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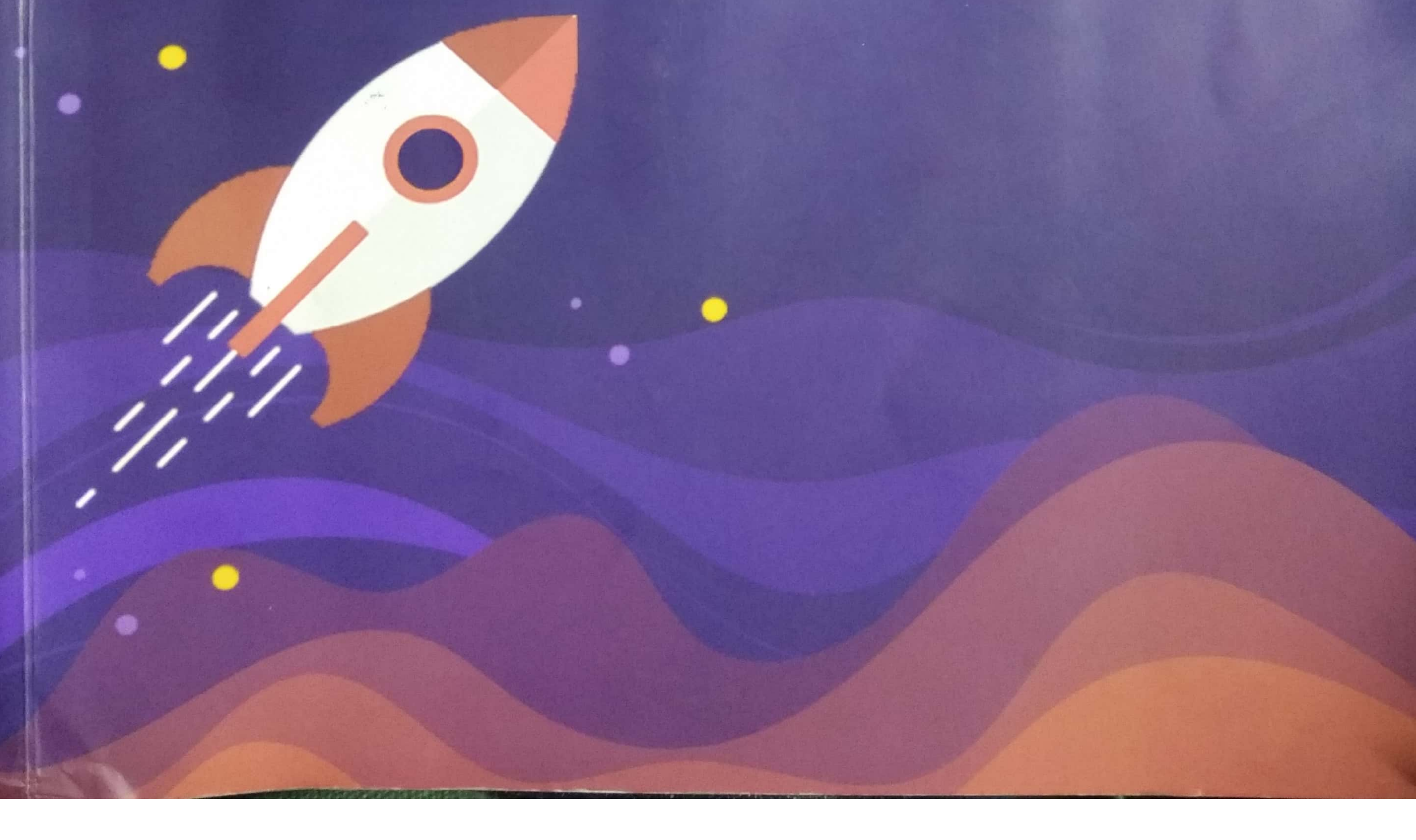

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Competitive Programming Module

Content

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About the contributors :

Prateek Narang

Fondly referred to as *Prateek Bhaiya*, Prateek Narang is one of the founding members of Coding Blocks. Having completed his Computer Engineering from DTU with exceptional grades, he is now pursuing a Masters in Machine Learning from IIT Delhi. He is the lead mentor for Competitive Programming and Data Structures-Algorithms at Coding Blocks - for both classroom and online courses. His interactive CV (www.prateeknarang.com) is extremely popular in the tech world internationally.

Deepak Aggarwal

Popular for his creative thinking style, Deepak tries to offer creative solutions to problems. He is a successful competitive programmer, who qualified for ACM-ICPC Regionals. He had won the International Problem Solving Competition by Dyalog APL. He likes to spend his free time reading. He is mentoring students in Competitive Programming and Data Structure courses at Coding Blocks.

Other contributors :

Rajkishor Ranjan (IIT Patna), **Vishal Panwar** (DTU), **Aditya Agrawal** (DTU), **Mayank Ladia** (JIIT), **Saransh Gupta** (IIT Delhi) **Shubham Rawat** (DTU), **Hardik Agrawal** (DTU)

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1 Generic Programming, Standard Template Library

Before we start with C++ Standard Template Library, we must know some basics about generic programming in C++ and how templated classes are made. Let's us start our journey with C++ templates.

C++ Templates

Templates are a feature of the C++ programming language that allows functions and classes to operate with generic types. This allows a function or class to work on many different data types without being rewritten for each one.

C++ STL has some containers (pre-build data structures) like **vectors, iterators, pairs** etc. These are all generic class which can be used to represent collection of any data type.

Generic Programming using C++ Templates

Every program we write revolves around 3 things - data, algorithms and containers.
Let us try to understand the role of each.

1. Data : Every program has some input and output data. For e.g.: A program to control the movement of self-driving car will have some input data like images,visual data from surrounding and output could be int/floating value denoting the required acceleration of the car.
2. Algorithm : Algorithm is the required logic which operates on the input to generate desired output. These algorithms can be made generic using templates in C++.
3. Container : A container is a holder object that stores a collection of other objects (its elements). They are implemented as class templates, which allows a great flexibility in the types supported as elements. The container manages the storage space for its elements and provides member functions to access them, either directly or through iterators. C++ provides various containers like list, vector(dynamic array), stack, queue etc which are used in algorithms for storing in data.

Making Generic Functions & Classes in C++

Let us consider a simple function to search an element in an array.

```
int search(int arr[], int n, int elementToSearch) {
  for (int pos = 0; pos < n; ++pos) {
     if (arr[pos] == elementToSearch) {
```

```
            return pos;
        }
    }
    return -1;
}
```

Observation :

The search function above works only for an *integer array*. However the functionality *search*, is logically separate from array and applicable to all data types, i.e. searching a char in a char array or searching is applicable in searching in a linked list.

Hence, data, algorithms and containers are logically separate but very strongly connected to each other.

Generic Programming enables programmer to achieve this logical separation by writing **general *algorithms that work on all containers with all data types.***

Separating Data

A function can be made general to all data types with the help of templates.

Templates are a blueprint based on which compiler generates a function as per the requirements

To create a template of search function, we *replace int with a type T* and tells compiler that *T is a t*... using the *statement* template <class T>

```
template <class T>
int search(T arr[], int n, T elementToSearch) {
    for (int pos = 0; pos < n; ++pos) {
        if (arr[pos] == elementToSearch) {
            return pos;
        }
    }
    return -1;
}
```

Now the search function runs for all types of arrays for which statement 4 is defined.

Now the search function runs for all types of arrays for which statement 4 is defined.

```
int arrInt[100];
char arrChar[100];
float arrFloat[100];
Book arrBook[100], X;      //X is a book
```

```
search(arrInt, 100, 5);     //T is replaced by int
search(arrChar, 100, 'A'); //T is replaced by char
search(arrFloat, 100, 1.24); //T is replaced by float
search(arrBook, 100, X); //T is replaced by Book.
```

So, one function can be run on different data types. This makes our function *general* for all types of data.

Note

In the actual code that is produced after compilation, 4 different functions will be produced based on the template with T replaced accordingly.

You could use <typename T> or <class T> in the template statement 1. The keywords typename and class serve the same purpose.

Separating Algorithm

The search function will not work for Book objects if computer doesn't know how to compare 2 books. So our function is limited in some sense. To work it for all data types, we use a concept of *comparator* (also see predicate).

Let's rewrite the templated search function again

```
template <class T, class Compare>
int search(T arr[], int n, T elementToSearch, Compare obj) {
//compare is a class that has () operator overloaded
  for (int pos = 0; pos < n; ++pos) {
     if (obj(arr[pos], elementToSearch) == true) {
//obj compares elements of type T
        return pos;
     }
  }
  return -1;
}
```

To use the search function for integers, you shall now write:

```
//defining a class compare
class compareInt {
public:
  bool operator()(int a, int b) {
```

```
    return a == b ? true : false;
  }
};
//calling search templated function
compareInt obj;
        //obj is the object of class compareInt
search(arrInt, 100, 20, obj); //T replaced with int
        //Compare replaced with compareInt
```

Line 4 works since obj(xInt, yInt) is defined by the class compareInt.

To use *search function for a book class*, we should write a compareBook class.

```
class compareBook{
public:
bool operator()(const Book& B1, const Book& B2){
return B1.getIsbn () == B2.getIsbn();
}
search(arrBook, 100, X, compareBook);
        //calling search function
```

The same search function now operators for Books just by writing a small compare class.

However, search function still works only for arrays. However the functionality of searching extend to list equally. To make it general for all containers(here array) we introduce a concept of iterators in our discussion.

Separating Containers

Iterators

Visualise iterators as an entity using which you can access the data within a container with certain restrictions.

These are classified into 5 categories

Input Iterator : An entity through which you can read from container and move *ahead*.

Que : *What sort of container will posses an input iterator???*

Sol : A keyboard.

Output Iterator An entity through which you can write into the container and move ahead. *Container like printer or monitor will have such an iterator.*

Forward Iterator Iterator with functionalities of both Input and Output iterator in single direction.
Singly linked list will posses a forward entity since we can read/write only in the forward direction.

Bidirectional Iterator Forward iterator that can move in both the directions.
Doubly linked list will posses a bidirectional iterator.

Random Access Iterator Iterator that can read/write in both directions plus can also take jumps.
An array will have random access iterator. Since, you can jump by writing arr[5], which means jump to the 5th element.

Entity that does this, behaves like a pointer in some sense.

To write search function that is truly independent of data and the underlying container, we use iterator.

```
template<class ForwardIterator, class T, class Compare>
ForwardIterator search(ForwardIter beginOfContainer, ForwardIter endOfContainer,
T elementToSearch, Compare Obj) {
   while (beginOfContainer  != endOfContainer) {
        if (obj((*beginOfContainer), elementToSearch) == true) break; //iterators are like
pointers!
      ++beginOfContainter;
   }
   return beginOfContainer;
}
```

Here, beginOfContainer is a *ForwardIterator*, i.e., beginOfContainer must know how to read/write and move in *forward* direction.

So, if a container has at least ForwardIterator, the algorithm works. Hence, it works for list, doubly linked list and array as well thus achieving generality over container.

```
//search for book in an array
search(arr, arr + n, X, compareBook);
//search for book in a list
list<Book> lb;  // see list
search(lb.begin(), lb.end(), X, compareBook);
//begin and end are member function of the class list.
```

Summary

1. *Using templates, we achieve freedom from data types*
2. *Using comparators, we achieve freedom from operation(s) acting on data*
3. *Using iterators, we achieve freedom from underlying data structure (container).*

Working with C++ STL Containers

C++ provides a powerful Standard Template Library(STL) template library,which is a set of C++ template classes to provide common programming data structures and functions.

1. **Algorithms** : There are inbuilt algorithms for tasks like sorting,searching etc. We will discuss these in the later part of the chapter.

2. **Containers** : Containers or container classes store objects and data. They are divided into following categories.

Sequence Containers : Implement data structures which can be accessed in a sequential manner.

(a) Vector (b) List (c) Deque (d) Arrays

(e) Forward list (Introduced in C++11)

Container Adaptors : provide a different interface for sequential containers.

(f) queue (g) priority queue (h) stack

Associative Containers : Implement sorted data structures that can be quickly searched (O(log n) complexity).

(i) Set (j) Multiset (k) Map (l) Multimap

String

C++ provides a powerful alternative for the **char *** . It is not a built-in data type, but is a container class in the **Standard Template Library**. String class provides different string manipulation functions like concatenation, find, replace etc. Let us see how to construct a string type.

```
string s1; // s1 = ""
string s2 (s1); // s2 = "Hello"
string s3 (s1, 1, 2); // s3 = "el"
string s4 ("Hello World", 5); // s4 = "Hello"
string s5 (5, '*'); // s5 = "*****"
string s6 (s1.begin(), s1.begin()+3); //s6= "Hel"
```

Here are Some Member Functions :

- **append():** Inserts additional characters at the end of the string. (can also be done using '+' or '+=' operator). It's **time complexity** is **O(N)** where N is the size of the new string.
- **begin():** Returns an iterator pointing to the first character. Its **time complexity** is **O(1)**.
- **clear():** Erases all the contents of the string and assign an empty string ("") of length zero. Its **time complexity** is **O(1)**.
- **compare():** Compares the value of the string with the string passed in the parameter and returns an integer accordingly. Its **time complexity** is **O(N + M)** where N is the size of the first string and M is the size of the second string.

- **copy():** Copies the substring of the string in the string passed as parameter and returns the number of characters copied. Its **time complexity** is **O(N)** where N is the size of the copied string.
- **empty():** Returns a boolean value, true if the string is empty and false if the string is not empty. Its time complexity is **O(1).**
- **end():** Returns an iterator pointing to a position which is next to the last character. Its **time complexity** is **O(1)**.
- **erase():** Deletes a substring of the string. Its **time complexity** is **O(N)** where N is the size of the new string.
- **find():** Searches the string and returns the first occurrence of the parameter in the string. Its **time complexity** is **O(N)** where N is the size of the string.
- **insert():** Inserts additional characters into the string at a particular position. Its time complexity is O(N) where N is the size of the new string.
- **length():** Returns the length of the string. Its **time complexity** is **O(1).**
- **size():** Returns the length of the string. Its **time complexity** is **O(1)**.
- **substr():** Returns a string which is the copy of the substring. Its **time complexity** is **O(N)** where N is the size of the substring.

Vector

Vectors are sequence containers that have dynamic size. In other words, vectors are dynamic arrays. Just like arrays, vector elements are placed in contiguous storage location so they can be accessed and traversed using iterators. To traverse the vector we need the position of the first and last element in the vector which we can get through **begin() and end()** or we can use indexing from **0 to size().**

```
vector<int> a; // empty vector of ints
vector<int> b (5, 10); // five ints wit h
                        value 10
vector<int> c (b.begin(),b.end());
                  // iterating through second
vector<int> d (c); // copy of c
```

Some of the Member Functions of Vectors are :

- **at() :** Returns the reference to the element at a particular position (can also be done using '[ ]' operator). It's time complexity is O(1).
- **back() :** Returns the reference to the last element. It's time complexity is O(1).
- **begin() :** Returns an iterator pointing to the first element of the
- **Vector :** It's time complexity is O(1).
- **clear() :** Deletes all the elements from the vector and assign an empty vector. It's time complexity is O(N) where N is the size of the vector.

- ❑ **empty()** : Returns a boolean value, true if the vector is empty and false if the vector is not empty. Its time complexity is O(1).
- ❑ **end()** : Returns an iterator pointing to a position which is next to the last element of the vector. Its time complexity is O(1).
- ❑ **erase()** : Deletes a single element or a range of elements. It's time complexity is O(N + M) where N is the number of the elements erased and M is the number of the elements moved.
- ❑ **front()** : Returns the reference to the first element. It's time complexity is O(1).
- ❑ **insert()** : Inserts new elements into the vector at a particular position. ts time complexity is O(N + M) where N is the number of elements inserted and M is the number of the elements moved.
- ❑ **pop_back()** : Removes the last element from the vector. It's time complexity is O(1).
- ❑ **push_back()** : Inserts a new element at the end of the vector. It's time complexity is O(1).
- ❑ **resize()** : Resizes the vector to the new length which can be less than or greater than the current length. It's time complexity is O(N) where N is the size of the resized vector.
- ❑ **size()** : Returns the number of elements in the vector. It's time complexity is O(1).

List

List is a sequence container which takes constant time in inserting and removing elements. List in STL is implemented as Doubly Link List. The elements from List cannot be directly accessed. For example to access element of a particular position, you have to iterate from a known position to that particular position.

```
list <int> LI;
list<int> LI(5, 100)
//here LI will have 5 int element s of
  value 100
```

Some of the Member Function of List:

- ❑ **begin( )** : It returns an iterator pointing to the first element in List. Its time complexity is O(1).
- ❑ **end( )** : It returns an iterator referring to the theoretical element(doesn't point to an element) which follows the last element.Its time complexity is O(1).
- ❑ **empty( )** : It returns whether the list is empty or not.It returns 1 if the list is empty otherwise returns 0.It's time complexity is O(1).
- ❑ **back( )** : It returns reference to the last element in the list.It's time complexity is O(1).
- ❑ **assign( )** : It assigns new elements to the list by replacing its current elements and change it's size accordingly.It time complexity is O(N).
- ❑ **erase( )** : It removes a single element or the range of element from the list. It's time complexity is O(N).

- **front( ) :** It returns reference to the first element in the list.It's time complexity is O(1).
- **push_back( ) :** It adds a new element at the end of the list, after its current last element. It's time complexity is O(1).
- **push_front( ) :** It adds a new element at the beginning of the list, before its current first element. It's time complexity is O(1).
- **remove( ) :** It removes all the elements from the list, which are equal to given element. Its time complexity is O(N).
- **pop_back( ) :** It removes the last element of the list, thus reducing its size by 1. Its time complexity is O(1).
- **pop_front( ) :** It removes the first element of the list, thus reducing its size by 1. It's time complexity is O(1).
- **insert( ) :** It insert new elements in the list before the element on the specified position. It's time complexity is O(N).
- **reverse ( ) :** It reverses the order of elements in the list. It's time complexity is O(N).
- **size( ) :** It returns the number of elements in the list. It's time complexity is O(1).

Pair

Pair is a container that can be used to bind together a two values which may be of different types. Pair provides a way to store two heterogeneous objects as a single unit.

```
pair <int, char> p1; // default
pair <int, char> p2 (1, 'a'); // value initialization
pair <int, char> p3 (p2); // copy of p2
```

We can also initialize a pair using **make_pair()** function. **make_pair(x, y)** will return a pair with first element set to x and second element set to **y**.

```
p1 = make_pair(2, 'b');
```

To access the elements we use keywords, **first** and **second** to access the first and second element respectively.

```
cout << p2.first << ' ' << p2.second << endl;
```

Set and Multiset

Sets are containers which store only **unique values** and permit easy look ups. The values in the sets are stored in some specific order (like ascending or descending). Elements can only be inserted or deleted, *but cannot be modified.* We can access and traverse set elements using iterators just like vectors.

Multisets are containers that store elements following a specific order, and where **multiple elements can have equivalent values.**

In a multiset, the value of an element also identifies it (the value is itself the key, of type T). The value of the elements in a multiset cannot be modified once in the container (the elements are always constant), but they can be inserted or removed from the container.

```
set<int> s1; // Empty Set
int a[]= {1, 2, 3, 4, 5, 5};
set<int> s2 (a, a + 6); // s2 = {1, 2, 3, 4, 5}
set<int> s3 (s2); // Copy of s2
set<int> s4 (s3.begin(), s3.end()); // Set created using iterators
multiset<int> first; // empty multiset of ints
int myints[]= {10,20,30,20,20};
multiset<int> second (myints,myints+5); // pointers used as iterators
multiset<int> third (second); // a copy of second
multiset<int> fourth (second.begin(), second.end()); //multiset created using iterators
```

Some of the Member Functions of Set are :

- **begin()** : Returns an iterator to the first element of the set. It's time complexity is O(1).
- **clear()** : Deletes all the elements in the set and the set will be empty . Its time complexity is O(N) where N is the size of the set.
- **count()** : Returns 1 or 0 if the element is in the set or not respectively. Its time complexity is O(logN) where N is the size of the set.
- **empty()** : Returns true if the set is empty and false if the set has at least one element. Its time complexity is O(1).
- **end()** : Returns an iterator pointing to a position which is next to the last element. Its tim complexity is O(1).
- **Erase ()** : Deletes a particular element or a range of elements from the set. Its time complexit s O(N) where N is the number of element deleted.
- **Find ()** : Searches for a particular element and returns the iterator pointing to the element element is found otherwise it will return the iterator returned by end(). Its time comple O(logN) where N is the size of the set.
- **Insert ()** : insert a new element. Its time complexity is O(logN) where N is the size of the
- **size()** : Returns the size of the set or the number of elements in the set. Its time complexity

Set Example :

```
#include<iostream>
#include<set>
using namespace std ;
int main(){

	//Create a Set - Set Stores unique entries in sorted order
	set<int> s;
	//Insert in Set
	s.insert(10);
	s.insert(12);
	s.insert(10);
	s.insert(3);
	s.insert(8);
	s.insert(12);
	//Deletion
	s.erase(12);
	//Searching
	auto f  = s.find(3);
	if(f!=s.end()){
		cout<<"3 exists"<<endl; // 3 exists
	}
	//Iterating using a for each loop
	for(auto no:s){
			cout<<no<<" "; // 3 8 10
	}

}
```

Maps

Maps are containers which store elements by mapping their **value against a particular key**. It stores the combination of key value and mapped value following a specific order. Here key value are used to uniquely identify the elements mapped to it. The data type of key value and mapped value can be different. Elements in map are always in sorted order by their corresponding key and can be accessed directly by their key using bracket operator ([ ]).

