CS2040S
Data Structures and Algorithms

Welcome!

Last Time: Sorting, Part I

- Sorting algorithms
  - BubbleSort
  - SelectionSort
  - InsertionSort
  - MergeSort introduction
- Properties
  - Running time
  - Space usage
  - Stability

Today: Sorting, Part II

MergeSort

- Divide-and-Conquer
- Analysis

QuickSort

- Divide-and-Conquer
- Paranoid QuickSort
- Randomized Analysis

Sorting

Problem definition:

Input: array A[1..n] of words / numbers

Output: array B[1..n] that is a permutation of A such that:

$B[1] \leq B[2] \leq \ldots \leq B[n]$

Example:

$A = [9, 3, 6, 6, 6, 4] \rightarrow [3, 4, 6, 6, 6, 9]$

MergeSort

Step 1:
Divide array into two pieces.

MergeSort(A, n)

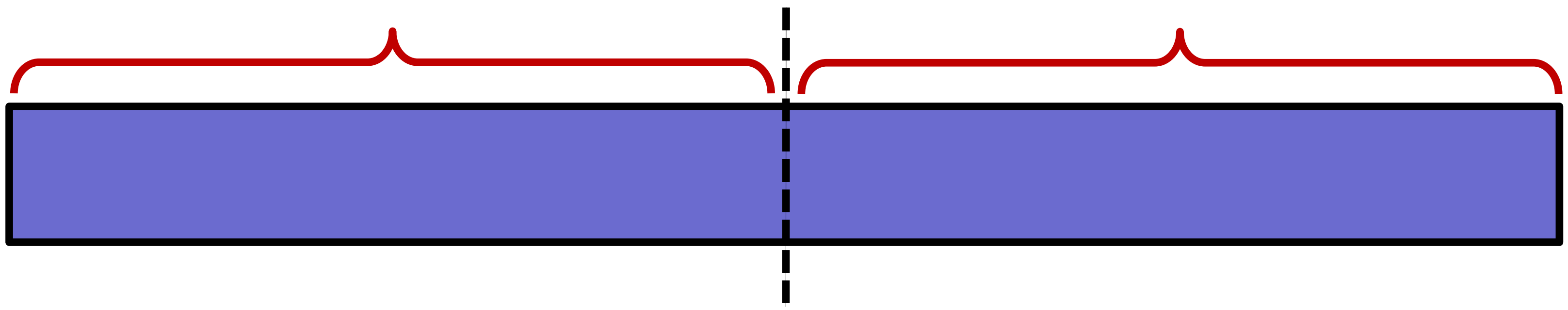
if (n=1) **then return;**

else:

X ←MergeSort**(**A[1..n/2], n/2**);**

Y ←MergeSort**(**A[n/2+1, n], n/2**);**

return Merge **(**X,Y, n/2**);**



MergeSort

Step 2:
Recursively sort the two halves.

MergeSort(A, n)

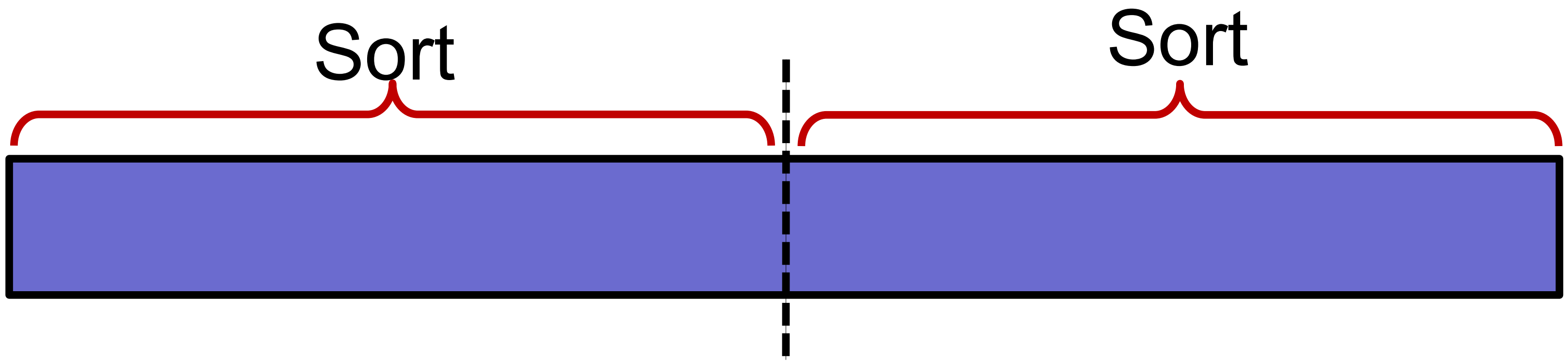
if (n=1) **then return;**

else:

X ←MergeSort**(**A[1..n/2], n/2**);**

Y ←MergeSort**(**A[n/2+1, n], n/2**);**

return Merge **(**X,Y, n/2**);**



MergeSort

Step 3:
Merge the two halves into one sorted array.

MergeSort(A, n)

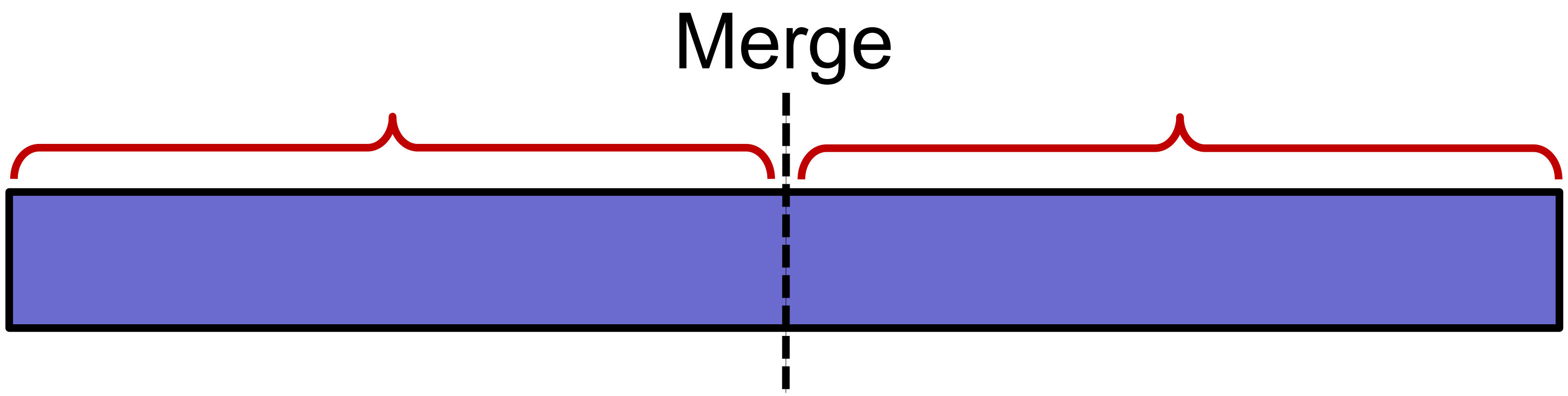
if (n=1) **then return;**

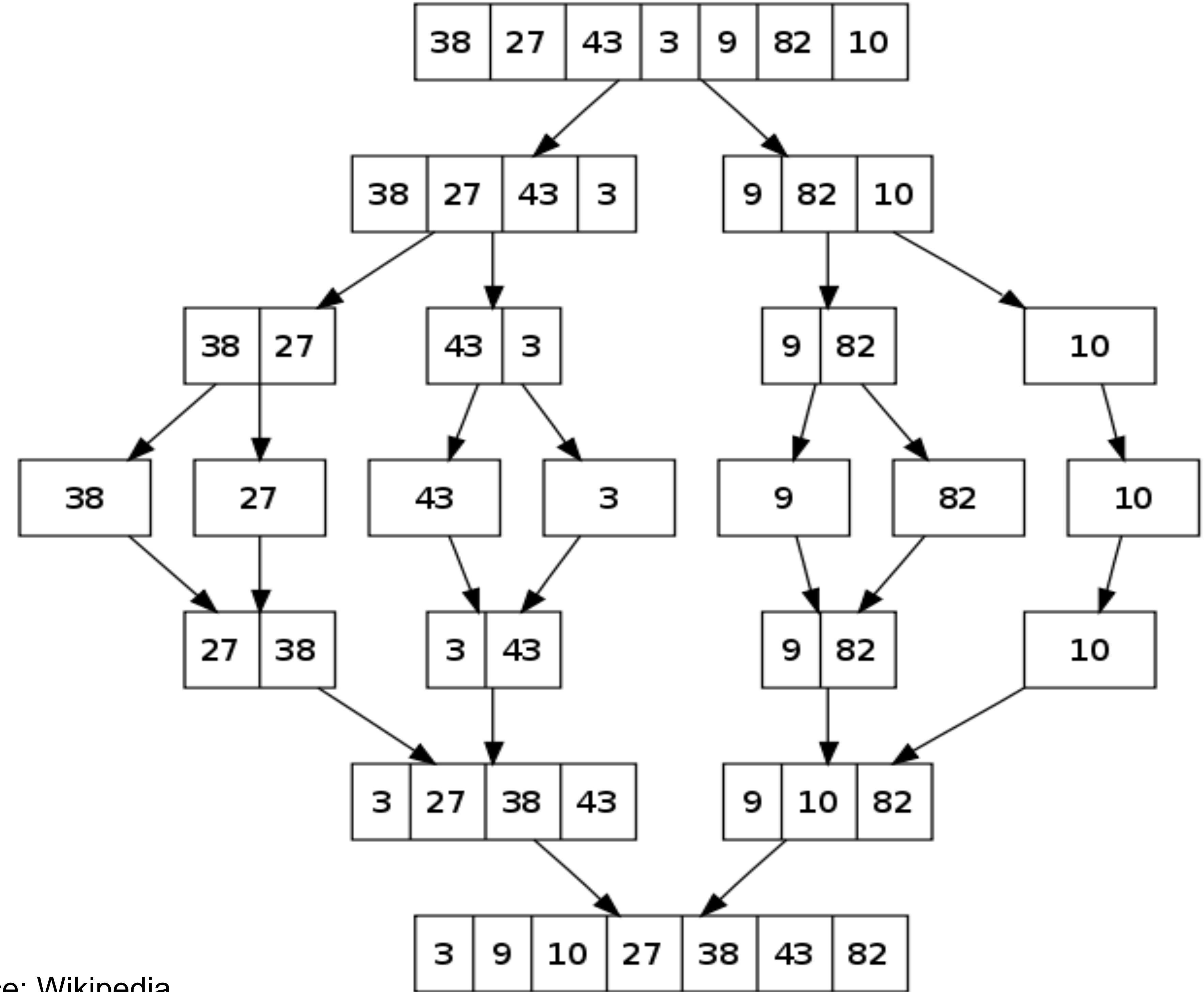
else:

X ←MergeSort(A[1..n/2], n/2);

Y ←MergeSort(A[n/2+1, n], n/2);

return Merge (X,Y, n/2);





Source: Wikipedia

MergeSort, Bottom Up

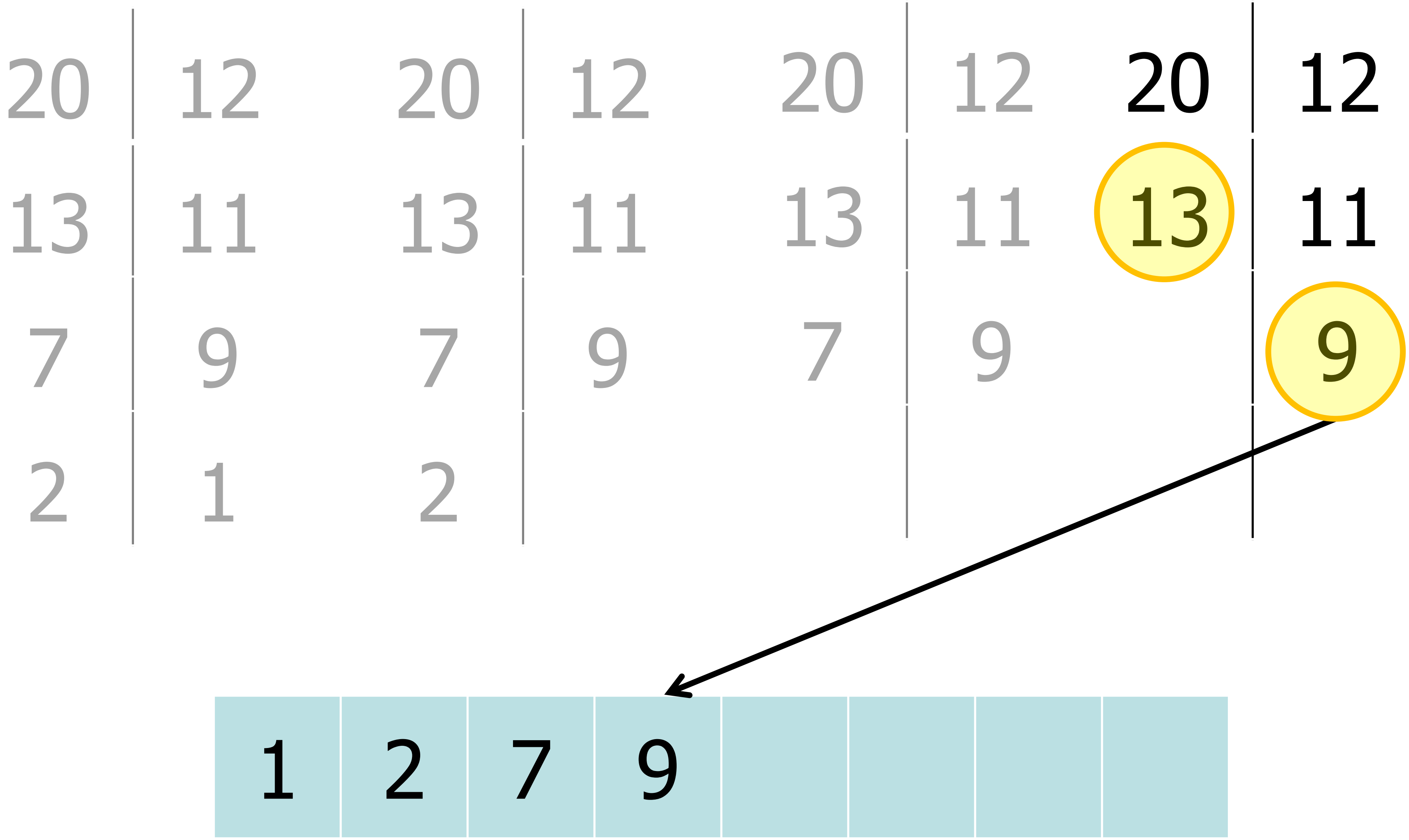
15	7	9	2	6	12	13	4	1	8	10	5	3	14	11	16

Merging Two Sorted Lists

Key subroutine: Merge

- How to merge?
- How fast can we merge?

Merging Two Sorted Lists



Merge: Running Time

Given two lists:

- A of size n/2
- B of size n/2

Total running time: $O(n) = cn$

- In each iteration, move *one* element to final list.
- After n iterations, all the items are in the final list.
- Each iteration takes $O(1)$ time to compare two elements and copy one.

Merge-Sort Analysis

Let $T(n)$ be the worst-case running time for an array of n elements.

MergeSort(A, n)

if $(n=1)$ **then return;** ←---------------- $\theta(1)$

else:

$X \leftarrow$ Merge-Sort**(**...**);** ←------------ $T(n/2)$

$Y \leftarrow$ Merge-Sort**(**...**);** ←------------ $T(n/2)$

return Merge **(**X,Y, n/2**);** ←------------ $\theta(n)$

MergeSort Analysis

Let $T(n)$ be the worst-case running time for an array of n elements.

$$
\begin{aligned}
T(n) &= \theta(1) && \textbf{if } (n=1) \\
&= 2T(n/2) + cn && \textbf{if } (n>1)
\end{aligned}
$$

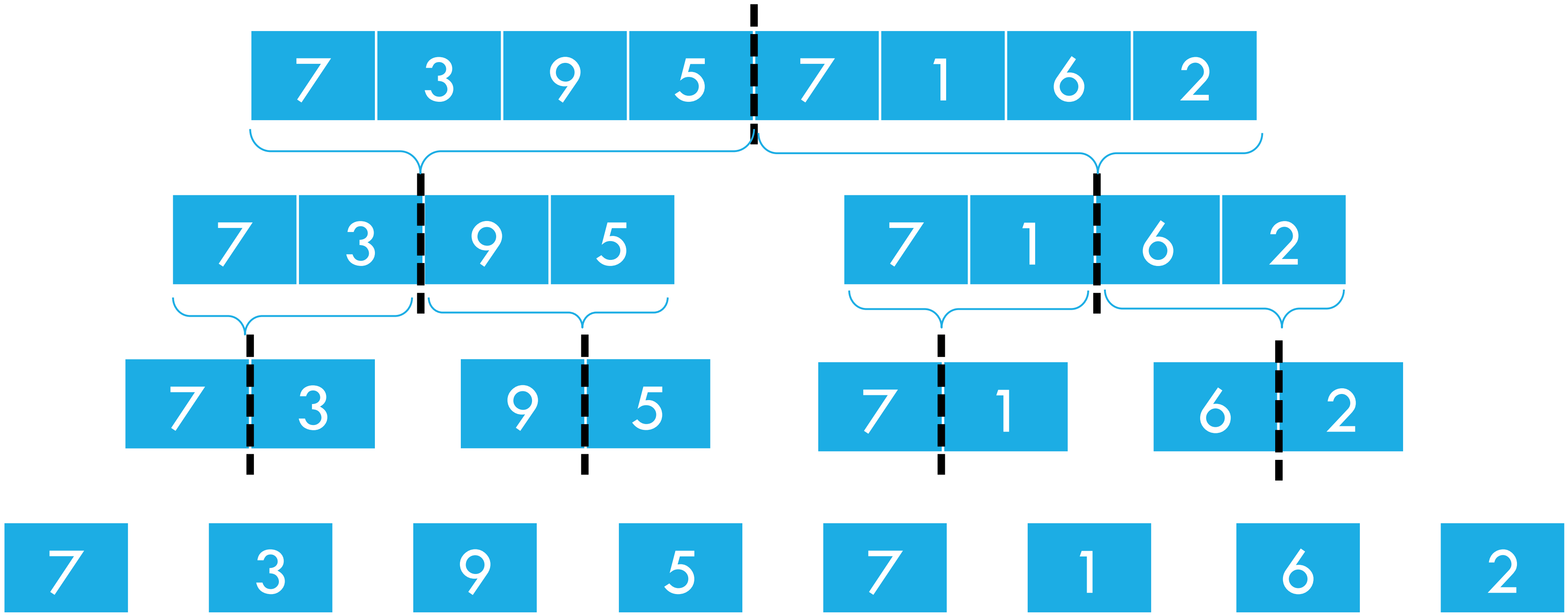
Techniques for Solving Recurrences

1. Guess and verify (via induction).

2. Draw the recursion tree.

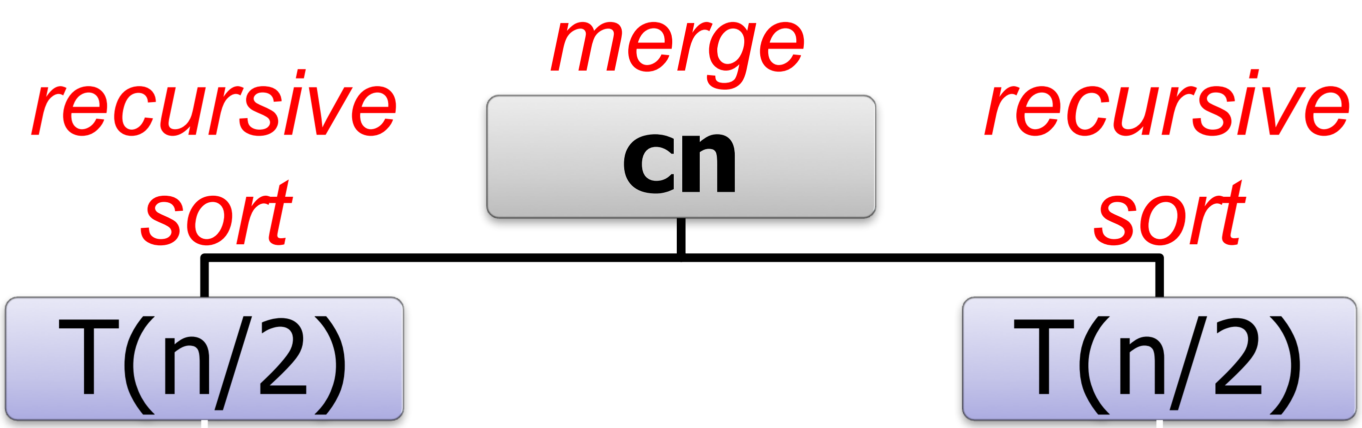
3. Use the Master Theorem (see CS3230) or the Akra–Bazzi Method, or other advanced techniqus.

MergeSort: Recurse "downwards"



MergeSortAnalysis

$T(n) = 2T(n/2) + cn$



MergeSortAnalysis

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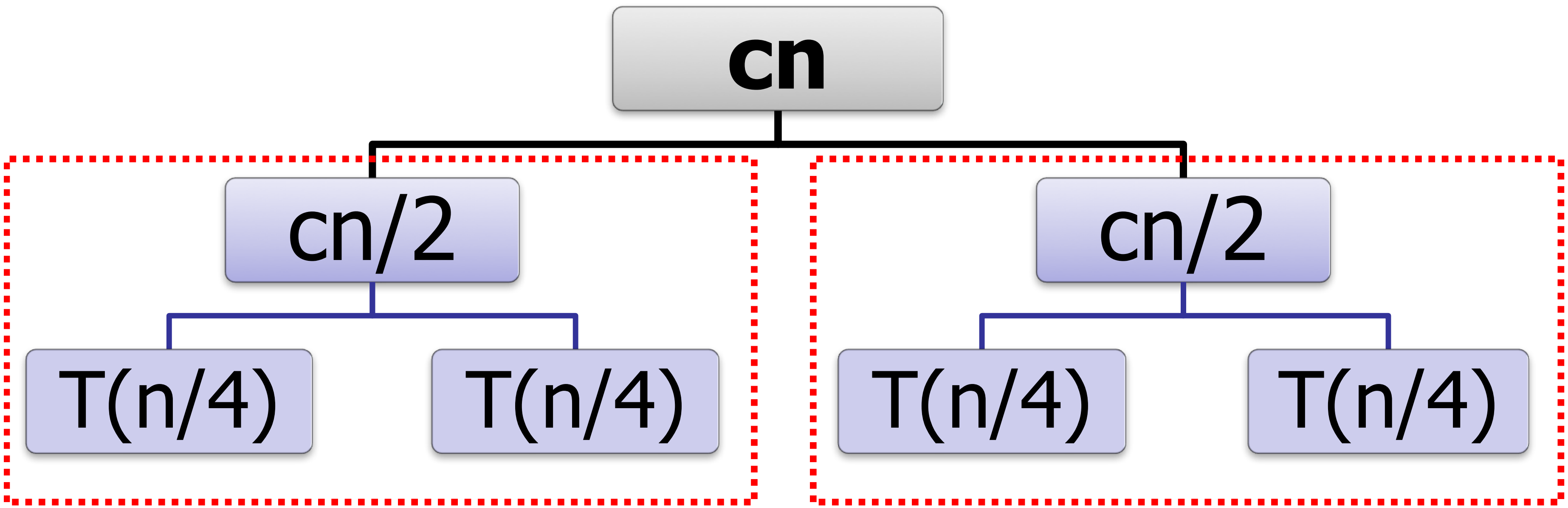
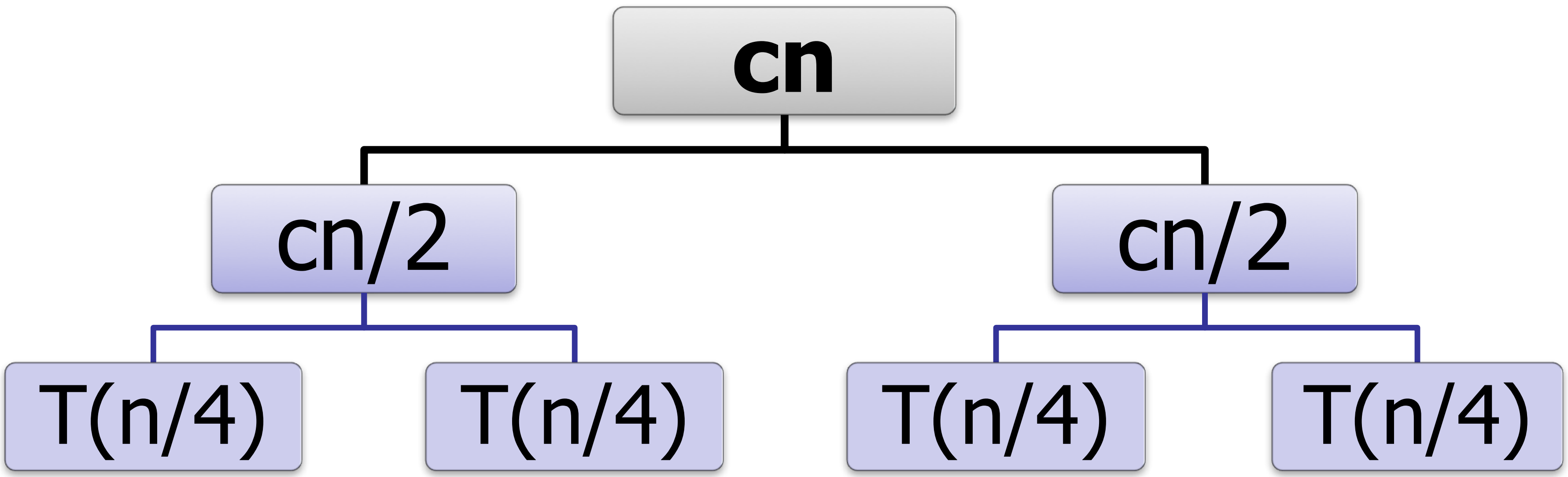


MergeSortAnalysis

$T(n) = 2T(n/2) + cn$



MergeSort Analysis

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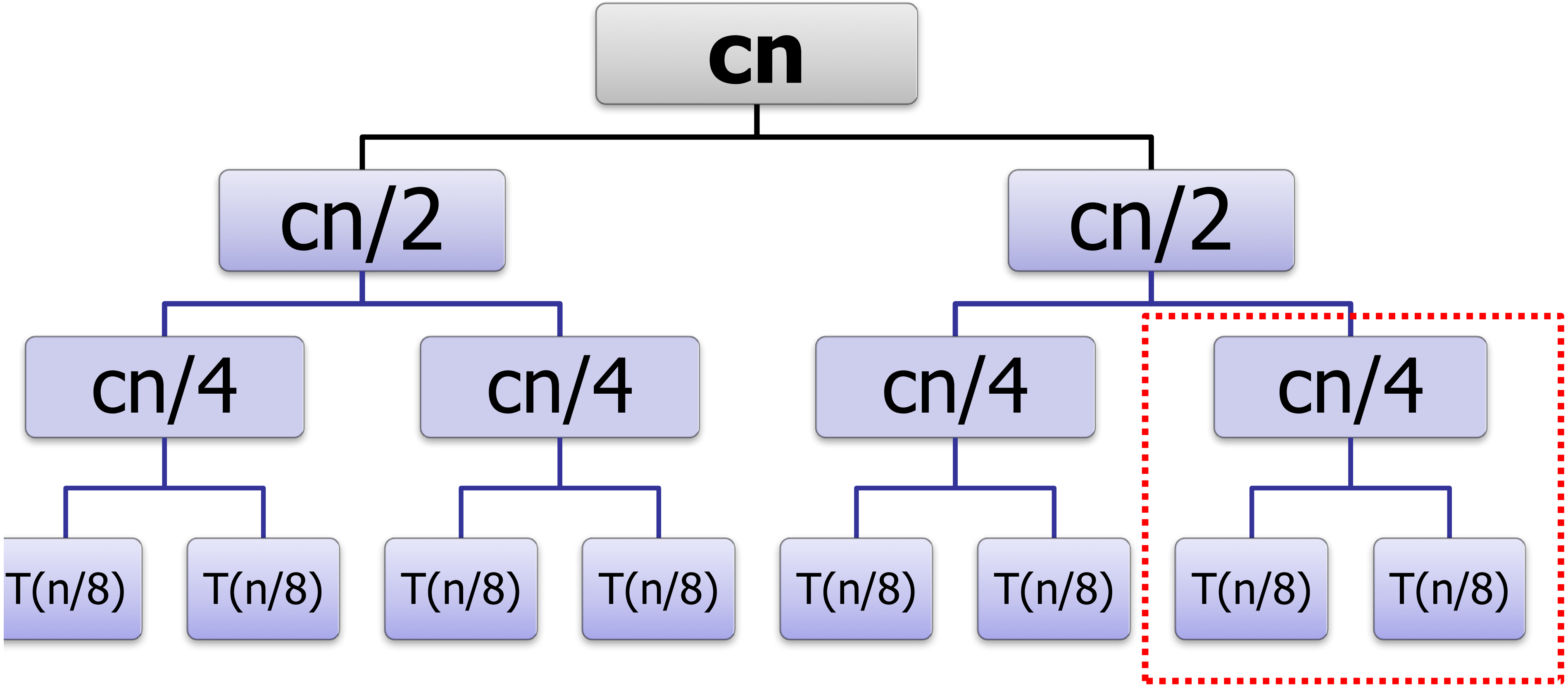
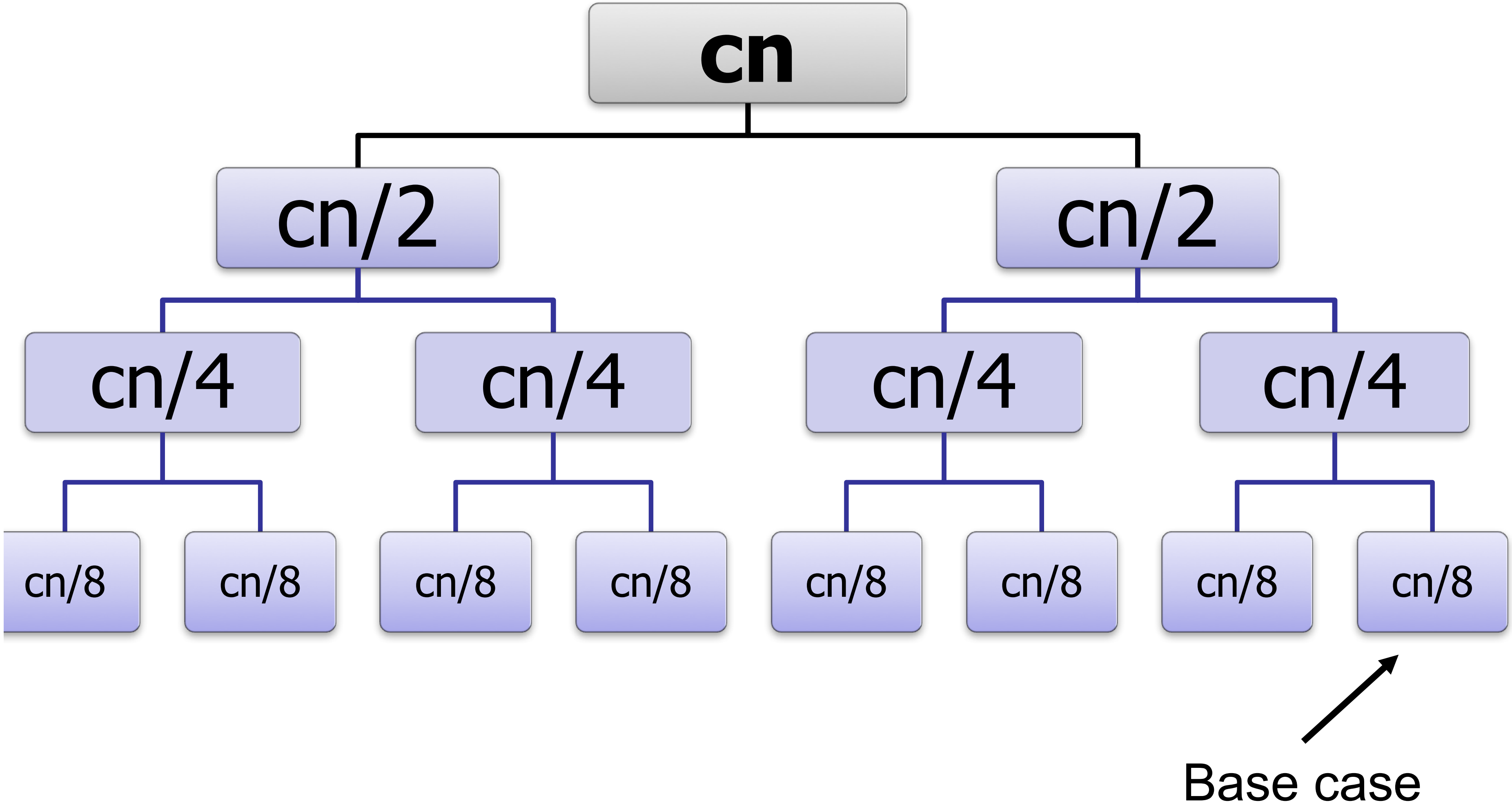


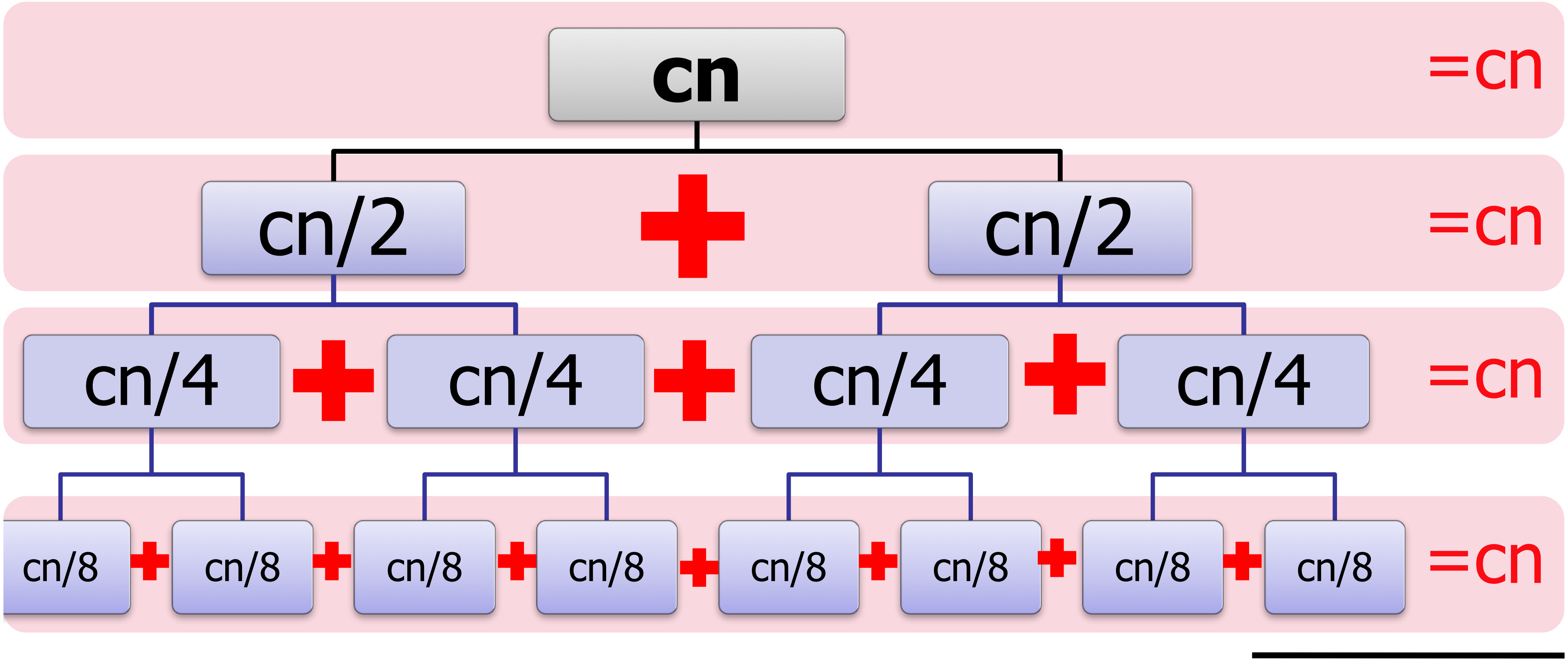
MergeSort Analysis

$T(n) = 2T(n/2) + cn$



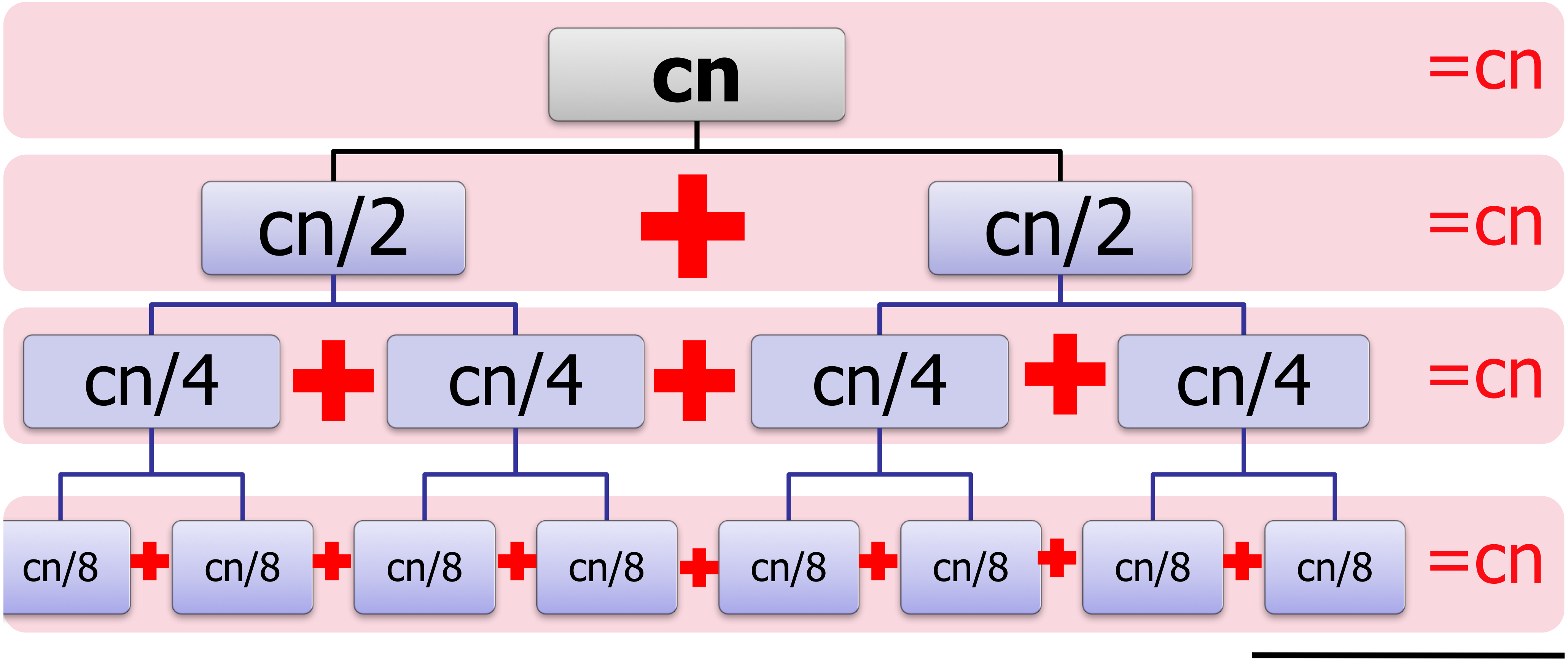
MergeSort Analysis

$T(n) = 2T(n/2) + cn$



MergeSort Analysis

$T(n) = 2T(n/2) + cn$



Key question: how many levels?

