Logical Rules (usually not cited)

Modus Ponens

Def. if p is true and p implies q, then q is true.

Logical Notation: $p, \underset{\text{p implies q}}{p \to q} \therefore q$

Example: p = "It is raining", q = "The ground is wet"

Given: "It is raining" and "It is raining ∴ the ground is wet",

Conclusion: The ground is wet

Modus Tollens

Def. If not q is true and p implies q, then p is not true.

Logical Notation: $\neg q, \ p \to q \therefore \neg p$

Example: p = "it is raining", q = "the ground is wet"

Given: "the ground is not wet" and "it is raining implies the ground is wet"

Conclusion: it is not raining

Hypothetical Syllogism

→ if p implies q and q implies r, then p implies r

Logical Notation: $(p \to q), (q \to r) \therefore (p \to r)$

Disjunctive Syllogism

→ If not p is true and p or q is true, then q is true.

logical Notation: $\neg p, \ (p \underset{\text{or}}{\vee} q) \therefore q$

(non-exclusive, could be both)

it has to be one or the other, i can prove it has to be (no neither/3rd option) and I can disprove one, so the other must be true

Addition

→ If p is true, then p or q is true

$p \therefore (p \vee q)$

Simplification

If p and q are true, then p is true

$(p \wedge q) \therefore p$

Conjunction

⇒ If p is true and q is true, then q and p are true

$p, q \therefore (p \wedge q)$

Mathematical Sets

↳ A set is a collection of objects

$\{\ \}$

i.e. {All Triangles}

Definitions ...

- verbal: „A is the set of all pos. even numbers"

$\{2, 4, 6, 8, \dots\}$ (this pattern repeats)

{Set Builder Notation}

$\{x \mid x \text{ is even and } x > 0\}$

{what is in set | condition to be in the set}

$x \in A$

is an element of

$3 \notin$ — A = set, a = element of set —

{Important Sets}

- $\mathbb{N}$ nat.: $\{1, 2, 3, \dots\}$
- $\mathbb{W}$ nat + 0: $\{0, 1, 2, 3, \dots\}$
- $\mathbb{Z}$ Integers: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- $\mathbb{Q}$ rational Numbers: $\left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\}$

  alle Zahlen die als Bruch $\left(\frac{p}{q}\right)$ geschrieben werden können, müssen ganze Zahlen sein,
  q ist nicht 0 (man kann nicht durch 0 teilen)

- $\mathbb{I}$ irrational Numbers (commonly written as $\mathbb{R} \setminus \mathbb{Q}$): $\{x \mid x \in \mathbb{Q}, x \in \mathbb{R}\}$
- $\mathbb{R}$ reelle Zahlen: $\{x \mid x \text{ can be written as a decimal}\} = \{x \mid x \in \mathbb{Q} \text{ OR } x \in \mathbb{I}\}$
- $\mathbb{C}$ complex / imaginary numbers $\{a + bi \mid a, b \in \mathbb{R}, i = \sqrt{-1}\}$

$\emptyset$ empty set: $\{\}$

$\mathbb{U}$ universal set: {All objects currently under discussion}

The Cardinality of a set is the number of distinct elements in the set.

$|A|$ is the Cardinality of set A

- sets don't care about order or number of distinct elements (repetition), they are just a collection of stuff =)

$\{1, 2, 2, 3, 3, 3, 4, 5\} = \{4, 3, 2, 1\}$ etc.

Set A and B are equal, $(A = B)$ if

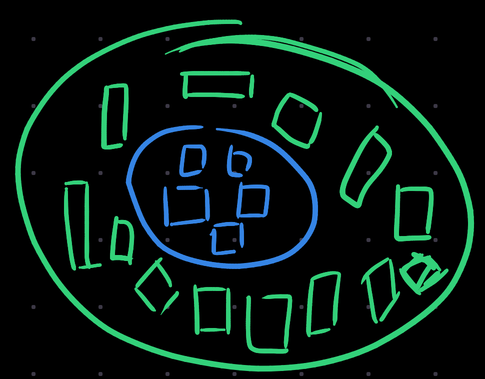
① every element of A is an element of B
② every element of B is an element of A
(~ Both contain exactly the same elements)

$A \subseteq B$ $\qquad$ $B \subseteq A$

is a subset

Subset means that every element of a is also an element of B however not every element of the set is in the subset

not every rectangle is a square, but every square is a rectangle -> squares are a subset of rectangles



$\mathbb{N} \subseteq \mathbb{W}$
$\mathbb{W} \not\subseteq \mathbb{N}$

The Power Set of A, $P(A)$, is the set of all possible subsets

$\{0, 1\}$

$P(\{0,1\}) = \{\{0\}, \{1\}, \{0,1\}, \{\}\}$

$\{0\} \subseteq \{0,1\}$ etc
~ every element is in there

Complement, A^c or A' of a set a A is the set of all the elements in the universal set that are not elements of A.