In map, key and mapped value have a pair type combination, i.e both key and mapped value can be accessed using pair type functionalities with the help of iterators.

```
map <char ,int > mp;
mp['b'] = 1;
```

In map mp , the values be will be in sorted order according to the key.

Some Member Functions of Map :

- at( ) : Returns a reference to the mapped value of the element identified with key.Its time complexity is O(logN).
- Count( ) : Searches the map for the elements mapped by the given key and returns the number of matches. As map stores each element with unique key, then it will return 1 if match if found otherwise return 0.Its time complexity is O(logN).
- clear( ) : Clears the map, by removing all the elements from the map and leaving it with its size 0.Its time complexity is O(N).
- begin( ) : Returns an iterator(explained above) referring to the first element of map.Its time complexity is O(1).
- end( ) : Returns an iterator referring to the theoretical element(doesn't point to an element) which follows the last element.Its time complexity is O(1).
- empty( ) : Checks whether the map is empty or not. It doesn't modify the map.It returns 1 if the map is empty otherwise returns 0. It's time complexity is O(1).
- erase( ) : Removes a single element or the range of element from the map.
- find( ) : Searches the map for the element with the given key, and returns an iterator to it, if it is present in the map otherwise it returns an iterator to the theoretical element which follows the last element of map.Its time complexity is O(logN).
- insert( ) : Insert a single element or the range of element in the map.Its time complexity is O(logN), when only element is inserted and O(1) when position is also given.

Unordered Maps

Unordered maps fall under the subset of associative containers that use a pair of a key and a mapped value to store the corresponding elements. In an unordered map, the key value is usually used to uniquely identify the element, while the mapped value stores the content associated to this key. These data structures allow for fast retrieval of individual contained elements based on their mapped keys. Internally, the elements in the unordered_map are not sorted in any particular order with respect to either their key or mapped values, but organized into buckets depending on their hash values to allow for fast access to individual elements directly by their key values (with a constant average time complexity on average.

Note : Maps and Unordered Maps have almost the same functions, but have different underlying implementations. We say unordered maps take **O(1) time for search, insert and erase in average case, hence are very useful.**

	Map	Unordered_map
Element ordering	strict weak	n/a
common implementation	balanced tree or red-black tree	hash table
search time	log(n)	O(1) if no has collisions, upto to O(n) if there are hash collisions, O(n) when hash is same for any key
Insertion time	log(n) + rebalance	Same as search
Deletion time	log(n) + rebalance	Same as search
needs comparators	only operator <	only operator --
needs has function	**no**	yes
common use case	when good has is not possible or too slow. Or when order is required	In most other cases

- **find( )** : Searches the container for an element with *k* as key and returns an iterator to it if found, otherwise it returns an iterator to unordered_map::end.
- **rehash( )** : Sets the number of buckets in the container to *n*.
- **insert( )** : inserts a new key-value pair into the container.
- **erase( )** : Removes from the unordered_map container either a single element or a range of elements
- **count( )** : This function returns 1 if an element with that key exists in the container, and zero otherwise.
- **load_factor( )** : This function returns a floating value denoting current load factor in the unordered_map container.

  load_factor = current_size / bucket_count

- **clear( )** : Clears the map, by removing all the elements from the map and leaving it with its size 0.Its time complexity is O(N).
- **begin( )** : Returns an iterator(explained above) referring to the first element of map.Its time complexity is O(1).
- **end( )** : Returns an iterator referring to the theoretical element(doesn't point to an element) which follows the last element.Its time complexity is O(1).
- **Operator[ ]** : If *k* matches the key of an element in the container, the function returns a reference to its mapped value.

Example :

```
#include<iostream>
#include<unordered_map>
using namespace std;
/*
class Fruit{
        price;
        color;
        sweetness;
        state;
        id;
        vendor;
}
*/
int main(){
        unordered_map<string,int> h;
        //unordered_map<string,Fruit> h2;
        //Insertion
        h["Mango"]  = 100;
        //Updation
        h["Mango"] = 80;
        //Print the value if  Mango Exists
        if(h.count("Mango")!=0){
                        cout<<h["Mango"]<<endl;
        }
        //Another Way to insert
        h.insert(make_pair("Kiwi",170));
        //Searching for a given fruit
        string f;
        cin>>f;
        if(h.count(f)){
                cout<<"Fruit costs"<<h[f]<<endl;
        }
        else{
                cout<<"Fruit doesn't exist";
        }
        //Deleting a Fruit(key)
        h.erase("Mango")
```

```
    //Print all the elements
    for(auto p:h){
            cout<<p.first<<" and "<<p.second<<endl;
    }
    return 0;
}
```

Stack

Stack is a container which follows the LIFO (Last In First Out) order and the elements are inserted and deleted from one end of the container. The element which is inserted last will be extracted first.

```
stack <int> s;
```

Some of the Member Functions of Stack are :

- push( ) : Insert element at the top of stack. Its time complexity is O(1).
- pop( ) : Removes element from top of stack. Its time complexity is O(1).
- top( ) : Access the top element of stack. Its time complexity is O(1).
- empty( ) : Checks if the stack is empty or not. Its time complexity is O(1).
- size( ) : Returns the size of stack. Its time complexity is O(1).

Queue

Queue is a container which follows FIFO order (First In First Out) . Here elements are inserted at one end (rear) and extracted from another end(front).

```
queue <int> q;
```

Some member function of Queues are:

- push( ) : Inserts an element in queue at one end(rear). It's time complexity is O(1).
- pop( ) : Deletes an element from another end if queue(front). It's time complexity is O(1).
- front( ) : Access the element on the front end of queue. It's time complexity is O(1).
- empty( ) : Checks if the queue is empty or not. It's time complexity is O(1).
- size( ) : Returns the size of queue. Its time complexity is O(1).

Priority Queue

A priority queue is a container that provides constant time extraction of the largest element, at the expense of logarithmic insertion. It is similar to the heap in which we can add element at any time but only the maximum element can be retrieved. In a priority queue, an element with high priority is served before an element with low priority.

```
priority_queue<int> pq;
```

To make a min-priority queue, declare priority queue as:

```
#include <functional> //for greater <int>
//min priority queue
priority_queue < int, vector < int >,greater <int> > pq;
```

Some Member Functions of Priority Queues are :

- **empty()** : Returns true if the priority queue is empty and false if the priority queue has at least one element. Its time complexity is O(1).
- **pop()** : Removes the largest element from the priority queue. Its time complexity is O(logN) where N is the size of the priority queue.
- **push()** : Inserts a new element in the priority queue. Its time complexity is O(logN) where N is the size of the priority queue.
- **size()** : Returns the number of element in the priority queue. Its time complexity is O(1).
- **top()** : Returns a reference to the largest element in the priority queue. Its time complexity is O(1).

Example :

```
#include<iostream>
#include<queue>
#include<vector>
#include<functional>
#include<cstring>
using namespace std;
//To Compare Integers
class myComparison{
public:
  bool operator()(int a,int b){
    return a<b;
  }
};

class Person{
public:
  char name[20];
  int money;
```

```
Person(){
   name[0] = '\0';
   money = 0;
}
Person(char *n,int m){
   money =m;
   strcpy(name,n);
}
void print(){
   if(money>1000){
      cout<<name<<" is Rich"<<endl;
   }
```

DeQue

Double-ended queues are sequence containers with dynamic sizes that can be expanded or contracted on both ends (either its front or its back).

```
deque<int> first; // empty deque of integer
deque<int> second (4,100); // four ints with value 100
deque<int> third (second.begin(),second.end()); // iterating through second
deque<int> fourth (third); // a copy of third
```

Some Member Functions of Deque are:

- **assign()** : Assigns new contents to the deque container, replacing its current contents, and modifying its size Accordingly.
- **at(n)** : Returns a reference to the element at position n in the deque container object.
- **back()** : Returns a reference to the last element in the container.
- **begin()** : Returns an iterator pointing to the first element in the deque container.
- **empty()** : Returns whether the deque container is empty (i.e. whether its size is 0).
- **end()** : Returns an iterator referring to the past-the-end element in the deque container.
- **erase()** : Removes from the deque container either a single element (position) or a range of elements ([first,last)).
- **front()** : Returns a reference to the first element in the deque container.
- **pop_back()** : Removes the last element in the deque container, effectively reducing the container size by one.

- pop_front() : Removes the first element in the deque container, effectively reducing its size by one.
- push_back() : Adds a new element at the end of the deque container, after its current last element.
- push_front() : Inserts a new element at the beginning of the deque container, right before its current first element.
- size() : Returns the number of elements in the deque container.

Iterator

An iterator is any object that, points to some element in a range of elements (such as an array or a container) and has the ability to iterate through those elements using a set of operators (with at least the increment (++) and dereference (*) operators).

For Vector:

vector <int>::iterator it;

For List:

list <int>::iterator it;

etc....

TIME TO TALK ABOUT ALGORITHMS !

<algorithm>

The header <algorithm> defines a collection of functions especially designed to be used on ranges of elements.

binary_search(first,last,val)

Returns true if any element in the range [first,last) is equivalent to val, and false otherwise.

```
binary_search (v.begin(), v.end(), 3))
            //v is a vector
```

find(first,last,val)

Returns an iterator to the first element in the range [first,last) that compares equal to val. If no such element is found, the function returns last.

```
it = find (myvector.begin(), myvector.end(), 30); //it is an iterator
```

lower_bound(first,second,val)

Returns an iterator pointing to the first element in the range [first,last) which does not compare less than val.

```
it = lower_bound (v.begin(), v.end(), 20); //v is a vector
```

upper_bound(first,second,val)

Returns an iterator pointing to the first element in the range [first,last) which compares greater than val.

it = upper_bound (v.begin(), v.end(), 20); //v is a vector

max(a,b)

Returns the largest of a and b. If both are equivalent, a is returned.

cout << max(a,b);

min(a,b)

Returns the smallest of a and b. If both are equivalent, a is returned.

cout << min(a,b);

reverse(first,last)

Reverses the order of the elements in the range [first,last).

reverse(myvector.begin(),myvector.end());

rotate(first,middle,last)

Rotates the order of the elements in the range [first,last), in such a way that the element pointed by middle becomes the new first element.

rotate(myvector.begin(),myvector.begin()+3,myvector.end());

sort(first,last)

Sorts the elements in the range [first,last) into ascending order.

sort(v.begin(),v.end());

sort(a,a+n);

sort(v.begin(),v.end(),comparator);

sort(a,a+n,comparator);

swap(a,b)

Exchanges the values of a and b.

swap(a,b);

next_permutation(first,last)

Rearranges the elements in the range [first,last) into the next lexicographically greater permutation.

next_permutation(v.begin(),v.end());

SOME EXAMPLES

1. Sort Game, HackerBlocks

(#sorting, #vectors, #pairs)

Sanju needs your help! He gets a list of employees with their salary. The maximum salary is 100. Sanju is supposed to arrange the list in such a manner that the list is sorted in decreasing order of salary. And if two employees have the same salary, they should be arranged in lexicographical manner such that the list contains only names of those employees having salary greater than or equal to a given number x.

Help Sanju prepare the same!

Input : On the first line of the standard input, there is an integer x. The next line contains integer N, denoting the number of employees. N lines follow, which contain a string and an integer, denoting the name of the employee and his salary.

```
79
4
Eve 78
Bob 99
Suzy 86
Alice 86
```

Output :

```
Bob 99
Alice 86
Suzy 86
```

Solution :

Code :

```
#include<iostream>
#include<algorithm>
#include<cstring>
using namespace std;
bool myCompare(pair<string,int> p1,pair<string,int> p2){
  ///first = Name, second = Salary
  /// Preference Salary > Name
  if(p1.second==p2.second){
    return p1.first < p2.first;
}
return p1.second > p2.second;
}
void Sort(emp[]){
```

```
    for(int i=0;i<n;i++){
        if( myCompare(emp[i],emp[i+1])){
          swap(emp[i],emp[i+1]);
      }
    }
}
int main(){
    int min_salary,n;
    pair<string,int>  emp[100005];
    cin>>min_salary;
    cin>>n;
    string name;
    int salary;
    for(int i=0;i<n;i++){
       cin>>name>>salary ;
       emp[i].first = name;
       emp[i].second = salary;
    }
    sort(emp,emp+n,myCompare);
    ///Print
    for(int i=0;i<n;i++){
        if(emp[i].second>=min_salary){
        cout<<emp[i].first <<" "<<emp[i].second<<endl;
      }
    }
  return 0;
}
```

2. **ArraySub, Spoj**

(#sliding-window, #deque)

Given an array and an integer k, find the maximum for each and every contiguous subarray of size k

Input : The number n denoting number of elements in the array then after a new line we have the numbers of the array and then k in a new line

9

1 2 3 1 4 5 2 3 6

3

Output : 3 3 4 5 5 5 6

Solution :

Code :

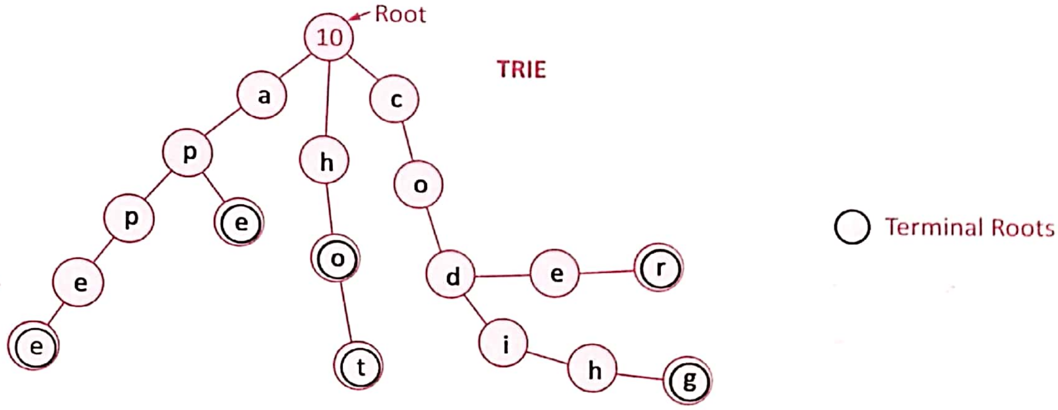
```
#include<iostream>
#include<cstdio>
#include<deque>
using namespace std;
int a[1000001];
int main(){
int n,k,i;
scanf("%d",&n);
for(i=0;i<n;i++)
  {
      scanf("%d",&a[i]);
  }
scanf("%d",&k);
deque<int> Q(k);
for(i=0;i<k;i++)
  {
   while(!Q.empty()&&a[i]>=a[Q.back()])
  Q.pop_back();
  Q.push_back(i);
 }
for(;i<n;i++)
{
 printf("%d ",a[Q.front()]);
  while((!Q.empty())&&(Q.front()<=i-k))
    Q.pop_front();
  while(!Q.empty()&&a[i]>=a[Q.back()])
    Q.pop_back();
Q.push_back(i);
}
printf("%d\n",a[Q.front()]);
return 0;
}
```

Some more Data Structures

Data Structures like Trie, Graphs are not directly available in STL but can be implemented easily using other data structures.

Trie Data Structure :

1. Trie is an information retrieval data structure.
2. It is also called radix/prefix tree.
3. It is used for efficient searching of keys in the container. If the keys are strings, the a particular string can be search in O(n) length where n denotes the length of the string to be searched.
4. Each node of the trie has multiple branches, a node where a word ends in marked with is Terminal = true.



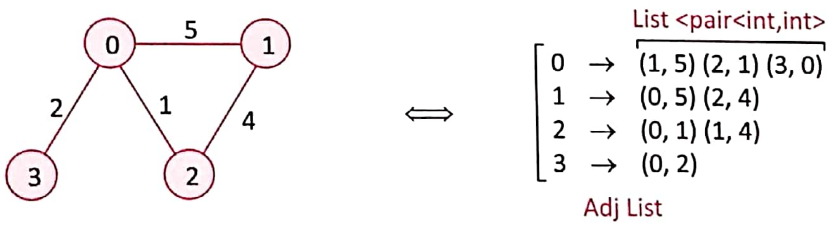

```
#include<iostream>
#include<unordered_map>
using namespace std;
#define hashmap unordered_map<char,node*>
class node{
public:
  char data;
  hashmap h;
  bool isTerminal;
  node(char d){
    data = d;
    isTerminal = false;
  }
};
class Trie{
```

```
    node*root;
  public:
    Trie(){
        root = new node('\0');
    }
    void addWord(char *word){
        node*temp = root;
        for(int i=0;word[i]!='\0';i++){
            char ch = word[i];
            if(temp->h.count(ch)==0){
                node* child = new node(ch);
                temp->h[ch] = child;
                temp = child;
            }
            else{
                temp = temp->h[ch];
            }
        }
        temp->isTerminal = true;
    }
    bool search(char *word){
        node*temp = root;
        for(int i=0;word[i]!='\0';i++){
            char ch = word[i];
            if(temp->h.count(ch)){
                temp = temp->h[ch];
            }
            else{
                return false;
            }
            cout<<temp->data<<" ";
        }
        return temp->isTerminal;
    }
```

```
};
int main(){
 char word[10][100] = {"apple","ape","coder","coding blocks","no"};
 Trie t;
 for(int i=0;i<5;i++){
   t.addWord(word[i]);
 }
 char searchWord[100];
 cin.getline(searchWord,100);
 if(t.search(searchWord)){
   cout<<searchWord<<" found "<<endl;
 }
 else{
   cout<<"not found !"<<endl;
 }
 return 0;
}
```

Graph Data Structure

An adjacency list implementation of graph can be easily represented using a vector. Example of DFS traversal on weighted graph.




```
#include <iostream>
#include <vector>
#include <algorithm>
using namespace std;
vector < pair < int,int > > graph[100005]; //graph ( using array of vectors >
int visited[100005]; //visited array
void dfs(int cur)
{    //dfs method
     if(visited[cur])
      return;
```

```
    cout<<cur<<" "; //printing current node
    visited[cur] = 1; //setting current node as visited
    for(int i=0;i<graph[cur].size();++i)
    {
        dfs(graph[cur][i].first);
    }
}
int main()
{
    int n,m;
    int a,b,w;
    cin>>n>>m;
    for(int i=0;i<m;++i)
    {
        cin>>a>>b>>w; //edge between a and b with weight w
        graph[a].push_back(make_pair(b,w));
        graph[b].push_back(make_pair(a,w));
    }
    cin>>a;
    dfs(a);
    return 0;
}
```

SELF STUDY NOTES

2 Mathematics

Birthday Paradox - Warmup Problem!

What is the minimum number of people that should be present in a room so that there's 50% chance of two people having same birthday ?

In a room of just __ people there's a 50-50 chance of two people having the same birthday. In a room of __ there's a 99.9% chance of two people matching.

HINT :

If there are two people in a room, Probability that two will have same birthday

= 1/365 = 0.00274 = 0.274%

Probability that two will have different birthdays = 1 – (probability that two have same birthday) = 1 – 0.00274 = 0.9973 = 99.73%. Now take your time and think about the approach.

SOLUTION:

23 for 50% probability

70 for 99.9% probability

Code:

```
#include<iostream>
using namespace std;
int main(){
        float p = 1;
        //p denotes prob of 2 ppl having different birthday
        // same bday = 1 - p
        float num = 365;
        float denom = 365;
        int people = 0;
        while(p>0.5){
                p *= (num/denom);
                num–;
                people++;
```

```
            cout<<"Probability is "<<p<<" and people are "<<people<<endl;
        }
        return 0;
}
```

Types of Problems in Mathematics

- Adhoc/Formula Based/Brute Force
- Big Integers
- Exponentiation
- Number Systems/Series
- Pigeonhole Principle
- Inclusion-Exclusion Principle
- Probability & Expectation
- Combinatorics

ADHOC/ BRUTE FORCE/ COMPLETE SEARCH

These are relatively simpler problems based upon some formula or complete search.

Let us see one example.

German Lotto :

In the German Lotto you have to select 6 numbers from the set {1,2,...,49}. A popular strategy to play Lotto - although it doesn't increase your chance of winning — is to select a subset *S* containing *k* (*k* > 6) of these 49 numbers, and then play several games with choosing numbers only from *S*.

Input :

For example, for *k* = 8 and *S* = *f*1; 2; 3; 5; 8; 13; 21; 34 there are 28 possible games: [1,2,3,5,8,13],

Output :

Your job is to write a program that reads in the number *k* and the set *S* and then prints all possible games choosing numbers only from *S*.

[1,2,3,5,8,21], [1,2,3,5,8,34], [1,2,3,5,13,21], ..., [3,5,8,13,21,34].

Solution :

Brute Force, Sort the array and use 6 Loops to pick all possible combinations 6 numbers.

