PHSX 315 Assignment 6

Due by Monday May 8th at 1 PM (this is the official final exam time for this class)

Learning Goals: Generation of random variates from particular probability distributions. Use of random numbers to simulate simple experiment. Estimation of integrals using Monte Carlo method. Estimation of integrals using numerical methods. Numerical solution of initial-value type differential equations.

Problem 1 Use the acceptance-rejection technique to sample 1,000 random values in the range $[0,\ 2\pi]$ proportional to the following function,

$$f(x) = x^4 \cos^6 x \ .$$

You should keep track of the efficiency of your method, ie. how many random numbers do you need to throw in order to produce 1,000 random variates distributed according to this distribution.

Problem 2 Simple simulation example. Let's imagine that each of the random variates is the result of a (biased) wheel of fortune spin in dollars. So if the selected random variate is 3.2 it is worth 3.2$. Sum the 1,000 random variates from problem 1 and report the total value associated with spinning your (biased) wheel of fortune 1,000 times. Now do this "experiment" 1000 more times (each experiment should be independent of the others). What is the distribution of the total value for each experiment. Where does your initial experiment lie in this distribution. What is the mean? What is the rms? What is the maximum possible value? What is the minimum possible value? You need to make sure that you don't reuse the same random numbers...

Problem 3 Evaluate the following integral by acceptance-rejection Monte Carlo integration. You can probably reuse the method of problem 1.

$$I = \int_0^{2\pi} x^4 \cos^6 x \ \mathrm{d}x$$

and report the uncertainty.

Problem 4 Evaluate the following integral using the basic theorem of Monte Carlo integration. See Numerical Recipes 3rd edition p398, and class notes. Also report the uncertainty.

$$I = \int_0^{2\pi} x^4 \cos^6 x \ \mathrm{d}x$$

Problem 5 Evaluate the following integral using both the extended Simpson's rule and the extended Boole's rule and report the uncertainty. See ClassExamples/Quadrature for examples.

$$I = \int_0^{2\pi} x^4 \cos^6 x \, \mathrm{d}x$$

Uncertainties can be estimated by comparing results with half the step size.

Problem 6 Use both the Euler method and the RK2 method (see ClassExamples/ODEs) to set up the coupled first-order system of ordinary differential equations (ODEs) to solve the initial-value problem for projectile motion including the effects of quadratic drag in Earth's atmosphere. The 4-cponent state vector should consist of the 2-d position, $\mathbf{r}$, and 2-d velocity $\mathbf{v}$ of the projectile and should be a simple application of writing,

$$\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t = \mathbf{F}/m$$

and

$$\mathrm{d}\mathbf{r}/\mathrm{d}t = \mathbf{v} \, .$$

One can assume that the forces acting on the cannon-ball are gravity and a drag force opposing the motion and given by,

$$F_D = \frac{1}{2}\rho_F \, C_D u^2 A \, .$$

This means that at each point in the trajectory, one needs to align the drag force opposite to the velocity. We will take the projectile as a solid sphere with drag coefficient, C_D of 0.47 like a cannon-ball. The cannon-ball's mass is 5.0 kg and its diameter is 10.6 cm. The frontal area of the cannon-ball, A, can be taken as πr^2 (the area of air occluded by the projectile). The velocity of the projectile with respect to the fluid (ie. air) is u, and finally ρ_F is the density of the air. Take g as 9.80 m s^{-2}. The air density falls off approximately exponentially as a function of altitude, h, and can be taken as

$$\rho_F = \rho_0 \exp(-h/H_n)$$

with $H_n = 10.4$ km and $\rho_0 = 1.204$ kg m^{-3}. For such a cannon-ball launched at 150 m/s at a launch angle of 45°, what is the range and the hang-time? How different is the range if one assumes that the air density does not depend on altitude? What is the best launch angle for maximum range? Graphs for various launch angles would be good. You should also give uncertainty estimates. It may be helpful to first set up the problem with the drag force set to zero and check that you get the text-book answers.

You should upload to your repository a summary of your findings, a copy of your code, example results from running the code, and appropriate figures.