$$\boxed{A^c = \{x \mid x \in \mathbb{U} \text{ and } x \notin A\}}$$

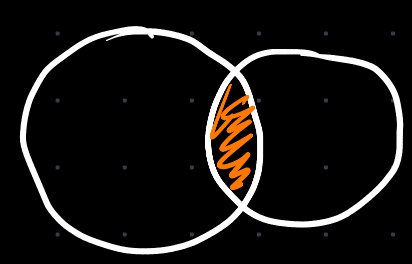
Set difference
All elements in $\mathbb{U}$ "minus" all elements in A

The Union of two sets, $A \cup B$, is the set containing all the elements from either A or B.

$$\boxed{A \cup B = \{x \mid x \in A \text{ OR } x \in B\}}$$

Intersection $A \cap B$, : elements that are in both sets

$$A \cap B = \{x \mid x \in A \text{ AND } x \in B\}$$

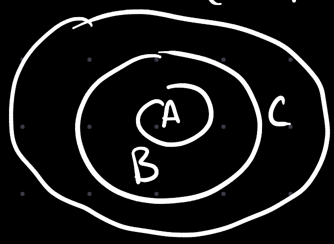


$A \; \{1, 3, 5\}$ $\qquad$ $B \; \{1, 4, 6, 5\}$

$\rightarrow A \cup B \; \{1, 3, 4, 5, 6\}$ $\quad$ $\hookrightarrow A \cap B \; \{1, 5\}$

if no intersection: $\{\}$ -> „A and B are disjoint"

Prove that: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$



Theorem (transitive property)

(Mine)

If $A \subseteq B$ and $B \subseteq C$, but $A \not\subseteq C$, then A would have to have elements not in C, but B is a subset of C and therefore has the same elements, so A necessarily has to only have elements of B which are by definition elements of C, or A is not a subset of B.

if $a \in A$ then $a \in C$

let $a \in A \Rightarrow a \in B$

$\Rightarrow a \in C$ (since $a \in B \subseteq C$) $\Rightarrow A \subseteq C$ ■ =)

Quantifiers

ideas we encounter so often, they have names

Universal Quantifier	Existential Quantifiers
"For all"	"there exists"
$\forall$	$\exists$
$\forall y \in \mathbb{R},\ y^2 \geq 0$	$\exists x$ such that $x + 3 = 5$

Negations (falsifying the statement)

$\sim$ $\neg$ $'$

$\neg(A \text{ or } B) = \neg A \text{ and } \neg B$

$\neg(A \text{ and } B) = \neg A \text{ or } \neg B$

$\neg(\text{if } A, \text{ then } B) = A \text{ and } \neg B$ (if a happening implies B happening, the negation is A happens and B doesn't)

$\neg(\forall x, y) = \exists x$ such that $\neg y$

$\neg(\exists x$ such that $y) = \forall x, \neg y$

$\exists \xleftrightarrow{\text{negate}} \forall$

{Practice Problem}

① For all M in the Real Numbers there exists x in the real numbers such that the absolute value of f(x) is greater than or equal to M

$(\forall M \in \mathbb{R})\ (\exists x \in \mathbb{R})$ such that $[\,|f(x)| \geq M\,]$

Negation

$(\exists M \in \mathbb{R})$ such that $(\forall x \in \mathbb{R})\ [\,|f(x)| < M\,]$

②

$(\forall M \in \mathbb{R})(\exists x \in \mathbb{R})$ such that $(\forall y > x)\ [f(y) > M]$

$(\exists M \in \mathbb{R})$ such that $(\forall x \in \mathbb{R})\underset{\exists}{(\forall y > x)}[f(y) \leq M]$

For every γ greater than 0, there exists a delta greater than 0 such that the absolute value of $f(x)$ minus $f(x$ initial$)$ is less than γ when the absolute value of x minus x initial is less than delta

$$(\forall \gamma, \gamma > 0)\ (\exists \Delta, \Delta > 0) \text{ s.t. } (|f(x)| - f(x_i) < \gamma) \text{ when } (|x - x_i|) < \Delta$$

$$(\exists \gamma, \gamma > 0) \text{ s.t. } (\forall \Delta, \Delta > 0)\ (|f(x)| - f(x_i) > \gamma) \text{ when } (|x - x_i|) < \Delta$$

$$(\forall \varepsilon > 0)\ (\exists \delta > 0) \text{ s.t. } (\forall x \in \mathbb{R})\ [|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon]$$