Code :

```
#include <iostream>
using namespace std;
int main() {
        //Numbers from 1 to 49
    //Choose a subset of 6 Numbers
    int a[] = {1,2,4,5,6,7,8,10,12}; //assuming the array after sorting it
    int n = sizeof(a)/sizeof(int);
    for(int i=0;i<n-5;i++){
        for(int j=i+1;j<n-4;j++){
            for(int k=j+1;k<n-3;k++){
                for(int l=k+1;l<n-2;l++){
                    for(int m = l+1;m<n-1;m++){
                        for(int o= m+1;o<n;o++){
            cout<<a[i]<<",”<<a[j]<<",”<<a[k]<<",”<<a[l]<<",”<<a[m]<<",”<<a[o]<<endl;
                        }
                    }
                }
            }
        }
    }
    return 0;
}
```

Big Integers

Problems involving big integers are quite common in online competitions. In Java, Python it is easy to work with big integers but in C++ it's difficult because the *long long int* datatype can store only at max 18 digits.

So, for problems involving Big Numbers(containing 100's of digits) we either use ***Java Big Integer Class*** or Python or we use Arrays in C++ ! Let us see one example.

Note : There is a BOOST C++ Library which allows us to work with big integers as well.

Computing Large Factorials in C++

Code :

```
#include<iostream>
using namespace std;
void multiply(int *a,int &n,int no){
        int carry = 0;
        for(int i=0;i<n;i++){
                int product = a[i]*no + carry;
                a[i] = product%10;
                carry = product/10;
        }
        while(carry){
                a[n] = carry%10;
                carry = carry/10;
                n++;
        }
}
void big_factorial(int number){
        //Assuming max 1000 digits
        int *a = new int[1000]{0};
        a[0] = 1;
        int n = 1; //n denotes the array index
        for(int i=2;i<=number;i++){
                multiply(a,n,i);
        }
        for(int i=n-1;i>=0;i–){
                cout<<a[i];
        }
        cout<<endl;
}
int main(){
        big_factorial(100);
return 0;
}
```

The Java Big Integer Class

In Java, the Big Integer class is very powerful and supports lots of operations on big numbers (having 100's of digits) like :

1. Modular Arithmetic
2. Base Conversion
3. GCD Calculation
4. Power Calculation
5. Prime Generation
6. Bit-masking, Bitwise Operations
7. Other Miscellaneous Tasks

It is important to learn about this class, to make our work easy in Programming Contests

Examples :

Code :

```
import java.math.BigInteger;
import java.util.Scanner;
public class Main{
    static void playWithInt(){
        String s;
        Scanner sc = new Scanner(System.in);
        String s1 = sc.next();
        String s2 = sc.next();
        //The second parameter denotes the base
        BigInteger one = new BigInteger(s1,2);
        BigInteger two = new BigInteger(s2,2);
        System.out.println(one);
        System.out.println(two);
        //Number of Set Bits
        System.out.println(one.bitCount());
        //Number of total bits
        System.out.println(one.bitLength());
        //To add we use add()
        one = one.add(two);
        //To multiply
        one = one.multiply(two);
        System.out.println(one);
```

```
        //Computing Factorial
        //Computing GCD
        BigInteger b1 = new BigInteger("15");
        BigInteger b2 = new BigInteger("6");
        System.out.println(b1.gcd(b2));
        System.out.println(b1.add(b2));
        System.out.println(b1.multiply(b2));
        //Next probable prime - Generates the next available prime
        BigInteger b3 = new BigInteger("25");
        System.out.println(b3.nextProbablePrime());
        //Power Function
        BigInteger b4 = new BigInteger("3");
        System.out.println(b4.pow(5));
        //value of  - Int/Long Int to Big Integer
        BigInteger b5 = BigInteger.valueOf(100);
        System.out.println(b5);
        //Base Conversion, interprets 1001 in base 2
        BigInteger b6 = new BigInteger("1001",2);
        System.out.println(b6);
    }
    public static void main(String [] args){
        playWithInt();
    }
}
```

Factorial of Big Number in Java

Code :

```
import java.math.BigInteger;
import java.util.Scanner;
public class Main {
    static BigInteger fact(int N){
        BigInteger b = new BigInteger("1");
```

```
        for(int i=2;i<=N;i++){
            b = b.multiply(BigInteger.valueOf(i));
        }
        return b;
    }
    public static void main(String args[]) {
        int N = 100;
        System.out.println(fact(N));
    }
}
```

Python Code for Factorial :

```
def fact(n):
    ans = 1
    for i in range(1,n+1):
        ans = ans*i
    return ans
print fact(100)
```

Time to Try

Problem - *Klaudia and Natalia have 10 apples together, but Klaudia has two apples more than Natalia. How many apples does each of he girls have?*

Julka said without thinking: Klaudia has 6 apples and Natalia 4 apples. The teacher tried to check if Julka's answer wasn't accidental and repeated the riddle every time increasing the numbers. Every time Julka answered correctly. The surprised teacher wanted to continue questioning Julka, but with big numbers she couldn't solve the riddle fast enough herself. Help the teacher and write a program which will give her the right answers.

Problem Statement : http://www.spoj.com/problems/JULKA/

Solution : http://cb.lk/code/JULKA

Number Series and Sequences

- Most of the sequences are based upon some formula or some recurrence.
- The sequence may contain - AP, GP, HP, Polynomial Sequence, Linear Recurrence etc.
- Other commonly used sequences are - Fibonacci Sequence, Binomial Series, Catalan Numbers etc.

OEIS - Online Sequence Finder !!!

You can always refer https://oeis.org/ to find out any sequence and its formula.

So You only need to generate output for small inputs and then search for that sequences at OEIS (The Online Encyclopedia of Integer Sequences) I Try to search it for following sequences.

Try Searching these on OEIS

Example 1 : 1 , 2, 6, 24

Example 2 : 1, 2, 6, 10

Example 3 : 1, 2, 5, 14, 42, 132, 429

Binomial Coefficients

Binomial Coefficient ${}^nC_k$ denotes the number of ways of selecting k items from n items.

$$C(n,k) = n! / (n-r)!\, r!$$

Computing C(n,k) becomes difficult when n and k are large. We might prefer to use dynamic programming or Pascal's Triangle to Compute some are all values of C(n, k)

$$C(n, k) = C(n-1,k) + C(n-1,k-1)$$

Using this formula we can also build the Pascal's Triangle in a bottom up way

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
.
.

And so on, Binomial Coefficients are frequently used in problems involving Combinatorics.

Catalan Numbers (Very Important Series)

Let's us start with one example.

How many ways are there to construct a Binary Search Tree with 'n' nodes numbered from 1 to N ?

Hint :

Make every possible i^{th} node as the root node and recursively count for number of BST's it its left *half and right-half.* Do it for every value of i (1<=i<=n) and sum it up.

The formula generating after adding the series is the nth Catalan Number !!

It is defined using binomial coefficient notation $^{n}C_{k}$ as :

$\text{Cat}(n) = {}^{2n}C_{n}/(n+1)$

$\text{Cat}(0) = 1$

Using the above formula , the first few terms of series are :

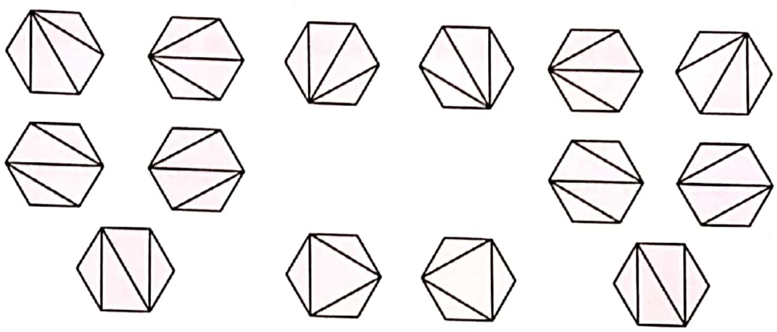
1, 1, 2, 5, 14, 42, 132, 429, 1430

Another Recursive Formula is :

$C_0 = 1$ and $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ for $n \geq 0$;

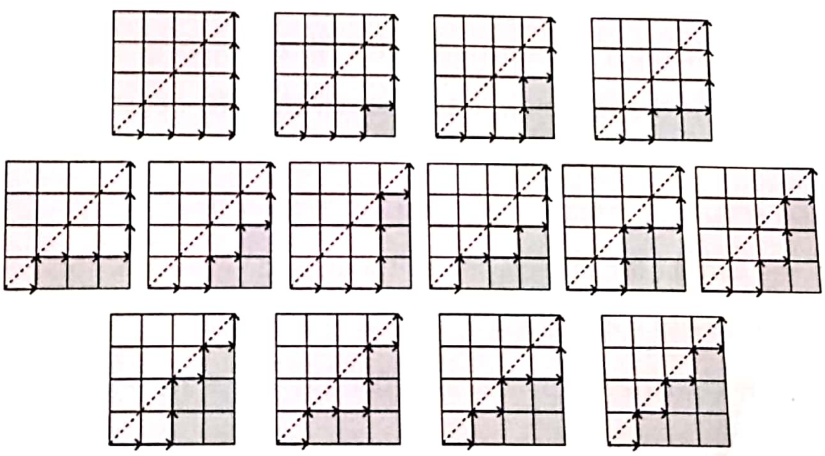
Applications of Catalan Numbers :

- Number of possible Binary Search Trees with n keys.
- Number of expressions containing n pairs of parentheses which are correctly matched. For n = 3, possible expressions are ((())), ()(()), ()()(), (())(), (()())
- Number of Ways n + 1 factors can be completely parenthesized, for e.g. N = 3 and 3 + 1 factors : {a, b, c, d}, we have: (ab)(cd), a(b(cd)), ((ab)c)d, (a(bc))(d) and a((bc)d).
- Number of ways a convex polygon of n+2 sides can split into triangles by connecting vertices.



- Number of different Unlabelled Binary Trees can be there with n nodes

  The number of paths with 2n steps on a rectangular grid from bottom left, i.e., (n – 1, 0) to top right (0, n-1) that do not cross above the main diagonal.



- Number of ways to form a "mountain ranges" with n upstrokes and n down-strokes that all stay above the original line. The mountain range interpretation is that the mountains will never go below the horizon

```
n = 0    *                                        1 way

n = 1    /\                                       1 way

n = 2          /\                                 2 ways
         /\/\, /  \

                                        /\
                                       /  \       5 ways
                          /\   /\/\   /    \
n = 3                /\  /  \, /    \, /      \
         /\/\/\, /\/  \, /  \/\, /
```

Mountain Ranges

Solving Linear Recurrences

The problem is generally asking you the n-th term of a linear recurrence. It is possible to solve with dynamic programming if n is small, problem arises when n is very large.

Linear Recurrence :

A linear recurrence relation is a function or a sequence such that each term is a linear combination of previous terms. Each term can be described as a function of the previous terms.

A famous example is the Fibonacci sequence: f(i) = f(i – 1) + f(i – 2). Linear means that the previous terms in the definition are only multiplied by a constant (possibly zero) and nothing else. So, this sequence: f(i) = f(i – 1) * f(i – 2) is not a linear recurrence.

Problem :

Given f, a function defined as a linear recurrence relation. Compute f(N). N may be very large.

How to Solve ?

Break the problem in four steps. Fibonacci sequence will be used as an example

Step 1 : Determine K, the number of terms on which f(i) depends

More precisely, K is the minimum integer such that f(i) doesn't depend on f(i – M), for al M > K.

For Fibonacci sequence, because the relation is: f(i) = f(i – 1) + f(i – 2), therefore, K = 2.

In this way, be careful for missing terms though, for example, this sequence:

f(i) = 2f(i – 2) + f(i – 4) has K = 4,

because it can be rewritten explicitly as: f(i) = 0f(i – 1) + 2f(i – 2) + 0f(i – 3) + 1f(i – 4).

Step 2 : Determine the F1 vector the initial values

If each term of a recurrence relation depends on K previous terms, then it must have the first K terms defined, otherwise the whole sequence is undefined. For Fibonacci sequence (K = 2), the well-known initial values are:

$f(1) = 1$

$f(2) = 1$

Note: We are indexing Fibonacci from 1, f(0) = 0.

$$F_1 = \begin{bmatrix} f(1) \\ f(2) \\ \vdots \\ f(k) \end{bmatrix}$$

We define a column vector Fi as a K x 1 matrix whose first row is f(i), second row is f(i + 1), and so on, until K-th row is f(i + K – 1). The initial values of f are given in column vector F1 that has values f(1) through f(K):

Step 3 : Determine T, the transformation matrix. Construct a K x K matrix T, called transformation matrix, such that

$$TF_i = F_i + 1$$

Suppose,

$$f(i) = c_1 f(i-1) + c_2 i(i-2) + c_3 f(i-3) + \ldots + c_k f(i-k)$$

$$f(i) = \sum_{i=1}^{k} c_j f(i-j)$$

Putting i = k + 1

$$f(k+1) = c_1 f(k) + c_2 f(k-1) + c_3 f(k-2) + \ldots + c_k f(1)$$

Hence, the transformation matrix is:

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_K & c_{K-1} & c_{K-2} & c_{K-3} & \cdots & c_1 \end{bmatrix}$$

Example For Fibonacci :

$C_1 = 1, C_2 = 1$

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Step 4: Determine F(n)

$$F_2 = TF_1$$
$$F_3 = TF_2 = T^2F_1$$
$$F_n = T^{n-1}F_1$$

Therefore, the original problem is now (almost) solved: compute FN as above, and then we can obtain f(N): it is exactly the first row of FN. In case of our Fibonacci sequence, the N-th term in Fibonacci sequence is the first row of:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{N-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Use Fast Exponentiation!

To compute T^N-1 use exponentiation by squaring method that works in O(log N) time, with this recurrence:

- $A^p = A$, if $p = 1$,
- $A^p = A * A^{p-1}$, if p is odd
- $A^p = X^2$, where $X = A^{p/2}$, otherwise.

Multiplying two matrices takes $O(K^3)$ time using standard method, so the overall time complexity to solve a linear recurrence is $O(K^3 \log N)$.

RECURSIVE SEQUENCE (SPOJ)

http://www.spoj.com/problems/SEQ/

Sequence (a_i) of natural numbers is defined as follows:

$$a_i = b_i \text{ (for } i \le k)$$
$$a_i = c_1a_{i-1} + c_2a_{i-2} + \dots + c_ka_{i-k} \text{ (for } i > k)$$

where b_j and c_j are given natural numbers for $1 \le j \le k$. Your task is to compute a_n for given n and output it modulo 10^9.

Solution:

Code :

```
#include <iostream>
#include <vector>
using namespace std;
#define ll long long
#define MOD 1000000000
```

```
ll k;
vector<ll> a,b,c;
//Multiply two matrices
vector<vector<ll> > multiply(vector<vector<ll> > A,vector<vector<ll> > B ){
    //third matrix mei result store
     vector<vector<ll> > C(k+1,vector<ll>(k+1));
    for(int i=1;i<=k;i++){
        for(int j=1;j<=k;j++){
            for(int x=1;x<=k;x++){
                C[i][j] = (C[i][j] + (A[i][x]*B[x][j])%MOD)%MOD;
            }
        }
    }
    return C;
}
vector<vector<ll> >  pow(vector<vector<ll> > A,ll p){
    //Base case
    if(p==1){
        return A;
    }
    //Rec Case
    if(p&1){
        return multiply(A, pow(A,p-1));
    }
    else{
        vector<vector<ll> > X = pow(A,p/2);
        return multiply(X,X);
    }
}
ll compute(ll n){
    //Base case
    if(n==0){
        return 0;
    }
```

```
//Suppose n<=k
if(n<=k){
    return b[n-1];
}
//Otherwise we use matrix exponentiation, indexing 1 se
vector<ll> F1(k+1);
for(int i=1;i<=k;i++){
    F1[i] = b[i-1];
}
//2. Transformation matrix
vector<vector<ll> > T(k+1,vector<ll>(k+1));
// Let init T
for(int i=1;i<=k;i++){
    for(int j=1;j<=k;j++){
        if(i<k){
            if(j==i+1){
                T[i][j] = 1;
            }
            else{
                T[i][j] = 0;
            }
            continue;
        }
        //Last Row - store the Coefficients in reverse order
        T[i][j] = c[k-j];
    }
}
// 3. T^n-1
T = pow(T,n-1);
// 4. multiply by F1
ll res = 0;
for(int i=1;i<=k;i++){
    res = (res + (T[1][i]*F1[i])%MOD)%MOD;
}
```

```
    return res;
}
int main() {
    ll t,n,num;
    cin>>t;
    while(t–){
        cin>>k;
        //Init Vector F1
        for(int i=0;i<k;i++){
            cin>>num;
            b.push_back(num);
        }
        //Coefficients
        for(int i=0;i<k;i++){
            cin>>num;
            c.push_back(num);
        }
        // the value of n
        cin>>n;
        cout<< compute(n)<<endl;
        b.clear();
        c.clear();
    }
    return 0;
}
```

Variation :

The recurrence relation may include a constant i.e., the function is of the form

$$f(i) - \sum_{j=1}^{k} c_j f(i-j) + d$$

In this variant, the F vector is enhanced to remember the value of d.

It is of size (K + 1) x 1 now :

$$F_i = \begin{bmatrix} f(i) \\ f(i+1) \\ \vdots \\ f(i+k-1) \\ d \end{bmatrix}$$

We now need to construct the T matrix, of size (K x 1)(K x 1) such that

$TF_i = F_{i+1}$

$$[T]\begin{bmatrix} f(i) \\ f(i+1) \\ \vdots \\ f(i+k-1) \\ d \end{bmatrix}\begin{bmatrix} f(i+1) \\ f(i+2) \\ \vdots \\ f(i+k) \\ d \end{bmatrix}$$

Hence, the transformation matrix is :

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_K & c_{K-1} & c_{K-2} & c_{K-3} & \cdots & c_1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

TIME TO TRY

1. Generate the Transformation matrix for the given sequence.

   $f(i)=2f(i-1)+3f(i-2)+5$

2. Generate the Transformation matrix for the given sequences and write code to compute nth term.

   $f(i)=f(i-1)+2i^2+3i+5$

   $f(i)=f(i-1)+2i^2+5$

3. **Fibonacci Number (HackerBlocks) :**

   Write an efficient code to compute nth Fibonacci Number where N<=10^18.

4. **Recursive Sequence - Version-II (Spoj) :**

   Read the problem statement at Spoj.

   http://www.spoj.com/problems/SPP/

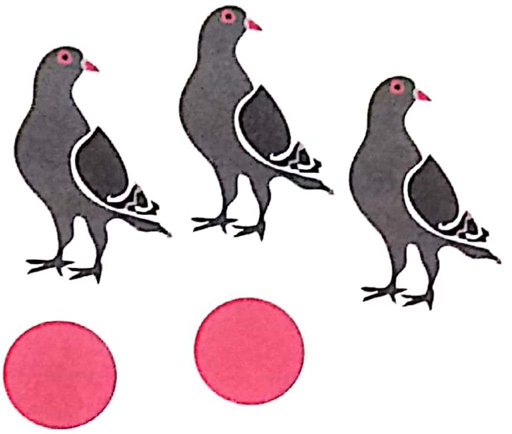
5. **Fast Ladders (HackerBlocks) :**

   Given a ladder of containing N steps, a person standing at the foot of the ladder can take at max a jump of K steps at every point. Find out the number of ways to reach the top of that Ladder.

6. **Fast Tiling Problem (HackerBlocks) :**

   Given n, and grid of size 4 x n, you have to find out the number of ways of filling the grid using 1 x 4 tiles.

Pigeonhole Principle



The pigeonhole principle is a fairly simple idea to grasp. Say that you have 7 pigeons and 6 pigeonholes. So, now you decide to start putting the pigeons one by one into each pigeonhole.

|p| |p| |p| |p| |p| |p| |p|

So, now, you have one pigeon left, and you can put it into any of the pigeonholes.

|pp| |p| |p| |p| |p| |p| |p|

The point is that when the number of pigeons > number of pigeonholes, there will be at least one pigeonhole with at least two pigeons.

Hair counting problem :

If the amount of hair is expressed in terms of the number of hair strands, the average human head has about 150,000 hair strands. It is safe to assume, then, that no human head has more than 1,000,000 strands of hair. Since the population of Delhi is more than 1,000,000 at least two people in Delhi have the same amount of hair.

EXAMPLES :

1. **Divisible Subset(Codechef) :**

   To find a non-empty subset of the given multiset with the sum of elements divisible by the size of original multiset.

   https://www.codechef.com/problems/DIVSUBS

How to use Pigeonhole Principle ?

$a \% N = x$

$b \% N = x$

Then, $(b - a) \% N = (b \% N - a \% N) = (x - x) = 0$

Let's denote $a_1 + a_2 + \dots + a_k$ by $\mathbf{b_k}$. So, we obtain:

$b_0 = 0$

$b_1 = a_1$

$b_2 = a_1 + a_2$

.

. . .

$b_n = a_1 + a_2 + a_3 + a_4 + \dots\dots + a_N$

So, $a_L + a_{L+1} + \dots\dots + a_R = b_R - b_{L-1}$

Therefore, if there are two values with equal residues modulo N among $b_0, b_1, \dots, b_n$ then we take the first one for L-1 and the second one for R and the required subsegment is found.

