Assignment 3 - Solutions

Due: October 16, in class
No late assignments accepted

Issued: October 9, 2013

Problem 1

(a) Find a basis for the column space and row space of matrix A given by

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(b) Construct a matrix B such that the null space of B is identical to the row space of A.

(a) We have the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

A basis for the column space of A can be found in various ways. (Finding a basis for the row space is similar.) One way is to perform Gaussian Elimination (GE) on A^T, then find the independent rows of A^T (corresponding to the independent columns of A).
Alternatively, one can also do a GE on A and then use the "subtle theorem" discussed in class. Specifically, GE gives us the matrix U. The independent columns in U are the columns with the nonzero pivots. Recall that the "subtle theorem" tells that the independent columns in A are the columns in those same positions. This would give us the choices of columns of A as a basis for the column space.
We can also make a dimensional argument to justify our choices of columns of A as a basis: we select independent columns of A and we select as many as the number of the rank of A, i.e. the dimension of the column space of A. Now we perform GE on A

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{row 2} - \text{ row 1}} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{row 3} - \text{ row 2}} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

From GE, we have the upper triangular matrix U. Since we have 2 nonzero rows (or 2 pivots), so the rank of A is 2. A basis for its column space can be its 2^{nd} and 4^{th} column:

$$\vec{a}_1 = [1\ 1\ 0]^T \quad \vec{a}_2 = [3\ 4\ 1]^T.$$

A basis for its row space can be the first and second row of A:

$$\vec{r}_1^T = [0\ 1\ 2\ 3\ 4] \quad \vec{r}_2^T = [0\ 1\ 2\ 4\ 6].$$

(b) We would need $B\vec{r}_1 = \vec{0}$ and $B\vec{r}_2 = \vec{0}$. B should have five columns, so $n = 5$. Its null space has dimension 2. For any matrix, the dimension of the nullspace equals $(n - r)$, where r is the rank of B. Here $n - r = 2$ and $n = 5$, hence B must be of rank $r = 3$. We might as well make B a 3×5 matrix. Note that the nullspace of a matrix and the row space of that same matrix are orthogonal to each other. So here we need B to be a 3×5 matrix with 3 independent rows that are orthogonal to $\vec{r}_1$, $\vec{r}_2$. For any row $\vec{b}^T$ of matrix B, we must have

$$\begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \end{bmatrix} \vec{b} = \vec{0}, \quad \text{so} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \end{bmatrix} \vec{b} = \vec{0}$$

We have to find the solution $\vec{b}$ of the above equation. These give two equations for 5 unknowns. As $\vec{r}_1$ and $\vec{r}_2$ are independent, we have $5 - 2 = 3$ degrees of freedom. In order to find the solution of $\begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \end{bmatrix} \vec{b} = \vec{0}$, we perform GE on matrix $\begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \end{bmatrix}$,

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \end{bmatrix} \xrightarrow{\text{row 2} - \text{ row 1}} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Let the solution $\vec{b} = [b_1 \ b_2 \ b_3 \ b_4 \ b_5]^T$. Then we have

$$b_4 + 2b_5 = 0, \text{ and } b_2 + 2b_3 + 3b_4 + 4b_5 = 0$$

The first equation gives $b_4 = -2b_5$. Substituting this into the second equation, we have $b_2 + 2b_3 - 2b_5 = 0$. So, $b_2 = -2b_3 + 2b_5$. Therefore, we have the solution in the form

$$\vec{b} = \begin{bmatrix} b_1 \\ -2b_3 + 2b_5 \\ b_3 \\ -2b_5 \\ b_5 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b_5 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

So, the solution space is of dimension 3. The solution space is spanned by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

We can choose three independent vectors from the solution space to form the 3 rows of matrix B. For example, we can let

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 \end{bmatrix}.$$

For each row $\vec{b}_i^T, i = 1, 2, 3$ of B, it is easy to check that $\begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \end{bmatrix} \vec{b}_i = 0$, which implies that $B\vec{r}_1 = B\vec{r}_2 = 0$. And B has rank 3 because the 1st, 3rd and last column of B makes the basis for R^3. Thus the null space of B is spanned by $\vec{r}_1$, $\vec{r}_2$, which is identical to the row space of A.

Problem 2

Let A be an $m \times n$ matrix with rank $r \leq \min\{m, n\}$. Depending on $m,\ n$ and r, a system $A\vec{x} = \vec{b}$ can have none, one, or infinitely many solutions.
For what choices of $m,\ n$ and r do each of the following cases hold? If no such $m,\ n$ and r can be found explain why not.

(a) $A\vec{x} = \vec{b}$ has no solutions, regardless of $\vec{b}$

(b) $A\vec{x} = \vec{b}$ has exactly 1 solution for any $\vec{b}$

(c) $A\vec{x} = \vec{b}$ has infinitely many solutions for any $\vec{b}$

(**Hint**: think of what conditions the column vectors and column space of A should satisfy.)

