Assignment 5 - Solutions

Due: November 6, in class
No late assignments accepted

Issued: October 30, 2013

Problem 1

True or False? Motivate your answers clearly.

(a) For a non-singular $n \times n$ matrix A, the *Jacobi* method for solving $A\vec{x} = \vec{b}$ will always converge to the correct solution, if it converges.

(b) The *Gauss-Seidel* iteration used for solving $A\vec{x} = \vec{b}$ will always converge for any $n \times n$ invertible matrix A.

(a) True.
Let's look at the general equation for a stationary iteration based on matrix splitting:

$$M\vec{x}^{(k+1)} = N\vec{x}^{(k)} + \vec{b}, \quad M - N = A$$

In *Jacobi* method, M would be the diagonal of A. If the iteration converges, then $\vec{x}^{(k+1)}$ and $\vec{x}^{(k)}$ will both converge to an $\vec{x}^*$. Then $M\vec{x}^{(k+1)} = N\vec{x}^{(k)} + \vec{b}$ leads to $M\vec{x}^* = N\vec{x}^* + \vec{b}$, or $(M - N)\vec{x}^* = \vec{b}$. And because $M - N = A$, we have $A\vec{x}^* = \vec{b}$.

(b) False.
We give a counter example here.

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

with exact solution $\vec{x}^* = [1\ 1\ 1]^T$. *Gauss-Seidel* gives $M\vec{x}^{(k+1)} = N\vec{x}^{(k)} + \vec{b}$, with

$$M = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

We get

$$M^{-1}N = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 4 & 2 \\ 0 & 8 & 4 \end{bmatrix}, \quad M^{-1}\vec{b} = \begin{bmatrix} -1 \\ -5 \\ -11 \end{bmatrix}$$

Noticing that the Frobenius norm of $M^{-1}N$ is about 10.2, which is larger than 1, we may suspect that the iteration will diverge.
Starting with $\vec{x}^{(0)} = \vec{b}$, we quickly diverge:

$$\vec{x}^{(1)} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 4 & 2 \\ 0 & 8 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \\ -11 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 17 \end{bmatrix}$$

$$\vec{x}^{(2)} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 4 & 2 \\ 0 & 8 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ 17 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \\ -11 \end{bmatrix} = \begin{bmatrix} 17 \\ 65 \\ 129 \end{bmatrix}$$

Problem 2

(a) Find an invertible 2×2 matrix for which the *Jacobi* method does not converge. Please find a matrix not already shown in class or the workshop.

(b) Find an invertible 10×10 non-diagonal matrix for which the *Jacobi* method converges very quickly. Please find a matrix not already shown in class or the workshop.

(a) Take a diagonally inferior matrix (as opposed to diagonally dominant). For example, we would take:

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

To show it does not converge, we can look at the iteration matrix $M^{-1}N$. For *Jacobi* we have

$$M^{-1}N = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

It is easy to see that the Frobenius norm of this amplification matrix is larger than 1, and we expect trouble. We can set up an iteration to show that this iteration does not converge, as long as we don not take the initial guess to be equal to the exact solution to $A\vec{x} = \vec{b}$. For example, we can try to solve for $\vec{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, which has an exact solution $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We get as iterates (starting with $\vec{x}^{(0)}$ equal to $\vec{b}$).

$$\vec{x}^{(1)} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

$$\vec{x}^{(2)} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 6 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \end{bmatrix}, \quad \text{etc.}$$

which is clearly diverging.

(b) We expect that for a matrix that is strongly diagonally dominant, the *Jacobi* method would converge very quickly. For example, we take

$$A = \begin{bmatrix} 100 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 100 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 100 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 100 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 100 \end{bmatrix}_{10\times 10}$$

For this matrix, the amplification matrix is

$$M^{-1}N = \begin{bmatrix} 0.01 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0.01 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0.01 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0.01 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0.01 \end{bmatrix}_{10\times 10} \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix}_{10\times 10}$$

$$= \begin{bmatrix} 0 & -0.01 & 0 & \cdots & 0 & 0 \\ -0.01 & 0 & -0.01 & \cdots & 0 & 0 \\ 0 & -0.01 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -0.01 \\ 0 & 0 & 0 & \cdots & -0.01 & 0 \end{bmatrix}_{10\times 10}$$

which has a Frobenius norm much smaller than 1 (actually the Frobenius norm is around 0.04). Hence, the convergence is guaranteed and will be quite fast.

Problem 3

In this assignment, we'll be making much use of the 1-norm. We define the 1-norm of a vector $\vec{x}$ as $\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$, i.e. the sum of the magnitudes of the entries. The 1-norm of an $m \times n$ matrix A is defined as $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$, i.e. the largest column sum. It can be shown that for these norms, $\|A\vec{x}\|_1 \le \|A\|_1 \|\vec{x}\|_1$ (we will not prove this here, but you can use it as given, whenever needed). Compute the 1-norm of the following matrices and vectors

(a) $\vec{x} = (1, 2, 3, \ldots, n)^T$.

(b) $\vec{x} = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$.

(c) αI where I is the $n \times n$ identity matrix.