There are **N + 1** values of bi and **N** possible residues for N. So, according to the pigeonhole principle the required subsegment will always exist.

2. **The Gray Similar Code(Codechef) :**

Given 'N' 64 bit integers such that any two successive numbers differ at exactly '1' bit. We have to find out 4 integers such that their XOR is equal to 0.

https://www.codechef.com/problems/GRAYSC

Hint: If we take XOR of any two successive numbers, we will get a number with only 1 set bit and all others will be 0.

How to use Pigeonhole Principle ?

For N = 130, we have '65' pairs i.e. $\{X_1, X_2\}, \{X_3, X_4\}, \{X_5, X_6\} \dots \{X_{129}, X_{130}\}$. But there exists only 64 possible position for the set bit '1', by pigeonhole principle at least two bits will be set at same positions say $\{X_i, X_i + 1\}$ and $\{X_j, X_j + 1\}$. If we take x or of pair of these four numbers, we will get 0. Thus, by pigeonhole principle for all n >= 130, we will always find 4 integers such that their XOR is 0. For n < 130, we can iterate for 3 values of A[i], A[j], A[k] and do a binary search to find 4th number which is (A[i] ^ A[j] ^ A[k])

3. **Holiday Accommodation (Spoj) - Graph + Pigeonhole :**

Given a weighted tree, consider there are N people in N nodes. You have to rearrange these N people such that everyone is in a new node, and no node contains more than one person under the constraint that the distance travelled for each person must be maximized. There are N cities having N-1 highways connecting them.

http://www.spoj.com/problems/HOLI/

HINT: In order to maximize cost:

- All edges will be used to travel around.
- We need to maximize the use of every edge used. Once we know how many time each edge is used, we can calculate the answer.

How to apply Pigeonhole principle ?

Now for any edge E_i, we can partition the whole tree into two subtrees, if one side has n nodes, the other side will have N - n nodes. Also, note that, min(n, N-n) people will be crossing the edge from each side. Because if more people cross the edge, then by pigeon-hole principle in one side, we will get more people than available node which is not allowed in the problem statement. So, E_i will be used a total of 2 * min(n, N-n) times.

$$cost = \sum 2 * \min(n_i, N - n_i) * \text{weight}(E_i)$$

for every edge E_i

Code :

```
#include<bits/stdc++.h>
using namespace std;
class Graph{
        int V;
        list<pair<int,int> > *l;
public:
        Graph(int v){
                V = v;
                l = new list<pair<int,int> >[V];
        }
        void addEdge(int u,int v,int cost,bool bidir=true){
                l[u].push_back(make_pair(v,cost));
                if(bidir){
                        l[v].push_back(make_pair(u,cost));
                }
        }
        int dfsHelper(int node,bool *visited,int *count,int &ans){
                visited[node] = true;
                count[node] = 1;
                for(auto neighbour:l[node]){
                        int v = neighbour.first;
                        if(!visited[v]){
```

```
                    count[node] += dfsHelper(v,visited,count,ans);
                    ans += 2*min(count[v],V-count[v])*neighbour.second;
                }
            }
            return count[node];
        }
        int dfsMain(){
            bool *visited = new bool[V]{0};
            int *count = new int[V]{0};
            int ans = 0 ;
            dfsHelper(0,visited,count,ans);
            return ans;
        }
};
int main(){
    Graph g(4);
    g.addEdge(0,1,3);
    g.addEdge(1,2,2);
    g.addEdge(3,2,2);
    cout<<g.dfsMain();
}
```

TRY IT YOURSELF !

1. **Divisible Subarrays(HackerBlocks)**

   Find the number subarrays of the given multiset with the sum of elements divisible by the size of original multiset in linear time.

The Inclusion-Exclusion Principle

Every group of objects(or set) A can be associated with a quantity - denoted $|A|$ - called the number of elements in A or cardinality of A.

If $X = A \cup B$ and $A \cap B = \emptyset$, then $|X| = |A| + |B|$.

If A and B are not disjoint, we get the simplest form of the Inclusion-Exclusion Principle:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

In the more general case where there are n different sets Ai, the

EXAMPLES :

1. **Prime Looking Numbers - HackerBlocks :**

   How many number are there < 1000 such that they are Prime Looking

   i.e. composite but not divisible by 2,3 or 5 (Ex- 49, 77, 91). Given that there are 168 primes upto 1000.

   For any positive number N and m, the number of integers divisible by m which are less than N is floor((N-1)/m).

Solution :

divisible by 2 = floor(999/2) = 449 ; divisible by 3 = floor(999/3) = 333

divisible by 5 = floor(999/5) = 199 ; divisible by 2.3 = floor(999/6) = 166

divisible by 2.5 = floor(999/10) = 99

divisible by 3.5 = floor(999/15) = 66

divisible by 2.3.5 = floor(999/30) = 33

$|2 \cup 3 \cup 5| = |2| + |3| + |5| - |2 \cap 3| - |2 \cap 5| - |3 \cap 5| + |2 \cap 3 \cap 5|$

$= 499 + 333 + 199 - 166 - 99 - 66 + 33 = 733$

So, there exists 733 integers upto 1000 which have at-least 2,3 or 5 as divisor. This includes {2, 3, 5}.

Total number not having 2, 3 or 5 as divisor = 999 – 733 = 266.

Note that this set does not include 2, 3 or 5.

Since there are 168 prime numbers upto 1000, but we have already excluded 2, 3 and 5, number of prime looking numbers upto 1000 = 266 – 165 – 1(Since 1 is neither prime nor composite) = 100

The above generalised can be implemented using the following method :

Inclusion-Exclusion Using Bitmasks :

```
#include<iostream>
using namespace std;
int countBits(int n){
        int ans =0 ;
        while(n){
                n = n&(n-1);
                ans++;
        }
        return ans;
}
int main(){
        //Given a array of numbers of size k
```

```
    //We are finding the number of numbers which are divisible by 2,3 and 5
    int a[] = {2,3,5};
    int k = 3;
    int n = 999;
    int ans = 0;
    for(int i=1;i<8;i++){
            int mask = i;
            int bits = countBits(mask);
              int temp = 1;
              int pos = 0;
                while(mask>0){
                        int lastBit = (mask&1);
                        if(lastBit){
                                temp = temp*a[pos];
                        }
                        mask = mask>>1;
                        pos++;
                }
                cout<<endl;
                if(bits&1){
                        ans += n/temp;
                }
                else{
                        ans -= n/temp;
                }
    }
    cout<<ans<<endl;
return 0;
}
```

1. **Sereja & LCM - Codechef (Hard, Long Contest 9th Question) :**

   We have to find the possible number of arrays: A[1], A[2], A[3],...,A[N] such that A[i] >=1 nd A[i] <= M and LCM(A[1], A[2], ..., A[N]) is divisible by D. We have to find the sum of the ans rs with D = L, L+1, ...,R modulo $10^9 + 7$.

   A/Q we have to find the number of array whose LCM is a multiple of a given number(say 'x')

Using negation calculate the number of arrays whose LCM is not a multiple of x (say 'y').

Hence, ans = (possible array with m numbers) - y.

Note: The maximum value of the array elements can be 1000, the maximum number of distinct prime factors possible is 4 (2 * 3 * 5 * 7 * 11 > 1000).

Let the prime factors of x be p,q,r,s

x = (p^a) * (q^b) * (r^c) * (s^d)

p^a: P

q^b: Q

r^c: R

s^d: S

To calculate y :

None of element of array have any prime factor that x has OR it may have some of it missing.

So, calculate the number of arrays such that either (P or its multiple are not present) OR (Q or its multiple are not present) OR (R or its multiple are not present) OR (S or its multiple are not present).

y = |not(P) U not(Q) U not(R) U not(S)|

Applying Principle of Inclusion-Exclusion Principle :

A = power(m - m/P,n) + power(m - m/Q,n) + power(m - m/R,n) + power(m - m/S,n);

B = power(m - m/P - m/Q + m/(P*Q), n) + power(m - m/Q - m/R + m/(Q*R), n) . . .

C = power(m- m/P -m/Q - m/R + m/(P*Q) + m/(Q*R) + m/(P*R) - m/(P*Q*R),n) . . .

D = powmod(m - m/P - m/Q - m/R - m/S + m/(P*Q) + m/(Q*R) + m/(P*R) + m/(R*S) + m/(P*S) + m/(Q*S) - m/(P*Q*R)- m/(Q *R*S) - m/(P*Q*S) - m/(P*R*S) + m/(P*Q*R*S),n);

Final Answer will be:

y = A - B + C - D

ans = m ^ n - y

= m ^ n - A + B - C + D

Mathematical Expectation and Bernoulli Trial

Mathematically, for a discrete variable X with probability function P(X), the expected value E(X) is given by $\sum x_iP(x_i)$ the summation runs over all the distinct values xi that the variable can take.

For example, for a dice-throw experiment, the set of discrete outcomes is {1,2,3,4,5,6} and each of this outcome has the same probability 1/6. Hence, the expected value of this experiment will be 1/6*(1+2+3+4+5+6) = 21/6 = 3.5.

For a continuous variable X with probability density function P(x) , the expected value E(X) is given by $\int xP(x)dx$.

- Mathematical expectation is some sort of average value of your random variable.
- Expected value is not same as "most probable value" - rather, it need not even be one of the probable values. For example, in a dice-throw experiment, the expected value, viz 3.5 is not one of the possible outcomes at all.
- **Rather the most probable value is the value with max probability.**
- The rule of **"linearity of the expectation"** says that $E[ax_1 + bx_2] = aE[x_1] + bE[x_2]$.

Bernoulli Trial

In the theory of probability and statistics, a Bernoulli trial (or binomial trial) is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

1. **What is the expected number of coin flips for getting a head?**

   Let the expected number of coin flips be x. Then we can write an equation for it :

   (a) If the first flip is the head, then we are done. The probability of this event is 1/2 and the number of coin flips for this event is 1.

   (b) If the first flip is the tails, then we have wasted one flip. Since consecutive flips are independent events, the solution in this case can be recursively framed in terms of x - The probability of this event is 1/2 and the expected number of coins flips now onwards is x. But we have already wasted one flip, so the total number of flips is x + 1.

   The expected value x is the sum of the expected values of these two cases. Using the rule of linearity of the expectation and the definition of Expected value, we get

   $$x = (1/2)(1) + (1/2)(1+x)$$

   Solving, we get x = 2.

   Thus the expected number of coin flips for getting a head is 2.

2. **What is the expected number of coin flips for getting two consecutive heads?**

   Let the expected number of coin flips be x. The case analysis goes as follows:

   (a) If the first flip is a tails, then we have wasted one flip. The probability of this event is 1/2 and the total number of flips required is x + 1.

   (b) If the first flip is a heads and second flip is a tails, then we have wasted two flips. The probability of this event is 1/4 and the total number of flips required is x + 2.

   (c) If the first flip is a heads and second flip is also heads, then we are done. The probability of this event is 1/4 and the total number of flips required is 2.

Adding, the equation that we get is $x = (1/2)(x + 1) + (1/4)(x + 2) + (1/4)2$

Solving, we get $x = 6$.

Thus, the expected number of coin flips for getting two consecutive heads is 6.

3. **What is the expected number of coin flips for getting N consecutive heads, given N?**

Let the exepected number of coin flips be x. Based on previous exercises, we can wind up the whole case analysis in two basic parts

(a) If we get 1st, 2nd, 3rd,...,n'th tail as the first tail in the experiment, then we have to start all over again.

(b) Else we are done.

For the 1st flip as tail, the part of the equation is $(1/2)(x+1)$

For the 2nd flip as tail, the part of the equation is $(1/4)(x+2)$

...

For the k'th flip as tail, the part of the equation is $(1/(2^k))(x+k)$

...

For the N'th flip as tail, the part of the equation is $(1/(2^N))(x+N)$

The part of equation corresponding to case (b) is $(1/(2^N))(N)$

Adding, x = (1/2)(x+1) + (1/4)(x+2) + ... + (1/(2^k))(x+k) + .. + (1/(2^N))(x+N) + (1/(2^N))(N)

Solving this equation is left as an exercise to the reader. The entire equation can be very easily reduced to the following form: $x = 2^{N+1} - 2$

Thus, the expected number of coin flips for getting N consecutive heads is $(2^{N+1} - 2)$.

4. **Candidates are appearing for interview one after other. Probability of each candidate getting selected is 0.16. What is the expected number of candidates that you will need to interview to make sure that you select somebody?**

This is very similar to Q1, the only difference is that in this case the coin is biased. (The probability of heads is 0.16 and we are asked to find number of coin flips for getting a heads).

Let x be the expected number of candidates to be interviewed for a selection. The probability of first candidate getting selected is 0.16 and the total number of interviews done in this case is 1. The other case is that the first candidate gets rejected and we start all over again. The probability for that is $(1 - 0.16)*(x + 1)$. The equation thus becomes : $x = 0.16 + (1-0.16)*(x+1)$.

Solving, $x = 1/0.16$, i.e. $x = 6.25$

5. **(Generalized version of Q4) - The queen of a honey bee nest produces off-springs one-after-other till she produces a male offspring. The probability of producing a male offspring is p. What is the expected number of off-springs required to be produced to produce a male offspring?**

This is same as the previous question, except that the number 0.16 has been replaced by p. Observe that the equation now becomes : $x = p + (1 - p)*(x + 1)$

Solving, $x = 1/p$

Thus, observe that in the problems where there are two events, where one event is desirable and other is undesirable, and the probability of desirable event is p, then the expected number of trials done to get the desirable event is 1/p.

Generalizing on the number of events - If there are K events, where one event is desirable and all others are undesirable, and the probability of desirable event is p, then also the expected number of trials done to get the desirable event is 1/p.

The next question uses this generalization.

6. What is the expected number of dice throws required to get a "four"?

Let the expected number of throws be x. The desirable event (getting 'four') has probability 1/6 (as each face is equiprobable). There are 5 other undesirable events (K=5). Note that the value of the final answer does not depend on K. The answer is thus 1/(1/6) i.e. 6.

7. Candidates are appearing for interview one after other. Probability of k-th candidate getting selected is 1/(k+1). What is the expected number of candidates that you will need to interview to make sure that you select somebody?

The result will be the sum of infinite number of cases.

Case 1: First candidate gets selected. The probability of this event is 1/2 and the number of interviews is 1.

Case 2: Second candidate gets selected. The probability of this event is 1/6 (= 1/2 of first candidate not getting selected and 1/3 of second candidate getting selected, multiplied together gives 1/6) and the number of interviews is 2.

Case 3: Third candidate gets selected. The probability of this event is 1/2 * 2/3 * 1/4 = 1/12 (= first not getting selected and second not getting selected and third getting selected) and the number of interviews is 3.

...

Case k: k'th candidate gets selected. The probability of this event is 1/2 * 2/3 * 3/4 * ... * (k-1)/k * 1/(k+1). (The first k-1 candidates get rejected and the k'th candidate is selected). This evaluates to 1/(k*(k+1)) and the number of interviews is k.

...

[Note that similar to problem 4, here we can't just say - if the first candidate is rejected, then we will start the whole process again. This is not correct, because the probabilty of each candidate depends on it's sequence number. Hence sub-experiments are not same as the parent experiment. This means that all the cases must be explicitly considered.]

The resultant expression will be :

$x = 1/(1*2) + 2/(2*3) + 3/(3*4) + 4/(4*5) + \dots + k/(k*(k+1)) + \dots$

$= 1/2 + 1/3 + 1/4 + \dots$

This is a well-known divergent series, which means that sum doesnot converge, and hence the expectation doesnot exist.

8. **A random permutation P of [1...n] needs to be sorted in ascending order. To do this, at every step you will randomly choose a pair (i,j) where i < j but P[i] > P[j], and swap P[i] with P[j]. What is the expected number of swaps needed to sort permutation in ascending order. (Idea: Topcoder)**

   This is a programming question, and the idea is simple - since each swap has same probability of getting selected, the total number of expected swaps for a permutation P------- is

   $$E[P] = (1/cnt) * \sum (E[P_s] + 1)$$

   where cnt is the total number of swaps possible in permutation P, and P_s --------is the permutation generated by doing swap 's'. Since all swaps are equiprobable, we simply sum up the expected values of the resultant permutations (of course add 1 to each to account for the swap done already) and divide the result by the total number of permutations. The base case will be for the array that has been already sorted - and the expected number of permutations for a sorted array is 0.

9. **A fair coin flip experiment is carried out N times. What is the expected number of heads?**

   Consider an experiment of flipping a fair coin N times and let the outcomes be represented by the array $Z = \{a_1, a_2, \dots, a_n\}$ where each a_i is either 1 or 0 depending on whether the outcome was heads or tails respectively. In other words, for each i we have ai = if the i'th experiment gave head then 1 else 0.

   Hence we have: Number of heads in $z = a_1 + a_2 + \dots + a_n$

   Hence E[number of heads in z] $= E[a_1 + a_2 + \dots + a_n] = E[a_1] + E[a_2] + \dots + E[a_n]$

   Since ai corresponds to a coin-toss experiment, the value of E[ai] is 0.5 for each i. Adding this n times, the expected number of heads in Z comes out to be n/2.

10. **(Bernaulli Trials) n students are asked to choose a number from 1 to 100 inclusive. What is the expected number of students that would choose a single digit number?**

    This question is based on the concept of **bernaulli trials**. An experiment is called a bernaulli trial if it has exactly two outcomes, one of which is desired. For example - flipping a coin, selecting a number from 1 to 100 to get a prime, rolling a dice to get 4 etc. The result of a bernaulli trial can typically be represented as "yes/no" or "success/failure". We have seen in Q5 above that if the probability of success of a bernaulli trial is p then the expected number of trials to get a success is 1/p. is

This question is based on yet another result related to bernaulli trials - If the probability of a success in a bernaulli trial is p then the expected number of successes in n trials is n*p. The proof is simple :

The number of successes in n trials = (if 1st trial is success then 1 else 0) + ... + (if nth trial is success then 1 else 0)

The expected value of each bracket is 1*p + 0*(1-p) = p. Thus the expected number of successes in n trials is n*p.

In the current case, "success" is defined as the experiment that chooses a single digit number. Since all choices are equiprobable, the probability of success is 9/100. (There are 9 single digit numbers in 1 to 100). Since there are n students, the expected number of students that would contribute to success (i.e the expected number of successes) is n*9/100.

11. What is the expected number of coin flips to ensure that there are atleast N heads?

The solution can easily be framed in a recursive manner :

N heads = if 1st flip is a head then N-1 more heads, else N more heads.

The probability of 1st head is 1/2. Thus E[N] = (1/2)(E[N-1]+1)+ (1/2)(E[N] + 1)

Note that each term has 1 added to it to account for the first flip.

The base case is when N = 1 : E[1] = 2 (As discussed in Q2)

Simplifying the recursive case, E[N] = (1/2)(E[N-1] +1 + E[N] + 1) = (1/2)(E[N-1] + E[N] + 2)

⇒ 2 * E[N] = (E[N-1] + E[N] + 2) ⇒ E[N] = E[N-1] + 2

Since E[1] = 2, E[2] = 4, E[3] = 6,..., in general E[N] = 2N. Thus, the expected number of coin flips to ensure that there are atleast N heads in 2N.

The next problem discusses a generalization :

12. What is the expected number of bernaulli trials to ensure that there are at least N successes, if the probability of each success is p?

The recursive equation in this case is E[N] = p(E[N – 1] + 1)+ (1 – p)(E[N] + 1)

Solving, E[N] – E[N – 1] = p. Writing a total of N-1 equations:

E[N]-E[N-1] = 1/p

E[N-1]-E[N-2] = 1/p

E[N-2]-E[N-3] = 1/p

...

E[2]-E[1] = 1/p

Adding them all, E[N] – E[1] = (n – 1)/p. But E[1] is 1/p (lemma -1). Hence E[N] = n/p.

Moral: If probability of success in a Bernoulli trial is p, then the expected number of trials to guarantee N successes is N/p.

This completes the discussion on problems on Mathematical Expectation.

Reference : Codechef

TRY IT YOURSELF !

1. A game involves you choosing one number (between 1 to 6 inclusive) and then throwing three fair dice simultaneously. If none of the dice shows up the number that you have chosen, you lose \$1. If exactly one, two or three dice show up the number that you have chosen, you win \$1, \$3 or \$5 respectively. What is your expected gain?
2. There are 10 flowers in a garden, exactly one of which is poisonous. A dog starts eating all these flowers one by one at random. whenever he eats the poisonous flower he will die. What is the expected number of flowers he will eat before he will die?
3. A bag contains 64 balls of eight different colours, with eight of each colour. What is the expected number of balls you would have to pick (without looking) to select three balls of the same colour?
4. In a game of fair dice throw, what is the expected number of throws to make sure that all 6 outcomes appear atleast once?
5. What is the expected number of bernaulli trials for getting N consecutive successes, given N, if the probability of each success is p?

Coupon Collector Problem

Problem Statement: A certain brand of cereal always distributes a coupon in every cereal box. The coupon chosen for each box is chosen randomly from a set of 'n' distinct coupons. A coupon collector wishes to collect all 'n' distinct coupons. What is the expected number of cereal boxes must the coupon collector buy so that the coupon collector collects all 'n' distinct coupons?