MergeSort Analysis

$T(n) = 2T(n/2) + cn$

level	number
0	1
1	2
2	4
3	8
4	16
…	…
h	??

$$\text{number} = 2^{\text{level}}$$

MergeSort Analysis

$T(n) = 2T(n/2) + cn$

Level	**Number**
0	1
1	2
2	4
3	8
4	16
…	…
h	*n*

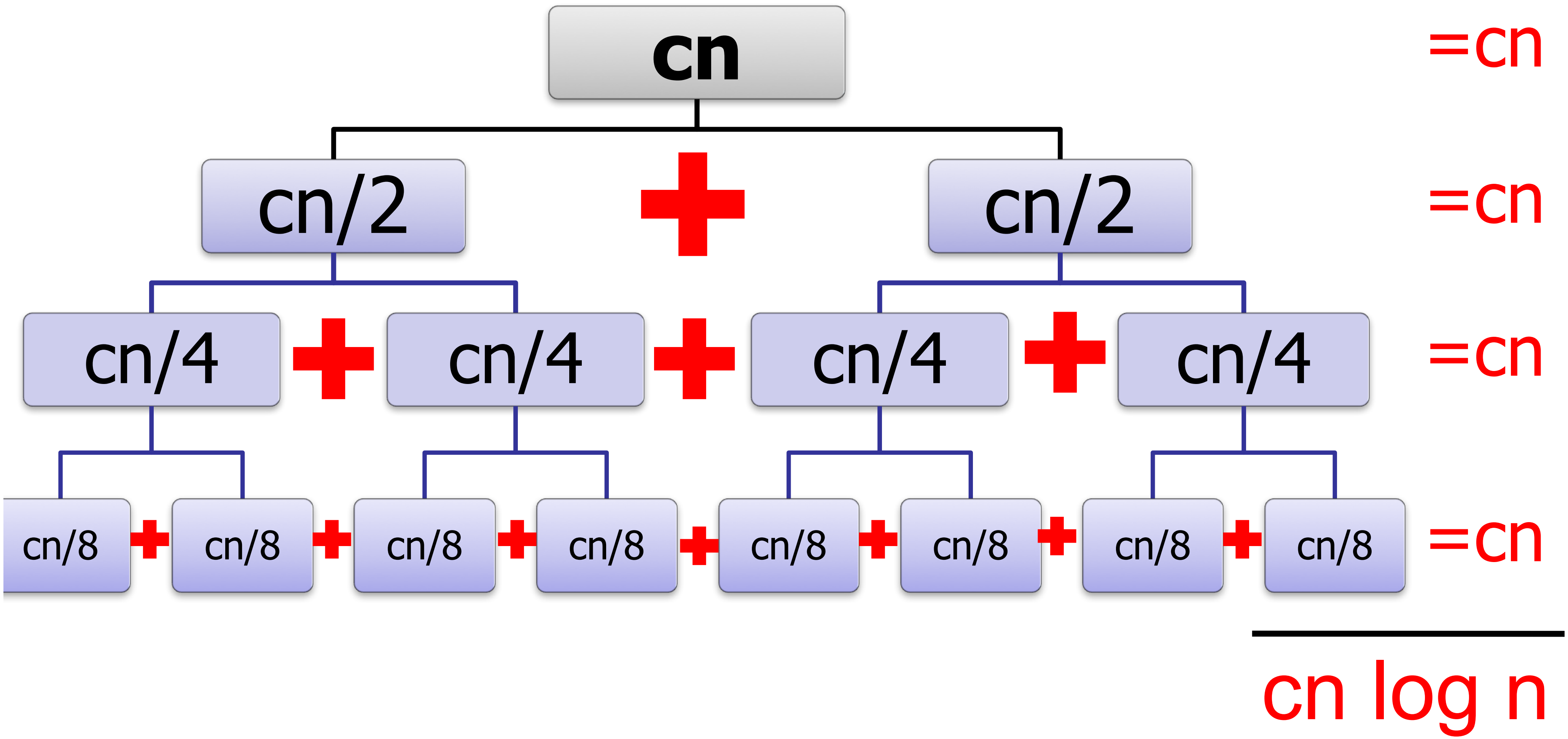
$$\text{number} = 2^{\text{level}}$$

$$n = 2^h$$

$$\log n = h$$

MergeSort Analysis

$$T(n) = 2T(n/2) + cn$$



MergeSortAnalysis

$T(n) = O(n \log n)$

MergeSort(A, n)

if (n=1) **then return;**

else: ← $\theta(1)$

X ←MergeSort**(**...**);**

Y ←MergeSort**(**...**);** ← $T(n/2)$

return Merge **(**X,Y, n/2**);** ← $T(n/2)$

← $\theta(n)$

Techniques for Solving Recurrences

1. Guess and verify (via induction).

2. Draw the recursion tree.

3. Use the Master Theorem (see CS3230) or the Akra–Bazzi Method, or other advanced techniqus.

Guess: $T(n) = O(n \log n)$

Recurrence being analyzed:

$T(n) = 2T(n/2) + c \cdot n$

$T(1) = c$

Guess: $T(n) = c \cdot n \log n$

More precise guess:
Fix constant c.

<u>Recurrence being analyzed:</u>

$T(n) = 2T(n/2) + c \cdot n$

$T(1) = c$

Guess: $T(n) = c \cdot n \log n$

Induction:
Base case

$T(1) = c$

Recurrence being analyzed:

$T(n) = 2T(n/2) + c \cdot n$

$T(1) = c$

Guess: $T(n) = c \cdot n \log n$

Induction:
Assume true for all smaller values.

$T(1) = c$

$T(x) = c \cdot x \log x$ for all $x < n$.

<u>Recurrence being analyzed:</u>

$T(n) = 2T(n/2) + c \cdot n$

$T(1) = c$

Guess: T(n) = c·n log n

Induction:
Prove for n.

T(1) = c

T(x) = c·x log x for all x < n.

$$T(n) \quad = \quad 2T(n/2) + cn$$

Recurrence being analyzed:

T(n) = 2T(n/2) + c·n

T(1) = c

Guess: T(n) = c·n log n

Induction:
Prove for n.

T(1) = c

T(x) = c·x log x for all x < n.

induction!

$$\begin{aligned} T(n) &= 2T(n/2) + cn \\ &= 2(c(n/2)\log(n/2)) + cn \end{aligned}$$

<u>Recurrence being analyzed:</u>

T(n) = 2T(n/2) + c·n

T(1) = c

Guess: T(n) = c·n log n

Induction:
Prove for n.

T(1) = c

T(x) = c·x log x for all x < n.

$$
\begin{aligned}
T(n) &= 2T(n/2) + cn \\
&= 2(c(n/2)\log(n/2)) + cn \\
&= cn\log(n/2) + cn
\end{aligned}
$$

Recurrence being analyzed:

T(n) = 2T(n/2) + c·n

T(1) = c

Guess: T(n) = c·n log n

Induction:
Prove for n.

T(1) = c

T(x) = c·x log x for all x < n.

$$\begin{aligned}
T(n) &= 2T(n/2) + cn \\
&= 2(c(n/2)\log(n/2)) + cn \\
&= cn\log(n/2) + cn \\
&= cn\log(n) - cn\log(2) + cn
\end{aligned}$$

<u>Recurrence being analyzed:</u>

T(n) = 2T(n/2) + c·n

T(1) = c

Guess: T(n) = c·n log n

Induction:
Prove for n.

T(1) = c

T(x) = c·x log x for all x < n.

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T(n) &= 2T(n/2) + cn \\
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<u>Recurrence being analyzed:</u>

T(n) = 2T(n/2) + c·n

T(1) = c

Guess: T(n) = c·n log n

T(1) = c

T(x) = c·x log x for all x < n.

$$
\begin{aligned}
T(n) &= 2T(n/2) + cn \\
&= 2(c(n/2)\log(n/2)) + cn \\
&= cn\log(n/2) + cn \\
&= cn\log(n) - cn\log(2) + cn \\
&= cn\log(n)
\end{aligned}
$$

Induction:
It works!

<u>Recurrence being analyzed:</u>

T(n) = 2T(n/2) + c·n

T(1) = c

When is it better to use InsertionSort instead of MergeSort?

A. When there is limited space?
B. When there are a lot of items to sort?
C. When there is a large memory cache?
D. When there are a small number of items?
E. When the list is mostly sorted?

MergeSort

When the list is mostly sorted:

- InsertionSort is fast!
- MergeSort is O(n log n)

How "close to sorted" should a list be for InsertionSort to be faster?

MergeSort

Small number of items to sort:

- MergeSort is slow (because of recursion overhead)!
- Caching performance, branch prediction, etc.
- User InsertionSort for $n < 1024$, say.

Base case of recursion:

- Use slower sort.

Run an experiment and post on the forum what the best switch-over point is for your machine.

MergeSort

Space usage:

- Need extra space to do merge.
- Merge copies data to new array.
- How much extra space?

Challenge of the Day:

How much space does MergeSort need to sort n items?

(Use the version presented today.)

Design a version of MergeSort that minimizes the amount of extra space needed.

MergeSort

Is it stable?

MergeSort

Is it stable?

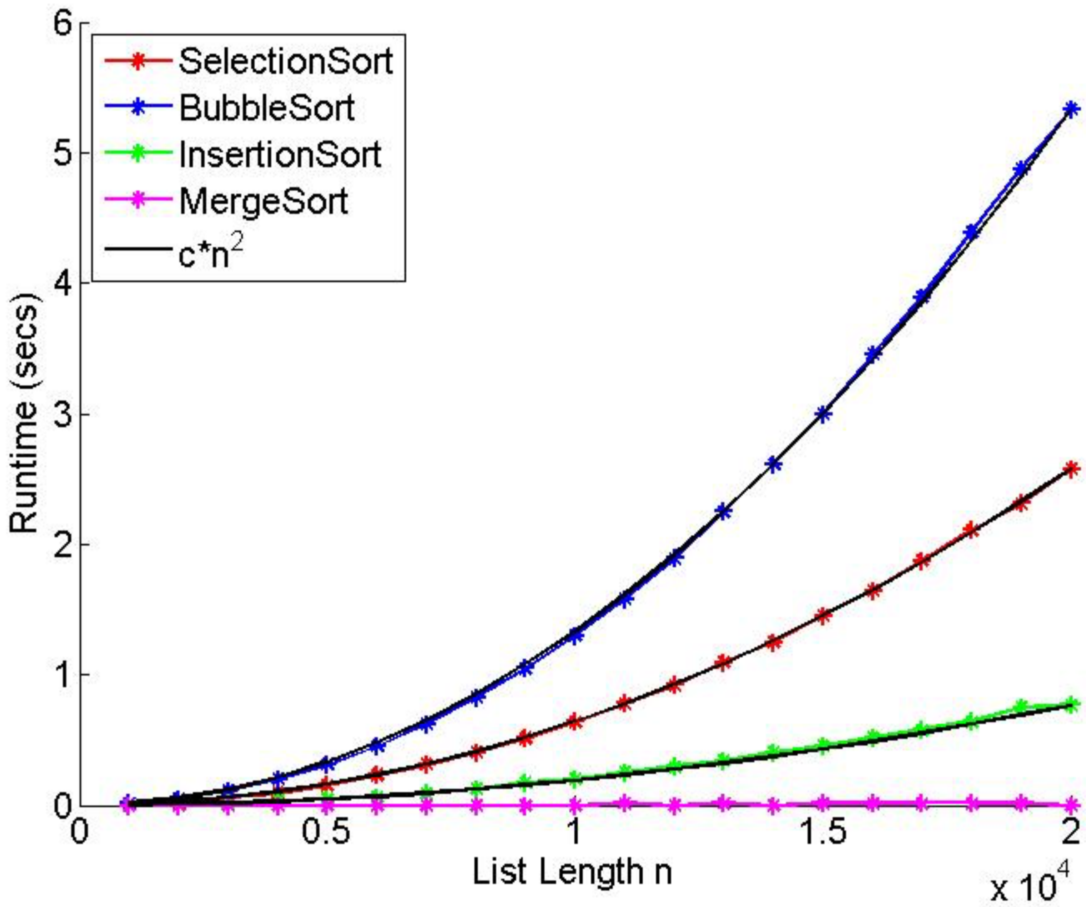
MergeSort is stable if “merge” is stable.

Merge is stable if carefully implemented.

Summary

Name	Best Case	Average Case	Worst Case	Extra Memory	Stable?
Bubble Sort	$O(n)$	$O(n^2)$	$O(n^2)$	$O(1)$	Yes
Selection Sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(1)$	No
Insertion Sort	$O(n)$	$O(n^2)$	$O(n^2)$	$O(1)$	Yes
Merge Sort	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	$O(n)$	Yes

real world performance



http://www.cs.toronto.edu/~jepson/csc148/2007F/notes/sorting.html]

Performance Profiling

(*Dracula* vs. *Lewis & Clark*)

Version	Change	Running Time
Version 1		4,311.00s
Version 2	Better file handling	676.50s
Version 3	Faster sorting	6.59s
Version 4	No sorting!	2.35s

V.2 → V.3 was using MergeSort instead of SelectionSort.

Today: Sorting, Part II

QuickSort

- Divide-and-Conquer
- Paranoid QuickSort
- Randomized Analysis

QuickSort

History:

- Invented by C.A.R. Hoare in 1960
  - Turing Award: 1980
- Visiting student at Moscow State University
- Used for machine translation (English/Russian)

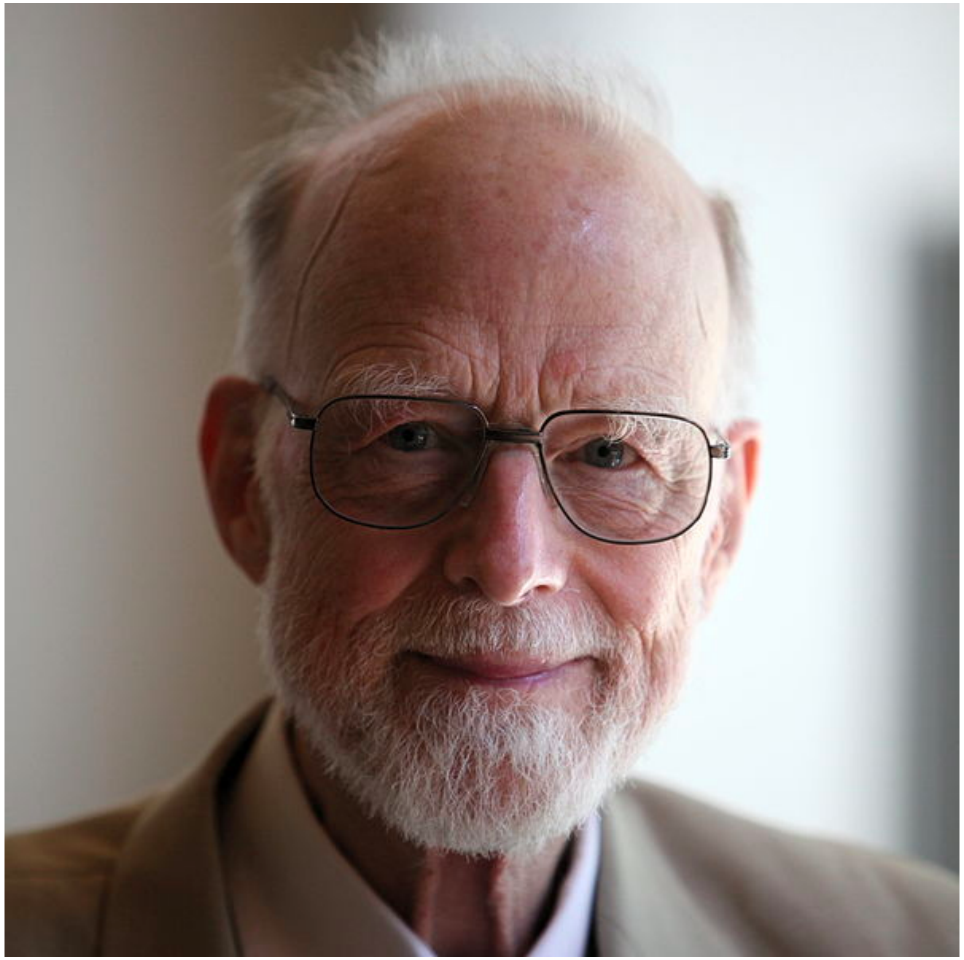


Photo: Wikimedia Commons (Rama)

Hoare

Quote:

“There are two ways of constructing a software design:

One way is to make it so simple that there are obviously no deficiencies, and the other way is to make it so complicated that there are no obvious deficiencies.

The first method is far more difficult.”

QuickSort

History:

- Invented by C.A.R. Hoare in 1960
- Used for machine translation (English/Russian)

In practice:

- Very fast
- Many optimizations
- In-place (i.e., no extra space needed)
- Good caching performance
- Good parallelization

QuickSort Today

1960: Invented by Hoare

1979: Adopted everywhere (e.g., Unix qsort)

1993: Bentley & McIlroy improvements

"Engineering a sort function"

Yet in the summer of 1991 our colleagues Allan Wilks and Rick Becker found that a qsort run that should have taken a few minutes was chewing up hours of CPU time. Had they not interrupted it, it would have gone on for weeks. They found that it took n^2 comparisons to sort an 'organ-pipe' array of 2n integers: 123..nn.. 321.

QuickSort Today

1960: Invented by Hoare

1979: Adopted everywhere (e.g., Unix qsort)

1993: Bentley & McIlroy improvements

"Ok, QuickSort is done," said everyone.

Every algorithms class since 1993:

Punk in the front row:

“But what if we used more pivots?”

Every algorithms class since 1993:

Punk in the front row:

"But what if we used more pivots?"

Professor:

"Doesn't work. I can prove it.
Let's get back to the syllabus...."

In 2009:

Punk in the front row:

"But what if we used more pivots?"

Professor:

"Doesn't work. I can prove it.
Let's get back to the syllabus...."

Punk in the front row:

"Huh... let me try it. Wait a sec, it's faster!"

QuickSort Today

1960: Invented by Hoare

1979: Adopted everywhere (e.g., Unix qsort)

1993: Bentley & McIlroy improvements

2009: Vladimir Yaroslavskiy

- Dual-pivot Quicksort !!!
- Now standard in Java
- 10% faster!

QuickSort Today

1960: Invented by Hoare

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2009: Vladimir Yaroslavskiy

- Dual-pivot Quicksort !!!
- Now standard in Java
- 10% faster!

2012: Sebastian Wild and Markus E. Nebel

- "Average Case Analysis of Java 7's Dual Pivot…"
- Best paper award at ESA

Moral of the story:

1) Don’t just listen to me. Go try it!

2) Even “classical” algorithms change. QuickSort in 5 years may be different than QuickSort I am teaching today.

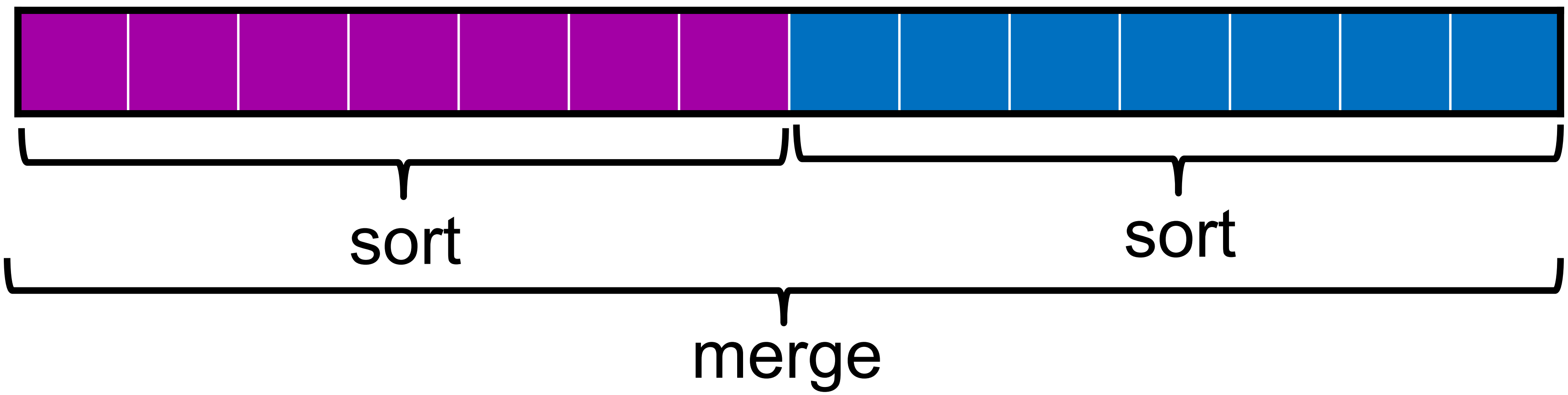
QuickSort

In class:

- Easy to understand! (divide-and-conquer…)
- Moderately hard to implement correctly.
- Harder to analyze. (Randomization…)
- Challenging to optimize.

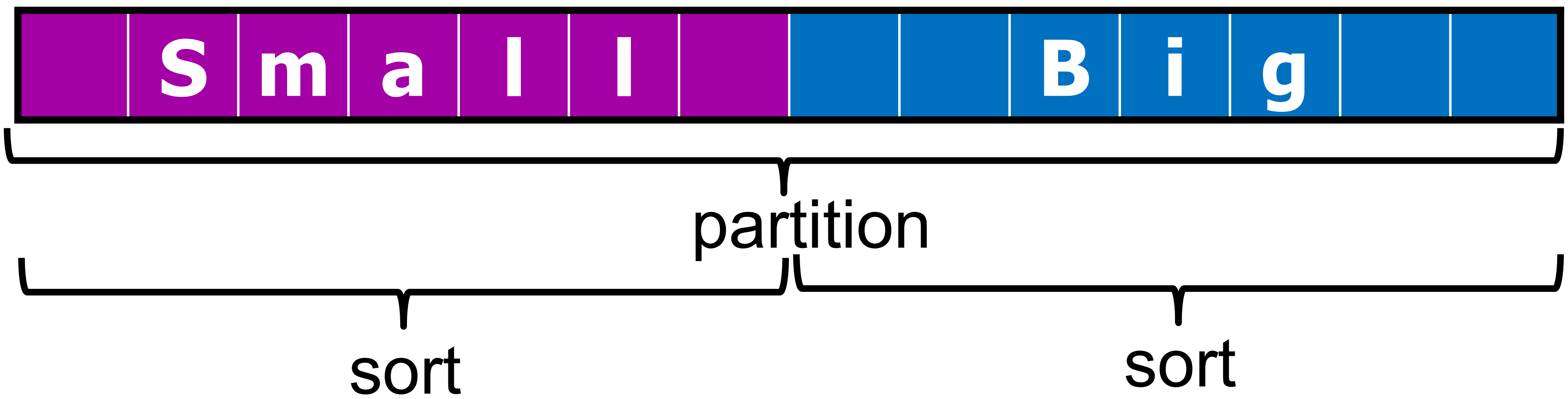
Recall: MergeSort

```
MergeSort(A[1..n], n)
    if (n==1) then return;
    else
        x = MergeSort(A[1..n/2], n/2)
        y = MergeSort(A[n/2+1..n], n/2)
        return merge(x, y, n/2)
```



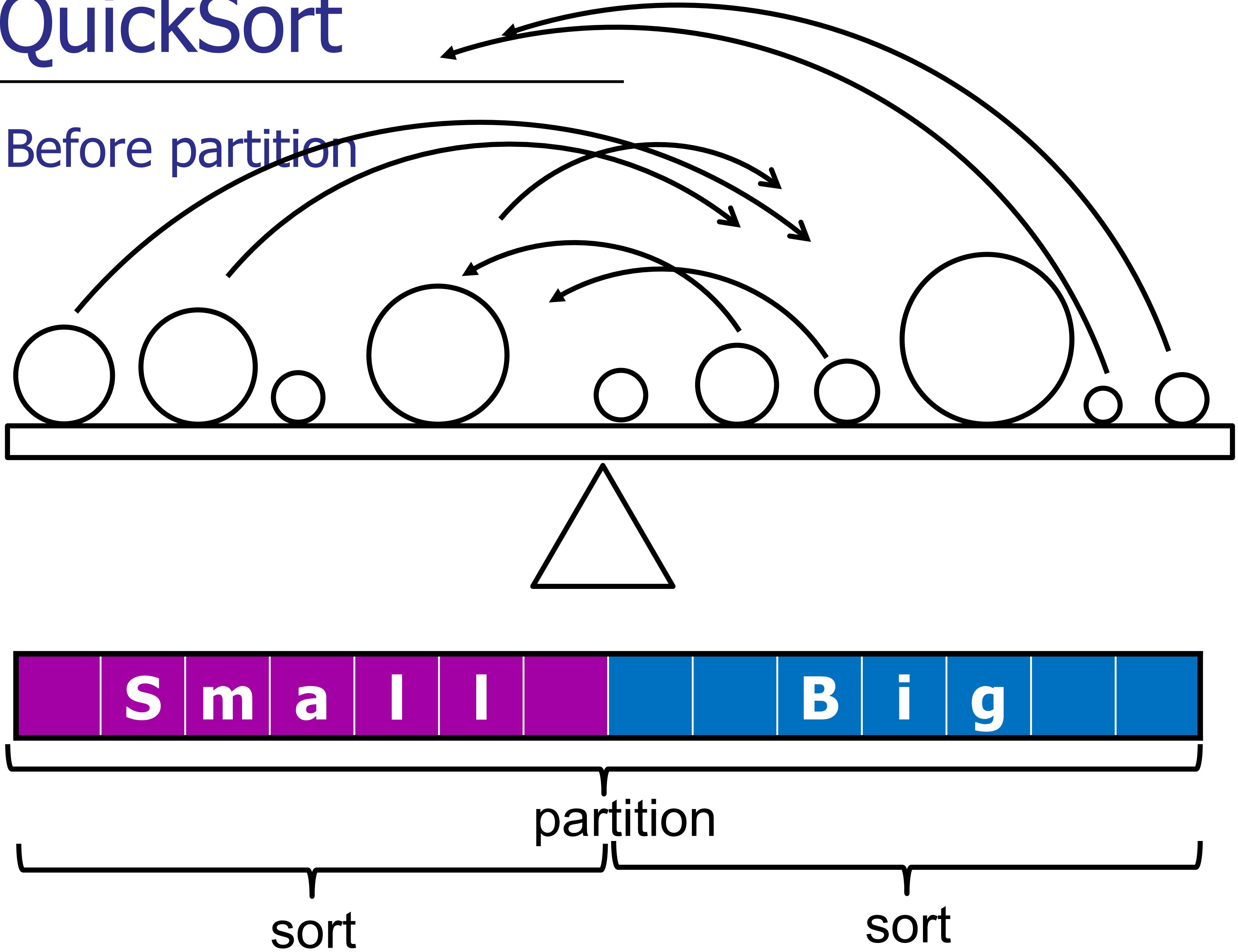
QuickSort

```
QuickSort(A[1..n], n)
    if (n==1) then return;
    else
        p = partition(A[1..n], n)
        x = QuickSort(A[1..p-1], p-1)
        y = QuickSort(A[p+1..n], n-p)
```



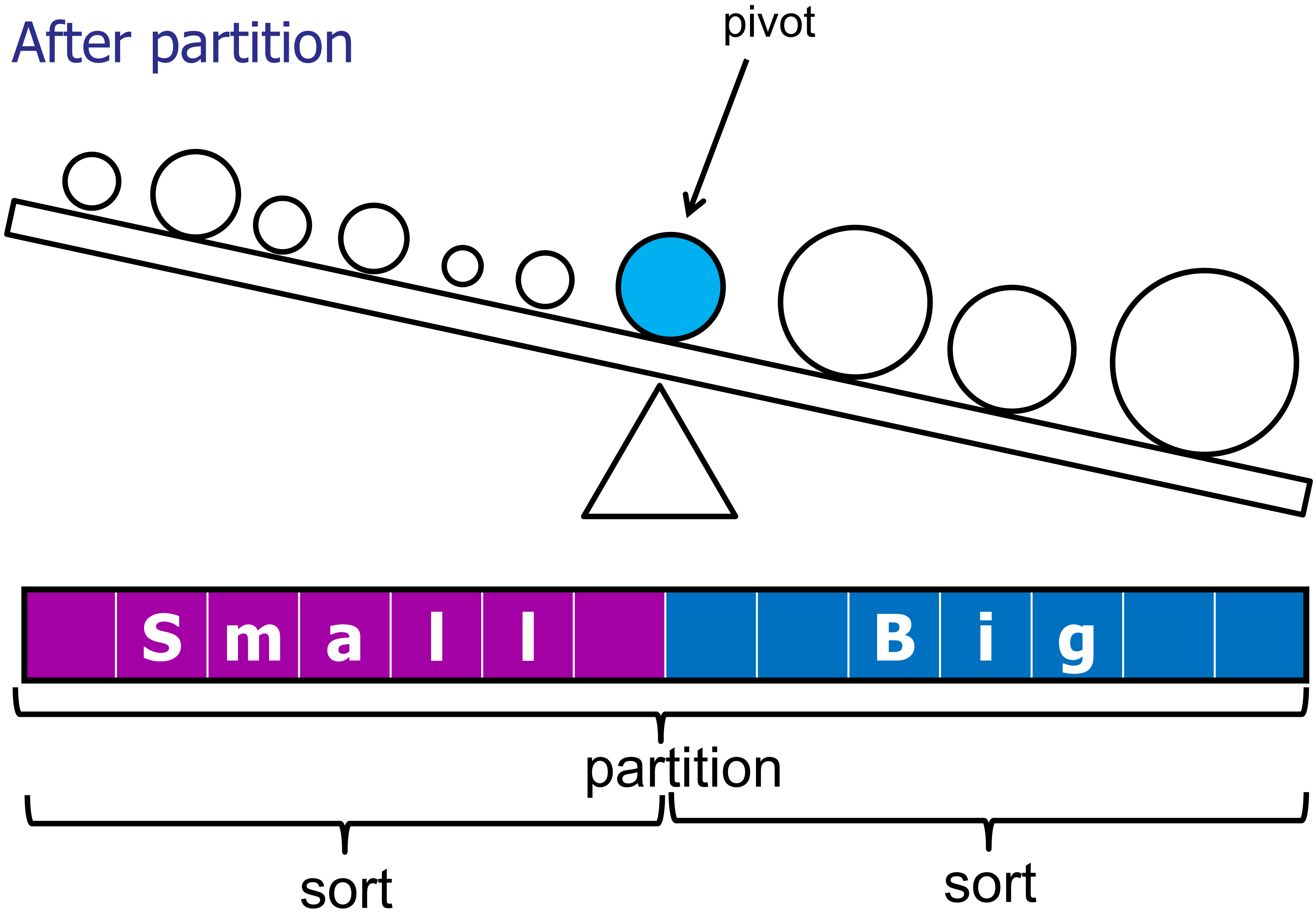
QuickSort

Before partition

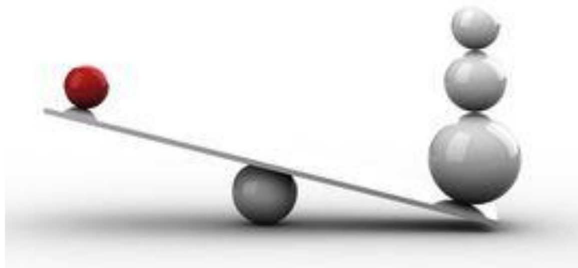


QuickSort

After partition



QuickSort



QuickSort(A[1..n], n)

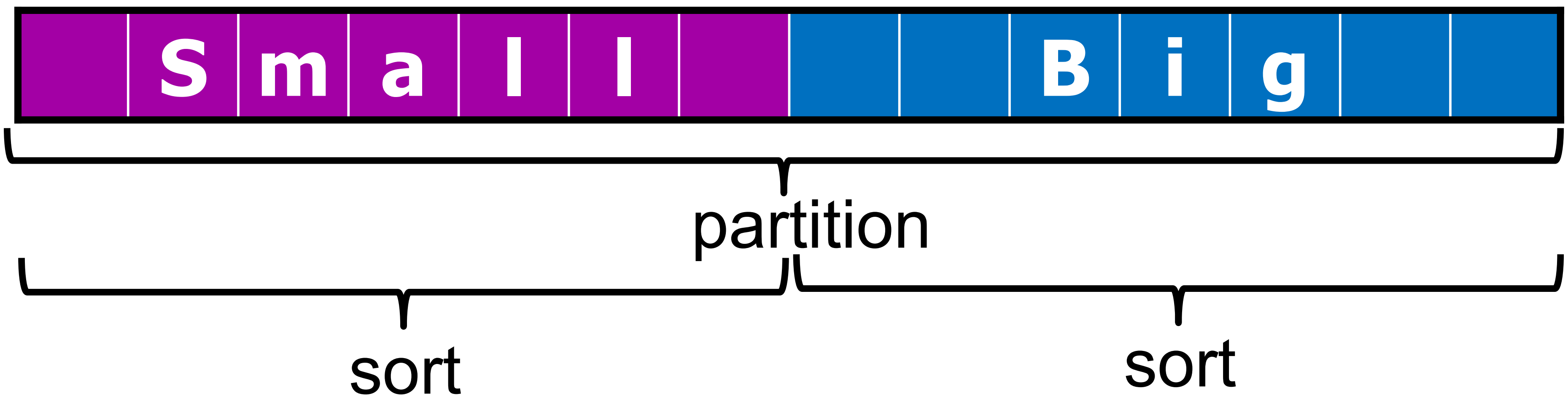
    if (n==1) **then** return;

    else

        p = partition(A[1..n], n)

        x = QuickSort(A[1..p-1], p-1)

        y = QuickSort(A[p+1..n], n-p)



QuickSort

Given: n element array $A[1..n]$

1. Divide: Partition the array into two sub-arrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper sub-array.

$< x$	x	$> x$

2. Conquer: Recursively sort the two sub-arrays.
3. Combine: Trivial, do nothing.

Key: efficient *partition* sub-routine

Partitioning an Array

Three steps:

1. Choose a pivot.
2. Find all elements smaller than the pivot.
3. Find all elements larger than the pivot.

$< x$	x	$> \mathbf{x}$

Quicksort

Example:

6 3 9 8 4 2

Quicksort

Example:

6	3	9	8	4	2
3	4	2	6	9	8

Quicksort

Example:

6	3	9	8	4	2
3	4	2	6	9	8
2	3	4			

Quicksort

Example:

6	3	9	8	4	2
3	4	2	6	9	8
2	3	4		8	9

Quicksort

Example:

6	3	9	8	4	2
3	4	2	$\boxed{6}$	9	8
2	3	4	6	8	9

Quicksort

Example:

