Module 2

APPLICATIONS TO SOLVE INITIAL AND BOUNDARY VALUE PROBLEMS INVOLVING ORDINARY DIFFERENTIAL EQUATIONS

Applications of Laplace Transforms

In this article we shall see how Laplace transforms can be profitably used to solve differential equations. For the use of Laplace transforms to solve differential equations we need the following results. If Laplace transform of y *i.e.* $L(y)$ is denoted by $\bar{y}$ then [See (1), (2), (3) of § 14, page 1-52] we have

$$\boxed{\begin{aligned} L(y') &= s\bar{y} - y(0) \\ L(y'') &= s^2\bar{y} - sy(0) - y'(0) \\ L(y''') &= s^3\bar{y} - s^2y - sy'(0) - y''(0) \end{aligned}} \quad \text{.................... (1)}$$

Example 1 : Solve using Laplace transforms $3\dfrac{dy}{dt} + 2y = e^{3t}$, $y = 1$ at $t = 0$. **(M.U. 2002)**

Sol. : Taking Laplace transforms of both sides,

$$3L(y') + 2L(y) = L(e^{3t})$$

$$3[s\bar{y} - y(0)] + 2\bar{y} = \frac{1}{s-3}. \quad \text{But } y(0) = 1$$

$$\therefore \quad 3[s\bar{y} - 1] + 2\bar{y} = \frac{1}{s-3}$$

$$\therefore \quad (3s+2)\bar{y} = \frac{1}{s-3} + 3 = \frac{3s-8}{s-3}$$

$$\bar{y} = \frac{3s-8}{(s-3)(3s+2)}$$

$$\therefore \quad \bar{y} = \frac{30}{11}\cdot\frac{1}{3s+2} + \frac{1}{11}\cdot\frac{1}{(s-3)} \qquad \text{[By partial fractions]}$$

$$\therefore \quad \bar{y} = \frac{10}{11}\cdot\frac{1}{s+(2/3)} + \frac{1}{11}\cdot\frac{1}{s-3}$$

Taking inverse Laplace transforms

$$y = \frac{10}{11}L^{-1}\left[\frac{1}{s+(2/3)}\right] + \frac{1}{11}L^{-1}\left[\frac{1}{s-3}\right] = \frac{10}{11}e^{-(2/3)t} + \frac{1}{11}e^{3t}.$$

Example 2 : Solve using Laplace transforms $\dfrac{dy}{dx} + 3y = 2 + e^{-t}$, if $y = 1$ at $t = 0$.

Sol. : Taking Laplace transforms of both sides,

$$L(y') + 3L(y) = L(2) + L(e^{-t})$$

$$s\bar{y} - y(0) + 3\bar{y} = 2\frac{1}{s} + \frac{1}{s+1}. \quad \text{But } y(0) = 1$$

$$\therefore \quad (s+3)\bar{y} = \frac{2}{s} + \frac{1}{s+1} + 1 = \frac{s^2+4s+2}{s(s+1)}$$

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$$\therefore \quad \bar{y} = \frac{s^2 + 4s + 2}{s(s+1)(s+3)}$$

$\therefore$ By partial fractions $\quad \bar{y} = \frac{2}{3} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s+1} - \frac{1}{6} \cdot \frac{1}{s+3}$

Taking inverse Laplace transforms, $\quad y = \frac{2}{3} + \frac{1}{2} \cdot e^{-t} - \frac{1}{6} e^{-3t}$

Example 3 : Solve using Laplace transforms $R\frac{dQ}{dt} + \frac{Q}{C} = V$, $Q = 0$ when $t = 0$. **(M.U. 2016)**

Sol. : Taking Laplace transforms of both sides,

$$RL(Q') + \frac{1}{C} L(Q) = VL(1)$$

$$R\left[s\bar{Q} - Q(0)\right] + \frac{1}{C}\bar{Q} = \frac{V}{s}. \quad \text{But } Q(0) = 0$$

$$\therefore \left(Rs + \frac{1}{C}\right)\bar{Q} = \frac{V}{s} \quad \therefore \bar{Q} = \frac{V}{s} \cdot \frac{1}{[Rs + (1/C)]} = \frac{VC}{s(RCs+1)}$$

$$\therefore \bar{Q} = VC\, L^{-1}\frac{1}{s} - VRC^2\, L^{-1}\frac{1}{RCs+1} = VC\, L^{-1}\frac{1}{s} - VC\, L^{-1}\left[\frac{1}{s + 1/(RC)}\right]$$

Taking inverse Laplace transforms,

$$Q = VC - VC\, e^{-t/RC} = VC\left(1 - e^{-t/RC}\right)$$

Example 4 : Solve using Laplace transforms $L\frac{dI}{dt} + RI = E\, e^{-at}$, where $I(0) = 0$.

