Applications to solve Initial and Boundary value problems involving Ordinary Differential Equations

Module 2

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Basics Revision:

Definition:

$$\phi(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\boxed{L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt}$$

u.v generalized rule:

$$uv_1 + u'v_2 + u''v_3 + \cdots$$

Laplace transforms of standard functions.

1	$1/s$	$\sin at$	$\frac{a}{s^2+a^2}$
t	$1/s^2$	$\cos at$	$\frac{s}{s^2+a^2}$
t^2	$2/s^3$	$\sinh at$	$\frac{a}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$ or $\frac{\Gamma(n+1)}{s^{n+1}}$	$\cosh at$	$\frac{s}{s^2-a^2}$
e^{at}	$1/(s-a)$	$\text{erf}\sqrt{t}$	$\frac{1}{\sqrt{s}\sqrt{s+1}}$
e^{-at}	$1/(s+a)$		

Inverse Laplace transform by Partial Fractions:

- Linear Factors
- Quadratic Factors
- Repeated Factors

Convolution Theorem:

$$L^{-1}\{\phi(s)\} = \int_0^t \phi_1(u) \cdot \phi_2(t-u)\, du$$

Applications of Laplace Transform

$$L(y'') = s^2\bar{y} - s\,y(0) - y'(0)$$

$$L(y') = s\,\bar{y} - y(0)$$

$$L(y''') = s^3\bar{y} - s^2 y - s\,y'(0) - y''(0)$$

Example 1 : Solve using Laplace transforms $3\dfrac{dy}{dt} + 2y = e^{3t}$, $y = 1$ at $t = 0$.

Sol. : Taking Laplace transforms of both sides,

$$3L(y') + 2L(y) = L(e^{3t})$$

$$3[s\bar{y} - y(0)] + 2\bar{y} = \frac{1}{s-3}. \quad \text{But } y(0) = 1$$

$$\therefore \quad 3[s\bar{y} - 1] + 2\bar{y} = \frac{1}{s-3}$$

$$\therefore \quad (3s+2)\bar{y} = \frac{1}{s-3} + 3 = \frac{3s-8}{s-3}$$

$$\bar{y} = \frac{3s-8}{(s-3)(3s+2)}$$

$$\therefore \quad \bar{y} = \frac{30}{11}\cdot\frac{1}{3s+2} + \frac{1}{11}\cdot\frac{1}{(s-3)} \qquad \text{[By partial fractions]}$$

$$\therefore \quad \bar{y} = \frac{10}{11}\cdot\frac{1}{s+(2/3)} + \frac{1}{11}\cdot\frac{1}{s-3}$$

Taking inverse Laplace transforms

$$y = \frac{10}{11}L^{-1}\left[\frac{1}{s+(2/3)}\right] + \frac{1}{11}L^{-1}\left[\frac{1}{s-3}\right] = \frac{10}{11}e^{-(2/3)t} + \frac{1}{11}e^{3t}.$$

Example 2:

Example 7 : Solve $(D^2 - D - 2)\,y = 20 \sin 2t$, with $y(0) = 1$ and $y'(0) = 2$. **(M.U. 2005, 13)**

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') - L(y') - 2L(y) = 20\,L(\sin 2t)$$

But $L(y') = s\,\bar{y} - y(0) = s\,\bar{y} - 1$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - s - 2$

$\therefore$ The equation becomes,

$$(s^2\bar{y} - s - 2) - (s\bar{y} - 1) - 2\bar{y} = 20\,\frac{2}{s^2+4}$$

$$\therefore\ (s^2 - s - 2)\,\bar{y} = \frac{40}{s^2+4} + s + 1 = \frac{s^3 + s^2 + 4s + 44}{s^2+4}$$

$$\therefore\ \bar{y} = \frac{s^3 + s^2 + 4s + 44}{(s^2+4)(s^2 - s - 2)} \qquad \therefore\ \bar{y} = -\frac{8}{3}\cdot\frac{1}{s+1} + \frac{8}{3}\cdot\frac{1}{s-2} + \frac{s-6}{s^2+4}$$

