Local Sine and Cosine Bases of Coifman and Meyer and the Construction of Smooth Wavelets

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Abstract. *We give a detailed account of the local cosine and sine bases of Coifman and Meyer. We describe some of their applications; in particular, based on an approach by Coifman and Meyer, we show how these local bases can be used to obtain arbitrarily smooth wavelets. The understanding of this material requires only a minimal knowledge of the Fourier transform and classical analysis. It is our intention to make this presentation accessible to all who are interested in Wavelets and their applications.*

§1. Introduction.

It is often useful to focus on local properties of a signal. In precise mathematical language this means that if we are given a function f on $\mathbb{R}$ and want to consider its properties on a finite interval I, we can analyze the function multiplied by χ_I, the characteristic function of I. We can, for example, form the Fourier series of $\chi_I f$ with respect to a complete orthonormal system for $L^2(I)$. An example of such a system is

$$\left\{\sqrt{\frac{2}{|I|}}\chi_I(x)\sin\frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha)\right\}, \tag{1.1}$$

where α is the left end point of I, $k = 0, 1, 2...$(further discussion of this and other systems will be given in §3). If $-\infty < \cdots < \alpha_j < \alpha_{j+1} < \cdots < \infty$, with $\alpha_j \to \pm\infty$ as $j \to \pm\infty$, and $I_j = [\alpha_j, \alpha_{j+1}]$, we obtain an orthonormal system

$$\varphi_{j,k}(x) = \sqrt{\frac{2}{|I_j|}}\chi_{I_j}(x)\sin\frac{2k+1}{2}\frac{\pi}{|I_j|}(x-\alpha_j), \tag{1.2}$$

where j ranges through the integers $\mathbf{Z}$ and k through the non-negative integers $\mathbf{Z}^+$, that is a basis for $L^2(\mathbb{R})$. Expansions in terms of such bases are referred to

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as "windowed" or "short time" Fourier transforms. Though such systems are appropriate for focusing on local properties (*i.e.* what happens on the interval I_j), the abrupt "cutoff" effected by the multiplication by the characteristic functions χ_{I_j} involve some undesirable artifacts. In [1], Coifman and Meyer introduced othonormal bases of this general type that involve an arbitrarily smooth cut off. It is the purpose of this note to present the construction of such bases and, in addition, show some of their uses. In particular, we shall describe, in §4, how the smooth wavelets of Lemarié and Meyer [2] can be obtained from these bases. We hasten to add that this application was also pointed out to us by Coifman and Meyer.

Let us begin by trying to construct a projection P_I, given an interval I, that is similar to the one obtained by multiplication by χ_I but is "smoother". Clearly, P_I cannot have the form $f \to \rho_I f$, with $\rho = \rho_I$ a smooth function with support "close" to I, since the requirement that P_I be idempotent forces ρ to have values that are either 0 or 1. Perhaps a smooth version of χ_I can be "corrected" near the end points of I so that we have a projection. In order to reduce the problem to only one end point, let us try to carry out this idea on the infinite ray $I = [0, \infty)$ and let ρ be a smooth non-negative function supported on $[-\epsilon, \infty)$ such that $\rho(x) = 1$ if $x \geq \epsilon$. Let us also assume that ρ shares with χ_I the property

$$\rho(x) + \rho(-x) = 1 \tag{1.3}$$

for all $x \in \mathbb{R}$. In order to perform the above "correction" of the operator $f \to \rho f$ so as to obtain an orthogonal projection we pose the question: can we find a function t such that

$$(P_I f)(x) = (Pf)(x) \equiv \rho(x) f(x) + t(x) f(-x)$$

is an orthogonal projection? One immediate calculation shows that P is self-adjoint if and only if $t(x) = \overline{t(-x)}$. If, for the sake of simplicity, we assume t is real-valued, this relation becomes, simply, that t is an even function. The idempotent property $P^2 = P$, because of (1.3), becomes

$$\{\rho^2(x) + t(x)t(-x)\} f(x) + t(x) f(-x) = (P^2 f)(x) =$$

$$(Pf)(x) = \rho(x) f(x) + t(x) f(-x)$$

for all $f \in L^2(\mathbb{R})$. Again using (1.3) this equality holds if and only if $t(x)t(-x)$ $= \rho(x)(1 - \rho(x)) = \rho(x)\rho(-x)$. Since t is even this is equivalent to

$$t(x) = \pm\sqrt{\rho(x)\rho(-x)}.$$

This shows that, under these assumptions on ρ and t, P is a projection if and only if

$$(Pf)(x) = \rho(x) f(x) \pm \sqrt{\rho(x)\rho(-x)}\, f(-x). \tag{1.4}$$

It is not hard to make an explicit construction of such functions ρ that are as smooth as desired: we begin by choosing an even non-negative function ψ with $\operatorname{supp} \psi \subset [-\epsilon, \epsilon]$ so normalized that

$$\int_{\mathbb{R}} \psi = \frac{\pi}{2}.$$

Then let

$$\theta(x) = \int_{-\infty}^{x} \psi(t) dt.$$

An immediate consequence of the fact that ψ is even is that

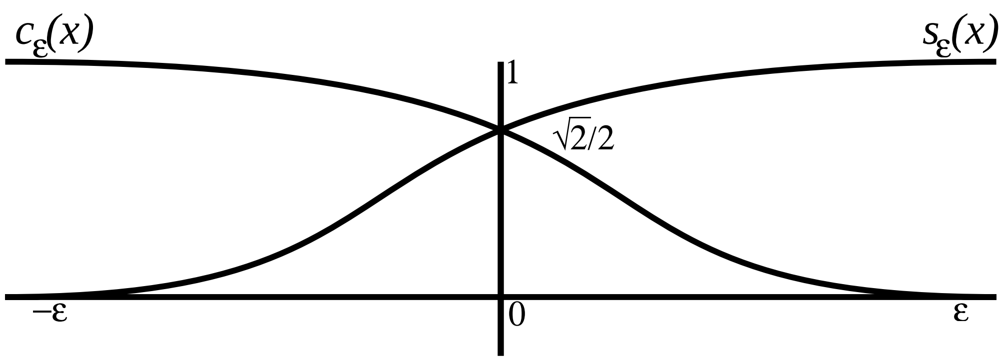
$$\theta(x) + \theta(-x) = \frac{\pi}{2}. \tag{1.5}$$

We now put $s_\epsilon(x) \equiv \sin \theta(x)$ and $c_\epsilon(x) \equiv \cos \theta(x)$. It follows from (1.5) that $c_\epsilon(x) = \cos[\frac{\pi}{2} - \theta(-x)] = s_\epsilon(-x)$. That is,

$$c_\epsilon(x) = s_\epsilon(-x). \tag{1.6}$$

Thus, the graph of c_ϵ is the mirror image, through the vertical axis $x = 0$, of the graph of s_ϵ. We also have the relation

$$s_\epsilon^2(x) + c_\epsilon^2(x) = 1. \tag{1.7}$$



The graphs of s_ε *and* c_ε.

If we now let $\rho(x) = s_\epsilon^2(x)$ we see that ρ enjoys the above properties; in particular, (1.3) is an immediate consequence of (1.6) and (1.7). Equality (1.4) then becomes

$$(Pf)(x) \equiv (P_0 f)(x) = s_\epsilon^2(x) f(x) \pm s_\epsilon(x) c_\epsilon(x) f(-x), \tag{1.8}$$

where we use the notation P_0 to indicate that P is associated with the ray $[0,\infty)$ (of course, P_0 also depends on $\epsilon > 0$ and, in a few occasions, we show this additional dependence by writing $P_{0,\epsilon}$ instead of P_0).

