CME 200 Workshop 4

October 18, 2013

1. Questions and solutions

1. Let A be the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & -2 \end{bmatrix}$$

Find (i) its rank, (ii) a basis for its row space, and (iii) a basis for its column space.
We row-reduce this matrix to obtain

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We deduce (i) A has rank 2, (ii) the first two rows of A form a basis for its row space

$$\mathrm{row}(A) = \mathrm{span}\left(\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}\right),$$

and (iii) the first and third columns of A form a basis for its column space

$$\mathcal{R}(A) = \mathrm{span}\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}\right\}\right).$$

We know (ii) and (iii) because the first two rows of the reduced matrix are pivot rows and the first and third columns of the reduced matrix are pivot columns.
The first two rows of the row-reduced matrix also form a basis for the row space (of A). Row reduction takes linear combinations of the rows, so the row space is preserved by row reducing. The same cannot be said of the column space, so the first and third columns of the row-reduced matrix are NOT a basis for the columns space.

2. Find two matrices A_1 and A_2 such that (i) $\mathrm{cond}(A_1) \approx 10^6$ and $\det(A_1) = 1$ and (ii) $\mathrm{cond}(A_2) = 2$ and $\det(A_2) = 10^{-6}$.

Recall (from the first assignment) that the condition number is invariant under scalar multiplication, but the determinant is not. To find A_1 and A_2, we first find two matrices with the correct condition number, and then multiply these two matrices by a scalar to obtain the desired determinant. Two matrices with the correct condition numbers are

$$\begin{bmatrix} 10^6 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We multiply both matrices by 10^{-3} to obtain A_1 and A_2:

$$A_1 = 10^{-3} \begin{bmatrix} 10^6 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A_2 = 10^{-3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3. Find an orthonormal basis for $\mathbf{R}^2$ with Gram-Schmidt orthogonalization starting with the vectors

$$a_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The first vector in our orthonormal basis is simply a_1 normalized:

$$q_1 = \frac{a_1}{\|a_1\|} = \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}.$$

To compute the second vector in our orthonormal basis, we first remove the component of a_2 along q_1:

$$w_2 = a_2 - (a_2^T q_1) q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{28}{25} \\ \frac{21}{25} \end{bmatrix}.$$

We then normalize q_2 to obtain the second vector in our orthonormal basis:

$$q_2 = \frac{w_1}{\|w_2\|} = \frac{5}{7} \begin{bmatrix} \frac{28}{25} \\ \frac{21}{25} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}.$$

2. A geometric interpretation of the determinant

Let $u \in \mathbf{R}^2$ and $v \in \mathbf{R}^2$ be the columns of a 2×2 matrix A, *i.e.*

$$A = \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}.$$

$|\det(A)|$ is the area of the parallelogram whose vertices are the origin, u, v, and $u + v$ (see Figure 1).

Proof. If we take the line segment between the origin and u to be the base of the parallelogram, then the area of the parallelogram is

$$\text{area} = (\text{base})(\text{height}) = (\|u\|)(\|v\| \sin(\theta)),$$

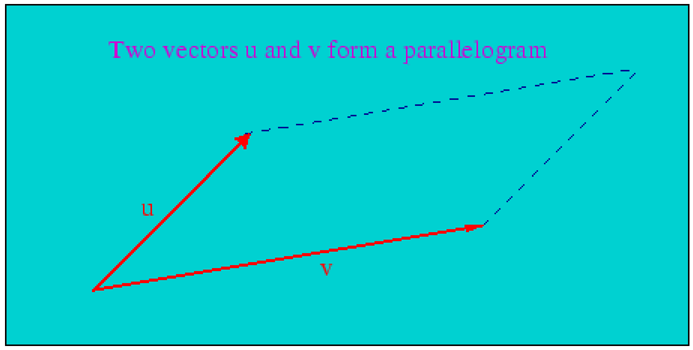

Fig 1: The parallelogram formed by two vectors u and v.

where θ is the angle between u and v. We square both sides to obtain

$$\begin{aligned}
\text{area}^2 &= \|u\|^2 \, \|v\|^2 \sin^2(\theta) \\
&= \|u\|^2 \, \|v\|^2 \, (1 - \cos^2(\theta)) \\
&= \|u\|^2 \, \|v\|^2 \left(1 - \left(\frac{u^T v}{\|u\| \, \|v\|} \right)^2 \right) \\
&= \|u\|^2 \, \|v\|^2 - (u^T v)^2.
\end{aligned}$$

We expand this expression in terms of the components of u and v to obtain

$$\begin{aligned}
\text{area}^2 &= (u_1^2 + u_2^2)(v_1^2 + v_2^2) - (u_1 v_1 + u_2 v_2)^2 \\
&= u_1^2 v_2^2 + u_2^2 v_1^2 - 2 u_1 v_1 u_2 v_2 \\
&= (u_1 v_1 - u_2 v_2)^2 \\
&= \det(A)^2.
\end{aligned}$$

We take the square root of both sides to deduce area $= |\det(A)|$. $\square$