Solution:

Let random variable Xi be the **number of boxes it takes for the coupon collector to collect the i-th new coupon after the i-1th coupon has already been collected.** (Note: this does NOT mean assign numbers to coupons and then collect the i-th coupon. Instead, this means that after Xi boxes, the coupon collector would have collected i distinct coupons, but with only Xi-1 boxes, the coupon

collector would have only collected i-1distinct coupons.)

Clearly $E(X_1)=1$, because the coupon collector starts off with no coupons. Now consider the i-th coupon. After the i-1-th coupon has been collected, then there are n-(i-1) possible coupons that could be the new i-th coupon. Each trial of buying another cereal box, "success" is getting any of the n – (i – 1) uncollected coupons, and "failure" is getting any of the already collected i-1 coupons. From this point of view, we see that p = (n-(i-1) /n.

This is a bernaulli trial with probability of success p and failure (1-p). In bernaulli trial, the expected number of trials for i-th success is 1/p i.e. 1/(success of the i-th outcome).

$$E(X_i) = 1/p = n/n - (i - 1).$$

To compute the number of cereal boxes X, required by the coupon collector to collect all n distinct coupons:

$$E(X) = E(X_1 + X_2 + X_3 + X_4 + \dots\dots\dots + X_n)$$

$$E(X) = E(X_1) + E(X_2) + \dots\dots + E(X_n)$$

$$E(X) = n(1 + 1/2 + 1/3 + 1/4 + \dots. + 1/n)$$

EXAMPLE

Favorite Dice (Spoj)

What is the expected number of throws of N sided dice so that each number is rolled at least once?

Statement - http://www.spoj.com/problems/FAVDICE

Code :

```
#include <iostream>
#include<iomanip>
using namespace std;
int main() {
    int t;
    int n;
    cin>>t;
    while(t-){
        cin>>n;
        double ans = 0;
        for(int i=1;i<=n;i++){
            ans += n/(i*1.0);
        }
        cout<<fixed<<setprecision(6)<<ans<<endl;
    }
    return 0;
}
```

TIME TO TRY

Fibonacci Sum (Spoj)

Given two non-negative integers N and M, you have to calculate the sum (F(N) + F(N + 1) + ... + F(M)) mod 1000000007 where F(N) denotes the nth Fibonacci Number.

http://www.spoj.com/problems/FIBOSUM/ (Matrix Exponentiation)

Modulo Sum (Codeforces) :

You are given a sequence of numbers $a_1, a_2, \ldots, a_n$, and a number m.

Check if it is possible to choose a non-empty subsequence a_{ij} such that the sum of numbers in this subsequence is divisible by m.

http://codeforces.com/contest/577/problem/B

Tavas and SaDDas (Codeforces) :

You are given a lucky number *n*. Lucky numbers are the positive integers whose decimal representations contain only the lucky digits 4 and 7. For example, numbers 47, 744, 4 are lucky and 5, 17, 467 are not.

If we sort all lucky numbers in increasing order, what's the 1-based index of *n*?

http://codeforces.com/problemset/problem/535/B (Maths, Counting)

Count the Binary Trees (HackerBlocks) :

Given n, you have to find the number of possible binary trees that can be made using N nodes.

Summing Sums (Spoj) :

Refer Spoj for the problem statement.

http://www.spoj.com/problems/SUMSUMS/ (Mathematics)

Marbles (Spoj) :

Refer Spoj for the problem statement.

http://www.spoj.com/problems/MARBLES/ (Maths)

SELF STUDY NOTES

3 Number Theory

In this chapter we are going to talk about important mathematical concepts, theorems and tricks. Let's get started.

Greatest Common Divisor(GCD) using Euclid's Algorithm

The Greatest Common Divisor (GCD) of two integers (a, b) denoted by gcd(a,b), is defined as the largest positive integer d such that d | a and d | b where x | y implies that x divides y.

Example of GCD: gcd(4, 8) = 4, gcd(10, 5) = 5, gcd(20,12) = 4.

```
Gcd(A,B) = Gcd(B,A%B) // recurrence for gcd
Gcd(A,0) = A // base case
```

Proof : If $a = bq + r$.

Let d be any common divisor of a and b which implies $d|a$ and $d|b \Rightarrow d|(a - bq) \Rightarrow d|r$.

Let e be any common divisor of b and $r \Rightarrow e|b$, $e|r \Rightarrow e|(bq + r) \Rightarrow e|a$. hence any common divisor of a and b must also be a common divisor of r and any common divisor of b and r must also be a divisor of a $\Rightarrow$ d is a common divisor of a and b iff d is a common divisor of b and r.

Similarly, the LCM of two integers (a, b) denoted by lcm(a,b), is defined as the smallest positive integer l such that a |l and b | l. Example of LCM : lcm(4,8) = 8, lcm(10,5) = 10, lcm(20,12) = 60.

gcd(a,b) * lcm(a,b) = a*b

```
int gcd(int a, int b) {
  return (b == 0 ? a : gcd(b, a % b));
  }
int lcm(int a, int b) {
  return (a * (b / gcd(a, b)));
} // divide before multiply!
```

The GCD of more than 2 numbers, e.g. gcd(a,b,c) is equal to gcd(a,gcd(b,c)), etc, and similarly for LCM

Both GCD and LCM algorithms run in $O(log(N))$, where n = max(a,b).

Extended Euclid's Algorithm

Extended Euclid's is used to find out the solution of equations of the form Ax + By = C , where C is a multiple of divisor of A and B. Extended Euclid's works in the same manner as the euclid's algorithm. Ax + By = 1 (we will find solutions of this equation let them be x' and y' given that gcd(a,b) is 1 then the solutions of equation Ax + By = k where k is a multiple of gcd(A,B) are given by k*x' and k*y'.

$Ax + By = 1$ ....(1)

$Bx' + (A\%B)y' = 1$ ....(2) // using euclid's algo gcd(a,b) = gcd(b,a%b)

Compare coefficients of (1) and (2)

$x = y'$

$y = x' - [A/B]y'$

Hence we can recursively calculate x and y in the following manner:

```
void eeuclid(ll a, ll b){
    if(b==0){
        ex=1;ey=0;ed=a;
    }
    else{
        eeuclid(b,a%b);
        ll temp=ex;
        ex=ey;
        ey=temp-(a/b)*ey;
    }
}
```

Application of Extended Euclidean Algorithm :

1. **To calculate multiplicative modulo inverse of a w.r.t. m.**

Let's see :

$$x \equiv a^{-1} (\text{mod } m)$$

$$a \,.\, a^{-1} \equiv a \,.\, x (\text{mod } m)$$

$$x \,.\, a \equiv 1 (\text{mod } m)$$

$$\Rightarrow \quad ax - 1 = qm$$

$$\Rightarrow \quad ax - qm = 1$$

This equation has solutions only if a and m are co-prime that is gcd(a,m) = 1.

We can calculate x and q using extended euclid's algorithm where x is the inverse of a modulo m.

Sieve of Eratosthenes

It is easy to find if some number (say N) is prime or not – you simply need to check if at least one number from numbers lower or equal sqrt(n) is divisor of N. This can be achieved by simple code:

```
boolean isPrime( int n )
{
 if ( n == 1 )
   return false; // by definition, 1 is not prime number
 if ( n == 2 )
   return true; // the only one even prime
 for ( int i = 2; i * i <= n; ++i )
   if ( n%i == 0 )
     return false;
 return true;
}
```

So it takes sqrt(n) steps to check this. Of course you do not need to check all even numbers, so it can be "optimized" a bit:

```
boolean isPrime( int n )
{
  if ( n == 1 )
    return false; // by definition, 1 is not prime number
  if ( n == 2 )
    return true; // the only one even prime
  if ( n%2 == 0 )
    return false; // check if is even
  for ( int i = 3; i * i <= n; i += 2 ) // for each odd number
    if ( n%i == 0 )
      return false;
  return true;
}
```

So let say that it takes 0.5sqrt(n) steps*. That means it takes 50,000 steps to check that 10,000,000,000 is a prime.

Time Complexiety :

If we have to check numbers upto N, we have to check each number individually. So time complexity will be *O*(*Nsqrt*(*N*)).

Can we do better?

Ofcourse! we can use a sieve of numbers upto N. For all prime numbers <= sqrt(N), we can make their multiple non-prime i.e. if p is prime, 2p, 3p, ..., floor(n/p)*p will be non-prime.

Sieve code :

```
void primes(int *p)
{
    for(int i = 2;i<=1000000;i++)
      p[i] = 1;
    for(int i = 2;i<=1000000;i++)
    {
        if(p[i])
        {
            for(int j = 2*i;j<=1000000;j+=i)
            {
              p[j] = 0;
            }
        }
    }
    p[1] = 0;
    p[0] = 0;
    return;
}
```

Can we still do better?

Yeah sure! Here we don't need to check for even numbers. Instead of starting the non-prime loop from 2p we can start from p^2.

Optimised code :

```
void primes(bool *p)
{
    for(int i = 3;i<=1000000;i += 2)
    {
        if(p[i])
        {
            for(int j = i*i;j <= 1000000; j += i)
            {
              p[j] = 0;
```

```
            }
        }
    }
    p[1] = 0;
    p[0] = 0;
    return;
}
T = O(NloglogN)
```

Hence, we have significantly reduced our complexity from N*sqrt(N) to approx linear time.

Optimizations!

we know that all the numbers which are even are non prime except 2.Hence we can mark only odd numbers as non prime in our sieve and jump to odd numbers always.

Code :

```
#define lim 10000000
vector <bool> mark(lim+2,1);
vector <ll> primes;
void sieve() // we need primes upto 10^8
{
        //ll times = 0;
        for(ll i=3;i<=lim;i+=2)
        {
                //times++;
                if(mark[i] == 1)
                {
                for(ll j=i*i ; j <= lim ;j += 2*i) //skip to odd numbers as i*i is odd
                        {
                                mark[j] = 0;
                        }
                }
        }

        primes.pb(2);
        for(ll i=3;i<=lim;i+=2)
        {
                if(mark[i])
                        primes.pb(i);
        }
}
```

Factorization of a Number using this Sieve :

```
vector <ll> factorize(ll m)
{
    vector <ll> factors;
    factors.clear();
    ll i = 0;
    ll p = primes[i];
    while(p*p <= m)
    {
        if(m%p == 0)
        {
            factors.pb(p);
            while(m%p == 0)
                m = m/p;
        }
        i++;
        p = primes[i];
    }
    if(m!=1)
        factors.pb(m);
    return factors;
}
```

Segmented Sieve

We use this sieve when array of size N does not fit in memory and we want to compute prime numbers between a range *l* and *r*. Example : $l = 10^8$, $r = 10^9$.

```
void sieve()
{
 for(int i = 0;i<=1000000;i++)
   p[i] = 1;
 for(int i = 2;i<=1000000;i++)
 {
   if(p[i])
   {
     for(int j = 2*i;j<=1000000;j+=i)
       p[j] = 0;
```

```
    }
  }
// for(int i=2;i<=20;i++)cout<<i<<" "<<p[i]<<endl;
}
int segmented_sieve(long long a,long long b)
{
 sieve();
 bool pp[b-a+1];
 memset(pp,1,sizeof(pp));
 for(long long i = 2;i*i<=b;i++)
 {
   for(long long j = a;j<=b;j++)
   {
    if(p[i])
    {
     if(j == i)
      continue;
     if(j % i == 0)
       pp[j-a] = 0;
    }
   }
  }
  int res = 1;
  for(long long i = a;i<b;i++)
     res += pp[i-a];
 return res;
}
```

Division

Let a and b be integers. We say a divides b, denoted by a|b, if there exists an integer c such that b= ac.

Linear Diophantine Equations

A Diophantine equation is a polynomial equation, usually in two or more unknowns, such that only the integral solutions are required. An Integral solution is a solution such that all the unknown variables take only integer values. Given three integers a, b, c representing a linear equation of the form : ax + by = c. Determine if the equation has a solution such that x and y are both integral values.

General solution (Infinitely many solutions)

(x, y) = (xo + b/d *t, yo — a/d *t)

We can use *Extended Euclidean Method* above to find the xo,yo.

Chinese Remainder Theorem

Typical problems of the form "Find a number which when divided by 2 leaves remainder 1, when divided by 3 leaves remainder 2, when divided by 7 leaves remainder 5" etc can be reformulated into a system of linear congruences and then can be solved using Chinese Remainder theorem.

For example, the above problem can be expressed as a system of three linear congruences:

$$x \equiv 1 \text{ (mod 2)}, \quad x \equiv 2 \text{ mod(3)}, \quad x \equiv 5 \text{ mod (7)}.$$

x % num[0] = rem[0],

x % num[1] = rem[1],

....

x % num[k-1] = rem[k-1]

A Naive Approach is to find x is to start with 1 and one by one increment it and check if dividing it with given elements in num[] produces corresponding remainders in rem[]. Once we find such a x, we return it

Chinese remainder theorem

$$x = \sum_{0 \le i \le n-1} (rem[i]*pp[i]*inv[i])) \% prod$$

```
rem[i] is given array of remainders
prod is product of all given numbers
prod = num[0] * num[1] * ... * num[k-1]

pp[i] is product of all but num[i]
pp[i] = prod / num[i]
inv[i] = Modular Multiplicative Inverse of
pp[i] with respect to num[i]
```

Code :

```
ll chinese_remainder_theorem(vector <ll> num,vector <ll> rem)
{
    // find pp vector
    vector <ll> pp; // product of all num array except num[i]
    pp.clear();
```

```
    ll prod = 1ll;
    for(ll i=0;i<num.size();++i)
        prod *= num[i];
    for(ll i=0;i<num.size();++i)
        pp.pb(prod/num[i]);
    // find inv[] vector
    // inv[i] is modular inverse of pp[i] with respect to num[i]
    vector <ll> inv;
    inv.clear();
    for(ll i=0;i<pp.size();++i)
        inv.pb(modular_inverse(pp[i],num[i]-2,num[i]));
    // (a^-1)%m when m is prime is (a^(m-2))%m using fermat's
    // now use the sum formula
    ll ans = 0ll;
    for(ll i=0;i<pp.size();++i)
    {
        ans = ans%prod + ( ((rem[i]*pp[i])%prod)*(inv[i])%prod )%prod;
        ans %= prod;
    }
    return ans;
}
```

Euler Phi Function

Euler's Phi function (also known as totient function, denoted by φ) is a function on natural number that gives the count of positive integers coprime with the corresponding natural number.

Thus , φ(8) = 4, φ(9) = 6

The value φ(n) can be obtained by Euler's formula :

Let $n = p_1^{a_1} * p_2^{a_2} * \dots \dots \dots * p_k^{a_k}$ be the prime factorization of n. Then

$$\varphi(n) = n * \left(1 - \frac{1}{p_1}\right) * \left(1 - \frac{1}{p_2}\right) * \dots * \left(1 - \frac{1}{p_k}\right)$$

Code :

```
int phi[] = new int[n+1];
for(int i=2; i <= n; i++)
 phi[i] = i; //phi[1] is 0
```

```
for(int i=2; i <= n; i++)
  if( phi[i] == i )
    for(int j=i; j <= n; j += i )
      phi[j] = (phi[j]/i)*(i-1);
```

Properties :

1. If P is prime then $\varphi(p^k) = (p-1)\, p^{(k-1)}$.
2. φ function is multiplicative, i.e. if (a,b) = 1 then $\varphi(ab) = \varphi(a)\varphi(b)$.
3. Let $d_1, d_2, \ldots d_k$ be all divisors of n (including n). Then $\varphi(d1) + \varphi(d2) + \ldots + \varphi(dk) = n$

   For Example: The divisors of 18 are 1,2,3,6,9 and 18.

   Observe that $\varphi(1) + \varphi(2) + \varphi(3) + \varphi(6) + \varphi(9) + \varphi(18) = 1 + 1 + 2 + 2 + 6 + 6 = 18$

4. Number of divisors of $n = p_1^{a_1} * p_2^{a_2} * \ldots\ldots * p_n^{a_n}$:

   $$d(n) = (a_1 + 1) * (a_2 + 1) * \ldots\ldots * (a_n + 1)$$

5. Sum of divisors:

   $$S(n) = \frac{p_1^{a_1 - 1}}{p_1 - 1} * \frac{p_2^{a_2 - 1}}{p_2 - 1} * \ldots * \frac{p_n^{a_n - 1}}{p_n - 1}$$

Wilson's Theorem

If p is a prime, then (p — 1)! = –1 (mod p)

Problem : DCEPC11B (SPOJ)

Hint :

This can be solved using Wisdom theorem

1. If n > = p ans would be 0
2. Else we have to use Wilson's theorem

   $$(p-1)! \equiv -1 (\text{mod } p)$$

   $1*2*3* \ldots\ldots *(n-1)*(n)* \ldots\ldots *(p-1) \equiv -1 \ (\text{mod } p)$

   $n!\,(n+1)* \ldots\ldots *(p-1) \equiv -1 \ (\text{mod } p)$

   $n! \equiv -1*[(n+1)* \ldots\ldots (p-2)*(p-1)^{\wedge} -1 \ (\text{mod } p)$

Lucas Theorem

In number theory, Lucas's theorem expresses the remainder of division of the binomial coefficient ^mC_n by a prime number p in terms of base p expansions of integers m and n.

Formulation :

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod p,$$

where $m = m_k p^k + m_{k-1} p^{k-1} + \ldots + m_1 p + m_0$ and $n = n_k p^k + n_{k-1} p^{k-1} + \ldots + n_1 p + n_0$

Problem : Compute ^nC_r % p.

Given three numbers n, r and p, compute the above value of ^nC_r % p.

Using Lucas Theorem ^nC_r % p

Lucas theorem basically suggests that the value of ^nC_r can be computed by multiplying results of $^{n_i}C_{r_i}$ where n_i and r_i are individually same-positioned digits in base p representations of n and r respectively.

The idea is to one by one compute $^{n_i}C_{r_i}$ for individual digits n_i and r_i in base p.

Code :

```
#include<bits/stdc++.h>
using namespace std;
int Cal_nCr_mod_p(int n, int r, int p)
{


    int C[r+1];
    memset(C, 0, sizeof(C));
    C[0] = 1;
    for (int i = 1; i <= n; i++)
    {
        for (int j = min(i, r); j > 0; j-)
                C[j] = (C[j] + C[j-1])%p;
    }
    return C[r];
}
```

```
int LucasApproach(int n, int r, int p)
{
    if(r==0)
    return 1;
    else
    {
        int n_i = n%p, r_i = r%p;
         int result = (LucasApproach(n/p, r/p, p)*Cal_nCr_mod_p(n_i,r_i,p))%p;
        return result;
    }
}
int main()
{
    int n,r,p;
    cin>>n>>r>>p;
    int result = LucasApproach(n,r,p);
    cout<<"nCr mod p is "<<result;
}
```

Fermat's little Theorem

Fermat's little theorem states that if *p* is a prime number, then for any integer *a*, the number $a^p - a$ is an integer multiple of *p*. In the notation of modular arithmetic, this is expressed as

$$a^p \equiv a \pmod{p}.$$

For example, if $a = 2$ and $p = 7$, $2^7 = 128$, and 128 " 2 = 7 × 18 is an integer multiple of 7.

If *a* is not divisible by *p*, Fermat's little theorem is equivalent to the statement that $a^{p-1} - 1$ is an integer multiple of *p*, or in symbols

$$a^{p-1} = 1 \pmod{p}$$

For example, if $a = 2$ and $p = 7$ then $2^6 = 64$ and 64 " 1 = 63 is thus a multiple of 7.

Problem based on Fermat's Theorem (Light's New Car) :

Statement : Given A and B, we have to find (A power B) % (10^9 + 7) where A, B<=10^100000

First let's deal with the base A. Now what we have to find is A%(10^9 + 7). This is beacause, let's suppose B = n (where n is an integer). A can be expressed as

```
int LucasApproach(int n, int r, int p)
{
    if(r==0)
    return 1;
    else
    {
        int n_i = n%p, r_i = r%p;
        int result = (LucasApproach(n/p, r/p, p)*Cal_nCr_mod_p(n_i,r_i,p))%p;
        return result;
    }
}
int main()
{
    int n,r,p;
    cin>>n>>r>>p;
    int result = LucasApproach(n,r,p);
    cout<<"nCr mod p is "<<result;
}
```

Fermat's little Theorem

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$A = [a * (mod) + b]$ (where mod = $10^9 + 7$, a and b are integers). This implies that

$(A \uparrow n) \% mod = [(a * (mod) + b) \uparrow n]\%mod = \{[(a * (mod) + b) \uparrow \% mod] * [(a * (mod) + b) \uparrow \%mod] *$

.... n times % mod $= b \uparrow n$ (where b is nothing but A % mod)

Using modulo properties :

$(a * b) \%m = ((a\%m)\% (a + b)\%m = (a\%m + b\%m)\%m$

Now A% $(10^9 + 7)$ can be found by iterating over the string A and generating an integer from it but at the same time taking it's modulo with $(10^9 + 7)$ to prevent overflow.