Sol. : Taking Laplace transforms of both sides,

$$L\,L[I'] + RL(I) = EL(e^{-at})$$

$$L[s\bar{I} - I(0)] + R\bar{I} = \frac{E}{s+a}. \quad \text{But } I(0) = 0$$

$$\therefore (Ls + R)\bar{I} = \frac{E}{s+a}$$

$$\therefore \bar{I} = \frac{E}{(s+a)(Ls+R)} \quad \text{[By partial fractions]}$$

$$= \frac{E}{R - La} \cdot \frac{1}{s+a} - \frac{EL}{R - La} \cdot \frac{1}{Ls + R}$$

Taking inverse Laplace transforms

$$I = \frac{E}{R - La} e^{-at} - \frac{E}{R - La} e^{-(R/L)t} = \frac{E}{R - La}\left[e^{-at} - e^{-(R/L)t}\right]$$

Example 5 : Using Laplace transform solve the following differential equation.

$$\frac{dx}{dt} + x = \sin \omega t, \ x(0) = 2. \qquad \textbf{(M.U. 1993)}$$

Sol. : Let $\bar{x}$ be the Laplace transform of x *i.e.* let $L(x) = \bar{x}$.

Taking Laplace transform of both sides of the given equation, ... (1)

$$L(x') + L(x) = L \sin(\omega t)$$

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But $L(x') = s(\bar{x}) - x(0) = s\bar{x} - 2$

Hence, the equation (1) becomes

$$s\bar{x} - 2 + \bar{x} = \frac{\omega}{s^2 + \omega^2} \quad \therefore \ (s+1)\bar{x} = 2 + \frac{\omega}{s^2 + \omega^2}$$

$$\therefore \ (s+1)\bar{x} = \frac{2s^2 + 2\omega^2 + \omega}{s^2 + \omega^2}$$

$$\therefore \ \bar{x} = \frac{2s^2 + 2\omega^2 + \omega}{(s^2 + \omega^2)(s+1)} \qquad \text{[By partial fractions]}$$

$$= \frac{2\omega^2 + \omega + 2}{1 + \omega^2} \cdot \frac{1}{s+1} + \frac{-\omega s + \omega}{(1+\omega^2)(s^2+\omega^2)}$$

$$= \frac{2\omega^2 + \omega + 2}{1 + \omega^2} \cdot \frac{1}{s+1} - \frac{\omega}{1+\omega^2} \cdot \frac{s}{s^2+\omega^2} + \frac{\omega}{1+\omega^2} \cdot \frac{1}{s^2+\omega^2}$$

Taking inverse Laplace transforms,

$$x = \frac{2\omega^2 + \omega + 2}{1 + \omega^2} L^{-1}\left(\frac{1}{s+1}\right) - \frac{\omega}{1+\omega^2} \cdot L^{-1}\left(\frac{s}{s^2+\omega^2}\right) + \frac{\omega}{1+\omega^2} L^{-1}\left(\frac{1}{s^2+\omega^2}\right)$$

$$= \frac{2\omega^2 + \omega + 2}{1 + \omega^2} \cdot e^{-t} - \frac{\omega}{1+\omega^2} \cdot \cos\omega t + \frac{\omega}{1+\omega^2} \cdot \frac{1}{\omega} \sin\omega t$$

$$= \frac{1}{1+\omega^2}\left[(2\omega^2 + \omega + 2)e^{-t} - \omega\cos\omega t + \sin\omega t\right]$$

Example 6 : Solve $(D^2 - 3D + 2)y = 4e^{2t}$, with $y(0) = -3$ and $y'(0) = 5$. **(M.U. 2004, 07, 08, 14, 17, 18, 19)**

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') - 3L(y') + 2L(y) = 4L(e^{2t})$$

But $L(y') = s\bar{y} - y(0) = s\bar{y} + 3$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} + 3s - 5$

$\therefore$ The equation becomes,

$$(s^2\bar{y} + 3s - 5) - 3(s\bar{y} + 3) + 2\bar{y} = 4\frac{1}{s-2}$$

$$(s^2 - 3s + 2)\bar{y} = \frac{4}{s-2} + 14 - 3s = \frac{-3s^2 + 20s - 24}{s-2}$$

$$\therefore \ \bar{y} = \frac{-3s^2 + 20s - 24}{(s-2)(s^2 - 3s + 2)} = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$$

By partial fractions, $\bar{y} = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

Taking inverse Laplace transform,

$$y = -7L^{-1}\left(\frac{1}{s-1}\right) + 4L^{-1}\frac{1}{s-2} + 4L^{-1}\frac{1}{(s-2)^2}$$

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$$\therefore \quad y = -7e^{t} L^{-1}\frac{1}{s} + 4e^{2t} L^{-1}\frac{1}{s} + 4e^{2t} L^{-1}\frac{1}{s^2} = -7e^{t} + 4e^{2t} + 4te^{2t}$$

$\therefore$ The solution is $y = -7e^{t} + 4e^{2t} + 4te^{2t}$.