Had we chosen the half ray $(-\infty, 0]$ instead of $[0,\infty)$, completely analogous reasoning would lead us to the orthogonal projection P^0 given by the formula

$$(P^0 f)(x) = c_{\epsilon'}^2(x) f(x) \pm c_{\epsilon'}(x) s_{\epsilon'}(x) f(-x), \tag{1.9}$$

where, again, we do not explicitly denote the dependence of P^0 on ϵ'. We should also observe that, in each case, the projections also depend on the choice of sign before the second summand. Thus, we have introduced the four projections

$$P_+^{0,\epsilon'}, P_-^{0,\epsilon'}, P_{0,\epsilon}^+, P_{0,\epsilon}^-. \tag{1.10}$$

§2. The construction of the projections P_I associated with the interval $I = [\alpha, \beta]$.

We begin by translating P_0 and P^0 to any two points α, β in $\mathbb{R}$. Letting t_γ be the translation operator defined by $(t_\gamma f)(x) = f(x-\gamma)$, we introduce the *translates* (by α and γ) of P_0 and P^0 by letting

$$P_\alpha \equiv t_\alpha P_0 t_{-\alpha} \quad \text{and} \quad P^\beta \equiv t_\beta P^0 t_{-\beta}. \tag{2.1}$$

It is easy to check that

$$\begin{aligned} (P_\alpha f)(x) &= s_\epsilon^2(x-\alpha) f(x) \pm s_\epsilon(x-\alpha) c_\epsilon(x-\alpha) f(2\alpha - x), \\ (P^\beta f)(x) &= c_{\epsilon'}^2(x-\beta) f(x) \pm c_{\epsilon'}(x-\beta) s_{\epsilon'}(x-\beta) f(2\beta - x). \end{aligned} \tag{2.2}$$

Since $t_\gamma^* = t_{-\gamma} = t_\gamma^{-1}$, we see immediately that P_α and P^β are orthogonal projections for each α and β. Observe that x and $2\gamma - x$ are symmetric with respect to the line $x = \gamma$ (they lie on opposite sides of γ and at a distance $|x-\gamma|$ from γ). We say that a function g is *even with respect to* γ if $g(2\gamma - x) = g(x)$ for all x. It is an immediate consequence of (2.2) that $gP_\alpha = P_\alpha g$ (and $gP^\beta = P^\beta g$) when g is even with respect to $\alpha(\beta)$. Using this commutativity with $g = \chi_{[\alpha-\epsilon,\alpha+\epsilon]}$ and $g = \chi_{[\beta-\epsilon',\beta+\epsilon']}$, the properties of $s_\epsilon, c_\epsilon, s'_\epsilon, c'_\epsilon$, and (2.2) we see that

$$P_\alpha P^\beta f = \chi_{[\alpha-\epsilon,\alpha+\epsilon]} P_\alpha f + \chi_{[\alpha+\epsilon,\beta-\epsilon']} f + \chi_{[\beta-\epsilon',\beta+\epsilon']} P^\beta f = P^\beta P_\alpha f \tag{2.3}$$

as long as $\alpha + \epsilon \le \beta - \epsilon'$. This allows us to define $P_I = P[\alpha, \beta]$ by letting

$$P_{[\alpha,\beta]} \equiv P_\alpha P^\beta = P^\beta P_\alpha \tag{2.4}$$

whenever $-\infty < \alpha < \beta < \infty$. Because of this commuting property it is clear that P_I is an orthogonal projection. In view of (1.10) we remark that P_I depends on the choices of $+$ and $-$ in P_α, P^β and on ϵ, ϵ' (as long as $\alpha+\epsilon \le \beta-\epsilon'$).

We shall discuss the importance of this dependency a little later. Before doing this we introduce the *bell over I*. This is the function b_I that depends on $\alpha, \beta, \epsilon, \epsilon'$, but *not* on the choice of signs, defined by

$$b_I(x) \equiv s_\epsilon(x-\alpha)c_{\epsilon'}(x-\beta) \tag{2.5}$$

for all $x \in \mathbb{R}$. We list the basic properties of this bell function; each is an easy consequence of (2.5) and the properties of the functions $s_\epsilon, c_\epsilon, s_{\epsilon'}, c_{\epsilon'}$ developed in §1 (in particular (1.6) and (1.7)):

The function b_I satisfies

(1) Supp $b_I = [\alpha-\epsilon, \beta+\epsilon']$.

On $[\alpha-\epsilon, \alpha+\epsilon]$:

(2) $b_I(x) = s_\epsilon(x-\alpha)$;

(3) $b_I(2\alpha - x) = s_\epsilon(\alpha - x) = c_\epsilon(x-\alpha)$;

(4) $\beta_I^2(x) + b_I^2(2\alpha - x) = 1$.

(5) Supp $b_I(x)b_I(2\alpha-x) = [\alpha-\epsilon, \alpha+\epsilon]$.

(6) $b_I(x) = 1$ *when* $x \in [\alpha+\epsilon, \beta-\epsilon']$. (2.6)

On $[\beta-\epsilon', \beta+\epsilon']$:

(7) $b_I(x) = c_{\epsilon'}(x-\beta)$;

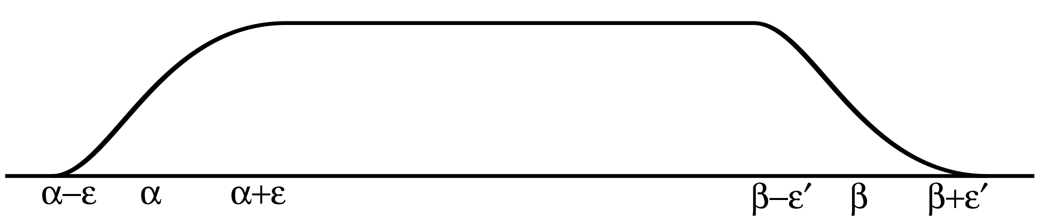
(8) $b_I(2\beta - x) = c_{\epsilon'}(\beta - x) = s_{\epsilon'}(x-\beta)$;

(9) $b_I^2(x) + b_I^2(2\beta - x) = 1$.

(10) *Supp* $b_I(x)b_I(2\beta-x) = [\beta-\epsilon', \beta+\epsilon']$.

(11) *When* $x \in$ Supp $b_I = [\alpha-\epsilon, \beta+\epsilon']$

$$b_I^2(x) + b_I^2(2\beta - x) + b_I^2(2\alpha - x) = 1.$$



*The bell **b** over* I=[α,β].

The last property, obviously an immediate consequence of (4), (6) and (9), is perhaps, the most useful. Observe that the most important feature of these properties is the focus on the behaviour of b_I on the three intervals $[\alpha-\epsilon, \alpha+\epsilon], [\alpha+\epsilon, \beta-\epsilon']$ and $[\beta-\epsilon', \beta+\epsilon']$.

