Assignment 8

Due: December 4, in class
No late assignments accepted

Issued: November 20, 2013

Important:

- Give complete answers: Do not only give mathematical formulae, but explain what you are doing. Conversely, do not leave out critical intermediate steps in mathematical derivations.
- Write your **name** as well as your **Sunet ID** on your assignment. **Please staple pages together.**
- Questions preceded by $\star$ are harder and/or more involved.
- **Include code with your assignment**
- Comment any graphs and plots on the same page as the graph or plot itself.

Problem 1

Consider the symmetric matrix

$$A = \begin{bmatrix} 9 & -7 \\ -7 & 17 \end{bmatrix}$$

(a) Compute the eigenvalues and eigenvectors of A. Do this by hand.

(b) Show that the eigenvectors of A are orthogonal.

Problem 2

True or False? If true, prove the statement. If false, you need only provide a counterexample.

(a) If the $n \times n$ matrix B is formed from the $n \times n$ matrix A by swapping two rows of A, then B and A have the same eigenvalues.

(b) Any invertible $n \times n$ matrix A can be diagonalized.

(c) A singular matrix must have repeated eigenvalues.

(d) If the $n \times n$ matrices A and B are diagonalizable, so is the matrix AB.

(e) Let A be an $n \times n$ matrix.

   (i) The eigenvalues of A and A^T are the same.

   (ii) The eigenvalues and eigenvectors of $A^T A$ and AA^T are the same.

Problem 3

A rabbit population r and a wolf population w are related according to the following system of differential equations:

$$\begin{aligned}\frac{dr}{dt} &= 5r - 2w \\ \frac{dw}{dt} &= r + 2w\end{aligned}$$

Here t denotes time.

(a) If initially (at $t = 0$) there were 100 rabbits and 50 wolves, find the numbers of rabbits and wolves as a function of time t.

(b) Design a matrix A for the above system such that the populations coverge to a finite non-zero limit as t goes to infinity.

Problem 4

Heat Equation. We look at the matrix

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

(a) Find the eigenvalues of A and the corresponding eigenvectors. A is symmetric, so the eigenvectors should be orthogonal. Check that this is the case.

(b) Give the algorithm of the power method that can be used to find the largest eigenvalue (in absolute value) of A.

(c) Execute this algorithm, either by hand or with matlab. As initial vector take one of the unit vectors, produce a figure showing the computed approximations as a function of iteration step. Repeat with an initial vector equal to an eigenvector corresponding to either of the two smallest eigenvalues. Observe that the algorithm still converges. but slower than before (round-off error accumulation helps in this case as we discussed in class).

   Note: The matrix is well-conditioned and the round-off error will not accumulate particularly fast. Make sure you force the iteration to run for a while (we have found at least 60 iterations works) because you are only testing the relative error in successive iterates of the eigenvalues.

(d) Instead of looking at this 3×3 matrix, take 10×10 (or larger if you like) matrix with the same tridiagonal structure (-2 on the main diagonal and 1 on the subdiagonals). Find all eigenvalues of this larger matrix using the QR iterations. Check your answers against the eigenvalues computed by matlab using the `eig` command. Again, motivate your convergence criterion.

(e) Since the heat equation discretization was an example of numerically solving a differential equation, the plotting of solutions was rather important. Since we just computed eigenvectors, we might be interested to see what the eigenvectors look like when plotted. Consider 102×102 discretization for the heat equation so that the matrix A is 100×100. Use the `eig` command in matlab to obtain the eigenvectors $\vec{v}^{(1)}, \ldots, \vec{v}^{(100)}$ for $-A$ (this is $-A$ so there are positive 2s on the diagonal). For $j = 1, \ldots, 5$, plot $\sqrt{101}\vec{v}^{(j)}$ as we have done in previous homeworks (remember the boundary values).

Now verify for yourself that the functions $u_j(t) = \sqrt{2}\sin(j\pi t)$ for $j = 1, 2, 3, \ldots$ satisfy the differential equation,

$$-\frac{d^2 u_j}{dt^2} = \lambda u_j, \quad \lambda = (j\pi)^2, \quad u_j(0) = u_j(1) = 0, \quad \int_0^1 u_j(t)^2 dt = 1.$$

How do the eigenvectors $\vec{v}^{(j)}$ of $-A$ compare to the "eigenfunctions" $u_j(t)$ of the differential equation? What significance does the factor $\sqrt{101}$ have when plotting $\sqrt{101}\vec{v}^{(j)}$ (think about a Riemann sum approximation of the above integral)?

Problem 5

In this problem we consider a small mass-spring system, consisting of 2 masses that are connected to each other and to fixed walls by three identical springs as shown in the figure.

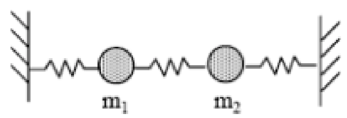

Figure 1: Mass-Spring System.

At time $t = 0$, we gtive the first mass m_1 a small displacement. As a result the mass-spring system will start to move in an oscillatory fashion. If the masses are both equal to 1, the system of equation that describes the displacement of the masses is given by

$$\frac{d^2\vec{u}}{dt^2} = A\vec{u},$$

where

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

with u_1 and u_2 are the displacements of m_1 and m_2, respectively, and

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

We take the initial conditions (we need two of them for a second order differential equation) to be

$$\vec{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \frac{d\vec{u}}{dt}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

(a) Find the matrices Y and Λ such that $A = Y\Lambda Y^{-1}$, with Y orthogonal, and Λ diagonal.

(b) Using this decomposition of A, show that we can transform the system of differential equations to

$$\frac{d^2\vec{z}}{dt^2} = \Lambda\vec{z}, \quad \vec{z} = Y^T\vec{u}.$$

The system for $\vec{z}$ is now "decoupled" so you can solve each component individually since they do not depend on each other. Solve the system of equations for $\vec{z}$ and from it find the displacements u_1 as a function of time t.

(c) If the masses are not equal, the system of differential equations describing the motion of the mass-spring system is instead given by

$$M\frac{d^2\vec{u}}{dt^2} = A\vec{u}, \quad M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}. \tag{1}$$

We guess solution will be of the form $\vec{u}(t) = e^{i\omega t}\vec{v}$, where ω and $\vec{v}$ are quantities determined by M and A. Plug in $\vec{u}(t) = e^{i\omega t}\vec{v}$ and show that finding ω and $\vec{v}$ corresponds to the so-called "generalized eigenvalue problem"

$$A\vec{v} = \lambda M\vec{v},$$

with $\lambda = (i\omega)^2$.

Now set $m_1 = 1$ and $m_2 = 2$. The characteristic polynomial for the generalized eigenvalue problem is given by $\det(A - \lambda M) = 0$. Solve the characteristic polynomial to find the two generalized eigenvalues λ_1, λ_2 (or if you wish, the two values of ω since $\lambda = (i\omega)^2$) and the two generalized eigenvectors $\vec{v}_1, \vec{v}_2$. Then give the solution to the differential equation in (1).