Example 7 : Solve $(D^2 - D - 2)\,y = 20\sin 2t$, with $y(0) = 1$ and $y'(0) = 2$. **(M.U. 2005, 13)**

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') - L(y') - 2L(y) = 20\,L(\sin 2t)$$

But $L(y') = s\bar{y} - y(0) = s\bar{y} - 1$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - s - 2$

$\therefore$ The equation becomes,

$$(s^2\bar{y} - s - 2) - (s\bar{y} - 1) - 2\bar{y} = 20\,\frac{2}{s^2+4}$$

$$\therefore \quad (s^2 - s - 2)\,\bar{y} = \frac{40}{s^2+4} + s + 1 = \frac{s^3 + s^2 + 4s + 44}{s^2+4}$$

$$\therefore \quad \bar{y} = \frac{s^3 + s^2 + 4s + 44}{(s^2+4)(s^2 - s - 2)} \qquad \therefore \quad \bar{y} = -\frac{8}{3}\cdot\frac{1}{s+1} + \frac{8}{3}\cdot\frac{1}{s-2} + \frac{s-6}{s^2+4}$$

Taking inverse Laplace transform,

$$y = \frac{-8}{3}L^{-1}\left(\frac{1}{s+1}\right) + \frac{8}{3}L^{-1}\left(\frac{1}{s-2}\right) + L^{-1}\frac{s}{s^2+4} - 6L^{-1}\frac{1}{s^2+4}$$

$$= -\frac{8}{3}e^{-t} + \frac{8}{3}e^{-2t} + \cos 2t - 3\sin 2t.$$

Example 8 : Using Laplace Transform solve $(D^2 + 3D + 2)\,y = e^{-2t}\sin t$, $y(0) = 0$, $y'(0) = 0$.

Sol. : Let $L(y) = \bar{y}$. Then taking the Laplace transform of both sides

$$L(y'') + 3L(y') + 2L(y) = L(e^{-2t}\sin t)$$

But $L(y') = s\bar{y} - y(0) = s\bar{y}$

and $L(y'') = s^2\bar{y} - s\,y(0) - y'(0) = s^2\bar{y}$

And $L(e^{-2t}\sin t) = \dfrac{1}{(s+2)^2 + 1}$

$\therefore$ The equation becomes,

$$s^2\bar{y} + 3s\bar{y} + 2\bar{y} = \frac{1}{s^2 + 4s + 5}$$

$$\therefore \quad (s^2 + 3s + 2)\,\bar{y} = \frac{1}{s^2 + 4s + 5} \qquad \therefore \quad \bar{y} = \frac{1}{(s^2 + 3s + 2)(s^2 + 4s + 5)}$$

Let $\dfrac{1}{(s^2 + 3s + 2)(s^2 + 4s + 5)} = \dfrac{a}{s+1} + \dfrac{b}{s+2} + \dfrac{cs + d}{s^2 + 4s + 5}$

Solving this, we get $a = \dfrac{1}{2}$, $b = -1$, $c = \dfrac{1}{2}$, $d = \dfrac{1}{2}$

$$\therefore \quad \bar{y} = \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\left(\frac{s+1}{s^2 + 4s + 5}\right)$$

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Taking inverse Laplace transform,

$$y = \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}L^{-1}\left[\frac{(s+2)-1}{(s+2)^2+1}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}L^{-1}\left[\frac{s+2}{(s^2+2)^2+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}\cdot e^{-2t}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{2}e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}\cdot e^{-2t}\cos t - \frac{1}{2}e^{-2t}\sin t$$

Example 9 : Solve $(D^2 + 3D + 2)\,y = 2\,(t^2 + t + 1)$ with $y(0) = 2$ and $y'(0) = 0$.

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') + 3L(y') + 2L(y) = 2L(t^2 + t + 1)$$

But $L(y') = s\bar{y} - y(0) = s\bar{y} - 2$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 2s$

$\therefore$ The equation becomes,

$$(s^2\bar{y} - 2s) + 3(s\bar{y} - 2) + 2\bar{y} = 2\left(\frac{2}{s^3} + \frac{1}{s^2} + \frac{1}{s}\right)$$

$$\therefore \quad (s^2 + 3s + 2)\,\bar{y} = 2\frac{(2 + s + s^2)}{s^3} + 2s + 6 = \frac{2(s^4 + 3s^3 + s^2 + s + 2)}{s^3}$$

By partial fractions,

$$\therefore \quad \bar{y} = \frac{2(s^4 + 3s^3 + s^2 + s + 2)}{s^3(s^2 + 3s + 2)} = \frac{3}{s} - \frac{2}{s^2} + \frac{2}{s^3} - \frac{1}{s+2}$$

Taking inverse Laplace transform,

$$y = 3L^{-1}\frac{1}{s} - 2L^{-1}\frac{1}{s^2} + 2L^{-1}\frac{1}{s^3} - L^{-1}\frac{1}{s+2}$$

$$y = 3 - 2t + t^2 - e^{-2t}$$

Example 10 : Use Laplace transform to solve,

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = 1 \text{ where, } y(0) = 0,\ y'(0) = 1.$$ **(M.U. 1994, 2009, 10, 18)**

