Assignment 1 - Background review

Due: October 2, in class
No late assignments accepted

Issued: September 26, 2013

Important:

- Give complete answers: Do not only give mathematical formulae, but explain what you are doing. Conversely, do not leave out critical intermediate steps in mathematical derivations. Point values are in parentheses.
- Write your **name** as well as your **Sunet ID** on your assignment. **Please staple pages together.** Points will be docked otherwise.
- Questions preceded by * are harder and/or more involved.

Problem 1 (10)

Indicate whether the following statements are TRUE or FALSE and motivate your answers clearly. To show a statement is false, it is sufficient to give one counterexample. If a statement is true, provide a general proof.

(a) If $A^2 + A = I$ then $A^{-1} = I + A$

(b) If all diagonal entries of A are zero, then A is singular.

Problem 2 (10)

The product of two $n \times n$ lower triangular matrices is again lower triangular (all its entries above the main diagonal are zero). Prove it in general and confirm this with a 3-by-3 example.

Problem 3 (10)

If $A = A^T$ and $B = B^T$, which of these matrices are certainly symmetric? Justify your answer.

(a) $A^2 - B^2$

(b) $(A + B)(A - B)$

(c) ABA

(d) $ABAB$

Problem 4 (10)

A *skew-symmetric* matrix is a matrix that satisfies $A^T = -A$. Prove that if A is a skew-symmetric matrix, then for any vector x, we must have $x^T Ax = 0$.

Problem 5 (10)

Suppose A is an invertible matrix, and you exchange its first two rows to create a new matrix B. Is the new matrix B necessarily invertible? If so, how could you find B^{-1} from A^{-1}? If not, why not?

Problem 6 (10)

Let A be an invertible $n \times n$ matrix. Prove that A^m is also invertible and that

$$(A^m)^{-1} = \left(A^{-1}\right)^m$$

for $m = 1, 2, 3, \ldots$

Problem 7 (10)

Let A be a 2×2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $a_{11} \neq 0$ and let $\alpha = a_{21}/a_{11}$. Show that A can be factored into a product of the form

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & b \end{pmatrix}$$

What is the value of b?

Problem 8* (20: 10 points each part)

(a) Show that if the $n \times n$ matrices A and B are invertible, and if the matrix $A + B$ is also invertible, then the matrix $B^{-1} + A^{-1}$ is also invertible.

(b) Assume that C is a skew-symmetric matrix and that D is a matrix defined as

$$D = (I + C)(I - C)^{-1}$$

Prove that $D^T D = DD^T = I$.

Problem 9* (10)

Given an $n \times n$ matrix A with column vectors $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}$, construct a matrix B such that the matrix AB has the columns $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}$ with the following properties:

(i) $\vec{u_i} = \vec{a_i}, \quad i \neq j$

(ii) $\vec{u_j} = \sum_{k=1}^{j} \alpha_k \vec{a_k}$,

where j is a fixed integer with $1 \leq j \leq n$ and $j \neq 0$.