(a) $A\vec{x} = \vec{b}$ has no solutions, regardless of $\vec{b}$
Answer: Impossible. Since for $\vec{b} = 0$ we always have a solution $\vec{x} = 0$.

(b) $A\vec{x} = \vec{b}$ has exactly 1 solution for any $\vec{b}$
Answer: For any $\vec{b} \in \mathsf{R}^m$ there is exactly one solution to $A\vec{x} = \vec{b}$, this happens if and only if the columns of A form a basis of space R^m. That is, if and only if the column space of A is the full space R^m and the columns are independent. That is, if and only if $m = n = r$. (Hence if and only if A is square and nonsingular.)

(c) $A\vec{x} = \vec{b}$ has infinitely many solutions for any $\vec{b}$
Answer: For any $\vec{b} \in \mathsf{R}^m$ there are infinite solutions to $A\vec{x} = \vec{b}$. That is, for any $\vec{b} \in \mathsf{R}^m$ we can find a solution to $A\vec{x} = \vec{b}$ and add to this solution any element of the nullspace of A. Therefore we need the nullspace of A to have an infinite number of vectors in it. In other words, the nullspace dimension should be at least 1, i.e. $n - r \geq 1$. Since there is at least one solution for any $\vec{b}$, the columns of A must span the full space R^m. Note that the columns of A live in R^m. So we need $r = m$.
On the other hand, if the column space of A is the full space ($r = m$) and the columns are dependent ($r < n$), then for any $\vec{b} \in \mathsf{R}^m$ there are infinite solutions to $A\vec{x} = \vec{b}$. So the converse is also true.
Therefore a necessary and sufficient condition is that the column space of A is the full space ($m = r$) and the columns are dependent ($r < n$). That is, the statement is true if and only if $m = r < n$.

Remark
Note that the equation $A\vec{x} = \vec{b}$ could have

- exactly 1 solution for any $\vec{b}$ (part (b))
- infinity many solutions for any $\vec{b}$ (part (c))
- no solution for some $\vec{b}$ and has 1 solution for some $\vec{b'}$
- no solution for some $\vec{b}$ and has infinitely many solutions for some $\vec{b'}$

Exercise: Think about the conditions that make the 3$^{\text{rd}}$ and 4$^{\text{th}}$ cases true.

Problem 3

Consider a matrix product AB, where A is $m \times n$ and B is $n \times p$. Show that the column space of AB is contained in the column space of A. Give an example of matrices $A,\ B$ such that those two spaces are not identical.
Definition. A vector space U is *contained* in another vector space V (denoted as $U \subseteq V$) if every vector

$\vec{u} \in U$ ($\vec{u}$ in vector space U) is also in V.
Definition. We say that two vector spaces are *identical* (equal) if $U \subseteq V$ **and** $V \subseteq U$.
(e.g. V is identical to itself since $V \subseteq V$ and $V \subseteq V$.)

Let $B = [\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p]$ and $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$. We have to show that $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$. Since the column space of AB is the span of columns of AB and column space of A is the span of columns of A, it suffices to show that any column $j = 1, 2, \ldots, p$ of AB is in $\mathcal{R}(A)$. Consider that any column j of AB is $A\vec{b}_j$. We have to show that $A\vec{b}_j$ can be written as a linear combination of columns of A. This is indeed the case since any column j of AB is

$$A\vec{b}_j = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]\vec{b}_j = \sum_{k=1}^{n} b_{kj}\vec{a}_k,$$

which is a linear combination of columns of A. So $A\vec{b}_j \in \mathcal{R}(A)$ for all $j = 1, \ldots, p$. Since every column of AB is in $\mathcal{R}(A)$, so $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$.

In order to find an example, we consider that if the two spaces are not identical, then the column space of AB can only be strictly smaller than the column space of A. So when trying to construct examples, we want $\text{rank}(AB) < \text{rank}(A)$.
For example, we can take A to be $m \times m$ and full rank, while B is $m \times p$ with $p < m$. Since B has only p columns we know that $\text{rank}(AB) < \text{rank}(A)$.
As another example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 4

An $n \times n$ matrix A has a property that the elements in each of its rows sum to 1. Let P be any $n \times n$ permutation matrix. Prove that $(P - A)$ is singular.

