Assignment 3

Due: October 16, in class
No late assignments accepted

Issued: October 9, 2013

Important:

- Give complete answers: Do not only give mathematical formulae, but explain what you are doing. Conversely, do not leave out critical intermediate steps in mathematical derivations.
- Write your **name** as well as your **Sunet ID** on your assignment. **Please staple pages together.** Points will be docked otherwise.
- Questions preceded by $\star$ are harder and/or more involved.

Problem 1

(a) Find a basis for the column space and row space of matrix A given by

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(b) Construct a matrix B such that the null space of B is identical to the row space of A.

Problem 2

Let A be an $m \times n$ matrix with rank $r \leq \min\{m, n\}$. Depending on m, n and r, a system $A\vec{x} = \vec{b}$ can have none, one, or infinitely many solutions.
For what choices of m, n and r do each of the following cases hold? If no such m, n and r can be found explain why not.

(a) $A\vec{x} = \vec{b}$ has no solutions, regardless of $\vec{b}$

(b) $A\vec{x} = \vec{b}$ has exactly 1 solution for any $\vec{b}$

(c) $A\vec{x} = \vec{b}$ has infinitely many solutions for any $\vec{b}$

(**Hint**: think of what conditions the column vectors and column space of A should satisfy.)

Problem 3

Consider a matrix product AB, where A is $m \times n$ and B is $n \times p$. Show that the column space of AB is contained in the column space of A. Give an example of matrices A, B such that those two spaces are not identical.

Definition. A vector space U is *contained* in another vector space V (denoted as $U \subseteq V$) if every vector $\vec{u} \in U$ ($\vec{u}$ in vector space U) is also in V.

Definition. We say that two vector spaces are *identical* (equal) if $U \subseteq V$ **and** $V \subseteq U$.
(e.g. V is identical to itself since $V \subseteq V$ and $V \subseteq V$.)

Problem 4

An $n \times n$ matrix A has a property that the elements in each of its rows sum to 1. Let P be any $n \times n$ permutation matrix. Prove that $(P - A)$ is singular.

Problem 5

Let V and W be 3 dimensional subspaces of $\mathbb{R}^5$. Show that V and W must have at least one nonzero vector in common.

Problem 6

(a) The nonzero column vectors $\vec{u}$ and $\vec{v}$ have n elements. An $n \times n$ matrix A is given by $A = \vec{u}\vec{v}^T$ (**Note:** this is different from the innerproduct (also sometimes known as the dot product), which we would write as $\vec{v}^T\vec{u}$). Show that the rank of A is 1.

(b) Show that the converse is true. That is, if the rank of a matrix A is 1, then we can find two vectors $\vec{u}$ and $\vec{v}$, such that $A = \vec{u}\vec{v}^T$.