Sol. : Let $\bar{y}$ be the Laplace transform of y *i.e.* let $L(y) = \bar{y}$

Taking Laplace transform of the both sides,

$$L(y'') + 4L(y') + 8L(y) = L(1) \qquad \text{.......... (1)}$$

Now, $L(y') = s\bar{y} - y(0) = s\bar{y}$

$$L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1 \text{ and } L(1) = \frac{1}{s}$$

$\therefore$ The equation (1) becomes

$$s^2\bar{y} - 1 + 4s\bar{y} + 8\bar{y} = \frac{1}{s} \qquad \therefore \quad \bar{y}(s^2 + 4s + 8) = \frac{1}{s} + 1 = \frac{s+1}{s}$$

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$$\therefore \bar{y} = \frac{s+1}{s(s^2+4s+8)} \qquad \therefore \quad y = L^{-1}(\bar{y}) = L^{-1}\frac{s+1}{s(s^2+4s+8)}$$

We obtain the L^{-1} by partial fractions

$$\therefore \quad y = L^{-1}\left[\frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s}{s^2+4s+8} + \frac{1}{2}\cdot\frac{1}{s^2+4s+8}\right]$$

$$= \frac{1}{8}L^{-1}\left(\frac{1}{s}\right) - \frac{1}{8}L^{-1}\frac{(s+2)-2}{(s+2)^2+2^2} + \frac{1}{2}L^{-1}\frac{1}{(s+2)^2+2^2}$$

$$= \frac{1}{8}\cdot 1 - \frac{1}{8}e^{-2t}L^{-1}\frac{s}{s^2+2^2} + \frac{6}{8}e^{-2t}L^{-1}\frac{1}{s^2+2^2}$$

$$\therefore \quad y = \frac{1}{8} - \frac{1}{8}e^{-2t}\cos 2t + \frac{3}{8}e^{-2t}\sin 2t.$$

Example 11 : Using Laplace transform solve $\frac{d^2y}{dt^2} + y = t,\ y(0) = 1,\ y'(0) = 0.$

(M.U. 1995, 2009, 13, 14, 15, 16)

Sol. : Let $\bar{y}$ be the Laplace transform of y *i.e.* let $L(y) = \bar{y}$.

Taking Laplace transform of both sides,

$$L(y'') + L(y) = L(t)$$

Now, $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - s$ and $L(t) = \frac{1}{s^2}$.

$\therefore$ The equation (1) becomes

$$s^2\bar{y} - s + \bar{y} = \frac{1}{s^2} \qquad \therefore \quad s^2\bar{y} + \bar{y} = s + \frac{1}{s^2} = \frac{s^3+1}{s^2}$$

$$\therefore \quad (s^2+1)\bar{y} = \frac{s^3+1}{s^2} \qquad \therefore \quad \bar{y} = \frac{s^3+1}{s^2(s^2+1)}$$

Let $\frac{s^3+1}{s^2(s^2+1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{cs+d}{s^2+1}$

$$\therefore \quad s^3 + 1 = a\,s(s^2+1) + b(s^2+1) + (cs+d)s^2$$

$$= (a+c)s^3 + (b+d)s^2 + as + b$$

Equating like powers of s,

$$a + c = 1,\ b + d = 0,\ a = 0,\ b = 1 \qquad \therefore \quad a = 0,\ b = 1,\ c = 1,\ d = -1$$

$$\therefore \quad \bar{y} = \frac{1}{s^2} + \frac{s-1}{s^2+1} = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

Taking inverse Laplace transform

$$y = L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{s}{s^2+1}\right) - L^2\left(\frac{1}{s^2+1}\right) = t + \cos t - \sin t.$$

Example 12 : Solve by using Laplace transform $(D^2 + 2D + 5)\,y = e^{-t}\sin t$, when $y(0) = 0$, $y'(0) = 1$.

(M.U. 1995, 2003, 05, 06, 07, 11, 14, 15, 19)

Sol. : Let $L(y) = \bar{y}$. Then taking Laplace transform of both sides,

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$$L(y'')+2L(y')+5L(y)=L(e^{-t}\sin t)$$

But $L(y')=s\bar{y}-y(0)=s\bar{y}$. And $L(y'')=s^2\bar{y}-sy(0)-y'(0)=s^2\bar{y}-1$.

