Midterm 2013

75 minutes, closed book

In class, October 30, 11 - 12:15am

Important:

- Start by writing your name on this sheet, as well as you SUNETID. Please write clearly.
- By writing your name, you agree to work by the Stanford Honor Code.
- Write your answers on these sheets.
- Always *show and explain your work to receive full credit* for each problem.
- Problems indicated with $*$ are harder.

Problem 1

(40 points total, 10 points each)
We are given the LU decomposition of a matrix A as

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. What is the rank $r(A)$ of matrix A? Explain

b. Find a basis for the null space of A. Show your work

c. Find a basis for the column space of A. Show your work

d. For $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, do solution(s) to $A\vec{x} = \vec{b}$ exist? Clearly motivate your answer. Note that it is not necessary to compute the solution(s).

Problem 2

(30 points total, 10 points each) TRUE/FALSE
Indicate whether the following statements are TRUE or FALSE and motivate your answer clearly.

a. If the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$ in $\Re^n$ span a subspace S of $\Re^n$, the dimension of S is m.

b*. If for the $m \times n$ matrix A, the equation $A\vec{x} = \vec{b}$ always has at least one solution for every $\vec{b}$, then the only solution to $A^T\vec{y} = \vec{0}$ is $\vec{y} = \vec{0}$.

c*. If the m vectors $\vec{x}_i, i = 1, \dots m$ are orthogonal, they are also independent. Note that here $\vec{x}_i \in \Re^n$ and $m < n$.

Problem 3

(30 points total, and two cookie problems)

a. (i) (10 points) Using Gram-Schmidt orthogonalization, create an orthonormal basis for $\Re^2$ from the vectors

$$\vec{a}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad \text{and} \quad \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(ii) (10 points) Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & 1 & -1 \end{bmatrix},$$

where Q is a 2×2 matrix and R is 2×3.

b. (10 points) The system $A\vec{x} = \vec{b}$, with $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ has the solution $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The vector $\vec{b}$ is now perturbed. The new vector is equal to $\vec{b} + \delta\vec{b}$ with $\delta\vec{b} = \begin{bmatrix} 0.001 \\ 0 \end{bmatrix}$. Estimate the maximum perturbation we can expect in the solution $\vec{x}$, measured in the vector 2-norm. For matrix-norms, you may use the Frobenius norm.

c**. (Question for a cookie (no extra credit for midterm). Each student who gives the correct answers will get a Coupa Cafe Cookie Coupon)

We are given an $n \times n$ matrix A. Show that the 2-norm and the Frobenius norm of the matrix satisfy that

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2.$$

d. (Question for a cookie (no extra credit for midterm). The student who gives the most creative answer will get a tin of cookies)

When my son was little, I used to ask him to come up with an exam bonus cookie question. Here's my favorite, recycled from 8 years ago: "How fast does the nullspace shuttle fly?".