Taking inverse Laplace transform,

$$y = \frac{-8}{3}L^{-1}\left(\frac{1}{s+1}\right) + \frac{8}{3}L^{-1}\left(\frac{1}{s-2}\right) + L^{-1}\frac{s}{s^2+4} - 6\,L^{-1}\frac{1}{s^2+4}$$

$$= -\frac{8}{3}e^{-t} + \frac{8}{3}e^{-2t} + \cos 2t - 3\sin 2t.$$

By Partial Fractions

$$\frac{s^3+s^2+4s+44}{(s^2+4)(s-2)(s+1)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+4}$$

$$\therefore s^3+s^2+4s+44 = A(s-2)(s^2+4) + B(s+1)(s^2+4) + (Cs+D)(s+1)(s-2)$$

$$s^3+s^2+4s+44 = A(s^3+4s-2s^2-8) + B(s^3+4s+s^2+4) + (Cs+D)(s^2-2s+s-2)$$

$$s^3+s^2+4s+44 = As^3+4As-2As^2-8A+Bs^3+4Bs+Bs^2+4B + Cs^3-2Cs^2+Cs^2-2Cs+Ds^2-2Ds+Ds-2D$$

$$s^3+s^2+4s+44 = (A+B+C)s^3 + (-2A+B-C+D)s^2 + (4A+4B-2C-D)s + (-8A+4B-2D)$$

$A+B+C=1$ — (1)

$-2A+B-C+D=1$ — (2)

$4A+4B-2C-D=4$ — (3)

$-8A+4B-2D=44$

$-4A+2B-D=22$ (Divide by 2) — (4)

Subtract eq^n (1) $\times$ 4 and (3)

$$\begin{array}{r} 4A+4B+4C=4 \\ -\quad 4A+4B-2C-D=4 \\ \hline 6C+D=0 \end{array} \qquad \boxed{\therefore D=-6C}$$

Substitute $D=-6C$ in eq^{ns} (4) & (2)

$-4A+2B+6C=22 \quad \therefore -2A+B+3C=11$ — (5)

$-2A+B-7C=1$ — (6)

Subtract (5) & (6)

$10C=10 \qquad \boxed{\therefore C=1}$

$\because D=-6C \qquad \boxed{\therefore D=-6}$

Substitute value of C & D in eq^{ns} (3) & (4) and Add

$$\begin{array}{r} 4A+4B=0 \\ -4A+2B=16 \\ \hline 6B=16 \end{array} \qquad \boxed{\therefore B=8/3}$$

From (1), $A+B+C=1 \quad \therefore A=1-8/3-1$

$\boxed{\therefore A=-8/3}$

Example 3

$$L(y'') = s^2\bar{y} - sy(0) - y'(0)$$

$$L(y') = s\bar{y} - y(0)$$

$$L(y''') = s^3\bar{y} - s^2y - sy'(0) - y''(0)$$

Example 11 : Using Laplace transform solve $\frac{d^2y}{dt^2} + y = t,\ y(0) = 1,\ y'(0) = 0.$

(M.U. 1995, 2009, 13, 14, 15, 16)

Sol. : Let $\bar{y}$ be the Laplace transform of y *i.e.* let $L(y) = \bar{y}$.

Taking Laplace transform of both sides,

$$L(y'') + L(y) = L(t)$$

Now, $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - s$ and $L(t) = \frac{1}{s^2}$.

$\therefore$ The equation (1) becomes

$$s^2\bar{y} - s + \bar{y} = \frac{1}{s^2} \qquad \therefore\ s^2\bar{y} + \bar{y} = s + \frac{1}{s^2} = \frac{s^3+1}{s^2}$$

Example 3

$$\therefore\ (s^2+1)\,\bar{y} = \frac{s^3+1}{s^2} \qquad \therefore\ \bar{y} = \frac{s^3+1}{s^2(s^2+1)}$$

Let $$\frac{s^3+1}{s^2(s^2+1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{cs+d}{s^2+1}$$

$$\therefore\ s^3+1 = a\,s(s^2+1) + b(s^2+1) + (cs+d)\,s^2$$

$$= (a+c)\,s^3 + (b+d)\,s^2 + as + b$$

Equating like powers of s,

$$a+c=1,\ b+d=0,\ a=0,\ b=1 \qquad \therefore\ a=0,\ b=1,\ c=1,\ d=-1$$

$$\therefore\ \bar{y} = \frac{1}{s^2} + \frac{s-1}{s^2+1} = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

Taking inverse Laplace transform

$$y = L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{s}{s^2+1}\right) - L^{2}\left(\frac{1}{s^2+1}\right) = t + \cos t - \sin t.$$