And $L\,e^{-t}\sin t=\dfrac{1}{(s+1)^2+1}$

$\therefore$ The equation becomes

$$(s^2\bar{y}-1)+2s\bar{y}+5\bar{y}=\frac{1}{(s+1)^2+1}$$

$$\therefore\quad (s^2+2s+5)\bar{y}=1+\frac{1}{s^2+2s+2}=\frac{s^2+2s+3}{s^2+2s+2}$$

$$\therefore\quad \bar{y}=\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

Let $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}=\dfrac{as+b}{(s^2+2s+5)}+\dfrac{cs+d}{(s^2+2s+2)}$

After simplification, we get

$$s^2+2s+s=(a+c)s^3+(2a+b+2c+d)s^2+(2a+2b+5c+2d)s+(2b+5d)$$

Equating the coefficients of like powers of s, we get,

$$a+c=0,\ 2a+b+2c+d=1,\ \ 2a+2b+5c+2d=2,\ \ 2b+5d=3$$

$$\therefore\quad a=0,\ \ b=\frac{2}{3},\ \ c=0,\ \ d=\frac{1}{3}$$

$$\therefore\quad \bar{y}=\frac{2}{3}\cdot\frac{1}{s^2+2s+5}+\frac{1}{3}\cdot\frac{1}{s^2+2s+2}$$

$$=\frac{2}{3}\cdot\frac{1}{(s+1)^2+2^2}+\frac{1}{3}\cdot\frac{1}{(s+1)^2+1^2}$$

Taking inverse Laplace transform

$$y=\frac{2}{3}\cdot e^{-t}\cdot L^{-1}\left[\frac{1}{s^2+2^2}\right]+\frac{1}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$=\frac{2}{3}e^{-t}\cdot\frac{1}{2}\sin 2t+\frac{1}{3}e^{-t}\sin t=\frac{e^{-t}}{3}(\sin 2t+\sin t).$$

Example 13 : Solve using Laplace transform $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+y=3te^{-t}$ given $y(0)=4$ and $y'(0)=2$. **(M.U. 2002, 15, 19)**

Sol. : Let $L(y)=\bar{y}$.

Taking Laplace transform of both sides,

$$L(y'')+2L(y')+L(y)=L(3te^{-t})$$

But $L(y')=s\bar{y}-y(0)=s\bar{y}-4$

and $L(y'')=s^2\bar{y}-sy(0)-y'(0)=s^2\bar{y}-4s-2$

$$L(e^{-t})=\frac{1}{s+1}\qquad\therefore\quad L[te^{-t}]=-\frac{d}{ds}\left(\frac{1}{s+1}\right)=\frac{1}{(s+1)^2}$$

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$\therefore$ The equation becomes,

$$(s^2\bar{y}-4s-2)+2(s\bar{y}-4)+\bar{y}=3\cdot\frac{1}{(s+1)^2}$$

$$\therefore\ (s^2+2s+1)\bar{y}-4s-10=\frac{3}{(s+1)^2}$$

$$\therefore\ (s+1)^2\bar{y}=\frac{3}{(s+1)^2}+4s+10$$

$$\therefore\ \bar{y}=\frac{3}{(s+1)^4}+\frac{4s+10}{(s+1)^2}=\frac{3}{(s+1)^4}+\frac{4s+4}{(s+1)^2}+\frac{6}{(s+1)^2}$$

$$=\frac{3}{(s+1)^4}+\frac{4}{s+1}+\frac{6}{(s+1)^2}$$

Taking the inverse Laplace transform of both sides,

$$y=L^{-1}\left[\frac{3}{(s+1)^4}\right]+4L^{-1}\left[\frac{1}{s+1}\right]+6L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$=3e^{-t}L^{-1}\left[\frac{1}{s^4}\right]+4e^{-t}L^{-1}\left[\frac{1}{s}\right]+6e^{-t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$=3e^{-t}\cdot\frac{t^3}{3!}+4e^{-t}\cdot 1+6e^{-t}\cdot t=3e^{-t}\left[\frac{t^3}{2}+6t+4\right]$$

Example 14 : Solve using Laplace transform $\frac{d^2y}{dt^2}+9y=18t$, given that $y(0)=0$ and $y(\pi/2)=0$.

(M.U. 1996, 98, 2004, 05, 07, 13, 19)

Sol. : Let $L(y)=\bar{y}$. Then taking Laplace transforms of both sides

$$L(y'')+9L(y)=18L(t) \qquad \text{.....................} (1)$$

But $L(y'')=s^2\bar{y}-sy(0)-y'(0)$; $L(t)=\frac{1}{s^2}$

Since, $y'(0)$ is not given let us assume $y'(0)=A$. **[Note this]**

Hence, (1) becomes $s^2\bar{y}-A+9\bar{y}=\frac{18}{s^2}$

$$\therefore\ (s^2+9)\bar{y}=\frac{18}{s^2}+A \qquad \therefore\ \bar{y}=\frac{18}{s^2(s^2+9)}+\frac{A}{s^2+9}$$

$$\therefore\ \bar{y}=\frac{18}{9}\left[\frac{1}{s^2}-\frac{1}{s^2+9}\right]+\frac{A}{s^2+9}=\frac{2}{s^2}+\left(\frac{A-2}{s^2+9}\right)$$

$$\therefore\ y=2L^{-1}\left(\frac{1}{s^2}\right)+(A-2)L^{-1}\frac{1}{s^2+9} \qquad \therefore\ y=2t+\frac{A-2}{3}\sin 3t$$

To find A we put $t=\frac{\pi}{2}$ and use that $y\left(\frac{\pi}{2}\right)=0$.

$$\therefore\ 0=2\cdot\frac{\pi}{2}+\left(\frac{A-2}{3}\right)\sin\left(\frac{3\pi}{2}\right) \qquad \therefore\ 0=\pi-\frac{(A-2)}{3}$$

$$\therefore\ 0=3\pi-A+2 \qquad \therefore\ A=3\pi+2$$

$$\therefore\ y=2t+\pi\sin 3t.$$

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Example 15 : Solve $(D^3 - 2D^2 + 5D)y = 0$, with $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$.

