Workshop Problems
for weeks 1 - 4

A selection of these problems will be discussed in the workshops.
Worked solutions will be made available online after the workshop.

Problems indicated by * are harder

WEEK 1

This week, we discuss proofs and true/false questions to get you familiar and comfortable with such mathematical questions. Try the problems yourself before you attend the workshops.

If you find these problems hard, attend office hours early on, and ask the TAs or instructor to take you through a few more examples.

1. For each of the following problems, give complete and well motivated answers:

(a) Prove that $(AB)^T = B^T A^T$. First demonstrate this property for some 3×2 matrix A and 2×2 matrix B (you may use MATLAB), then prove it for the general $m \times n$ matrix A and $n \times l$ matrix B.

(b) Let $C = AA^T$, for some $m \times n$ matrix A. Show that C is symmetric.

(c) In general, the statement $AA^T = A^T A$ is false. Show this by providing a counter example.

(d) Prove that the difference of any matrix and its transpose is a skew-symmetric matrix.

2. A lower-triangle matrix is one which has zero elements above the main diagonal. A strictly lower-triangular matrix is a lower-triangular matrix whose diagonal elements are zero. For each of the following problems, give complete and well motivated answers:

(a) Prove that the product of two lower-triangular matrices is lower-triangular also.

(b) If the diagonal elements of two lower-triangular matrices are equal to one, what are the diagonal elements of their product?

*(c) Prove that the inverse of a lower-triangular matrix is a lower-triangular matrix. Hint: Use induction.

(d) Prove that the product of a strictly lower-triangular matrix S and a lower-triangular matrix T is strictly lower-triangular.

*(e) Let S be an n x n strictly lower-triangular matrix. Prove that $S^n = 0$. Hint: use induction.

(f) Let S be an n x n strictly lower triangular matrix. Show that

$$(I - S)^{-1} = I + S + S^2 + \ldots + S^{n-1}$$

*3. The n x n matrix A has column vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$. Construct a matrix B such that AB has the columns $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$, for which

$$\vec{u}_i = \vec{a}_i, \quad i \neq j,$$

$$\vec{u}_j = \sum_{k=1}^{j} \beta_k \vec{a}_k .$$

where j is some non-zero integer between 1 and n, that you may choose.

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WEEK 2

4. Solve, if possible, the following systems of equations by Gaussian elimination, and give at least one solution. Give the upper triangular form of the system of equations as part of your solution.

(a)

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 1 \end{aligned}$$

(b)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ 2x_1 + x_2 - x_3 &= 2 \\ 5x_2 - 3x_3 &= 0 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 - 3x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 3x_1 - 2x_3 &= 1 \end{aligned}$$

(d)

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 5 \\ 2x_1 + 4x_2 + x_3 &= 7 \\ 3x_1 + 16x_2 + x_3 &= 21 \end{aligned}$$

5.

(a) Construct a system of five equations in two unknowns with a unique solution.

(b) Construct two equations in five unknowns that are inconsistent.

6. Determine α so that the following system has a solution, and obtain one solution.

$$\begin{aligned} &2a - c = -1 \\ &a + 3b - 14c = \alpha - 25 \\ &a - b + 4c = 11 \end{aligned}$$

7. Using elementary row operations, construct the inverses of the following matrices, if possible.

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 8 & 5 & 0 \\ 2 & 1 & -1 \\ 6 & 4 & 3 \end{bmatrix}$$

8. The matrix A is given by

$$A = \begin{bmatrix} b_1 & c_1 & 0 & 0 \\ a_1 & b_2 & c_2 & 0 \\ 0 & a_2 & b_3 & c_3 \\ 0 & 0 & a_3 & b_4 \end{bmatrix}$$

Show that

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1/r_1 & 1 & 0 & 0 \\ 0 & a_2/r_2 & 1 & 0 \\ 0 & 0 & a_3/r_3 & 1 \end{bmatrix} \begin{bmatrix} r_1 & c_1 & 0 & 0 \\ 0 & r_2 & c_2 & 0 \\ 0 & 0 & r_3 & c_3 \\ 0 & 0 & 0 & r_4 \end{bmatrix}$$

where $r_1 = b_1, r_{i+1} = b_{i+1} - a_i c_i / r_i, \mathit{for}\, i \ge 1$.

Replace 4 by n and generalize this result.

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WEEK 3

9. Factor the given matrix A into LU, where L is a lower triangular matrix and U is an upper triangular matrix.

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & 0 & 4 \end{pmatrix}$$

*10. We define the norm of a matrix by

$$\|A\|^2 = \sum_i \sum_j a_{ij}^2$$

Use the Schwarz's inequality to prove that $\|Ax\| \le \|A\| \times \|x\|$

11.

(a) Let L be the set of vectors $\vec{x}$ in R^4 for which $x_1 + x_2 + x_3 = 0$. Find a basis for L. What is the dimension of L?

(b) Show that $[2,-1,1]^T$, $[1,2,-3]^T$ and $[3,1,-2]^T$, $[-1,3,-4]^T$ span the same subspace of R^3.

(c) Find a basis for the subspace of R^4 spanned by the vectors $[0,2,3,0]^T$, $[0,0,1,0]^T$, $[2,1,0,0]^T$, $[0,1,1,0]^T$.

(d) Let M be a subspace of R^4 consisting of all vectors x such that $x_1 - x_2 = 0$, $x_2 - x_3 = 0$, and $x_1 - 2x_2 + x_4 = 0$. Find a basis for M. What is the dimension of M?

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WEEK 4

12. For each of the following matrices, which we all call A, determine

(a) A basis for the row space of A.

(b) A basis for the column space of A.

(c) The rank of A.

(d) The general solution of $Ax = 0$.

(e) The dimension and a basis for the null space of A.

$$\begin{pmatrix} 2 & 10 & 6 \\ -1 & -5 & -3 \\ 3 & -7 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 8 & -10 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & -2 \end{pmatrix}$$

13. Show that the area of a triangle whose vertices are the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\frac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

if the vertices are numbered 1, 2, 3 in the counter-clockwise direction.

14.

(a) If a matrix is close to being singular, its condition number is very high. Give an example of a 4×4 matrix A for which this is the case.

(b) For such matrices, the solution to $A\vec{x} = \vec{b}$ computed using Gaussian Elimination may be quite erroneous, as we have seen in class. Can pivoting help? In other words, can you re-order the matrix A to reduce errors resulting from Gaussian Elimination? Motive your answer clearly.

(c) The determinant of a singular matrix is 0. Does this mean that if a matrix has a determinant that is almost 0, the matrix is close to singular and therefore has a very high condition number? You may use example(s) to motivate your answer.

15. Using Gram-Schmidt orthogonalization, create an orthonormal basis for R^2 from the basis vectors

$$a_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \text{ and } a_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$