CS2040S
Data Structures and Algorithms

Welcome!

Summary

Monday:

Algorithm Analysis:

- *Asymptotic notation*
- *Models of computation*

Binary Search:

- *Finding an element in array.*
- *Invariants*

Peak Finding

- *Finding a (local) maximum*
- *1d-algorithm*
- *2d-algorithm*

Today: Optimization

Finding a maximum / minimum

- *Key aspect of machine learning*
- *Binary search isn't always good*

Newton's method

- *Second order approach*

Gradient descent

- *First order approach*

Announcements / Reminders

Problem sets:

Problem Set 1 was due Monday.

Problem Set 2 is due next Thursday. (Happy Lunar New Year!)

Optional Practice Problems

- Good practice!
- Make sure you understand the techniques.
- Extra coding experience.
- Extra algorithm experience.

Announcements / Reminders

Tutorials:

Read your e-mail / announcements and follow instructions.

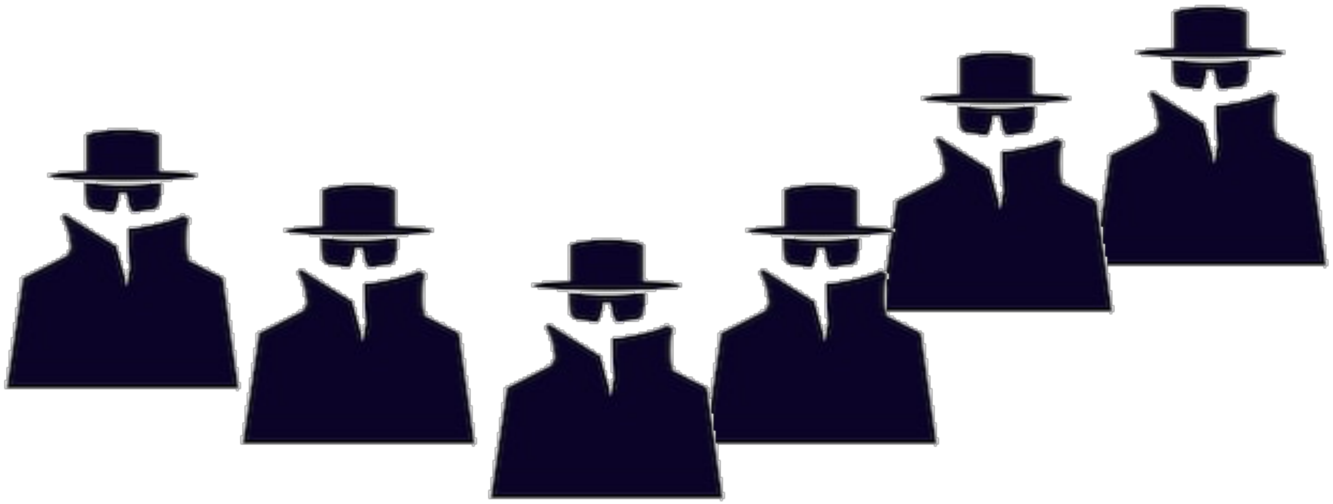
A mutually-agreed swap is approved by default.

Recitations:

We will get to it, after tutorials are resolved.

Puzzle of the Week

There are N students in CS2040S and K of them are spies. Your job is to identify all the spies.



Puzzle of the Week

<u>To catch a spy:</u>

You can send some students on a mission. 

If all K spies are on the mission, they will meet.

You learn if the meeting occurred or not.

You learn nothing else.  

Puzzle of the Week

Find:

The best strategy you can to catch all the spies. (Write a program!)

Announcements / Reminders

Competition:

Find the spies!

Open on Coursemology now:

- Optional.
- Write a program to implement your best spy catching strategy.
- We will test it on various spy rings.
- Fewest queries / fewest dollars wins!
- (And a small bonus for participating.)

Binary Search

O(log n) time

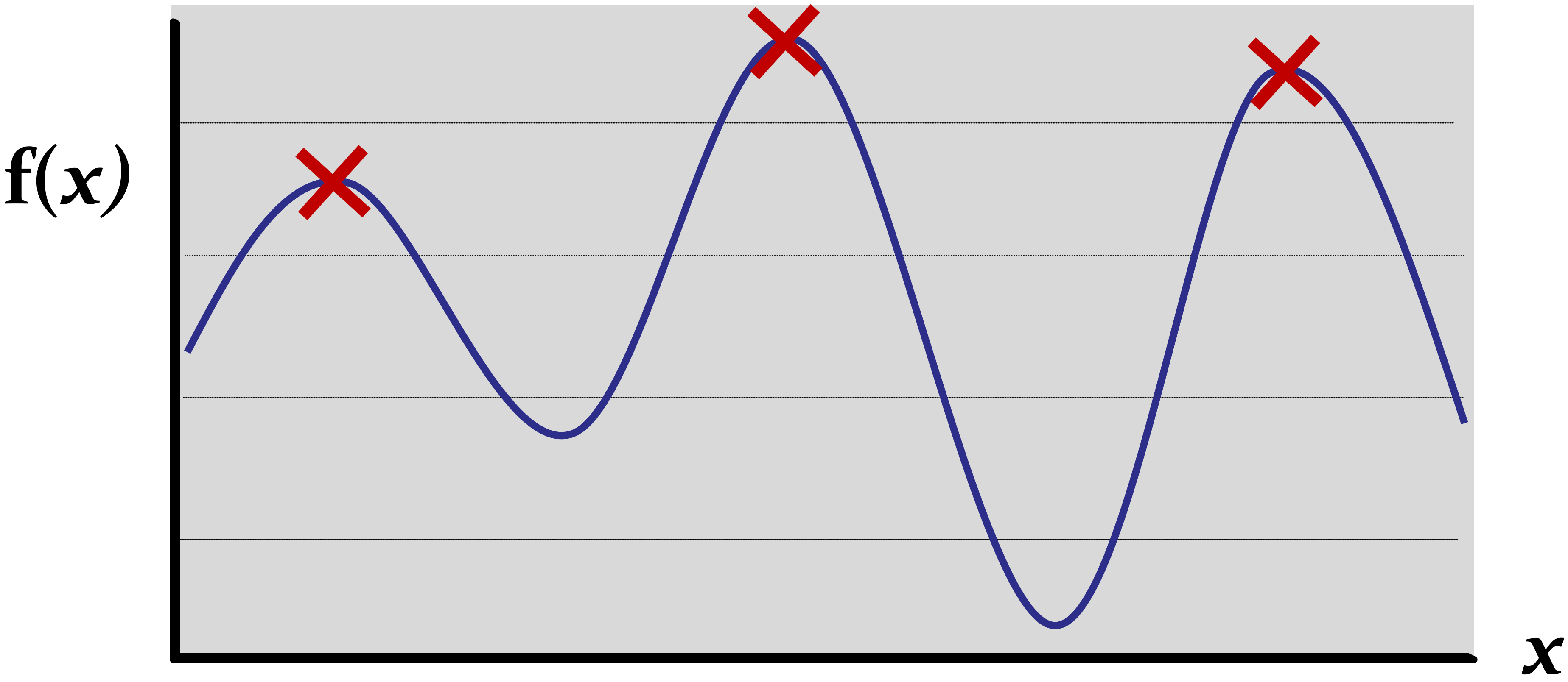
Sorted array: `A[1..n]`

2	4	4	5	6	7	8	9	11	17	23	28

```
Search(A, key, n)
    begin = 0
    end = n-1
    while begin < end do:
        mid = begin + (end-begin)/2;
        if  key <= A[mid] then
            end = mid
        else begin = mid+1
    return A[begin]
```

Peak Finding

Input: Some function f(x)



Output: local maximum

Peak Finding

O(log n) time

Input: Some array A[1..n]

2	4	9	2	11	6	23	4	6	8	17	5

FindPeak(A, n)

  if A[n/2] is a peak **then return** n/2

  else if A[n/2+1] > A[n/2] **then**

    FindPeak (A[n/2+1..n], n/2)

  else if A[n/2–1] > A[n/2] **then**

    FindPeak (A[1..n/2-1], n/2)

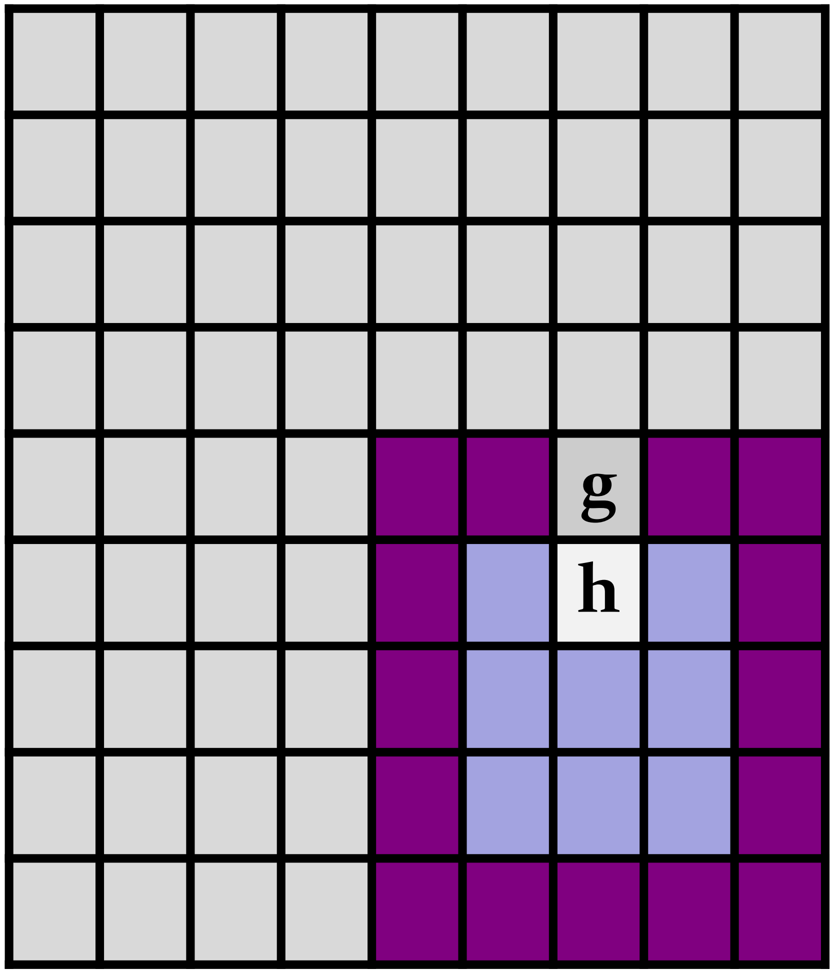
2D Algorithm

O(m + n) time

Divide-and-Conquer

1. Find MAX element on border + cross.
2. If found a peak, DONE.
3. Else:

Recurse on quadrant containing element bigger than MAX.



Binary Search

Sorted array: A[1..n]

2	4	4	5	6	7	8	9	11	17	23	28

Not just for searching arrays:

- Assume a complicated function:

  int complicatedFunction(int s)

- Assume the function is always increasing:

  complicatedFunction(i) $<$ complicatedFunction(i+1)

- Find the minimum value j such that:

  complicatedFunction(j) $>$ 100

Finding a maximum / minimum

Convex / concave function : f[1..n]

Problem Set 2

2	3	4	5	6	7	8	7	6	5	4	3

Not just for arrays:

- Assume a complicated function:

  int complicatedFunction(int s)

- Assume the function is convex (or concave)

- Find the maximum / minimum value:

  $\min_j$(complicatedFunction(j))

Why function maximization?

Two short answers:

Optimization is everywhere

- Maximize revenue
- Minimize cost
- Minimize time
- …

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Two short answers:

Optimization is everywhere

- Maximize revenue
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Machine learning

- All about optimization!
- Find the best model for your data
- Find the best neural network for your problem.

Machine Learning Example

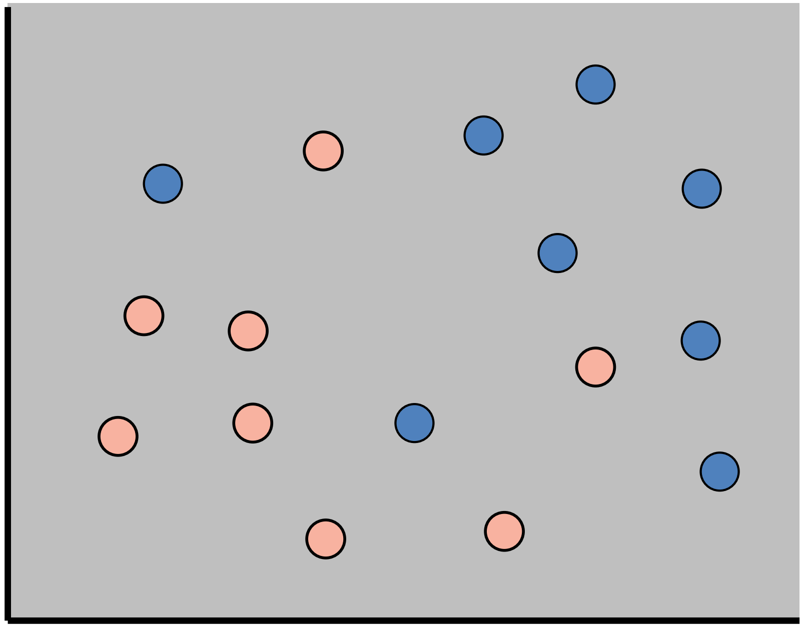
Classifier:

Input:

- A set of data points
- Each point has a label: 0 or 1

Output:

- A function that classifies points.



Machine Learning Example

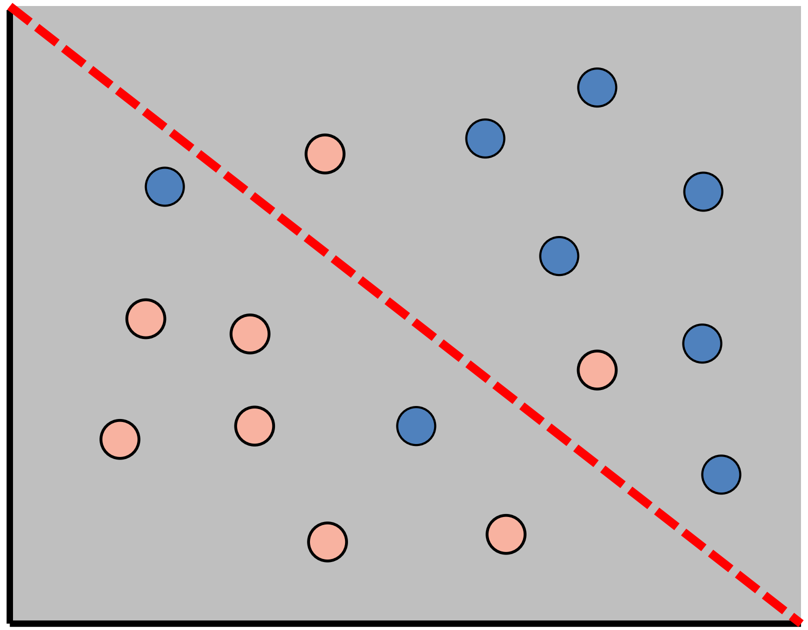
Linear classifier:

Idea:

- Find a line: $ax + by = c$
- To classify point (x_j, y_j):

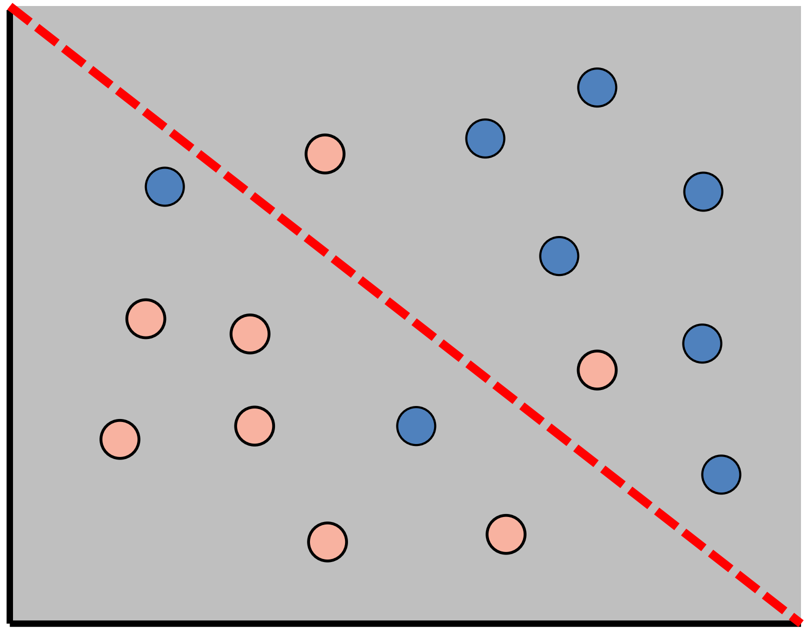
if $(ax_j + by_j > c)$: BLUE

if $(ax_j + by_j < c)$: RED



Machine Learning Example

Linear classifier:



Idea:

- Find a line: $ax + by + c = 0$
- To classify point (x_j, y_j):

  if $(ax_j + by_j + c > 0)$: BLUE

  if $(ax_j + by_j + c < 0)$: RED

color for point (x,y)

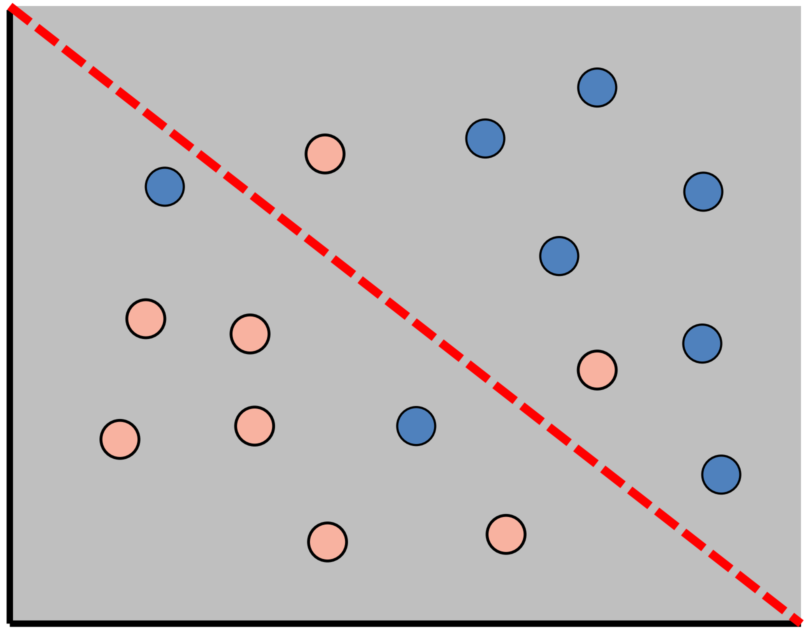
Question:

Given data $(x_1, y_1, z_1), (x_2, y_2, z_1), \ldots, (x_n, y_n, z_1)$

Find *best* values of (a, b, c)

Machine Learning Example

Linear classifier:



Idea:

Find a line: $ax + by + c = 0$

One way to make that precise:

Given data $(x_1, y_{1,} z_1), (x_2, y_{2,} z_1), \ldots, (x_n, y_{n,} z_1)$

$$\max_{a,b,c} \prod_{i=1}^{n} h_{a,b,c}(x_i, y_i)^{z_i} \cdot (1 - h_{a,b,c}(x_i, y_i))^{1-z_i}$$

<u>Likelihood function:</u>

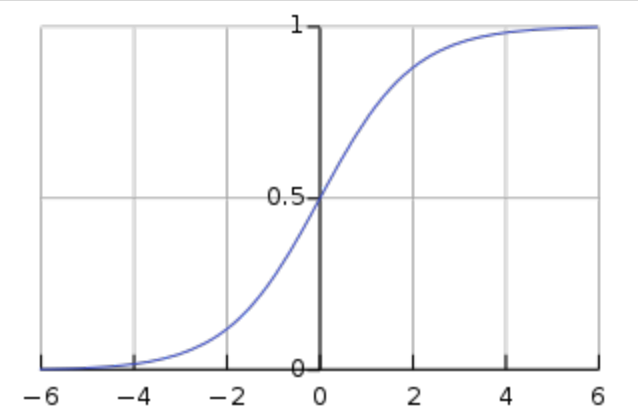
h(x, y) = probability that point (x,y) is blue/1.

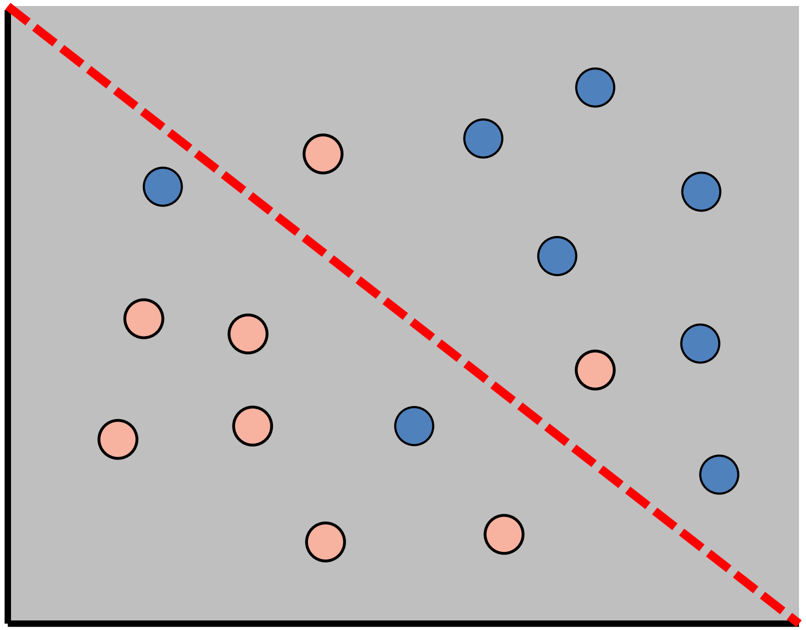
1 – h(x,y) = probability that point (x,y) is red/0.