The projection $P_I = P_{[\alpha,\beta]}$ has a simple expression in terms of the bell function b_I:

$$(P_I f)(x) = b_I^2(x)f(x) \pm b_I(x)b_I(2\alpha-x)f(2\alpha-x) \pm b_I(x)b_I(2\beta-x)f(2\beta-x). \tag{2.7}$$

This formula is an immediate consequence of (2.3) and (2.6). It exhibits very clearly the dependence of P_I on the choice of signs associated with the endpoints α and β of the interval I. When ϵ and ϵ' are fixed we are dealing with four projections that are dependent on the two *polarities* (that is, the choice of signs) at each endpoint. The polarities are particularly important when we want to study the properties of P_I and P_J when I and J are adjacent intervals. The dependence of these projections on the choice of ϵ and ϵ' is also important in these considerations. Let us examine this in detail.

Two adjacent intervals $I = [\alpha, \beta]$ and $J = [\beta, \gamma]$ are *compatible* and have *bells that are compatible* if $\alpha - \epsilon < \alpha < \alpha + \epsilon \le \beta - \epsilon' < \beta < \beta + \epsilon' \le \gamma - \epsilon'' < \gamma < \gamma + \epsilon''$ and

$$b_I(x) = s_\epsilon(x-\alpha)c_{\epsilon'}(x-\beta), \qquad b_J(x) = s_{\epsilon'}(x-\beta)c_{\epsilon''}(x-\gamma).$$

Clearly if we apply (2.6)(3) to b_J (with J, β, ϵ' replacing I, α, ϵ) we have $b_J(2\beta - x) = c_{\epsilon'}(x-\beta)$ when $x \in [\beta - \epsilon', \beta + \epsilon']$; by (2.6)(7), on the other hand, $b_I(x) = c_{\epsilon'}(x-\beta)$ when $x \in [\beta - \epsilon', \beta + \epsilon']$ (assuming I and J are compatible). Thus, when I and J are compatible,

$$b_I(x) = b_J(2\beta - x) \tag{2.8}$$

if $x \in [\beta - \epsilon', \beta + \epsilon']$. A similar use of (2.7)(6) and (9) gives us

$$b_I^2(x) + b_J^2(x) = 1 \tag{2.9}$$

when $x \in [\alpha + \epsilon, \gamma - \epsilon'']$. This last relation extends to the equality

$$\sqrt{b_I^2(x) + b_J^2(x)} = s_\epsilon(x-\alpha)c_{\epsilon''}(x-\gamma)$$

for all x, which is equivalent to

$$b_I^2 + b_J^2 = b_{I\cup J}^2 \tag{2.10}$$

whenever I and J are compatible adjacent intervals. As mentioned before, the bell functions b_I are independent of the choices of sign associated with the endpoints of I. The polarity of two adjacent intervals I and J, however, plays an important role if we desire that the projections P_I and P_J satisfy an additive property analogous to (2.10). More precisely we have the following result:

Proposition (2.11). *Suppose $I = [\alpha, \beta]$ and $J = [\beta, \gamma]$ are adjacent compatible intervals and P_I, P_J have opposite polarity at β. Then $P_I + P_J$ is the orthogonal projection $P_{I\cup J}$:*

$$P_I + P_J = P_{I\cup J}. \tag{2.12}$$

Moreover, P_I and P_J are orthogonal to each other:

$$P_I P_J = P_J P_I = 0. \tag{2.13}$$

Proof: Equality (2.12) is an immediate consequence of (2.7), (2.8), and (2.10) (since the terms involving the end point β cancel each other). Equality (2.13) is a consequence of the general result in Hilbert space theory: if P and Q are orthogonal projections such that $P+Q$ is an orthogonal projection, then $PQ=0$. Here is a simple proof of this fact. If $(P+Q)^2 = P+Q$, the idempotent properties of P and Q give us $PQ = -QP$. Thus, $PQ = P^2Q = -PQP = QP^2 = QP$. Since $PQ = -QP$ and $PQ = QP$ it follows that $PQ = QP = O$. ■

Another consequence of formula (2.7) is a simple characterization of the image $\mathcal{H}_I = P_I L^2(\mathbb{R}^n)$ of P_I. Let us say that a function f is *even (odd) with respect to* α *on* $[\alpha-\epsilon, \alpha+\epsilon]$ if and only if $f(2\alpha - x) = f(x)$ $(f(2\alpha-x) = -f(x))$ when $x \in [\alpha-\epsilon, \alpha+\epsilon]$. Observe that (2.7) can be written in the form

$$(P_I f)(x) = b_I(x)S(x), \tag{2.14}$$

where $S(x) = b_I(x)f(x) \pm b_I(2\alpha - x)f(2\alpha - x) \pm b_I(2\beta - x)f(2\beta - x)$. But this function is odd (even) with respect to α on $[\alpha-\epsilon, \alpha+\epsilon]$ if $-(+)$ is chosen before the second term in (2.7) and is odd (even) with respect to β on $[\beta-\epsilon', \beta+\epsilon']$ if $-(+)$ is chosen before the third term. If this odd/even property corresponds, at an end point of I, with the $-/+$ polarity of I at this end point, we say that f has the *same polarity as* I has at this point. Our characterization of $\mathcal{H}_I$, then, is given by:

Theorem (2.15). *$f \in \mathcal{H}_I = P_I L^2(\mathbb{R})$ if and only if $f = bS$, where S is a function in $L^2(\mathbb{R})$ having the same polarity as I at its end points.*

Proof: From (2.14) we see that each element of $\mathcal{H}_I$ has the form $f = bS$. Now suppose f has this form. Apply (2.7) to $f = bS$ and we obtain:

$$(P_I bS)(x) = b^2(x)b(x)S(x) \pm b(x)b^2(2\alpha - x)S(2\alpha - x) \pm b(x)b^2(2\beta - x)S(2\beta - x)$$

$$= \chi_{[\alpha-\epsilon,\alpha+\epsilon]}(x)b(x)S(x)\{b^2(x) + b^2(2\alpha - x)\} + \chi_{[\alpha+\epsilon,\beta-\epsilon']}(x)b(x)S(x) +$$

$$\chi_{[\beta-\epsilon',\beta+\epsilon']}(x)b(x)S(x)\{b^2(x) + b^2(2\beta - x)\} = b(x)S(x)$$

by (2.6)(4), (5), (6), (9), and (10). ■

Finally, let us show how to use these projections to decompose $L^2(\mathbb{R})$ into a direct sum of mutually orthogonal subspaces that are images under such projections:

$$L^2(\mathbb{R}) = \bigoplus_{k \in \mathbf{Z}} \mathcal{H}_k. \tag{2.16}$$

We do this as follows: choose a sequence $\{\alpha_k\}_{k\in\mathbf{Z}}$ of reals and accompanying positive numbers $\{\epsilon_k\}$ such that

$$\alpha_k + \epsilon_k < \alpha_{k+1} - \epsilon_{k+1}$$

for all k. Thus, each pair of adjacent intervals $I_{k-1} = [\alpha_{k-1}, \alpha_k]$ and $I_k = [\alpha_k, \alpha_{k+1}]$ is a compatible pair. Let $b_k = b_{I_k}$ be the bell over I_k and $P_k \equiv P_{I_k}$. Let us also choose these projections so that they have opposite polarity at α_k. We then have $\mathbb{R} = \bigcup_{k\in\mathbf{Z}} I_k$ if $\lim_{k\to\pm\infty} \alpha_k = \pm\infty$. Since

$$\bigcup_{k=-N}^{N} I_k = [\alpha_{-N}, \alpha_{N+1}],$$

it follows from (2.12) that

$$\sum_{k=-N}^{N} P_k = P_{[\alpha_{-N},\alpha_{N+1}]} \tag{2.17}$$

Letting $\mathcal{H}_k \equiv P_k L^2(\mathbb{R})$, (2.13) assures us that $\mathcal{H}_k \perp \mathcal{H}_l$ if $k \neq l$. From (2.3) we see that

$$P_{[-\alpha_N,\alpha_{N+1}]} f = \chi_{[\alpha_{-N}+\epsilon_{-N},\alpha_{N+1}-\epsilon_{N+1}]} f + E_N f,$$

where $E_N f$ is a function supported in the two intervals

$$[\alpha_{-N} - \epsilon_{-N}, \alpha_{-N} + \epsilon_{-N}] \text{ and } [\alpha_{N+1} - \epsilon_{N+1}, \alpha_{N+1} + \epsilon_{N+1}].$$