6	3	9	8	4	2
3	4	2	6	9	8
2	3	4	6	8	9

Partitioning an Array

Example: partition around 22

22	1	6	40	32	10	18	50	4

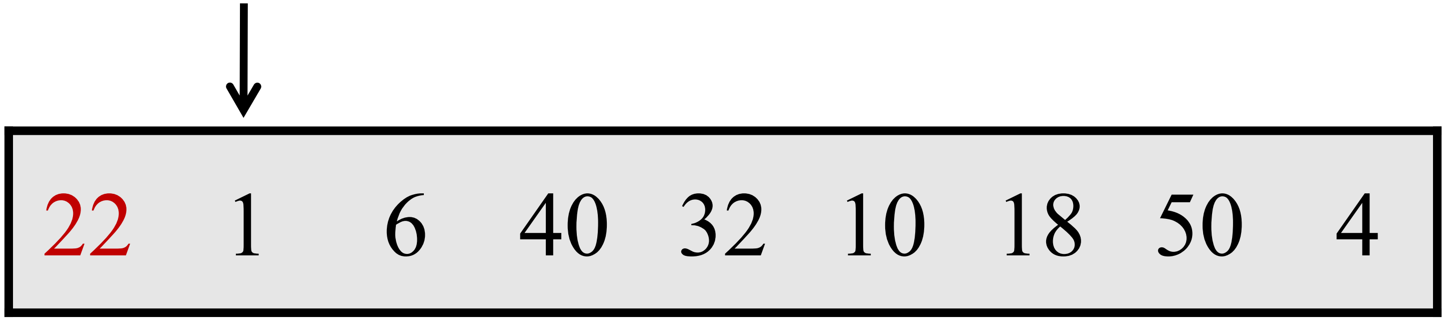
Output array:

low
< 22

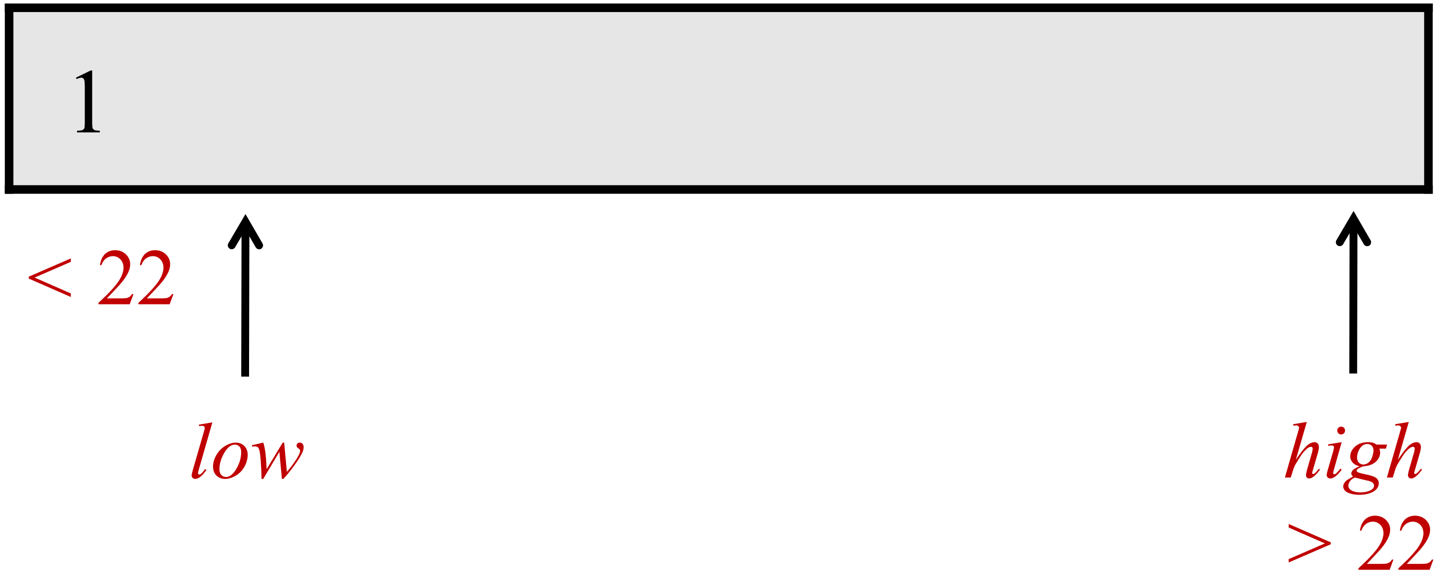
high
> 22

Partitioning an Array

Example: partition around 22

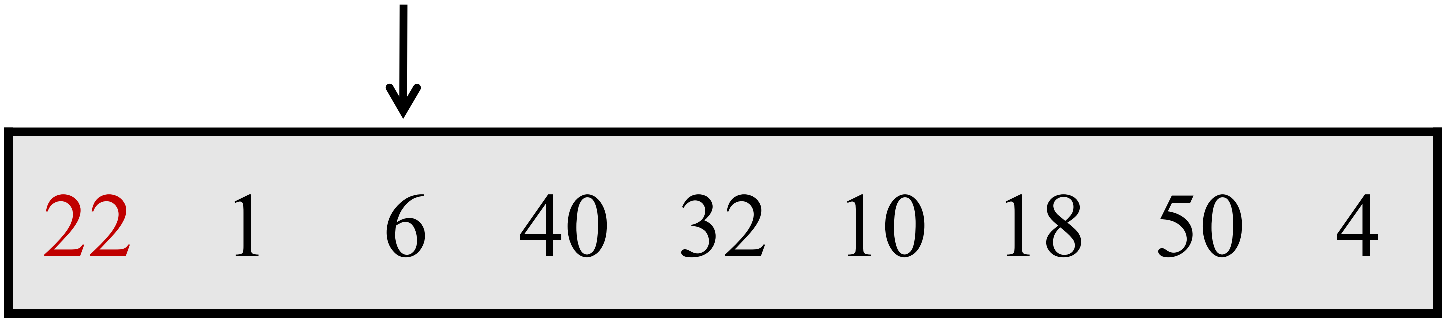


Output array:

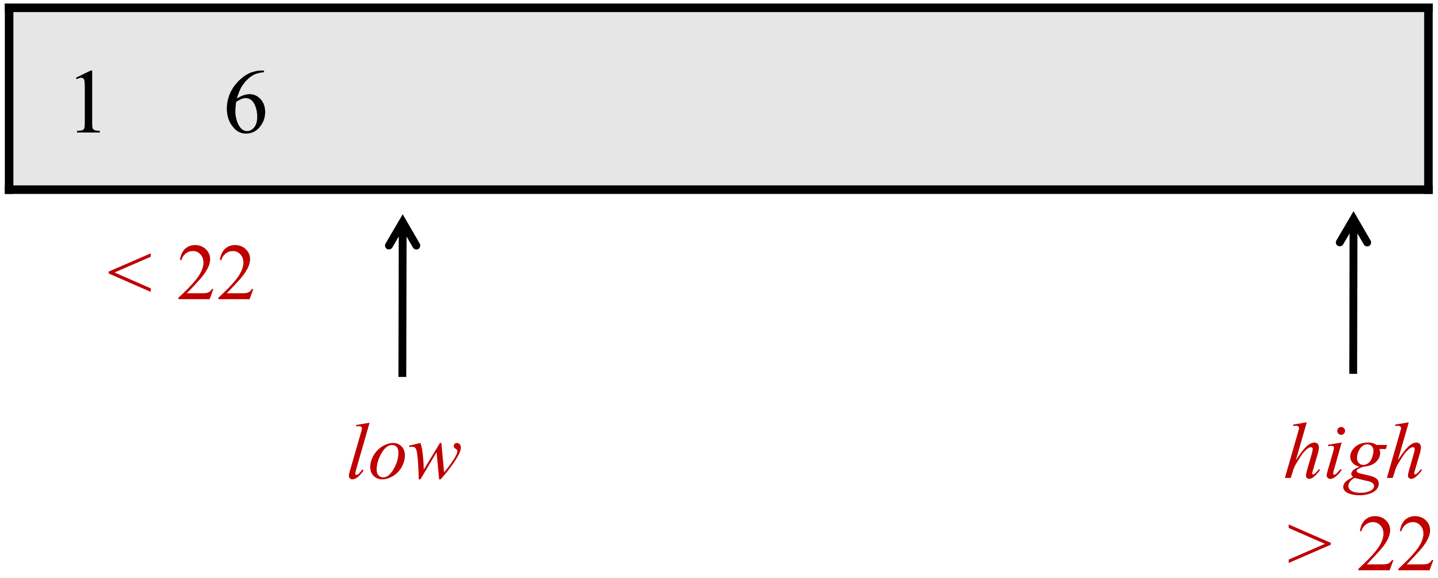


Partitioning an Array

Example: partition around 22

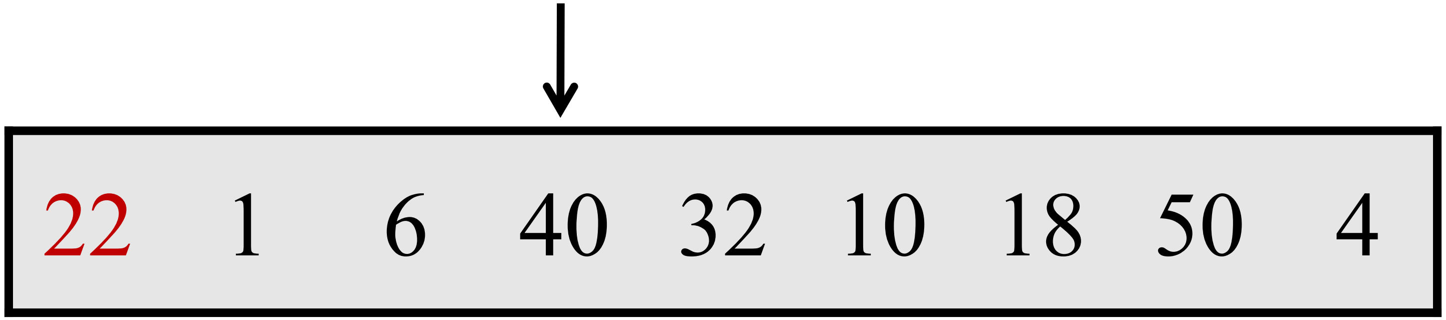


Output array:

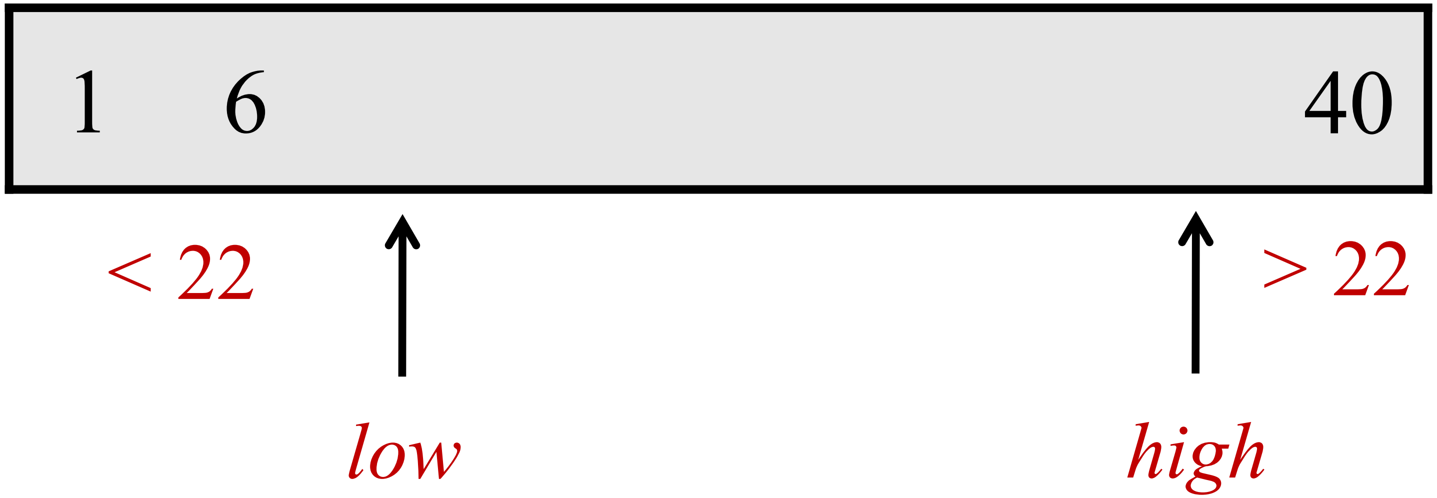


Partitioning an Array

Example: partition around 22

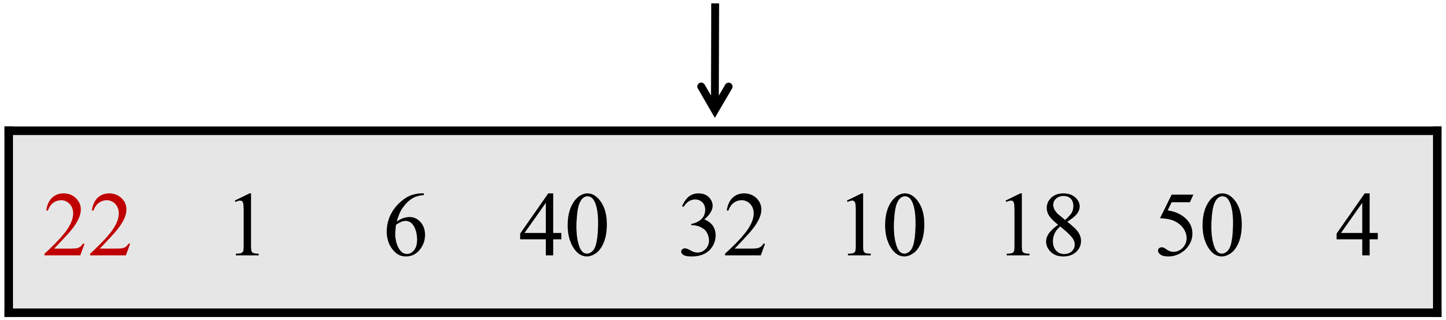


Output array:

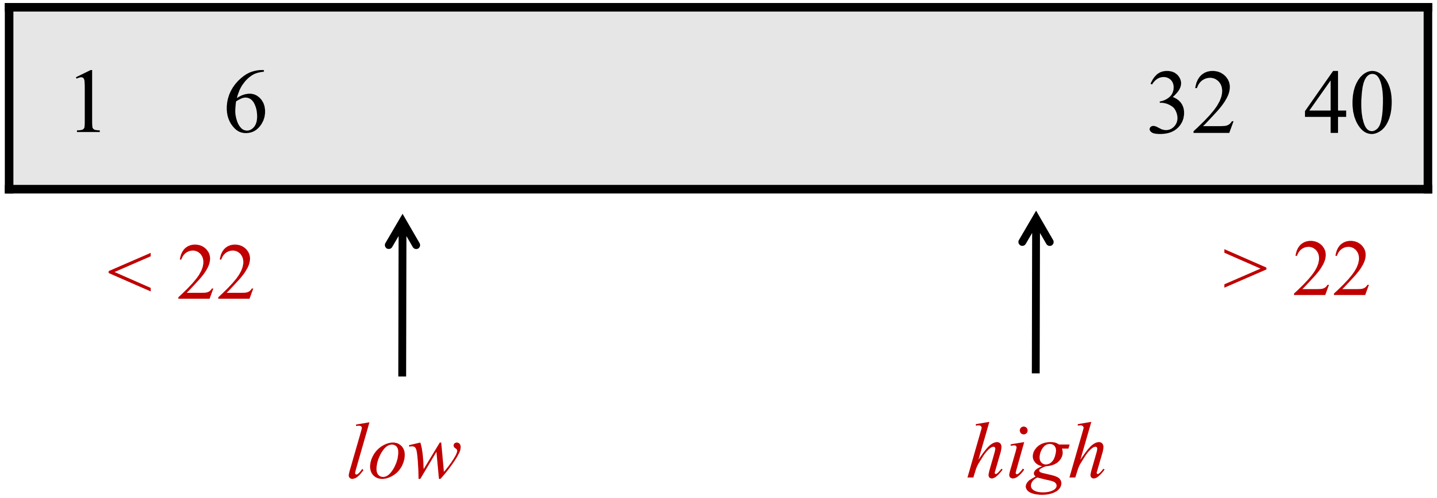


Partitioning an Array

Example: partition around 22

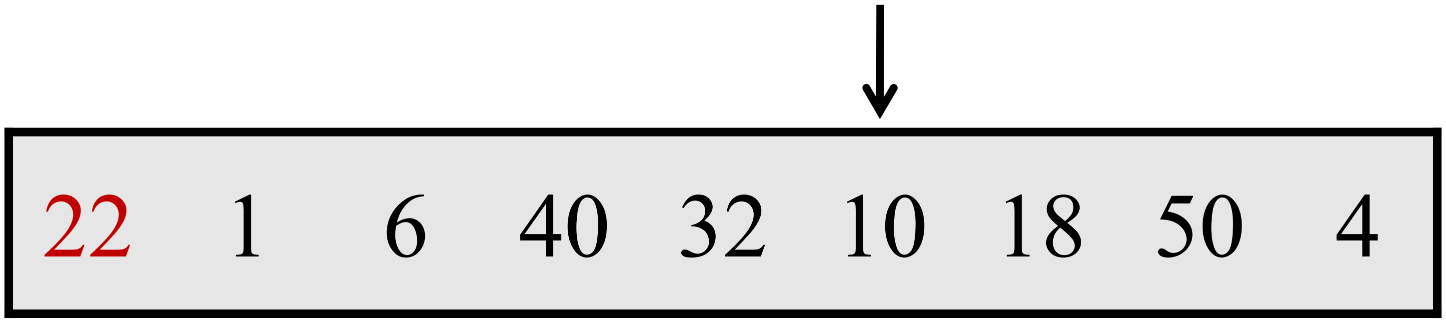


Output array:

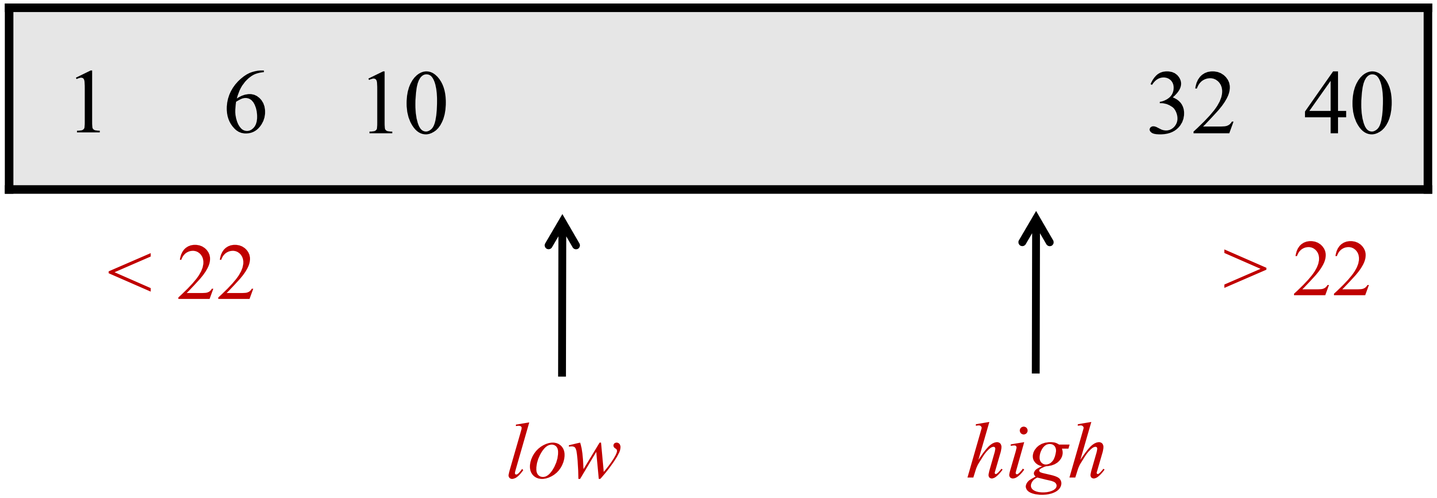


Partitioning an Array

Example: partition around 22

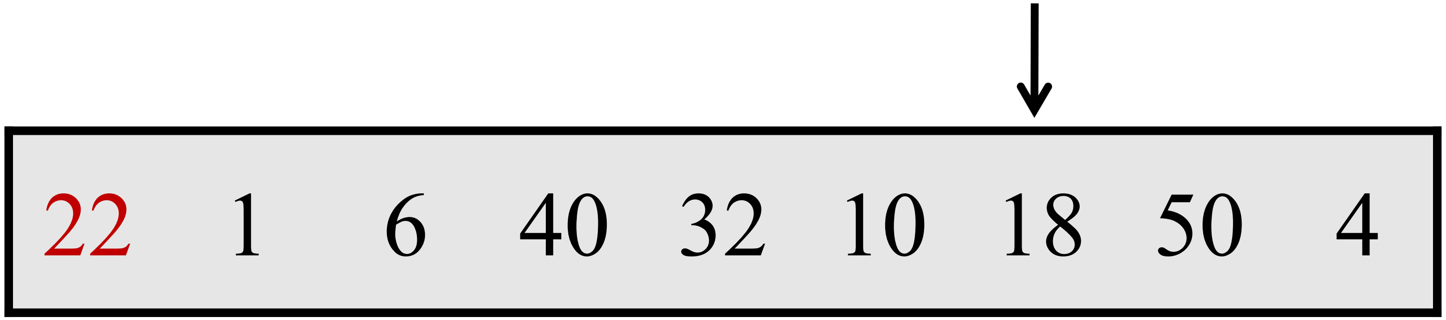


Output array:

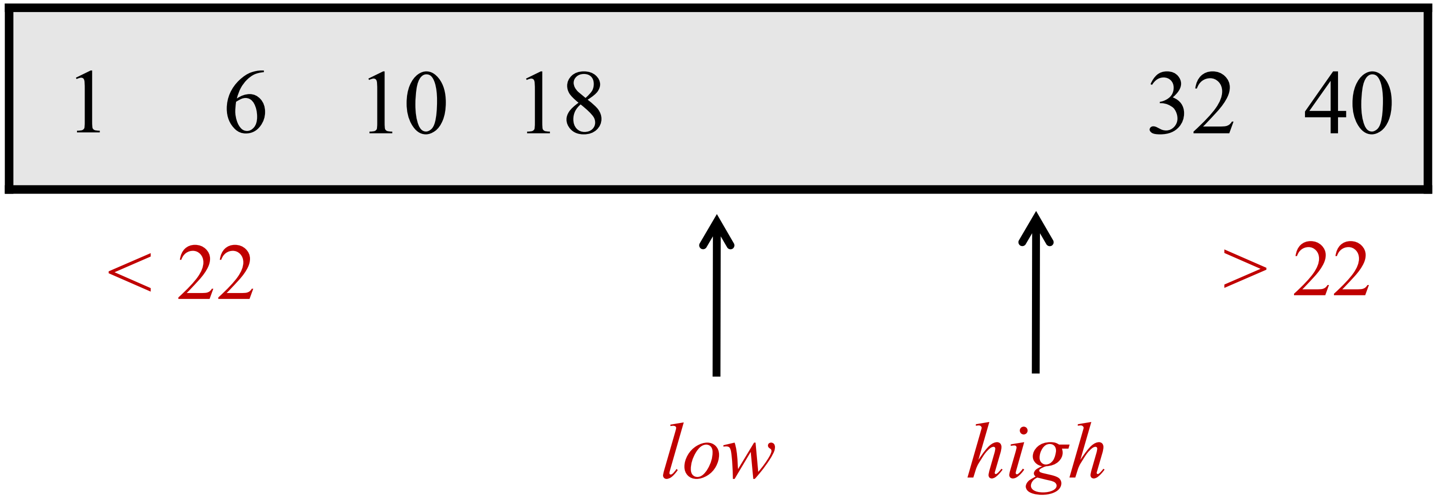


Partitioning an Array

Example: partition around 22

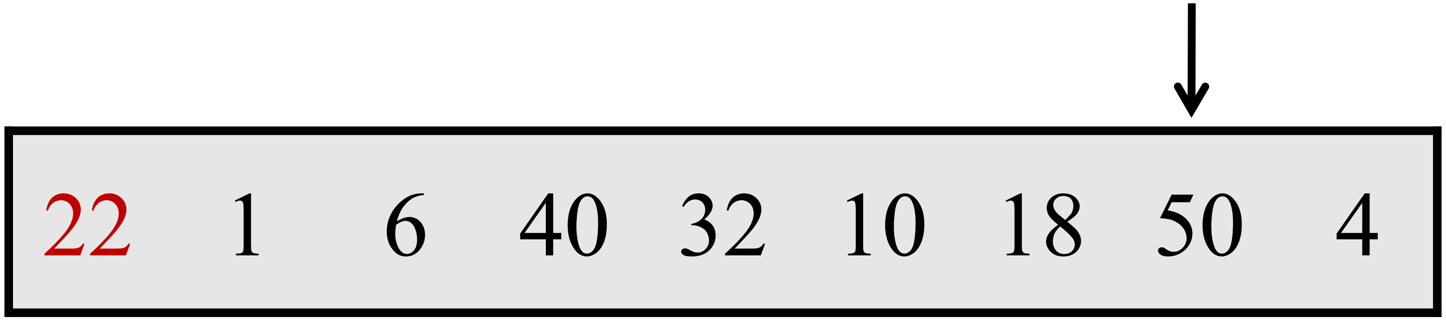


Output array:

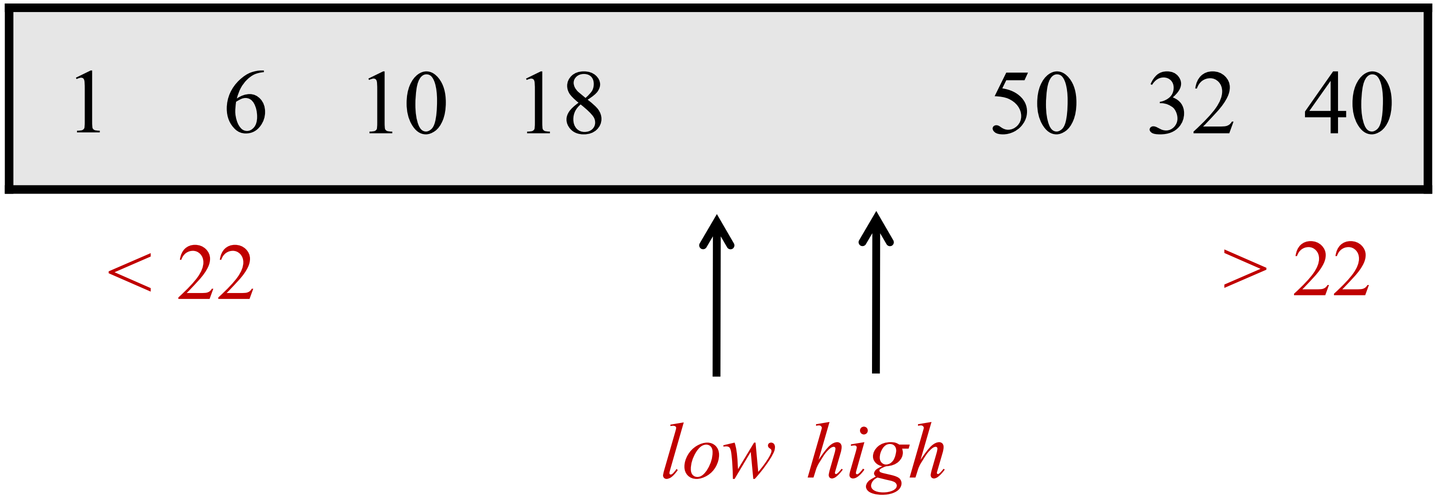


Partitioning an Array

Example: partition around 22

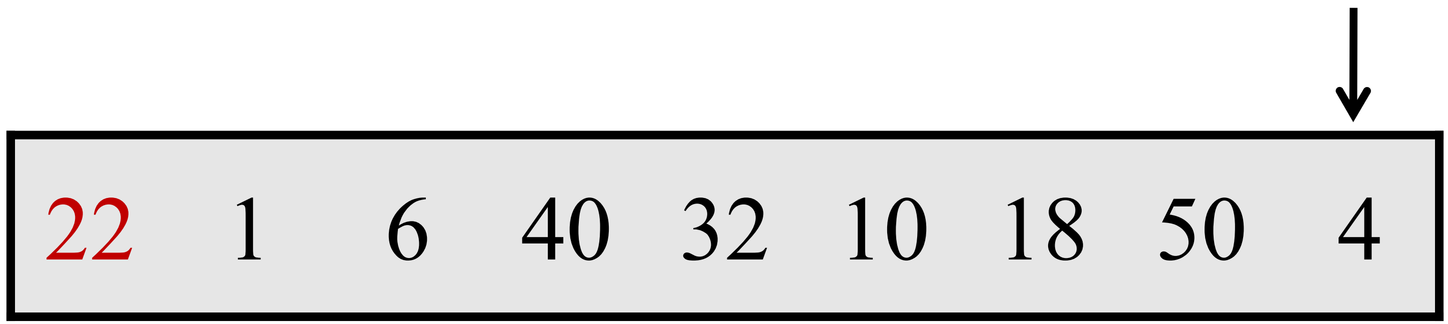


Output array:

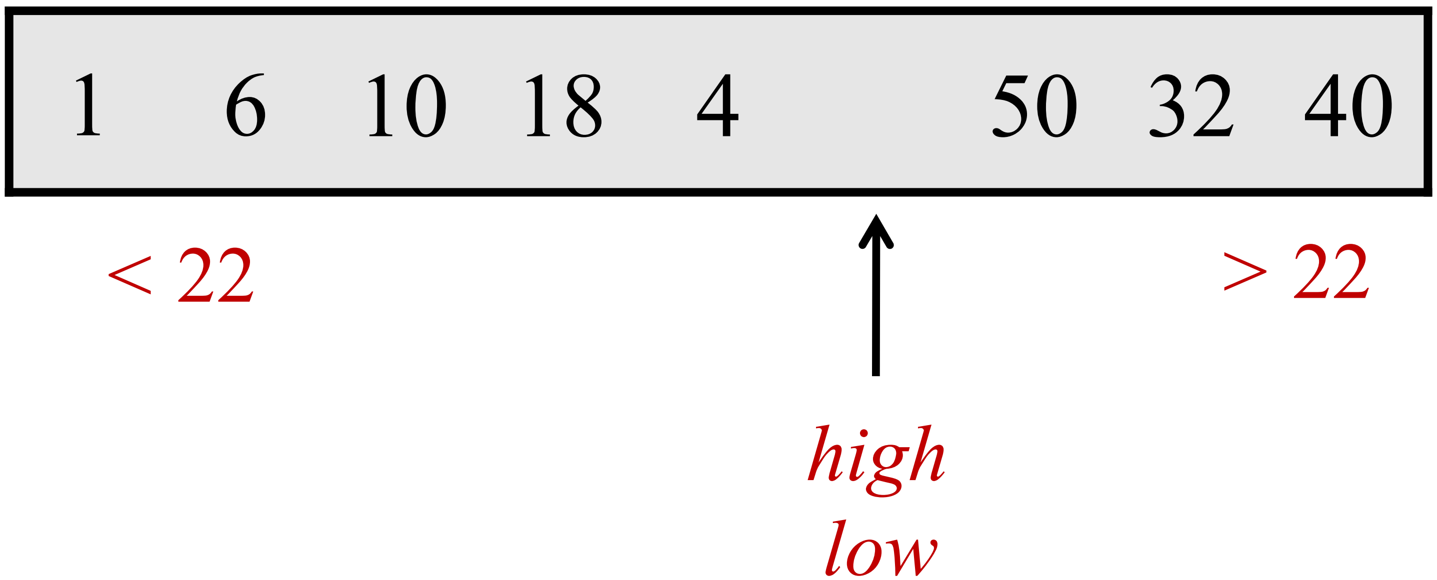


Partitioning an Array

Example: partition around 22



Output array:



Partitioning an Array

Example: partition around 22

22	1	6	40	32	10	18	50	4

Output array:

1	6	10	18	4	22	50	32	40

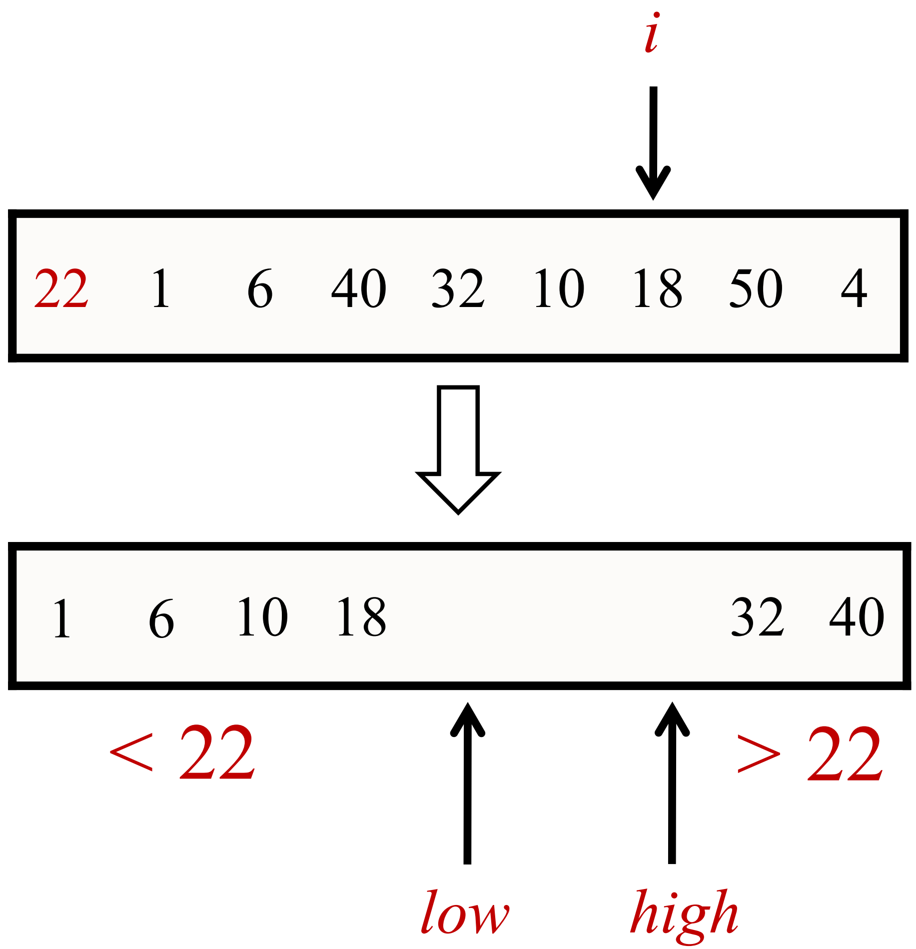
< 22

> 22

high
low

```
partition(A[2..n], n, pivot)
    B = new n element array
    low = 1;
    high = n;
    for (i = 2; i<= n; i++)
        if (A[i] < pivot) then
            B[low] = A[i];
            low++;
        else if (A[i] > pivot) then
            B[high] = A[i];
            high– – ;
    B[low] = pivot;
    return  < B, low >
```

// Assume no duplicates



Partition

Claim: array B is partitioned around the pivot

Proof:

Invariants:

1. For every $i < low : B[i] < pivot$
2. For every $j > high : B[j] > pivot$

In the end, every element from A is copied to B.

Then: $B[i] = pivot$

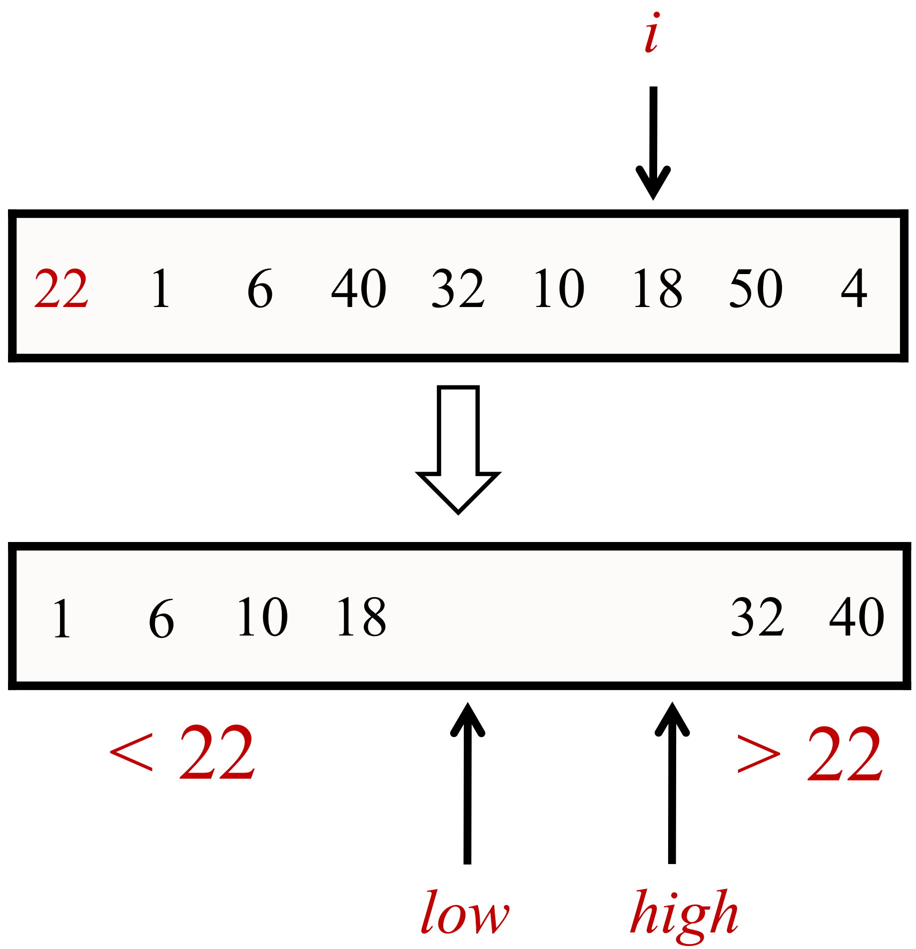
By invariants, B is partitioned around the pivot.

