Assignment 4

Due: October 23, in class
No late assignments accepted

Issued: October 16, 2013

Important:

- Give complete answers: Do not only give mathematical formulae, but explain what you are doing. Conversely, do not leave out critical intermediate steps in mathematical derivations.
- Write your **name** as well as your **Sunet ID** on your assignment. **Please staple pages together.**
- Questions preceded by $\star$ are harder and/or more involved.
- **Include code with your assignment.**
- Comment any graphs and plots on the same page as the graph or plot itself.

Problem 1

In assignment 2, we discretized the 1-dimensional heat equation with Dirichlet boundary conditions:

$$\frac{d^2T}{dx^2} = 0, 0 \leq x \leq 1$$
$$T(0) = 0, T(1) = 2$$

The discretization leads to the matrix-vector equation $At = b$, with

$$A = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -2 \end{pmatrix} \tag{1}$$

Here A is an $(N-1) \times (N-1)$ matrix.

a. (10 pts) Find the LU factorization of A for $N = 10$ using `Matlab`. Is `Matlab` using any pivoting to find the LU decomposition? Find the inverse of A also. As you can see the inverse of A is a dense matrix. **Note:** The attractive sparsity of A has been lost when computing its inverse, but L and U are sparse. Generally speaking, banded matrices have L and U with similar band structure. Naturally we then prefer to use the L and U matrices to compute the solution, and not the inverse. Finding L and U matrices with least "fill-in" ("fill-in" refers to nonzeros appearing at locations in the matrix where A has a zero element) is an active research area, and generally involves sophisticated matrix re-ordering algorithms.

b. (10 pts) Compute the determinants of L and U for $N = 1000$ using `Matlab`'s determinant command. Why does the fact that they are both nonzero imply that A is non-singular? How could you have computed these determinants really quickly yourself without `Matlab`'s determinant command?

Problem 2

a. (i) (10 pts) Use `Matlab` command `A=rand(4)` to generate a random 4-by-4 matrix and then use the function `qr` to find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$. Compute the determinants of A, Q and R.

(ii) (15 pts) Set `A=rand(n)` for at least 5 different n's in `Matlab` for computing the determinant of Q where Q is the orthogonal matrix generated by `qr(A)`. What do you observe about the determinants of the matrices Q? Show, with a mathematical proof, that the determinant of any orthogonal matrix is either 1 or -1.

(iii) (10 pts) For a square $n \times n$ matrix matrix B, suppose there is an orthogonal matrix Q and an upper-triangular matrix R such that $B = QR$. Show that if a vector x is a linear combination of the first k column vectors of B with $k \leq n$, then it can also be expressed as a linear combination of the first k columns of Q.

b*. (i) (10 pts) Assume $\{\vec{v_1}, \vec{v_2} \cdots \vec{v_n}\}$ is an orthonormal basis of $\mathbb{R}^n$. Suppose there exists a unit vector $\vec{u}$ such that $\vec{u}^T \vec{v_k} = 0$ for all $k = 2, 3 \cdots n$, show that $\vec{u} = \vec{v_1}$ or $\vec{u} = -\vec{v_1}$.

(ii) (10 pts) Prove that if C is non-singular and $C = QR$, where Q is an orthogonal matrix and R is an upper-triangular matrix with diagonal elements all positive, then the Q and R are unique.
Hint: Use proof by contradiction.

Problem 3

In class we have introduced the LU decomposition of A, where L is unit-lower triangular, in that it has ones along the diagonal, and U is upper triangular. However, in the case of symmetric matrices, such as the discretization matrix, it is possible to decompose A as LDL^T, where L is still unit-lower triangular and D is diagonal. This decomposition clearly shows off the symmetric nature of A.

a. (10 pts) Find the LDL^T decomposition for the matrix given in **Problem 1**. Show that L is bidiagonal. How do D and L relate to the matrix U in the LU decomposition of A?
Hint: Think about how D and L^T relate to U.
Note: Computing LDL^T this way does not work out for any symmetric matrix, it only happens to work for this matrix in particular.

b*. (i) (10 pts) To solve $A\vec{x} = \vec{b}$ we can exploit this new decomposition. We get $LDL^T\vec{x} = \vec{b}$ which we can now break into three parts: Solve $L\vec{y} = \vec{b}$ using forward substitution, now solve $D\vec{z} = \vec{y}$, and then solve $L^T\vec{x} = \vec{z}$ using back substitution. Write a `Matlab` code that does exactly this for arbitrary N for the A in **Problem 1**.

(ii) (5 pts) Solve a system of the same form as **Problem 1** for A of size 10 and of size 1000 with $\vec{b}$ having all zeros except 2 as the last entry in both cases, and verify the correctness of your solution using `Matlab`'s `A\b` operator and the `norm` command.