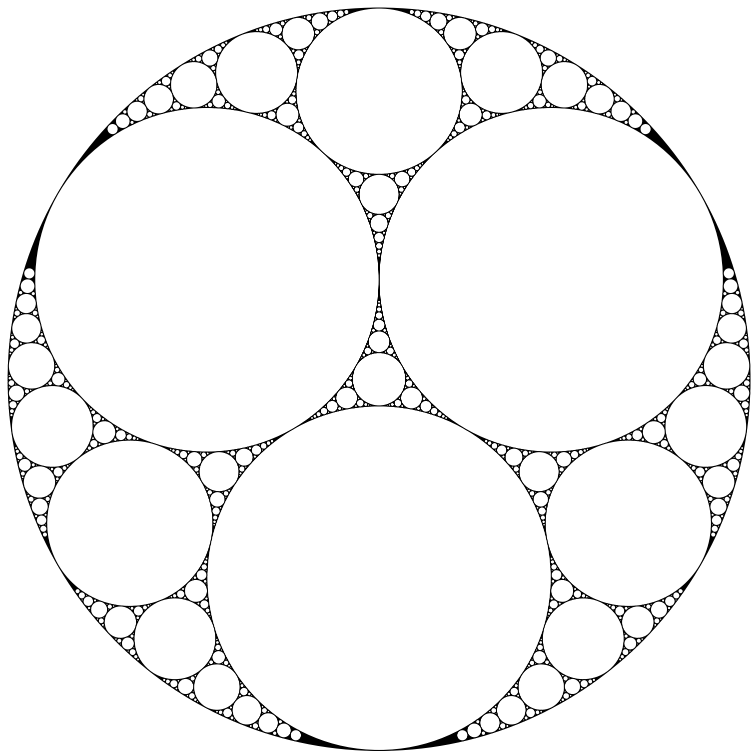

Figure 3: The extended set $\mathcal{S}^*$.

The extended set $\mathcal{S}^*$ does not contain any unmatched "corners". To the lowest order it has the topology of a tetrahedron, cf. Fig. 3. It is invariant under a discrete subgroup $\mathcal{G}$ of $\mathrm{SL}(2,\mathbf{C})/\mathbf{Z}_2$ containing the elements corresponding to the conformal mappings F_a, F_b and F_c. The same group, represented by SO(1,3) transformations, leaves the corresponding extended set of Q-vectors invariant.

Due to the non-compact nature of $\mathcal{G}$, no finite measure can be defined on $\mathcal{S}^*$, that is invariant under $\mathcal{G}$.

This group is also the symmetry group of a certain geometric structure, as will be shown in the next section.

9 Regular structures

Two dimensions

An icosahedron, e.g., can be viewed as a regular graph-like structure in a two-dimensional manifold of constant positive curvature ($\Leftrightarrow$ the unit sphere). Such two-dimensional highly symmetric structures can be labelled by two integers $k, l > 2$, with k defining the type of area element, and l the coordination number around a node. The dual structure is obtained by interchanging k and l.

The curvature of the embedding space depends on the sign of the expression:

$$\mu = \sin\frac{\pi}{k} - \cos\frac{\pi}{l}. \tag{45}$$

This is zero precisely when $(k-2)(l-2) = 4$, defining the planar case. For the positive curvature case, $\mu > 0$, the only possibilities are

- $(3,3) \Rightarrow$ tetrahedron,
- $(3,4) \Rightarrow$ octahedron,
- $(4,3) \Rightarrow$ cube,
- $(3,5) \Rightarrow$ icosahedron,
- $(5,3) \Rightarrow$ dodecahedron.

In the planar case, $\mu = 0$, we obtain

- $(3,6) \Rightarrow$ lattice of isosceles triangles,
- $(4,4) \Rightarrow$ square lattice,
- $(6,3) \Rightarrow$ honeycomb lattice.

For higher k and/or higher l, we have $\mu < 0$, and such structures must be embedded in a negatively curved space.

In particular, this is true for critical (barely self-avoiding) regular trees, which are defined by $k \to \infty$. Such a tree, when pictured in the Poincaré circle, asymptotically generate a dense set of points at the boundary $r = 1$, cf. Fig. 4 for the case of $l = 3$.