(M.U. 2004, 05)

Sol. : Let $L(y) = \bar{y}$.

Taking Laplace transform of both sides,

$$L(y''') - 2L(y'') + 5L(y') = 0$$

$$L(y') = s(\bar{y}) - y(0),\ L(y'') = s^2\bar{y} - s\,y(0) - y'(0)$$

$$L(y''') = s^3\bar{y} - s^2 y(0) - sy'(0) - y''(0)$$

From given conditions,

$$L(y') = s\bar{y},\ \ L(y'') = s^2\bar{y},\ \ L(y''') = s^3\bar{y} - 1.$$

$\therefore$ The equation becomes

$$s^3\bar{y} - 1 - 2s^2\bar{y} + 5s\,\bar{y} = 0 \qquad \therefore\ \ \bar{y} = \frac{1}{s^3 - 2s^2 + 5s}$$

Taking inverse Laplace transform

$$y = L^{-1}\left[\frac{1}{s^3 - 2s^2 + 5s}\right] = L^{-1}\left[\frac{1}{s(s^2 - 2s + 5)}\right] = L^{-1}\left[\frac{1}{s[(s-1)^2 + 2^2]}\right]$$

We obtain the inverse by convolution theorem.

Let $\quad \Phi_1(s) = \dfrac{1}{(s-1)^2 + 2^2}$ and $\Phi_2(s) = \dfrac{1}{s} \quad \therefore\ \Phi(s) = \Phi_1(s) \cdot \Phi_2(s)$

$$\therefore\ \ f_1(t) = L^{-1}\Phi_1(s) = L^{-1}\left[\frac{1}{(s-1)^2 + 2^2}\right] = e^t \cdot L^{-1}\frac{1}{s^2 + 2^2} = \frac{1}{2} \cdot e^t \cdot \sin 2t$$

$$f_2(t) = L^{-1}\Phi_2(s) = L^{-1}\left(\frac{1}{s}\right) = 1$$

$$\therefore\ \ f_1(u) = \frac{1}{2}e^u \sin 2u.$$

Then by Cor. (16A) page 2-15,

$$\therefore\ \ L^{-1}\Phi(s) = \int_0^t \frac{1}{2}e^u \sin 2u\, du = \frac{1}{2} \cdot \frac{1}{5} \cdot \left[e^u(\sin 2u - 2\cos 2u)\right]_0^t$$

$$\therefore\ \ y = \frac{1}{10}\left[e^t(\sin 2t - 2\cos 2t) + 2\right]$$

$\therefore$ The solution is $y = \dfrac{1}{5} - \dfrac{1}{5}e^t \cos 2t + \dfrac{1}{10}e^t \sin 2t.$

***Example 16 :** Solve the equation $y + \int_0^t y\, dt = 1 - e^{-t}$.

(M.U. 2

Sol. : Let $L(y) = \bar{y}$. Taking the Laplace transform of both sides, we get,

$$L(y) + L\left[\int_0^t y\, dt\right] = L(1) + L(e^{-t})$$

Since, $L\left[\int_0^t y\, dt\right] = \int_0^\infty e^{-st}\int_0^\infty y\, dt = \left[\int_0^t y\, dt \cdot \dfrac{e^{-st}}{s}\right]_0^\infty - \int_0^\infty -\dfrac{e^{-st}}{s} \cdot y\, dt$

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$$= 0 + \frac{1}{s}\int_0^{\infty} e^{-st} y\,dt = \frac{1}{s}L(y) = \frac{1}{s}\bar{y}$$

and $Le^{-t} = \frac{1}{s+1}$, the equation becomes $\bar{y} + \frac{\bar{y}}{s} = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}$

$$\therefore\ \bar{y}\frac{(s+1)}{s} = \frac{1}{s(s+1)} \qquad \therefore\ \bar{y} = \frac{1}{(s+1)^2}$$

$$\therefore\ y = L^{-1}\frac{1}{(s+1)^2} = e^{-t}L^{-1}\frac{1}{s^2} = e^{-t}\cdot t \qquad \therefore\ y = t\,e^{-t}.$$

***Example 17** : Solve the following equation by using Laplace transform $\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t$, given that $y(0) = 1$.

