Some Comments on Assignment 5

Issued: November 12, 2013

Problem 1 and 2

For the splitting methods that we have discussed in class, we split $A = M - N$. In order to find a solution of $A\vec{x} = \vec{b}$, we update

$$\vec{x}^{(k+1)} = M^{-1}N\vec{x}^{(k)} + M^{-1}\vec{b} \tag{1}$$

Suppose $\vec{x}^*$ is a solution to $A\vec{x} = \vec{b}$. Let $\vec{e}^{(k)} = \vec{x}^{(k)} - \vec{x}^*$ be the error at the k^{th} iteration. Then we have shown in class that

$$\vec{e}^{(k)} = G\vec{e}^{(k-1)} = G^k\vec{e}^{(0)}, \tag{2}$$

where $G = M^{-1}N$. Taking the norm, we get

$$\|\vec{e}^{(k)}\| \leq \|G\|^k\|\vec{e}^{(0)}\| \tag{3}$$

From equation (3), you can and cannot conclude the followings:

- We know that if $\|G\| < 1$, then we can guarantee the convergence of the method because $\|G\|^k \to 0$ as $k \to \infty$. So $\vec{e}^{(k)} \to 0$ as $k \to \infty$. So for problem 2(b), it suffices to show that $\|G\| \ll 1$. And you can guarantee that the method converges quickly ($\|\vec{e}^{(k)}\|$ goes to 0 very quickly).
- If $\|G\| > 1$, we CANNOT conclude anything about convergence of the method. This is because equation (3) does not tell us whether $\|e^{(k)}\|$ is increasing or decreasing as k increases. In particular, if $\|G\| > 1$, you may suspect that the method might diverge, but you cannot conclude that it will surely diverge. So, for problem 1(b) and 2(a), you will have to show the first few iterations to make sure that the methods (Jacobi or Guass-Seidel) diverge for your matrix A.
- Suppose you know that a method converges, you CANNOT conclude that $\|G\| < 1$.

Note that everything listed above is what equation (3) implies or does not imply. Some of them might be true. However, you will need some other ways to prove it.

One other comment is that when you look at the first few iterations $\vec{x}^{(0)}, \vec{x}^{(1)}, \vec{x}^{(2)}, \ldots$, if the entries of $\vec{x}^{(k)}$ are increasing, it is not clear that the method diverges. You will have to compare your $x^{(k)}$ to the exact solution. This is because it might be the case that your $x^{(k)}$ are increasing toward the exact solution which means the method converges.

Problem 3

Problem 4