Three dimensions

The above type of structures can be generalized to three dimensions: there we need three integers (k, l, m), all > 2. Then (k, l) defines the body type of the building block, and (l, m) the coordination of neighbour nodes. Thus, e.g., $(4, 3, 5)$ defines a structure with cubes as building blocks, and an icosahedrical arrangement of neigbour nodes. The dual structure is defined by (m, l, k).

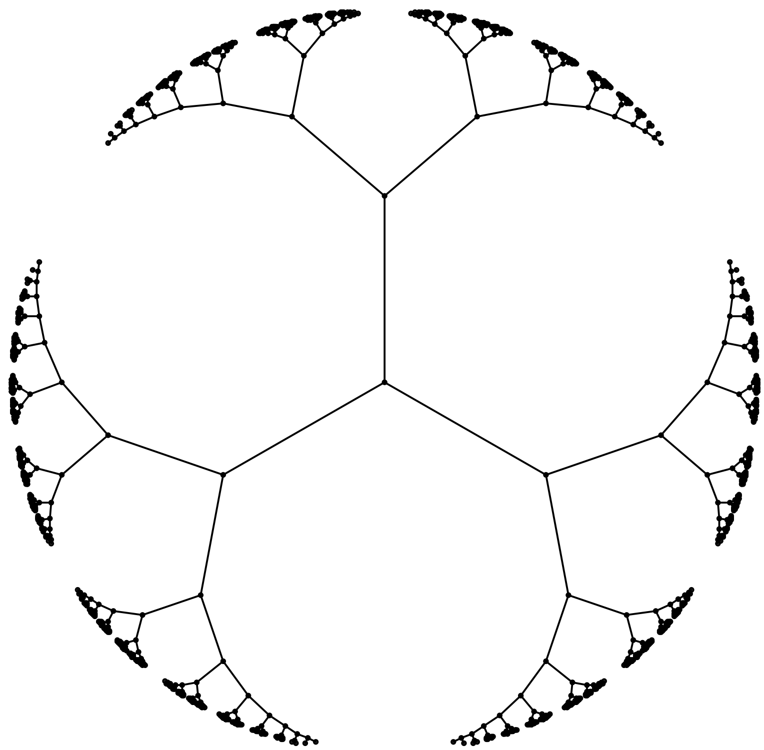

Figure 4: The Farey tree $(\infty, 3)$ in the Poincaré circle. It does a Farey organization of the boundary $r = 1$.

Also here, the curvature of the embedding space is given by the sign of a simple expression:

$$\nu = \sin\frac{\pi}{k}\sin\frac{\pi}{m} - \cos\frac{\pi}{l}. \tag{46}$$

Again, trees are defined by $k = \infty$, and must be embedded in a space with negative curvature.

The symmetry group $\mathcal{G}$ of the extended Apollonian set $\mathcal{S}^*$ is the symmetry group of such a regular graph-like structure in a three-dimensional manifold of constant negative curvature (such as the unit mass-shell in $\mathbf{M}_4$).

In this case, the "Apollonian tree", we have

$$(k, l, m) = (\infty, 3, 3), \tag{47}$$

and the graph is a tree with coordination number four. For every node of the tree, the neighbour nodes are tetrahedrically arranged. The dual structure consists of infinite hyperbolic tetrahedra.

The Apollonian group

The discrete symmetry group $\mathcal{G}$ of this structure (excluding reflections) is generated by three elements, Q, R and P, satisfying the following constraints:

$$Q^2 = R^3 = P^2 = (QR)^3 = (PR)^2 = 1, \tag{48}$$

where Q and R generate the tetrahedric little group of a node i, while P interchanges this node with the neighbour node j that is invariant under R.

In terms of these, A, B and C are given as

$$A = PQ \ , \ B = PQRQ \ , \ C = PRQR, \tag{49}$$

and they all map node i onto node j.

The group $\mathcal{G}$ has a covering group $\mathcal{G}'$ with a nontrivial center Z_2. The simplest representation of the corresponding elements in $\mathcal{G}'$ is given (mod a sign) by the 2×2 integer complex matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \ , \ B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ , \ C = \begin{pmatrix} 1+i & 1 \\ 1 & 1-i \end{pmatrix} \ . \tag{50}$$

The subgroup of $\mathcal{G}'$ generated by A and B alone is the modular group SL(2,$\mathbf{Z}$), related to the Farey organization of the rationals [5]. The corresponding factor group SL(2,$\mathbf{Z}$)/$\mathbf{Z}_2$ is the symmetry group of the subtree $(\infty, 3)$ of the Apollonian tree, displayed in Fig. 4.

