CME 200 Workshop 3

October 11, 2013

1. All bases of a subspace must have the same number of elements

Proof. Suppose not. Let $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_l\}$ be two bases of a subspace $S \subset \mathbf{R}^n$. Without loss of generality, we assume $l > k$.[1] $\mathcal{U}$ is a basis of S and $\{v_1, \ldots, v_l\} \subset S$, so we can express $v_1, \ldots, v_l$ in terms of $u_1, \ldots, u_k$:

$$v_1 = \sum_{i=1}^{k} \alpha_{i,1} u_i$$

$$\vdots$$

$$v_l = \sum_{i=1}^{k} \alpha_{i,l} u_i$$

In matrix notation:

$$V = UA.$$

$u_1, \ldots, u_k$ and $v_1, \ldots, v_{k+l}$ are the columns of $U \in \mathbf{R}^{n \times k}$ and $V \in \mathbf{R}^{n \times (k+l)}$, and $\alpha_{i,j}$ are the entries in $A \in \mathbf{R}^{k \times (k+l)}$. A has more columns than rows, so the columns of A are linearly dependent. This means there is a $z \in \mathbf{R}^{k+l}$ such that $Az = 0$. But this also means

$$Vz = UAz = 0.$$

This is a contradiction: we assumed $v_1, \ldots, v_{k+l}$ are linearly independent. $\square$

2. Verifying two bases span the same subspace

To show two sets A and B are the same, we must show $A \subset B$ and $B \subset A$. In this case, we must check

$$\operatorname{span}(\{u_1, \ldots, u_k\}) \subset \operatorname{span}(\{v_1, \ldots, v_k\})$$
$$\operatorname{span}(\{v_1, \ldots, v_k\}) \subset \operatorname{span}(\{u_1, \ldots, u_k\}).$$

To show

$$\operatorname{span}(\{u_1, \ldots, u_k\}) \subset \operatorname{span}(\{v_1, \ldots, v_k\}),$$

it suffices to show

$$\{u_1, \ldots, u_k\} \subset \operatorname{span}(\{v_1, \ldots, v_k\}).$$

[1] We simply define $\{u_1, \ldots, u_k\}$ to be the set with the smaller number of elements.

Why? Because if $u_1, \ldots, u_k$ are linear combinations of $v_1, \ldots, v_k$, then any linear combination of $u_1, \ldots, u_k$ is also a linear combination of $v_1, \ldots, v_k$:

$$\sum_{i=1}^{k} \alpha_i u_i = \sum_{i=1}^{k} \alpha_i \left(\sum_{j=1}^{k} \beta_{i,j} v_j \right) = \sum_{i=1}^{k} \left(\alpha_i \sum_{j=1}^{k} \beta_{i,j} \right) v_j.$$

We can use the same trick (just swap u_i and v_i) to show

$$\operatorname{span}(\{v_1, \ldots, v_k\}) \subset \operatorname{span}(\{u_1, \ldots, u_k\}).$$

3. Effect on x because of a perturbation to b and A

We are interested in how the solution x to the linear system $Ax = b$ changes when we perturb both A and b. Consider the perturbed linear system

$$(A + \Delta A)y = b + \Delta b.$$

The perturbed solution y can be expressed as $x + \Delta x$ for some Δx, so we have

$$(A + \Delta A)(x + \Delta x) = b + \Delta b.$$

Assuming $A + \Delta A$ is invertible, we solve for Δx to obtain

$$\Delta x = (A + \Delta A)^{-1}(b + \Delta b - (A + \Delta A)x) \tag{3.1}$$
$$= (A + \Delta A)^{-1}(b + \Delta b) - x. \tag{3.2}$$