Now let's deal with the power B. We can use the concept of fermat's little theorem as

$x^{p-1}\%p = 1$ (where p is a prime number)

B can be presented as $B = a*(p - 1) + b$ (where a and b are integers and $p = (10^9 + 7)$)

Hence $A^B\%p = A^{(a*(p-1)+b)}\%p = \{[A^{a*(p-1)}]\%p * [A^b]\%p\} \%p = (A^b)\%p$

Here b is nothing but B %(p – 1)

Using Fermat's little theorem and modulo properties

Now B%(p – 1) can found in the similar way as A % $(10^9 + 7)$

Finally

Let $x = A \%(10^9 + 7)$, $n = B \%(10^9 + 7 - 1)$

Required answer would be $x\%(10^9 + 7)$ which can be found out easily by fast modulo exponentiation in O(logn).

Miller - Rabin Primality Test

The Miller-Rabin primality test or Rabin-Miller primality test is a primality test: an algorithm which determines whether a given number is prime.

This method is a probabilistic method to find Prime number.

Input #1 : n > 3, an odd integer to be tested for primality;

Input #2 : k, a parameter that determines the accuracy of the test

Output: Composite if n is composite, otherwise probabily prime

write n – 1 as $2^r.d$ with d odd by factoring powers of 2 from n – 1

WitnessLoop : repeat k times : pick a random integer *a* in the range [2, n – 2]

$x \leftarrow a^d \bmod n$

if x = 1 or y = n – 1 then

continue WitnessLoop

repeat r – 1 times:

```
x ← x² mod n
if x = 1 then
return composite
if x = n − 1 then
continue WitnessLoop
return composite
return probably prime
```

Example :

Input: n = 13, k = 2

1. Computed d and r such that d*2r = n − 1,
   d = 3, r = 2.
2. Call millerTest k times.

1st Iteration:

1. Pick a random number 'a' in range [2, n − 2]
   Suppose a = 4
2. Compute: x = pow(a, d) % n
   x = 43 % 13 = 12
3. Since x = (n − 1), return prime.

IInd Iteration:

1. Pick a random number 'a' in range [2 + n − 2]
   Suppose a = 5
2. Compute: x = pow(a, d) % n
   x = 53 % 13 = 8
3. x neither 1 nor 12
4. Do following (r − 1) = 1 times
   (a) x = (x * x) % 13 = (8 * 8) % 13 = 12 (b) Since x = (n − 1), return true.

Since both iterations return true, we return prime.

POWPOW2, Spoj

Problem :

Given three integers a, b, n, 1d ≤ a,b,nd ≤ 10^{5}

$a^{(b(f(n))}$**mod 1000000007 , where** $f(n) = ({}^nC_0^2 + {}^nC_1^2 + \ldots\ldots + {}^nC_n^2)$

Dealing with f(n) :

The function f complicates the expression, but we can notice that $f(n) = {}^{2n}C_n$. It's easy to find proofs online, e.g. here, so I'll skip that.

Reducing the Exponents :

$b^{(2n,\,n)}$ is a huge number and we need to reduce it to a more tractable number.

Euler's theorem states that if a and m are coprime, then $a^{\varphi(m)} \equiv 1 (\mathrm{mod}\, m)$, where $\varphi(m)$ is Euler's totient function. This is useful because $a^{y} \equiv a^{(y \bmod \varphi(m))} (\mathrm{mod}\, m)$.

The repeated $\varphi(m)$factors in the exponent will yield a bunch of 1s).

$m = 10^9 + 7$ which is a prime number, so $\varphi(m) = m - 1 = 10^9 + 6 = 2 \times 500000003$

So, we have $a^{y \bmod 1000000006} \bmod 1000000007$.

The main difficulty of this problem is that our y is also an exponential, $y = b^{(2n,\,n)}$. In order to find the result, we need first to calculate $b^{(2n,\,n)} \bmod 1000000006$.

Finding $b^{(2n,\,n)} \bmod 1000000006$ when b is odd

Suppose b is odd. Then, we can apply Euler's theorem because and 1,000,000,006 are coprime (recall that $b \leq 10^5$ so the 500000003 factor will always be coprime with b).

$$b^{(2n,\,n)} \equiv b^{(2n,\,n) \bmod \varphi(1000000006)} \bmod 1000000006$$

$$\varphi(1000000006) = \varphi(2) \times (500000003) = (2-1) \times (500000003 - 1) = 500000002)$$

$$500000002 = 2 \times 41^2 \times 148721500000002 = 2 \times 41^2 \times 148721$$

So, we need to find (2n, n)mod 500000002 which is not prime. Therefore, we need to use another tool: the Chinese Remainder Theorem (CRT). We can calculate

(2n, n)mod 2

(2n, n)mod 41^2

(2n, n)mod 148721

and use CRT to get the result modulo 500000002.

Finding $b^{(2n,\,n)} \bmod 1000000006\, b^{(2n,\,n)}$ when b is even

Unfortunately, if b is even, b and 1000000006 are not coprime.

Therefore, we need CRT again. Our modulus is the product of two primes: 2 and 500,000,003. So, we shall find and and use CRT to get the result modulo 1,000,000,006.

Note that when b is even the result modulo 2 is always 0. So, we only need to calculate the result modulo 500000003 and $\varphi(500000003) = \varphi(1000000006)$, so this part is equal to the case when b is odd. The only difference is using CRT.

Adding everything together

After finding y = $b^{(2n, n)}$ mod 1000000006, we can calculate a y mod 1000000007 normally to get the final result.

Code :

```
#include<bits/stdc++.h>
#define ll long long int
int t;
ll a, b, n;
ll fact[200005];
ll md = 1000000007;
long long int c_pow(ll i, ll j, ll mod)
 {
   if (j == 0)
     return 1;
   ll d;
  d = c_pow(i, j / (long long)2, mod);
  if (j % 2 == 0)
    return (d*d) % mod;
  else
      return ((d*d) % mod * i) % mod;
 }
 ll InverseEuler(ll n, ll MOD)
 {
   return c_pow(n, MOD - 2,MOD);
 }
 ll fact_14[1700][1700];
 ll fact_B[150000];
 ll min1(ll a, ll b)
 {
  return a > b ? b : a;
 }
 void calc_fact()
 {
  fact[0] = fact[1] = 1;
  ll tmd = 148721;
   for (int i = 2; i < 200003; ++i)
   {
     fact[i] = (fact[i - 1] * i);
     if (fact[i] >= (tmd))
```

```
        fact[i] %= (tmd);
    }
}
ll fact_41[200005];
ll fact_41_p[200005];
void do_func()
{
    fact_41[0] = 1;
    fact_41_p[0] = 0;
    for (int i = 1; i < 200003; ++i)
    {
        ll y = i;
        fact_41_p[i] = fact_41_p[i - 1];
        while (y % 41 == 0)
        {
            y = y / 41;
            fact_41_p[i]++;
        }
        fact_41[i] = (y*fact_41[i - 1]) % 1681;
    }
}
ll fact_2[200005];
void do_func2()
{
    fact_2[0] = 1;
    for (int i = 1; i < 200005; ++i)
    {
        fact_2[i] = (i*fact_2[i - 1]) % 2;
    }
}
ll get_3rd(ll n, ll r, ll MOD)
{
    ll ans = (InverseEuler(fact[r], MOD)*InverseEuler(fact[n - r], MOD)) % MOD;
    ans = (fact[n] * ans) % MOD;
    return ans;
}
ll inverse2(ll m1, ll p1)
```

```
{
    ll i = 1;
    while (1)
    {
     if ((m1*i) % p1 == 1)
      return i;
     i++;
    }
}
ll chinese_remainder_2(ll n1, ll n2, ll n3)
{
    ll p1 = 2, p2 = 1681, p3 = 148721;
    ll m1, m2, m3;
    ll i1, i2, i3;
    ll m;
    ll ans;
    m = p1*p2*p3;
    m1 = m / p1; m2 = m / p2; m3 = m / p3;
    i1 = InverseEuler(m1, p1); i2 = inverse2(m2, p2); i3 = InverseEuler(m3, p3);
    //printf("i1 = %lld i2 = %lld\n",i1,i2);
    ans = (n1*m1*i1) % m + (n2*m2*i2) % m + (n3*m3*i3) % m;
    ans = ans%m;
    return ans;
    //printf("%d\n",ans);
}
int main()
{
    ios_base::sync_with_stdio(false);
    cin.tie(NULL);
    calc_fact();
    do_func();
    do_func2();
    cin >> t;
    while (t--)
    {
     cin >> a >> b >> n;
      if (a == 0 && b == 0)
      {
```

```
        cout << "1\n";
        continue;
      }

      if (b == 0)
      {
        cout << "1\n";
        continue;
      }
      ll a1 = (n == 0) ? 1 : 0;
      ll a2 = (fact_41[2 * n] * inverse2(fact_41[n],1681)) % 1681;
      a2 = (a2 * inverse2(fact_41[n], 1681)) % 1681;
      a2 = (a2 * c_pow(41, fact_41_p[2 * n] - 2 * fact_41_p[n], 1681)) % 1681;
      ll a3 = get_3rd(2 * n, n, 148721);
      //cout << a1 << " " << a2 << " " << a3 << "\n";
      ll ans = chinese_remainder_2(a1, a2, a3);
      if (ans == 0)ans = 500000002;
      ll y1 = c_pow(b, ans, md - 1);
      cout << y1 << "\n";
      ll z = c_pow(a, y1, md);
      cout << z << "\n";
    }
    return 0;
}
```

Best Method for ^nC_r

We want to compute C(n,r)%p where p is prime and N,R <=10^8 :

```
#include<iostream>
using namespace std;
#include<vector>
/* This function calculates (a^b)%MOD */
long long pow(int a, int b, int MOD)
{
 long long x=1,y=a;
 while(b > 0)
  {
   if(b%2 == 1)
```

```
    {
      x=(x*y);
     if(x>MOD) x%=MOD;
    }
   y = (y*y);
   if(y>MOD) y%=MOD;
   b /= 2;
  }
  return x;
}
/* Modular Multiplicative Inverse
Using Euler's Theorem
a^(phi(m)) = 1 (mod m)
a^(-1) = a^(m-2) (mod m) */
long long InverseEuler(int n, int MOD)
{
  return pow(n,MOD-2,MOD);
}
long long C(int n, int r, int MOD)
{
  vector<long long> f(n + 1,1);


  for (int i=2; i<=n;i++)
     f[i]= (f[i-1]*i) % MOD;
    return (f[n]*((InverseEuler(f[r], MOD) * InverseEuler(f[n-r],MOD))% MOD)) %
MOD;
  }
int main()
{
  int n,r,p;
  while (~scanf("%d%d%d",&n,&r,&p))
  {
    printf("%lld\n",C(n,r,p));
  }
}
```

Modular Exponentiation :

We can now clearly see that this approach is very inefficient, and we need to come up with something better. We can take care of this problem in $O(\log_2 b)$ by using a technique called exponentiation by squaring. This uses only $O(\log_2 b)$ squarings and $O(\log_2 b)$ multiplications. This is a major improvement over the most naive method.

Code :

```
ans=1                          //Final answer which will be displayed
 while(b !=0 ) {
   /*Finding the right most digit of 'b' in binary form, if it is 1, then multiply
the current value of a
    in  ans. */
        if(b&1) {             //rightmost digit of b in binary form is 1.
        ans = ans*a ;
        ans = ans%c;          //at each iteration if value of ans exceeds then
reduce it to modulo c.
      }
  a = a*a;                   /
  a %= c;                    //at each iteration if value of a exceeds then reduce
it to modulo c.
  b >>= 1;                   //Trim the right-most digit of b in binary form.
}
```

Questions :

1. **Find sum of divisors of all the numbers from 1 to n . n <= 10^5.**
   (SPOJ DIVSUM)

   Here n is relatively small so we can precompute all the divisor sum using sieve like approach which runs in O(NLOGN).

Code :

```
ll sum[5000005];
void sieve()
{
      F(i,1,5000002)
      {
            for(ll j=i;j<=5000002;j+=i) // every j has i as a divisor
            {
                  sum[j] += i;
```

```
        }
    }
    // subtract number itself from sum if proper divisors are required
    F(i,1,5000002)
        sum[i] -= i;
}
```

2. **Find sum of divisors of a number. n <= 10^16.**
   (SPOJ-DIVSUM2)

Explanation :

Here our previous approach will fail since we cannot create such large array. Hence we have to factorize n in the form of p1^k1 * p2^k2...

Now we can get the sum of divisors using the formula mentioned in this booklet above.

Code :

```
/*input
3
2
10
10000000000000000
*/
#include <bits/stdc++.h>
#include<stdio.h>
using namespace std;
#define F(i,a,b) for(ll i = a; i <= b; i++)
#define RF(i,a,b) for(ll i = a; i >= b; i–)
#define pii pair<ll,ll>
#define PI 3.14159265358979323846264338327950288
#define ll long long
#define ff first
#define ss second
#define pb(x) push_back(x)
#define mp(x,y) make_pair(x,y)
#define debug(x) cout << #x << " = " << x << endl
#define INF 1000000009
#define mod 1000000007
#define S(x) scanf("%d",&x)
#define S2(x,y) scanf("%d%d",&x,&y)
```

```
#define P(x) printf("%d\n",x)
#define all(v) v.begin(),v.end()
#define lim 100000002
vector <bool> mark(lim+2,1);
vector <ll> primes;
void sieve() // we need primes upto 10^8
{
        //ll times = 0;
        for(ll i=3;i<=lim;i+=2)
        {
                //times++;
                if(mark[i] == 1)
                {
                        for(ll j=i*i ; j <= lim ;j += 2*i)
                        {
                //              times++;
                                mark[j] = 0;
                        }
                }
        }
        //debug(times);
        primes.pb(2);
        for(ll i=3;i<=lim;i+=2)
        {
                if(mark[i])
                        primes.pb(i);
        }
}
ll power(ll a,ll b) // find a to the power b
{
        ll ans = 1ll;
        while(b > 0)
        {
                if(b&1)
                        ans = ans*a;
                a = a*a;
                b /= 2;
        }
        return ans;
```

```
}
ll factorize(ll n) // find multiplication of all (p^(k+1)-1)/(p-1) where k is
the power of p in n
{
        // exhaust powers of 2 first
        ll c = 0;
        while(n%2 == 0)
        {
                c++;
                n = n/2;
        }
        ll i=0 , ans = 1ll;
        if(c>0)
                ans = (power(2ll,c+1) - 1);
        ll times = 0;
        ll p = primes[0];
        while(p*p <= n)
        {
                //times++;
                if(n%p == 0) // if p is a prime factor of n
                {
                        ll cnt = 0; // find power of p
                        while(n%p == 0)
                        {
                                //times++;
                                n /= p;
                                cnt++;
                        }
                        // update ans;
                        ll numerator = power(p,cnt+1) - 1;
                        //debug(numerator);
                        ll denom = p - 1;
                        ll curans = numerator/denom;
                        ans = ans * (curans);
                }
                //debug(n);
                if(n == 1)
```

```
                break;
            i++;
            p = primes[i];
        }
        //debug(times);
        if(n != 1)
            ans = ans * (n+1);
        return ans;
}
int main()
{
        std::ios::sync_with_stdio(false);
        sieve();
        //cout<<primes.size(); //5761455 primes less than 10^8
        ll t;
        cin>>t;
        while(t--)
        {
            ll n;
            cin>>n;
            ll ans = factorize(n);
            ans -= n;
            cout<<ans<<endl;
        }
        return 0;
}
```

3. **Find the value of 1! * 2! * 3*... N! modulo P where P = 109546051211.**

   FACTMUL SPOJ, Chinese Remainder Thm

Explanation :

The naive approach for calculating this value under modulo p will fail since (a*b)%p will overfl coz p is itself large.

Here is the trick :-

P = p1*p2 where p1 = 186583 p2 = 587117

Let a = 1! * 2! * 3*... N!

x1 = a%p1

x2 = a%p2

This is a set of equations satisfying the CRT criteria hence we can calculate the value of a using CRT.

Code :

```
/*input
5
*/
#include <bits/stdc++.h>
#include<stdio.h>
using namespace std;
#define F(i,a,b) for(ll i = a; i <= b; i++)
#define RF(i,a,b) for(ll i = a; i >= b; i--)
#define pii pair<ll,ll>
#define PI 3.141592653589793238462643383279502884
#define ll long long
#define ff first
#define ss second
#define pb(x) push_back(x)
#define mp(x,y) make_pair(x,y)
#define debug(x) cout << #x << " = " << x << endl
#define INF 1000000009
#define mod 109546051211 // 186583*587117
#define S(x) scanf("%d",&x)
#define S2(x,y) scanf("%d%d",&x,&y)
#define P(x) printf("%d\n",x)
#define all(v) v.begin(),v.end()
ll power(ll a,ll b,ll m)
{
        ll ans = 1ll;
        while(b > 0)
        {
                if(b&1)
                        ans = ans*a;
                a = a*a;
                ans%=m;
                a%=m;
```

```
            b /= 2;
        }
        return ans;
}
int main()
{
        std::ios::sync_with_stdio(false);
        ll n;
        cin>>n;
        //use crt since MOD = p1*p2
        ll p1 = 186583ll;
        ll p2 = 587117ll;
        // find first value with respect to p1 and second value with respect to p2
        ll ans_p1 = 1ll , ans_p2=1ll , curfactorial_p1 = 1ll , curfactorial_p2 = 1ll;
        F(i,2,n)
        {
                curfactorial_p1 = curfactorial_p1*i;
                curfactorial_p1 %= p1;
                curfactorial_p2 = curfactorial_p2*i;
                curfactorial_p2 %= p2;
                ans_p1 = ans_p1*curfactorial_p1;
                ans_p1 %= p1;
                ans_p2 = ans_p2*curfactorial_p2;
                ans_p2 %= p2;
        }
        //debug(ans_p1);
        //debug(ans_p2);
        // num[0] = p1 , num[1] = p2
        // rem[0] = ans_p1 rem[1] = ans_p2
        // pp[0] = p2 pp[1] = p1
        // prod = p1*p2
        // inv[i] = mpdular multiplicative inverse of pp[i] with respect to num[i]
        // inv[0] = inverse of p2 w.r.t p1 , inv[1] = inverse of p1 w.r.t p2
        // first remainder is ans_p1 and second remainder is ans_p2
        // (x%p1) = ans_p1
        // (x%p2) = ans_p2 , we can combine these two to find x
```

```
    // x = rem[0]*inv[0]*pp[0] + rem[1]*inv[1]*pp[1]
    ll inv_zero = power(p2,p1-2ll,p1); // fermats
    ll inv_first = power(p1,p2-2ll,p2); // fermats
    ll ans = (((ans_p1*inv_zero)%mod)*(p2%mod))%mod +
(((ans_p2*inv_first)%mod)*p1%mod)%mod;
    ans %= mod;
    cout<<ans<<endl;
    return 0;
}
```

TRY YOURSELVES

http://www.spoj.com/problems/MAIN74/ //Find first few values and observe pattern
http://www.spoj.com/problems/DIVSUM/ // precomputation or multiplicative formula
http://www.spoj.com/problems/DIVSUM2/ // multiplicative formula
http://www.spoj.pl/problems/NDIVPHI/ // can be solved only using BIG INTEGER or in PYTHON
http://www.codechef.com/problems/THREEDIF // very simple
http://www.spoj.com/problems/LCPCP2/ // very simple
http://www.spoj.com/problems/GCD2/ // tricky
http://www.spoj.com/problems/FINDPRM/
http://www.spoj.com/problems/TDKPRIME/ // simple sieve and precompute
http://www.spoj.com/problems/TDPRIMES/ // same as TDKPRIME
http://www.spoj.com/problems/PRIME1/ //segmented sieve
http://www.spoj.com/problems/FACTMUL/ // CRT
http://www.spoj.com/problems/FACTCG2/ // factorization
http://www.spoj.com/problems/ALICESIE/ // formula
http://www.spoj.com/problems/AMR10C/ // factorization
http://www.spoj.com/problems/DCEPC11B/ // Wilson Theorem
http://www.codechef.com/problems/SPOTWO
http://www.spoj.com/problems/DCEPC13D/ // CRT + LUCAS + FERMAT
http://www.spoj.com/problems/CUBEFR/
http://www.spoj.com/problems/NFACTOR/
http://www.spoj.com/problems/CSQUARE/
http://www.spoj.com/problems/CPRIME/
http://www.spoj.com/problems/ANARC09C/
http://www.spoj.com/problems/GCDEX/ read this :-
https://discuss.codechef.com/questions/72953/a-dance-with-mobius-function

umber Theory

http://www.spoj.com/problems/AMR11E/ // easy sieve

https://www.hackerearth.com/challenge/competitive/code-monk-number-theory-i/problems/

https://www.hackerearth.com/challenge/competitive/code-monk-number-theory-ii/problems/

https://www.hackerearth.com/challenge/competitive/code-monk-number-theory-iii/problems/

Advanced Problem :- https://www.hackerrank.com/challenges/ncr/problem

4 Binary Search

Divide and Conquer

Searching an element in a Sorted Sequence

```
binary_search(A, target):
lo = 1, hi = size(A)
while lo <= hi:
     mid = lo + (hi-lo)/2
     if A[mid] == target:
          return mid
else if A[mid] < target:
        lo = mid+1
else:
       hi = mid-1
```

First Occurrence of an Element

```
FirstOccur(A, target):
    lo = 1, hi = size(A), result=-1;
  while lo <= hi:
            mid = lo + (hi-lo)/2
            if A[mid] == target:
                 result=mid; high=mid-1;
          else if A[mid] < target:
              lo = mid+1
          else:
        hi = mid-1
  return res;
```

Last Occcurrence of an Element

```
LastOccur(A, target):
  lo = 1, hi = size(A), result=-1;
 while lo <= hi:
   mid = lo + (hi-lo)/2
```

```
    if A[mid] == target:
      result=mid; low=mid+1;
    else if A[mid] < target:
      lo = mid+1
    else:
      hi = mid-1
  return res;
```

Beyond Sorted arrays

Binary search obviously works on searching for elements in a sorted array. But if you think about the reason why it works is because the array itself is monotonic (either increasing or decreasing). So, if you are at a particular position, you can make a definite call whether the answer lies in the left part of the position or the right part of it. But Most importantly you need to know the range [start,end].