The Apollonian tree can be generated by recursively applying the corresponding triple of SO(1,3) matrices, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$, as previously defined, to the starting point $(1, 0, 0, 0)$, and adding also the preimages (which can be obtained as simple rotations of the images).

The Poincaré sphere representation

In order to be able to display the tree in a compact manner, we need a compactified representation of the unit mass-shell.

The mass-shell, $\omega^2 - \vec{k}^2 = 1, \vec{k} \in \mathbf{R}^3$, $\omega > 0$, is a 3-dimensional manifold with a constant negative curvature. From the usual Minkowski metric we obtain the induced metric,

$$-ds^2 = -d\omega^2 + dk^2 + k^2 d\Omega^2 = dk^2/\omega^2 + k^2 d\Omega^2 \tag{51}$$

(with Ω representing the angles).

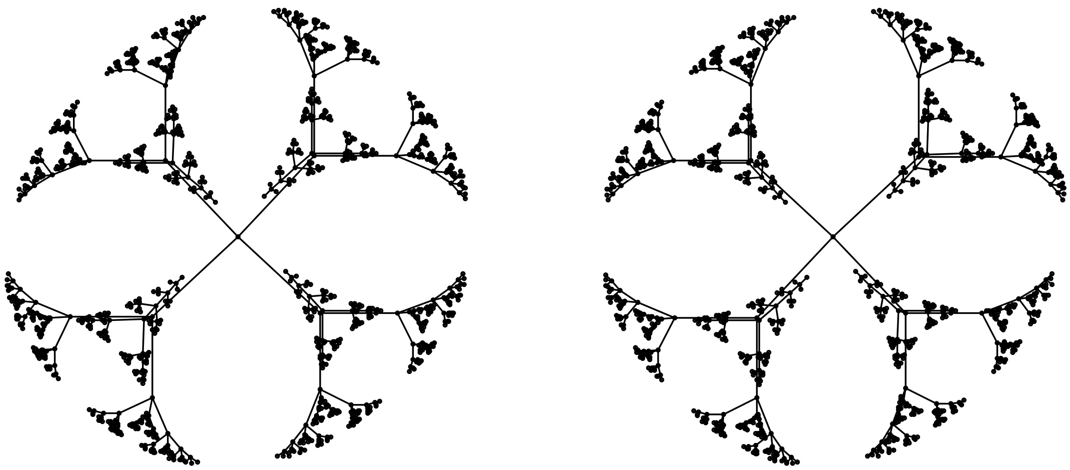

Figure 5: The Apollonian tree in the Poincaré sphere representation (stereo picture)

Now define $\vec{r} \in \mathbf{R}^3$, $r^2 < 1$, by:

$$\vec{r} = \frac{\vec{k}}{\omega + 1} \Rightarrow \vec{k} = \frac{2\vec{r}}{1 - r^2}, \; \omega = \frac{1 + r^2}{1 - r^2} \tag{52}$$

Then the metric in terms of $\vec{r}$ is conformal:

$$-ds^2 = \frac{4}{(1 - r^2)^2}(dr^2 + r^2 d\Omega^2) \tag{53}$$

and the curvature is -1.

This defines the Poincaré sphere representation, which is used to picture the Apollonian tree in Fig. 5. The result is striking: the asymptotic picture that emerges when $r \to 1$, is precisely a conformal mapping of the (extended) Apollonian set. The circular voids correspond to the non-compact building blocks of type $(\infty, 3)$, each of which occupy a circular part of the border $r = 1$.

There seems to be no way to define a finite scalar measure on such a tree, that transforms simply under the symmetry group, since the group (and the tree) is non-compact. This can only be done on a sub-tree, obtained by cutting the full tree at a node or, equivalently, at a link.

10 Discussion

The Apollonian packing has been shown to have many interesting properties, such as the possession of a high degree of symmetry, described by a discrete non-compact group $\mathcal{G}$, and the relation to a certain regular tree structure.