Machine Learning Example

Choice of h function?

$$h_{a,b,c}(x,y) = \frac{1}{1+e^{-(ax+by+c)}}$$





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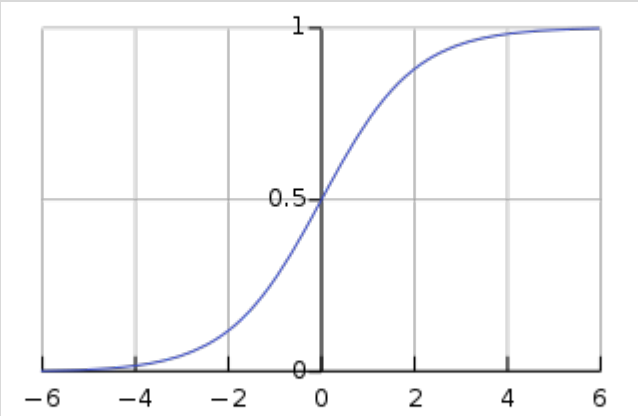

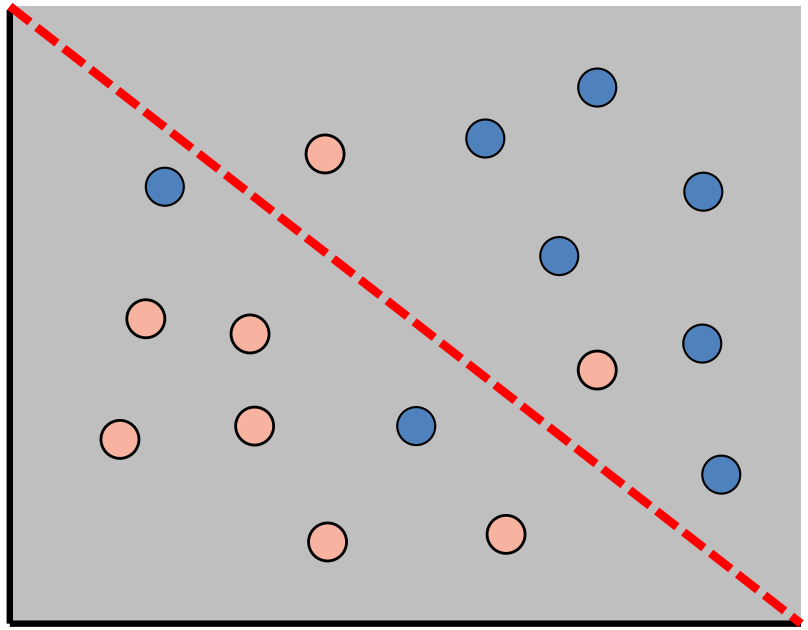

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$$\max_{a,b,c} \sum_{i=1}^{n} z_i \log(h_{a,b,c}(x_i, y_i)) + (1-z_i)\log(1-h_{a,b,c}(x_i,y_i))$$

Log-Likelihood function:
log of the likelihood function.

Machine Learning Example

Choice of h function?

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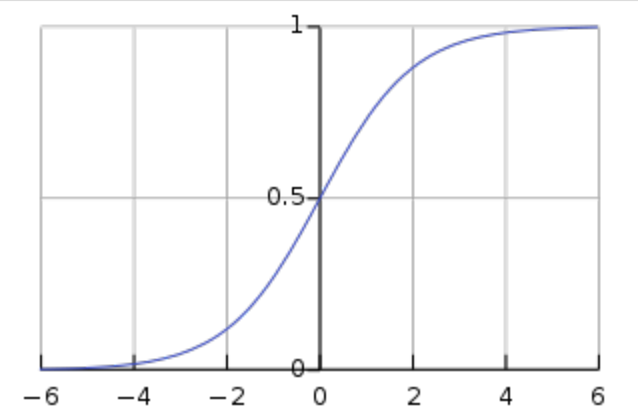

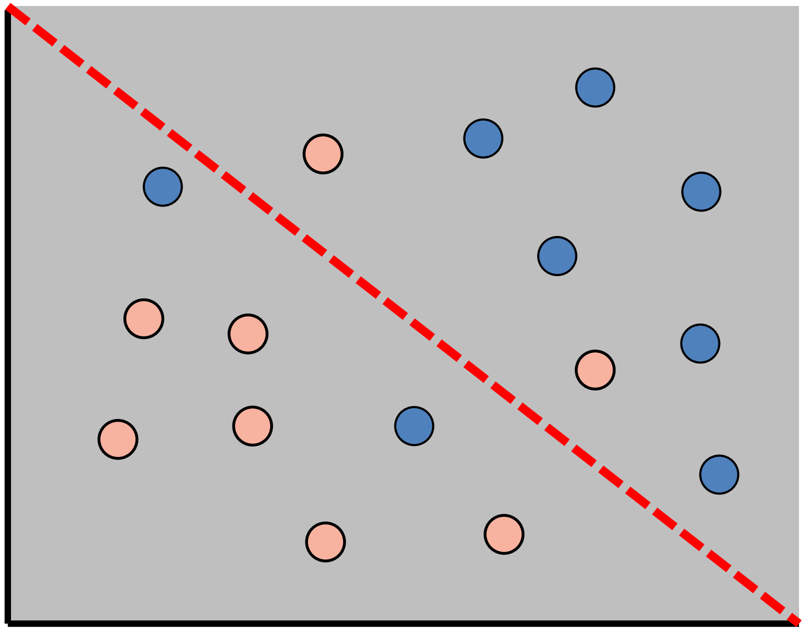

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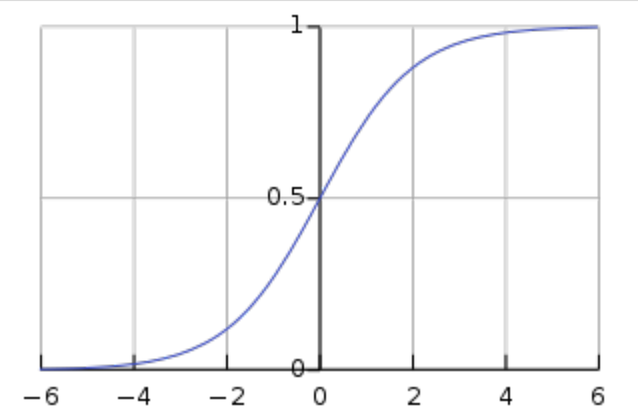
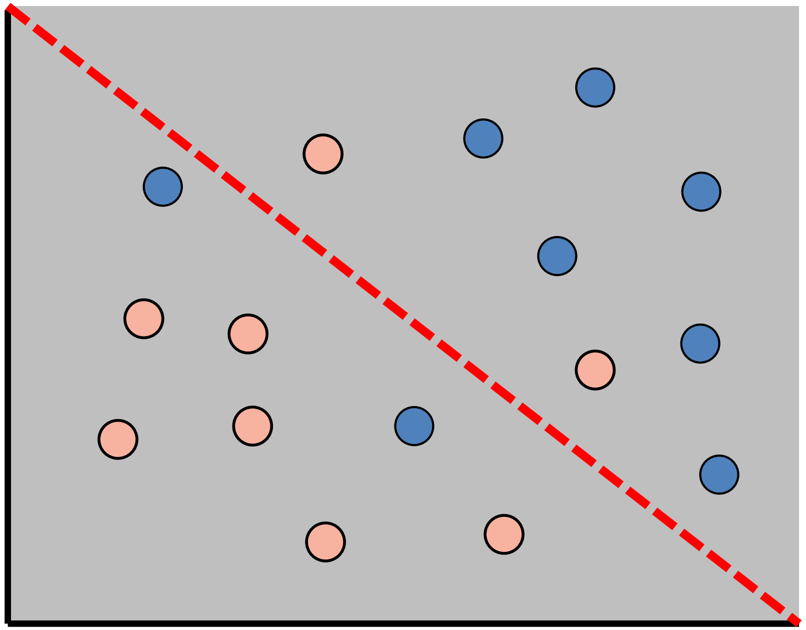
Logistic Regression

Machine Learning Example

Choice of h function?

$$h_{a,b,c}(x,y) = \frac{1}{1+e^{-(ax+by+c)}}$$





$$\max_{a,b,c} \sum_{i=1}^{n} z_i \log(h_{a,b,c}(x_i,y_i)) + (1-z_i)\log(1-h_{a,b,c}(x_i,y_i))$$

Solution: binary search?

Finding a maximum / minimum

Convex / concave function : f[1..n]

3	4	5	7
2	8	12	9
1	4	8	7
0	2	6	3

What happens in d dimensions?

Dimension	Time
1	$O(\log n)$
2	$O(n)$
3	$O(n^2)$
...	...
d	$O(n^{d-1})$

Maximum distance between two points in a d-dimensional cube with side length n::

$O(dn)$

Machine Learning Example

Choice of h function?

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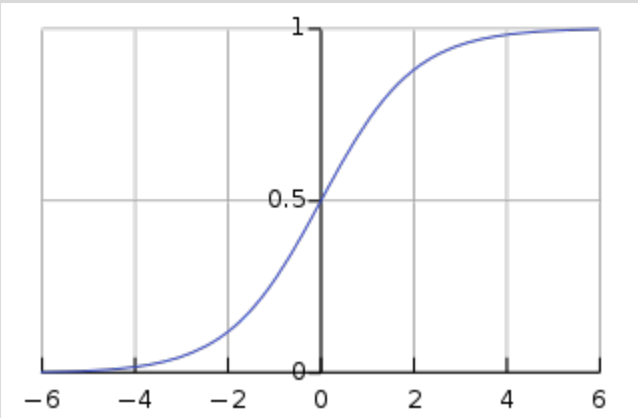

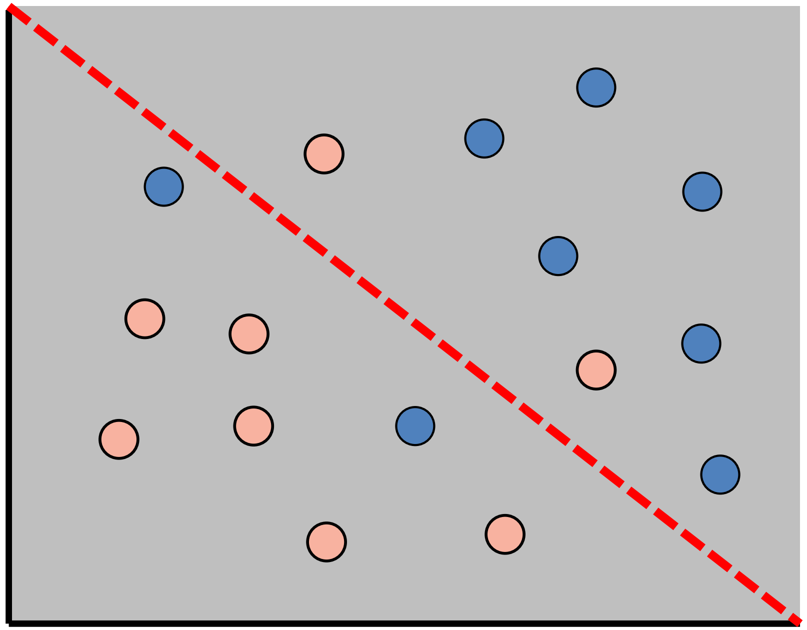

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Solution: binary search?

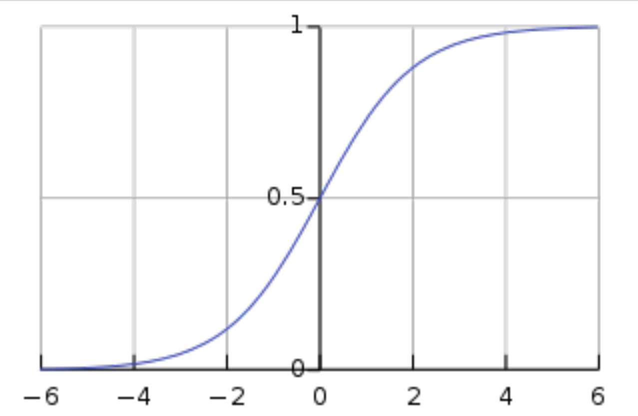
➔ Slow (as dimensions get large).

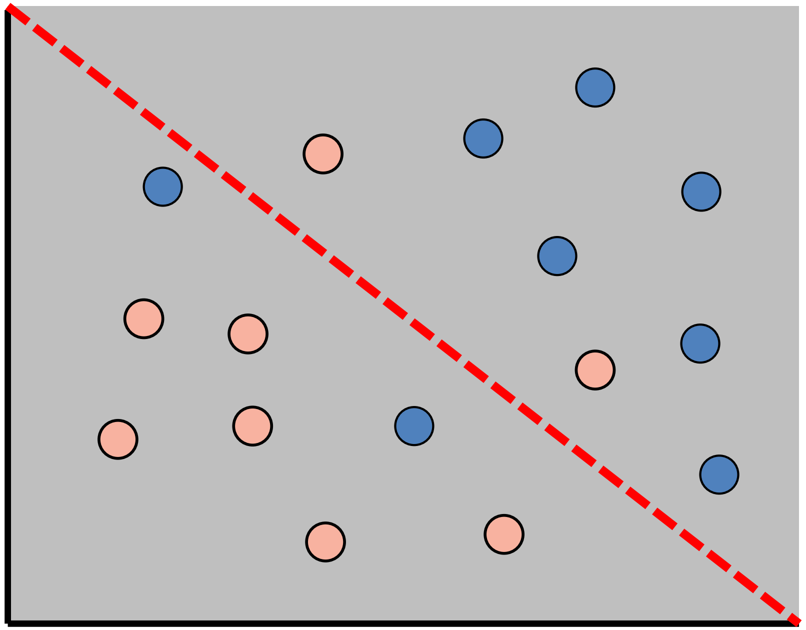
➔ Not very robust to noise.

Machine Learning Example

Choice of h function?

$$h_{a,b,c}(x,y) = \frac{1}{1+e^{-(ax+by+c)}}$$





Solution: set (partial) derivatives to zero

$$\sum_{i=1}^{n}[z_i - h_{a,b,c}(x_i,y_i)]x_i = 0$$

$$\sum_{i=1}^{n}[z_i - h_{a,b,c}(x_i,y_i)]y_i = 0$$

$$\sum_{i=1}^{n}[z_i - h_{a,b,c}(x_i,y_i)] = 0$$

➔ Hard algebraically.

➔ Can approximate numerically

➔ E.g., Newton's method

Machine Learning Example

Choice of h function?

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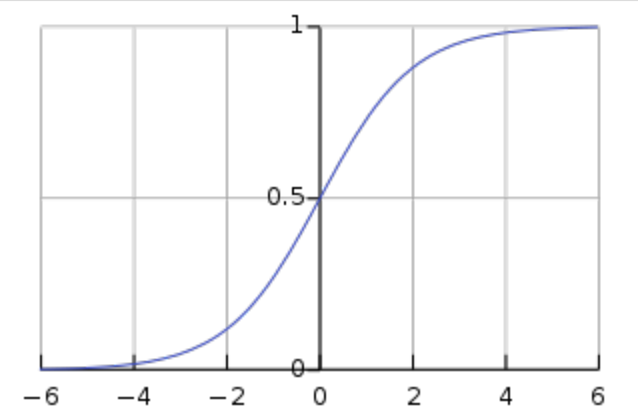

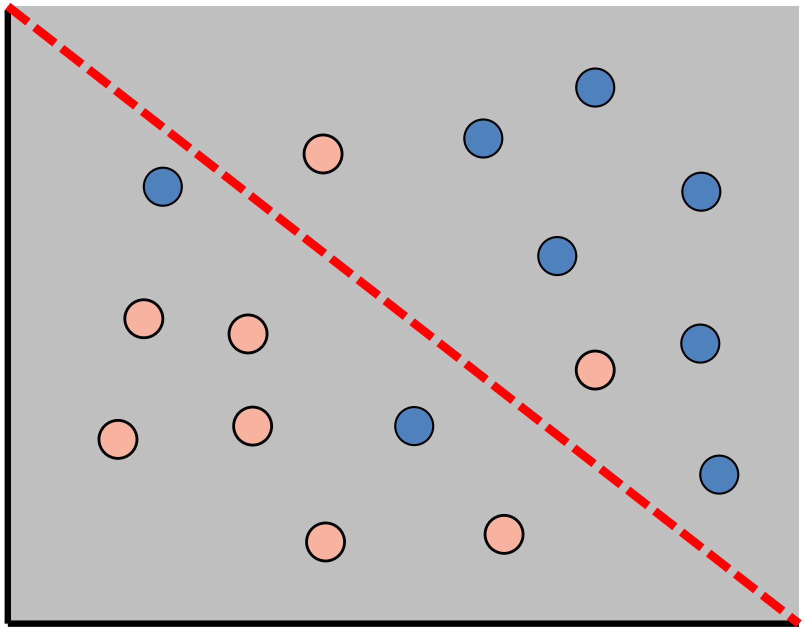

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Solution: approximate directly

- ➔ Hard algebraically.
- ➔ Can approximate numerically
- ➔ E.g., gradient descent

Summary

Today: Optimization

Finding a maximum / minimum

- *Key aspect of machine learning*
- *Binary search isn't always good*

Newton's method

- *Second order approach*

Gradient descent

- *First order approach*

When I say:
"Find x s.t. $f(x) = a^3 - x^{1/3} = 0$"

You can hear:
"Find parameters so that the partial derivatives of the log-likelihood function equal 0"

When I say:
"Find x to minimize
$f(x) = 2x^2 + 4x + 9$"

You can hear:
"Find parameters to maximize the log-likelihood function"

Pop Quiz

What does this code do?

```
float HIDDEN(float number)
{
        int i;
        float x2, y;
        float threehalfs = 1.5f;

        x2 = number*0.5f;
        y = number;
        i = Float.toIntBits(number);       // evil floating bit level hacking
        i = 0x5f3759df – (i >> 1);         // what the *bleep*?
        y = Float.intBitsToFloat(i);
        y = y * (threehalfs – (x2*y*y))    // first iteration
        return y;
}
```

Pop Quiz

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Original comments…

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```

Fake java:

Returns bit representation of a float.

IEEE float:

- 1 sign bit
- 8 (biased) exponent bits
- 23 significand bits

Fake java:

Turns bits back into a float.

Pop Quiz

What does this code do?

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Magic number!

