Assignment 7

Due: November 20, in class
No late assignments accepted

Issued: November 13, 2013

Important:

- Give complete answers: Do not only give mathematical formulae, but explain what you are doing. Conversely, do not leave out critical intermediate steps in mathematical derivations.
- Write your **name** as well as your **Sunet ID** on your assignment. **Please staple pages together.**
- Questions preceded by $\star$ are harder and/or more involved.

Problem 1

(a) From class we know that $P = A(A^TA)^{-1}A^T$ is an orthogonal projection matrix that projects onto the column space of A. Prove that P is symmetric and $P^2 = P^TP = P$.

(b) In general, a matrix which satisfies $P^2 = P$ is a projector. Show that if a projector matrix is symmetric, then it is an orthogonal projector.

(c) Show that regardless of the rank of A, the equation $A^TAx = A^Tb$ always has at least one solution.

Problem 2

We are interested in finding the fixed points (the points at which the time derivatives are zero) of the following system of equations:

$$\frac{dx_1}{dt} = x_1(a - bx_2)$$
$$\frac{dx_2}{dt} = -x_2(c - dx_1)$$

for $a = 3, b = 1, c = 2, d = 1$. We can use the Newton-Raphson method to find these fixed points, simply by setting the derivatives zero in the given system of equations.

(a) In the scalar case, Newton-Raphson breaks down at points at which the derivative of the nonlinear function is zero. In general, where can it break down for systems of nonlinear equations? For the system given above, find the troublesome points.

(b) Find the fixed points of the above system analytically.

(c) Find all fixed points using repeated application of the Newton-Raphson method. You will have to judiciously choose your starting points (but of course, you are not allowed to use the known roots as starting points!). You may use MATLAB to program the method if you like.

Problem 3

(a) Let A be an $n \times n$ symmetric matrix. Let $\vec{q}_i$ and $\vec{q}_j$ be the eigenvectors of A corresponding to the eigenvalues λ_i and λ_j respectively. Show that if $\lambda_i \neq \lambda_j$, then $\vec{q}_i$ and $\vec{q}_j$ are orthogonal.

(b) Let A be an $n \times n$ matrix. We say that A is **positive definite** if for any non-zero vector $\vec{x}$, the following inequality holds

$$\vec{x}^T A \vec{x} > 0.$$

Show that the eigenvalues of a positive definite matrix A are all positive.

(c) Let A be an $n \times n$ matrix. Show that

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (λ_i's do not have to be all different). You may assume that all eigenvalues of A are real numbers.

[Hint 1: One way to prove this is to use the fact that any square matrix with real eigenvalues can be decomposed in the following way (called Schur decomposition)

$$A = QRQ^T,$$

where R is an upper triangular matrix and Q is an orthogonal matrix.

Hint 2: You might find it useful to recall the properties of trace.]

Problem 4

(a) Find the best straight-line fit to the following measurements, and graph your solution:

$$y_1 = 2 \text{ at } t_1 = -1, \qquad y_2 = 0 \text{ at } t_2 = 0,$$
$$y_3 = -3 \text{ at } t_3 = 1, \qquad y_4 = -5 \text{ at } t_4 = 2.$$

What is the norm of the residual?

(b) Suppose that instead of a straight line, we fit the data above by the parabolic function:

$$y_i = a_2 x_i^2 + a_1 x_i + a_0.$$

Derive the over-determined system $A\vec{x} = \vec{b}$ to which least squares could be applied to find this quadratic fit.

(c) Let's look at the general problem of making n observations $y_i,\ i = 1, 2, \ldots, n$ at n different times t_i. You can extend what you did in the last two parts to find polynomial fits of degree k ($y_i = a_k t_i^k + a_{k-1} t_i^{k-1} + \cdots + a_1 t_i + a_0$) by using least squares. If $k < n - 1$, what would the over-determined system $A\vec{x} = \vec{b}$ look like for this general case?

(d) Prove that for $k = n - 1$, the system $A\vec{x} = \vec{b}$ will no longer be over-determined and we can find a unique fit by solving $A\vec{x} = \vec{b}$ instead of the normal equations.

(e) Consider the systems you solved for in part c and d. For $0 < k < n$, how does the norm of the residual change as we increase k?

Problem 5

(a) If $P^2 = P$, show that

$$e^P \approx I + 1.718P$$

(b) Convert the equation below to a matrix equation and then by using the exponential matrix find the solution in terms of $y(0)$ and $y(0)$:

$$y'' = 0$$

(c) Show that $e^{A+B} = e^A e^B$ is not generally true for matrices.