$(P - A)$ is a matrix such that the elements in each of its rows sum to 0. This is because the permutation matrix P has exactly one entry 1 in each row and 0's elsewhere.
Consider that $(P-A)$ is singular if any only if $\mathcal{N}(P-A)$ contains a nonzero vector. Recall that $\mathcal{N}(P-A) = \left\{\vec{x} \in \mathsf{R}^n | (P-A)\vec{x} = \vec{0}\right\}$. Therefore, in order to show that $\mathcal{N}(P-A)$ contains at least one nonzero vector, we have to find some vector $\vec{v} \neq \vec{0}$ such that $(P-A)\vec{v} = \vec{0}$.
We explore the fact that the elements in each of the rows of $(P-A)$ sum to 0. Let c_{ij} be the (i,j)-th entry of matrix $(P-A)$. Then we have

$$\vec{0} = \begin{bmatrix} \sum_{j=1}^{n} c_{1j} \\ \sum_{j=1}^{n} c_{2j} \\ \vdots \\ \sum_{j=1}^{n} c_{nj} \end{bmatrix} = (P-A)\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Now let $\vec{v} = [1, 1, \ldots, 1]^T$ be a vector with 1s in all its entries. Then $(P-A)\vec{v} = \vec{0}$, which means that $\vec{v}$ is in the null space of $(P-A)$. Since $\vec{v} \neq \vec{0}$, we found a nonzero vector in $\mathcal{N}(P-A)$, that is, $\dim \mathcal{N}(P-A) > 0$. Thus, $(P-A)$ is singular.

Problem 5

Let V and W be 3 dimensional subspaces of R^5. Show that V and W must have at least one nonzero vector in common.

Solution I

Let bases of V and W be $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, respectively. Then span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is a subspace of R^5. Since the dimension of R^5 is 5, the six vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}_1, \vec{w}_2, \vec{w}_3$ cannot be independent. Thus, there exist constants $\alpha_i, \beta_i,\ i = 1, 2, 3$ (these constants cannot all be zeros) such that

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \beta_1\vec{w}_1 + \beta_2\vec{w}_2 + \beta_3\vec{w}_3 = \vec{0} \in \mathsf{R}^5.$$

Rearranging the terms, we can write

$$\vec{u} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = -(\beta_1\vec{w}_1 + \beta_2\vec{w}_2 + \beta_3\vec{w}_3) \in \mathsf{R}^5.$$

If $\vec{u}$ is zero, we must have $\alpha_i = 0,\ i = 1, 2, 3$, since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is independent, and $\beta_i = 0,\ i = 1, 2, 3$, since $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is independent. However, by the argument above, we cannot have all the constants to be zeros, thus $\vec{u} \neq 0$.

We have $\vec{u} \in V$, since $\vec{u} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3$. Similarly $\vec{u} \in W$, since $\vec{u} = -(\beta_1\vec{w}_1 + \beta_2\vec{w}_2 + \beta_3\vec{w}_3)$. Thus, we have a nonzero vector $\vec{u} \in \mathsf{R}^5$, which is both in V and in W.

Solution II

We use proof by contradiction. Let bases of V and W be $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, respectively. We assume that V and W are disjoint. Then any linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ cannot be expressed as a linear combination of $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$. Equivalently, the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{w}_1 + c_5\vec{w}_2 + c_6\vec{w}_3 = 0$$

can only have a trivial solution $\vec{c} = [c_1, c_2, c_3, c_4, c_5, c_6]^T = \vec{0}$.

Let $A = [\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}_1, \vec{w}_2, \vec{w}_3]$. So A is a 5×6 matrix. The linear equation $A\vec{x} = \vec{0}$ is a homogeneous underdetermined system, so it must have a non-trivial solution. This contradiction proves that V and W cannot be disjoint.

Problem 6

(a) The nonzero column vectors $\vec{u}$ and $\vec{v}$ have n elements. An $n \times n$ matrix A is given by $A = \vec{u}\vec{v}^T$ (**Note:** this is different from the innerproduct (also sometimes known as the dot product), which we would write as $\vec{v}^T\vec{u}$). Show that the rank of A is 1.

(b) Show that the converse is true. That is, if the rank of a matrix A is 1, then we can find two vectors $\vec{u}$ and $\vec{v}$, such that $A = \vec{u}\vec{v}^T$.

(a) Since $\vec{u}$ is nonzero, suppose $u_i \neq 0$ for some i. Similarly suppose $v_j \neq 0$ for some j. Then the (i, j)-th element of matrix A is $a_{i,j} = u_i v_j \neq 0$. Therefore, the rank of A is at least one: $\mathrm{r}(A) \geq 1$.

On the other hand, from Problem 3 we know that the column space of $A = \vec{u}\vec{v}^T$ is contained in the column space of $\vec{u}$, which has at most one dimension since $\vec{u}$ is a vector. Thus, $\mathrm{r}(A) \leq 1$.

Therefore, we have $\mathrm{r}(A) = 1$.

(b) To show the converse is true, observe that since the column space of A is of dimension one, its basis has only one vector. Let this vector be $\vec{u}$. Then every column of A is in the space spanned by the vector $\vec{u}$. That is, for every $i = 1, 2, \ldots, n$, we can find a real number $v_i \in \mathsf{R}$, such that $\vec{a}_i = v_i\vec{u}$. Then we have

$$A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n] = [v_1\vec{u}, v_2\vec{u}, \ldots, v_n\vec{u}] = \vec{u}[v_1, v_2, \ldots, v_n] = \vec{u}\vec{v}^T.$$