Assignment 5

Due: November 6, in class
No late assignments accepted

Issued: October 30, 2013

Important:

- Give complete answers: Do not only give mathematical formulae, but explain what you are doing. Conversely, do not leave out critical intermediate steps in mathematical derivations.
- Write your **name** as well as your **Sunet ID** on your assignment. **Please staple pages together.** Points will be docked otherwise.
- Questions preceded by $\star$ are harder and/or more involved.

Problem 1

True or False? Motivate your answers clearly.

(a) For a non-singular $n \times n$ matrix A, the *Jacobi* method for solving $A\vec{x} = \vec{b}$ will always converge to the correct solution, if it converges.

(b) The *Gauss-Seidel* iteration used for solving $A\vec{x} = \vec{b}$ will always converge for any $n \times n$ invertible matrix A.

Problem 2

(a) Find an invertible 2×2 matrix for which the *Jacobi* method does not converge. Please find a matrix not already shown in class or the workshop.

(b) Find an invertible 10×10 non-diagonal matrix for which the *Jacobi* method converges very quickly. Please find a matrix not already shown in class or the workshop.

Problem 3

In this assignment, we'll be making much use of the 1-norm. We define the 1-norm of a vector $\vec{x}$ as $\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$, i.e. the sum of the magnitudes of the entries. The 1-norm of an $m \times n$ matrix A is defined as $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$, i.e. the largest column sum. It can be shown that for these norms, $\|A\vec{x}\|_1 \le \|A\|_1 \|\vec{x}\|_1$ (we will not prove this here, but you can use it as given, whenever needed). Compute the 1-norm of the following matrices and vectors

(a) $\vec{x} = (1, 2, 3, \ldots, n)^T$.

(b) $\vec{x} = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$.

(c) αI where I is the $n \times n$ identity matrix.

(d) $J - I$ where J is the $n \times n$ matrix filled with 1s and I is the $n \times n$ identity.

Problem 4

In the workshop, we will discuss page rank computations. The linear system is of the form $(I - \alpha P)\vec{x} = \vec{v}$. Here α is the fraction of a page's rank that it propagates to neighbors at each step and each entry in $\vec{v}$ is the amount of rank we give to each page initially. For this problem, we set $\alpha = 0.85$, and all elements v_i of the vector $\vec{v}$ equal to $v_i = 1/n$, where n is the number of total pages in the internet domain we are investigating. The matrices I and P are $n \times n$ matrices. For this problem (and also in general) we do not allow pages to link to themselves. Thus the diagonal elements of the matrix P are all 0.

(a) As we will prove in a later part of this problem, the matrix $I - \alpha P$ is invertible, so a unique solution to the pagerank system exists. We'll try to find this solution using the Jacobi iteration. Give the algorithm for the Jacobi iteration for the page rank equation.

(b) Lets assume the answer is the vector $\vec{x}^*$. Analyze the distance between $\vec{x}^*$ and successive iterates of the Jacobi iteration, that is, find the relationship between $\|\vec{x}^{(k+1)} - \vec{x}^*\|_1$ and $\|\vec{x}^{(k)} - \vec{x}^*\|_1$. Note that we measure distance in terms of the 1-norm here. Your analysis must hold for any $n \times n$ page rank matrix P. Form your analysis, show that α must be chosen to be smaller than 1.
(Hint: first compute the 1-norm of P.)

(c) Now, show that the matrix $I - \alpha P$ is invertible for any $n \times n$ page rank matrix P.
(Hint: don't forget that we have $0 < \alpha < 1$.)