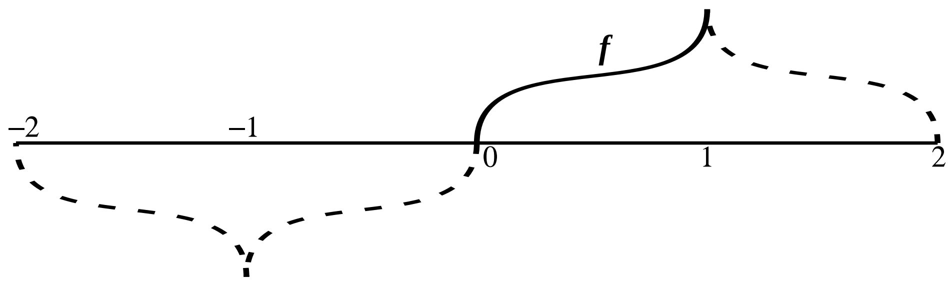
On the first interval $|(E_N f)(x)|$ is dominated by $|f(x)| + |f(2\alpha_{-N} - x)|$ and, on the second, by $|f(x)| + |f(2\alpha_{N+1} - x)|$. Hence, $\|E_N f\|_2 \to 0$ as $N \to \infty$ and it follows that

$$\lim_{N\to\infty} \|f - P_{[-\alpha_N,\alpha_{N+1}]} f\|_2 = 0.$$

These considerations clearly give us the decomposition (2.16).

§3. The local cosine and sine bases for $L^2(\mathbb{R})$.

Let us fix an interval I and consider the problem of constructing "natural" orthonormal bases for the spaces $\mathcal{H}_I = P_I L^2(\mathbb{R})$. There are four such spaces if we take into account the two possible polarities at each end point of I. We ask, first, the simpler question: if the projection is the one obtained by multiplication by χ_I, what are the "natural" bases of the image space $L^2(I)$ from the point of view of a harmonic analyst? Simplifying further, let us assume $I = [0,1]$. Motivated by the polarity properties we have been discussing, we seek some orthonormal bases of $L^2(0,1)$ that reflect these properties. Given $f \in L^2(0,1)$ let us extend it to the interval [0,2] so that it gives us a function $\tilde{f}$ that is even with respect to 1 on this larger interval. Analytically this means $\tilde{f}(x) = f(2-x)$ for $x \in [1,2]$. We then extend $\tilde{f}$ to an odd function on $[-2,2]$.



The extension of f on [0,1] to [-2,2] so that it is even at 1 and odd at 0.

This last function can then be developed into a Fourier series on $[-2,2]$ by means of the orthonormal basis

$$\{\frac{1}{2}, \frac{1}{\sqrt{2}}\sin\frac{k\pi x}{2}, \frac{1}{\sqrt{2}}\cos\frac{k\pi x}{2}\}, k=1,2,\ldots. \tag{3.1}$$

Since we are dealing with an odd function on $[-2,2]$, the even part $\{\frac{1}{2}, \frac{1}{\sqrt{2}}\cos\frac{k\pi x}{2}\}, k=1,2,\ldots$, plays no rôle in this expansion. For the same reason, among the remaining terms, only those that are even with respect to 1 on $[0,2]$ give us (possibly) non-zero coefficients in this Fourier series development. This shows that f can be expanded on $[0,1]$ in terms of the orthogonal family $\{\sin\frac{2k+1}{2}\pi x\}, k=0,1,2,\ldots$.

Had we extended f to obtain the other three pairs of polarities at the points 0 and 1, we would have obtained three other subcollections of the family (3.1). These considerations give us

Proposition (3.2). *Each of the following four systems forms an orthonormal basis for $L^2(0,1)$:*

(i) $\{\sqrt{2}\sin\frac{2k+1}{2}\pi x\}, k=0,1,2,\ldots;$
(ii) $\{\sqrt{2}\sin k\pi x\}, k=1,2,3,\ldots;$
(iii) $\{\sqrt{2}\cos\frac{2k+1}{2}\pi x\}, k=0,1,2,\ldots;$
(iv) $\{1, \sqrt{2}\cos k\pi x\}, k=1,2,3,\ldots.$

The polarities of each of the functions in the first basis are $(-,+)$ at $(0,1)$, in the second basis they are $(-,-)$, in the third they are $(+,-)$ and in the fourth they are $(+,+)$.

Let us now return to the study of the space $\mathcal{H}_I = \mathcal{H}_{[0,1]} = P_I L^2(\mathbb{R})$. To fix our ideas let us assume P_I is chosen with negative polarity at 0 and positive polarity at 1. The bell over $I=[0,1]$ in this case is $b(x) = b_I(x) = s_\epsilon(x)c_{\epsilon'}(x-1)$ and $P = P_I$ is given (see (2.7)) by

$$(Pf)(x) = b(x)\{b(x)f(x) - b(-x)f(-x) + b(2-x)f(2-x)\} \equiv b(x)S(x). \tag{3.3}$$

If we restrict S to $[0,1]$, using the first basis in Proposition (3.2), we have

$$S(x) = \sqrt{2}\sum_{k=0}^{\infty} c_k \sin\frac{2k+1}{2}\pi x, \tag{3.4}$$

where the equality may be interpreted in the norm of $L^2(I)$, and the coefficients c_k are given by the equality

$$c_k = \sqrt{2}\int_0^1 S(x)\sin\frac{2k+1}{2}\pi x dx. \tag{3.5}$$

But each of the functions $\sin\frac{2k+1}{2}\pi x$ satisfy the same polarity properties as S. It follows that equality (3.4) is valid on $[-\epsilon, 1+\epsilon']$ and the convergence can be taken in the norm of $L^2(-\epsilon, 1+\epsilon')$. Multiplying this new equality on both sides by $b(x)$ we see that any $f \in \mathcal{H}_I$ satisfies

$$f(x) = \sqrt{2}\sum_{k=0}^{\infty} c_k b(x)\sin\frac{2k+1}{2}\pi x \tag{3.6}$$

in $L^2(-\epsilon, 1+\epsilon')$.

