Assignment 7

Due: November 20, in class
No late assignments accepted

Issued: November 13, 2013

Important:

- Give complete answers: Do not only give mathematical formulae, but explain what you are doing. Conversely, do not leave out critical intermediate steps in mathematical derivations.
- Write your **name** as well as your **Sunet ID** on your assignment. **Please staple pages together.**
- Questions preceded by $\star$ are harder and/or more involved.

Problem 1

For this problem we assume that eigenvalues and eigenvectors are all real valued.

(a) Let A be an $n \times n$ symmetric matrix. Let $\vec{q}_i$ and $\vec{q}_j$ be the eigenvectors of A corresponding to the eigenvalues λ_i and λ_j respectively. Show that if $\lambda_i \neq \lambda_j$, then $\vec{q}_i$ and $\vec{q}_j$ are orthogonal.

(b) Let A be an $n \times n$ matrix. We say that A is **positive definite** if for any non-zero vector $\vec{x}$, the following inequality holds

$$\vec{x}^T A \vec{x} > 0.$$

Show that the eigenvalues of a positive definite matrix A are all positive.

(c) $\star\star$ Let A be an $n \times n$ matrix. Show that

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (λ_i's do not have to be all different).

[Hint 1: One way to prove this is to use the fact that any square matrix with real eigenvalues can be decomposed in the following way (called Schur decomposition)

$$A = QRQ^T,$$

where R is an upper triangular matrix and Q is an orthogonal matrix.]

[Hint 2: The following property of trace might be useful: given two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$, the trace of their product, $\mathrm{tr}(AB)$, is *invariant under cyclic permutations*, i.e. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
Note that this implies $\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)$ for any matrices A, B, C with appropriately chosen dimension.]

Problem 2

We are interested in finding the fixed points (the points at which the time derivatives are zero) of the following system of equations:

$$\frac{dx_1}{dt} = x_1(a - bx_2)$$
$$\frac{dx_2}{dt} = -x_2(c - dx_1)$$

for $a = 3,\ b = 1,\ c = 2,\ d = 1$. We can use the Newton-Raphson method to find these fixed points, simply by setting the derivatives zero in the given system of equations.

(a) In the scalar case, Newton-Raphson breaks down at points at which the derivative of the nonlinear function is zero. In general, where can it break down for systems of nonlinear equations? For the system given above, find the troublesome points.

(b) Find the fixed points of the above system analytically.

(c) Find all fixed points using repeated application of the Newton-Raphson method. You will have to judiciously choose your starting points (but of course, you are not allowed to use the known roots as starting points!). You may use MATLAB to program the method if you like.

Problem 3

(a) If $P^2 = P$, show that

$$e^P \approx I + 1.718P$$

(b) Convert the equation below to a matrix equation and then by using the exponential matrix find the solution in terms of $y(0)$ and $y'(0)$:

$$y'' = 0.$$

(c) Show that $e^{A+B} = e^A e^B$ is not generally true for matrices.