(M.U. 1999, 2006, 09, 14)

Sol. : Let $L(y) = \bar{y}$. Taking Laplace transform of both sides, we get

$$L(y') + 2L(y) + L\left[\int_0^t y\,dt\right] = L(\sin t)$$

But $L(y') = sL(y) - y(0) = s\bar{y} - 1$

$$L\left[\int_0^t y\,dt\right] = \frac{1}{s}L(y) = \frac{1}{s}\bar{y}, \quad L(\sin t) = \frac{1}{s^2+1}$$

$\therefore$ The equation becomes

$$s\bar{y} - 1 + 2\bar{y} + \frac{1}{s}\bar{y} = \frac{1}{s^2+1} \qquad \therefore\ \left(s + 2 + \frac{1}{s}\right)\bar{y} = \frac{1}{s^2+1} + 1 = \frac{s^2+1+1}{s^2+1}$$

$$\therefore\ \frac{(s^2+2s+1)}{s}\bar{y} = \frac{(s^2+2)}{s^2+1} \qquad \therefore\ \bar{y} = \frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

Let $\frac{s(s^2+2)}{(s+1)^2(s^2+1)} = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{cs+d}{s^2+1}$

$$\therefore\ s(s^2+2) = a(s+1)(s^2+1) + b(s^2+1) + (cs+d)(s+1)^2$$

Putting $s = -1$, $-3 = 2b$ $\quad\therefore\ b = -3/2$. Putting $s = 0$, $0 = a + b + d$.

Equating the coefficients of s^2 and s^3.

$$0 = a + b + 2c + d \text{ and } 1 = a + c$$

$\because\ b = -3/2,\ a + d = 3/2$ and $a + 2c + d = 3/2$.

But $a + d = 3/2 \quad \therefore\ 2c = 0 \quad \therefore\ c = 0$

$\because\ 1 = a + c$ and $c = 0 \quad \therefore\ a = 1$

$\because\ a + d = 3/2$ and $a = 1 \quad \therefore\ d = 1/2$

$\therefore\ a = 1,\ b = -3/2,\ c = 0,\ d = 1/2$

$$\therefore\ \bar{y} = \frac{1}{s+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$\therefore\ y = L^{-1}\left(\frac{1}{s+1}\right) - \frac{3}{2}e^{-t}L^{-1}\frac{1}{s^2} + \frac{1}{2}L^{-1}\frac{1}{s^2+1}$$

$$\therefore\ y = e^{-t} - \frac{3}{2}e^{-t}\cdot t + \frac{1}{2}\sin t.$$

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EXERCISE

Using Laplace transform solve the following differential equations with the given conditions.

1. $(3D+2)\,y = e^{3t}$, $y(0) = 1$. **(M.U. 2002)**
2. $\dfrac{dy}{dt} + 2y = e^{-3t}$, $y(0) = 1$
3. $\dfrac{dy}{dt} + y = e^{-2t}$, $y(0) = 0$
4. $\dfrac{dy}{dt} + 2y = 5$, $y(0) = 1$
5. $\dfrac{dy}{dt} + 2y = \sin t$, $y(0) = 0$
6. $\dfrac{dy}{dt} + y = \cos 2t$, $y(0) = 1$
7. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 0$; at $x = 0$, $y = 0$, $\dfrac{dy}{dx} = 4$.
8. $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} = -8t$; $x(0) = 0 = x'(0)$.
9. $(D^2 + 9)\,y = 18t$; $y(0) = 0$, $y'(0) = 0$.
10. $f''(t) + f'(t) = t$; $f(0) = 1$, $f'(0) = -1$.
11. $(D^2 - 3D + 2)\,y = 2e^{3t}$; $y = 2$, $y' = 3$ at $t = 0$.

*12. $(D^2 + 1)\,y = \sin t$; $y(0) = 1$, $y'(0) = -\dfrac{1}{2}$.

13. $(D^2 - 2D + 2)\,x = 0$; $x(0) = x'(0) = 1$.
14. $y'' - 2y' + y = e^t$; $y(0) = 2$, $y'(0) = -1$. (M.U. 2019)
15. $(D^2 + D - 2)\,x = 2(1 + t - t^2)$; $x = 0$, $Dx = 3$ for $t = 0$. (M.U. 2006)
16. $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 6y = 2$; $y(0) = 1$, $y'(0) = 0$.

*17. $(D^2 - 3D + 2)\,y = 4t + e^{3t}$ if $y = 1$, $Dy = -1$ at $t = 0$.

*18. $(D^3 + 2D^2 - D - 2)\,y = 0$ if $y = 1$, $Dy = 2$, $D^2y = 0$ at $y = 0$.