Similar thing can be done with monotonic functions (monotonically increasing or decreasing) as well.

Lets say we have f(x) which increases when x increases.

So, given a problem of finding x so that f(x) = p, I can do a binary search for x.

/Take Example/

Therefore

1. if f(current_position) > p, then I will search for values lower than current position.
2. if f(current_position) < p, then I will search for values higher than current position
3. if f(current_position) = p, then I have found my answer.

Optimisation Problems

Consider an optimisation problem where you need to minimise a quantity X satisfying some constraint Y such that:

- If X satisfies the constraint, anything more than X will satisfy the constraint too.
- For a given value of X, you know how to check whether the constraint is satisfied.

Painter's Partition Problem

You have to paint N boards of length $\{A_0, A_1, A_2, A_3 \ldots A_{N-1}\}$. There are K painters available and you are also given how much time a painter takes to paint 1 unit of board. You have to get this job done as soon as possible under the constraints that any painter will only paint contiguous sections of board.

/Take example/

Observations :

- The lowest possible value for costmax must be the maximum element in A (name this as lo).
- The highest possible value for costmax must be the entire sum of A, (name this as hi).

- As costmax increases, x decreases. The opposite also holds true.

```
int painterNum(vector<int> &C, long long X){
    long long sum = 0;
    long long num = 1;
    for(auto c: C){
        if(sum + c > X){
            sum = c;num++;
        }else sum += c;
    }
    return num;
}
int paint(int A, int B, vector<int> &C) {

    int high=0,low=0;
    for(int x: C){low = max(x, low); high += x;}
    while(low < high){
        long long mid = low + (high-low)>>1;
        if(painterNum(C, mid) <= A)high = mid;
        else low = mid+1;
    }

    return low;
}
```

Allocate Books

N number of books are given.

The ith book has Pi number of pages.

You have to allocate books to M number of students so that maximum number of pages alloted to a student is minimum. A book will be allocated to exactly one student. Each student has to be allocated at least one book. Allotment should be in contiguous order, for example: A student cannot be allocated book 1 and book 3, skipping book 2.

Return -1 if a valid assignment is not possible

Problem :

Aggressive Cows :

http://www.spoj.com/problems/AGGRCOW/

Approach:

Define the following function:

Func(x) = 1 if it is possible to arrange the cows in stalls such that the distance between any two cows is at least x

Func(x) = 0 otherwise

The problem satisfies the monotonicity condition necessary for binary search.Why??

Check that if Func(x)=0, Func(y)=0 for all y>x and if Func(x)=1,

Funx(y)=1 for all y < x

Find low and high values

Func(0)=1 trivially since the distance between any two cows is at least 0. Also, since we have at least two cows, the best we can do is push them towards the stalls at the end - so there is no way we can achieve better. Hence Func(maxbarnpos-minbarnpos+1)=0.

```
#include <bits/stdc++.h>
using namespace std;
int n,c;
int func(int num,int array[])
{
    int cows=1,pos=array[0];
    for (int i=1; i<n; i++)
    {
        if (array[i]-pos>=num)
        {
            cows++;
            if (cows==c)
                return 1;
            pos=array[i];
        }
    }
    return 0;
}
int bs(int array[])
```

```
{
    int ini=0,last=array[n-1],max=-1;
    while (last>ini)
    {
        int mid=(ini+last)/2;
        if (func(mid,array)==1)
        {
            if (mid>max)
                max=mid;
            ini=mid+1;
        }
        else
            last=mid;
    }
    return max;
}
int main()
{
    int t;
    scanf("%d",&t);
    while (t–)
    {
        scanf("%d %d",&n,&c);
        int array[n];
        for (int i=0; i<n; i++)
            scanf("%d",&array[i]);
        sort(array,array+n);
        int k=bs(array);
        printf("%dn",k);
    }
    return 0;
}
```

Time to Try

- ❑ EKO (Spoj)
- ❑ PRATA (Spoj)
- ❑ BEAUTIFUL TRIPLETS (Hackerearth)

5 Greedy Algorithm

Greedy Algorithms are one of the most intuitive algorithms. Whenever we see a problem we first try to apply some greedy strategy to get the answer(we humans are greedy, aren't we :P ?).

Greedy approaches are quite simple and easy to understand/formulate.But many times the proving part might be difficult.

- Greedy Algorithm always makes the choice that looks best at **the moment.**
- You hope that by choosing a local optimum at each step, you will end up at a global optimum.

Counting Money

Problem Statement: Suppose you want to count out a certain amount of money,using the fewest possible notes or coins.

Greedy Approach

At each step, take the largest possible note or coin that does not overshoot the sum of money and include it in the solution.

Example **:**

To make Rs 39 with fewest possible notes or coins.

Rs 10 note, to make 29

Rs 10 note, to make 19

Rs 10 note, to make 9

Rs 5 note, to make 4

Rs 2 coin, to make 2

Rs 2 coin, to make 0

Total is 4 notes and 2 coins, which is the optimum solution.

Note : For Indian currency, the greedy algorithm, even after Demonetization always gives the optimum solution.

Is this true for any currency?

Now that Donald Trump is the new President of US, he decides to do something extraordinary like PM Modi. So he decides to change all the currency notes to 1 dollars, 7 dollars and 10 dollars.

Suppose you went to US and used the same greedy approach to count minimum numbers of notes and coins in exchange for a Burger which costs $15.

To make $15:

1 $10 note

5 $1 notes

Total = 6 notes

Can we do better than 6?

$7 + $7 +$1 – Only 3 notes required!

PROBLEMS

- **BUSYMAN**

Activity Scheduling Problem

http://www.spoj.com/problems/BUSYMAN/

```
#include<iostream>
#include<algorithm>
#include<vector>
using namespace std;
struct cmpr
{
    bool operator()(pair<int,int> l, pair<int,int> r)
   {
    return l.second < r.second;
   }
}cmp;
int main()
{
   int t,n,s,e,res;
   scanf("%d",&t);
   while(t--)
  {
   res = 1;
   vector<pair<int,int> > activity;
   scanf("%d",&n);
   for(int i = 0;i<n;i++)
   {
       scanf("%d %d",&s,&e);
```

```
        activity.push_back(make_pair(s,e));
    }
    //Sort the activities according to their finish time
        sort(activity.begin(),activity.end(),cmp);
    //fin denotes the finish time of the choosen activity
    //i.e. activity with least finish time
     int fin = activity[0].second;
     for(int i = 1;i<activity.size();i++)
     {
        //To find the next compatible activity with
        //least finish time
        //Find the first activity whose start time
           //is more than finish time of previous activity
         if(activity[i].first >= fin)
         {
             //update the finish time as the finish time
            //of this activity
             fin = activity[i].second;
             //Since we have choosen a new activity,
             //increment the count by 1
               res++;
         }
     }
     printf("%d\n",res);
  }
  return 0;
}
```

❑ CONNECTING WIRES

- There are n white dots and n black dots, equally spaced, in a line.
- You want to connect each white dot with some one black dot, with a minimum total length of "wire".

Greedy Approach : Suppose you have a sorted list of the white dot positions and the black dot positions. Then you should match the i-th white dot with the i-th black dot.

Why should greedy work?

Let b0 and w0 be the first black and white balls respectively and let bi and wj be the next white and black balls.

Case 1: w0 and wj < b0

b0 bi

w0 wj

(i) C(w0, bi) + C(wj, b0) = (bi - w0) + (b0 - wj)

(ii) C(w0, b0) + C(wi, bj) = (b0 - w0) + (bi - wj)

Both are same.

Case 2: w0 < b0 and b0 < wj < bi

b0 bi

w0 wj

(i) C(w0, bi) + C(wj, b0) = (bi - w0) + (wj - b0)

(ii) C(w0, b0) + C(wi, bj) = (b0 - w0) + (bi - wj)

(i) - (ii) = 2(wj - b0) is extra

Case 3: w0 < b0 and bi < wj

b0 bi

w0 wj

(i) C(w0, bi) + C(wj, b0) = (bi - w0) + (wj - b0)

(ii) C(w0, b0) + C(wi, bj) = (b0 - w0) + (wj - bi)

(i) - (ii) = 2(bi - b0) is extra

Hence, in all the cases connecting 1st b to 1st w and 2nd b to 2nd w yeilds more optimal solution.

- **BIASED STANDINGS**

In a competition, each team is be able to enter their preferred place in the ranklist. Suppose that we already have a ranklist. For each team, compute the distance between their preferred place and their place in the ranklist. The sum of these distances will be called the badness of this ranklist. Find one ranklist with the minimal possible badness.

http://www.spoj.com/problems/BAISED/

This is question is similar to the connecting wires problem, where the dseired ranks denote the black balls and the actual ranks denote the white balls. Hence, we can apply the greedy approach by sorting the desired ranks and calculating the distance between i-th desired rank and i-th actula rank.

Code :

```
#include <bits/stdc++.h>
using namespace std;
#define abs(a,b) a > b ? a - b : b - a
#define ll long long
int main()
{
    int t;
    cin>>t;
    while(t--)
    {
        vector<int> v;
        string name;
        ll sum=0;
        cin>>n;
        for(int i=0;i<n;i++)
        {
            cin>>name;
            cin>>temp;
            //Insert the desired rank in vector v
            v.push_back(temp);
        }
        //Sort the vector containing the ranks
        sort(v.begin(),v.end());
        //This is the greedy approach
        for(int i=1;i<=n;i++)
        {
            //Take the difference between the i-th actual rank
            //and the i-th desired rank
            sum += abs(v[i-1],i);
        }
        printf("%lld\n",sum);
    }
    return 0;
}
```

T = O(nlogn)

Can we do better?

```
#include <bits/stdc++.h>
using namespace std;
#define abs(a,b) a > b ? a - b : b - a
#define ll long long
int arr[100000+10];
int main()
{
    int t,n,temp;
    cin>>t;
    while(t—)
    {
        memset(arr,0,sizeof arr);
        string name;
        ll sum=0;
        cin>>n;
        for(int i=0;i<n;i++)
        {
            cin>>name;
            cin>>temp;
            //Increment the desired rank index in array arr
            arr[temp]++;
        }
        int pos = 1;
        //This is the greedy approach
        for(int i=1;i<=n;i++)
        {
            //If the i-th rank was desired by atleast one team
            //Assign and increment the actual rank index 'pos'
            //till this rank is desired
            while(arr[i])
            {
```

```
        sum += abs(pos,i);
        arr[i]--;
       pos++;
      }
    }
    printf("%lld\n",sum);
   }
  return 0;
 }
```

T = O(n)

LOAD BALANCER

Rebalancing proceeds in rounds. In each round, every processor can transfer at most one job to each of its neighbors on the bus. Neighbors of the processor i are the processors i-1 and i+1 (processors 1 and N have only one neighbor each, 2 and N-1 respectively). The goal of rebalancing is to achieve that all processors have the same number of jobs.Determine the minimal number of rounds needed to achieve the state when every processor has the same number of jobs, or to determine that such rebalancing is not possible.

http://www.spoj.com/problems/BALIFE/

Greedy Approach: First find the final load that each processor will have. We can find it by sum(arr[1], arr[2],...,arr[n])/n, let us call it **load.**

So in the final state, each of the processor will have final load = load.

For each index i from 1 to n-1, create a partition of (1...i) and (i+1...n) and find the amount of load that is to be shared between these two partitions. The answer will be maximum of all the loads shared between all the partitions with i varying from 1 to n-1.

Example **:**

Initial state: 4, 8, 12, 16

Final state: 10 10 10 10

i = 1:

partition: (4) (8,12,16)

Load —> (4) needs 6 and (8,12,16) needs to give 6.

max_load = 6

i = 2:

partition: (4,8) (12,16)

Load—> (4,8) needs 8 and (12,16) needs to give 8.

max_laod = 8

i = 3:

partition: (4,8,12) (16)

Load—> (4,8,12) needs 6 and (16) needs to give 6.

max_laod = 6

Final answer = 8

Why does it works?

In all the partitions where less than max_load is transferred, we can internally transfer the lo between these partitions when max_laod is being transferred between the max_load partition.

t = 2s:

when load = 2 is transferred between (4,8) and (12,16)

we can transfer load = 2 between (8) —> (4) and (16)—> (12)

State: 6 8 12 14

t = 4s:

when load = 2 is transferred between (6,8) and (12,14)

we can transfer load = 2 between (8) —> (6) and (14) —> (12)

State: 8 8 12 12

t = 6s:

when load = 2 is transferred between (8,8) and (12,12)

State: 8 10 10 12

t = 8s:

when load = 2 is transferred between (8,10) and (10,12)

we can transfer load = 2 between (10) —> (8) and (12) —> (10)

State: 10 10 10 10

Code :

```
#include <bits/stdc++.h>
using namespace std;
int main()
{
```

```
int arr[9000],n, i, val, diff;
while(1)
{
  int max_load = 0, load = 0;
  cin>>n;
  if(n == -1)
    break;
  for(int i = 0;i<n;i++)
  {
     cin>>arr[i];
     load += arr[i];
  }
  //If we cannot divide the load equally
  if(load % n)
  {
     cout<<-1<<endl;
     continue;
  }
  //Find the load that is to be divided equally
  load /= n;
  //Greedy step
  for(int i = 0;i<n;i++)
  {
     //At each iteration, find the value
     //of difference between final load to be assigned
     //and current load
     //Keep adding this difference in 'diff'
     diff += (arr[i] - load);
     //If the net difference is negative i.e.
     //we need diff amount till i-th index
    if(diff < 0)
     max_load = max(max_load, -1*diff);
```

```
        //If diff is positive i.e. we have to
        //give diff amout to (n-i) processors
        else
          max_load = max(max_load, diff);
        //calculate the max of load that can be given or
        //taken at each iteration .
      }
      cout<<max_load<<endl;
    }
    return 0;
}
```

T = O(n)

❑ DEFENSE OF A KINGDOM, SPOJ

Problem Statement

Given H,W of a field, and location of towers which guard the horizontal and the vertical lines corresponding to their positions. Find out the largest unbounded area(white rect). Refer the diagram on spoj.

http://www.spoj.com/problems/DEFKIN/

Greedy Approach: Given w, h as width and height of the playing field, and the coordinates of the towers as (x1,y1) ... (xN,yN), split the coordinates into two lists x1...xN, y1...yN, sort both of those coordinate lists.

Then calculate the empty spaces, e.g. dx[] = { x1, x2-x1, ..., xN - xN-1, (w + 1) - xN }. Do the same for the y coordinates: dy[] = { y1, y2-y1,..., yN - yN-1, (h + 1) - yN }. Multiply max(dx)-1 by max(dy)-1 and you should have the largest uncovered rectangle. You have to decrement the delta values by one because the line covered by the higher coordinate tower is included in it, but it is not uncovered.

Code :

```
#include<iostream>
#include<algorithm>
using namespace std;
int point_x[40000+10], point_y[40000+10];
int main()
{
```

```
    int t,w,h,n,x,y;
    scanf("%d",&t);
    while(t--)
    {
        scanf("%d %d %d",&w,&h,&n);
        for(int i = 0;i<n;i++)
        {
            scanf("%d %d",&point_x[i],&point_y[i]);
        }
        //sort the x-coordinates of the list
        sort(point_x, point_x + n);
        //sort the y-coordinates of the list
        sort(point_y, point_y + n);
        //dx --> maximum uncovered tiles in x coordinate
        //dy --> maximum uncocered tiles in y coordinate
        //Initially dx and dy are the first guars's position
        int dx = point_x[0],dy = point_y[0];
        //calculate the maximum uncovered gap
        //in x and y coordinate
        for(int i = 1;i<n;i++)
        {
            dx = max(dx,point_x[i] - point_x[i-1]);
            dy = max(dy,point_y[i] - point_y[i-1]);
        }
        dx = max(dx, w + 1 - point_x[n-1]);
        dy = max(dy, h + 1 - point_y[n-1]);
        printf("%d\n",((dx-1) * (dy-1)));
    }
    return 0;
}
```

T = O(nlogn)

QUES. CHOPSTICKS

Given N sticks of length L[1], L[2], ..., L[N] and a positive integer D. Two sticks can be paired if the difference of their length is at most D.

Find the max number of pairs.

https://www.codechef.com/problems/TACHSTCK

Greedy Appraoch :

- Sort the list of sticks according to their lengths.
- If L[1] and L[2] cannot be paired, L[1] is useless.
- Else pair L[1] and L[2] and remove them from the list.
- Repeat steps 2-3 till the list become empty.

Why does this works?

- By pairing starting stick to its immediate next, we have max number of options left for next pairing.
- If L[1] and L[2] can be paired, but instead we pair (L[1], L[M]) and (L[2], L[N]).

Code :

```
#include <iostream>
#include <vector>
#include <algorithm>
using namespace std;
typedef long long ll;
int main()
{
    ll n,d,a;
    vector<ll> v;
    cin>>n>>d;
    for(int i = 0;i<n;i++)
    {
        cin>>a;
        v.push_back(a);
    }
    //Sort the list of sticks
    sort(v.begin(),v.end());
    int res = 0;
    for(int i = 0;i<n-1;)
    {
        //If the adjacent difference is less than d
        //we can make a pair
        //Now remove these two sticks from the list
```

```
        if(v[i+1] - v[i] <= d)
        {
          res++;
           i+=2;
         }
        //If we cannot make a pair with adjacent stick
         //This stick is useless, remove this stick from the list
         else
            i++;
      }
    cout<<res<<endl;
    return 0;
}
```

T = O(nlogn)

❑ EXPEDITION, SPOJ

A damaged truck needs to travel a distance of L. The truck leaks one unit of fuel for every unit distance it travels.There are N (<=10^4)fuel stops on the path, what is the minimum number of refueling stops that the truck needs to make given that it has P amout of fuel initially.

http://www.spoj.com/problems/EXPEDI/

The first observation is, if you want to reach the city, say, point L, you **have to ensure that every single point between the current position and city must also be reachable.**

Now, the task is to minimize the number of stoppages for fuel, which is at most 10000. So, we sort the fuel stations, and start from current **position. For every fuel station, if we want to reach it, we must have fuel f more than or equal to the distance d. Also, using the** larger capacities will always reduce the number of stations we must stop.

How does greedy work?

If we make sure that we only stop at the largest capicity fuel stations

upto the point where the fuel capacity of truck becomes 0, number of

stops will be minimized.

Example:

Initial capacity = 11

Fuel Stations: 3 8 10 12

Capacity: 4 10 2 3

Initially, we don't have enough fuel to reach 12.

So we can reach upto maximum 11th city.

In order to minimize our stops, we will stop at 8th city

where fuel capacity is maximum.

Code:

```
#include<iostream>
#include<vector>
#include<queue>
#include<algorithm>
using namespace std;
bool cmpr(pair<int,int> l,pair<int,int> r)
{
    return l.first > r.first;
}
int main()
{
    int n,t,x,d,f,D,F,prev = 0;
    scanf("%d",&t);
    while(t--)
    {
      int flag = 0,ans = 0;
      vector<pair<int,int> > v;
      priority_queue<int> pq;
      scanf("%d",&n);
      for(int i = 0;i<n;i++)
      {
          scanf("%d %d",&d,&f);
          //Insert the city index and fuel capacity
          v.push_back(make_pair(d,f));
      }
      //Sort the cities according to their location
          sort(v.begin(),v.end(),cmpr);
          scanf("%d %d",&D,&F);
          //Calculate the difference between the current city and the
            //destination i.e. v[i] = j means that we need to travel j
```

```
//units to rach our destination
for(int i = 0;i<n;i++)
{
v[i].first = D - v[i].first;
}
//prev denotes the previous city visited
prev = 0;
//x will denote the current city that we are in
x = 0;
while(x < n)
{
//cout<<x<<" "<<F<<" "<<v[x].first<<" "<<prev<<" "<<endl;
//If we have enough fuel to travel from prev to current city
//Push this fuel station to priority_queue
//Reduce the amount of fuel used
//update the previous city
if(F >= (v[x].first - prev))
{
F -= (v[x].first - prev);
pq.push(v[x].second);
prev = v[x].first;
}
//If we dont have enough fuel to visit
//the current city
//Find the max capacity fuel station between
//prev and current city and use it to refuel
else
{
//If no fuel station is left
//i.e. we have used all of
//the fuel stations and still not able
//to reach the city
//return FAIL!
if(pq.empty())
{
```

```
            flag = 1;
            break;
        }
    //Increment the fuel capacity of truck
    //by the maximum fuel station capacity
    F += pq.top();
    //Remove that fuel station from heap
    pq.pop();
    //Increment the number of used fuel
    //station by 1
    ans++;
    continue;
    }
    //If we have visited the current city
    //Visit next city
    x++;
    }
    if(flag)
    {
    printf("-1\n");
    continue;
    }
    //Find the distance between the destination
    //and last city
    //Check if it is possible to visit the destination
    //from the last city.
    D = D - v[n-1].first;
    if(F >= D)
    {
      printf("%d\n",ans);
      continue;
    }
    while(F < D)
    {
        if(pq.empty()){
```

```
            flag = 1;
            break;
            }
        F += pq.top();
        pq.pop();
        ans++;
    }
    if(flag){
        printf("-1\n");
        continue;
    }
    printf("%d\n",ans);
  }
  return 0;
}
```

T = O(NlogN)

GREEDY KNAPSACK PROBLEM, CODECHEF

You are given N items, each item has two parameters: the weight and the cost. Let's denote M as the sum of the weights of all the items. Your task is to determine the most expensive cost of the knapsack, for every capacity 1, 2, ..., M. The capacity C of a knapsack means that the sum of weights of the chosen items can't exceed C.

https://www.codechef.com/problems/KNPSK

The key observation here is that weight of each item is either 1 or 2.