What is wrong with the partition procedure?

1. There is a bug. It doesn't work.
2. It uses too much memory.
3. It is too slow.
4. It only works for integers.
5. It has poor caching performance.
6. It works perfectly.

```
partition(A[2..n], n, pivot)
    B = new n element array
    low = 1;
    high = n;
    for (i = 2; i<= n; i++)
        if (A[i] < pivot) then
            B[low] = A[i];
            low++;
        else if (A[i] > pivot) then
            B[high] = A[i];
            high– – ;
    B[low] = pivot;
    return < B, low >
```

// Assume no duplicates



Partitioning an Array "in-place"

Example: partition around 22

22	1	6	40	32	10	18	4	50

low
< 22

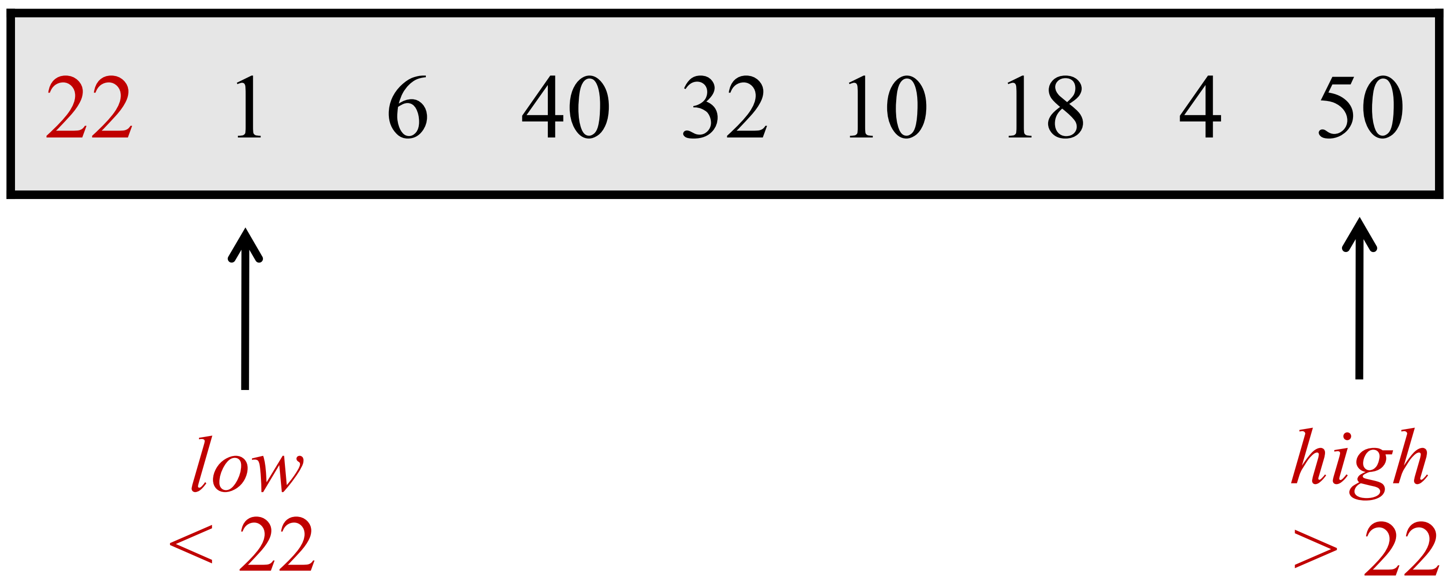
high
> 22

Move until it's bigger than the pivot

Move until it's less than the pivot

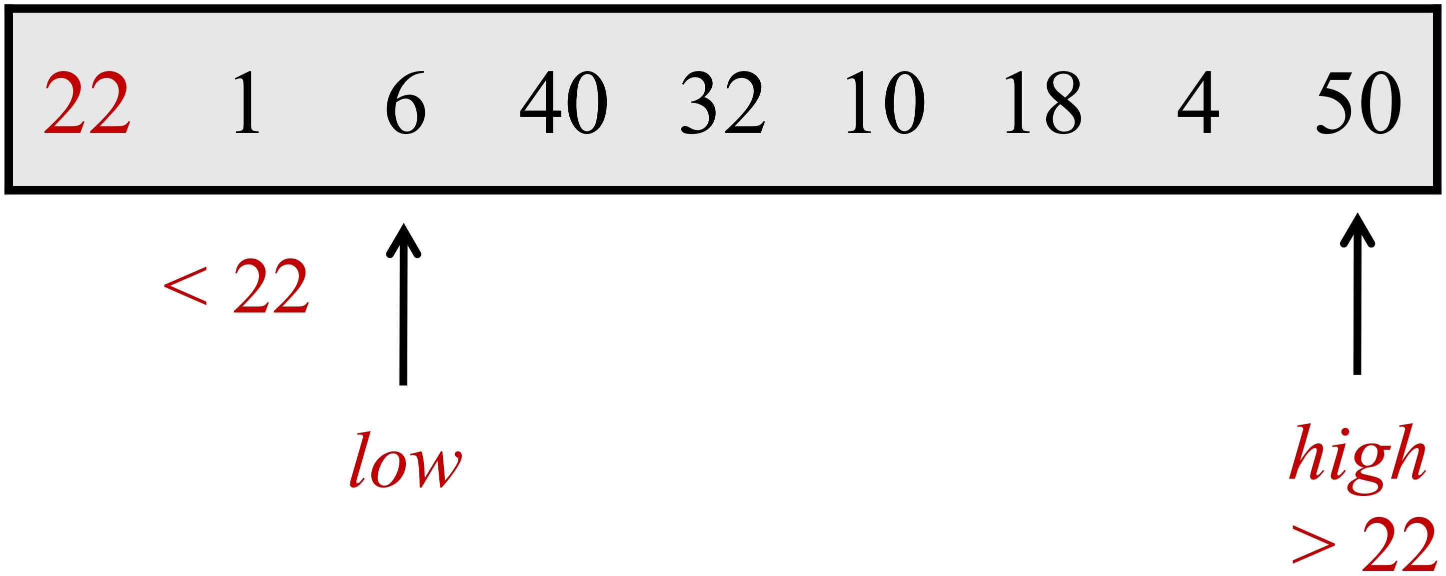
Partitioning an Array

Example: partition around 22



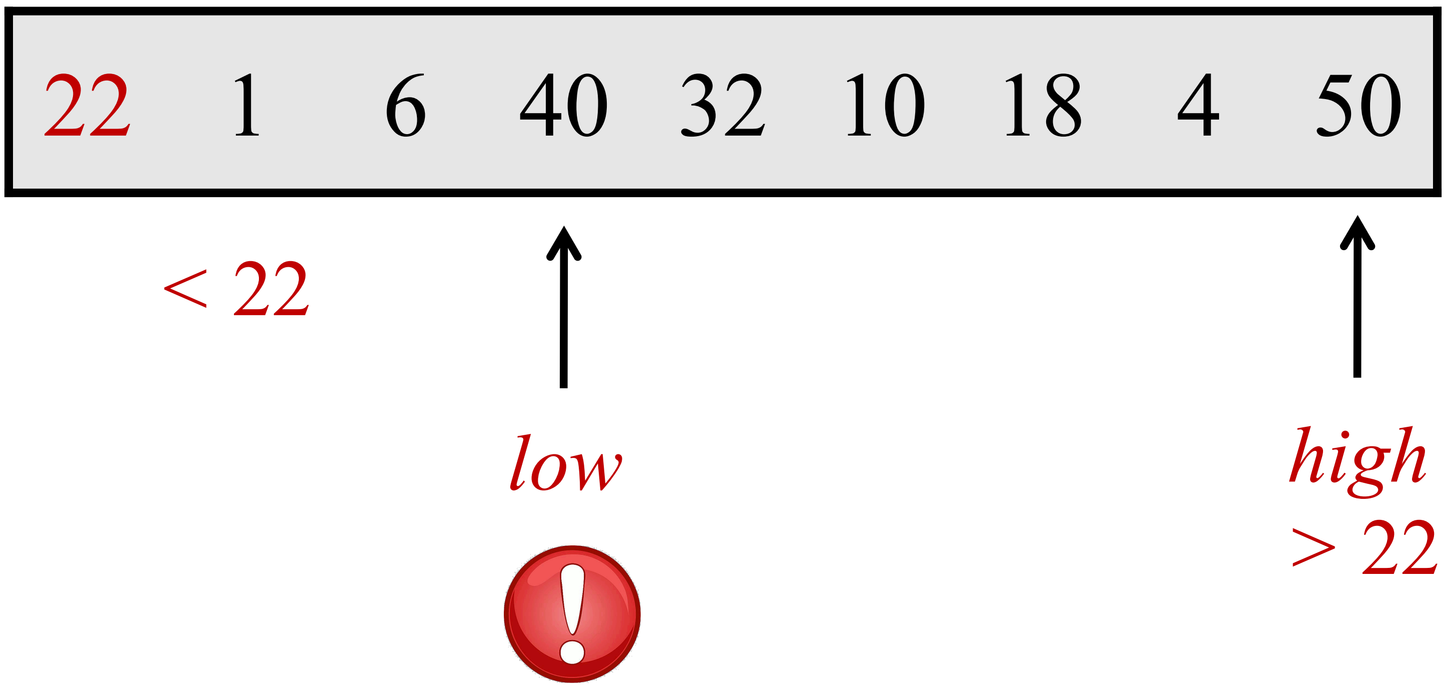
Partitioning an Array

Example: partition around 22



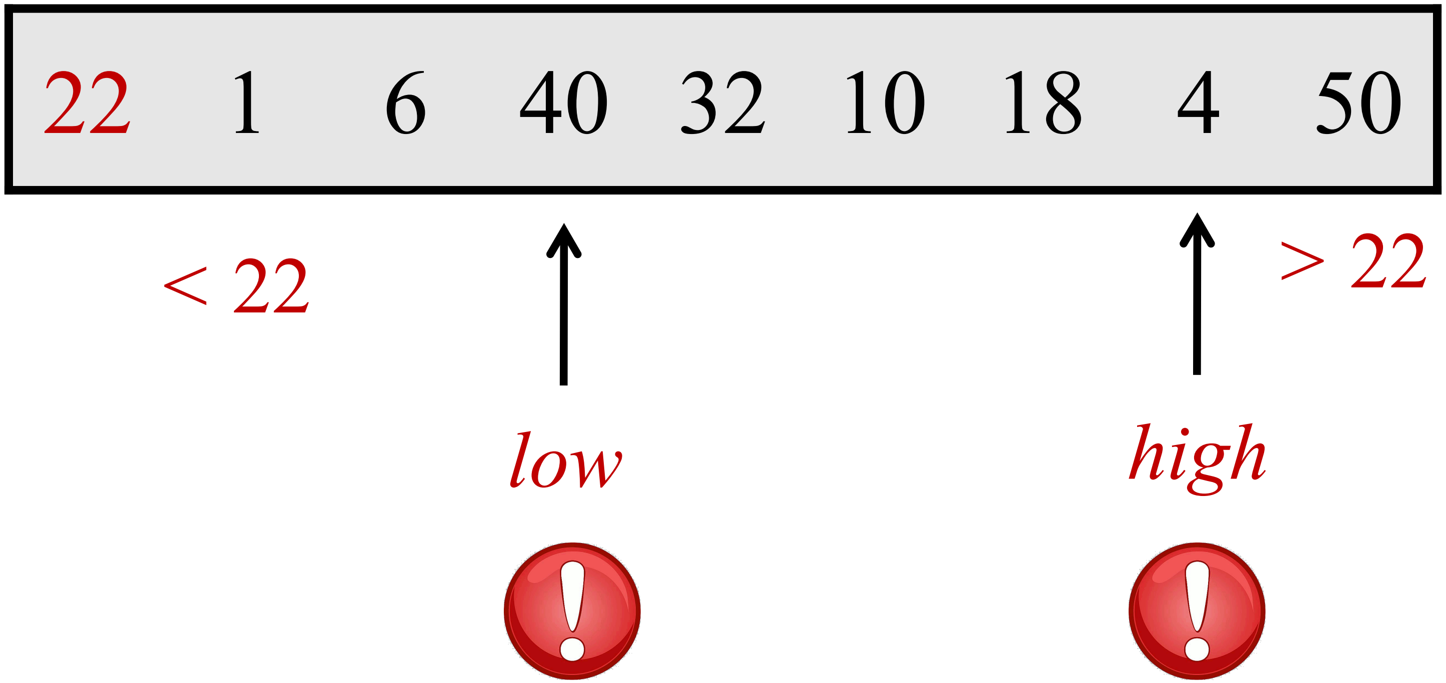
Partitioning an Array

Example: partition around 22



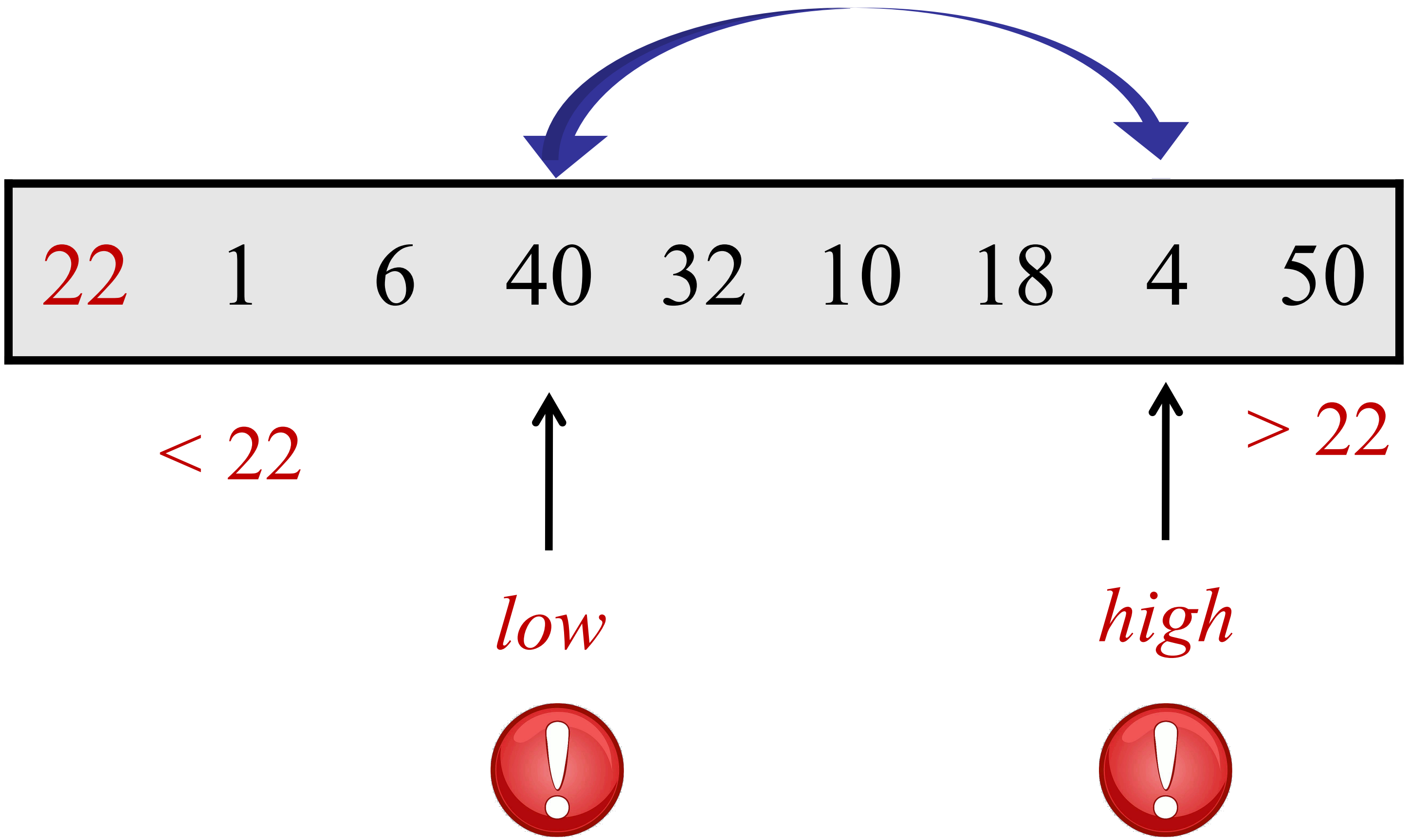
Partitioning an Array

Example: partition around 22



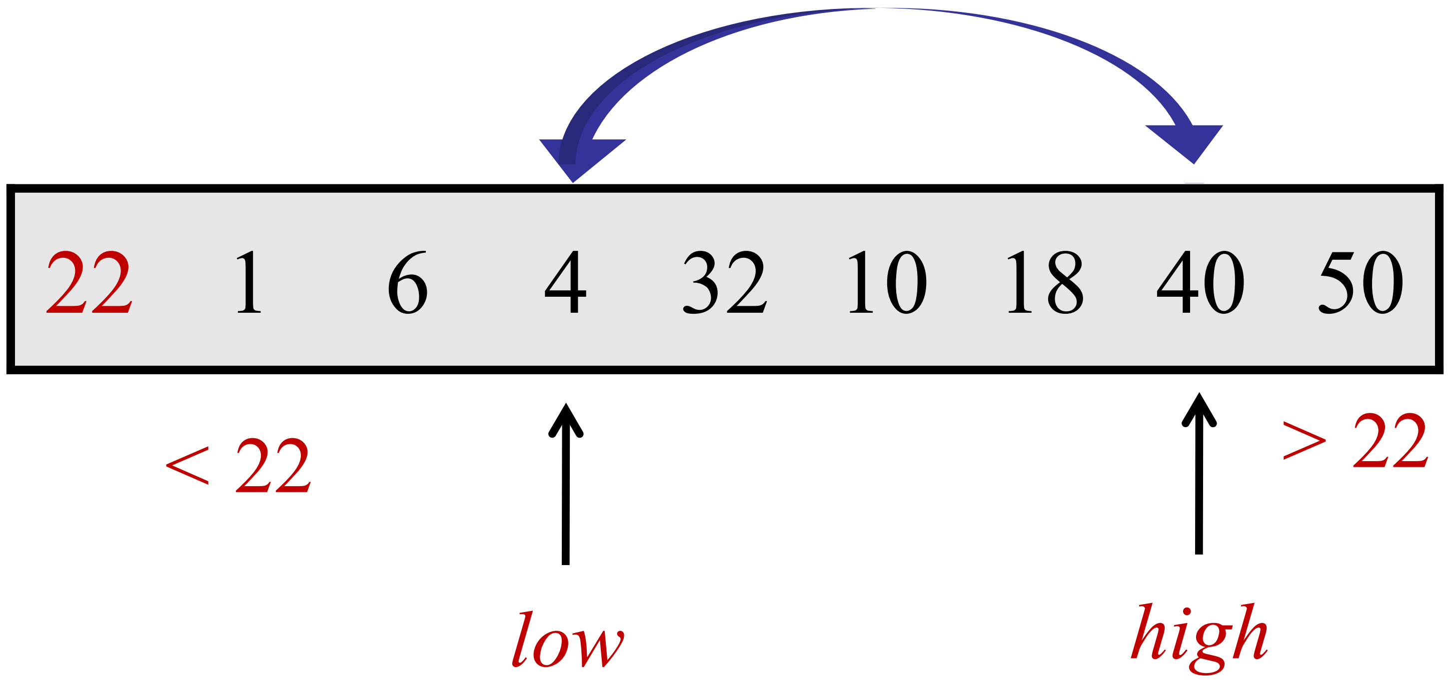
Partitioning an Array

Example: partition around 22



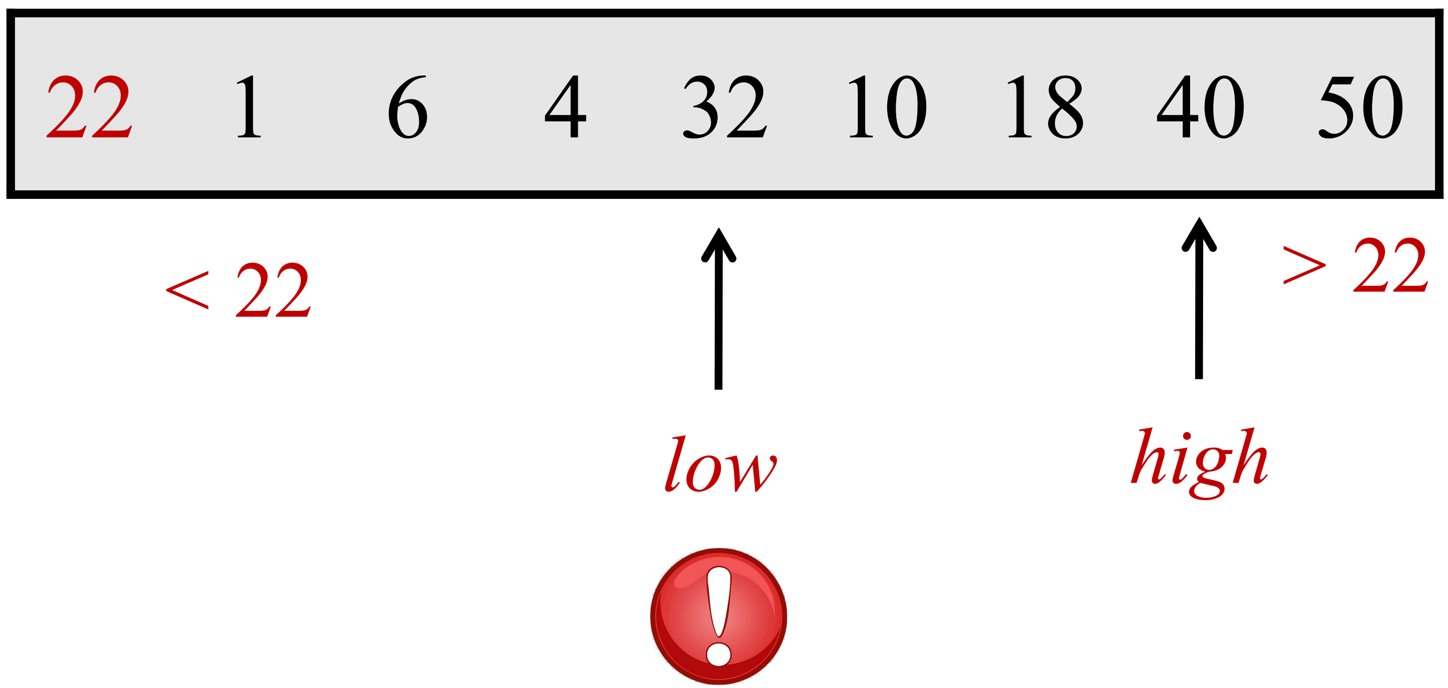
Partitioning an Array

Example: partition around 22



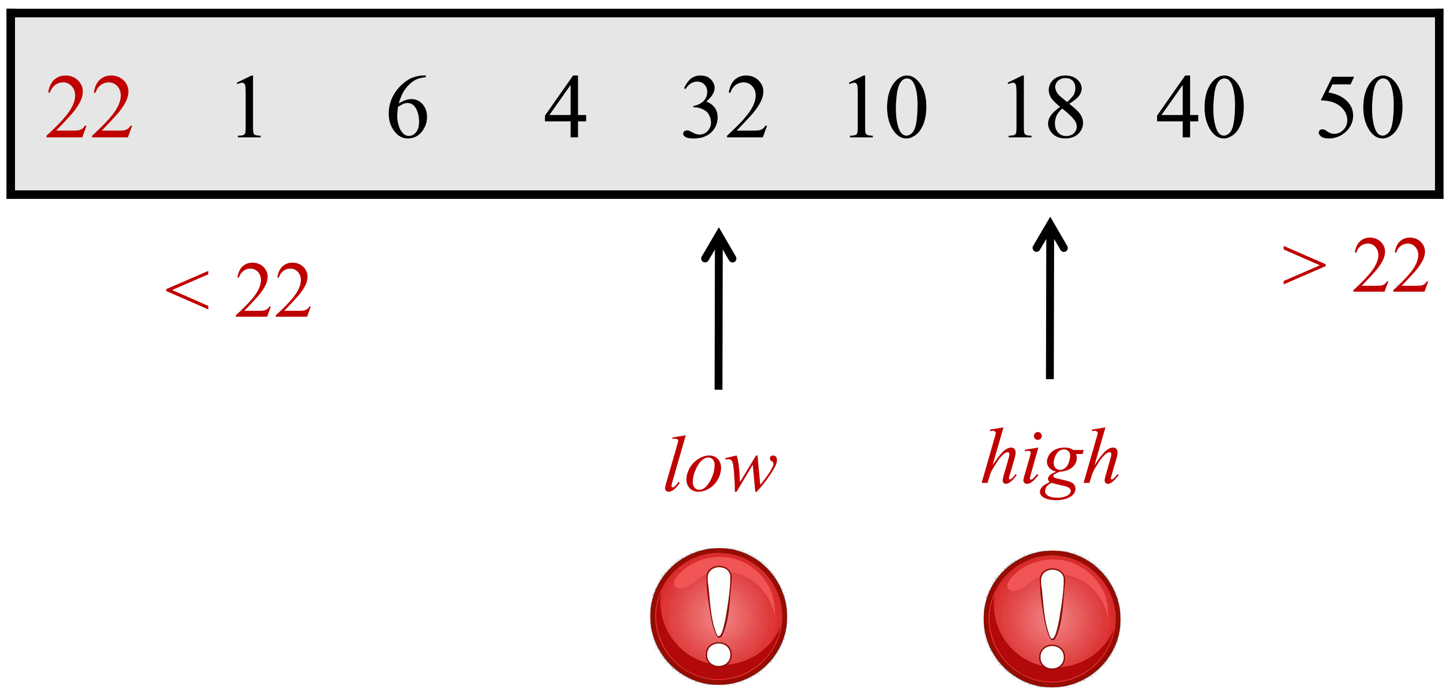
Partitioning an Array

Example: partition around 22



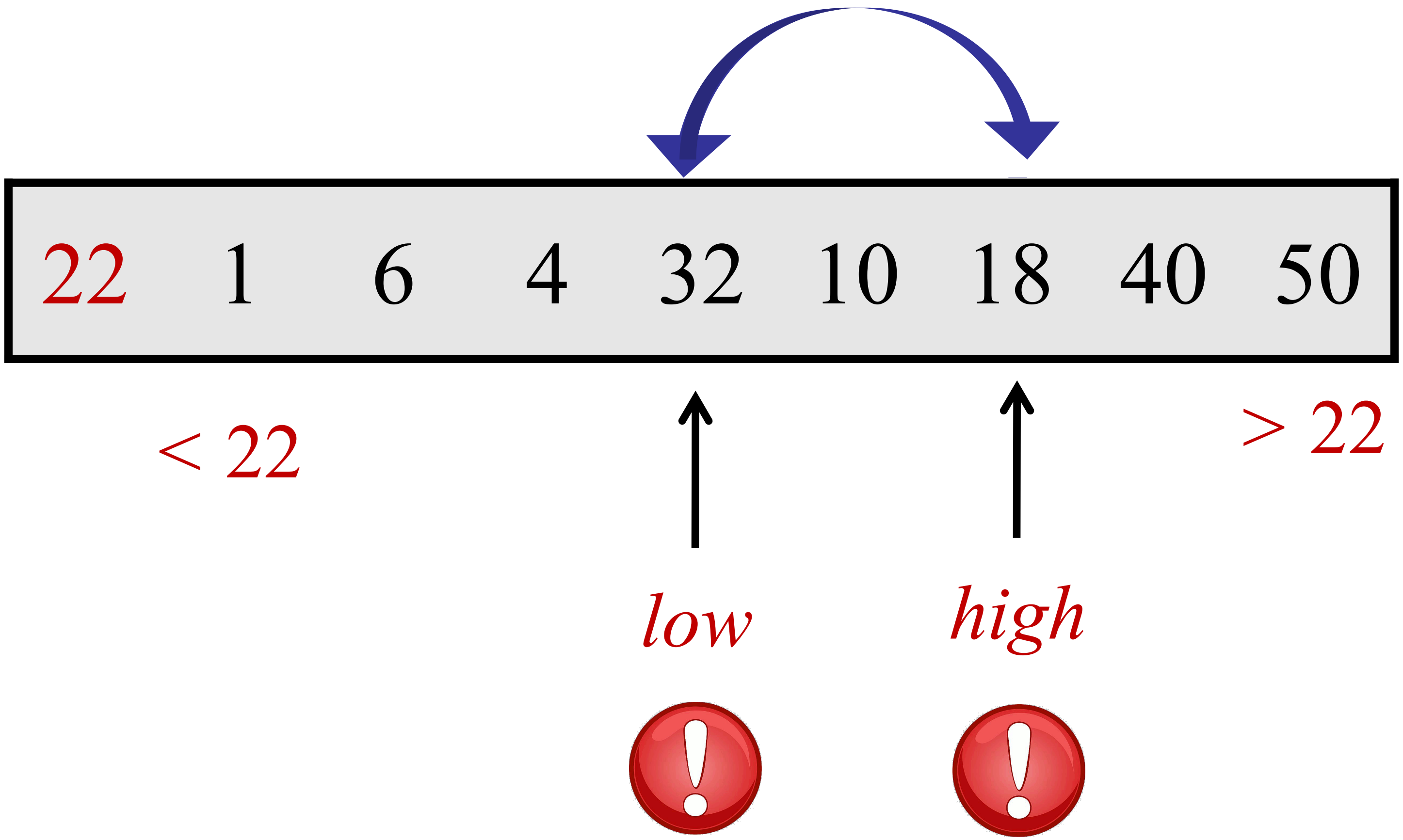
Partitioning an Array

Example: partition around 22



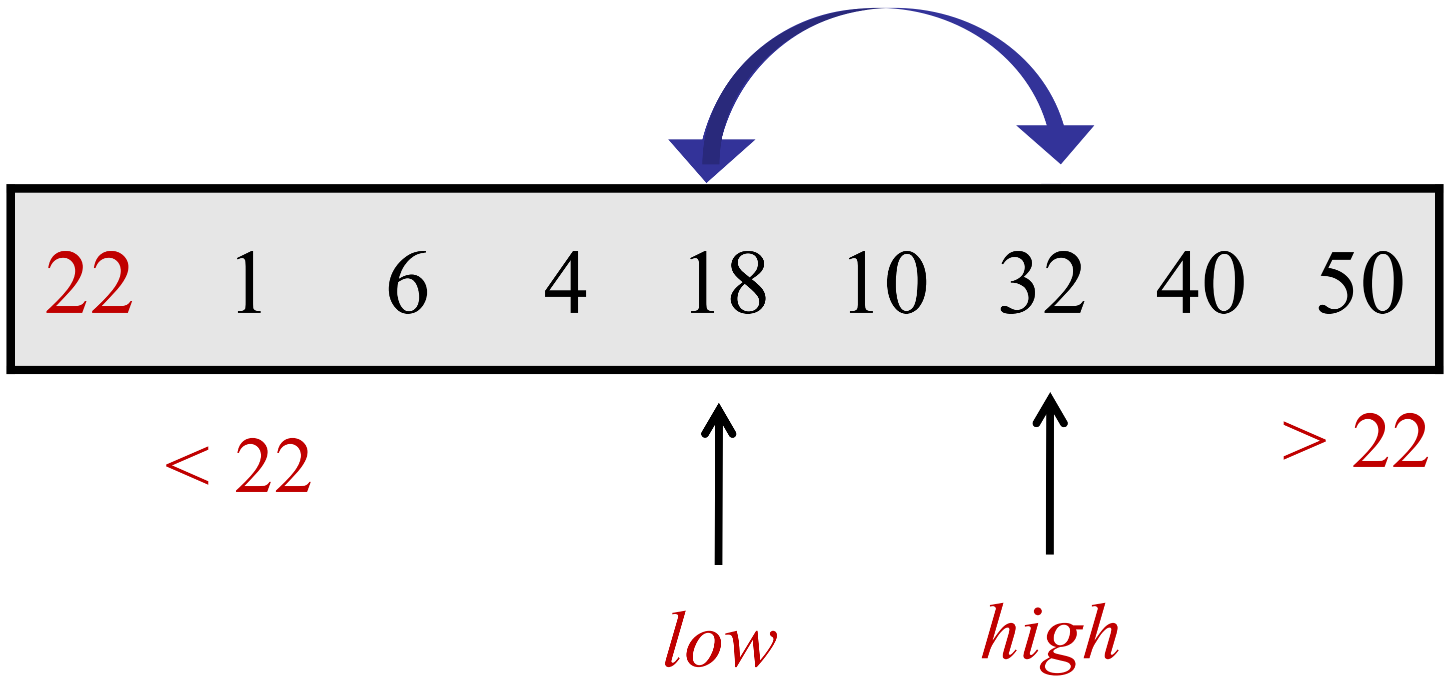
Partitioning an Array

Example: partition around 22



Partitioning an Array

Example: partition around 22



Partitioning an Array

Example: partition around 22

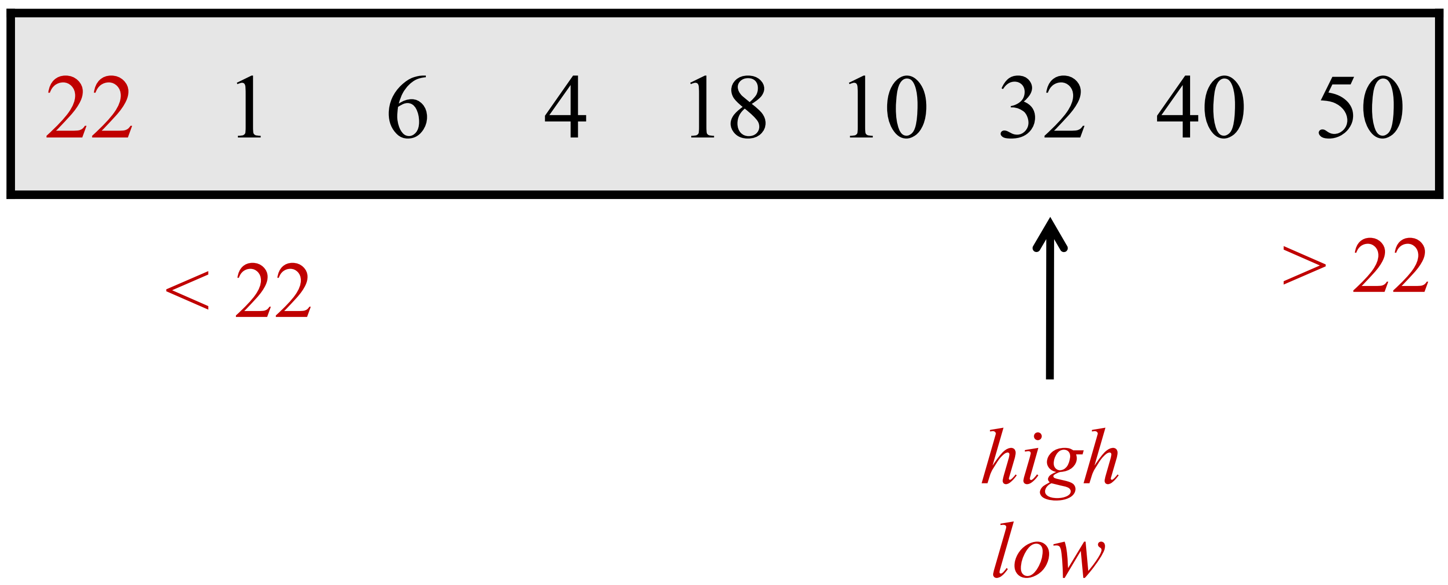
22	1	6	4	18	10	32	40	50

< 22   > 22

low *high*

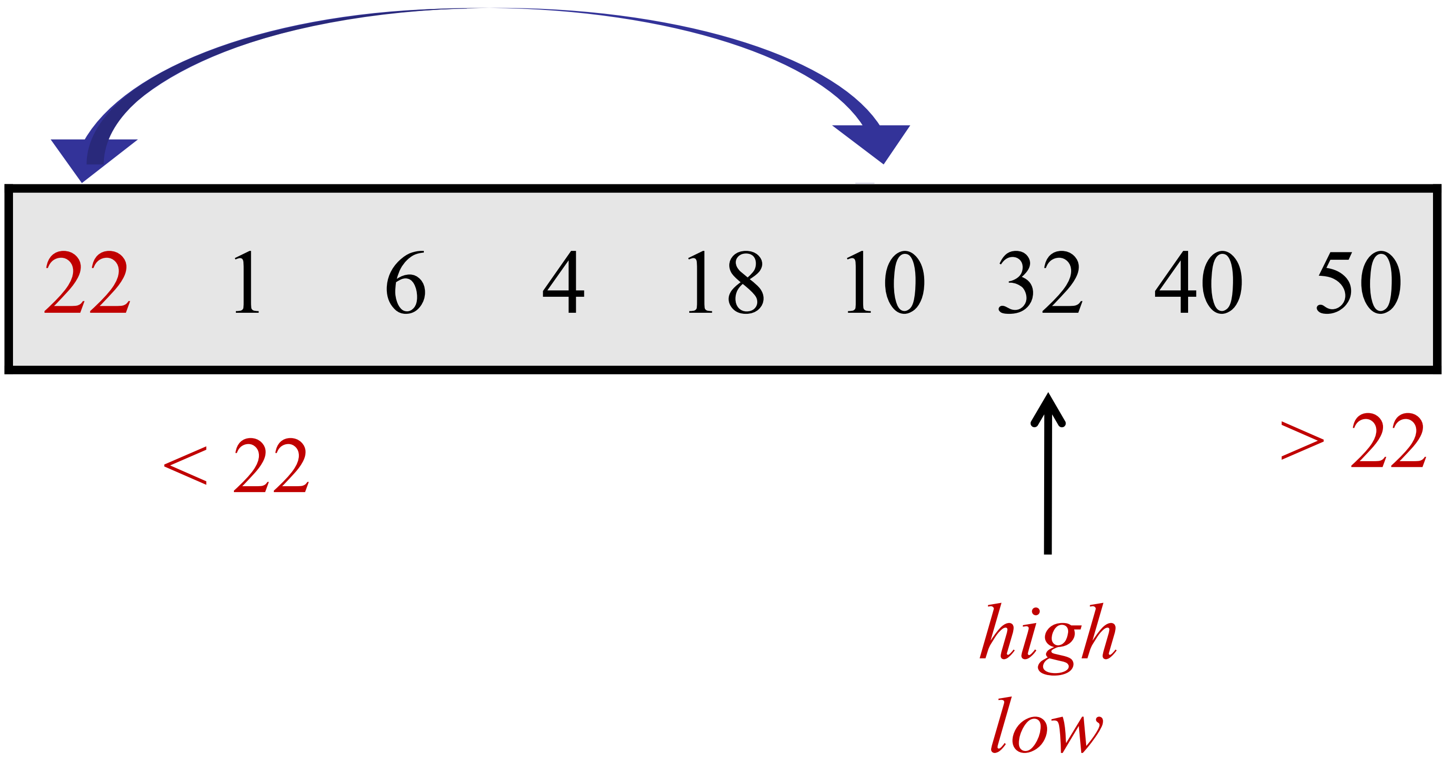
Partitioning an Array

Example: partition around 22



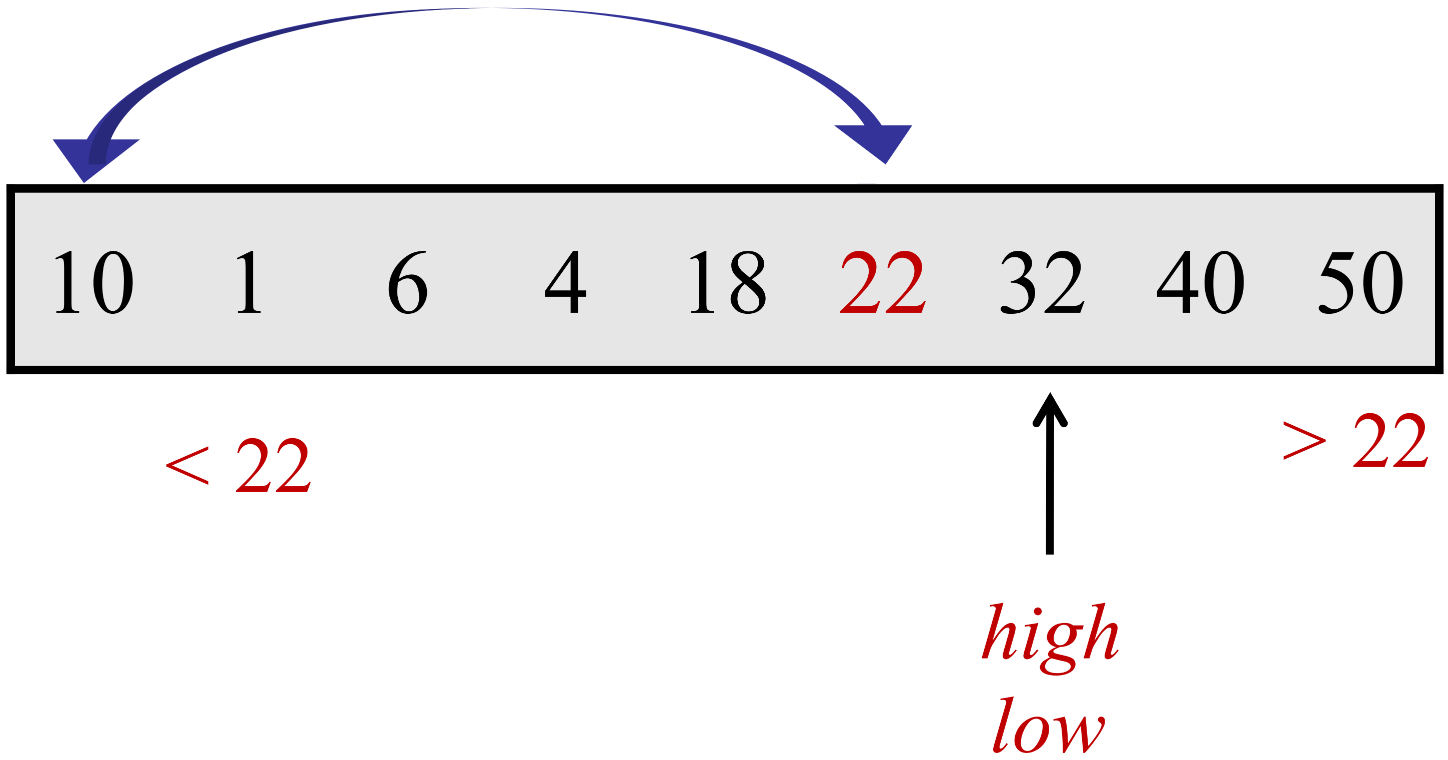
Partitioning an Array

Example: partition around 22



Partitioning an Array

Example: partition around 22



```
partition(A[1..n], n, pIndex)          // Assume no duplicates, n>1
    pivot = A[pIndex];                 // pIndex is the index of pivot
    swap(A[1], A[pIndex]);             // store pivot in A[1]
    low = 2;                           // start after pivot in A[1]
    high = n+1;                        // Define: A[n+1] = ∞
    while (low < high)
        while (A[low] < pivot) and (low < high) do low++;
        while (A[high] > pivot) and (low < high) do high– – ;
        if (low < high) then swap(A[low], A[high]);
    swap(A[1], A[low–1]);
    return  low–1;
```

Partition

Invariant: $A[high] > pivot$ at the end of each loop.

Proof:

Initially: true by assumption $A[n+1] = \infty$

Partition

Invariant: $A[high] > pivot$ at the end of each iter:

Proof: During loop:

- When exit loop incrementing low: $A[low] > pivot$

  If ($low > high$), then by **while** condition.

  If ($low = high$), then by inductive assumption.

- When exit loop decrementing high:

  $A[high] < pivot$ OR $low = high$

- If ($high == low$), then $A[high] > pivot$
- Otherwise, swap $A[high]$ and $A[low]$>pivot.

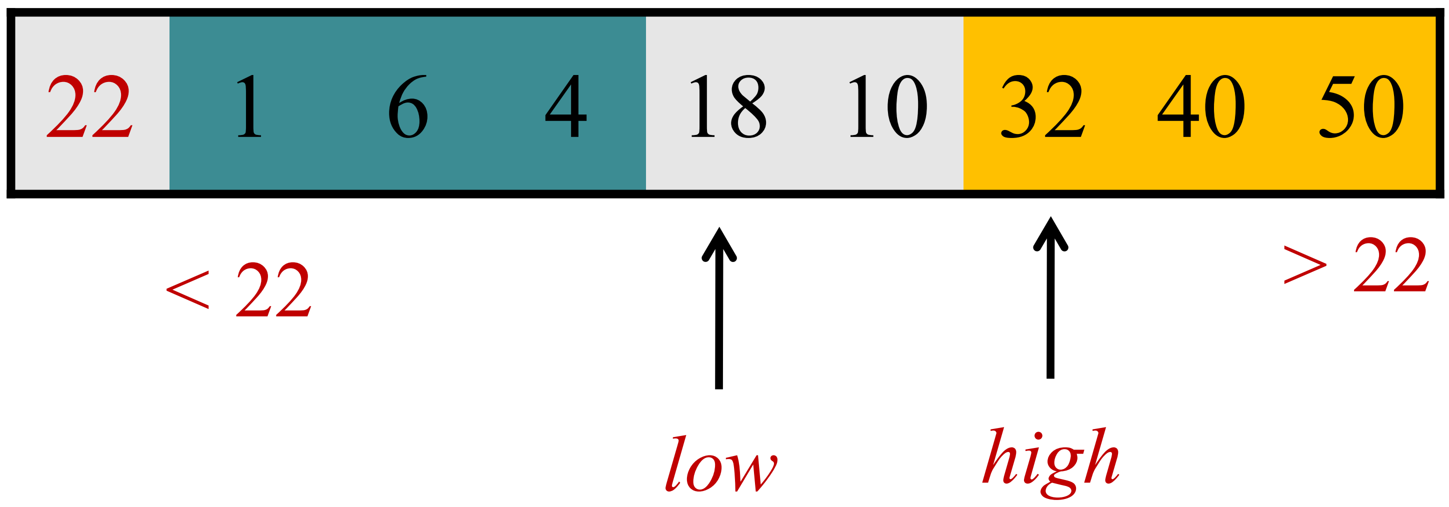
```
partition(A[1..n], n, pIndex)          // Assume no duplicates, n>1
    pivot = A[pIndex];                 // pIndex is the index of pivot
    swap(A[1], A[pIndex]);             // store pivot in A[1]
    low = 2;                           // start after pivot in A[1]
    high = n+1;                        // Define: A[n+1] = ∞
    while (low < high)
        while (A[low] < pivot) and (low < high) do low++;
        while (A[high] > pivot) and (low < high) do high– – ;
        if (low < high) then swap(A[low], A[high]);
    swap(A[1], A[low–1]);
    return  low–1;
```

Partition

Invariant: At the end of every loop iteration:

for all $i >= high,\ A[i] > pivot.$

for all $1 < j < low,\ A[j] < pivot.$

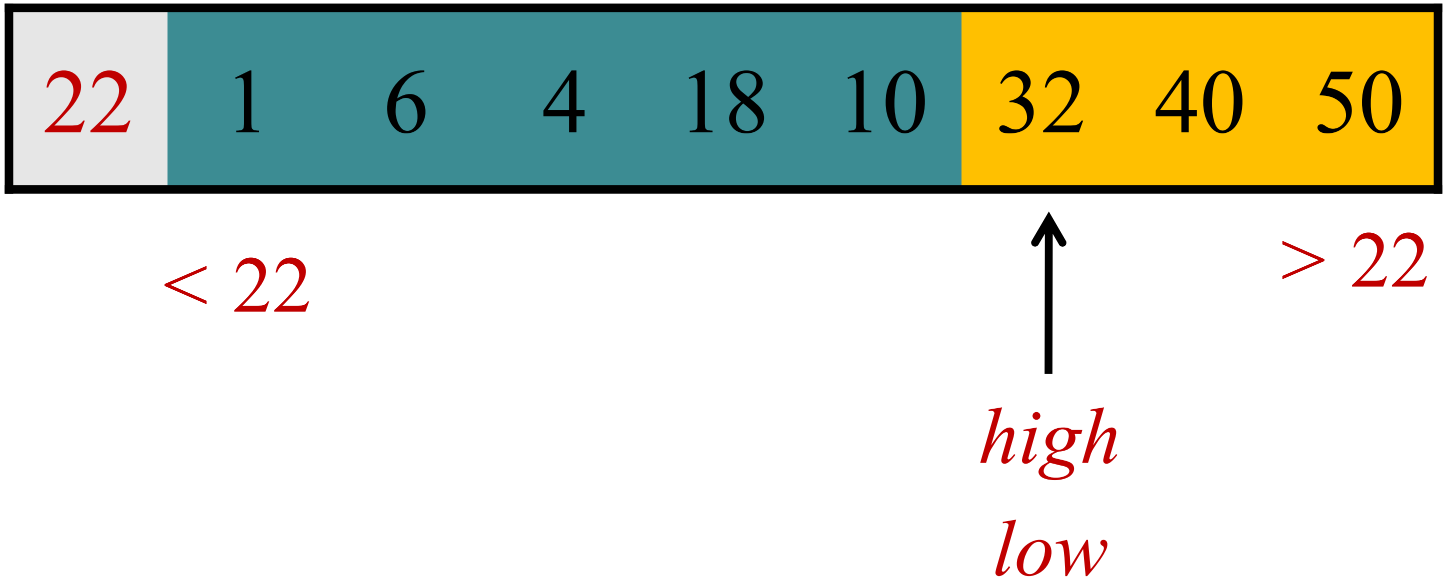


Partition

Invariant: At the end of every loop iteration:

for all $i >= high, A[i] > pivot.$