We claim the system $\{\sqrt{2}b(x)\sin\frac{2k+1}{2}\pi x\}$, $k = 0, 1, 2, \ldots$, is an orthonormal basis of $\mathcal{H}_I$ and $\{c_k\}$, given by (3.5), is the sequence of coefficients of $f \in \mathcal{H}_I$ with respect to this basis. It is clear from our discussion that in order to establish this claim all we need to show is that this system is orthonormal in $\mathcal{H}_I$. This is done by a simple calculation in which (2.6) and (2.7) play an important rôle: Let $f(x) = \sqrt{2}b(x)\sin\frac{2l+1}{2}\pi x$, then, by two changes of variables,

$$\begin{aligned} \int_{-\epsilon}^{\epsilon} f(x)\sqrt{2}b(x)\sin\frac{2k+1}{2}\pi x dx &= \\ \sqrt{2}\int_0^{\epsilon}\{f(x)b(x) - f(-x)b(-x)\}\sin\frac{2k+1}{2}\pi x dx &= \\ 2\int_0^{\epsilon}\{s_\epsilon^2(x) + c_\epsilon^2(x)\}(\sin\frac{2l+1}{2}\pi x)(\sin\frac{2k+1}{2}\pi x)dx &= \\ 2\int_0^{\epsilon}(\sin\frac{2l+1}{2}\pi x)(\sin\frac{2k+1}{2}\pi x)dx & \end{aligned}$$

and

$$\begin{aligned} \int_{1-\epsilon'}^{1+\epsilon'} f(x)\sqrt{2}b(x)\sin\frac{2k+1}{2}\pi x dx &= \\ \sqrt{2}\int_{1-\epsilon'}^{1}\{f(2-x)b(2-x) + f(x)b(x)\}\sin\frac{2k+1}{2}\pi x dx &= \\ 2\int_{1-\epsilon'}^{1}\{s_{\epsilon'}^2(x-1) + c_{\epsilon'}^2(x-1)\}(\sin\frac{2l+1}{2}\pi x)(\sin\frac{2k+1}{2}\pi x)dx &= \\ 2\int_{1-\epsilon'}^{1}(\sin\frac{2l+1}{2}\pi x)(\sin\frac{2k+1}{2}\pi x)dx. & \end{aligned}$$

Thus, since $b(x) = 1$ on $[\epsilon, 1-\epsilon']$,

$$\int_{-\epsilon}^{1+\epsilon'} f(x)\sqrt{2}b(x)\sin\frac{2k+1}{2}\pi x dx$$
$$= 2\int_0^1 (\sin\frac{2l+1}{2}\pi x)(\sin\frac{2k+1}{2}\pi x)dx = \delta_{kl}$$

the last equality being a consequence of (3.2)(i).

Had we chosen the other polarities, $(-,-),(+,-),(+,+)$, in the definitions of $P = P_{[0,1]}$, we reach the same conclusions if, instead of the system (3.2)(i), we use the systems (ii), (iii), and (iv) respectively. The case of the general interval $I = [\alpha,\beta]$ now follows from these results by simple translation and dilation arguments. More precisely, we obtain

Theorem (3.7). *If $P_I = P_{[\alpha,\beta]}$ is the projection associated with negative polarity at α and positive polarity at β, then the system*

$$(i)\left\{\sqrt{\frac{2}{|I|}}b_I(x)\sin\frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha)\right\}, k = 0,1,2,\ldots, \tag{3.8}$$

is an orthonormal basis for $\mathcal{H}_I = P_I L^2(\mathbb{R})$. If we choose the polarities $(-,-)$, $(+,-)$, and $(+,+)$ at (α,β), the same result is true if we use (respectively) the systems

$$(ii)\left\{\sqrt{\frac{2}{|I|}}b_I(x)\sin k\frac{\pi}{|I|}(x-\alpha)\right\}, k = 1,2,3,\ldots;$$
$$(iii)\left\{\sqrt{\frac{2}{|I|}}b_I(x)\cos\frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha)\right\}, k = 0,1,2,\ldots; \tag{3.8}$$
$$(iv)\left\{\frac{1}{\sqrt{|I|}}b_I(X), \sqrt{\frac{2}{|I|}}b_I(x)\cos k\frac{\pi}{|I|}(x-\alpha)\right\}, k = 1,2,3,\ldots,$$

instead of (3.8)(i).

We can now combine the results obtained at the end of §2 with this theorem to obtain the *local sine and cosine bases of Coifman and Meyer.* Let a sequence $\{\alpha_j\}$ be selected as described at the end of §2. Thus, $\alpha_j < \alpha_{j+1}$, $\lim\limits_{j\to\pm\infty}\alpha_j = \pm\infty$ and there exists an accompanying sequence $\{\epsilon_j\}$ such that

$$\alpha_j + \epsilon_j \le \alpha_{j+1} - \epsilon_{j+1}$$

for all $j \in \mathbf{Z}$. Let $P_j = P[\alpha_j, \alpha_{j+1}]$ be constructed with, say, negative polarity at α_j and positive polarity at α_{j+1}. Let

$$b_{jk}(x) = b_{[\alpha_j,\alpha_{j+1}]}(x)\sqrt{\frac{2}{\alpha_{j+1}-\alpha_j}}\sin\frac{2k+1}{2}\frac{\pi}{\alpha_{j+1}-\alpha_j}(x-\alpha_j), \tag{3.9}$$

$j \in \mathbf{Z}, k = 0, 1, 2, \ldots$. Our discussion (in particular (2.16) and (3.7) allows us to conclude that the *functions in (3.9) form an orthonormal basis for* $L^2(\mathbb{R})$. We can change the polarity for each projection P_j and arrive at the same conclusion as long as the polarity for P_{j-1} at α_j is opposite to that of P_j at α_j; moreover, each such choice must be accompanied by exchanging the basis (3.9) with the basis in Theorem (3.7) that corresponds to these choices in polarity.

Here are some further observations that follow easily from the results we have obtained:

Theorem (3.10). *Suppose $f \in \mathcal{H}_I$, where $I = [\alpha, \beta]$ is a finite interval in $\mathbb{R}$ and the polarities are, say, $-+$ at α and β, then the series*

$$\sqrt{\frac{2}{|I|}} \sum_{k=0}^{\infty} c_k b(x) \sin \frac{2k+1}{2} \frac{\pi}{|I|}(x-\alpha)$$

converges to $f(x)$ a.e. in $[\alpha - \epsilon, \beta + \epsilon']$. If $g \in L^2(\mathbb{R})$, then the development of g in terms of the basis (3.9) converges a.e. to $g(x)$.

Proof: If $f = b_I S_I = bS \in \mathcal{H}_I$ we have shown that,

$$f(x) = \sum_{k=0}^{\infty} c_k b(x) \sqrt{\frac{2}{|I|}} \sin \frac{2k+1}{2} \frac{\pi}{|I|}(x-\alpha) \tag{3.11}$$

with convergence in the $L^2(I)$ norm. But we observed that the coefficients

$$c_k = \sqrt{\frac{2}{|I|}} \int_{\alpha-\epsilon}^{\beta+\epsilon'} f(x) b(x) \sin \frac{2k+1}{2} \frac{\pi}{|I|}(x-\alpha) dx,$$

$k = 0, 1, 2, \ldots$, can also be calculated as the sine series coefficients

$$c_k = \sqrt{\frac{2}{|I|}} \int_{\alpha}^{\beta} S(x) \sin \frac{2k+1}{2} \frac{\pi}{|I|}(x-\alpha) dx. \tag{3.12}$$