Pop Quiz

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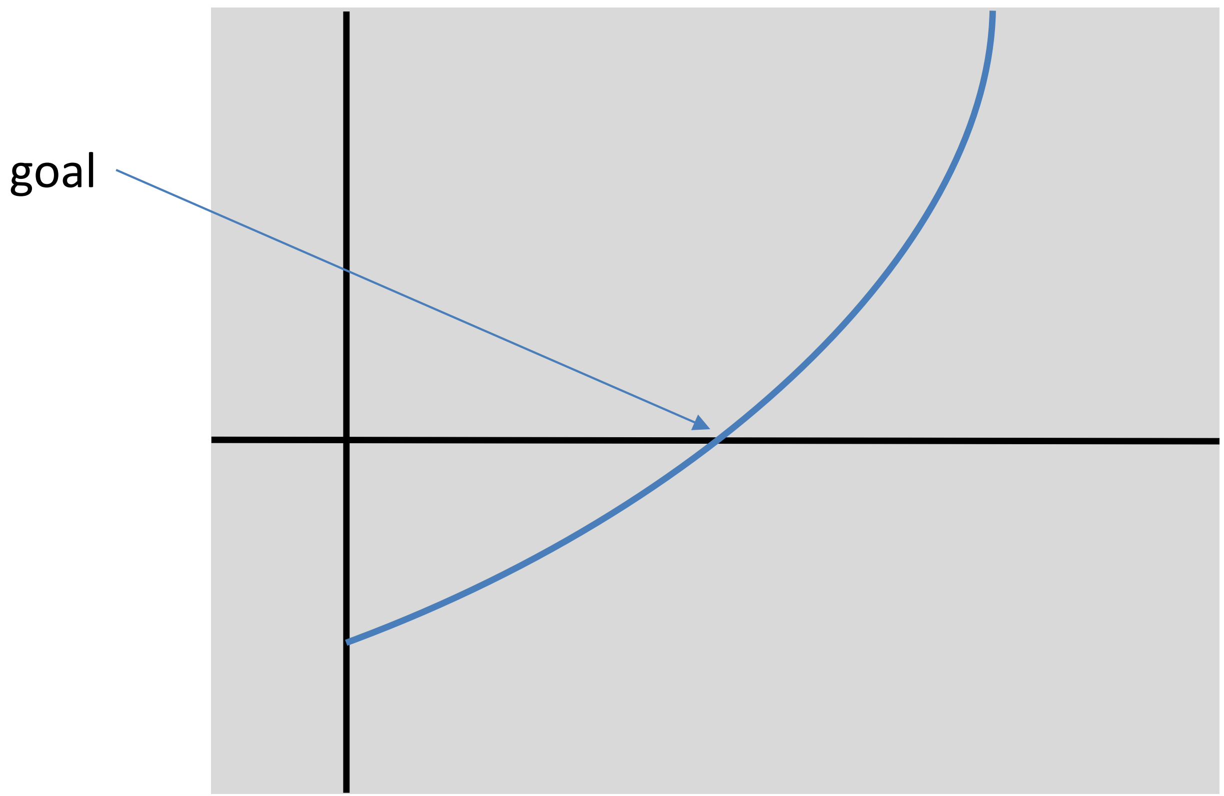
```

Newton's Method

Goal:

Find a root of f(x) = 0 via successive approximation:



Newton’s Method

Note: for this example in one dimension, we could use binary search!

Goal:

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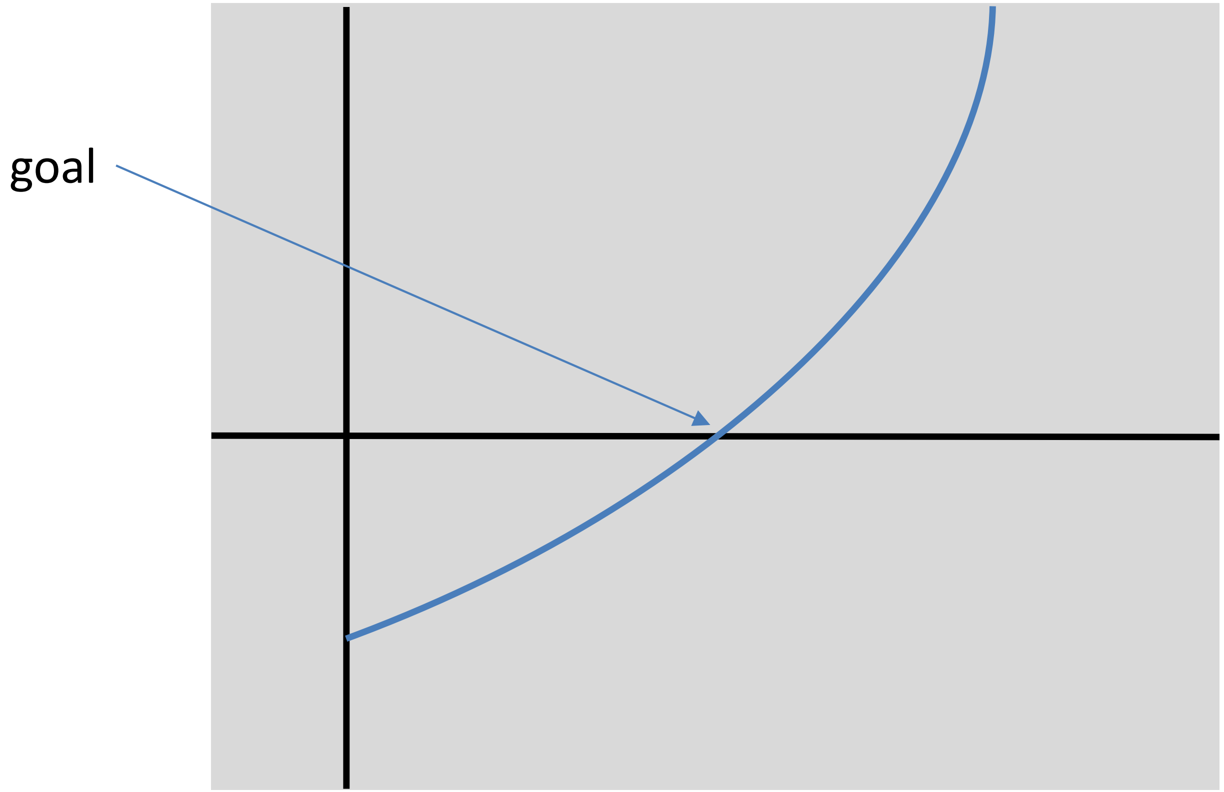

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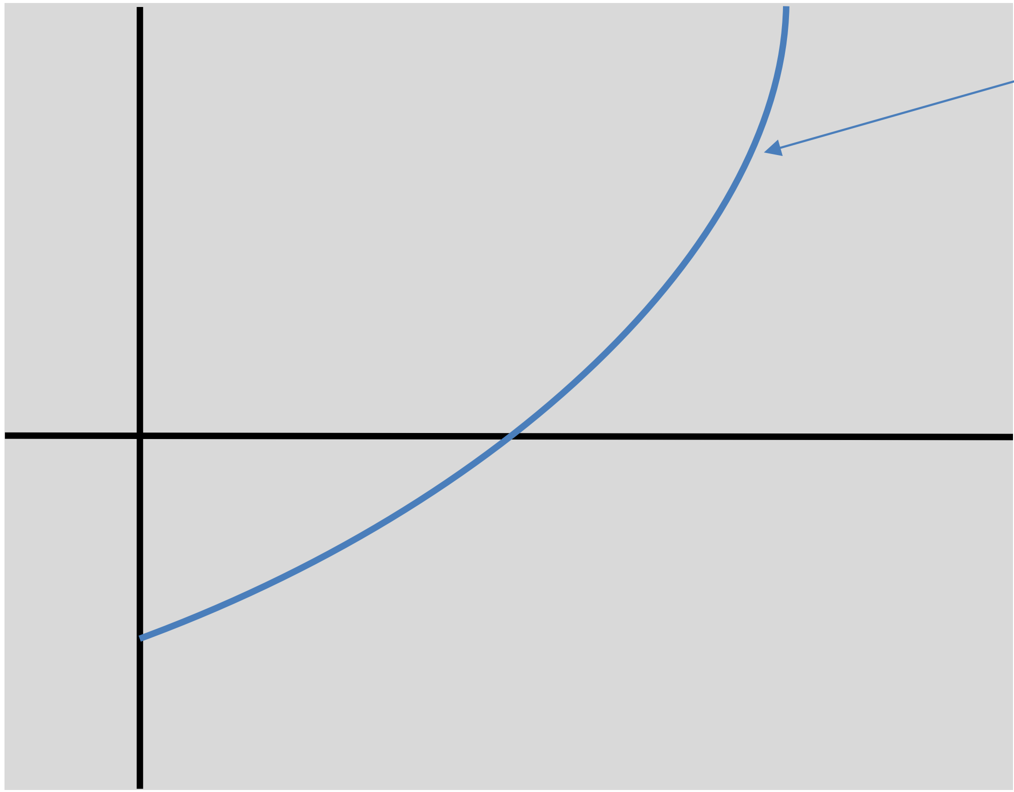

start here:
x_0

Newton's Method

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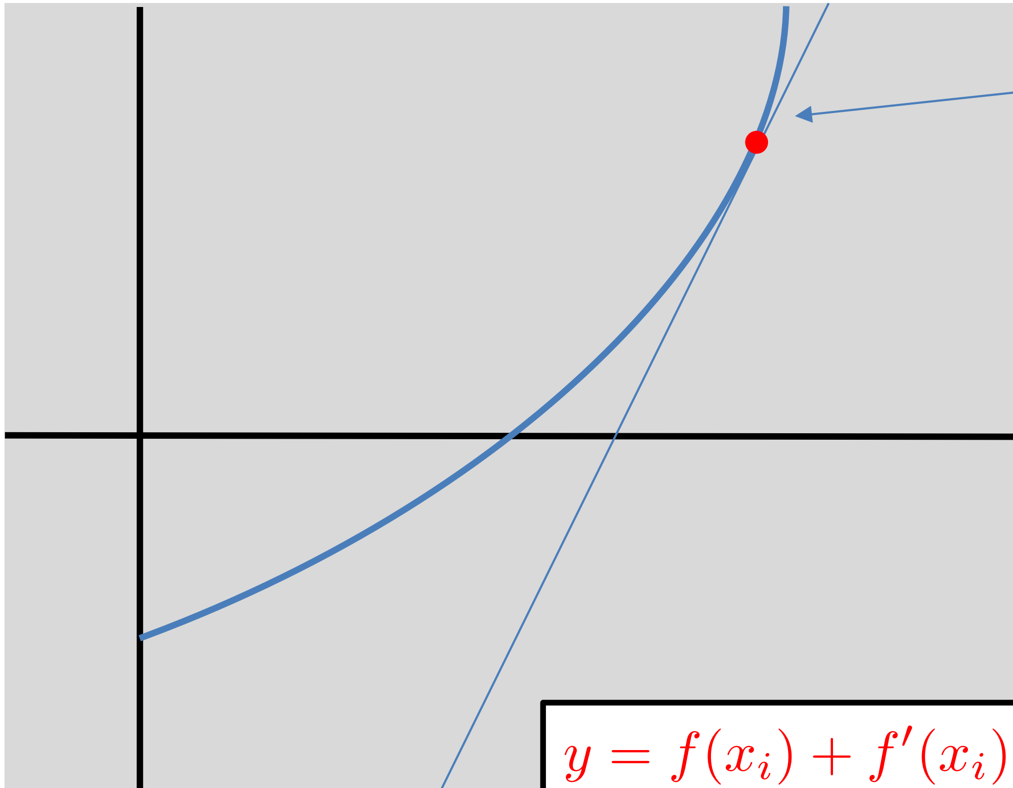

$$\boxed{y = f(x_i) + f'(x_i)(x - x_i)}$$

Newton's Method

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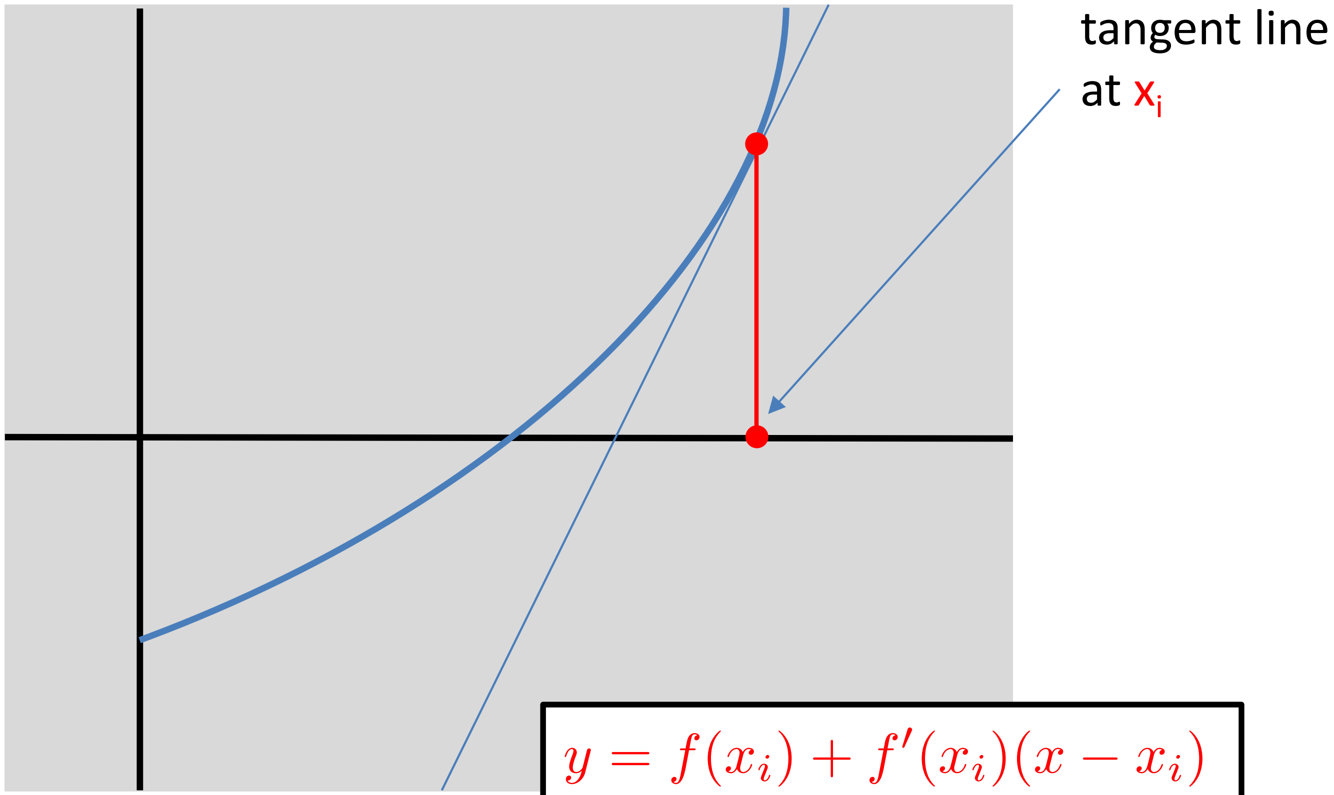

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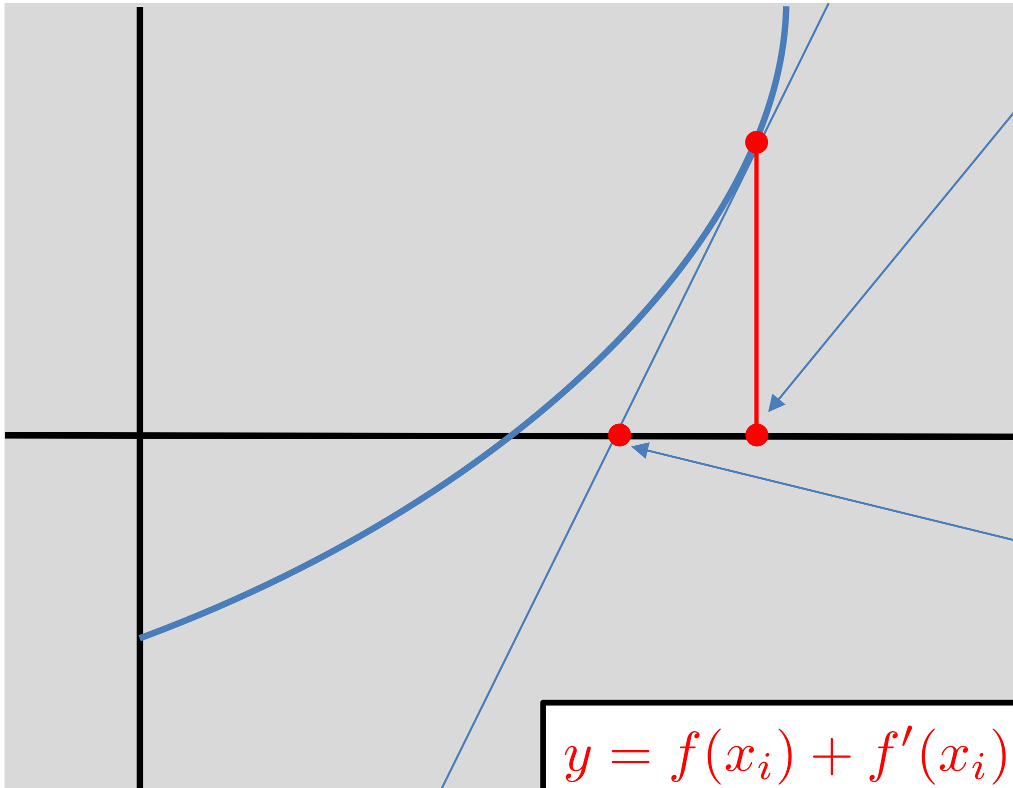

tangent line at x_i

x-intercept of tangent: x_{i+1}

$$\boxed{y = f(x_i) + f'(x_i)(x - x_i)}$$

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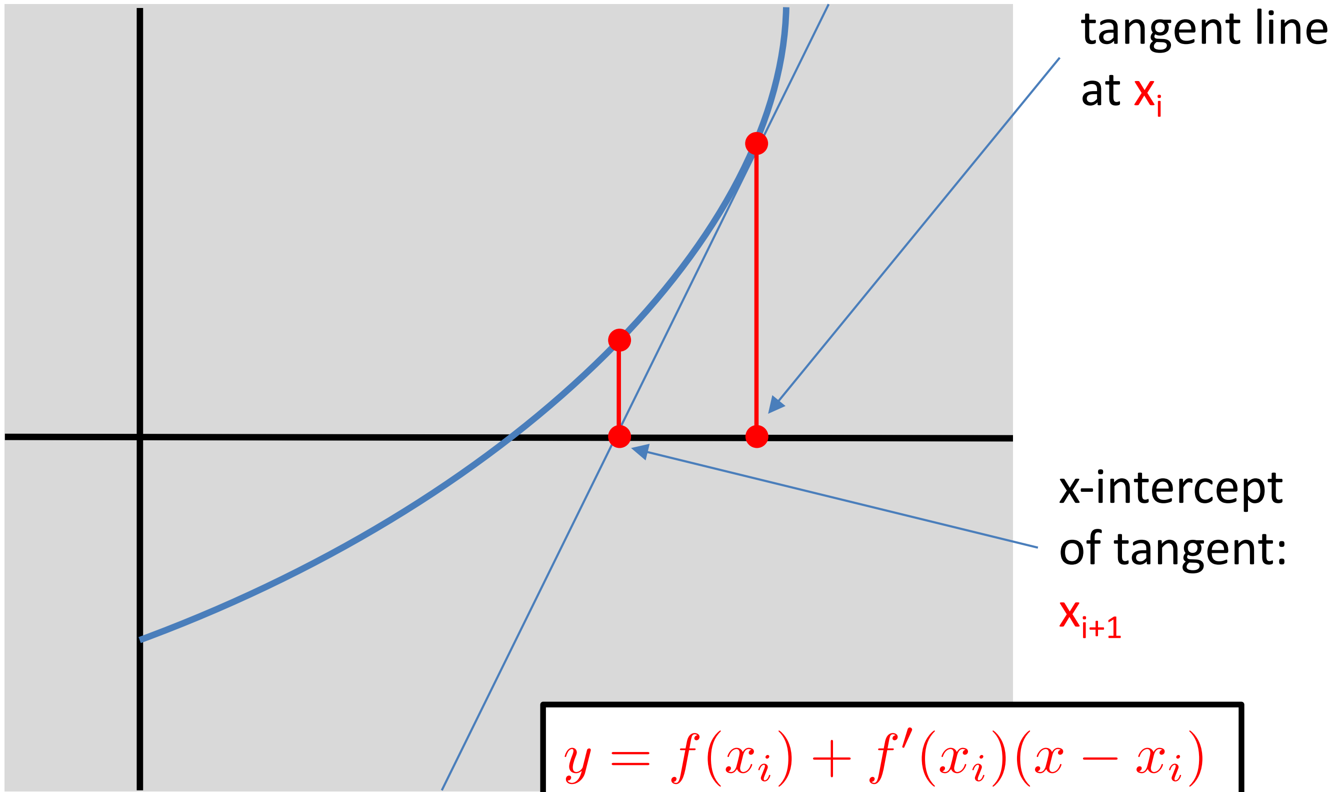


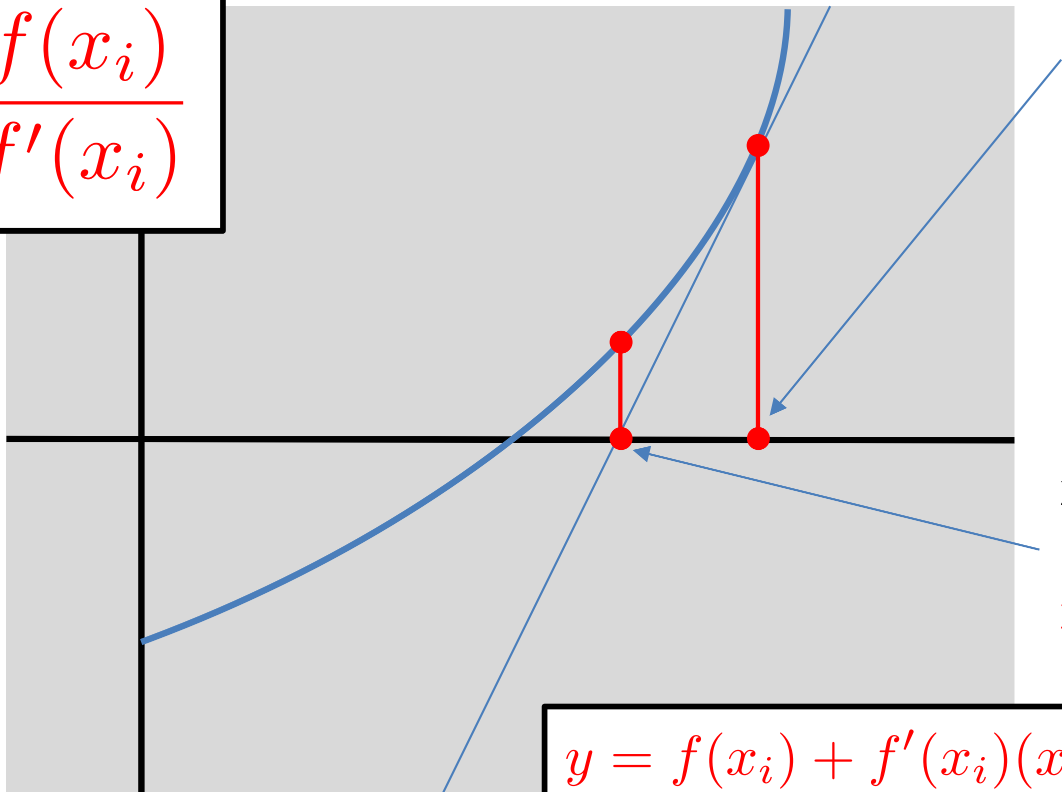
$$y = f(x_i) + f'(x_i)(x - x_i)$$

Newton's Method

Goal:

Find a root of f(x) = 0 via successive approximation:

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$



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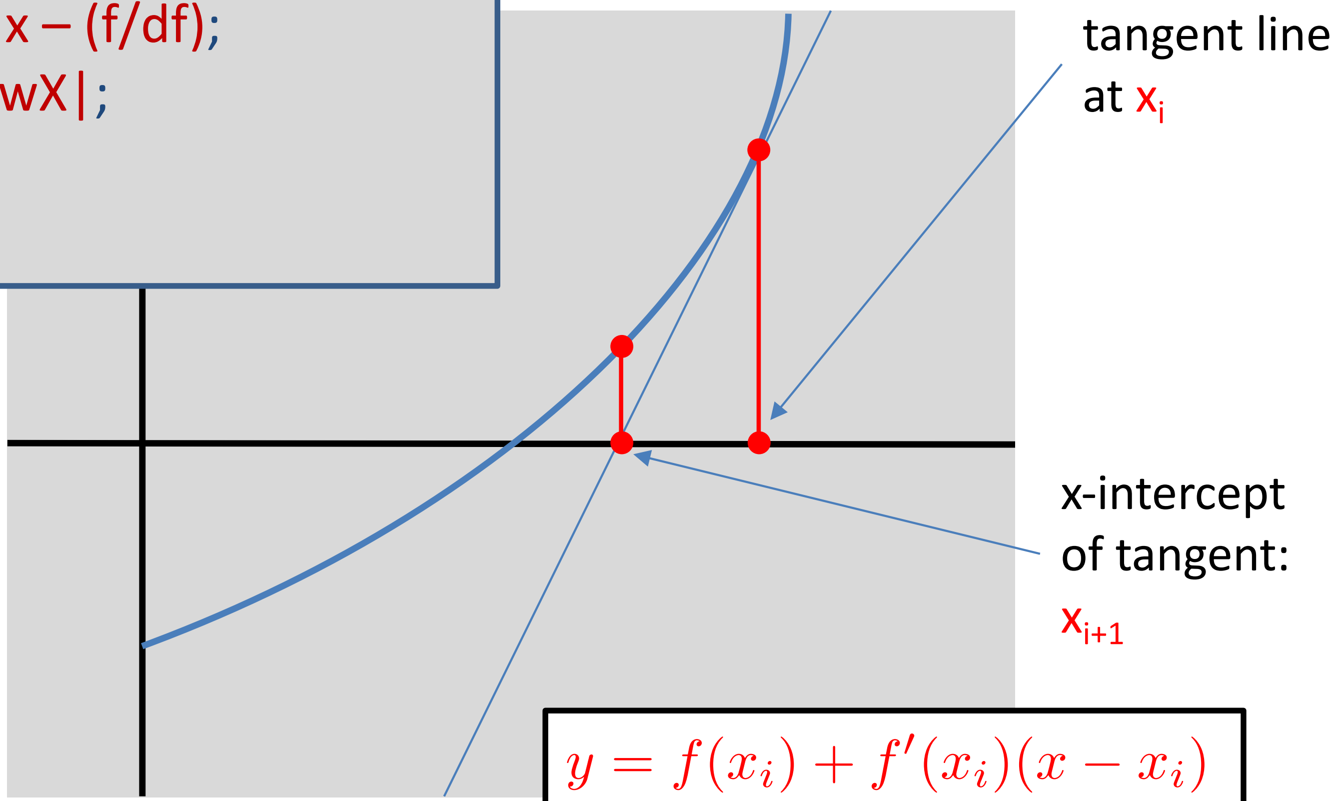
Newton's Method