for all $1 < j < low, A[j] < pivot.$

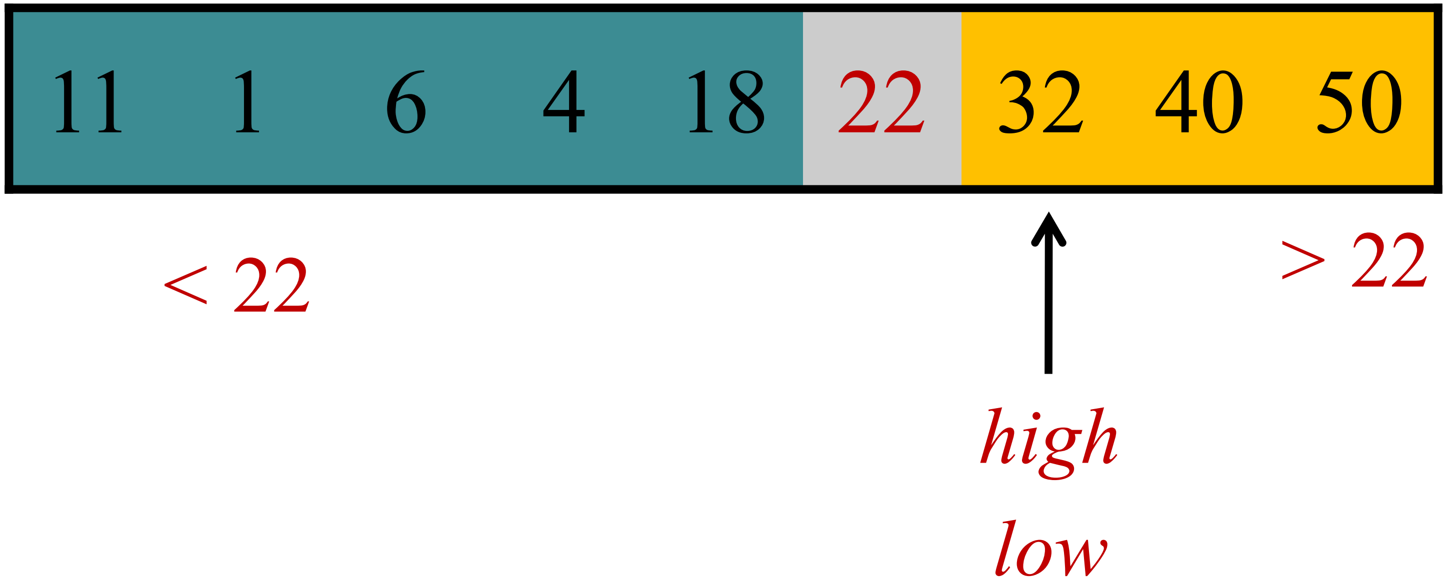


Partition

Claim: At the end of every loop iteration:

for all $i >= high$, $A[i] > pivot$.

for all $1 < j < low$, $A[j] < pivot$.



Claim: Array A is partitioned around the pivot

```
partition(A[1..n], n, pIndex)          // Assume no duplicates,  n>1
    pivot = A[pIndex];                 // pIndex is the index of pivot
    swap(A[1], A[pIndex]);             // store pivot in A[1]
    low = 2;                           // start after pivot in A[1]
    high = n+1;                        // Define: A[n+1] = ∞
    while (low < high)
        while (A[low] < pivot) and (low < high) do low++;
        while (A[high] > pivot) and (low < high) do high– – ;
        if (low < high) then swap(A[low], A[high]);
    swap(A[1], A[low–1]);
    return  low–1;
```

```
partition(A[1..n], n, pIndex)
    pivot = A[pIndex];
    swap(A[1], A[pIndex]);
    low = 2;
    high = n+1;
    while (low < high)
        while (A[low] < pivot) and (low < high) do low++;
        while (A[high] > pivot) and (low < high) do high– – ;
        if (low < high) then swap(A[low], A[high]);
    swap(A[1], A[low–1]);
    return  low–1;
```

Running time:

O(n)

QuickSort

QuickSort(A[1..n], n)

if (n == 1) **then** return;

else

Choose pivot index *pIndex*.

p = partition(A[1..n], n, *pIndex*)

x = QuickSort(A[1..p–1], p–1)

y = QuickSort(A[p+1..n], n–p)

Today: Sorting, Part II

QuickSort

- Divide-and-Conquer
- Partitioning
- Duplicates
- Choosing a pivot
- Randomization
- Analysis

QuickSort

What happens if there are duplicates?

Duplicates

QuickSort(A[1..n], n)

if (n==1) **then** return;

else

Choose pivot index $pIndex$.

p = partition(A[1..n], n, $pIndex$)

x = QuickSort(A[1..p–1], p–1)

y = QuickSort(A[p+1..n], n–p)

$< x$	x	x	x	$> x$

Quicksort

Example:

6 6 6 6 6 6

Quicksort

Example:

6	6	6	6	6	6
6	6	6	6	6	6

Quicksort

Example:

6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6

Quicksort

Example:

6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6

Quicksort

Example:

6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6

Quicksort

Example:

6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6

Quicksort

Example:

6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6

Quicksort

Example:

Running time:

$O(n^2)$

6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6
6	6	6	6	6	6

```
partition(A[1..n], n, pIndex)          // Assume no duplicates,  n>1
    pivot = A[pIndex];                 // pIndex is the index of pivot
    swap(A[1], A[pIndex]);             // store pivot in A[1]
    low = 2;                           // start after pivot in A[1]
    high = n+1;                        // Define: A[n+1] = ∞
    while (low < high)
        while (A[low] < pivot) and (low < high) do low++;
        while (A[high] > pivot) and (low < high) do high– – ;
        if (low < high) then swap(A[low], A[high]);
    swap(A[1], A[low–1]);
    return  low–1;
```

Duplicates

QuickSort(A[1..n], n)

if (n==1) **then** return;

else

Choose pivot index $pIndex$.

p = partition(A[1..n], n, $pIndex$)

x = QuickSort(A[1..p–1], p–1)

y = QuickSort(A[p+1..n], n–p)

Duplicates

QuickSort(A[1..n], n)

if (n==1) **then** return;

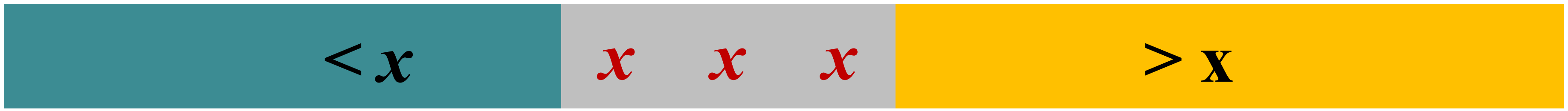
else

Choose pivot index *pIndex*.

p = partition(A[1..n], n, *pIndex*)

x = QuickSort(A[1..p–1], p–1)

y = QuickSort(A[p+1..n], n–p)



Pivot

Duplicates

3-Way Partitioning

– Option 1: two pass partitioning

1. Regular partition.
2. Pack duplicates.

Pack Duplicates

Example:

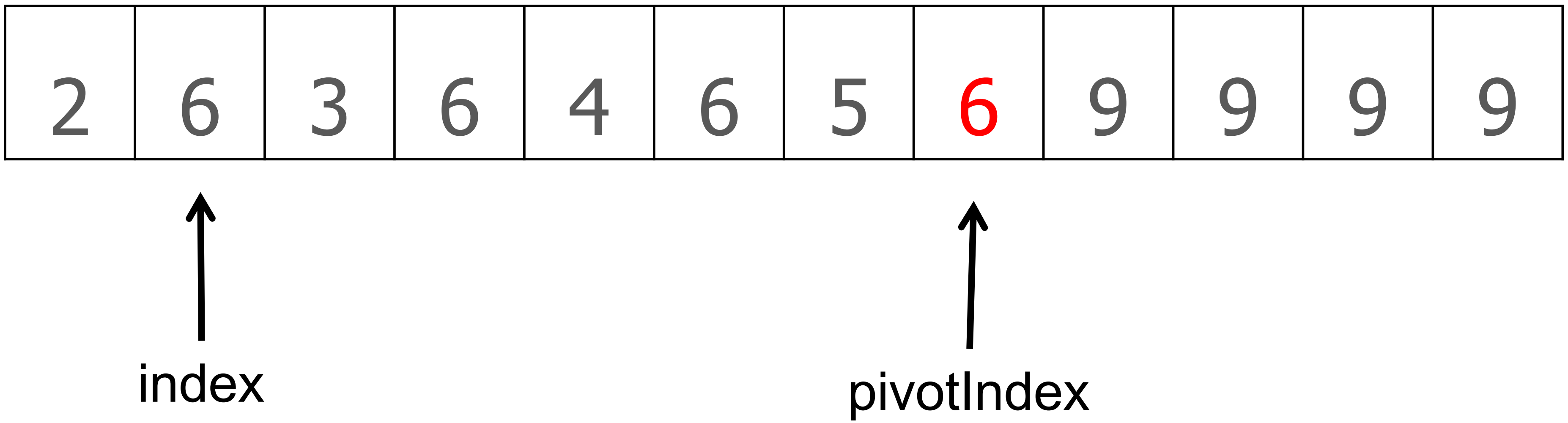
2	6	3	6	4	6	5	6	9	9	9	9

index

pivotIndex

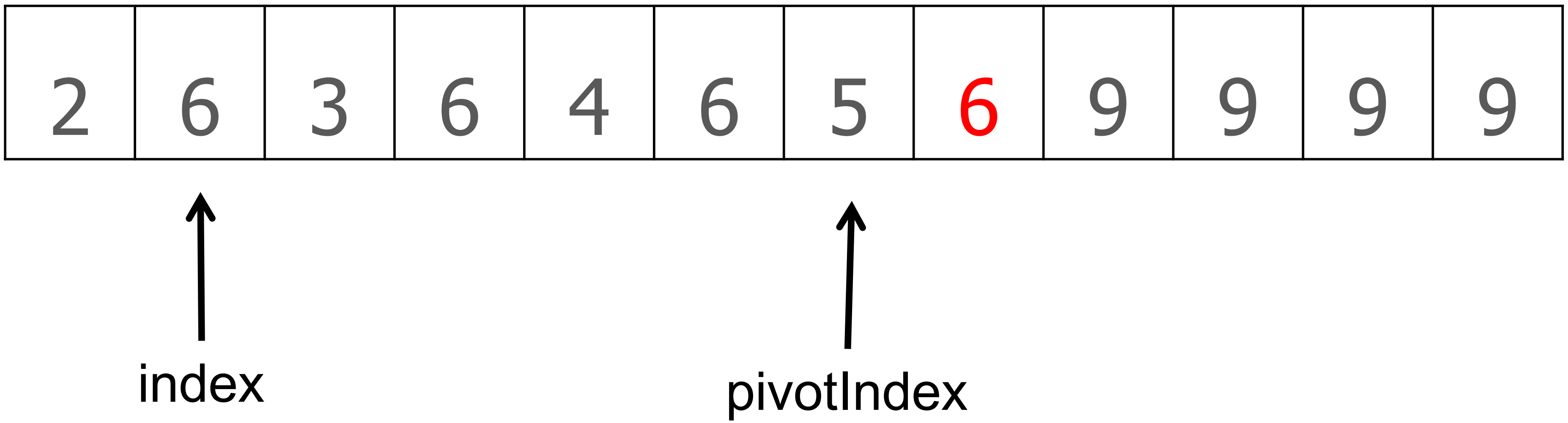
Pack Duplicates

Example:



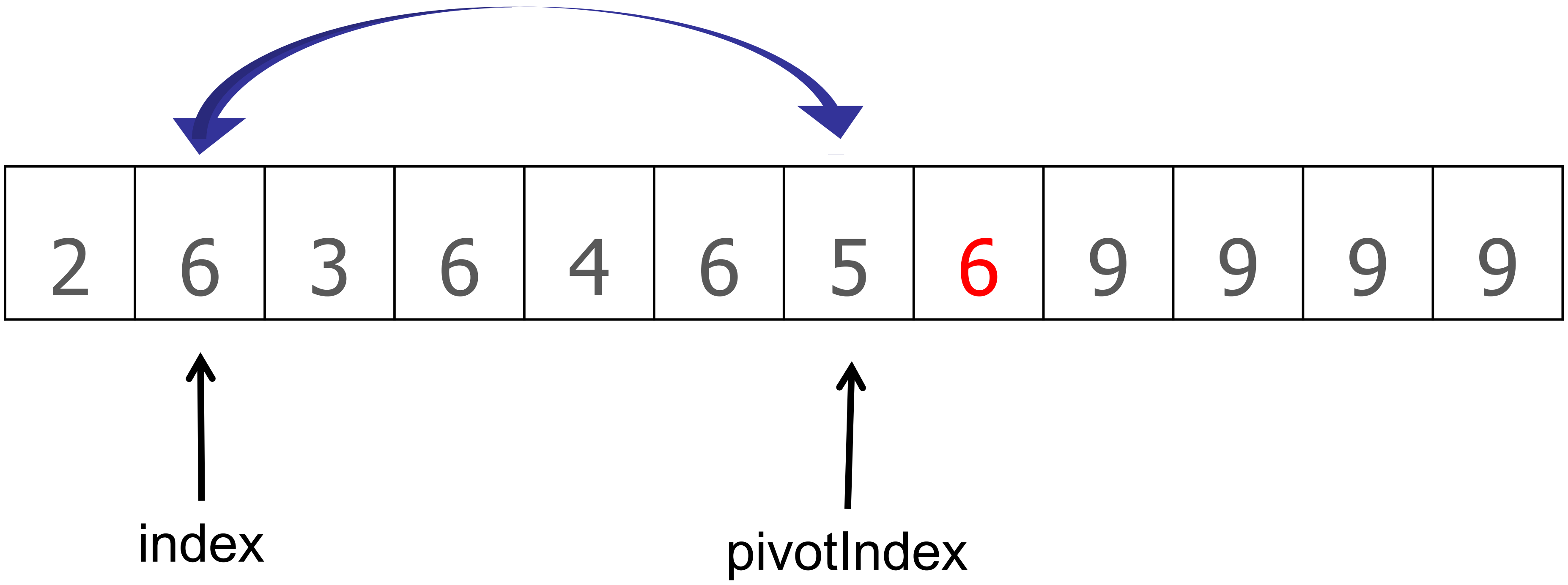
Pack Duplicates

Example:



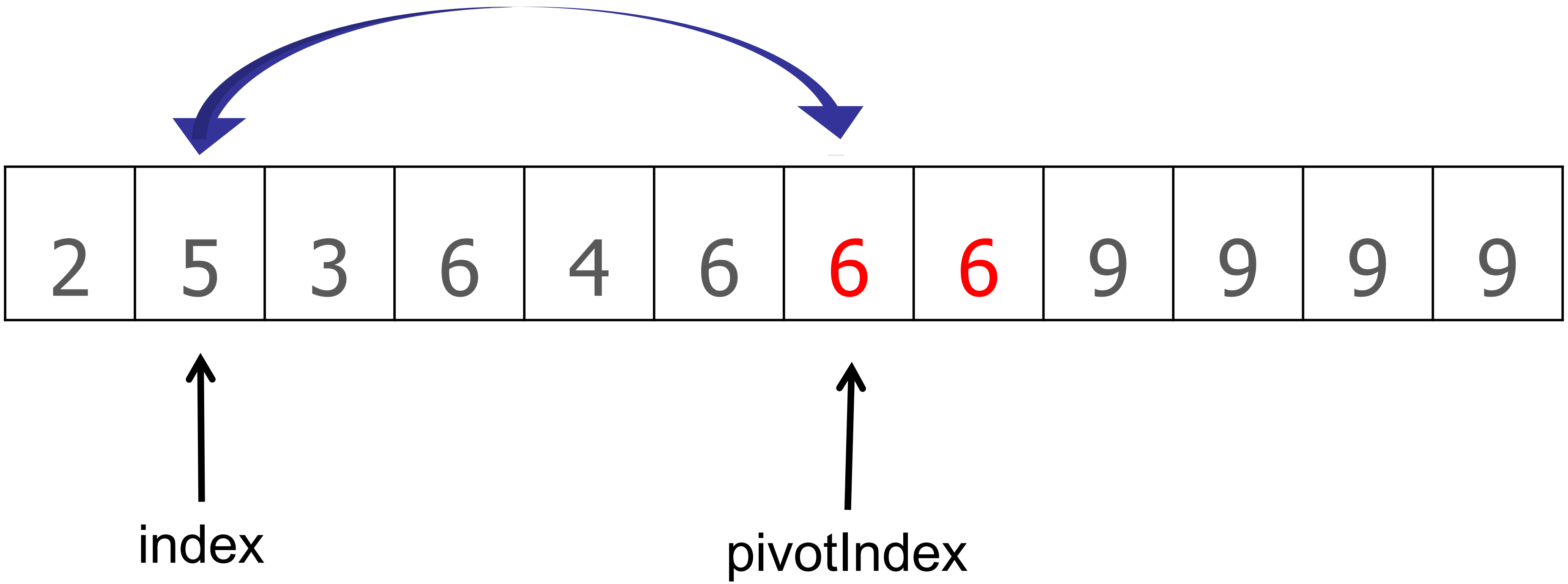
Pack Duplicates

Example:



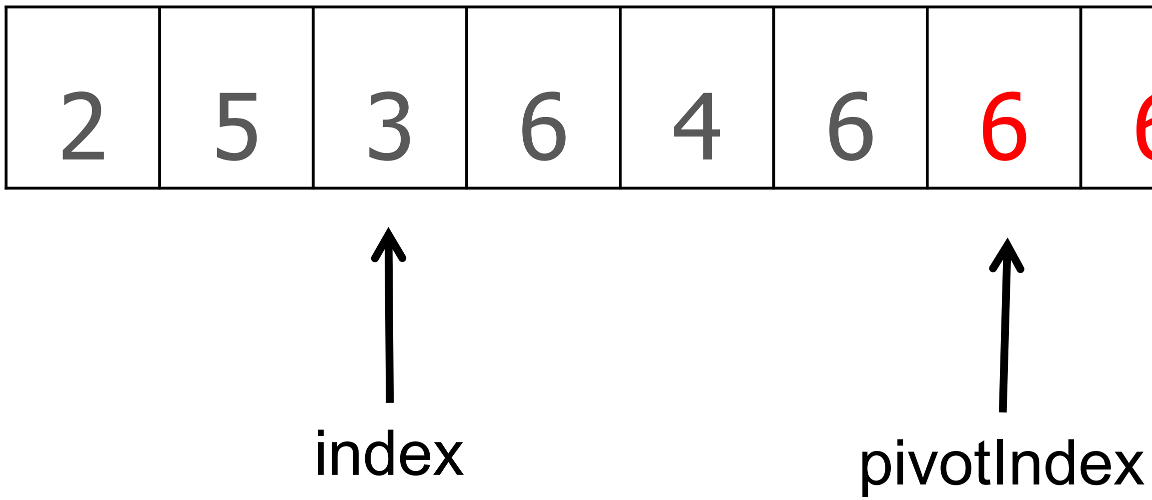
Pack Duplicates

Example:



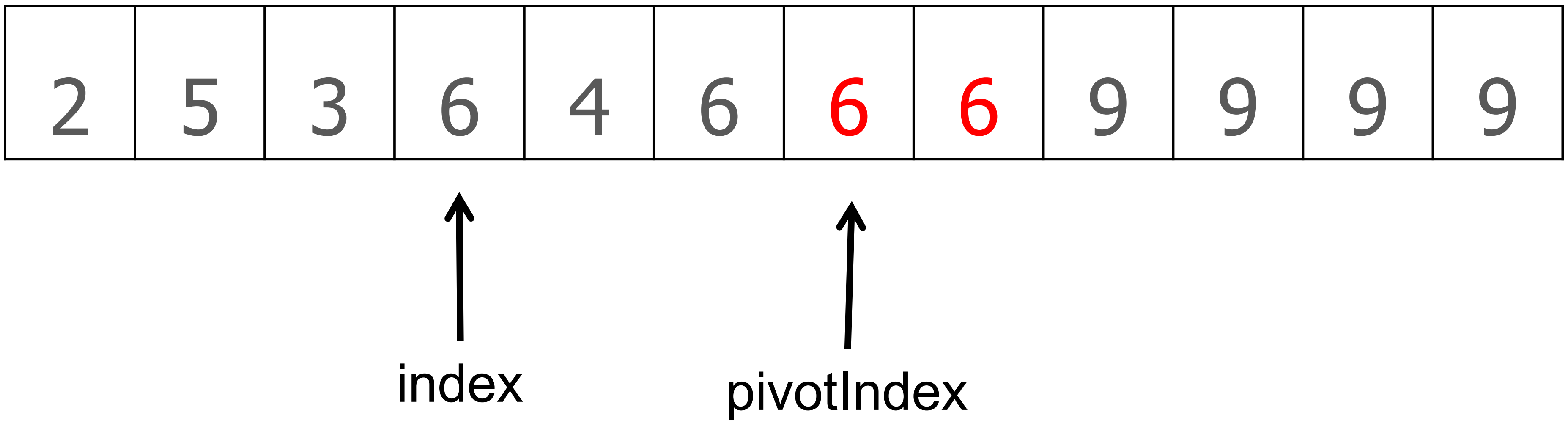
Pack Duplicates

Example:



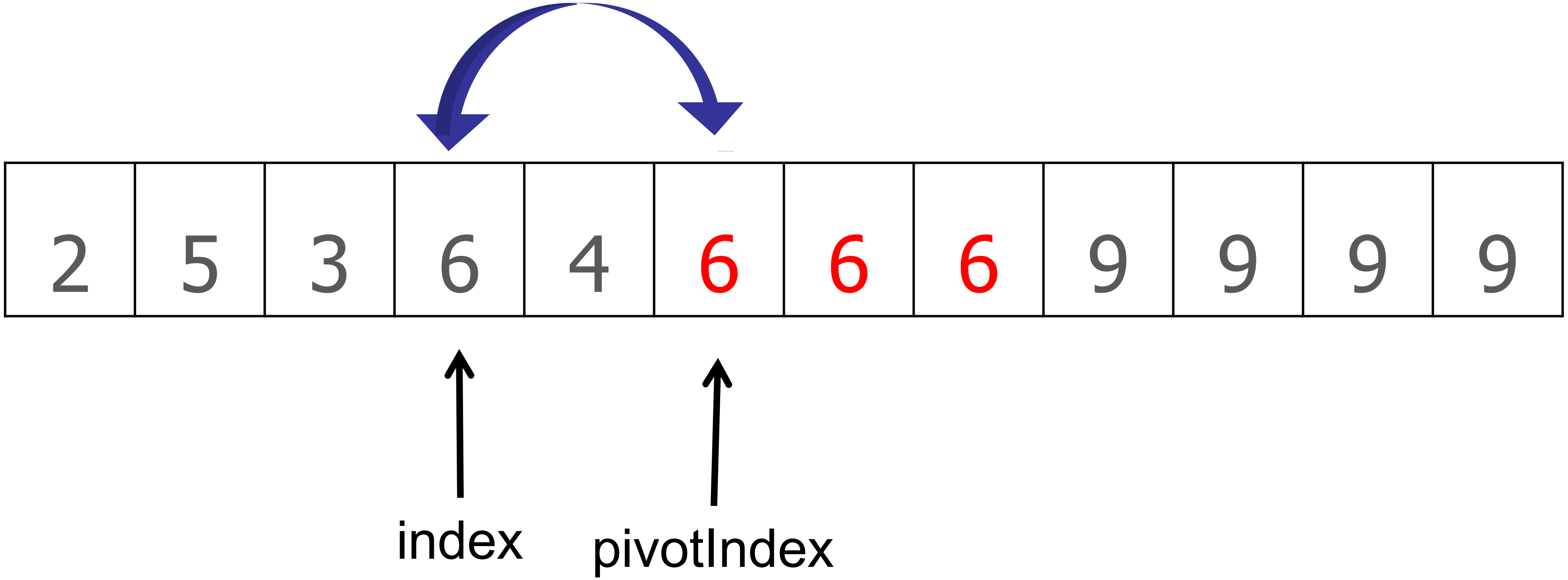
Pack Duplicates

Example:



Pack Duplicates

Example:



Pack Duplicates

Example:

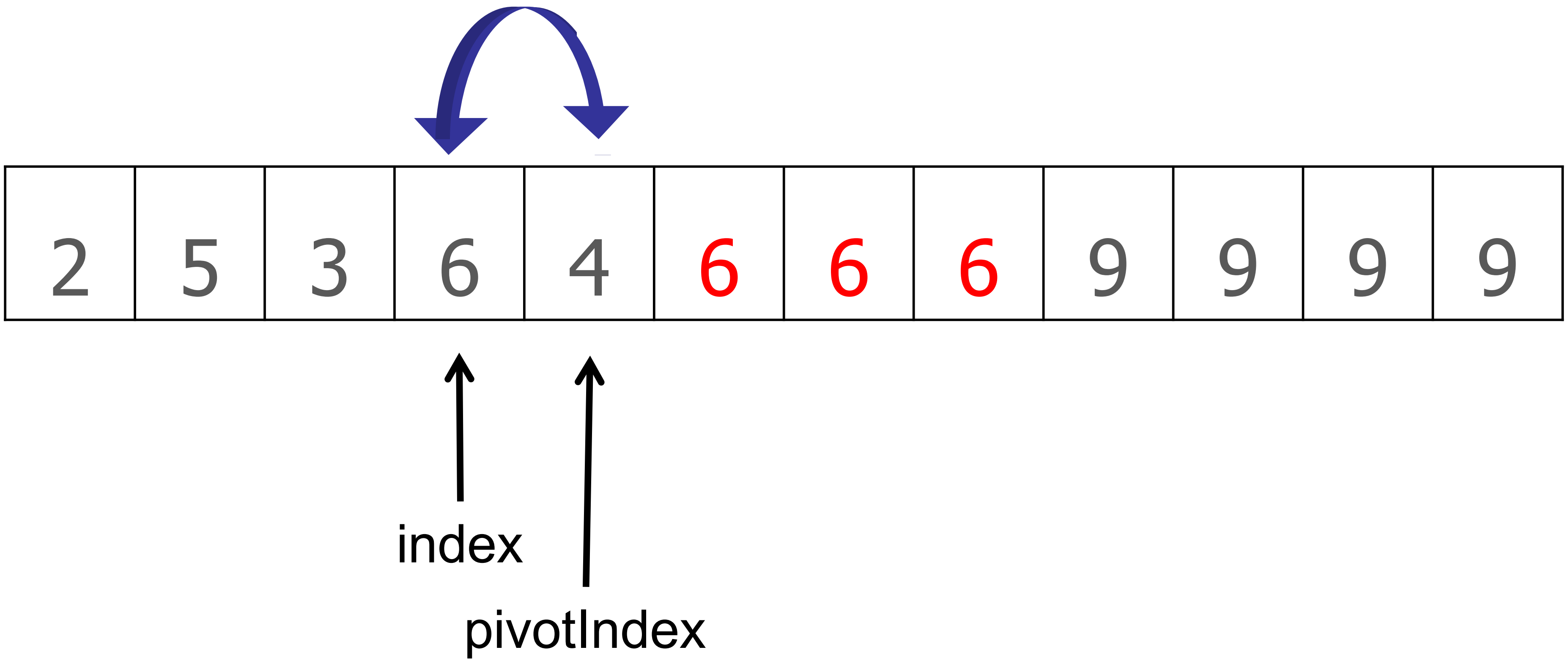
2	5	3	6	4	6	6	6	9	9	9	9

index

pivotIndex

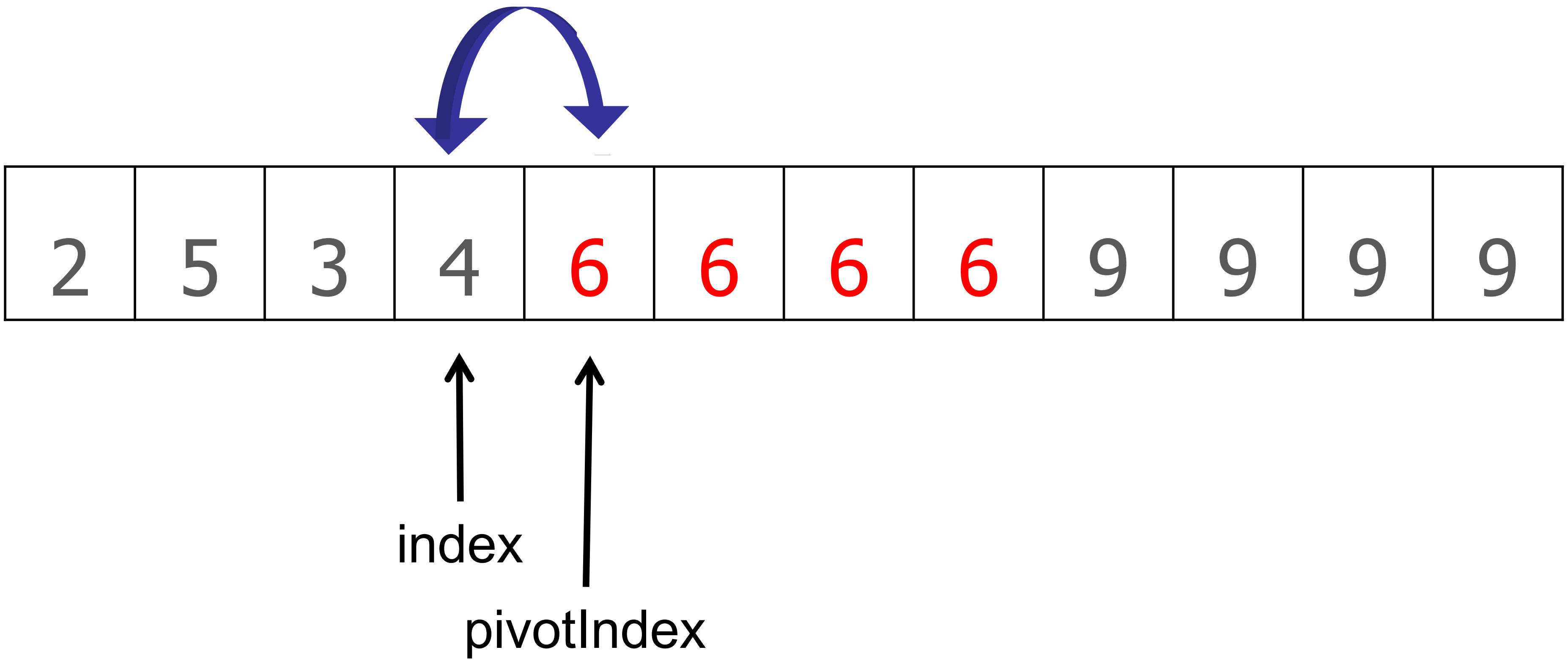
Pack Duplicates

Example:



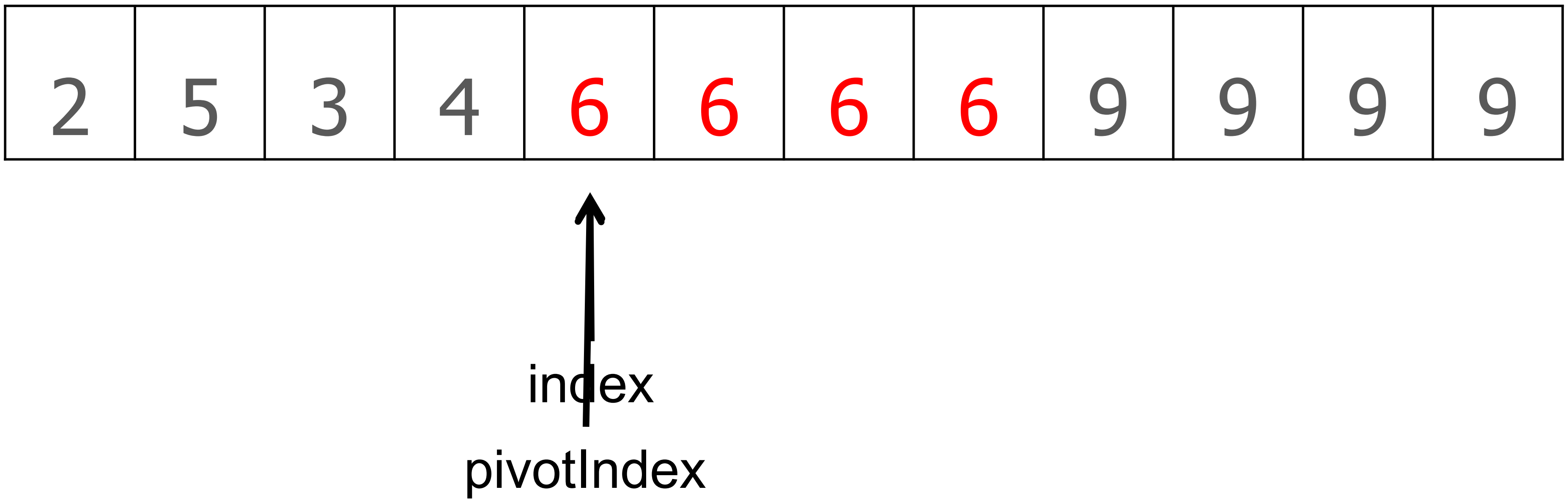
Pack Duplicates

Example:



Pack Duplicates

Example:



Duplicates

QuickSort(A[1..n], n)

if (n==1) **then** return;

else

Choose pivot index *pIndex*.

p = 3wayPartition(A[1..n], n, *pIndex*)

x = QuickSort(A[1..p–1], p–1)

y = QuickSort(A[p+1..n], n–p)

$< x$	x	x	x	> x

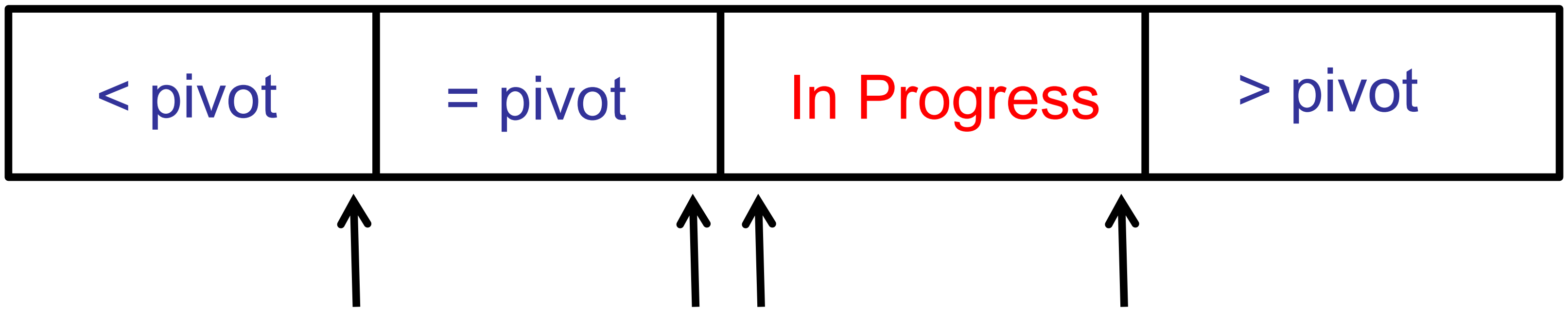
Duplicates

3-Way Partitioning

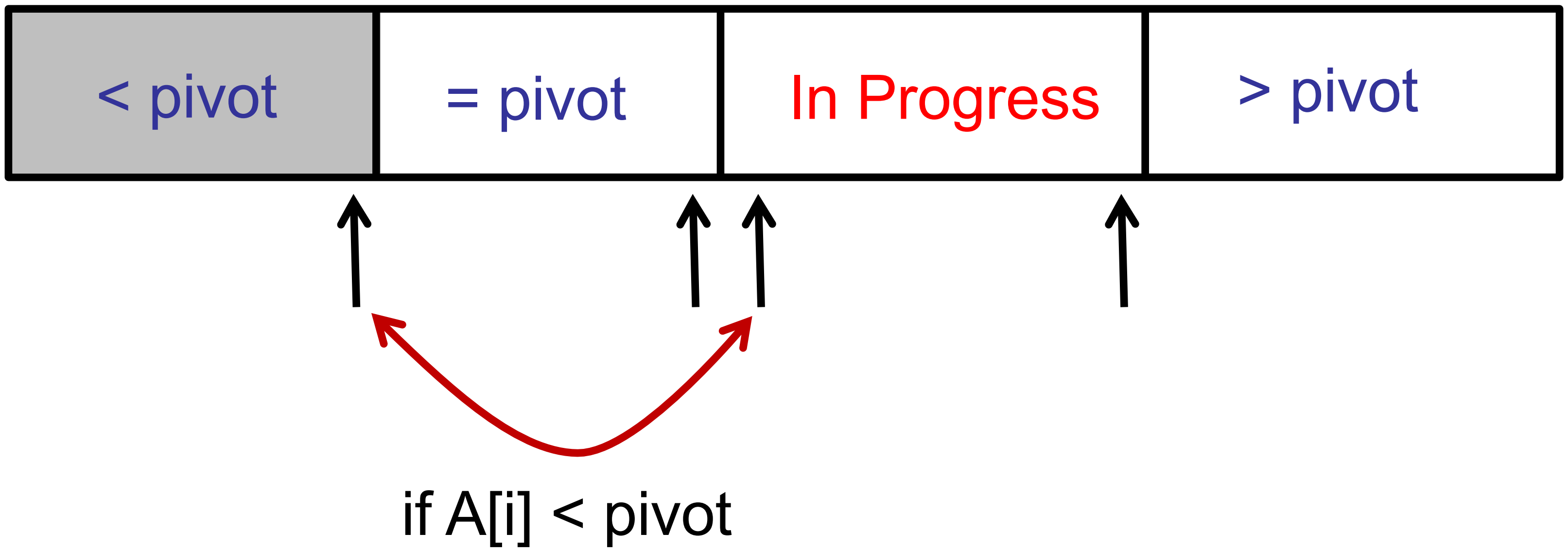
- Option 1: two pass partitioning
    1. Regular partition.
    2. Pack duplicates.

- Option 2: one pass partitioning
    - More complicated.
    - Maintain four regions of the array

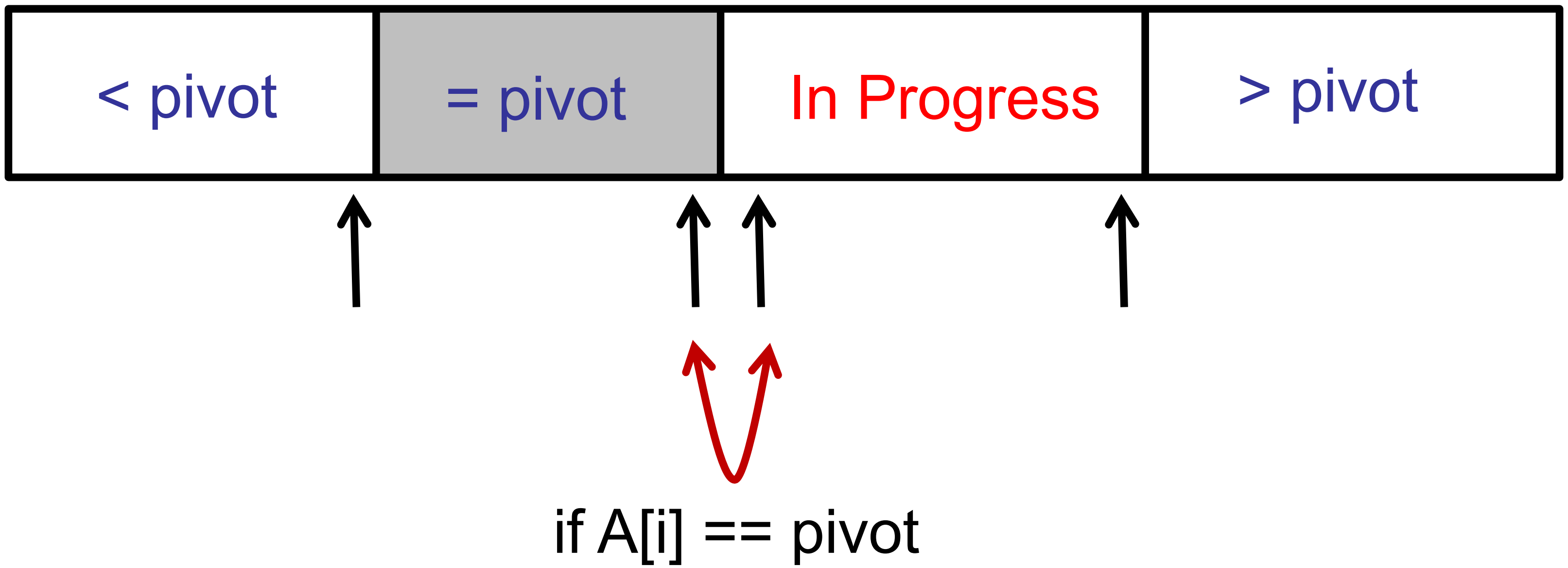
3-Way Partitioning



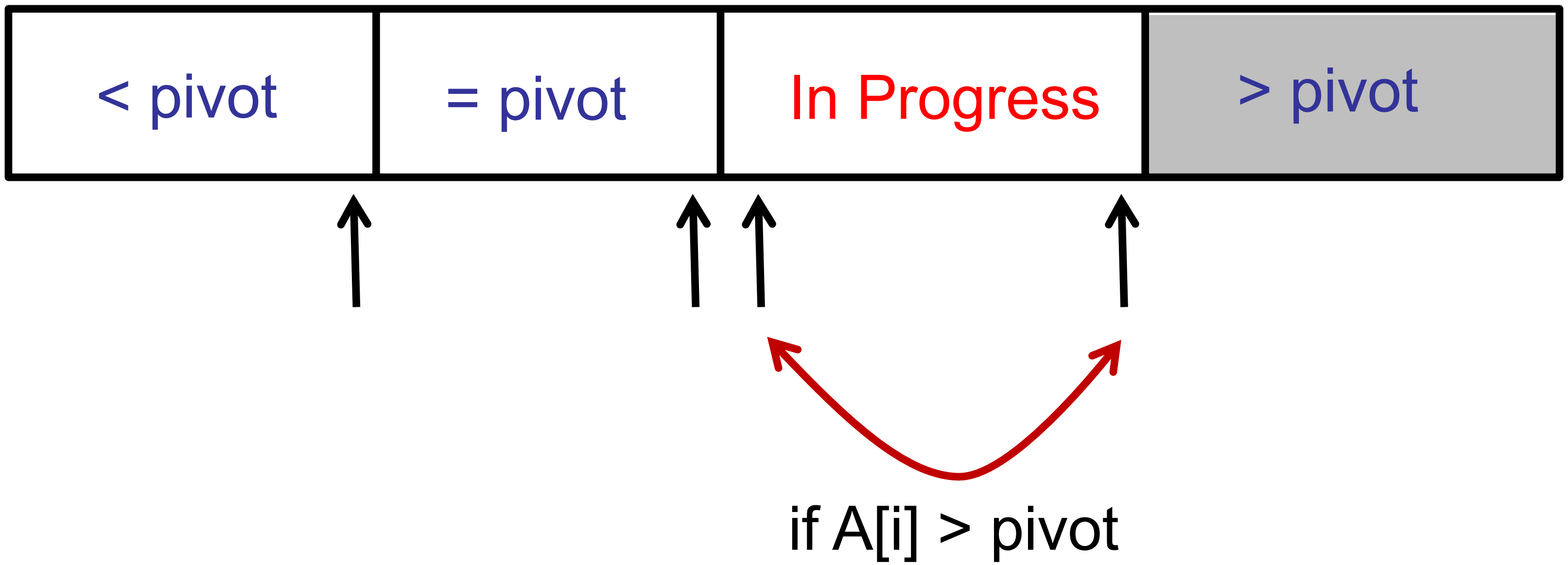
3-Way Partitioning



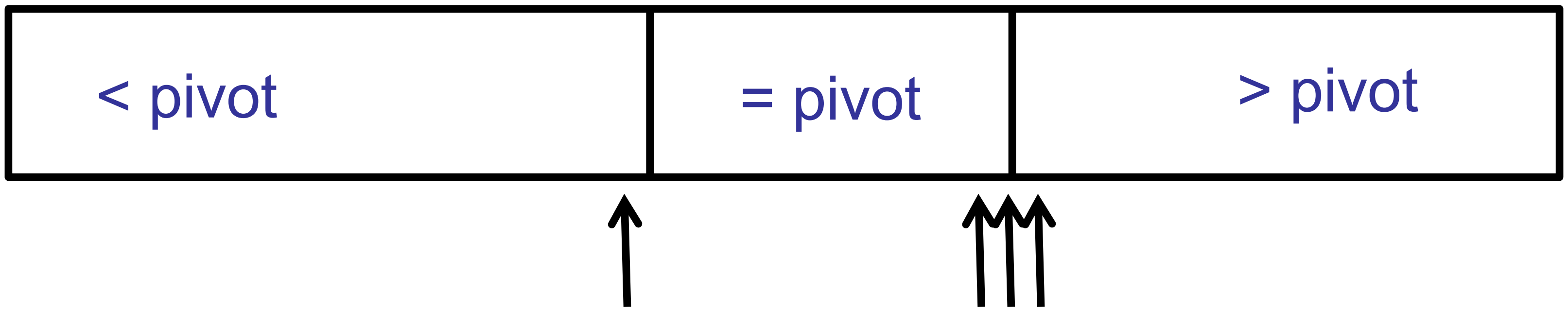
3-Way Partitioning



3-Way Partitioning



3-Way Partitioning



Duplicates

QuickSort(A[1..n], n)

if (n==1) **then** return;

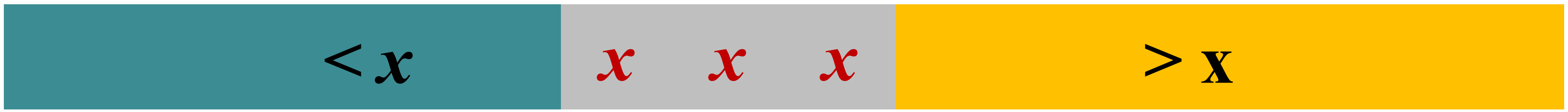
else

Choose pivot index *pIndex*.

p = 3wayPartition(A[1..n], n, *pIndex*)

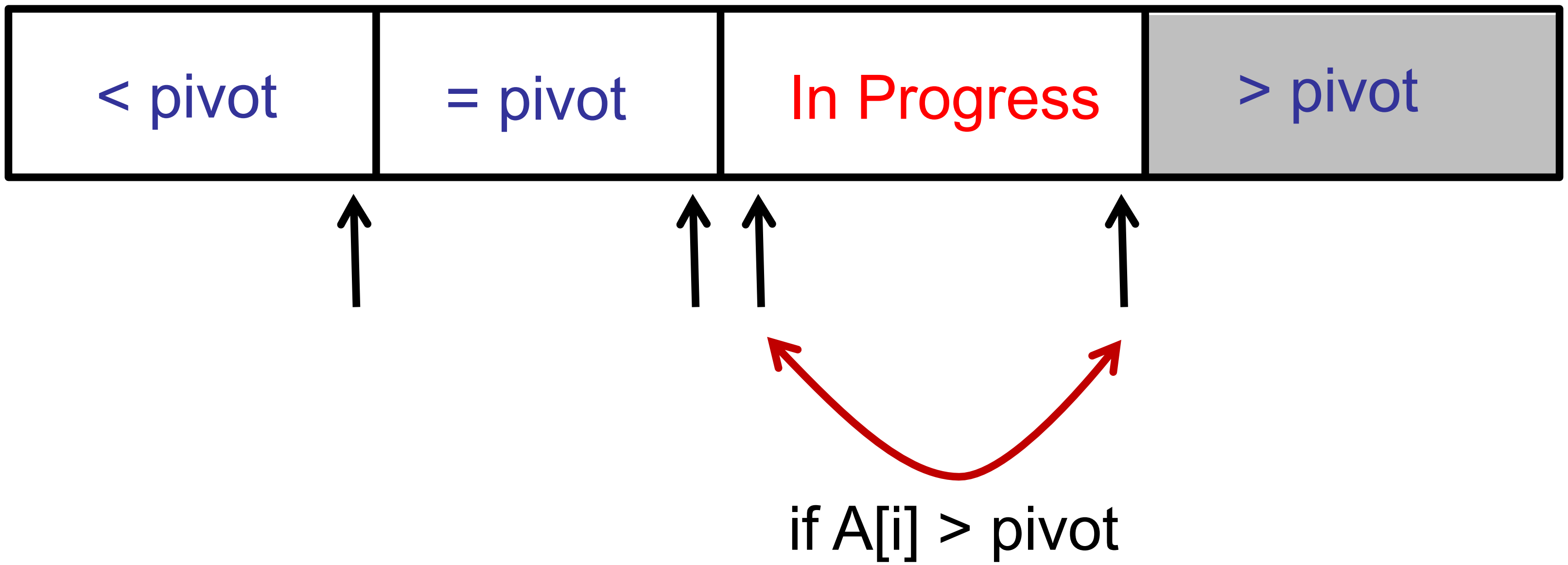
x = QuickSort(A[1..p–1], p–1)

y = QuickSort(A[p+1..n], n–p)



Is QuickSort stable?

QuickSort is not stable



Choice of Pivot

Options:

- first element: A[1]
- last element: A[n]
- middle element: A[n/2]
- median of (A[1], A[n/2], A[n])

Choice of Pivot

Options:

- first element: A[1]
- last element: A[n]
- middle element: A[n/2]
- median of (A[1], A[n/2], A[n])

In the worst case, it does not matter!

All options are equally bad.

Choice of Pivot

Choose $A[1]$ for pivot:

100	99	98	97	96	95	94	93	92

99	98	97	96	95	94	93	92	100

Choice of Pivot

Choose $A[1]$ for pivot:

99	98	97	96	95	94	93	92	100

98	97	96	95	94	93	92	99	100

Choice of Pivot

Choose $A[1]$ for pivot:

98	97	96	95	94	93	92	99	100

$$\Downarrow$$

97	96	95	94	93	92	98	99	100

Choice of Pivot

Choose $A[1]$ for pivot:

98	97	96	95	94	93	92	99	100

97	96	95	94	93	92	98	99	100

Choice of Pivot

Sorting the array takes n executions of partition.

- Each call to partition sorts one element.
- Each call to partition of size k takes: $\geq k$

Total: $n + (n-1) + (n-2) + (n-3) + \ldots = O(n^2)$

98	97	96	95	94	93	92	99	100

97	96	95	94	93	92	98	99	100

Deterministic QuickSort

QuickSort Recurrence:

$$T(n) = T(n-1) + T(1) + n$$

Cost of <u>partition</u> on n elements

Cost of <u>QuickSort</u> on 1 element

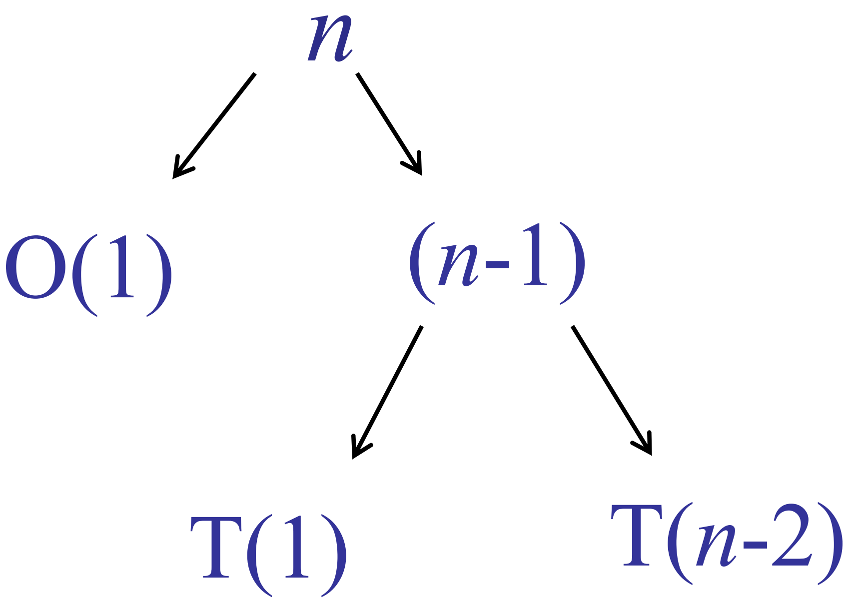
Cost of <u>QuickSort</u> on n – 1 elements

Cost of <u>QuickSort</u> on n elements

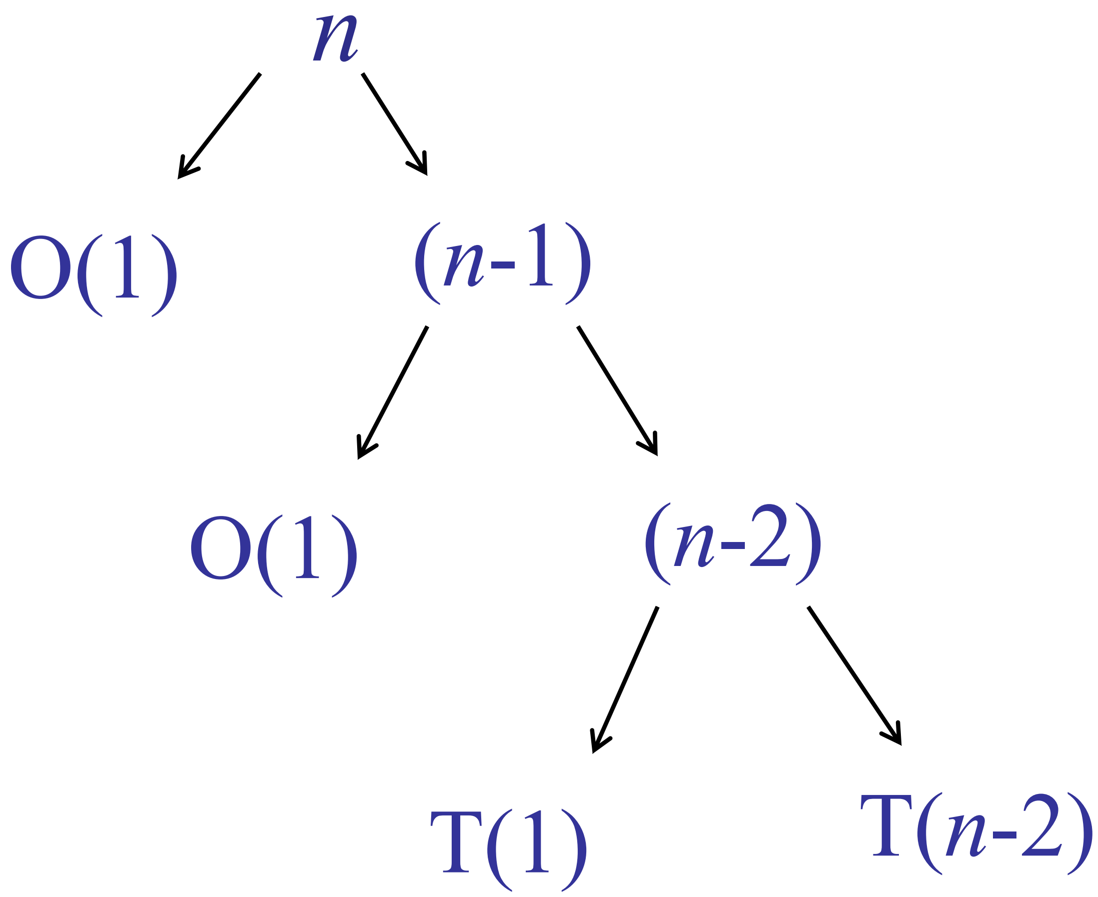
Deterministic QuickSort

$$\begin{tikzcd}[column sep=small] & n \arrow[dl] \arrow[dr] & \\ \mathrm{T}(1) & & \mathrm{T}(n\text{-}1) \end{tikzcd}$$

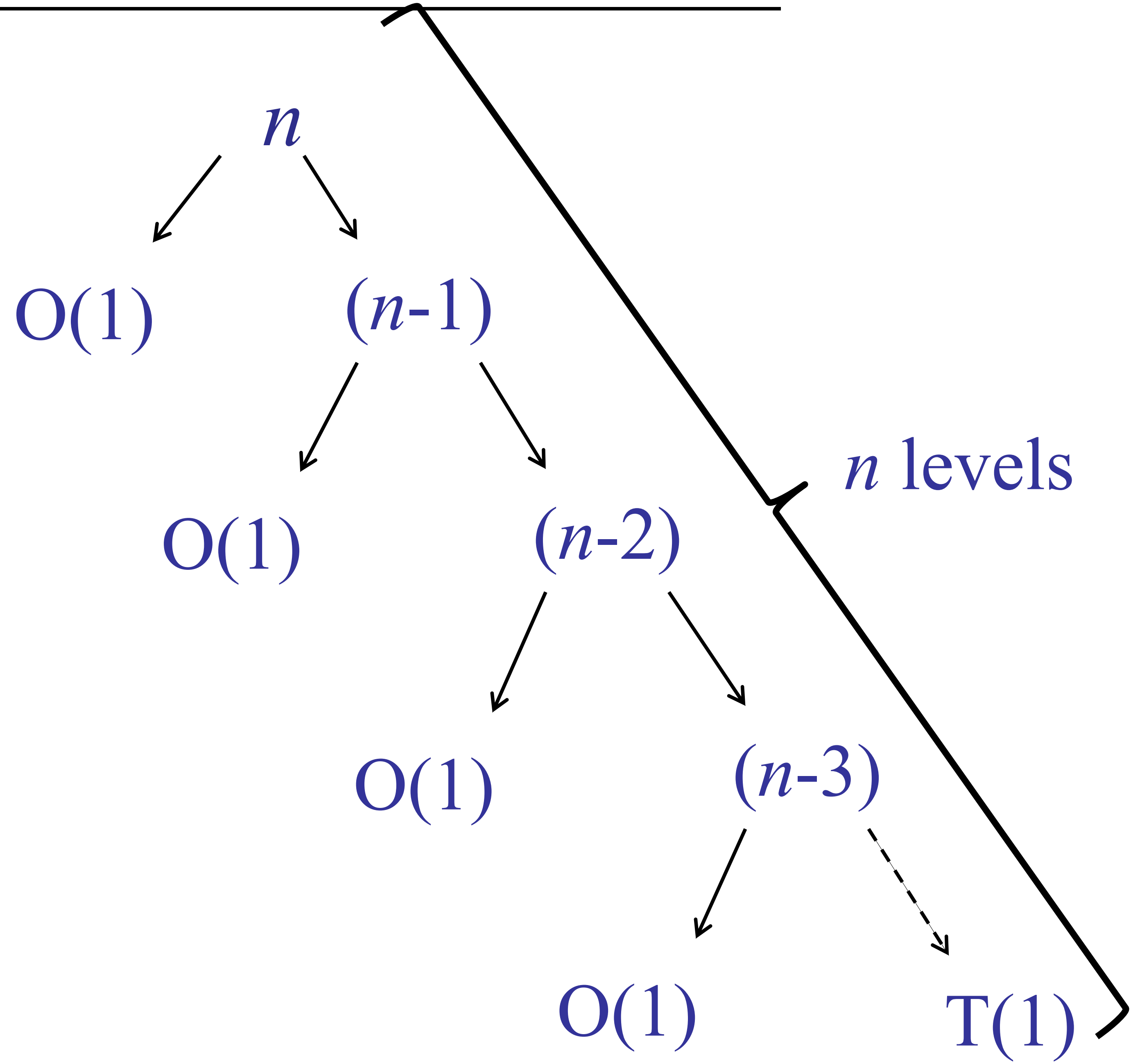
Deterministic QuickSort



Deterministic QuickSort

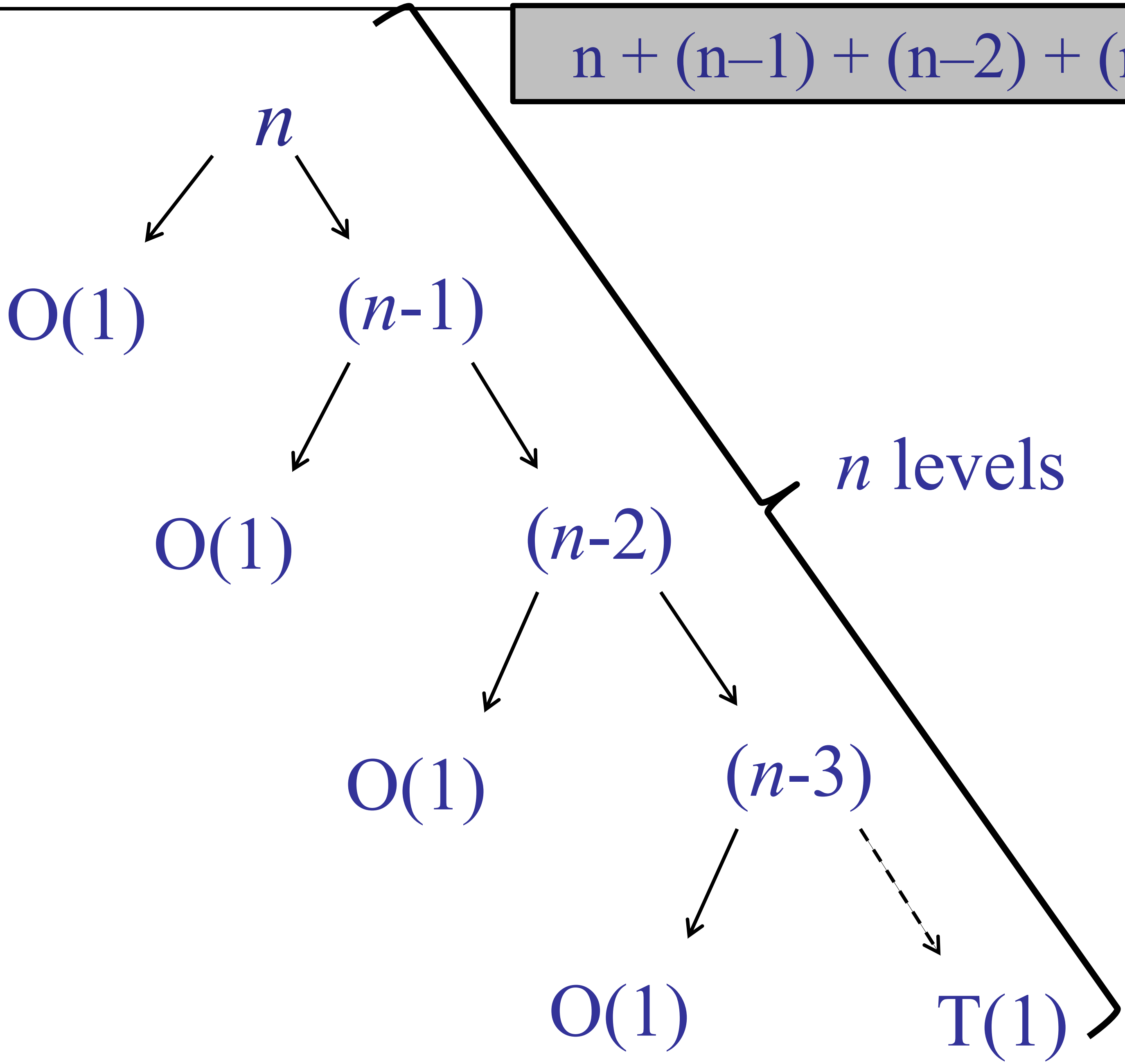


Deterministic QuickSort



Deterministic QuickSort

$$n + (n-1) + (n-2) + (n-3) + \ldots = \mathbf{O(n^2)}$$



QuickSort

QuickSort($A[1..n]$, n)

  if (n==1) **then** return;

  else

    Choose pivot index *pIndex*.

    p = partition($A[1..n]$, n, *pIndex*)

    x = QuickSort($A[1..p-1]$, $p-1$)

    y = QuickSort($A[p+1..n]$, $n-p$)

Better QuickSort

What if we chose the *median* element for the pivot?

$$T(n) = T(n/2) + T(n/2) + n$$

Cost of partition on n elements

Cost of QuickSort on *high* elements

Cost of QuickSort on *low* elements

Cost of QuickSort on n elements

Better QuickSort

If we split the array evenly:

$$
\begin{aligned}
T(n) &= T(n/2) + T(n/2) + cn \\
&= 2T(n/2) + cn \\
&= O(n \log n)
\end{aligned}
$$

QuickSort Summary

- If we choose the pivot as A[1]:
  - Bad performance: $\Omega(n^2)$

- If we could choose the median element:
  - Good performance: $O(n \log n)$

- If we could split the array (1/10) : (9/10)
  - ??