This is a consequence of (3.5) (after an appropriate translation and dilation). Since $f \in L^2(\mathbb{R}), S = b^{-1} f$ is square integrable over $[\alpha, \beta]$. It then follows from Carleson's theorem that the series

$$\sqrt{\frac{2}{|I|}} \sum_{k=0}^{\infty} c_k \sin \frac{2k+1}{2} \frac{\pi}{|I|}(x-\alpha) = S(x) \tag{3.13}$$

converges to $S(x)$ a.e. in $[\alpha, \beta]$. Multiplying both sides by $b(x)$ we then have the validity of (3.11) for a.e. $x \in [\alpha, \beta]$. Since both sides are odd with respect to α on $[\alpha - \epsilon, \alpha + \epsilon]$ and even with respect to β on $[\beta - \epsilon', \beta + \epsilon']$, this a.e. convergence extends to $[\alpha - \epsilon, \beta + \epsilon']$. ∎

Clearly there are versions of Theorem (3.10) for each choice of polarities associated with the interval $[\alpha, \beta]$. If the sequence $\{\alpha_j\}$ (and the accompanying sequence $\{\epsilon_j\}$) gives us a family of mutually orthogonal projections P_j whose ranges span $L^2(0,\infty)$ (by choosing $\alpha_j \to 0$ as $j \to -\infty$ and $\alpha_j \to \infty$ as $j \to \infty$), we obtain an orthonormal basis of those functions in $L^2(\mathbb{R})$ that are the Fourier transforms of the elements of the classical Hardy space H^2. In the next section we describe another application in which it is useful to think of these local bases as the Fourier transform images of interesting bases of $L^2(\mathbb{R})$.

We make one final comment before presenting the application that was just mentioned. The projections P_I are, indeed, "smoother" than the ones obtained by multiplication by χ_I. They are not, however, necessarily "smoothing." For example, let $f = \chi_{[0,1]}$ and $P_I = P_{[0,1]}$. If I has negative polarity at 0, $P_I f$ still has a jump at 0. Positive polarity at 0 on the other hand, "smooths out" f near 0.

§4. The Construction of the Lemarié-Meyer Smooth Wavelets.

Lemarié and Meyer [2] constructed a wavelet basis

$$\{w_{k,n}(x)\} = \{2^{-k/2} w(2^{-k}x - n)\}, \tag{4.1}$$

$k, n \in \mathbf{Z}$, where the "mother function" w belongs to $\mathcal{S}(\mathbb{R})$ (in fact, it is the restriction of an entire function on $\mathbb{C}$) and

$$\text{Supp } \widehat{w} \ \subset [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \frac{8\pi}{3}]. \tag{4.2}$$

This means, in particular, that $\{w_{k,n}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. This basis furnishes us with an example of a smooth "Multi Resolution Analysis" as described in [Ma] and [M]. We show that this construction can be carried out in an easy and natural way by using the local bases we have just presented.

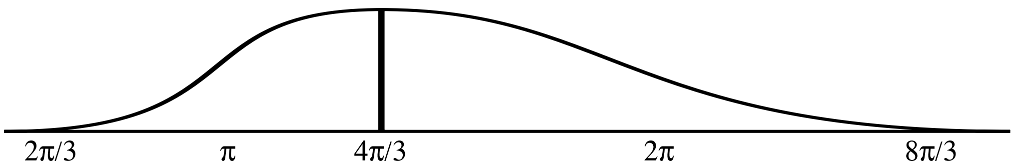
Let $I = [\pi, 2\pi], \epsilon = \frac{\pi}{3}, \epsilon' = 2\epsilon = \frac{2\pi}{3}$. Consider the orthogonal projection P_I with polarity $(+,-)$. The bell function $b = b_I$ associated with P_I will not have an interval of constancy since $\pi + \epsilon = 2\pi - \epsilon'$. Its construction is explicitly given by (2.5) with $\alpha = \pi, \beta = 2\pi, \epsilon = \frac{\pi}{3} = \frac{\epsilon'}{2}$. The range of this projection is the subspace generated by the orthonormal basis

$$\psi(n;\xi) = \sqrt{\frac{2}{\pi}}\, b(\xi) \cos \frac{2n+1}{2}(\xi - \pi),$$

$n = 0, 1, 2, \ldots$ (see Theorem (3.7) and, in particular, (3.8) (iii)). The dilations by $2^k, k \in \mathbf{Z}$, then give us the projections $P_k = P_{[2^{-k}\pi, 2^{-k+1}\pi]}$ with ranges spanned by the orthonormal basis

$$\psi_k(n;\xi) \equiv 2^{k/2}\psi(n; 2^k\xi), \tag{4.3}$$

$k \in \mathbf{Z}$. It follows from the material in §3. (see, in particular, the observation following Theorem (3.10), concerning the spanning of $L^2(0,\infty)$), that, with



Graph of $\boldsymbol{b}(\xi)$.

this choice of polarity, the collection defined by (4.3) forms an orthonormal basis of $L^2(0,\infty)$.

Let us also carry out the completely analogous construction based on the local sine basis (3.8) (i):

$$\varphi(n;\xi) = \sqrt{\frac{2}{\pi}}b(\xi)\sin\frac{2n+1}{2}(\xi-\pi),$$

$n = 0, 1, 2, \ldots$. We then obtain the orthonormal basis of $L^2(0,\infty)$

$$\varphi_k(n;\xi) = 2^{k/2}\varphi(n;2^k\xi), \tag{4.4}$$

$k \in \mathbf{Z}$.

In order to obtain an orthonormal basis for $L^2(\mathbb{R})$ we consider the even extensions of the functions (4.3) and the odd extensions of the functions (4.4) to all of $\mathbb{R}$. More precisely, let

$$\Psi_{k,n}(\xi) \equiv \frac{1}{\sqrt{2\pi}}2^{k/2}b(2^k|\xi|)\cos\frac{2n+1}{2}(2^k|\xi|-\pi)$$

and

$$\Phi_{k,n}(\xi) \equiv \frac{1}{\sqrt{2\pi}}2^{k/2}b(2^k|\xi|)(\mathrm{sgn}\xi)\sin\frac{2n+1}{2}(2^k|\xi|-\pi).$$

Theorem (4.5). *The collection of functions*

$$\alpha_{k,n} \equiv \Psi_{k,n} + i\Phi_{k,n} \quad \textit{and} \quad \beta_{k,n} = \Phi_{k,n} + i\Psi_{k,n},$$

$k = 0, \pm1, \pm2, \ldots, n = 0, 1, 2, \ldots,$ *is an orthonormal basis for* $L^2(\mathbb{R})$.