```
findRoot(float x, float error) {
        diff = error+1;
        while (diff > error){
            float f = function(x);
            float df = derivative(x);
            float newX = x – (f/df);
            diff = |x – newX|;
            x = newX;
        return x
```



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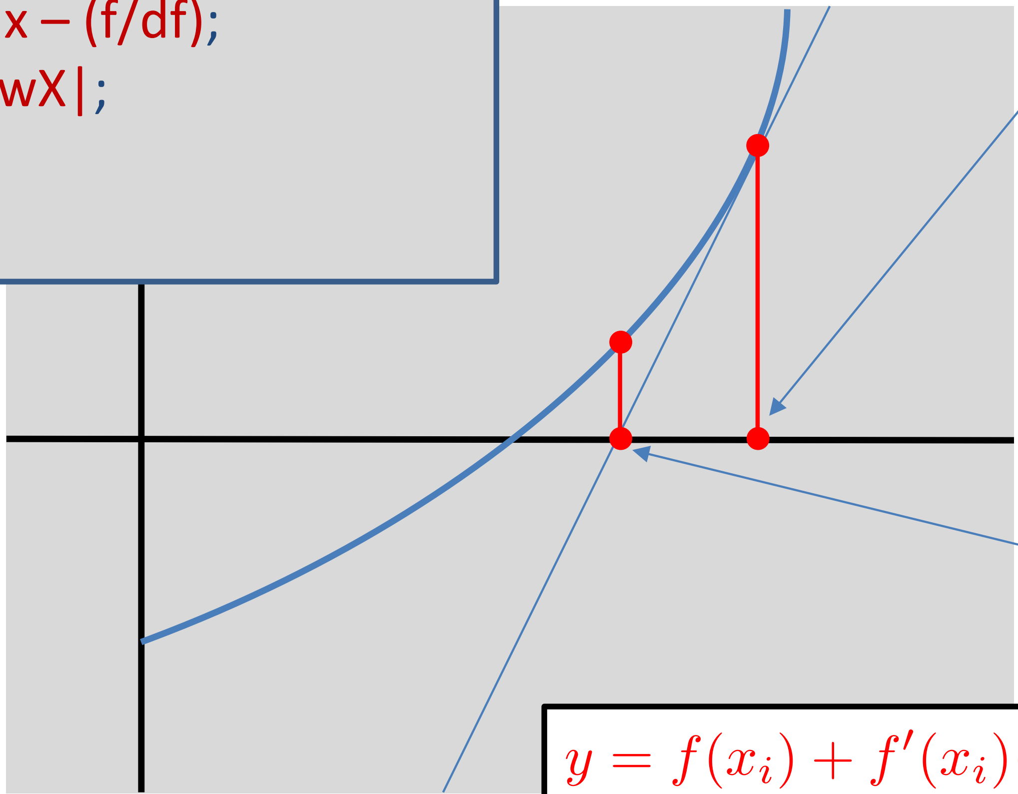
How accurate an answer do you want?



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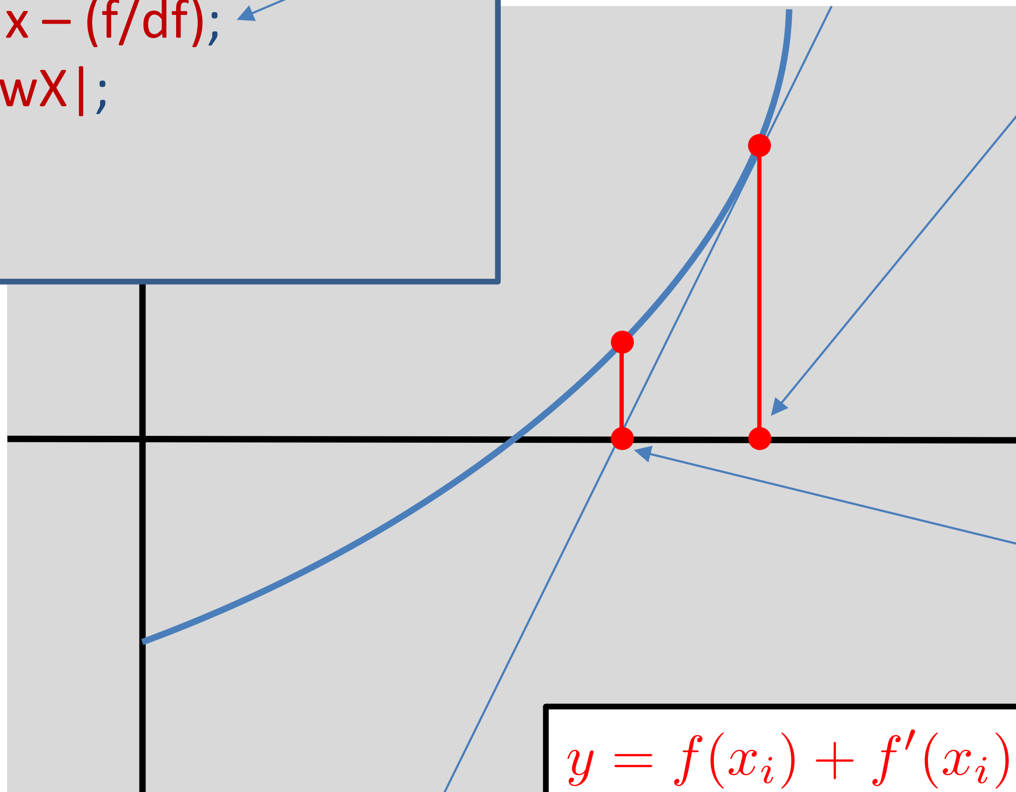
        return x
```

How accurate an answer do you want?

Newton step



$$y = f(x_i) + f'(x_i)(x - x_i)$$

Newton's Method

Example: compute $a^{1/3}$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Function: $f(x) = a^{1/3} - x$

Derivative: $f'(x) = -1$

Newton's Method

Example: compute $a^{1/3}$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Function: $f(x) = a^{1/3} - x$

Derivative: $f'(x) = -1$

Update: $x_{i+1} = x_i + (a^{1/3} - x_i) = a^{1/3}$

Correct, but not very useful…

Newton's Method

Example: compute $a^{1/3}$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Function: $f(x) = a - x^3$

Derivative: $f'(x) = -3x^2$

Newton's Method

Example: compute $a^{1/3}$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Function: $f(x) = a - x^3$

Derivative: $f'(x) = -3x^2$

Update: $$\begin{aligned} x_{i+1} &= x_i - \frac{a - x_i^3}{-3x_i^2} = x_i + \frac{a}{3x_i^2} - (2/3)x_i \\ &= \frac{2x_i}{3} + \frac{a}{3x_i^2} \end{aligned}$$

Newton's Method

How fast does it converge?

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = \frac{2x_i}{3} + \frac{a}{3x_i^2}$$

Function: $f(x) = a - x^3$

Derivative: $f'(x) = -3x^2$

<u>Quadratic convergence:</u>

Every iteration, the number of correct digits doubles.

Iteration	Value
0	2
1	2.6750000000000003
2	2.5333260546772647
3	2.5251076782078234
4	2.525080872118423
5	2.525080871833849
TRUE:	2.525080871833849

Newton's Method

How fast does it converge?

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} \quad = \quad \frac{2x_i}{3} + \frac{a}{3x_i^2}$$

Function: $f(x) = a - x^3$

Derivative: $f'(x) = -3x^2$

<u>Quadratic convergence:</u>

Every iteration, the number of correct digits doubles.

d digits of accuracy

➔

O(log d) iterations

error ≤ 10^{-67}

➔

7 iterations

Convergence analysis

Assume:

$$\epsilon_i < 1$$

$$x_i = a^{1/3}(1+\epsilon_i)$$

Newton's Method iteration

plug in approximation assumption

algebra

$$\begin{aligned}
x_{i+1} &= \frac{2x_i}{3} + \frac{a}{3x_i^2} \\
&= \frac{2a^{1/3}(1+\epsilon_i)}{3} + \frac{a}{3a^{2/3}(1+\epsilon_i)^2} \\
&= \frac{2a(1+\epsilon_i)^3 + a}{3a^{2/3}(1+\epsilon_i)^2} \\
&= a^{1/3} \cdot \left[\frac{2(1+\epsilon_i)^3 + 1}{3(1+\epsilon)^2}\right] \\
&= a^{1/3} \cdot \left[\frac{3(1+\epsilon)^2 + 2(1+\epsilon_i)^3 - 3(1+\epsilon)^2 + 1}{3(1+\epsilon)^2}\right] \\
&= a^{1/3} \cdot \left[1 + \frac{2(1+\epsilon_i)^3 - 3(1+\epsilon)^2 + 1}{3(1+\epsilon)^2}\right] \\
&= a^{1/3} \cdot \left[1 + \frac{2\epsilon_i^3 + 3\epsilon_i^2}{3(1+\epsilon)^2}\right]
\end{aligned}$$

quadratic convergence

$$\begin{aligned}
\epsilon_{i+1} &= \frac{2\epsilon_i^3 + 3\epsilon_i^2}{3(1+\epsilon)^2} \\
&\leq \left(\frac{5}{3}\right)\epsilon^2
\end{aligned}$$

Newton's Method

How fast does it converge?

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = \frac{2x_i}{3} + \frac{a}{3x_i^2}$$

Function: $f(x) = a - x^3$

Derivative: $f'(x) = -3x^2$

Quadratic convergence:

Every iteration, the number of correct digits doubles.

General conditions for quadratic convergence:

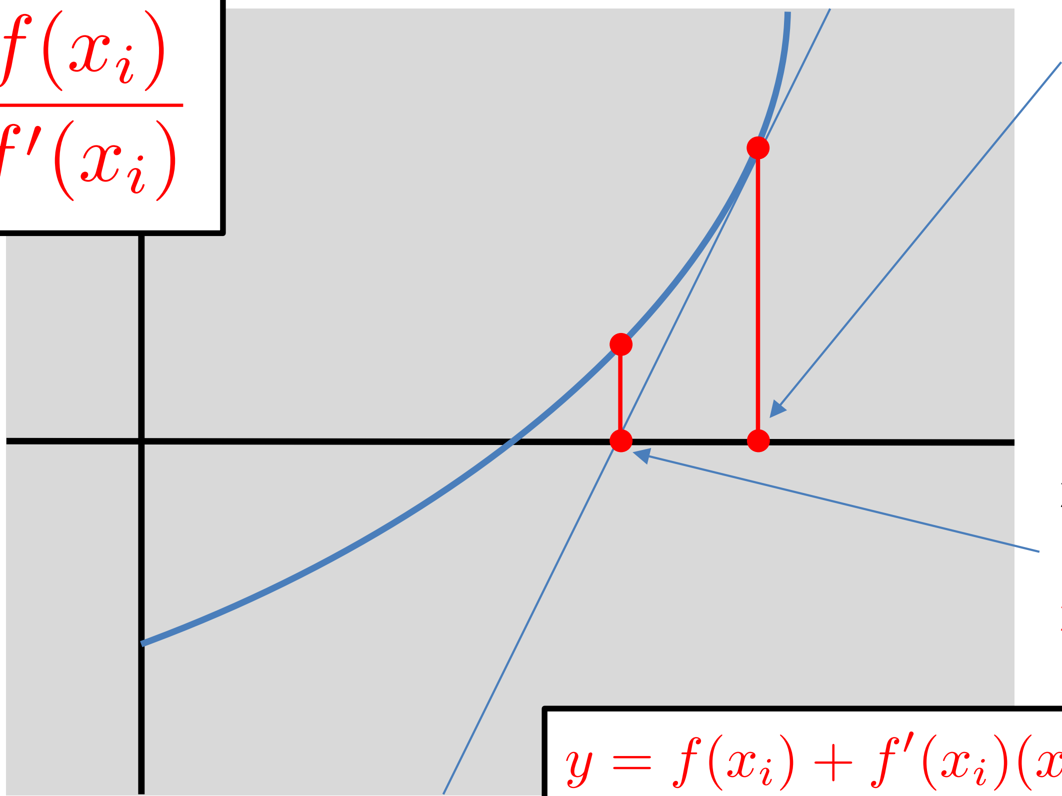
Function f continuously differentiable, derivative non-zero at root, has a second derivative at root.

Newton's Method

Goal:

Find a root of $f(x) = 0$ via successive approximation:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

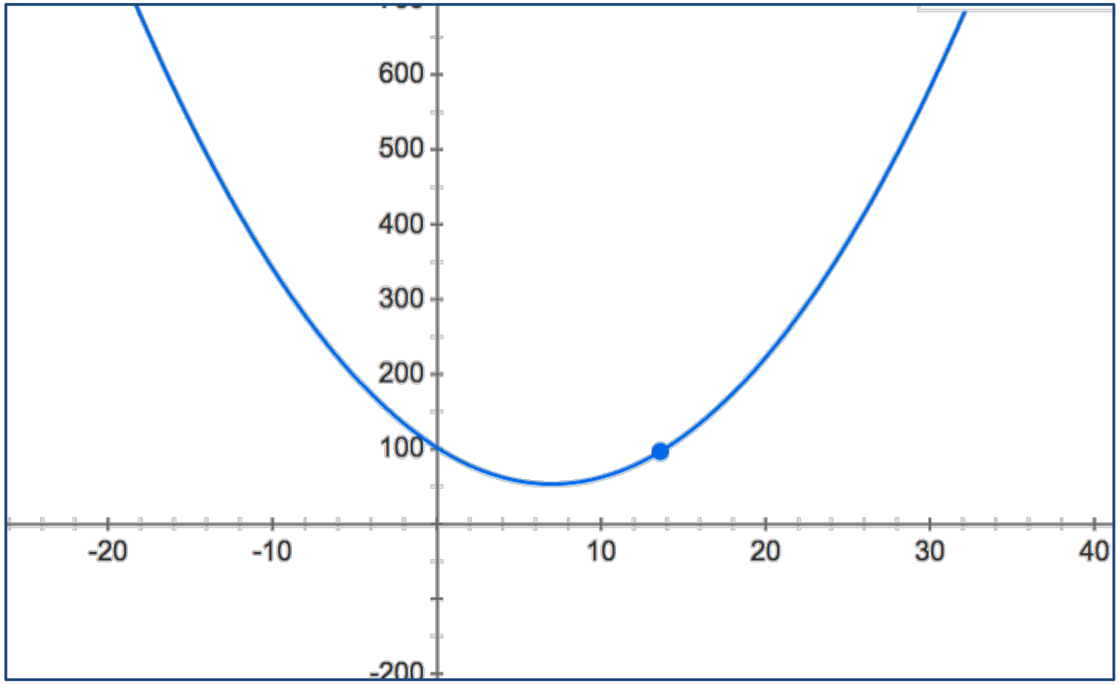


$$y = f(x_i) + f'(x_i)(x - x_i)$$

Newton's Method Part 2

Goal:

Find the minimum value of f(x) via successive approximation



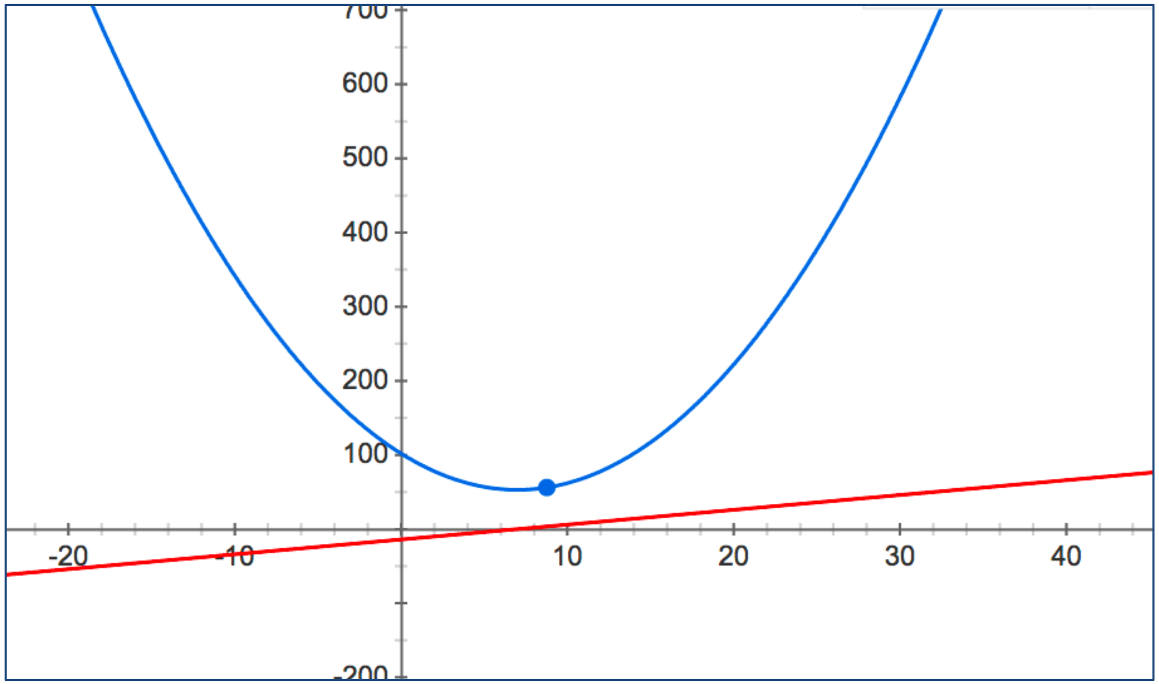
Newton's Method Part 2

Goal:

Find the minimum value of f(x) via successive approximation:

Reduction:

Find a root of f'(x) = 0.



Newton's Method Part 2

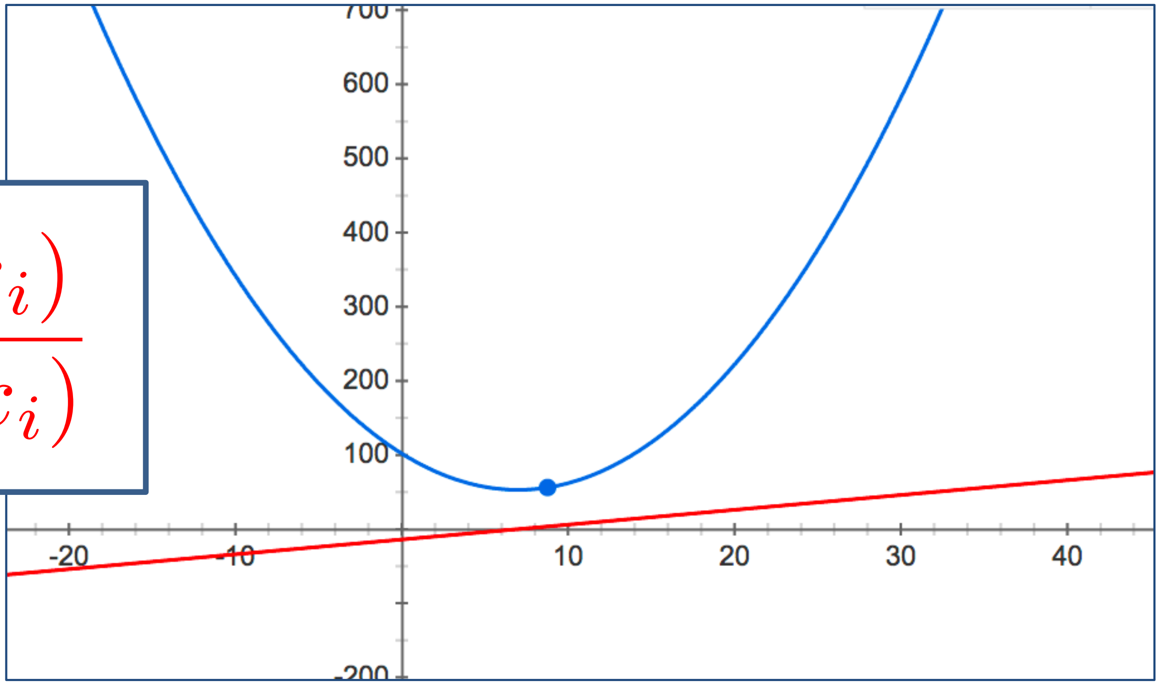
Goal:

Find the minimum value of f(x) via successive approximation:

Reduction:

Find a root of f'(x) = 0.