QuickSort Pivot Choice

Define sets L (low) and H (high):

- $L = \{A[i] : A[i] < pivot\}$
- $H = \{A[i] : A[i] > pivot\}$

What if the *pivot* is chosen so that:

1. L > n/10
2. H > n/10

QuickSort

$k = \min(|L|, |H|)$

QuickSort with interesting *pivot* choice:

$$T(n) = T(n-k) + T(k) + n$$

Cost of partition on n elements

Assume: $9n/10 > k > n/10$

Assume: $9n/10 > (n - \mathrm{k}) > n/10$

Cost of QuickSort on n elements

QuickSort

Tempting solution:

$$T(n) = T(n-k) + T(k) + n$$

$$< T(9n/10) + T(9n/10) + n$$

$$< 2T(9n/10) + n$$

$$< O(n \log n)$$

What is wrong?

QuickSort

Tempting solution:

$$
\begin{aligned}
T(n) &= T(n-k) + T(k) + n \\
&< T(9n/10) + T(9n/10) + n \\
&< 2T(9n/10) + n \\
&\cancel{< O(n \log n)} \\
&= O(n^{6.58})
\end{aligned}
$$

Too loose an estimate.

QuickSort Pivot Choice

Define sets L (low) and H (high):

– $L = \{A[i] : A[i] < pivot\}$

– $H = \{A[i] : A[i] > pivot\}$

What if the *pivot* is chosen so that:

1. $L = n(1/10)$

2. $H = n(9/10)$ (or *vice versa*)

QuickSort Analysis

$k = n/10$

$$n$$

$$T(n/10) \qquad\qquad T(9n/10)$$

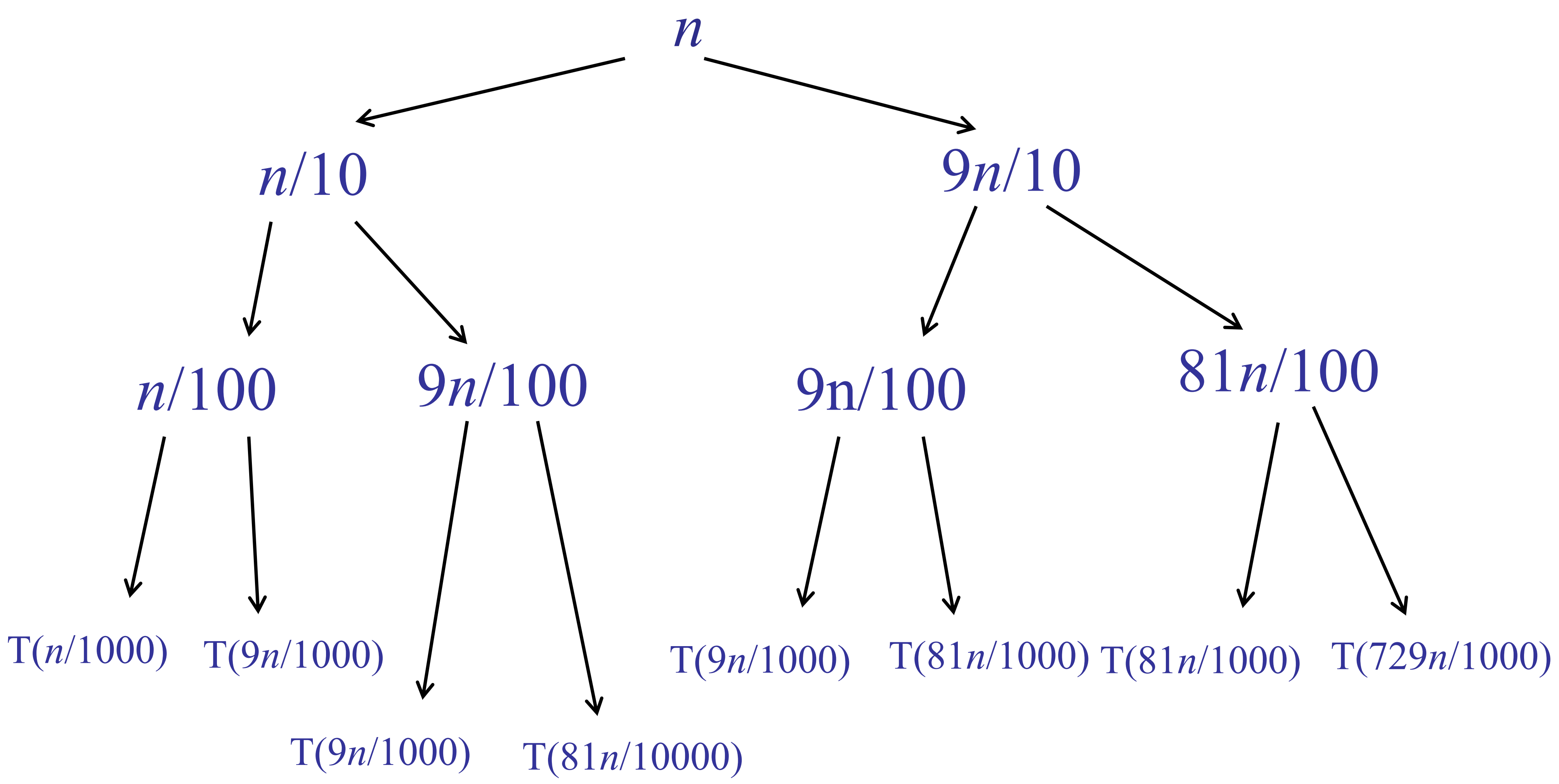
QuickSort Analysis

$$k = n/10$$

$$
\begin{tikzcd}
& & & n \arrow[dll] \arrow[drr] & & & \\
& n/10 \arrow[dl] \arrow[dr] & & & & 9n/10 \arrow[dl] \arrow[dr] & \\
\mathrm{T}(n/100) & & \mathrm{T}(9n/100) & & \mathrm{T}(9\mathrm{n}/100) & & \mathrm{T}(81n/100)
\end{tikzcd}
$$

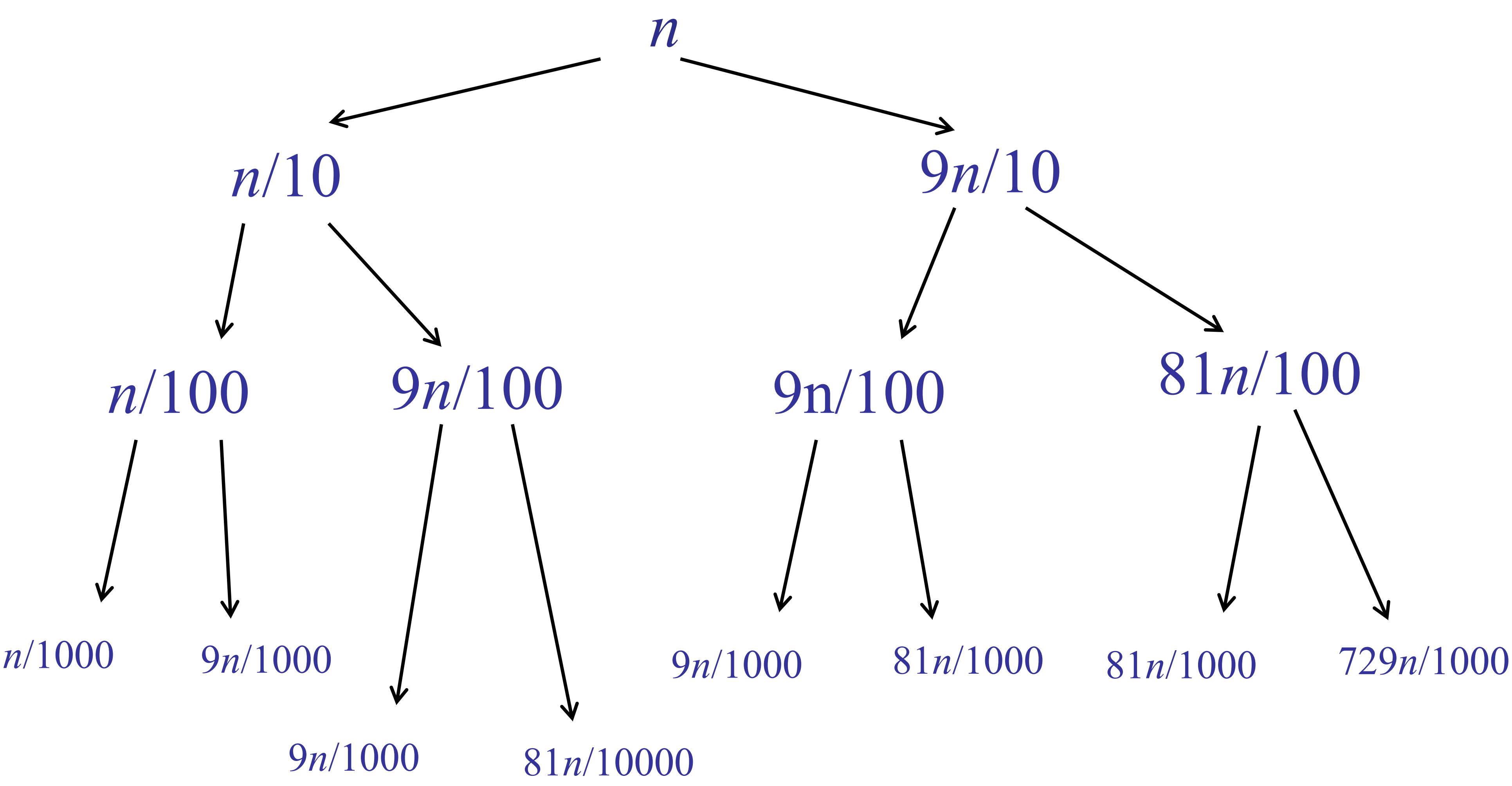
QuickSort Analysis

$k = n/10$

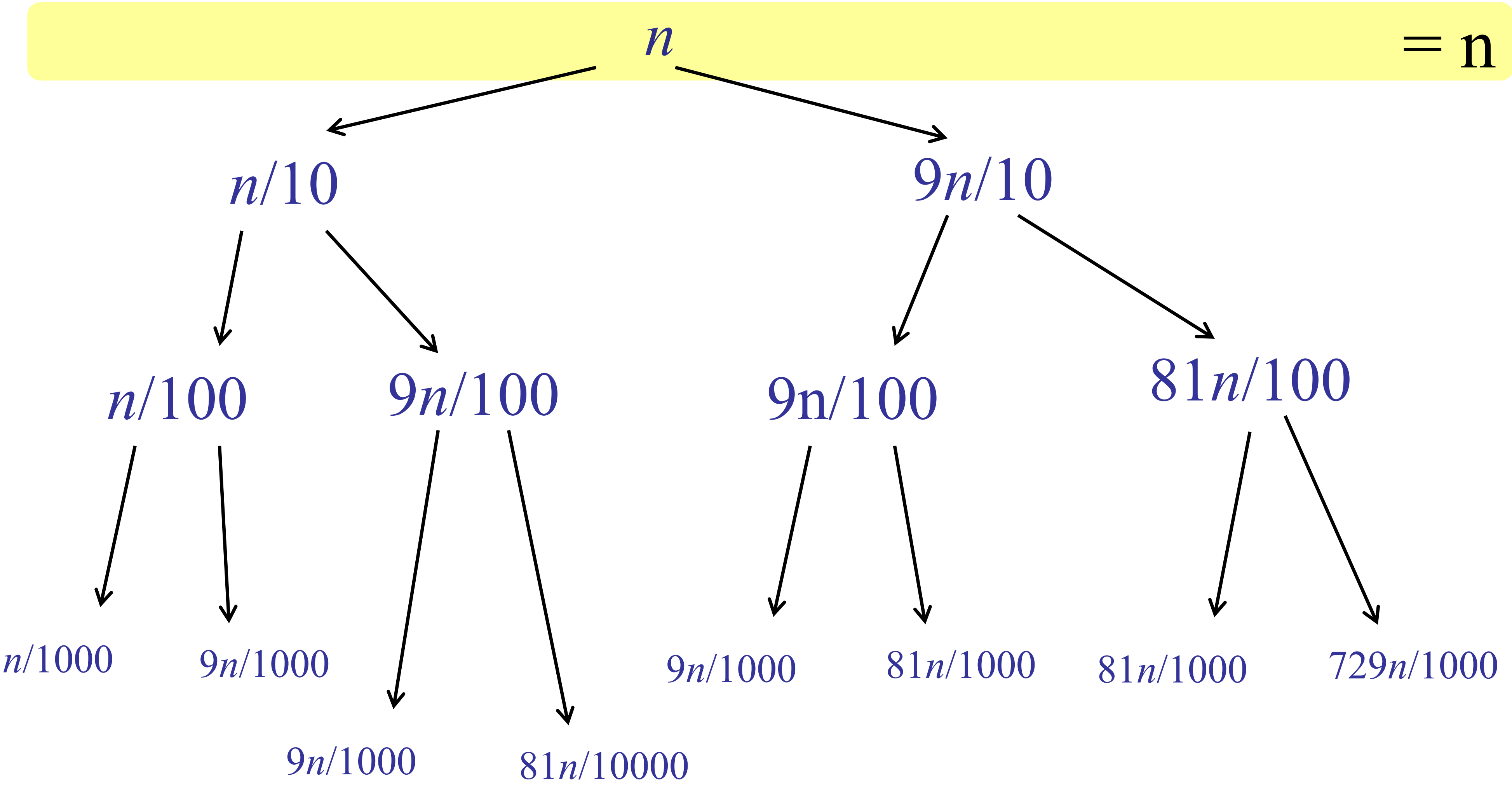


QuickSort Analysis

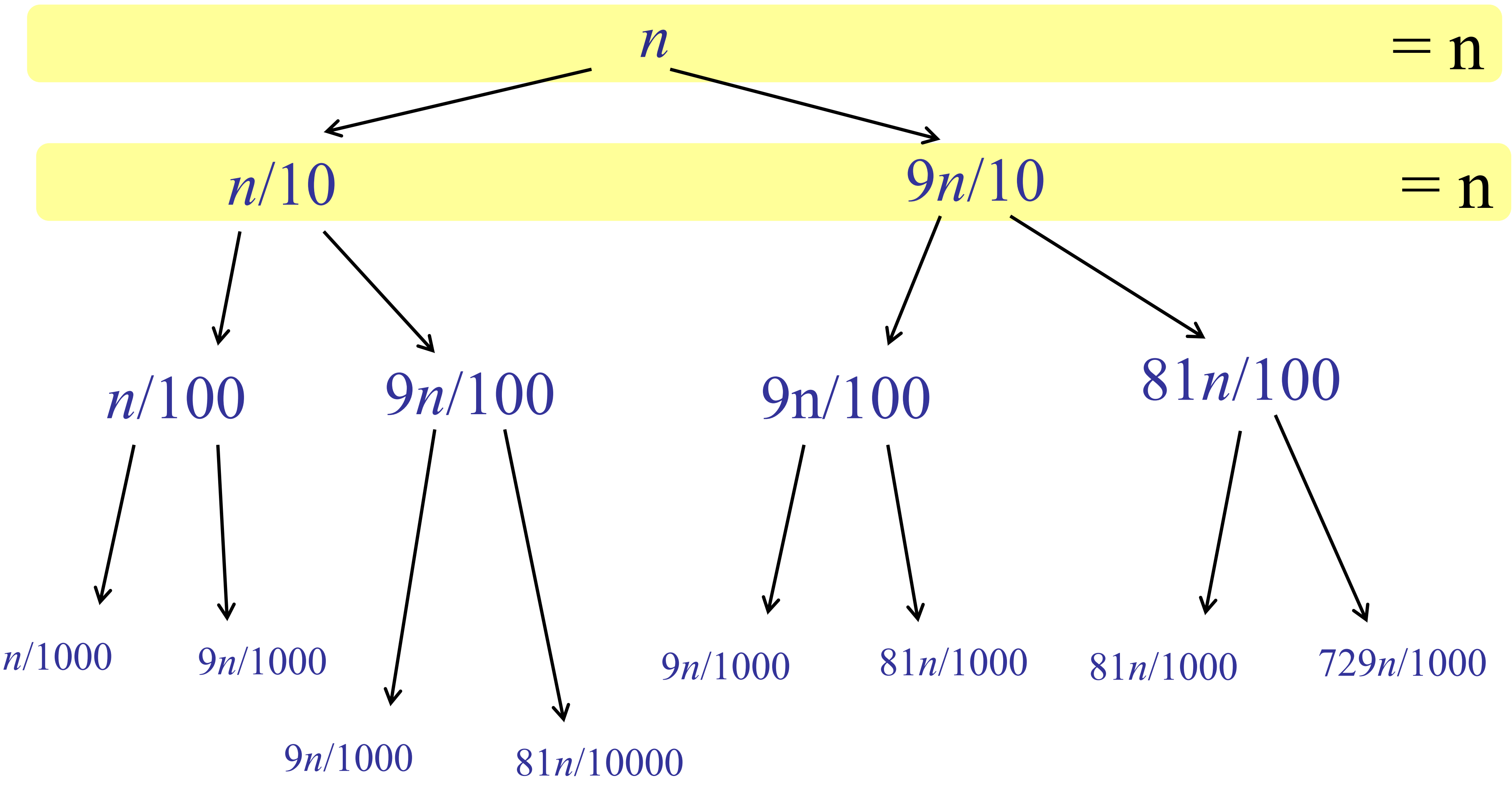
$k = n/10$



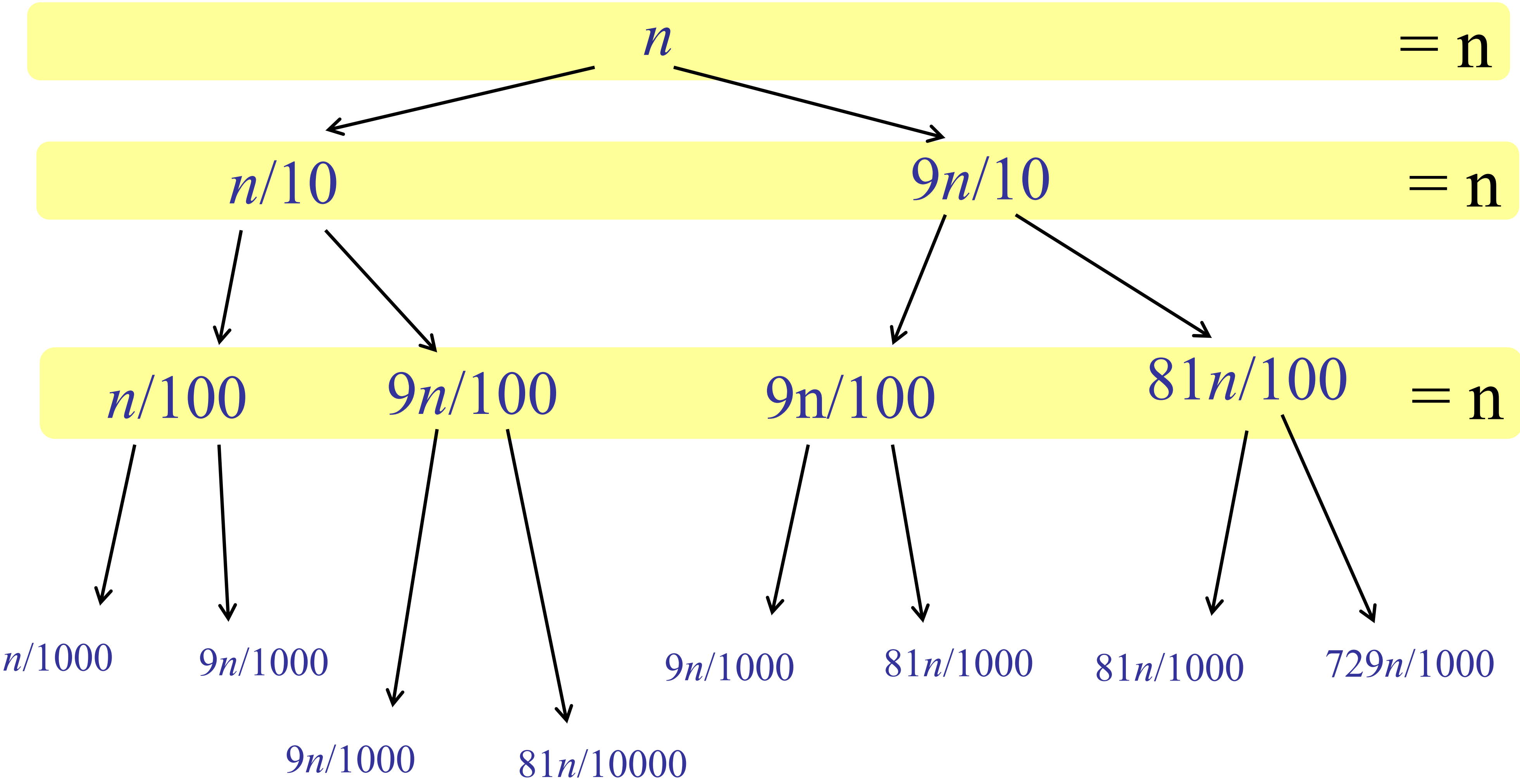
QuickSort Analysis



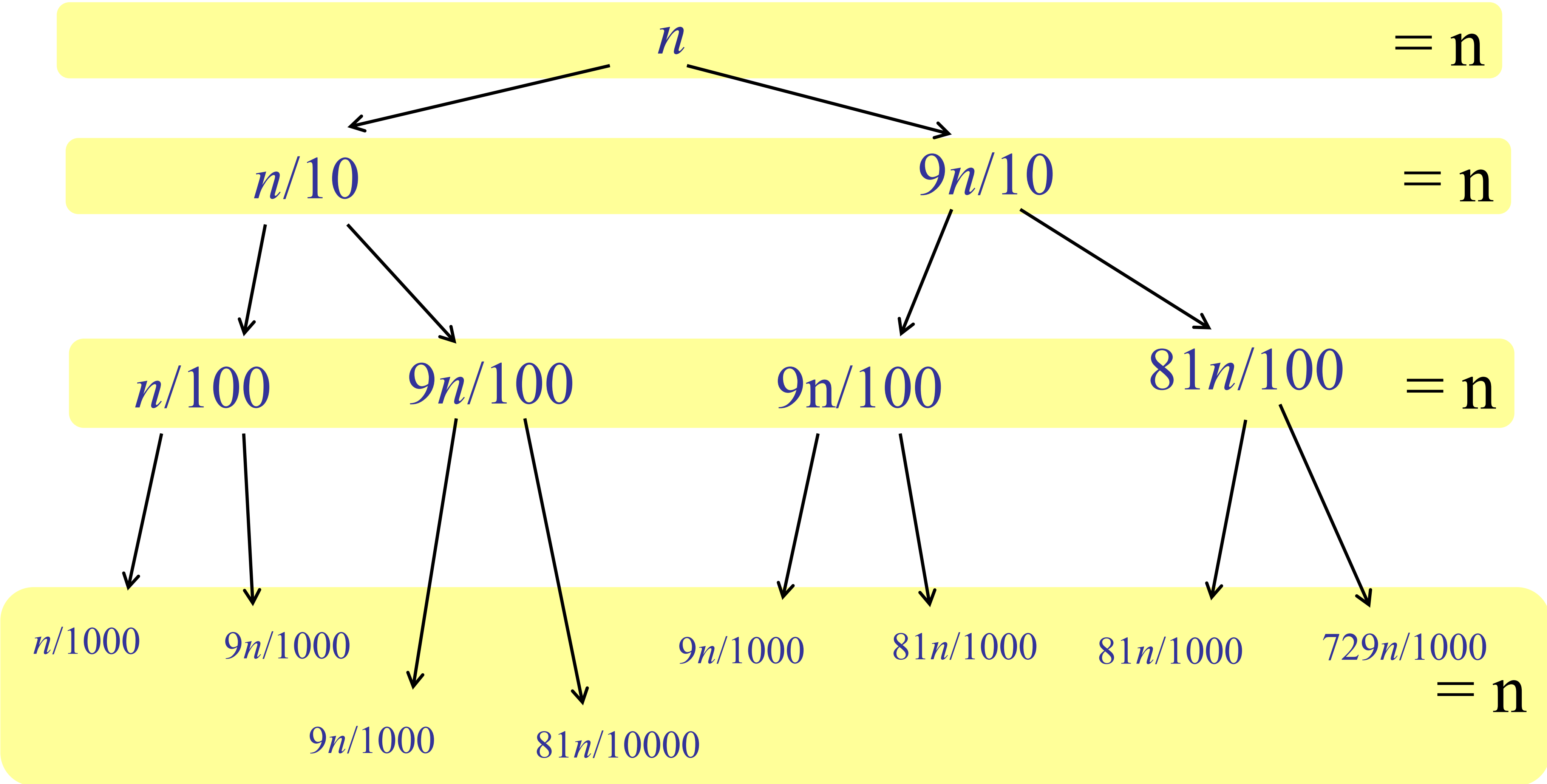
QuickSort Analysis



QuickSort Analysis

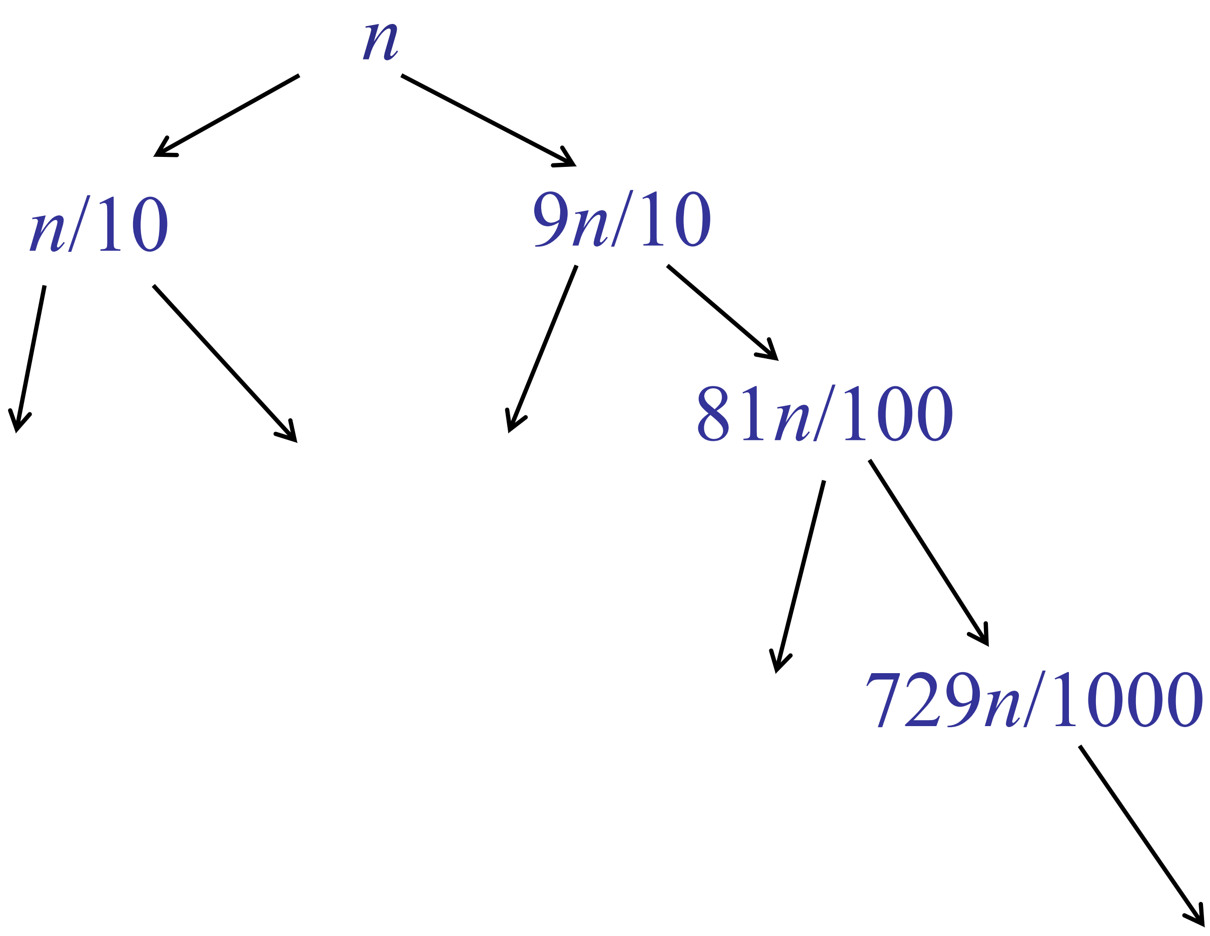


QuickSort Analysis



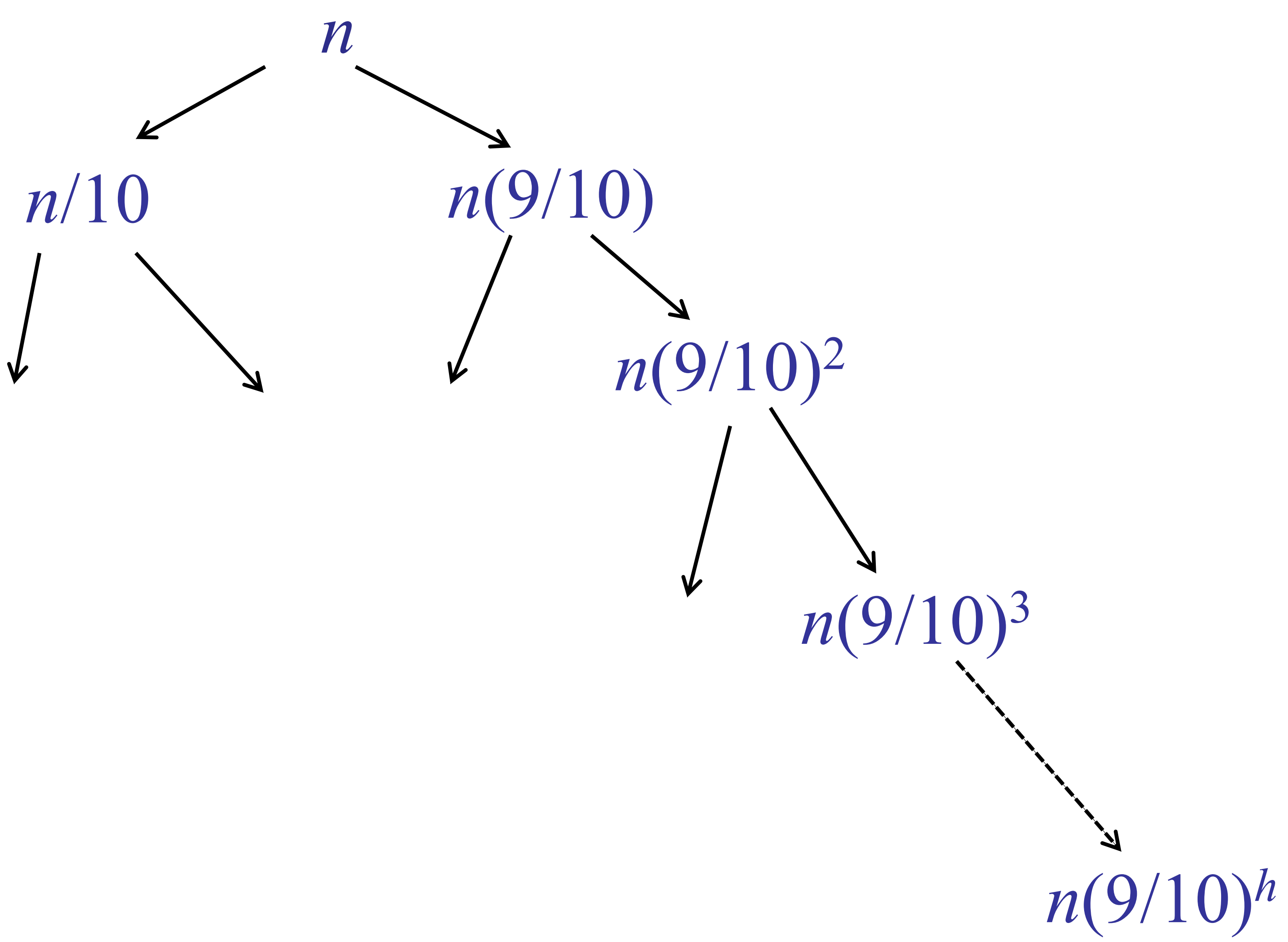
How many levels??

QuickSort Analysis



How many levels??

QuickSort Analysis



How many levels??