Proof: Let $\langle f, g\rangle = \int_{\mathbb{R}} f\overline{g}$ be the standard inner product in $L^2(\mathbb{R})$. The orthonormality relations

$$\langle \alpha_{k,n}, \alpha_{k',n'}\rangle = \delta_{kk'}\delta_{nn'} = \langle \beta_{k,n}, \beta_{k,'n'}\rangle$$

follow from the orthonormality relations satisfied by the families (4.3), (4,4) and the fact that each product of the form $\Phi_{k,n}\Psi_{k',n'}$ is an odd function. The orthogonality relations

$$\langle \alpha_{k,n}, \beta_{k,'n'}\rangle = 0$$

for all $k \in \mathbf{Z}$ and $n = 0, 1, 2, \ldots$, follow by the simple calculation:

$$\langle \alpha_{k,n}, \beta_{k,'n'} \rangle = \int_{\mathbb{R}} (\Psi_{k,n}\Phi_{k',n'} + \Phi_{k,n}\Psi_{k',n'}) + i(\Phi_{k,n}\Phi_{k',n'} - \Psi_{k,n}\Psi_{k',n'})$$

$$= 0 + \frac{i}{2}(\delta_{kk'}\delta_{nn'} - \delta_{kk'}\delta_{nn'}) = 0.$$

It remains to be shown that the above system is complete. In order to do this it clearly suffices to show that for each real valued $f \in L^2(\mathbb{R})$

$$f = \sum_{k\in\mathbf{Z}} \sum_{n=0}^{\infty} \{\langle f, \alpha_{k,n}\rangle \alpha_{k,n} + \langle f, \beta_{k,n}\rangle \beta_{k,n}\}, \tag{4.6}$$

where the convergence is in the $L^2(\mathbb{R})$ norm. But, writing $f = f^{(e)} + f^{(o)}$, with $f^{(e)}$ the even part and $f^{(o)}$ the odd part of f, we see that

$$\sum_{k\in\mathbf{Z}} \sum_{n=0}^{\infty} \langle f, \alpha_{k,n}\rangle \alpha_{k,n} =$$
$$\sum_{k\in\mathbf{Z}} \sum_{n=0}^{\infty} \{\langle f^{(e)}, \Psi_{k,n}\rangle - i\langle f^{(o)}, \Phi_{k,n}\rangle\}(\Psi_{k,n} + i\Phi_{k,n})$$

from which we obtain

$$\begin{aligned} &\sum_{k\in\mathbf{Z}} \sum_{n=0}^{\infty} \langle f, \alpha_{k,n}\rangle \alpha_{k,n} \\ &= \frac{1}{2} f + i \sum_{k\in\mathbf{Z}} \sum_{n=0}^{\infty} \{\langle f^{(e)}, \Psi_{k,n}\rangle \Phi_{k,n} - \langle f^{(o)}, \Phi_{k,n}\rangle \Psi_{k,n}\} \end{aligned} \tag{4.7}$$

A completely analogous argument gives us

$$\begin{aligned} &\sum_{k\in\mathbf{Z}} \sum_{n=0}^{\infty} \langle f, \beta_{k,n}\rangle \beta_{k,n} \\ &= \frac{1}{2} f + i \sum_{k\in\mathbf{Z}} \sum_{n=0}^{\infty} \{\langle f^{(o)}, \Phi_{k,n}\rangle \Psi_{k,n} - \langle f^{(e)}, \Psi_{k,n}\rangle \Phi_{k,n}\}. \end{aligned} \tag{4.8}$$

Adding equalities (4.7) and (4.8) we obtain the desired result (4.6). ■

We now show that by modifying this basis slightly (multiplying its members by scalars of absolute value 1) we obtain an orthonormal basis of $L^2(\mathbb{R})$ that is generated by the single function $\gamma(\xi) \equiv i\alpha_{0,0}(\xi)$. More precisely, we have

Theorem (4.9). *The functions*

$$\gamma_{k,n}(\xi) \equiv 2^{k/2} e^{-i2^k n\xi} \gamma(2^k \xi),$$

$k, n \in \mathbf{Z}$, *form an orthonormal basis of* $L^2(\mathbb{R})$, *where*

$$\gamma(\xi) = \frac{\operatorname{sgn} \xi}{\sqrt{2\pi}} e^{i\frac{\xi}{2}} b(|\xi|).$$

Proof: Let us put $\alpha_{k,-n}(\xi) \equiv \overline{\alpha_{k,n-1}(\xi)} = -i\beta_{k,n-1}$ for $k \in \mathbf{Z}$ and $n > 0$. It then follows from Theorem (4.5) that $\{\alpha_{k,n}\}$, $k, n \in \mathbf{Z}$, is an orthonormal basis for $L^2(\mathbb{R})$. Our theorem is established if we show

$$\alpha_{0,n}(\xi) = (-1)^n (-i) e^{in\xi} \gamma(\xi) \tag{4.10}$$

for all $n \in \mathbf{Z}$. To prove (4.10) assume, first, that $n \geq 0$, then

$$\alpha_{0,n}(\xi) = \frac{1}{\sqrt{2\pi}} b(|\xi|)[\cos \frac{2n+1}{2}(|\xi| - \pi) + i(\operatorname{sgn} \xi) \sin \frac{2n+1}{2}(|\xi| - \pi)]$$

$$= \frac{b(|\xi|)}{\sqrt{2\pi}} \begin{cases} e^{i(\xi-\pi)(2n+1)/2} & \text{if } \xi \geq 0 \\ e^{i(\xi+\pi)(2n+1)/2} & \text{if } \xi < 0 \end{cases} = (-i)(-1)^n e^{in\xi} \gamma(\xi).$$

If $n < 0$, then $\alpha_{0,n}(\xi) = \overline{\alpha_{0,-n-1}(\xi)} = i \frac{\operatorname{sgn}\xi}{\sqrt{2\pi}} (-1)^{-(n+1)} e^{i(n+1)\xi} e^{-i\frac{\xi}{2}} b(|\xi|) = (-i)(-1)^n e^{in\xi} \gamma(\xi)$. Thus, (4.10) holds for all integers n. ∎

The promised wavelet basis is now immediately obtained from the functions in theorem (4.9) by applying the inverse Fourier transform to them. Let us carry out the details. We adopt the following definition of the *Fourier transform* of a function $f \in L^1 \cap L^2$:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx.$$

The inverse Fourier transform formula is, in this case,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi = \frac{1}{2\pi} \lim_{N \to \infty} \int_{-N}^{N} \hat{f}(\xi) e^{ix\xi} d\xi,$$

the limit being an L^2-limit. A version of the Plancherd theorem, then tells us that if $f \in L^2(\mathbb{R})$, then

$$\|\sqrt{2\pi} f\|_2 = \|\hat{f}\|_2, \tag{4.11}$$

and, if g is another function in $L^2(\mathbb{R})$, we have

$$\langle f, g \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle. \tag{4.12}$$

Thus, if we define w by

$$w(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \gamma(\xi) d\xi$$

we have $\widehat{w} = \sqrt{2\pi}\gamma$,

$$w_{k,n}(x) = 2^{-k/2} w(2^{-k}x - n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \gamma_{k,n}(\xi) d\xi,$$

for $k, n \in \mathbf{Z}$, and the collection $\{w_{k,n}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. Since $\operatorname{Supp}\widehat{w} = \operatorname{Supp} b(|\xi|) = [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \frac{8\pi}{3}]$ we see that w is, indeed, a mother function of the type described at the beginning of this section. That is, w generates a Lemarié-Meyer wavelet basis.

§5. Lemarié-Meyer wavelet bases with more general band limitations.

In this section we shall examine to what extent we can generalize this construction if we consider other conditions on the support of $\widehat{w}$. We pose our problem in a way that is consistent with the basic properties of local sine/cosine bases. Instead of the interval $[\pi, 2\pi]$ let us consider the interval $[1, \lambda]$. Dilates by λ, then, will then give us a covering of the right half-line:

$$(0, \infty) = \bigcup_{k=-\infty}^{\infty} [\lambda^k, \lambda^{k+1}].$$

We construct a bell function b associated with the interval $[1, \lambda], \epsilon = \frac{\lambda - 1}{\lambda + 1}$ and $\epsilon' = \frac{\lambda(\lambda-1)}{(\lambda+1)} = \lambda\epsilon$. Observe that $1 + \epsilon = \lambda - \epsilon'$. The dilates $b(\lambda^{-k}\xi)$ are then bell functions corresponding to the intervals $[\lambda^k, \lambda^{k+1}], k \in \mathbf{Z}$. Observe that adjacent intervals are compatible.