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)}$$



Newton's Method Continued

Example:

Find the maximum value of f(x) via successive approximation:

$$\max_{a,b,c} \sum_{i=1}^{n} z_i \log(h_{a,b,c}(x_i, y_i)) + (1 - z_i) \log(1 - h_{a,b,c}(x_i, y_i))$$

$$\sum_{i=1}^{n} [z_i - h_{a,b,c}(x_i, y_i)] x_i = 0$$

$$\sum_{i=1}^{n} [z_i - h_{a,b,c}(x_i, y_i)] y_i = 0$$

$$\sum_{i=1}^{n} [z_i - h_{a,b,c}(x_i, y_i)] = 0$$

Newton's Method Continued

What about higher dimensions?

Find the minimum value of f(x) via successive approximation:

Reduction:

Find a root of f'(x) = 0.

$$\boxed{x_{i+1} = x_i - (\nabla f(x_i))(\nabla^2 f(x_i))^{-1}}$$

derivative ➔ gradient (vector)

second derivative ➔ Hessian (matrix)
(division ➔ inverse)

Newton's Method Summary

Pros:

Fast convergence.

Relatively simple.

<u>Compare to binary search:</u>

- $O(d^c)$ cost per iteration, d is the dimension
- Quadratic convergence ➔ few iterations

Cons:

Slow computation in higher dimensions.

Requires access to derivative, second derivative of f.

Example problem:
Computing the inverse of the Hessian…

Summary

Today: Optimization

Finding a maximum / minimum

- *Key aspect of machine learning*
- *Binary search isn't always good*

Newton's method

- *Second order approach*

Gradient descent

- *First order approach*

When I say:
"Find x s.t. $f(x) = a^3 - x^{1/3} = 0$"

You can hear:
"Find parameters so that the partial derivatives of the log-likelihood function equal 0"

When I say:
"Find x to minimize
$f(x) = 2x^2 + 4x + 9$"

You can hear:
"Find parameters to maximize the log-likelihood function"

Newton's Method ➔ Gradient Descent

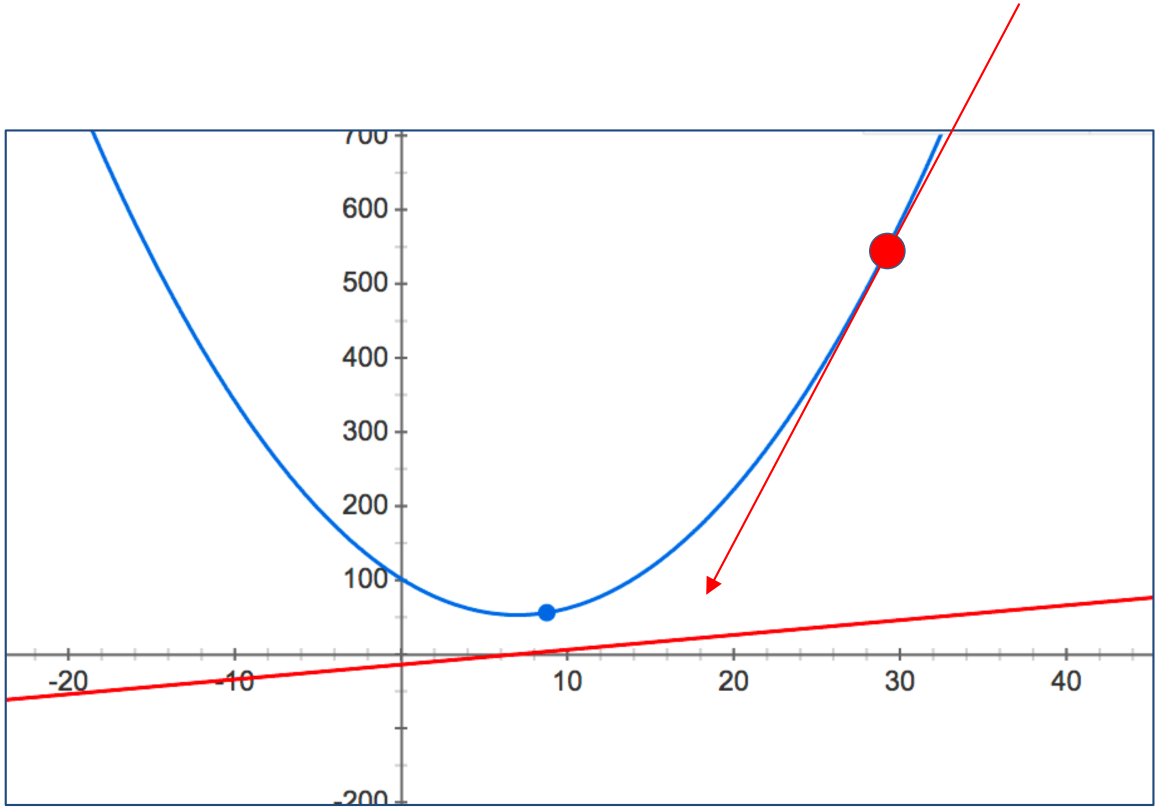
Goal:

Find the minimum value of f(x) via successive approximation:

Reduction:

Find a root of f'(x) = 0.

$$\boxed{x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)}}$$



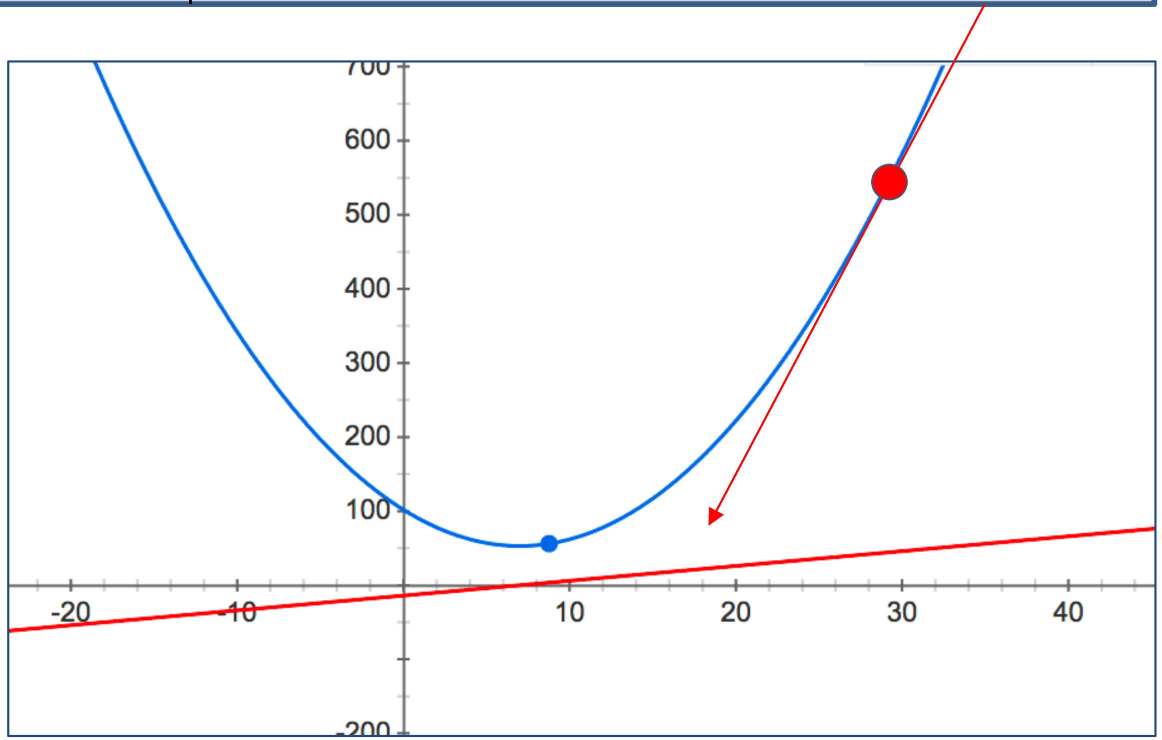
Newton's Method → Gradient Descent

Basic idea of Newton's Method:	Problem:
• Gradient tells you which direction to move. • Hessian tells you how far to move.	Hessian is expensive to compute. Solution: Replace it with a constant.

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)}$$

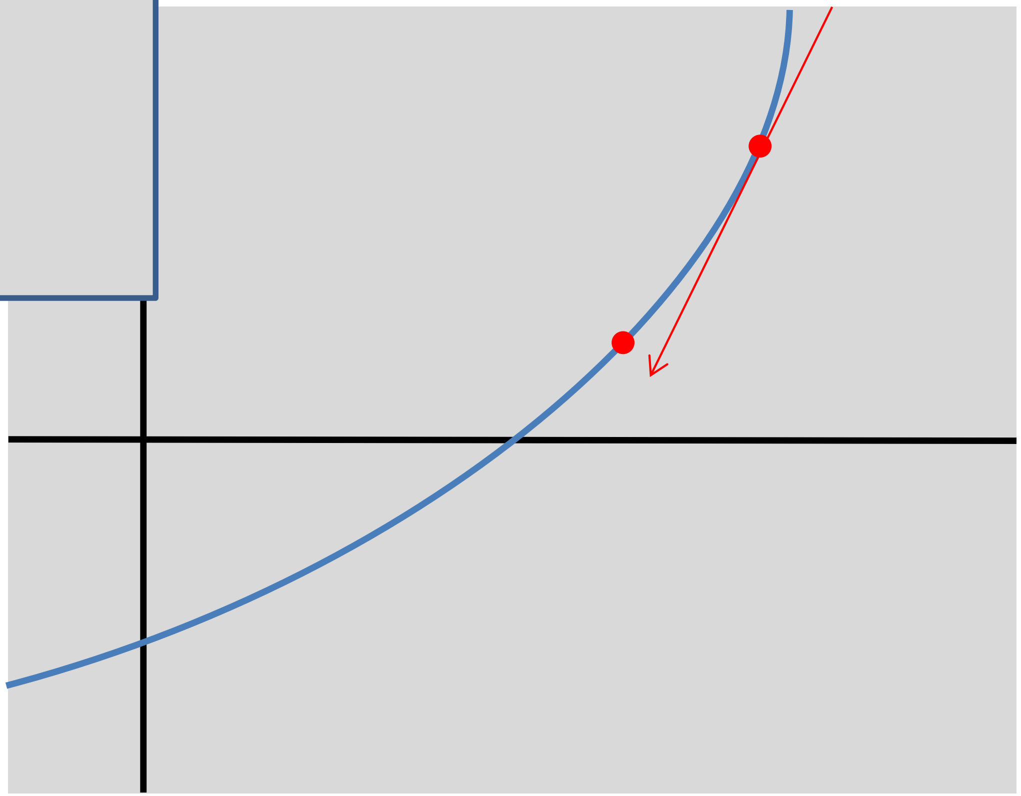
⇩

$$x_{i+1} = x_i - \gamma f'(x)$$



Gradient Descent

```
findMin(float x, float step, float error) {
      diff = error+1;
      while (diff > error){
          float df = derivative(x);
          float newX = x – (step*df);
          diff = |x – newX|;
          x = newX;
      return x
```



Gradient Descent

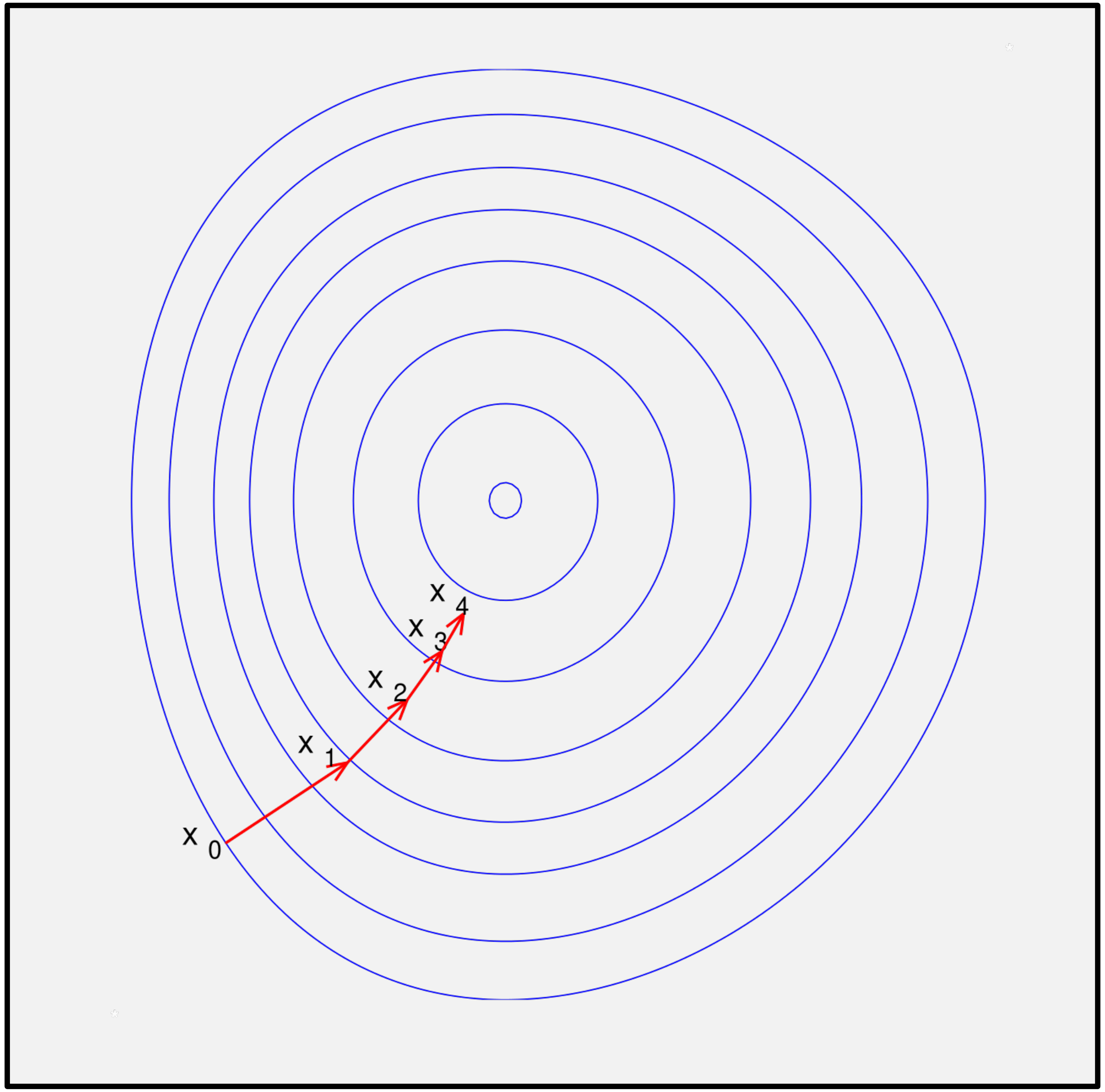


Image from:
Wikipedia

Gradient Descent

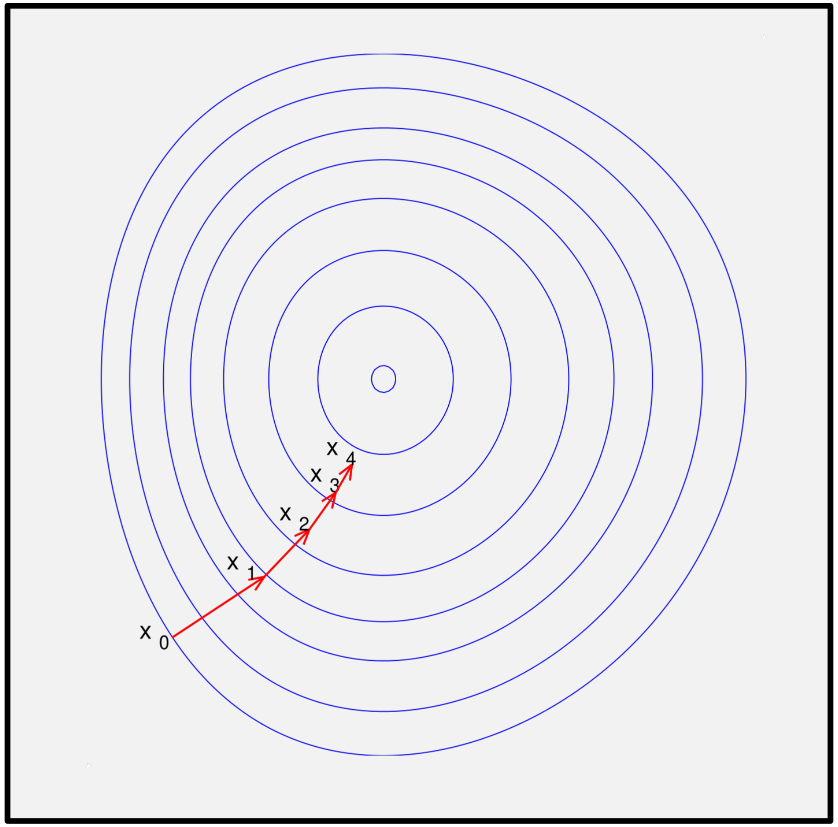
Goal: (in d dimensions)

Find the minimum value of f(x) via successive approximation:

$$\boxed{x_{i+1} = x_i - \gamma \nabla(f(x))}$$

d-dimensional vectors

scalar

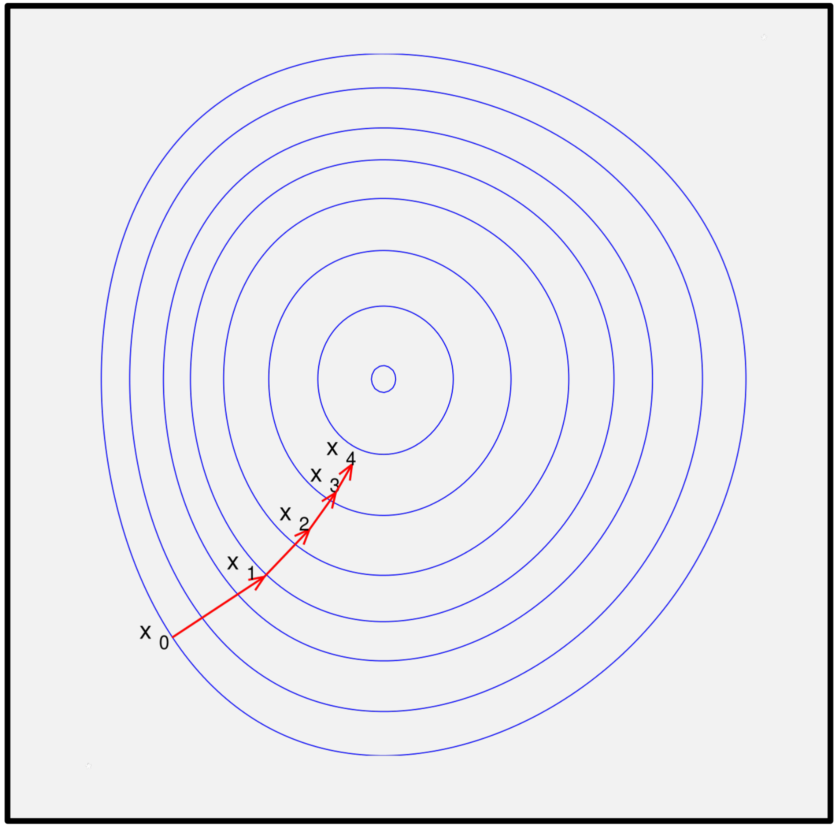


Gradient Descent

Goal: (in d dimensions)

Find the minimum value of f(a,b,c) via successive approximation:

$$(a,b,c)_{i+1} = (a,b,c) - \gamma \nabla (f(a,b,c))$$



<u>Notice:</u>
Each derivative computation may involve iterating over your entire data set!