QuickSort Analysis

Maximum number of levels:

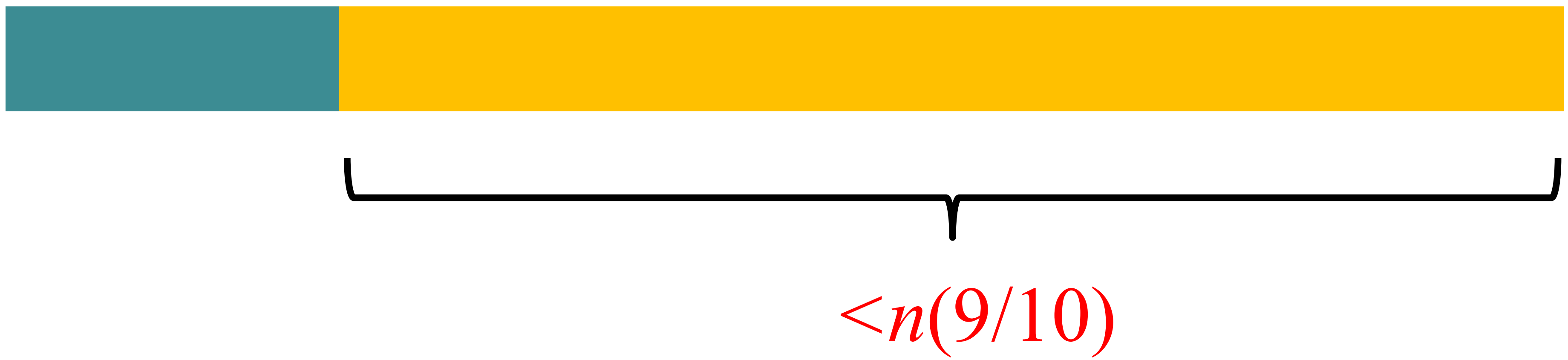
$$1 = n(9/10)^h$$

$$(10/9)^h = n$$

$$h = \log_{10/9}(n) = \mathrm{O}(\log \mathrm{n})$$

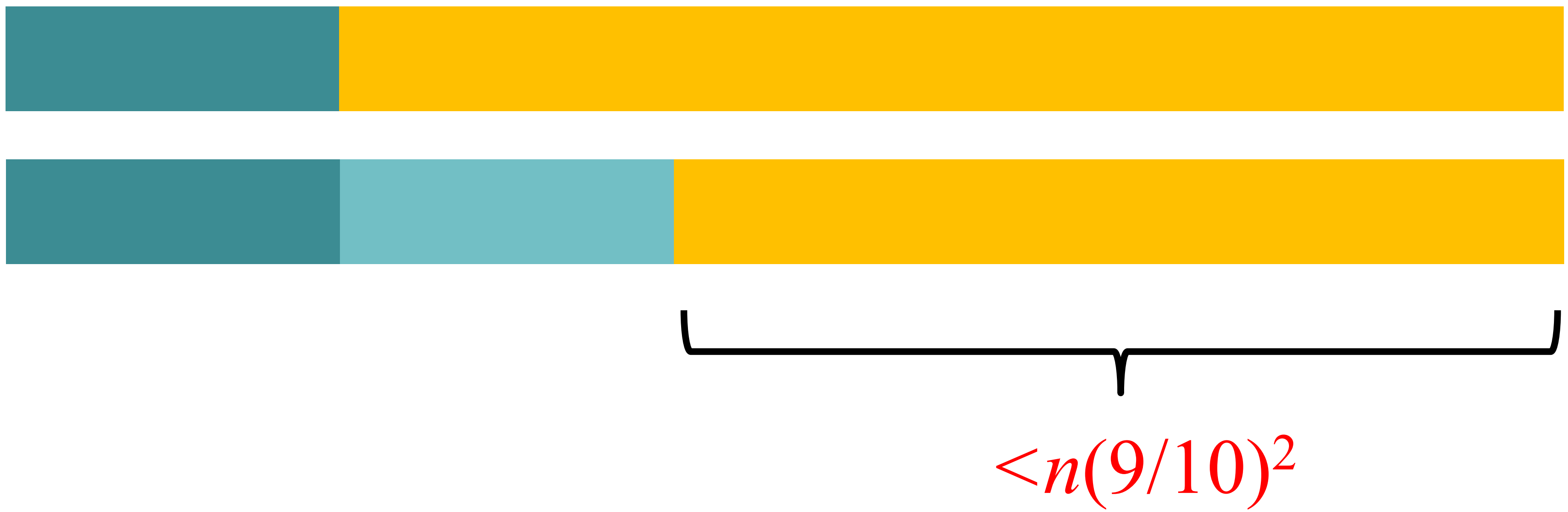
QuickSort Analysis

Assume larger part shrinks by at least 9/10 every iteration:



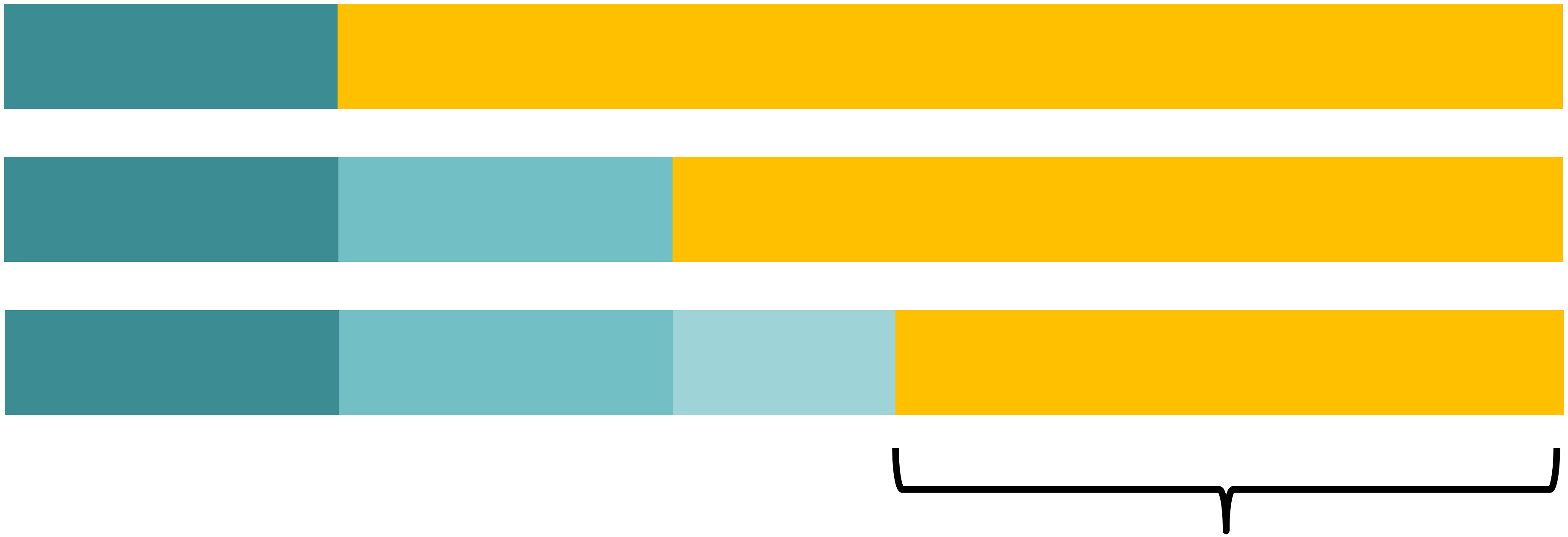
QuickSort Analysis

Assume larger part shrinks by at least 9/10 every iteration:



QuickSort Analysis

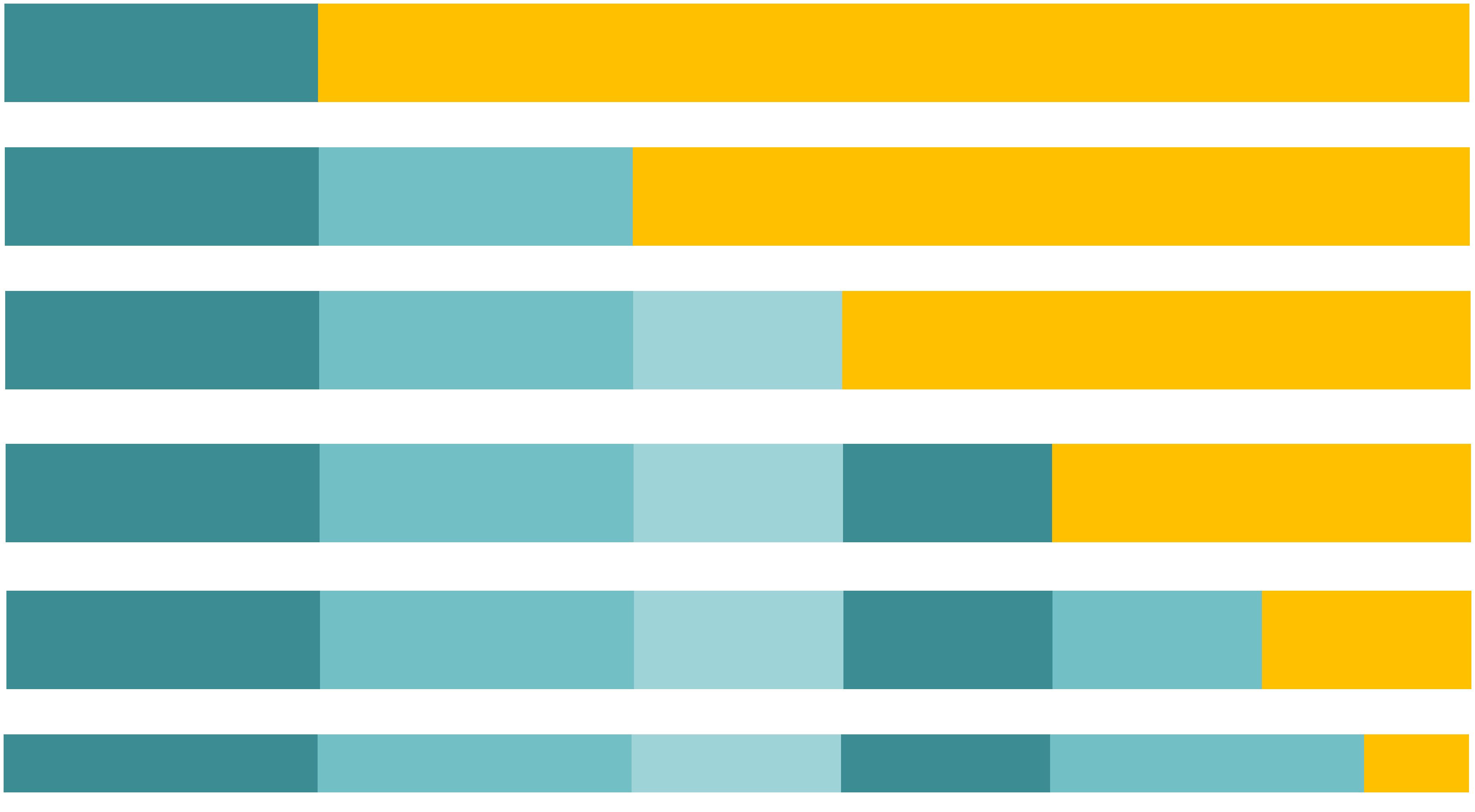
Assume larger part shrinks by at least 9/10 every iteration:



$< n(9/10)^3$

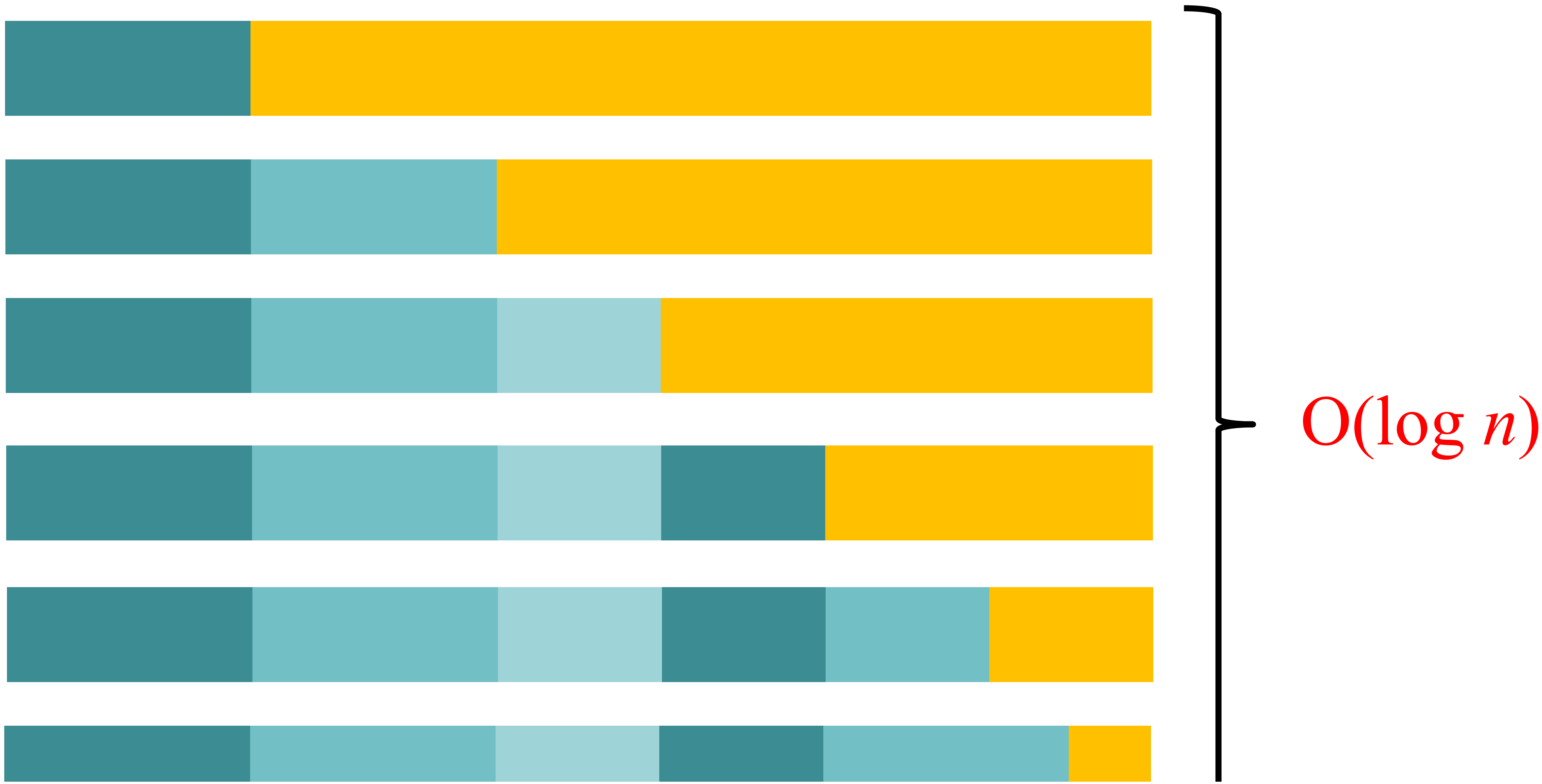
QuickSort Analysis

Assume larger part shrinks by at least 9/10 every iteration:



QuickSort Analysis

Assume larger part shrinks by at least 9/10 every iteration:



QuickSort Summary

- If we choose the pivot as A[1]:
  - Bad performance: $\Omega(n^2)$

- If we could choose the median element:
  - Good performance: $O(n \log n)$

- If we could split the array (1/10) : (9/10)
  - Good performance: $O(n \log n)$

QuickSort

QuickSort(A[1..n], n)

if (n==1) **then** return;

else

Choose pivot index *pIndex*.

p = partition(A[1..n], n, *pIndex*)

x = QuickSort(A[1..p–1], p–1)

y = QuickSort(A[p+1..n], n–p)

$< x$	x	> **x**

QuickSort

Key Idea:

- Choose the pivot at random.

Randomized Algorithms:

- Algorithm makes decision based on random coin flips.
- Can “fool” the adversary (who provides bad input)
- Running time is a *random variable*.

Randomization

What is the difference between:

- Randomized algorithms
- Average-case analysis

Randomization

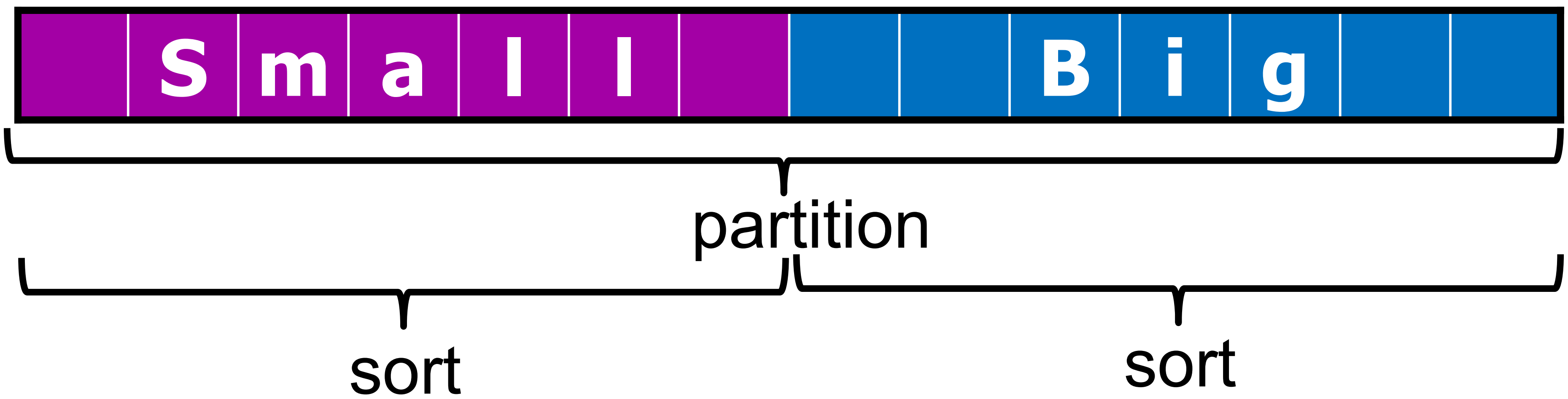
Randomized algorithm:

- Algorithm makes random choices
- For every input, there is a good probability of success.

Average-case analysis:

- Algorithm (may be) deterministic
- "Environment" chooses random input
- Some inputs are good, some inputs are bad
- For most inputs, the algorithm succeeds

```
QuickSort(A[1..n], n)
    if (n == 1) then return;
    else
      pIndex = random(1, n)
      p = 3WayPartition(A[1..n], n, pindex)
      x = QuickSort(A[1..p-1], p-1)
      y = QuickSort(A[p+1..n], n-p)
```



Paranoid QuickSort

```
ParanoidQuickSort(A[1..n], n)
    if (n == 1) then return;
    else
        repeat
            pIndex = random(1, n)
            p = partition(A[1..n], n, pIndex)
        until p > (1/10)n and p < (9/10)n

        x = QuickSort(A[1..p–1], p–1)
        y = QuickSort(A[p+1..n], n–p)
```

Paranoid QuickSort

Easier to analyze:

- Every time we recurse, we reduce the problem size by at least (1/10).
- We have already analyzed that recurrence!

Note: non-paranoid QuickSort works too

- Analysis is a little trickier (but not much).

Paranoid QuickSort

```
ParanoidQuickSort(A[1..n], n)
    if (n == 1) then return;
    else
        repeat
            pIndex = random(1, n)
            p = partition(A[1..n], n, pIndex)
        until p > (1/10)n and p < (9/10)

        x = QuickSort(A[1..p–1], p–1)
        y = QuickSort(A[p+1..n], n–p)
```

Paranoid QuickSort

T(n) → ParanoidQuickSort(A[1..n], n)

1 → **if** (n == 1) **then** return;

else

?? →
repeat
pIndex = **random**(1,
p = partition(A[1..n],
until p > (1/10)n **and** p < (

T(k) → x = QuickSort(A[1..p–1], p

T(n-k) → y = QuickSort(A[p+1..n], n

9n/10 > k > n/10

Paranoid QuickSort

T(n) → ParanoidQuickSort(A[1..n], n)

1 → **if** (n == 1) **then** return;

else

O(n) →
repeat

pIndex = **random**(1,

p = partition(A[1..n],

until p > (1/10)n **and** p < (

T(k) → x = QuickSort(A[1..p–1], p

T(n-k) → y = QuickSort(A[p+1..n], n

$9n/10 > k > n/10$

Paranoid QuickSort

Key claim:

- We only execute the **repeat** loop O(1) times (in expectation).

Then we know:

$$T(n) <= T(n/10) + T(9n/10) + n(\#\text{ iterations of } \textbf{repeat})$$

$$= O(n \log n)$$

Paranoid QuickSort

$T(n)$	ParanoidQuickSort(A[1..n], n)
1	**if** (n == 1) **then** return;
	else
	repeat
	pIndex = **random**(1,
$O(n)$	p = partition(A[1..n],
	until p > (1/10)n **and** p < (
$T(k)$	x = QuickSort(A[1..p–1], p
$T(n-k)$	y = QuickSort(A[p+1..n], n

9n/10 > k > n/10

Probability Theory

Probability Theory

Flipping a coin:

- Pr(heads) = ½
- Pr(tails) = ½

Coin flips are independent:

- Pr(heads → heads) = ½ * ½ = ¼
- Pr(heads → tails → heads) = ½ * ½ * ½ = 1/8

Probability Theory

Flipping a coin:

- Pr(heads) = ½
- Pr(tails) = ½

Set of uniform events (e_1, e_2, e_3, …, e_k):

- Pr(e_1) = 1/k
- Pr(e_2) = 1/k
- …
- Pr(e_k) = 1/k

Probability Theory

Events **A**, **B**:

- Pr(**A**), Pr(**B**)
- **A** and **B** are independent
  (e.g., unrelated random coin flips)

Then:

- Pr(**A** and **B**) = Pr(**A**)Pr(**B**)

How many times do you have to flip a coin before it comes up heads?

Probability Theory

Expected value:

- Weighted average

Example: event **A** has two outcomes:

- $\Pr(\mathbf{A} = 12) = 1/4$
- $\Pr(\mathbf{A} = 60) = 3/4$

Expected value of A:

$$E[A] = (1/4)12 + (3/4)60 = 48$$

Probability Theory

Flipping a coin:

- $Pr(heads) = ½$
- $Pr(tails) = ½$

In two coin flips: I <u>expect</u> one heads.

Probability Theory

Define event **A**:

- **A** = number of heads in two coin flips

In two coin flips: I <u>expect</u> one heads.

– Pr(heads, heads) = ¼	2 * ¼	=	½
– Pr(heads, tails) = ¼	1 * ¼	=	¼
– Pr(tails, heads) = ¼	1 * ¼	=	¼
– Pr(tails, tails) = ¼	0 * ¼	=	0
			1

Probability Theory

Flipping a coin:

- Pr(heads) = ½
- Pr(tails) = ½

In two coin flips: I expect one heads.

- If you repeated the experiment many times, on average after two coin flips, you will have one heads.

Goal: calculate expected time of QuickSort

Probability Theory

Set of outcomes for $X = (e_1, e_2, e_3, \ldots, e_k)$:

- $Pr(e_1) = p_1$
- $Pr(e_2) = p_2$
- …
- $Pr(e_k) = p_k$

Expected outcome:

$$E[X] = e_1p_1 + e_2p_2 + \ldots + e_kp_k$$

Probability Theory

Linearity of Expectation:

- $E[A + B] = E[A] + E[B]$

Example:

- A = # heads in 2 coin flips: $E[A] = 1$
- B = # heads in 2 coin flips: $A[B] = 1$
- $A + B$ = # heads in 4 coin flips

$$E[A+B] = E[A] + E[B] = 1 + 1 = 2$$

Probability Theory

Flipping an (unfair) coin:

- Pr(heads) = p
- Pr(tails) = (1 – p)

How many flips to get at least one head?

E[X]= expected number of flips to get one head

Example: X = 7

T T T T T T H

Probability Theory

Flipping an (unfair) coin:

- Pr(heads) = p
- Pr(tails) = (1 – p)

How many flips to get at least one head?

E[X]= Pr(heads after 1 flip)*1 +

Pr(heads after 2 flips)*2 +

Pr(heads after 3 flips)*3 +

Pr(heads after 4 flips)*4 +

…

Probability Theory

Flipping an (unfair) coin:

- Pr(heads) = p
- Pr(tails) = (1 – p)

How many flips to get at least one head?

E[X]= Pr(H)*1 +

Pr(T H)*2 +

Pr(T T H)*3 +

Pr(T T T H)*4 +

…

Probability Theory

Flipping an (unfair) coin:

- Pr(heads) = p
- Pr(tails) = (1 – p)

How many flips to get at least one head?

$$
\begin{aligned}
\mathbf{E}[X] = {} & p(1) + \\
& (1-p)(p)(2) + \\
& (1-p)(1-p)(p)(3) + \\
& (1-p)(1-p)(1-p)(p)(4) + \\
& \ldots
\end{aligned}
$$

Probability Theory

Flipping an (unfair) coin:

- Pr(heads) = p
- Pr(tails) = $(1 - p)$

How many flips to get at least one head?

$$\mathbf{E}[X] = (p)(1) + (1 - p)\,(1 + \mathbf{E}[X])$$

How many <u>more</u> flips to get a head?

Idea: If I flip "tails," the expected number of additional flips to get a "heads" is <u>still</u> $\mathbf{E}[X]$!!

Probability Theory

Flipping an (unfair) coin:

- Pr(heads) = p
- Pr(tails) = (1 – p)

How many flips to get at least one head?

$$\mathbf{E}[X] = (p)(1) + (1 - p)\,(1 + \mathbf{E}[X])$$

$$= p + 1 - p + 1\mathbf{E}[X] - p\mathbf{E}[X]$$

Probability Theory

Flipping an (unfair) coin:

- Pr(heads) = p
- Pr(tails) = $(1 - p)$

How many flips to get at least one head?

$$\mathbf{E}[X] = (p)(1) + (1 - p)(1 + \mathbf{E}[X])$$

$$= p + 1 - p + 1\mathbf{E}[X] - p\mathbf{E}[X]$$

$$\mathbf{E}[X] - \mathbf{E}[X] + p\mathbf{E}[X] = 1$$

Probability Theory

Flipping an (unfair) coin:

- Pr(heads) = p
- Pr(tails) = (1 – p)

How many flips to get at least one head?

$$\mathbf{E}[X] = (p)(1) + (1 - p)\,(1 + \mathbf{E}[X])$$

$$= p + 1 - p + 1\mathbf{E}[X] - p\mathbf{E}[X]$$

$$p\mathbf{E}[X] = 1$$

$$\mathbf{E}[X] = 1/p$$

Probability Theory

Flipping an (unfair) coin:

- Pr(heads) = p
- Pr(tails) = (1 – p)

How many flips to get at least one head?

If p = ½, the expected number of flips to get one head equals:

$$\mathbf{E}[X] = 1/p = 1/½ = 2$$

Paranoid QuickSort

```
ParanoidQuickSort(A[1..n], n)
    if (n == 1) then return;
    else
        repeat
            pIndex = random(1, n)
            p = partition(A[1..n], n, pIndex)
        until p > (1/10)n and p < (9/10)

        x = QuickSort(A[1..p–1], p–1)
        y = QuickSort(A[p+1..n], n–p)
```

QuickSort Partition

Remember:

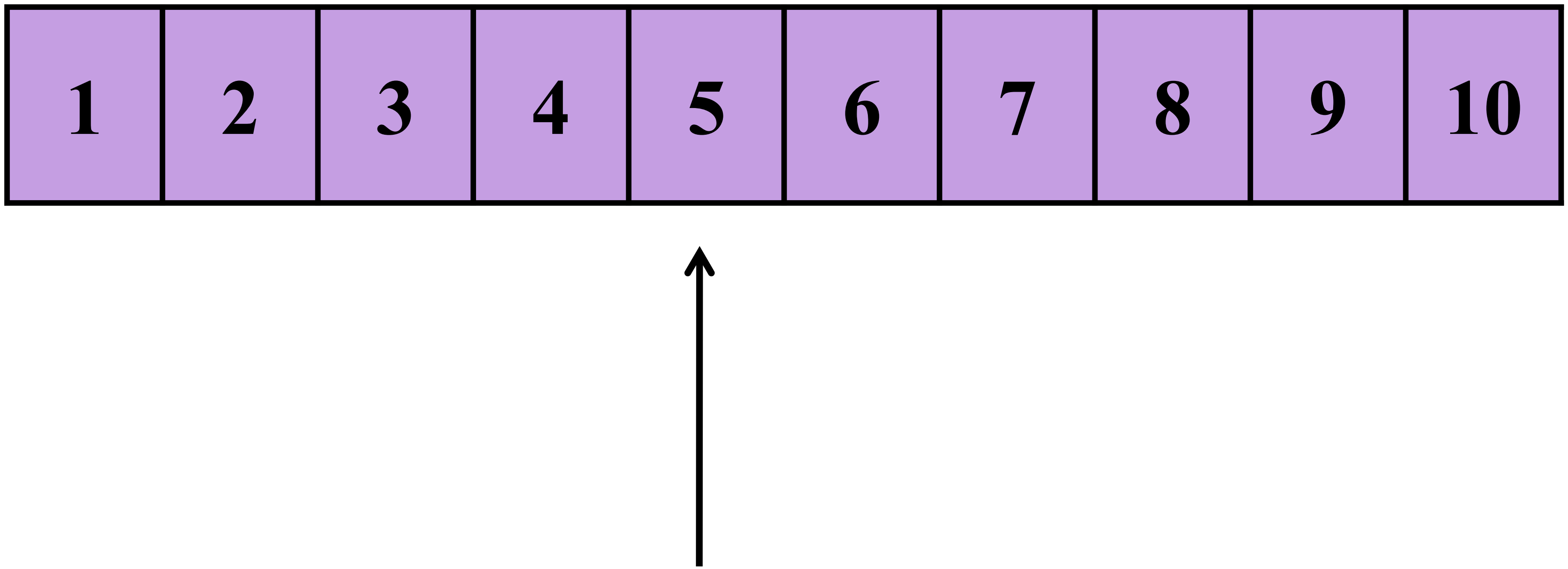
A *pivot* is **good** if it divides the array into two pieces, each of which is size at least $n/10$.

If we choose a pivot at random, what is the probability that it is good?

1. 1/10
2. 2/10
3. 8/10
4. $1/\log(n)$
5. $1/n$
6. I have no idea.

Choosing a Good Pivot

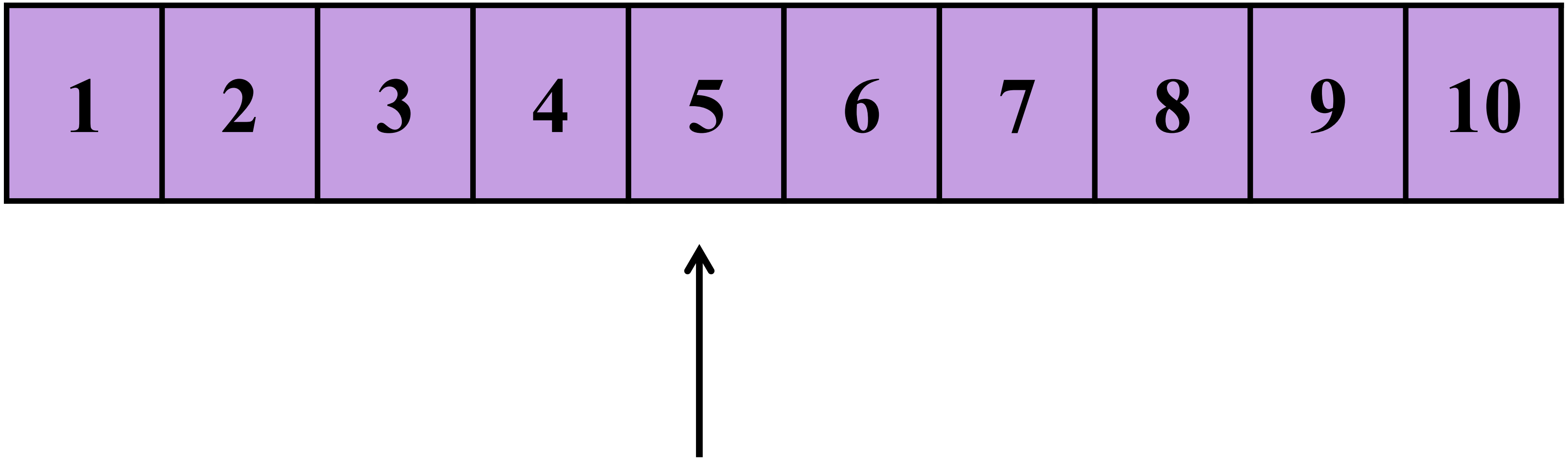
Imagine the array divided into 10 pieces:



Choose a random point at which to partition.

Choosing a Good Pivot

Imagine the array divided into 10 pieces:

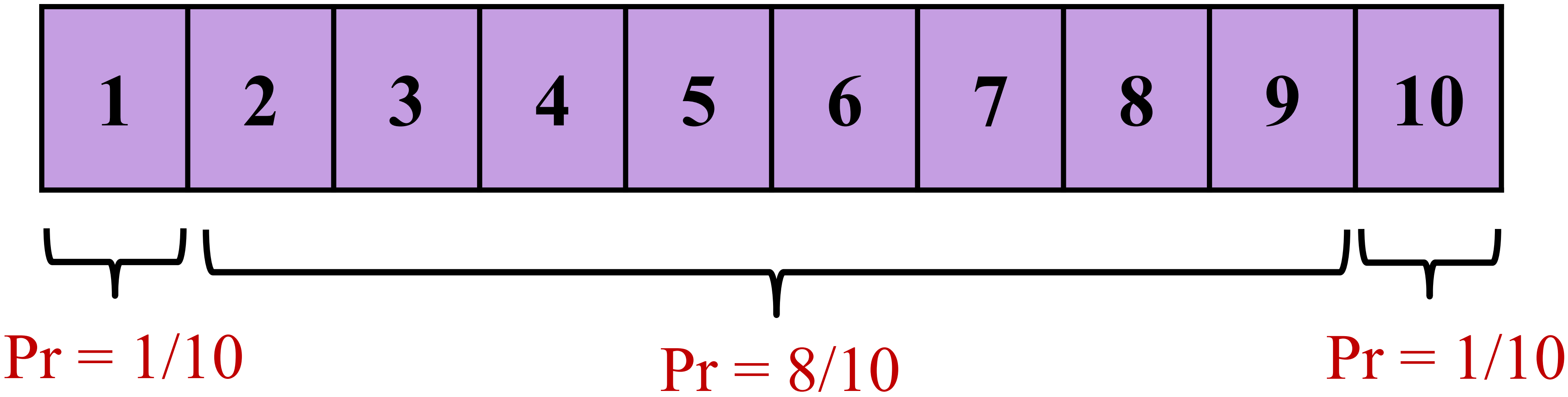


Choose a random point at which to partition.

- 10 possible events
- each occurs with probability 1/10

Choosing a Good Pivot

Imagine the array divided into 10 pieces:

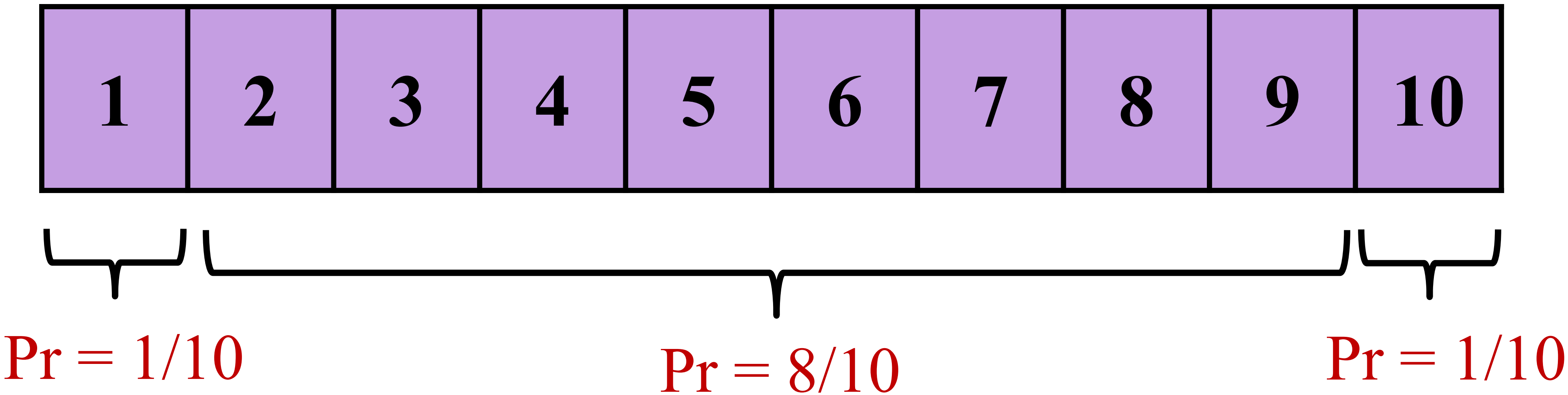


Choose a random point at which to partition.

- 10 possible events
- each occurs with probability 1/10

Choosing a Good Pivot

Imagine the array divided into 10 pieces:



Probability of a good pivot:

$p = 8/10$

$(1 - p) = 2/10$

Choosing a Good Pivot

Probability of a good pivot:

$$p = 8/10$$

$$(1 - p) = 2/10$$

Expected number of times to repeatedly choose a pivot to achieve a good pivot:

$$\mathbf{E}[\text{\# choices}] = 1/p = 10/8 < 2$$

Paranoid QuickSort

QuickSort(A[1..n], n)

if (n==1) **then** return;

else

repeat

$pIndex$ = **random**(1, n)

p = partition(A[1..n], n, $pIndex$)

until $p > n/10$ **and** $p < n(9/10)$

x = QuickSort(A[1..p–1], p–1)

y = QuickSort(A[p+1..n], n–p)

Paranoid QuickSort

Key claim:

We only execute the **repeat** loop < 2 times (in expectation).

Then we know:

$$\mathbf{E}[\mathrm{T}(n)] = \mathbf{E}[\mathrm{T}(k)] + \mathbf{E}[\mathrm{T}(n-k)] + \mathbf{E}[\#\text{ pivot choices}](n)$$

$$<= \mathbf{E}[\mathrm{T}(k)] + \mathrm{E}[\mathrm{T}(n-k)] + 2n$$

$$= \mathrm{O}(n \log n)$$

QuickSort Optimizations

Many, many optimizations and variants:

1. To save space, recurse into smaller half first.
   - Only need O(log n) extra space.

2. For small arrays, use InsertionSort.
   - Stop recursion at arrays of size MinQuickSort.
   - Do one InsertionSort on full array when done.

3. If array contains repeated keys, be careful!

Summary

QuickSort:

- Divide-and-Conquer
- Partitioning
- Duplicates
- Choosing a pivot
- Randomization
- Analysis

Next time: more sorting…