In complete analogy with the construction that led into Theorem (4.5), we let $\alpha^{\lambda}_{k,n} = \Psi^{\lambda}_{k,n} + i\Phi^{\lambda}_{k,n}$ for $k \in \mathbf{Z}, n \geq 0$, and $\alpha^{\lambda}_{k,n} = \overline{\alpha^{\lambda}_{k,-n-1}}$ if $k \in \mathbf{Z}$ and $n < 0$, where

$$\Psi^{\lambda}_{k,n}(\xi) = \sqrt{\frac{1}{2(\lambda - 1)}} \lambda^{\frac{k}{2}} b(\lambda^k |\xi|) \cos \pi (n + \frac{1}{2})(\frac{|\xi|\lambda^k - 1}{\lambda - 1}),$$

$k \in \mathbf{Z}, n \geq 0$,

$$\Phi^{\lambda}_{k,n}(\xi) = \sqrt{\frac{1}{2(\lambda - 1)}} \lambda^{\frac{k}{2}} b(\lambda^k |\xi|)(\operatorname{sgn} \xi) \sin \pi (n + \frac{1}{2})(\frac{\lambda^k |\xi| - 1}{\lambda - 1}),$$

$k \in \mathbf{Z}, n \geq 0$. The argument establishing Theorem (4.5) can be easily extended to show

Theorem (5.1). *The collection of functions* $\{\alpha^{\lambda}_{k,n}\}, k, n \in \mathbf{Z}$ *is an orthonormal basis for* $L^2(\mathbb{R})$.

Let us now examine if the functions $\alpha_{k,n} = \alpha^{\lambda}_{k,n}$ are "generated", via dilations (by integral powers of λ) and multiplications by $e^{i\pi n \frac{\xi}{\lambda-1}}$ $(n \in \mathbf{Z})$ applied to $\alpha_{0,0}$ as was the case for the analogous basis in Theorem (4.9). We begin by trying to establish a formula similar to (4.10). If $n \geq 0$ we have

$$\sqrt{2(\lambda-1)}\alpha_{0,n}(\xi) = b(|\xi|)\begin{cases} e^{i\pi(n+\frac{1}{2})\frac{(\xi-1)}{(\lambda-1)}}, \xi > 0 \\ e^{i\pi(n+\frac{1}{2})\frac{(\xi+1)}{(\lambda-1)}}, \xi < 0 \end{cases}$$

$$= \sqrt{2(\lambda-1)}\alpha_{0,0}(\xi)e^{i\pi n\xi/(\lambda-1)}\rho^{n \operatorname{sgn} \xi},$$

where $\rho = e^{-i\frac{\pi}{\lambda-1}}$. If $n < 0$ then

$$\alpha_{0,n}(\xi) = \overline{\alpha_{0,-n-1}(\xi)} = \overline{\alpha_{0,0}(\xi)\rho^{-(n+1)\operatorname{sgn}\xi}}e^{i\pi(n+1)\frac{\xi}{\lambda-1}}$$

$$= \frac{1}{\sqrt{2(\lambda-1)}}b(|\xi|)e^{-i\frac{\pi}{2}\frac{\xi}{\lambda-1}}e^{i\pi\frac{1}{2}\frac{\operatorname{sgn}\xi}{\lambda-1}}e^{-i\pi(n+1)\frac{\operatorname{sgn}\xi}{\lambda-1}}e^{i\pi(n+1)\frac{\xi}{\lambda-1}}$$

$$= \frac{1}{\sqrt{2(\lambda-1)}}\{b(|\xi|)e^{i\frac{\pi}{2}\frac{\xi}{\lambda-1}}e^{-1\frac{\pi}{2}\frac{\operatorname{sgn}\xi}{\lambda-1}}\}e^{i\pi n\frac{\xi}{\lambda-1}}e^{-i\pi n\frac{\operatorname{sgn}\xi}{\lambda-1}}$$

$$= \alpha_{0,0}(\xi)e^{i\pi n\frac{\xi}{\lambda-1}}\rho^{n \operatorname{sgn}\xi}.$$

This shows

$$\alpha_{0,n}(\xi) = \rho^{n \operatorname{sgn}\xi}e^{in\pi\frac{\xi}{\lambda-1}}\alpha_{0,0}(\xi), \tag{5.2}$$

for all $n \in \mathbf{Z}$

Thus, multiplication of $\alpha_{0,0}(\xi)$ by $e^{in\pi\frac{\xi}{\lambda-1}}$ gives us an orthonormal system, differing from $\alpha_{0,n}(\xi)$ by a constant multiple of absolute value 1, provided $\rho^{n \operatorname{sgn}\xi}$ is a constant of absolute value 1 for all ξ (and each $n \in \mathbf{Z}$). But this means that $\rho = e^{-i\frac{\pi}{\lambda-1}}$ must be real (either 1 or -1). Consequently, $\sin\frac{\pi}{\lambda-1} = 0$ and it follows that $\frac{\pi}{\lambda-1}$ must be an integral multiple of π. This and the condition $\lambda > 1$ force us to conclude that

$$\lambda = 1 + \frac{1}{m} \tag{5.3}$$

for m a positive integer. We thus obtain the following analog of Theorem (4.9):

Theorem (5.4). *The functions*

$$\gamma^{(\lambda)}_{k,n}(\xi) = \gamma_{k,n}(\xi) = \lambda^{\frac{k}{2}}e^{-in\pi\frac{\lambda^k\xi}{\lambda-1}}\gamma(\lambda^k\xi), \tag{5.5}$$

$k, n \in \mathbf{Z}$, *is an orthonormal basis of* $L^2(\mathbb{R})$, *where*

$$\gamma(\xi) = \frac{e^{\frac{-i\pi \operatorname{sgn}\xi}{2(\lambda-1)}}}{\sqrt{2(\lambda-1}}e^{i\pi\frac{\xi}{2(\lambda-1)}}b(|\xi|)$$

and $\lambda = 1 + \frac{1}{m}$, *m a positive integer.*

If we let $w(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi}\gamma(\xi)d\xi$, then w is a "mother function" that generates a wavelet basis (giving us a Multi Resolution Analysis)

$$\{w(x\lambda^{-k} - n\frac{\pi}{\lambda - 1})\lambda^{-\frac{k}{2}}\},$$

for all $k, n \in \mathbf{Z}$, whenever $\lambda = 1 + \frac{1}{m}, m$ a positive integer.

§6. Concluding remarks.

We repeat that the local bases we developed in §2. were introduced by Coifman and Meyer, and their use in obtaining the smooth wavelet bases were pointed out to us by these two authors. Some of the ideas involved in developing the properties of the local bases are also found in Malvar's paper [4]. The particular emphasis on the rôle played by the projections P_I, however, does not appear in Malvar's paper [5] and is not promminent in the exposition [1] of Coifman and Meyer. As mentioned in the Abstract, one of our goals is to make this material accessible to the largest possible audience. One can develop a parallel treatment connected with the discrete sine and cosine transforms. We shall present this and its connection with the corresponding discrete version of the smooth wavelet bases in a subsequent paper.

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