Stochastic GD

Example:

$$\sum_{i=1}^{n} [z_i - h_{a,b,c}(x_i, y_i)] x_i$$

$$\sum_{i=1}^{n} [z_i - h_{a,b,c}(x_i, y_i)] y_i$$

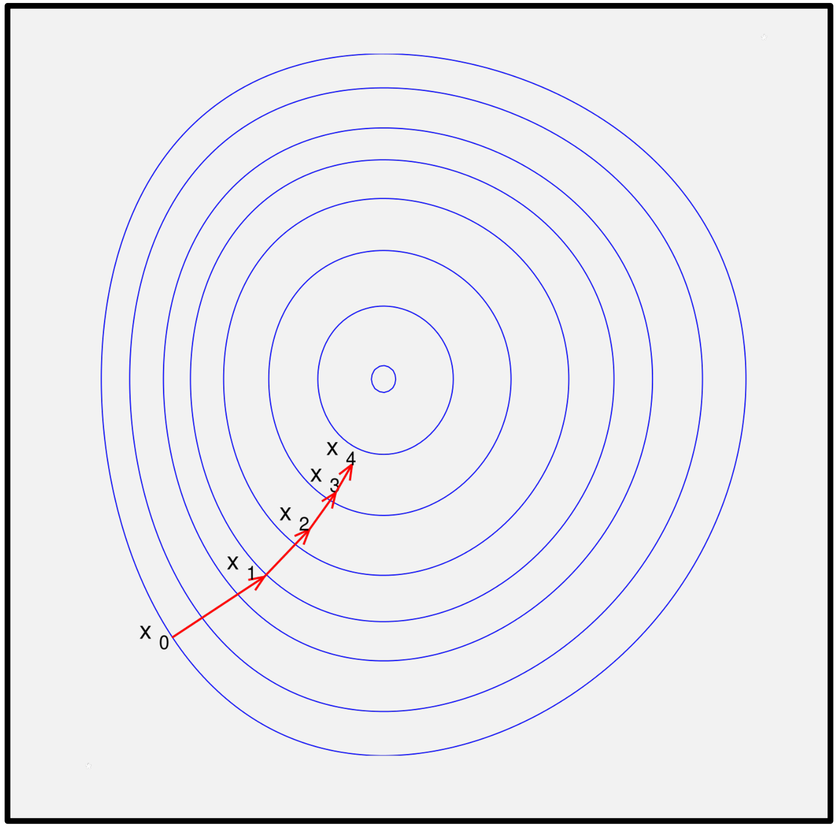
$$\sum_{i=1}^{n} [z_i - h_{a,b,c}(x_i, y_i)]$$

Gradient Descent

Goal: (in d dimensions)

Find the minimum value of f(x) via successive approximation:

$$\boxed{x_{i+1} = x_i - \gamma \nabla(f(x))}$$



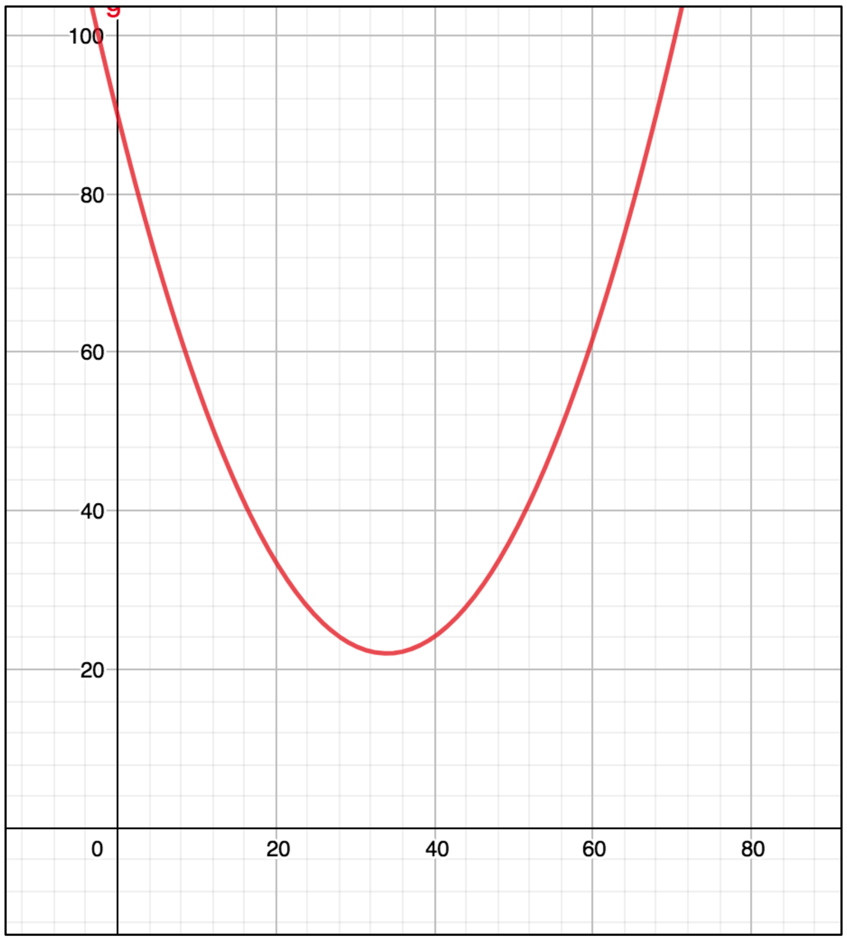
Cost per iteration:

- d computations of partial derivatives
- d multiplications
- d subtractions

Gradient Descent

Convergence:

A function f is convex if…

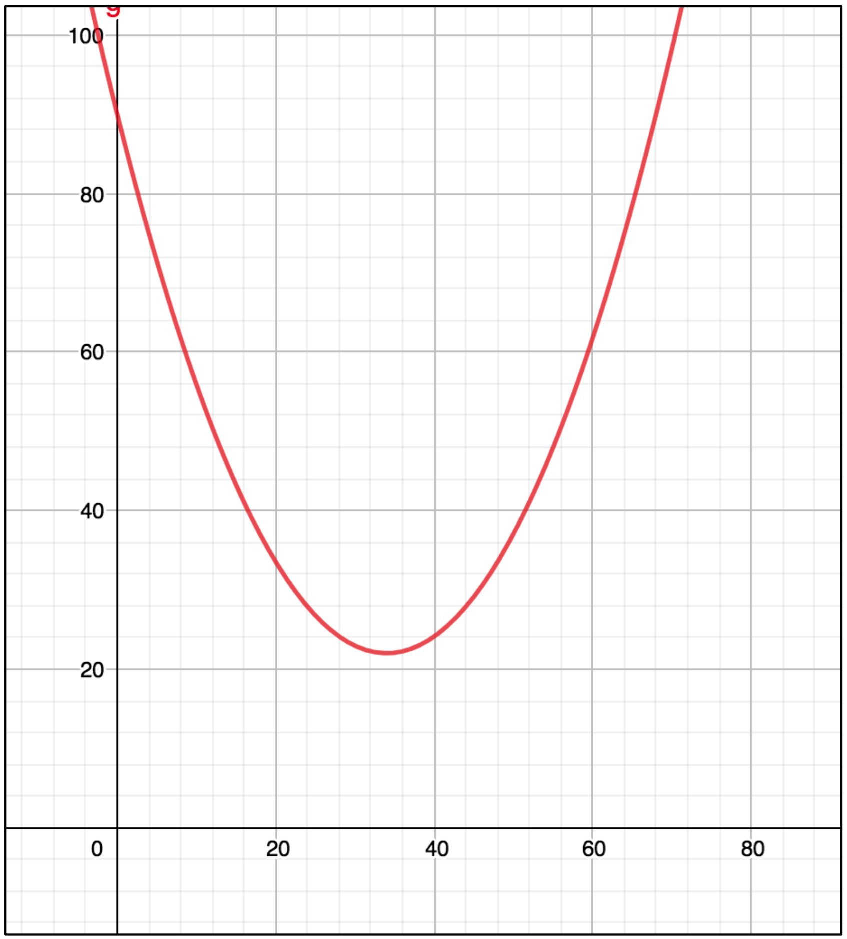


Gradient Descent

Convergence:

A function $\color{red}{f}$ is convex if…

➔ has a minimum



Gradient Descent

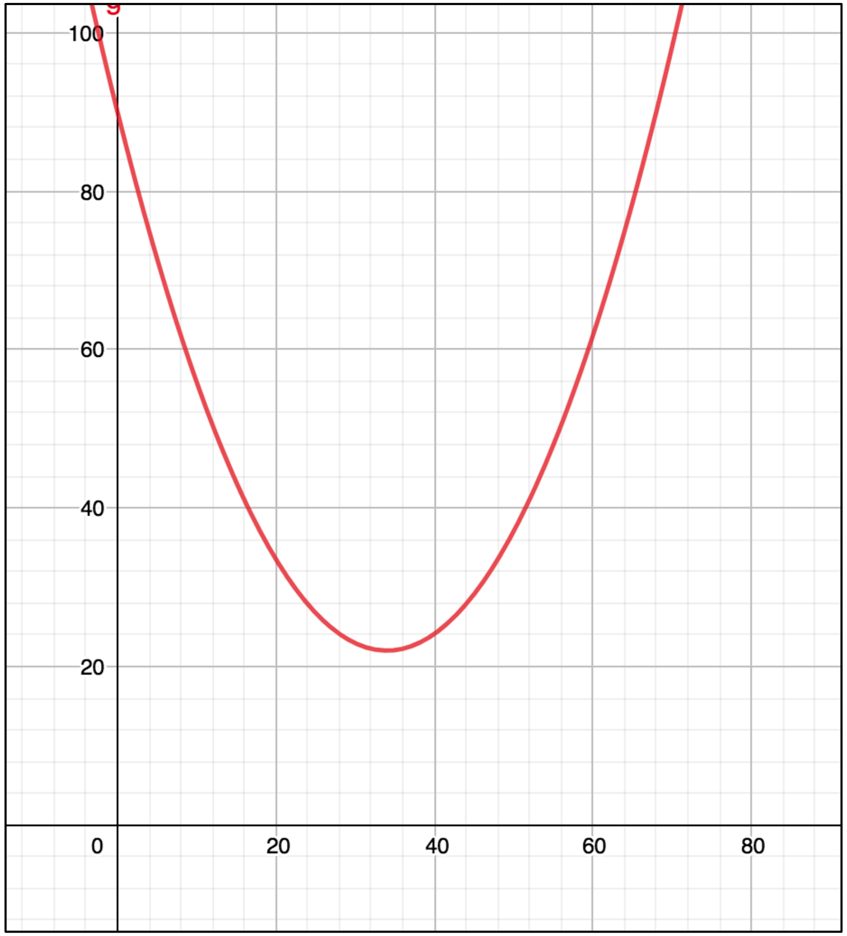
Convergence:

A function f is convex if…

A function f has a gradient that is L-Lipschitz if…

(gradient doesn't change too fast)

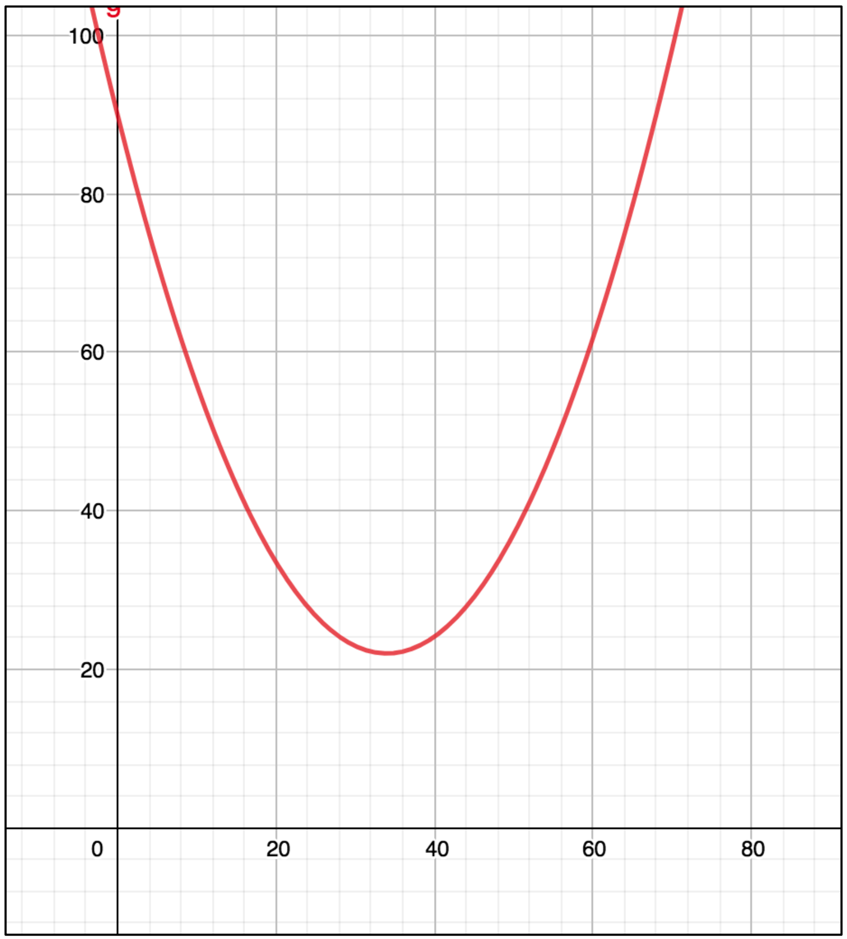
$$||\nabla f(x) - \nabla f(y)|| \leq L||x - y||$$



Gradient Descent

Convergence:

If f is convex and differentiable and its gradient is L-Lipschitz, and if step size $\gamma \leq 1/L$:

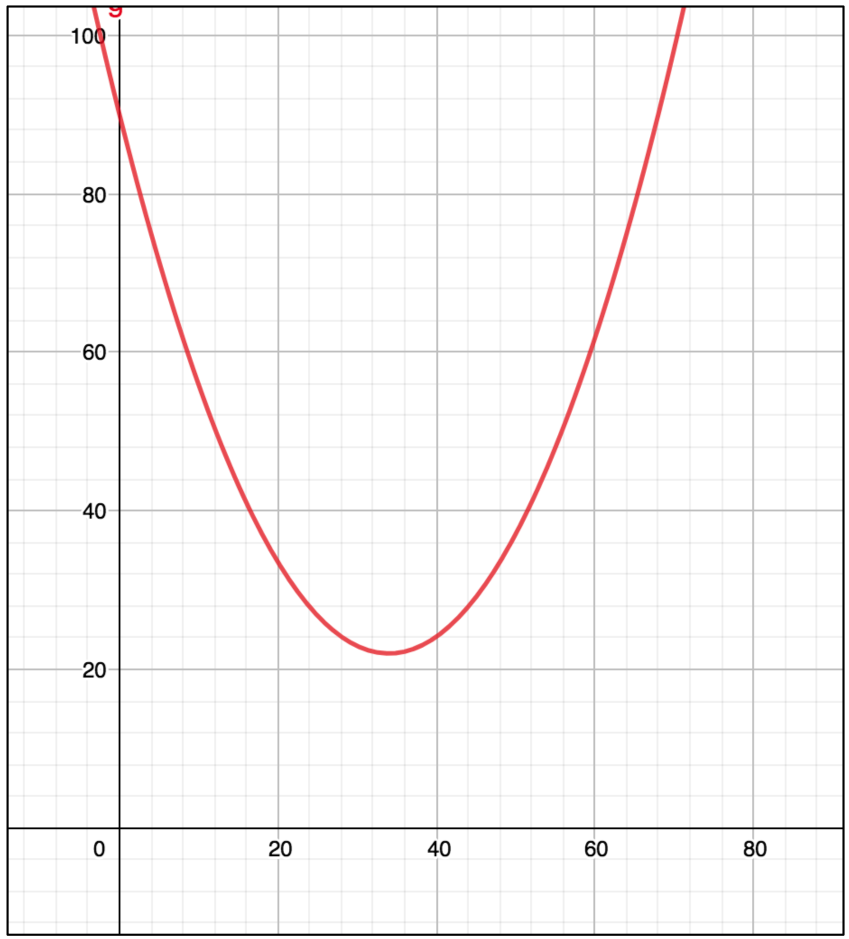


Gradient Descent

Convergence:

If f is convex and differentiable and its gradient is L-Lipschitz, and if step size **$\gamma \leq 1/L$**:

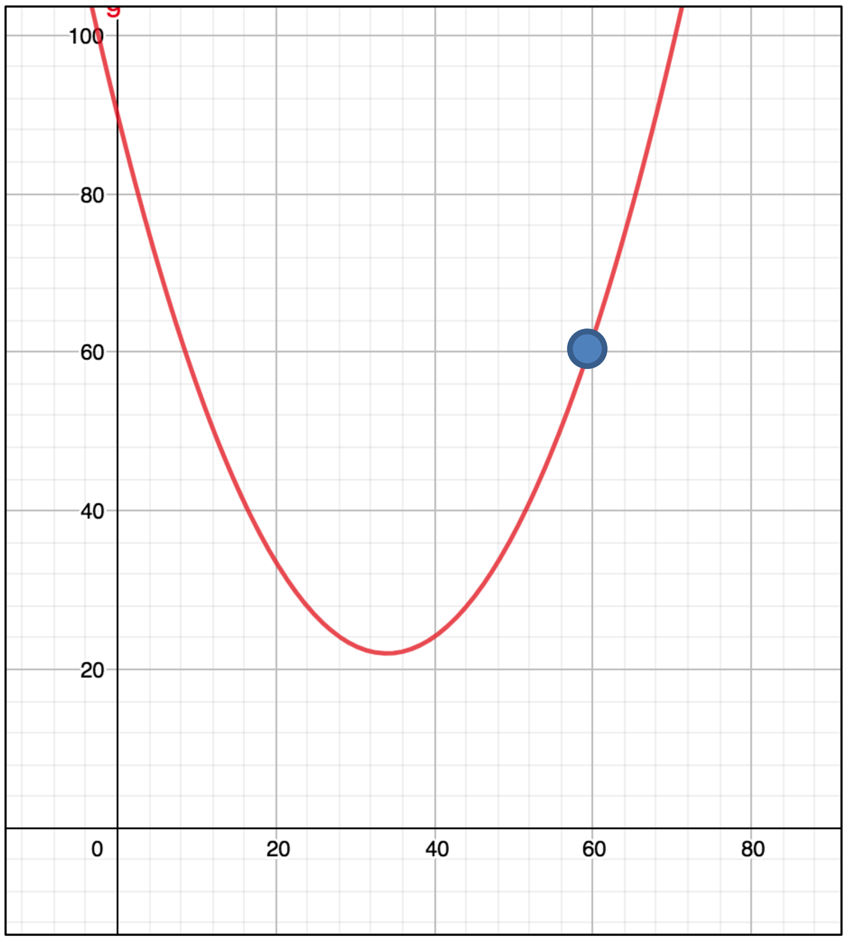
Gradient descent will converge!



Gradient Descent

Convergence:

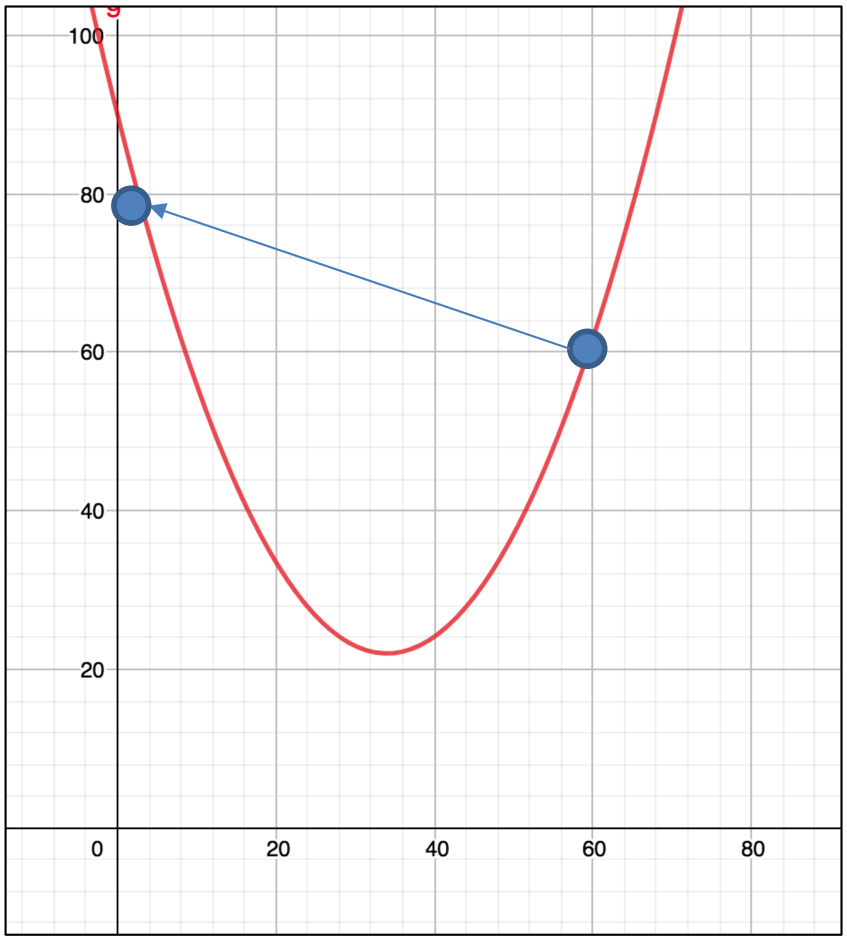
If step size is too large, may not converge:



Gradient Descent

Convergence:

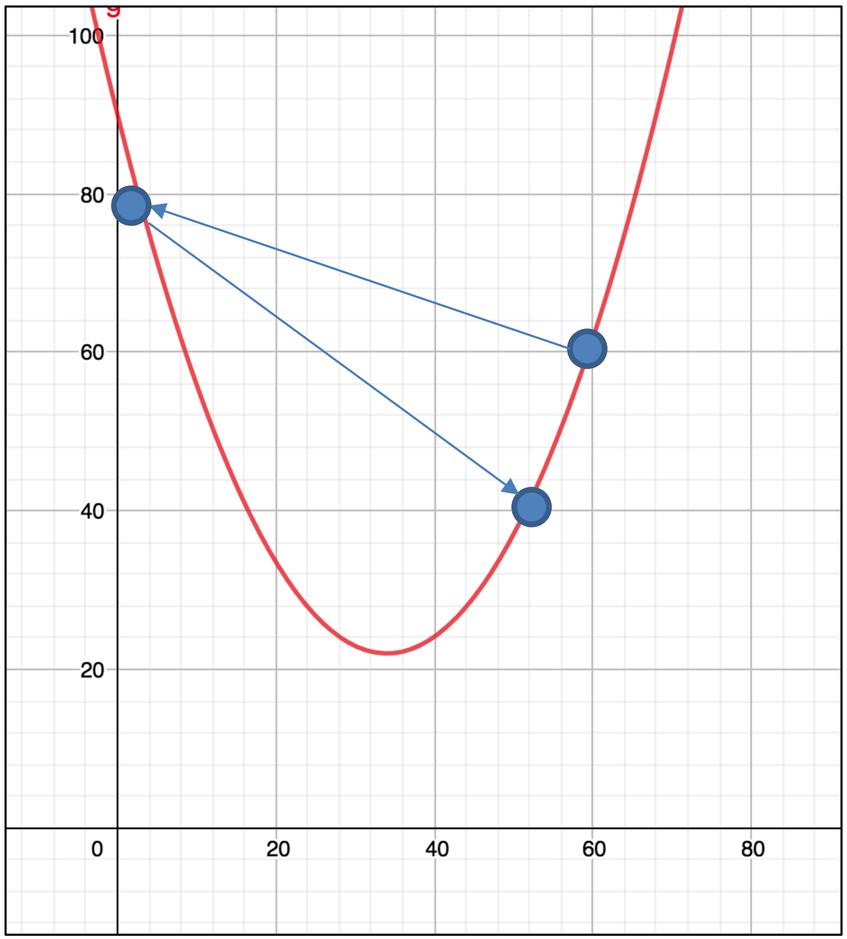
If step size is too large, may not converge:



Gradient Descent

Convergence:

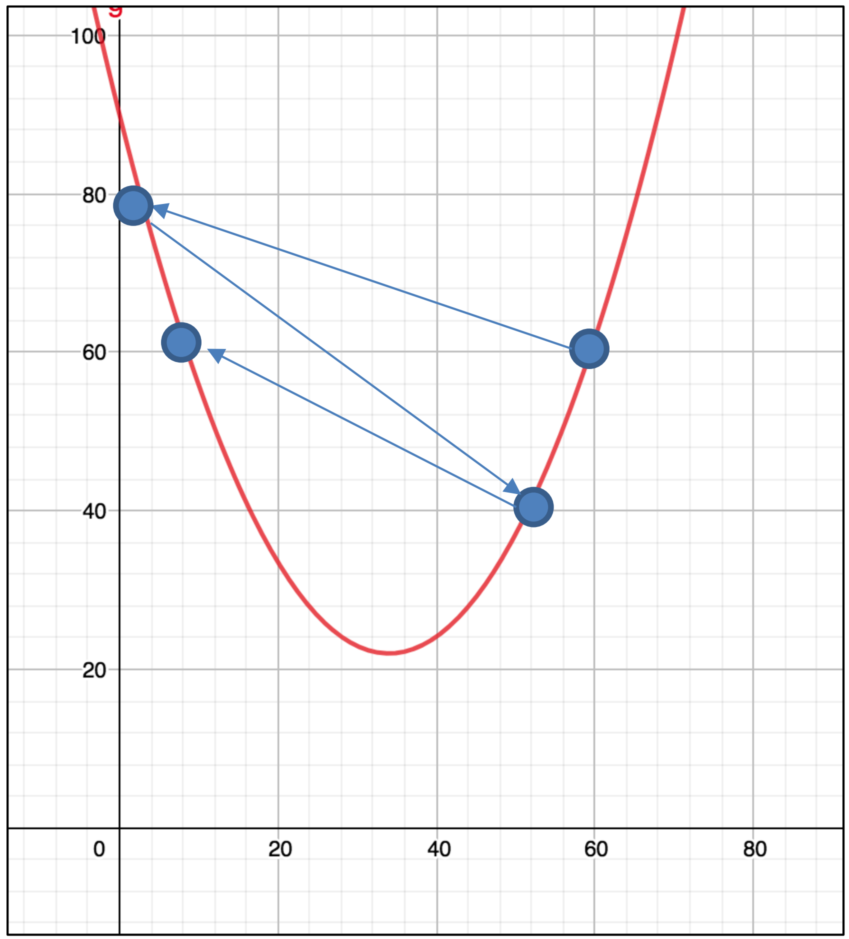
If step size is too large, may not converge:



Gradient Descent

Convergence:

If step size is too large, may not converge:

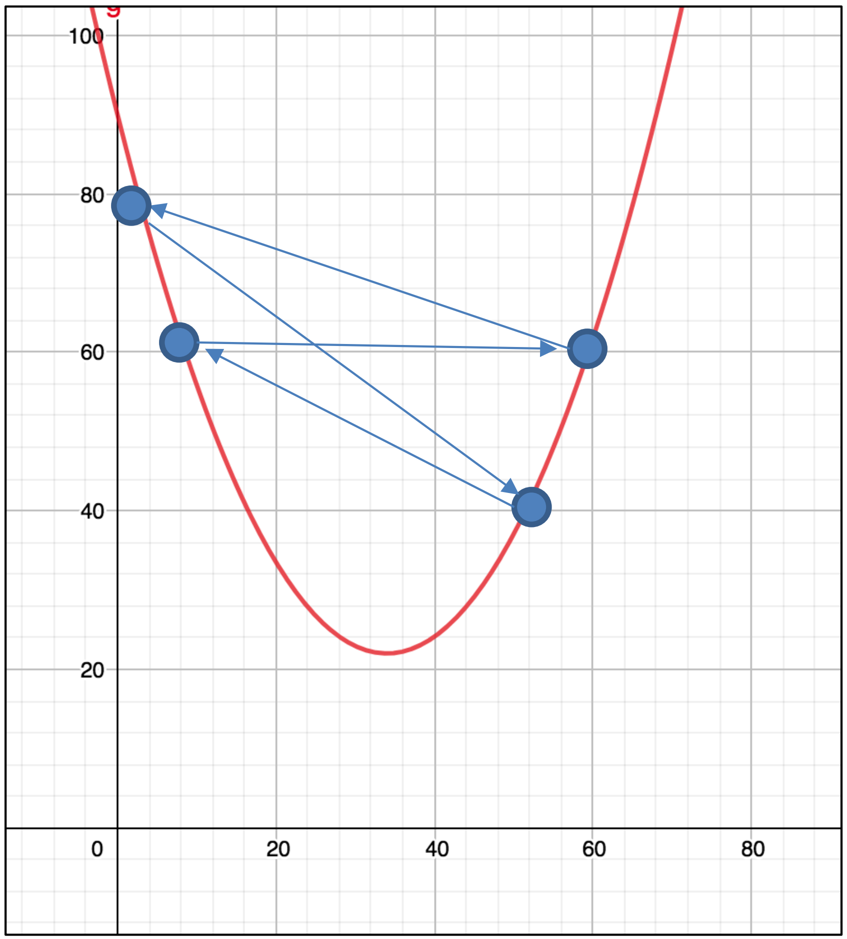


Gradient Descent

Convergence:

If step size is too large, may not converge:

(This is a fake example. Try it yourself!)



Gradient Descent

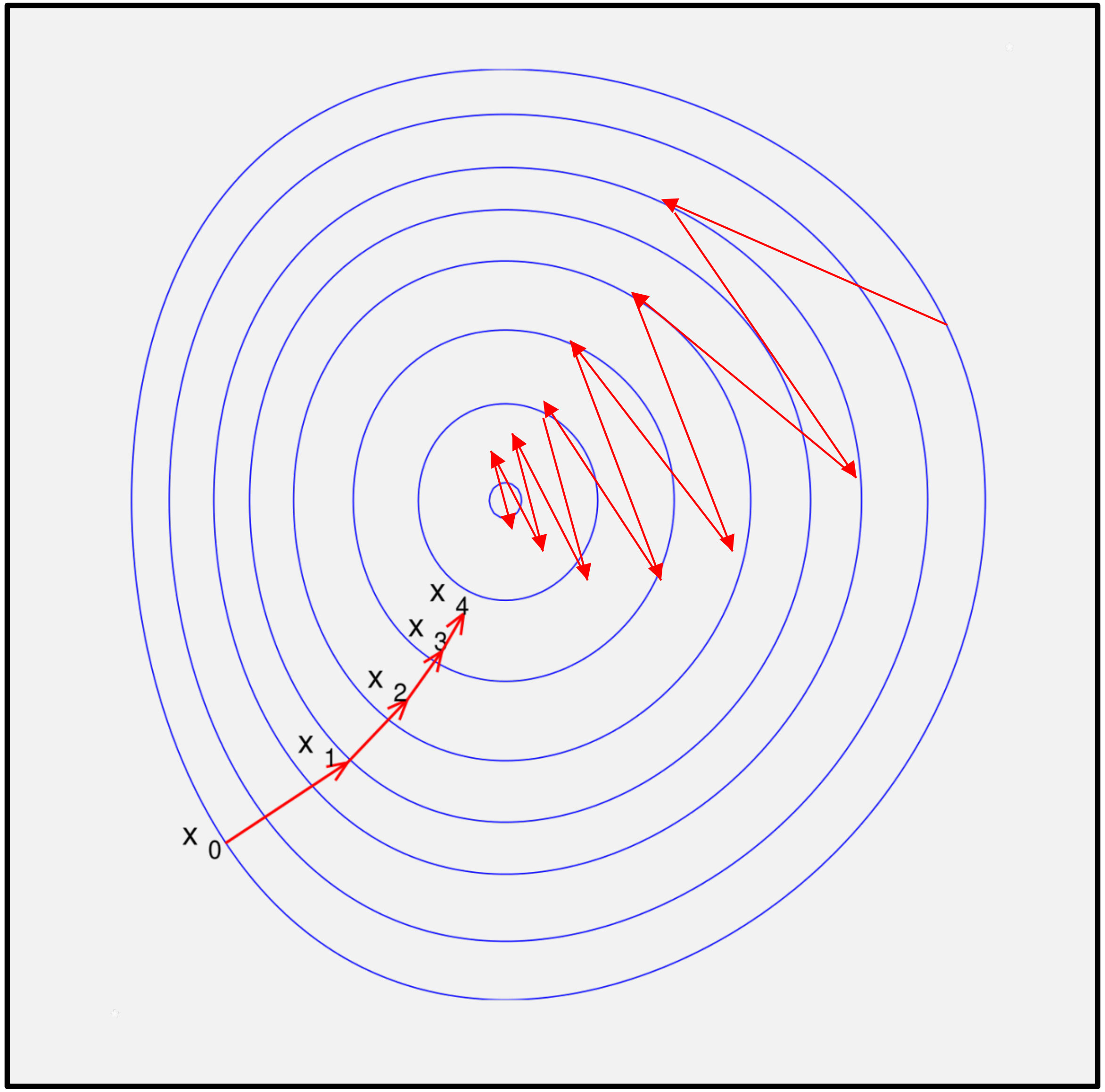


Image from:
Wikipedia

Gradient Descent

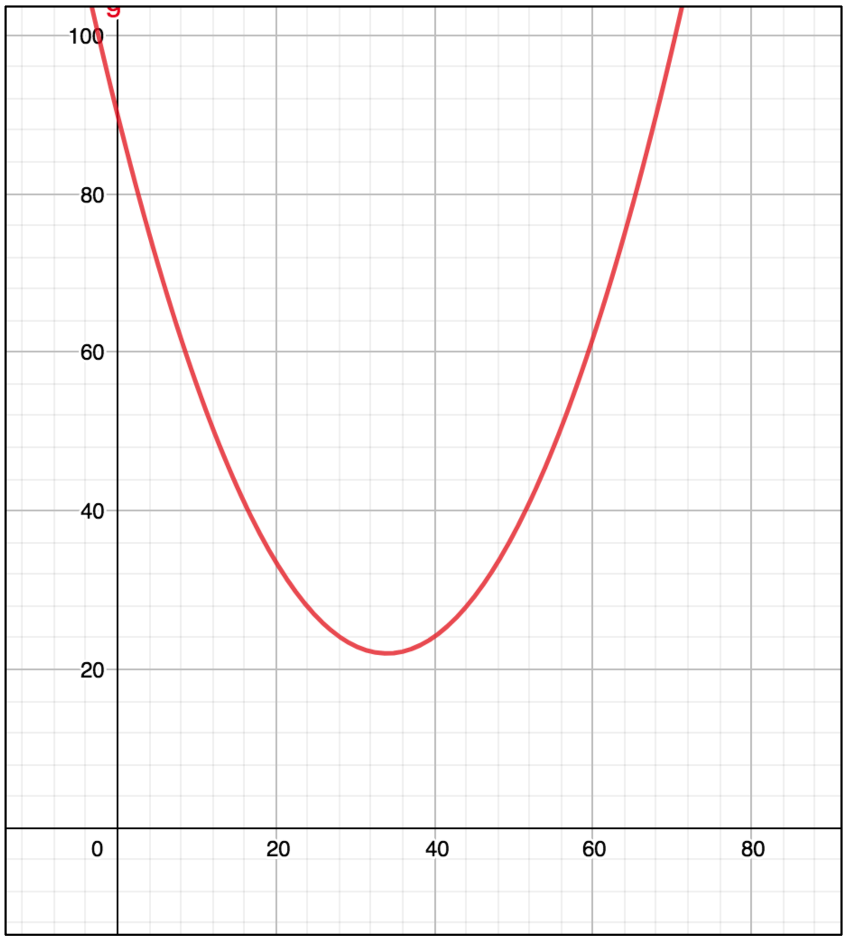
Convergence:

If f is convex and differentiable and its gradient is L-Lipschitz, and if step size **γ ≤ 1/L**:

Gradient descent will converge!

After t iterations:

$$f(x_t) - f(x^*) \leq \frac{||x_0 - x^*||^2}{2\gamma t}$$



Gradient Descent

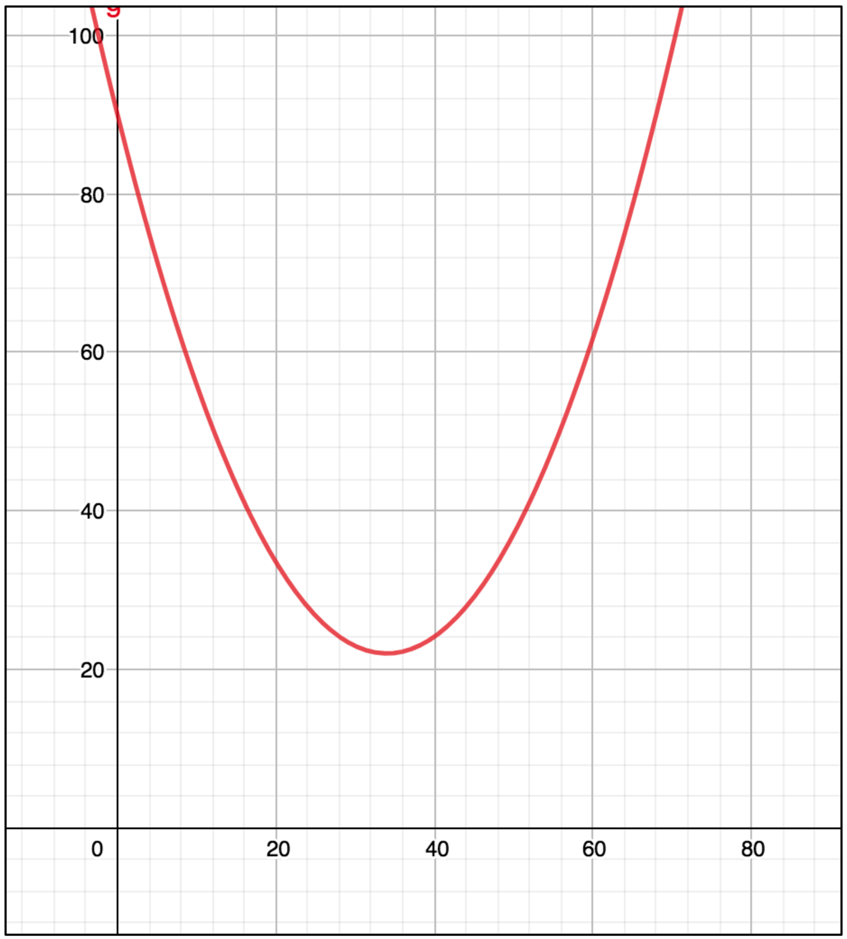
Convergence:

If f is convex and differentiable and its gradient is L-Lipschitz, and if step size **γ ≤ 1/L**:

Gradient descent will converge!

To achieve error ε:

$$t \geq \frac{||x_0 - x^*||^2}{2\gamma\epsilon} \geq \frac{L||x_0 - x^*||^2}{2\epsilon}$$



Gradient Descent

Convergence:

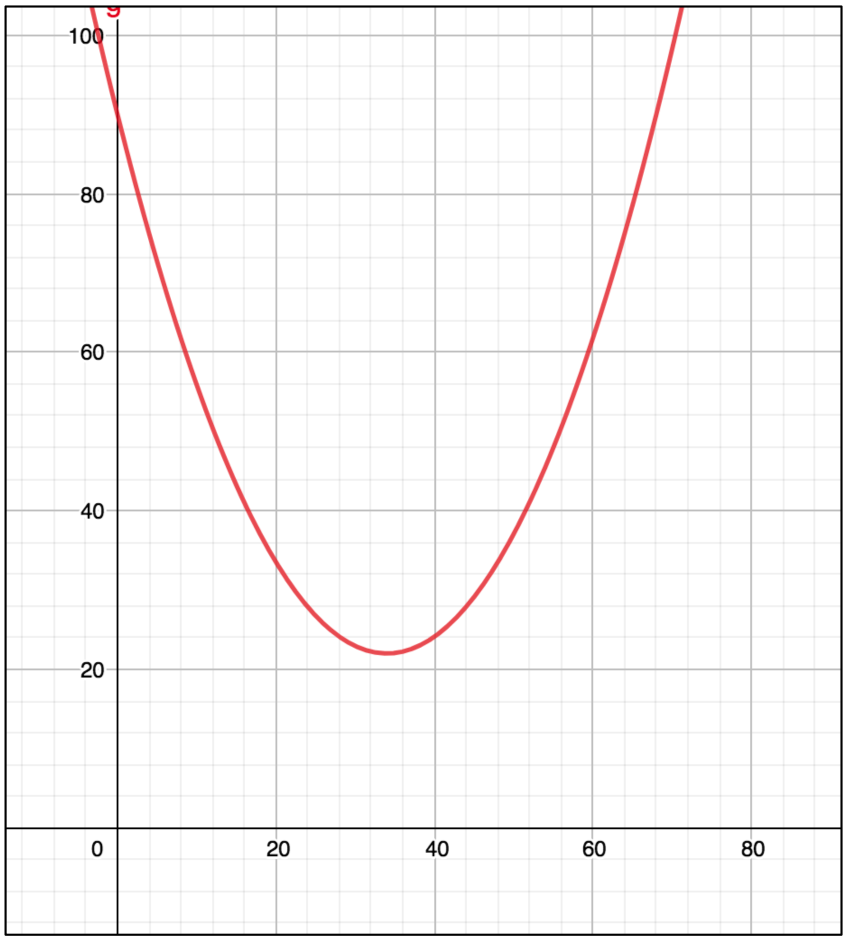
If f is convex and differentiable and its gradient is L-Lipschitz, and if step size **γ ≤ 1/L**:

Gradient descent will converge!

To achieve error ε:

$$t \geq \frac{||x_0 - x^*||^2}{2\gamma\epsilon} \geq \frac{L||x_0 - x^*||^2}{2\epsilon}$$

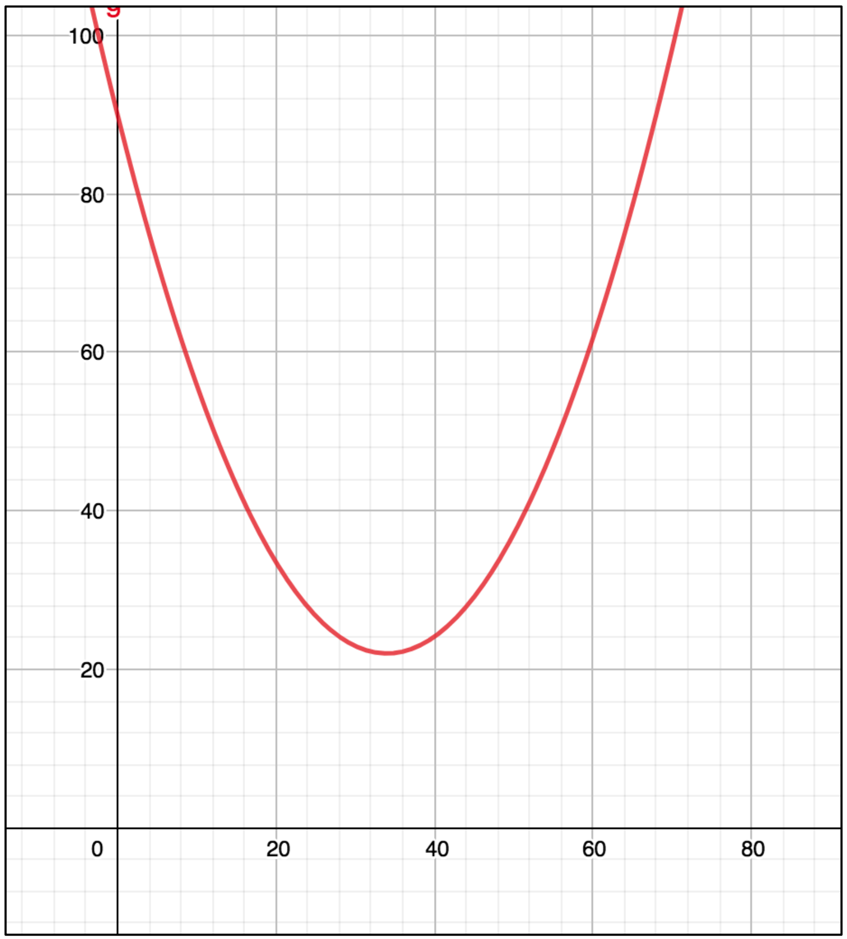
More iterations than Newton's method…
… but faster computation per iteration.



Gradient Descent

Convergence:

How to choose step size γ?

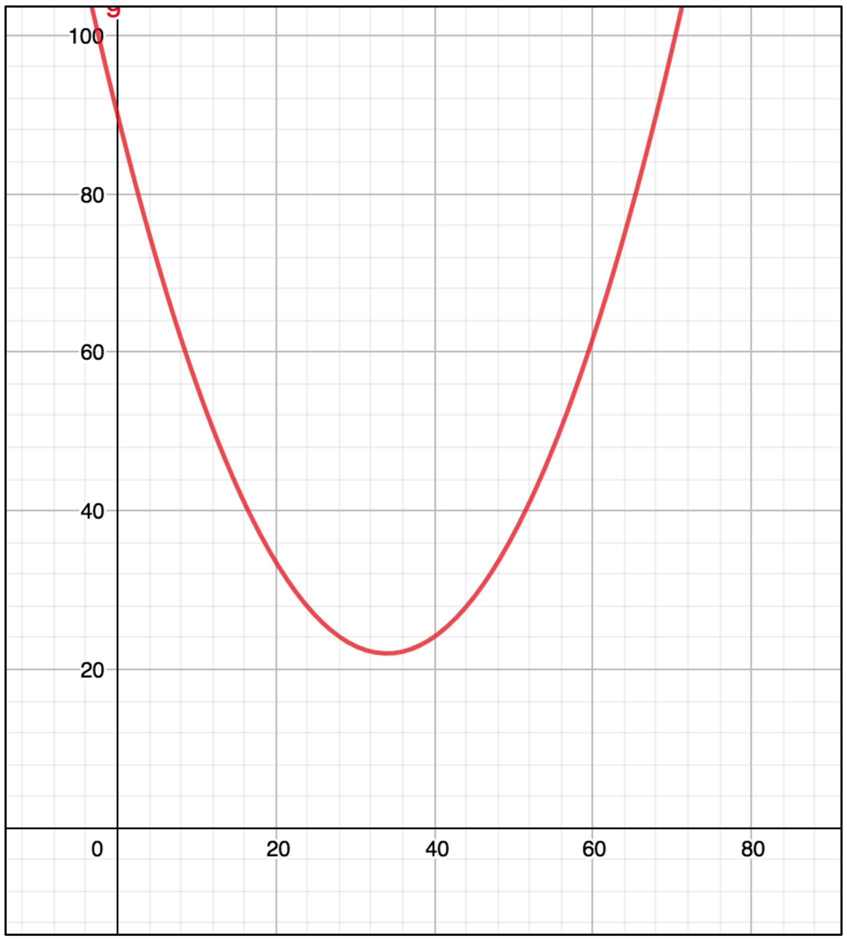


Gradient Descent

Convergence:

How to choose step size γ?