19. $(D^2 + 4D + 3)\,y = e^{-t}$; $y(0) = y'(0) = 1$.
20. $(D^2 + D)\,y = t^2 + 2t$ at $t = 0$, $y = 4$ and $Dy = 2$. (M.U. 1997)
21. $(D^2 - 2D + 1)\,x = e^t$ with the conditions $x = 2$, $Dx = -1$ at $t = 0$. (M.U. 1997, 2000, 19)
22. $(D + 1)^2 y = 6t\,e^{-t}$ with $y(0) = 2$, $y'(0) = 5$. (M.U. 1994)
23. $\dfrac{d^2y}{dx^2} + 16y = \delta(t)$ given that $y = 0$, $\dfrac{dy}{dt} = 0$ at $t = 0$.
24. $(D^2 - 3D + 2)\,y = 4e^{2t}$ at $t = 0$, $y = -3$ and $Dy = 5$. (M.U. 2004)
25. $(D^2 - 2D - 8)\,y = 4$, $y(0) = 0$ and $y'(0) = 1$. (M.U. 2003, 04, 13)
26. $(D^2 + 2D + 1)\,y = 3t\,e^{-t}$, $y(0) = 4$, $y'(0) = 2$. (M.U. 2002)
27. $(D^2 + D)\,y = t^2 + 2t$, $y(0) = 4$, $y'(0) = -2$. (M.U. 2004)
28. $(D^2 - 4)\,y = 3e^t$, $y(0) = 0$, $y'(0) = 3$. (M.U. 2004, 19)
29. $\dfrac{d^2y}{dt^2} + y = t$, $y(0) = 1$, $y'(0) = 0$. (M.U. 2006)

30. $2y'' + 5y' + 2y = e^{-2t}$, $y(0) = y'(0) = 1$. **(M.U. 2005)**
31. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 2y = e^{3t}$, given $y(0) = 0$ and $y'(0) = 1$. **(M.U. 2003)**
32. $\dfrac{d^2y}{dt} + 9y = \cos 2t$, $y(0) = 1$ and $y(\pi/2) = -1$. **(M.U. 2002)**
33. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 3y = \sin t$, $y(0) = 0$ and $y'(0) = 0$. **(M.U. 2014)**

Ans. : **(1)** $y = \dfrac{1}{11}e^{3t} + \dfrac{10}{11}e^{-2t/3}$, **(2)** $y = 2e^{-2t} - e^{-3t}$, **(3)** $y = e^{-t} - e^{-2t}$, **(4)** $y = \dfrac{5}{2} - \dfrac{3}{2}e^{-2t}$,

(5) $y = -\dfrac{1}{5}\cos t + \dfrac{2}{5}\sin t + \dfrac{1}{5}e^{-2t}$, **(6)** $y = \dfrac{1}{5}\cos 2t + \dfrac{2}{5}\sin 2t + \dfrac{4}{5}e^{-t}$,

(7) $y = e^x - e^{-3x}$, **(8)** $x = -\dfrac{1}{8} + \dfrac{1}{2}t - t^2 + \dfrac{e^{-4t}}{8}$, **(9)** $y = 2\left[t - \dfrac{1}{3}\sin 3t\right]$,

(10) $f(t) = 1 - t + \dfrac{t^2}{2}$, **(11)** $y = 2e^t - e^{2t} + e^{3t}$, **(12)** $y = \left(1 - \dfrac{t}{2}\right)\cos t$,

(13) $x = e^t \cos t$, **(14)** $y = e^t\left(2 - 3t + \dfrac{t^2}{2}\right)$, **(15)** $x = t^2 + e^t - e^{-2t}$,

(16) $y = -\dfrac{1}{3} + \dfrac{8}{15}e^{3t} + \dfrac{4}{5}e^{-2t}$, **(17)** $y = -\dfrac{1}{2}e^t - 2e^{2t} + \dfrac{1}{2}e^{3t} + 2t + 3$,

(18) $y = \dfrac{1}{3}(5e^t + e^{-2t}) - e^{-t}$, **(19)** $y = \dfrac{7}{4}e^{-t} - \dfrac{3}{4}e^{-3t} + \dfrac{1}{2}t\,e^{-t}$,

(20) $y = \dfrac{t^3}{3} + 2e^{-t} + 2$, **(21)** $x = 2e^t - 3e^t \cdot t + \dfrac{t^2}{2}e^t$,

(22) $y = e^{-t}(t^3 + 7t + 2)$, **(23)** $y = \dfrac{1}{4}\sin 4t$,

(24) $y = -7e^t + 4e^{2t} + 4t\,e^{2t}$, **(25)** $y = \dfrac{1}{6}[e^{4t} + 2e^{-2t} - 3]$,

(26) $y = e^{-t}\left(4 + 6t + \dfrac{t^3}{2}\right)$, **(27)** $y = 2 + 2e^{-t} + \dfrac{t^3}{3}$,

(28) $y = -e^t + \dfrac{3}{2}e^{2t} - \dfrac{1}{2}e^{-2t}$, **(29)** $y = t - \sin t + \cos t$,

(30) $y = \dfrac{20}{9}e^{-t/2} - \dfrac{11}{2}e^{-2t} - \dfrac{t}{3}e^{-2t}$, **(31)** $y = \dfrac{1}{6}e^t - \dfrac{4}{15}e^{-2t} + \dfrac{e^{3t}}{10}$,

(32) $y = \dfrac{1}{5}\cos 2t + \dfrac{4}{5}\cos 3t + \dfrac{4}{5}\sin 3t$. **(33)** $y = -\dfrac{1}{40}e^{-3t} + \dfrac{1}{8}e^t - \dfrac{1}{10}\cos t - \dfrac{1}{5}\sin t$

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