Generalizations

Alternatively, one may consider trees with cubical $(\infty, 4, 3)$, octahedrical $(\infty, 3, 4)$, etc., arrangement of neighbours, which are related to generalizations of the Apollonian tiling of the plane.

By varying the invariant link length of a tree, the cutting angle α of the circular voids is modified. This is equivalent to modification of the integer k describing the surface element in the corresponding regular structure.

Thus, with a real finite α, k formally will become imaginary, and a super-critical tree results: The branches, as seen in the Poincaré representation, diverge. The corresponding Apollonian set becomes completely disconnected, and the fractal dimension decreases.

For certain (imaginary) values of α, k will be integer: the result is a graph $(k, 3, 3)$ with k-fold loops. The kth power of the corresponding group elements A, B, C will become the identity element, and the symmetry group will be a factor group of $\mathcal{G}$. The corresponding Apollonian set is connected, and the fractal dimension larger.

For a small enough k, the graph becomes finite. The (extended) Apollonian set is replaced by a circular disk with a finite number of circular holes. The fractal dimension becomes two.

In $D = 2$, the case $(\infty, 3)$ is related to the Farey organization of rationals, as previously mentioned, cf. Fig 4. With an imaginary k, a topologically similar tree is obtained, but the branches diverge. At the boundary of the Poincaré circle a fractal occurs with a finite void for every rational, which is reminiscent of the Devil's staircase for the mode-locking in circle maps [5].

For $D = 4$, various tilings of $\mathbf{R}^3$ by spheres result, that can be used to model different kinds of space-filling packings by spheres, such as fractal foams or gravel. By modifying the cutting angle, also the packing of soft spheres can be described. Such packings are related to regular structures in curved four-space, described by three integers (k, l, m, n).

In all cases, and for every $D \geq 2$, there is a linear recurrence relation like eq. (17), by which all radii can be computed.

A different kind of generalization, space filling bearings, has been studied e.g. by Manna and Herrmann [2]. These are labeled by two integers, m, n, where each choice corresponds to a tree with two types of alternating nodes with local $\mathbf{Z}_{m+3}$ or $\mathbf{Z}_{n+3}$ symmetries. Also there, the recursive computation of radii can be linearized, in much the same way as in the case of Apollonian packing.

Combining this type of generalization with the kind described above, a very general class of circular packings can be defined.

Fractal properties

In a multi-fractal analysis [6] one can study e.g. the partition function

$$\mathcal{Z}_N(d) = \sum l^d, \tag{54}$$

where the sum runs over the subsets of $\mathcal{S}^{(N)}$, and l is some length measure of the subsets. The fractal dimension d_F is then given by the smallest d, such that $\mathcal{Z}_N(d)$ stays finite in the high level limit $N \to \infty$.

With l chosen as some linear combination (such as q or p) of the components of the vector Q associated with a subset, $\mathcal{Z}_N(d)$ can be computed analytically for every non-positive integer value of d, simply by finding the dominant eigenvalue of $A + B + C$ in the representations of $\mathcal{G}$, that are symmetric direct products of the vector representation, to which Q belongs. Unfortunately, that doesn't help much, since d_F is positive.

For numerical computations of d_F, the obvious choice for a length measure would be q or p. A toy-sized numerical computation of d_F was performed (on a DECstation 3100), using the ratio Z_{12}/Z_{11}. The result for d_F was 1.31415 using p as a length measure, and 1.29728 using q. The average of the two is 1.30571, quite close to the result 1.305684$\pm$.000010 of ref. [2].

Conclusion

Given their simple analytical and computational properties, the various generalizations of the Apollonian tiling might be useful as models of various fractal structures existing in nature.

The inherent symmetry properties in packings of this type, and the existence of simple vector representations with linear recurrence relations, gives – apart from mathematical beauty – good reason to hope for a better analytical understanding of their structure.

The (multi-) fractal properties of these packings have so far only been studied numerically. The fractal dimension d_F of a generalized Apollonian set is intimately related to the properties of the corresponding symmetry group $\mathcal{G}$, and one might hope that an understanding e.g. of the unitary representations of $\mathcal{G}$ could provide some means for an analytical determination of d_F.

Acknowledgement

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