1. Look at function and compute L.



Gradient Descent

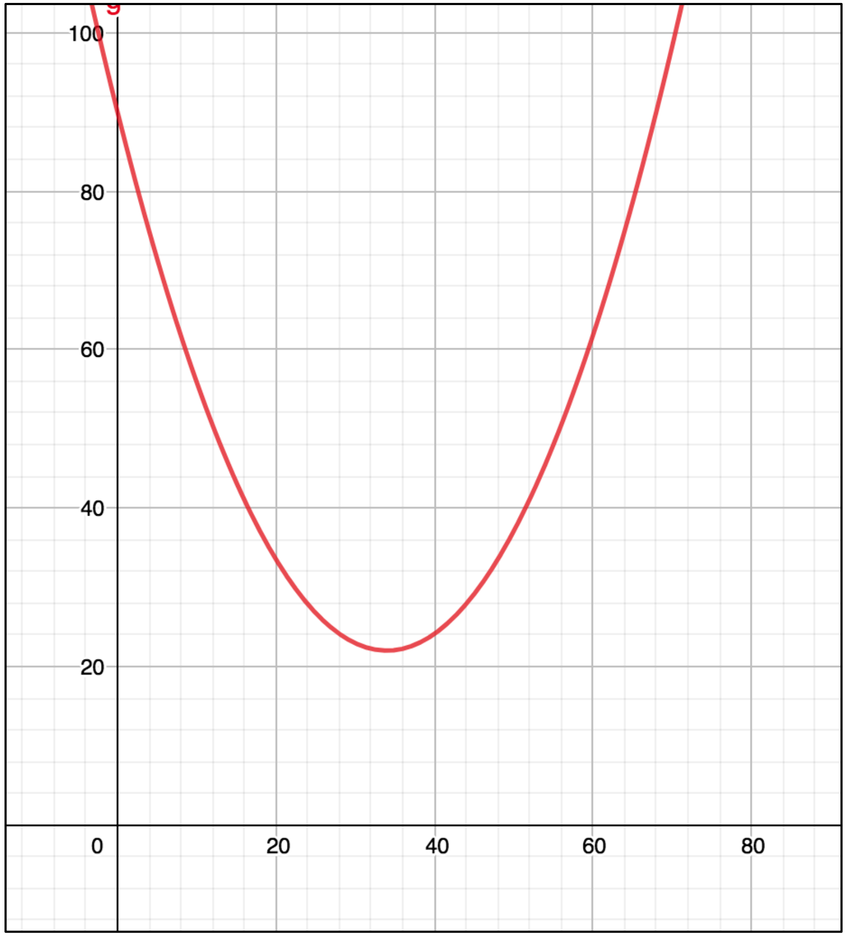
Convergence:

How to choose step size γ?

2. Adjust step size γ in each step:

> γ = very small value
> repeat:
>   Run t iterations of GD(γ).
>   If converged, stop.
>   Else $\gamma = \gamma/2$

Kind of like a binary search on γ!



Gradient Descent

Convergence:

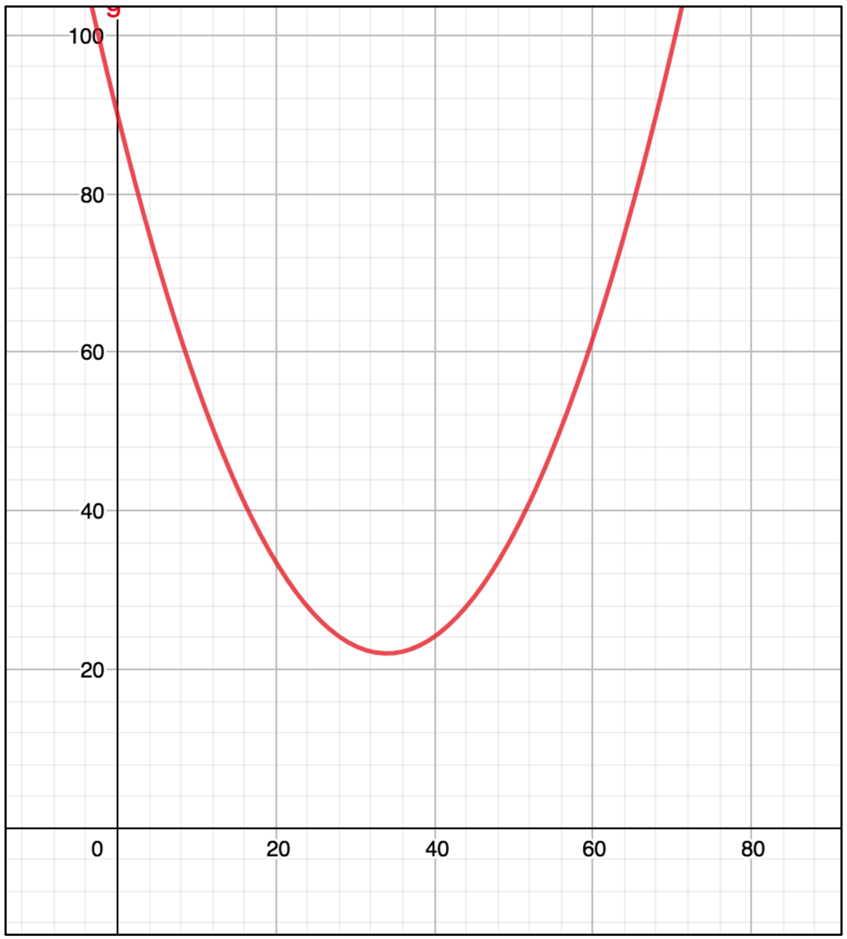
How to choose step size γ?

2. Adjust step size γ in each step:

γ = big value
repeat:
  Run t iterations of GD(γ).
  If converged, stop.
  Else $\gamma = \gamma/2$

What is a good value to start with?

How many iterations is enough?



Gradient Descent

Convergence:

How to choose step size γ?

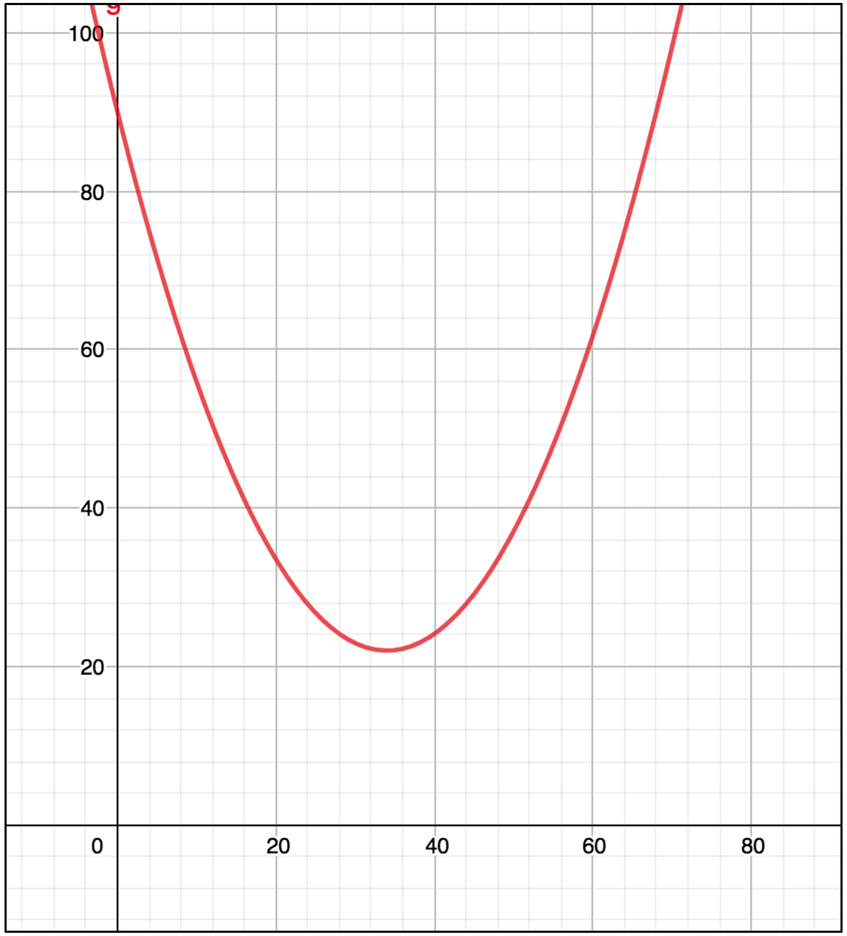
3. Compute an optimal step size for each iteration (*line search*):

$$\min_{\gamma} \left[f\left(x_i - \gamma \nabla f(x_i)\right) \right]$$

For (d=1), same as computing answer.

For (d > 1), it reduces to a one-dimensional minimization problem.

➔ line search



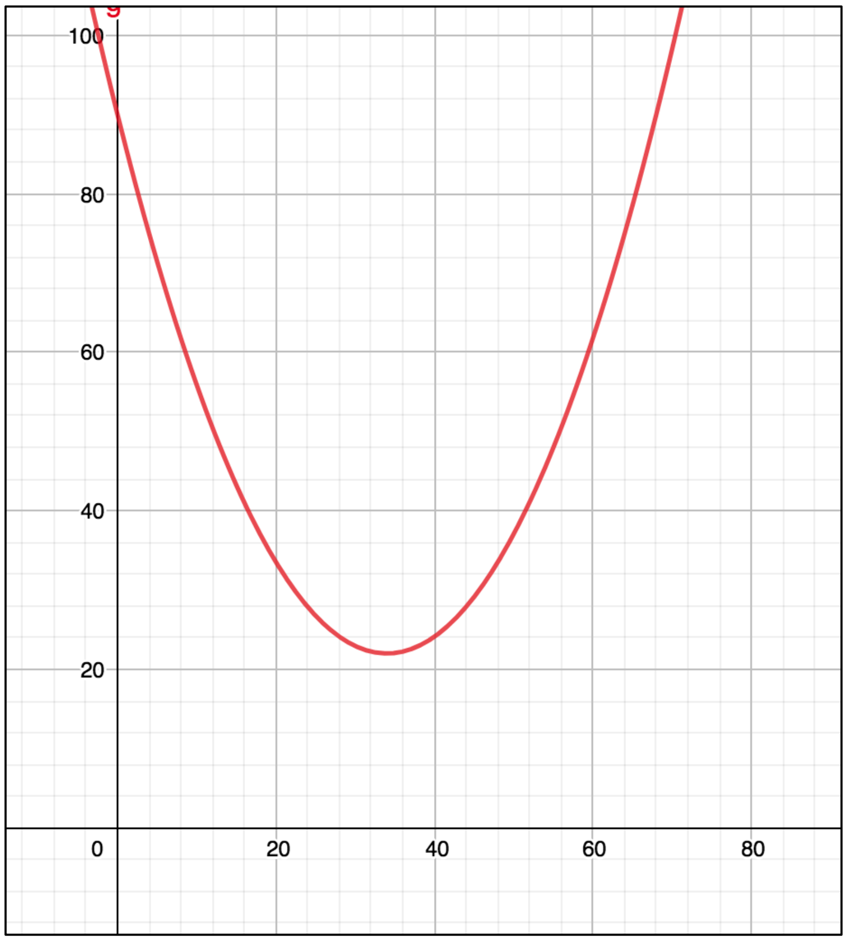
Gradient Descent

Convergence:

How to choose step size γ?

4. Backtracking
5. Other
6. Other
7. Other
8. Other
9. Compute the inverse Hessias

Much of the cleverness in implementing gradient descent lies in adjusting the step-size in a clever way.

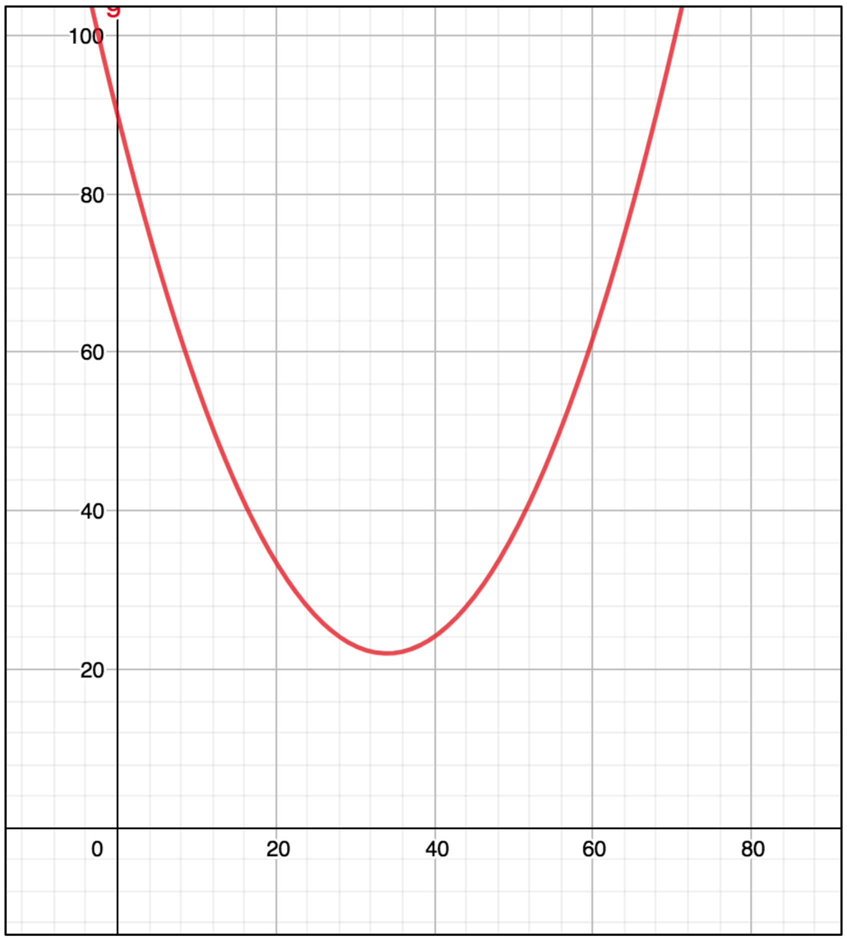


Gradient Descent

Convergence:

If function is strongly convex,
then convergence is much faster…

“Linear” ➔ rate = $\log 1/\epsilon$



Also see:

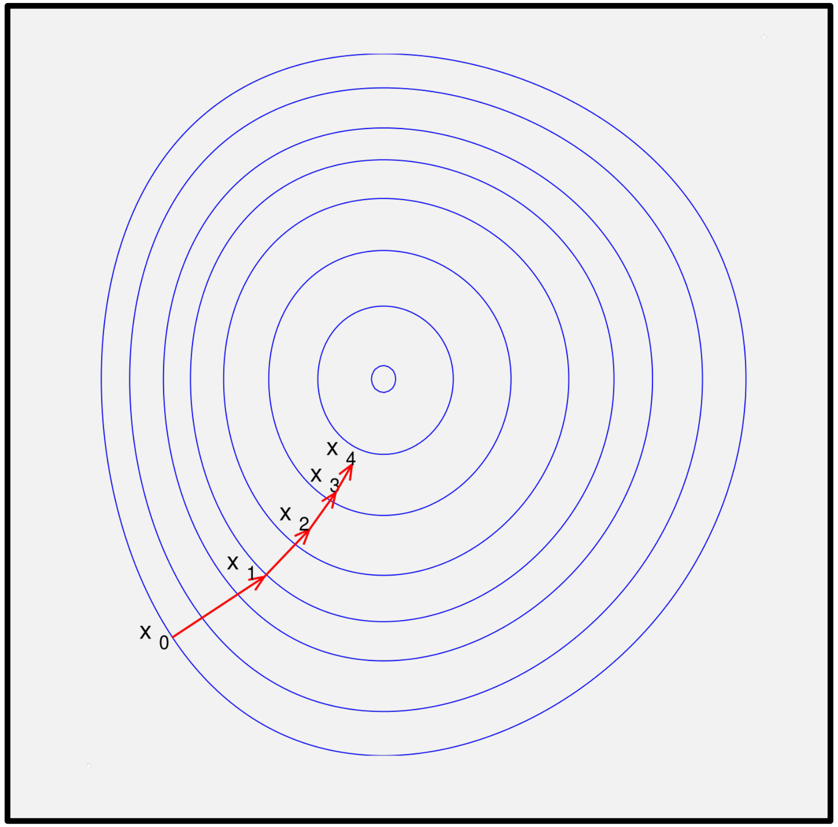
Accelerated gradient descent (Nesterov’s method), and many, many more variations

Gradient Descent

Goal: (in d dimensions)

Find the minimum value of f(x) via successive approximation:

$$\boxed{x_{i+1} = x_i - \gamma_i \nabla(f(x))}$$



Trade-Offs

Newton's Method

Fewer iterations

- *Convergence is quadratic*

More expensive computation

- *Requires inverse Hessian*

Requires access to second derivative

Gradient Descent

More iterations

- *Convergence is linear (when the function is strongly convex).*

Cheap computation

- *Just the gradient*

Requires choices of step size

- *Backtracking*
- *Line search*
- *Etc.*

Summary

Today: Optimization

Finding a maximum / minimum

- *Key aspect of machine learning*
- *Binary search isn't always good*

Newton's method

- *Second order approach*

Gradient descent

- *First order approach*

Next time:

Sorting

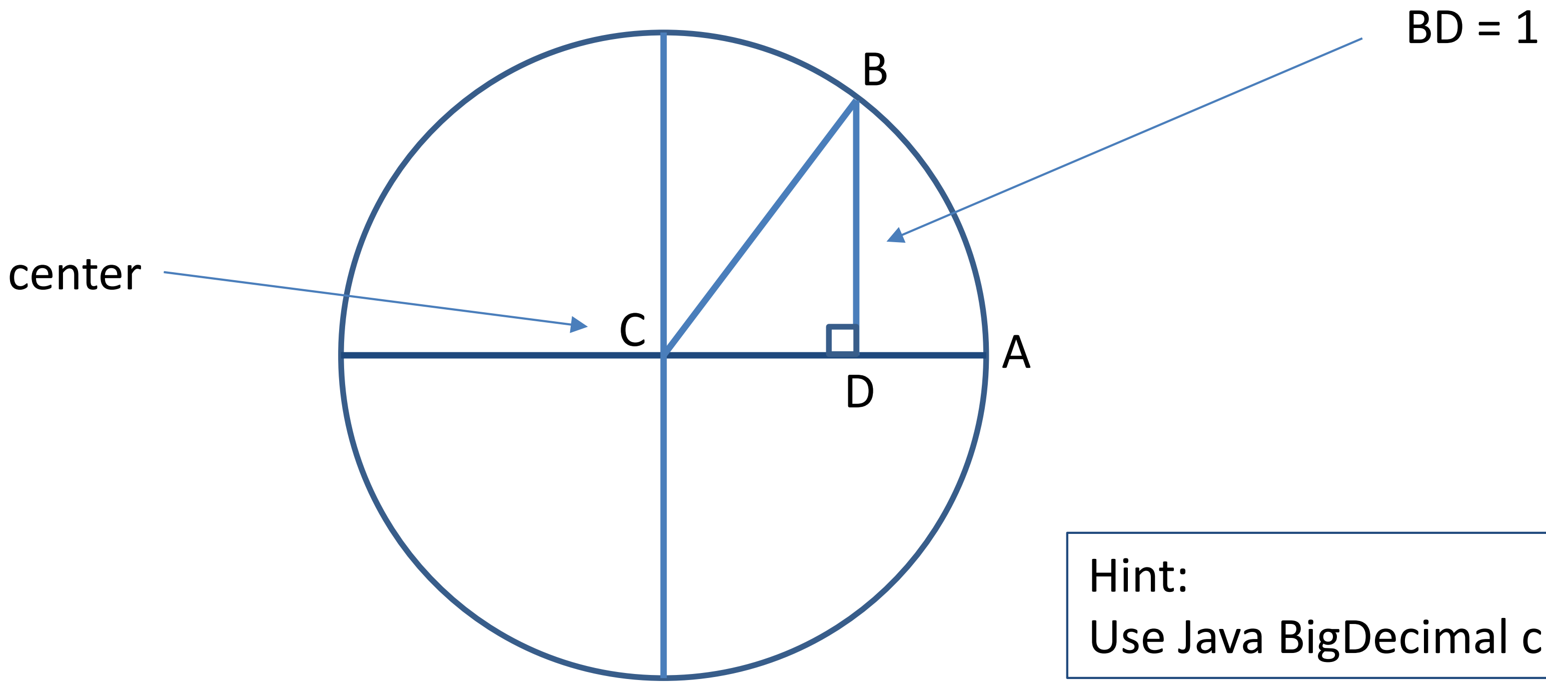
- *BubbleSort*
- *InsertionSort*
- *SelectionSort*
- *MergeSort*
- *QuickSort*

Properties of sorting algorithms

- *Stability*
- *Input-sensitive performance*
- *Etc.*

A Geometry Challenge

Compute AD (using, e.g., Newton's method, to 500 digits)



Hint:
Use Java BigDecimal class.

diameter = 1,000,000,000,000

What pattern do you see encoded in the result?