Assignment 2 - Numerically solving the steady state heat equation

Due: October 9, in class
No late assignments accepted

Issued: October 2, 2013

Important:

- Give complete answers: Do not only give mathematical formulae, but explain what you are doing. Conversely, do not leave out critical intermediate steps in mathematical derivations.
- Write your **name** as well as your **Sunet ID** on your assignment. **Please staple pages together.** Points will be docked otherwise.
- Questions preceded by $\star$ are harder and/or more involved.
- The theory needed for question 1 (e) (condition numbers) will be covered in class Monday, Oct 7.

Problem 1

In this assignment, we will solve 1D heat equation numerically with different boundary conditions. In 1D, the differential equation is of the form

$$\frac{d^2T}{dx^2} = f(x), \qquad \text{for } 0 \leq x \leq 1.$$

We interpret this equation as modeling the steady state solution of a heat diffusion problem in a laterally insulated rod with constant thermal conductivity, and a distributed heat source f. T denotes temperature. Note that this is the same equation that you've seen in class with $f(x) = 0$.

Set the source term $f(x) = 0$ with boundary conditions $T = 0$ at $x = 0$ and $T = 2$ at $x = 1$.

(a) Give the exact solution to this differential equation.

(b) Discretize the 1D equation using the central difference scheme we derived in class using a uniform grid with spacing $h = 1/N$ (so that you get the grid points $x_i = i_h$ for $i = 0 \dots N$ as shown in the class). Write an expression for each equation in the system. Be sure to explain your work; do not merely give the final set of equations.

(c) Set up the resulting matrix-vector equation $A\vec{x} = b$ for $N = 5$, and use Gaussian Elimination to check that the matrix A is nonsingular. (Do this by hand.)

(d) Now set the source $f(x) = -10\sin(3\pi x/2)$. Keep the boundary conditions as $T = 0$ at $x = 0$ and $T = 2$ at $x = 1$. Verify that $T(x) = (2 + 40/(9\pi^2))x + 40/(9\pi^2)\sin(3\pi x/2)$ is the exact solution that solves the differential equation with this source and boundary conditions.

Update the matrix-vector system $A\vec{x} = b$ for this source function. Solve this system $A\vec{x} = b$ using MATLAB for $N = 5$, $N = 10$, $N = 25$ and $N = 200$. Plot all computed solutions together in one figure with the exact solution. Clearly label your graph. Note that the curves may overlap. Use a legend.

Note: We want you to arrive at a discrete solution for N=200 points here not because we need it for accuracy, but because it will make it hard to construct the matrix in MATLAB element-by-element, which we do not want you to do. Use some of MATLAB's built in functions for creating these matrices. Use the diag command to construct tridiagonal matrices (consult help diag for usage). If you are ambitious, construct A as a sparse matrix using sp-diags, since the sparse representation is more efficient for matrices with few nonzero elements.

(e) Find the condition number of the matrix A using MATLAB for $N = 5$, $N = 10$, $N = 25$ and $N = 200$ (use Frobenius norm to compute the condition number, i.e. cond$A = \|A\|_F\|A^{-1}\|_F$. There is an in-built MATLAB function for that: cond(A,'fro')). Plot the condition numbers and comment on any pattern that you might find. Perturb the matrix A by ∂A for $\partial A = 0.1A$. Solve the perturbed system of equations $(A+\partial A)x = b$ and plot the solution. Comment on what you see. Can you relate it to the condition number of A?

(f) ⋆ Consider a non-zero source defined by the piecewise function

$$f(x) = \begin{cases} 0 & x < 1/3 \\ 100 & \text{otherwise} \end{cases}$$

With the same boundary conditions as in part (d), show that the exact solution is

$$T(x) = \begin{cases} -\frac{182x}{9} & x < 1/3 \\ \frac{-482x+50}{9} + \frac{100x^2}{2} & \text{otherwise} \end{cases}$$

Hint: Use the fact that if $f(x)$ is a constant, the equation $d^2T/dx^2 = f(x)$ can easily be solved.

Now solve the system $A\vec{x} = b$ using MATLAB for $N = 5$, $N = 10$, $N = 25$ and $N = 200$. Plot all computed solutions together with the exact solution in one figure.

Let T_N denote the solution vector using N grid points and $T_{exact,N}$ denote the vector of the exact solution at the N grid points. How big does N have to be for the relative error $\frac{\|T_{exact,N}-T_N\|_2}{\|T_{exact}\|_2}$ to be less than 0.01?

How can the problem of having grid points exactly on the discontinuity point of $f(x)$ be resolved?

(g) ⋆ With source $f(x) = 0$, perform the same steps as in (1b) and (1c) after replacing the Dirichlet boundary conditions with Neumann conditions $dT/dx = 0$ at $x = 0$, $x = 1$. That is,

- Give an expression for each equation in the system when it's of size N.
- Write out the resulting matrix-vector equation $A\vec{x} = b$ for $N = 5$.

- Use Gaussian Elimination for $N = 5$ (by hand) to show this matrix is singular.

How can you resolve this issue of a singular A without changing the boundary conditions?

(h) $\star$ With source $f(x) = 0$, Dirichlet boundary condition at x = 0 (i.e $T = 0$ at $x = 0$) and Neumann conditions $dT/dx = 0$ at $x = 1$,

- Give an expression for each equation in the system when it's of size N.
- Write out the resulting matrix-vector equation $A\vec{x} = b$ for $N = 5$.
- Check if the matrix A is singular when $N = 5$ (you can use MATLAB to answer the question).

Problem 2

Indicate whether the following statements are TRUE or FALSE and motivate your answers clearly. To show a statement false, it is sufficient to give one counter example. To prove a statement true, provide a general proof.

(a) $\|A\|_F^2 = \text{trace}(A^T A)$

(b) $\text{cond}(\alpha A) = \alpha \text{cond}(A)$, where $\text{cond}(A)$ denotes the condition number of A and $\alpha > 0$.