If ΔA is small, then $(A + \Delta A)^{-1} \approx A^{-1} - A^{-1}\Delta A A^{-1}$. We substitute this expression into (3.2)

$$\begin{aligned} \Delta x &\approx (A^{-1} - A^{-1}\Delta A A^{-1})(b + \Delta b) - x \\ &= A^{-1}\Delta b - A^{-1}\Delta A A^{-1}(b + \Delta b) \end{aligned}$$

and drop the second-order term $A^{-1}\Delta A A^{-1}\Delta b$

$$\begin{aligned} \Delta x &\approx A^{-1}\Delta b - A^{-1}\Delta A A^{-1} b \\ &= A^{-1}(\Delta b - \Delta A x). \end{aligned}$$

We take norms to obtain

$$\|\Delta x\| \le \left\|A^{-1}\right\| (\|\Delta b\| + \|\Delta A\| \, \|x\|).$$

We divide both sides by $\|x\|$

$$\frac{\|\Delta x\|}{\|x\|} \le \left\|A^{-1}\right\| \left(\frac{\|\Delta b\|}{\|x\|} + \|\Delta A\| \right)$$

and use the fact that $\frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|}$

$$\begin{aligned} \frac{\|\Delta x\|}{\|x\|} &\le \left\|A^{-1}\right\| \left(\|A\| \frac{\|\Delta b\|}{\|b\|} + \|\Delta A\| \right) \\ &= \kappa(A) \left(\frac{\|\Delta b\|}{\|b\|} + \frac{\|\Delta A\|}{\|A\|} \right). \end{aligned}$$

4. Questions

1. Let U and V be subspaces of $\mathbf{R}^n$.
   (a) The *intersection* of U and V is the set
   $$U \cap V := \{x \in \mathbf{R}^n \mid x \in U \text{ and } x \in V\}.$$
   Is $U \cap V$ a subspace for any U and V?
   Yes, since U and V are both closed under addition, the sum of any two vectors in both U and V must remain in both U and V. U and V are also both closed under scalar multiplication, so a scalar multiple of a vector in both U and V must also remain in both U and V.
   (b) The *union* of U and V is the set
   $$U \cup V := \{x \in \mathbf{R}^n \mid x \in U \text{ or } x \in V\}.$$
   Is $U \cup V$ a subspace for any U and V?
   No. For a counterexample, let $U \subset \mathbf{R}^2$ be the x-axis and $V \subset \mathbf{R}^2$ be the y-axis. The union of these two sets is closed under scalar multiplication, but not addition.
2. Suppose $u_1, \dots, u_k$ form a basis for a subspace S and $x \in S$. Then there are $\alpha_1, \dots, \alpha_k$ such that
   $$x = \sum_{i=1}^{k} \alpha_i u_i$$
   Are $\alpha_1, \dots, \alpha_k$ unique? Can you form x in terms of $u_1, \dots, u_k$ with a different set of coefficients?
   No the coefficients must be unique. If the coefficients are not unique, then $u_1, \dots, u_k$ are linearly dependent. To see this, suppose there are two sets of coefficients that give x:
   $$x = \sum_{i=1}^{k} \alpha_i u_i = \sum_{i=1}^{k} \beta_i u_i.$$
   If we subtract the second equation from the first, we obtain
   $$0 = \sum_{i=1}^{k} (\alpha_i - \beta_i) u_i.$$
   If $\alpha \neq \beta_i$ for all i, then we have found a nontrivial linear combination of the u_i's that yield zero. This contradicts the assumption that $u_1, \dots, u_k$ are linearly independent.
3. Problem 11 in Workshop Problems for Week 3

(a) Let L be the set of vectors x in $\mathbf{R}^4$ for which

$$x_1 + x_2 + x_3 = 0. \tag{4.1}$$

Find a basis for L. What is the dimension of L?

The dimension of L is 3. Any 3 linearly independent vectors that satisfy (4.1) would be a basis. An example is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(b) Find a basis for the subspace of $\mathbf{R}^4$ spanned by the vectors

$$\begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

To find a basis, we attempt to row reduce the matrix whose rows are these four vectors:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

We note that this matrix is already almost upper triangular (except the last row), and the last row is clearly a linear combination of the second and third rows. Hence this matrix has row-rank three and the first three rows form a basis for this subspace of $\mathbf{R}^4$.