(d) $J - I$ where J is the $n \times n$ matrix filled with 1s and I is the $n \times n$ identity.

(a) $\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

(b) $\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} \frac{1}{n} = n\left(\frac{1}{n}\right) = 1$.

(c) Every column has sum exactly equal to $|\alpha|$, thus the max column sum is $|\alpha|$.

(d) Every column sums to $n - 1$, so the max column sum is $n - 1$.

Problem 4

In the workshop, we will discuss page rank computations. The linear system is of the form $(I - \alpha P)\vec{x} = \vec{v}$. Here α is the fraction of a page's rank that it propagates ot neighbors at each step and each entry in $\vec{v}$ is the amount of rank we give to each page initially. For this problem, we set $\alpha = 0.85$, and all elements v_i of the vector $\vec{v}$ equal to $v_i = 1/n$, where n is the number of total pages in the internet domain we are investigating. The matrices I and P are $n \times n$ matrices. For this problem (and also in general) we do not allow pages to link to themselves. Thus the diagonal elements of the matrix P are all 0.

(a) As we will prove in a later part of this problem, the matrix $I - \alpha P$ is invertible, so a unique solution to the pagerank system exists. We'll try to find this solution using the Jacobi iteration. Give the algorithm for the Jacobi iteration for the page rank equation.

(b) Lets assume the answer is the vector $\vec{x}^*$. Analyze the distance between $\vec{x}^*$ and successive iterates of the Jacobi iteration, that is, find the relationship between $\|\vec{x}^{(k+1)} - \vec{x}^*\|_1$ and $\|\vec{x}^{(k)} - \vec{x}^*\|_1$. Note that we measure distance in terms of the 1-norm here. Your analysis must hold for any $n \times n$ page rank matrix P. Form your analysis, show that α must be chosen to be smaller than 1.
(Hint: first compute the 1-norm of P.)

(c) Now, show that the matrix $I - \alpha P$ is invertible for any $n \times n$ page rank matrix P.
(Hint: don't forget that we have $0 < \alpha < 1$.)

(a) We split the matrix $I - \alpha P = M - N$, where M is the diagonal to $I - \alpha P$. Remember that the diagonal of P has zeros only, so the diagonal of $I - \alpha P$ is exactly $M = I$. And hence $N = M - (I - \alpha P) = \alpha P$. The Jacobi iteration is the following:

$k = 0$
Choose $\vec{x}^{(0)}$ and compute $\vec{r}^{(0)} = \vec{v} - (I - \alpha P)\vec{x}^{(0)}$
while $k <$ *step_limit* and $\|\vec{r}^{(k)}\| \geq \epsilon \left(\|I - \alpha P\|\|\vec{x}^{(k)}\| + \|\vec{v}\|\right)$ **do**
  Compute $\vec{x}^{(k+1)} = \alpha P \vec{x}^{(k)} + \vec{v}$
  Compute the residual $\vec{r}^{(k+1)} = \vec{v} - (I - \alpha P)\vec{x}^{(k+1)}$
  Compute $\|\vec{x}^{(k+1)}\|$ and $\|\vec{r}^{(k+1)}\|$
  $k = k + 1$
end while

Note that every element of the new iterate can be computed independently of the others (the algorithm is parallelizable).

(b)

$$\begin{aligned}\|\vec{x}^{(k+1)} - \vec{x}^*\|_1 &= \|\alpha P\vec{x}^{(k)} + \vec{v} - (\alpha P\vec{x}^* + \vec{v})\|_1 \\ &= |\alpha|\|P(\vec{x}^{(k)} - \vec{x}^*)\|_1 \\ &\leq |\alpha|\|P\|_1\|\vec{x}^{(k)} - \vec{x}^*\|_1\end{aligned}$$

Remember that P has non-negative components, and the elements on each column of P add up to 1, indicating $\|P\|_1 = 1$. Also note that $\alpha > 0$.

$$\|\vec{x}^{(k+1)} - \vec{x}^*\|_1 \leq \alpha\|\vec{x}^{(k)} - \vec{x}^*\|_1$$

Note that $0 < \alpha < 1$, the distance between $\vec{x}^*$ and successive iterates decrease to 0.

(b) Consider any vector $\vec{x}$ such that $(I - \alpha P)\vec{x} = \vec{0}$, so $\vec{x} = \alpha P\vec{x}$. By taking the 1-norm on either side, we find

$$\|\vec{x}\|_1 = \|\alpha P\vec{x}\|_1 \leq |\alpha|\|P\|_1\|\vec{x}\|_1 = \alpha\|\vec{x}\|_1 \implies (1-\alpha)\|\vec{x}\|_1 \leq 0$$

Note that $\|P\|_1 = 1$ and $\alpha > 0$. But with $\alpha < 1$, so $1 - \alpha > 0$. This implies $\|\vec{x}\|_1 \leq 0$. Since $\|\vec{y}\|_1 \geq 0$ for any $\vec{y}$, this means that $\|\vec{x}\|_1 = 0$. This can only be true if $\vec{x} = \vec{0}$. We showed that $\mathcal{N}(I - \alpha P) = \{\vec{0}\}$. So, the matrix $I - \alpha P$ is non-singular.