Greedy Approach: Greedily pickup the most costliest item with weight <= 2, which can be taken in the knapsack.

Case 1: W is even

Select the most expensive item with sum of weight = 2. This can be done in two ways:

- Take the most expensive item of weight 2
- Or take at most two most expensive item of weight 1

Note that after picking up the most expensive item with sum of weight <= 2, we will remove the item taken and will recursively select the items to fill the most expensive elements.

Case 2: W is odd

We can simply select the most expensive item of weight 1. Now, we don't consider this item again. Now we have to select the most expensive weights from the remaining items. Since W-1 is even, we can solve this problem similar to case 1.

Example:

1 7 1 100 2 70

2 44 1 56 2 44

1 56 1 33 2 1

2 1 1 18

1 18

1 33

2 70

Case 1: Even

W = 2: (1,100) + (1,56) = 156

W = 4: 156 + (2,70) = 226

W = 6: 226 + (1,33) + (1,18) = 277

W = 8: 277 + (2,44) = 321

W = 10: 321 + (2,1) = 322

Case 2: Odd

W = 1: (1,100) = 100

W = 3: 100 + (1,56) + (1,33) = 189

W = 5: 189 + (2,70) = 259

W = 7: 259 + (2,44) = 303

W = 9: 321 + (1,18) = 321

Code :

```
#include<iostream>
#include<vector>
#include<algorithm>
#include<stdio.h>
using namespace std;
long long cost[200000+10];
int main()
{
    int n,w,c,m,W = 0;
```

```
 //Separate one and two for even and odd casses
 vector<int> one,two,One,Two;
scanf("%d",&n);
for(int i = 0;i<n;i++)
  {
    scanf("%d %d",&w,&c);
    //Insert weight 1 items in one vector
    if(w == 1)
       one.push_back(c);
     //Insert weight 2 items in two vector
     else
        two.push_back(c);
     W += w;
   }
  //sort the two vectors
   sort(one.begin(),one.end());
    sort(two.begin(),two.end());
   One = one;
   Two = two;
   long long sum = 0,cur = 0,res = 0;
     //even case
    for(int i = 2;i<=W;i+=2)
       {
            //res1 —> when we take atmost two 1 wight items
            //res2 —> when we take single 2 weight item
            long long res1 = 0, res2 = 0;
            //calc res2
        if(two.size() > 0)
          {
                res2 = two[two.size()-1];
          }
        //calculate res1
        if(one.size() > 1)
          {
              res1 = one[one.size()-1] + one[one.size()-2];
          }
```

```
else if(one.size() > 0)
    {
        res1 = one[one.size() - 1];
    }
    //if res2 > res1 remove the 2 weight item
    if(res2 > res1)
    {
        two.pop_back();
        res += res2;
    }
    //remove at most two 1 weight items
    else
    {
        if(one.size() > 1)
        {
            one.pop_back();
            one.pop_back();
        }
        else
            one.pop_back();
        res += res1;
    }
    //update the max cost for weight i
    cost[i] = res;
}
//odd case
//subtract 1 to make the weight sum even
res = 0;
//Find the most expensive 1 weight item
//and remove it from the list
if(One.size() > 0){
    res = One[One.size()-1];
    One.pop_back();
}
cost[1] = res;
//Similar to even case
```

```
for(int i = 3;i<=W;i+=2)
{
    long long res1 = 0, res2 = 0;
    if(Two.size() > 0)
        {
            res2 = Two[Two.size()-1];
        }
    if(One.size() > 1)
        {
            res1 = One[One.size()-1] + One[One.size()-2];
        }
        else if(One.size() > 0)
        {
            res1 = One[One.size()-1];
        }
        if(res2 > res1)
        {
            Two.pop_back();
            res += res2;
        }
        else
        {
            if(One.size() > 1)
            {
    One.pop_back();
    One.pop_back();
    }
    else
        One.pop_back();
    res += res1;
  }
 cost[i] = res;
}
for (int i = 1; i <= W; i++)
{
  if (i > 1)
```

```
        printf(" ");
      printf("%lld", cost[i]);
    }
  printf("\n");
  return 0;
}
```

T = O(nlogn)

Problems to Try :

1. **Fractional Knapsack** : Given weights and values of n items, we need put these items in a knapsack of capacity W to get the maximum total value in the knapsack.We can break items for maximizing the total value of knapsack.
2. **Huffman Coding** : Huffman coding is a lossless data compression algorithm. The idea is to assign variable-length codes to input characters, lengths of the assigned codes are based on the frequencies of corresponding characters. The most frequent character gets the smallest code and the least frequent character gets the largest code. Read about this problem and implement it yourself.
3. **Maximum Circles (HackerBlocks)** : There are *n* circles arranged on x-y plane. All of them have their centers on x-axis. You have to remove some of them, such that no two circles are overlapping after that. Find the minimum number of circles that need to be removed.
4. **Maximum Unique Segment(Codechef)** : You are given 2 arrays $W = (W_1, W_2, .., W_N)$ and $C = (C_1, C_2, .., C_N)$ with N elements each. A range [l, r] is *unique* if all the elements $C_l, C_{l+1}, .., C_r$ are unique (ie. no duplicates). The *sum* of the range is $W_l + W_{l+1} + ... + W_r$

   You want to find an *unique* range with the maximum *sum* possible, and output this sum.
5. **Station Balance - UVa 410**

Read and solve the problem statement Online

https://uva.onlinejudge.org/external/4/410.pdf

6. **DIE HARD (Spoj)**
7. **GERGOVIA (Spoj)**
8. **SOLDIER (Spoj)**
9. **CHOCOLA (Spoj)**
10. **CMIYC (Spoj)**

6 Recursion & Backtracking

Introduction :

Backtracking is a general algorithmic technique that considers searching every possible combination in order to solve an optimization problem. Backtracking is also known as depth-first search or branch and bound. Recursion technique is always used to solve backtracking problems. In this article, we will see the concept of recursion and backtracking.

Recursion :

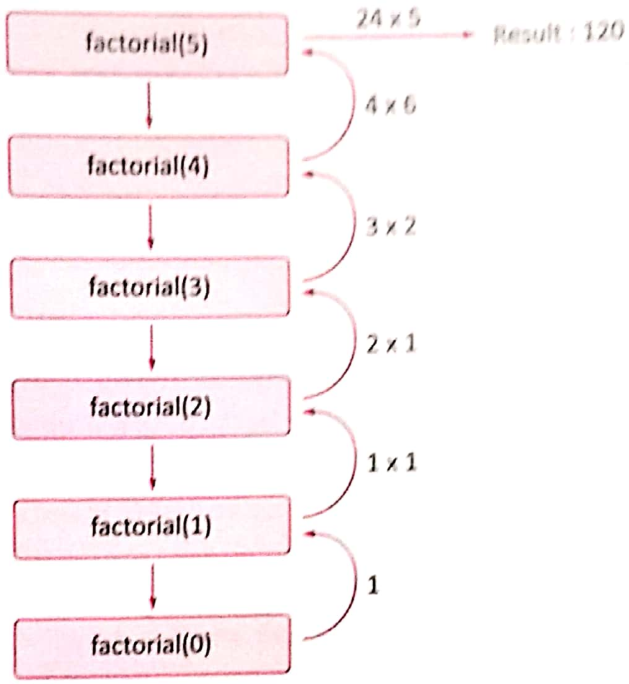
When a function calls itself, its called Recursion. It will be easier for those who have seen the movie Inception. Leonardo had a dream, in that dream he had another dream, in that dream he had yet another dream, and that goes on. So it's like there is a function called dream(), and we are just calling it in itself.

```
function  dream()
{
      print "Dreaming";
      dream();
}
```

Recursion is useful in solving problems which can be broken down into smaller problems of the same kind. But when it comes to solving problems using Recursion there are several things to be taken care of. Let's take a simple example and try to understand those. Following is the pseudo code of finding factorial of a given number X using recursion.

```
function factorial(x)
{
   if (x== 0)              // base case
      return 1;
   return x*factorial(x-1);      // break into smaller problem(s)
}
```

The following procedure shows how it works for factorial(5)



Base cases for Recusive Program :

Any recursive method must have a terminating condition. Terminating condition is one for which the answer is already known and we just need to return that. Ex. For the factorial problem, we know that factorial(0) = 1 , so when x is 0 we simply return 1, otherwise we break into smaller problem i.e. find factorial of (xd1). If we don't include a Base Case, the function will keep calling itself, and ultimately will result in stack overflow. For example, the dream() function given above has no base case. If you write a code for it in any language, it will give a runtime error.

Limitation of Recursive Call :

There is an upper limit to the number of recursive calls that can be made. To prevent this make sure that your base case is reached before stack size limit exceeds.

So, if we want to solve a problem using recursion, then we need to make sure that:

- The problem can broken down into smaller problems of same type.
- Problem has some base case(s).
- Base case is reached before the stack size limit exceeds.

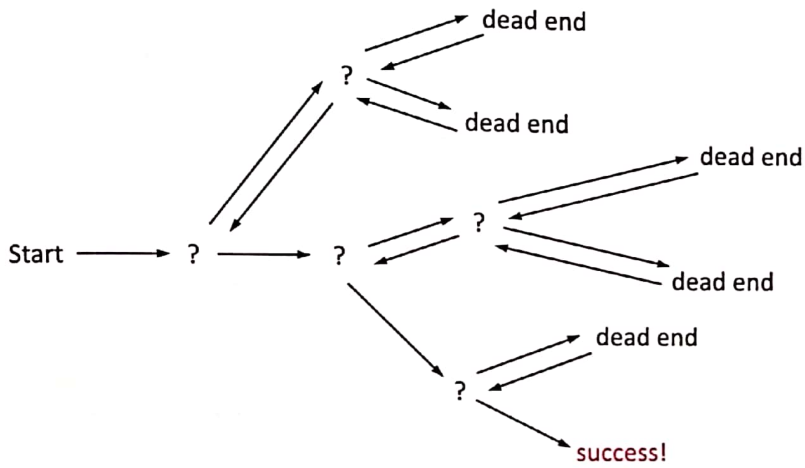
Backtracking :

When we solve a problem using recursion, we break the given problem into smaller ones. Let's say we have a problem **PROB** and we divided it into three smaller problems **P1, P2** and **P3**. Now it may be the case that the solution to PROB does not depend on all the three subproblems, in fact we don't even know on which one it depends.

Let's take a situation. Suppose you are standing in front of three tunnels, one of which is having a bag of gold at its end, but you don't know which one. So you'll try all three. First go in tunnel 1, if that is not the one, then come out of it, and go into tunnel 2, and again if that is not the one, come out of it and go into tunnel 3. So basically in backtracking we attempt solving a subproblem, and if we don't reach the desired solution, then undo whatever we did for solving that subproblem, and again try solving another subproblem.

Basically **Backtracking** is an algorithmic paradigm that tries different solutions until finds a solution that "works". Problems which are typically solved using backtracking technique have following property in common. These problems can only be solved by trying every possible configuration and each configuration is tried only once.

Note : We can solve this by using **recursion method.**



Problems based on Backtracking :

1. **N-Queen Problem :** Given a chess board having N×N cells, we need to place N queens in such a way that no queen is attacked by any other queen. A queen can attack horizontally, vertically and diagonally.

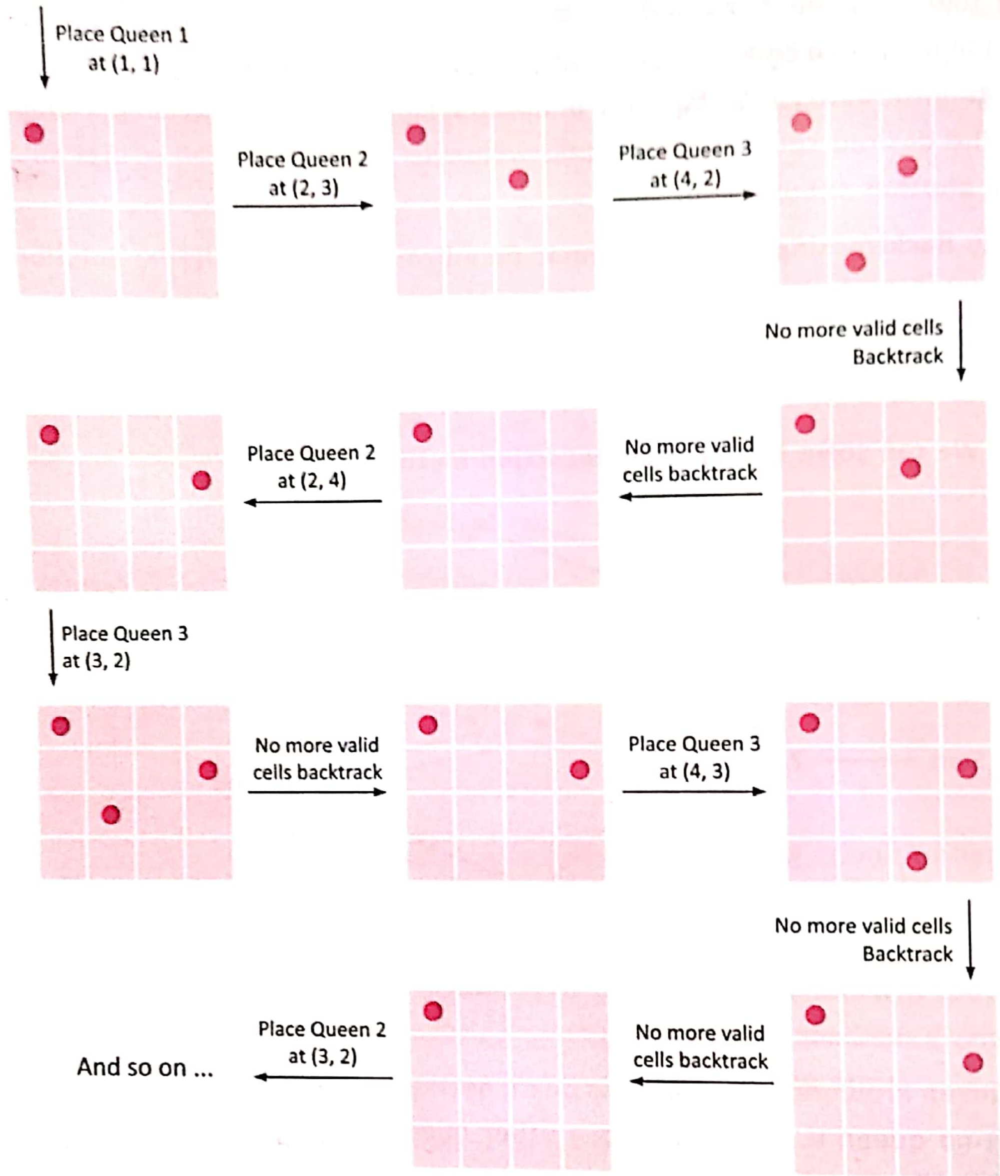
   Approach towards the Solution:

   We are having N × N unattacked cells where we need to place N- queens. Let's place the first queen at a cell (i, j), so now the number of unattacked cells is reduced, and number of queens to be placed is N – 1. Place the next queen at some unattacked cell. This again reduces the number of unattacked cells and number of queens to be placed becomes N – 2. Continue doing this, as long as following conditions hold.

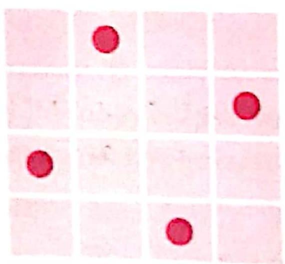
   - ❑ The number of unattacked cells is not equal to 0.
   - ❑ The number of queens to be placed is not equal to 0.

   If the number of queens to be placed becomes 0, then it's over, we found a solution. But if the number of unattacked cells become 0, then we need to backtrack, i.e. remove the last placed queen from its current cell, and place it at some other cell.

Let's N = 4, We have to place 4 queen in 4×4 cells.



So, final solution of this problem is

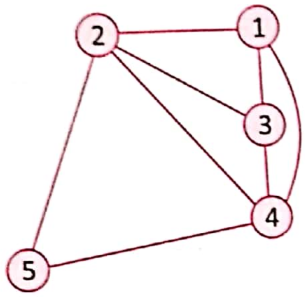


So, clearly, we tries solving a subproblem, if that does not result in the solution, it undo whatever changes were made and solve the next subproblem. If the solution does not exists (like N=2), then it returns false.

2. M Coloring Problems :

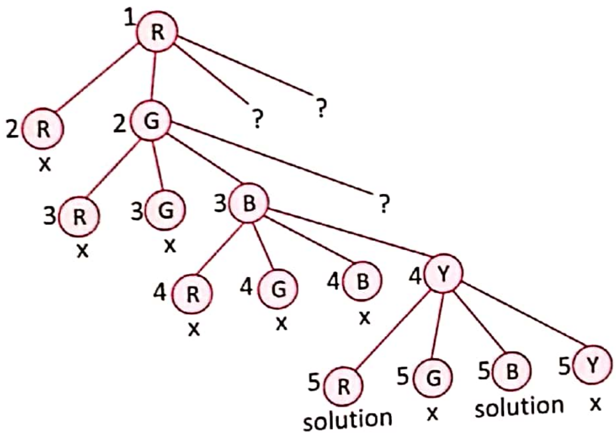
Given an undirected graph and a number m, determine if the graph can be colored with at most m colors such that no two adjacent vertices of the graph are colored with same color. Here coloring of a graph means assignment of colors to all vertices.

Suppose we have a graph G in the given figure and we need to color the vertex of this graph in such away that no two adjacent vertices have same color.



This graph coloring problem can be solved using the mechanism of Backtracking concept.

The state space tree for the above map is shown below :



Hence the above graph can be colored using four colors and four colors are Red, Green, Blue and Yellow.

3. Robots on Ice :

In this problem , we have given a grid size and a sequence of three check-in points. We need to find how many different tours are possible for to check-in all those points with equal time interval.

Let's try to understand this problem with example.

Suppose the given grid size is 3 × 6 and that the three check-in points , in order to visitation are (2,1), (2, 4) and (0, 4). Robot must start at (0, 0) and end at (0, 1) . It must visit location (2, 1) on step 4 (= #18/4#), location (2, 4) on step 9 (= #18/2#), and location (0, 4) on step 13 (= #3 × 18/4#) and at the end it must visit all 18 squares.

Approach towards the solution :

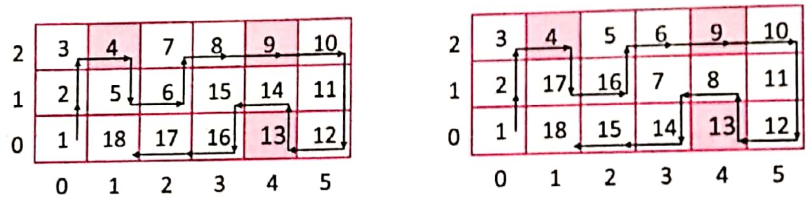
This problem has been break into three main subproblems. In first subproblem we have to check all those path via we can reach to square (2, 1) starting from (0, 0) in 4 steps.

After this we have to check all those path via we can reach to square (2,4) starting from (2,1) in 5 (= 9-4) steps. Then we have to check all those paths via we can reach to square (0,4) starting from (2,4) in 4 (=13-9) steps. Now finally we need to rached at squares (0, 1) in 5 (=18 – 13) steps.

In above process, we have to remind that Robot must visit every square only once.

This is a backtracking problem where we have to check all possible paths to check-in all three given points.

In this given example , there are only two ways to do this.



4. Rat in a Maze :

Given a maze, NxN matrix. A rat has to find a path from source to des-ti-na-tion. maze[0][0] (left top corner)is the source and maze[N-1][N-1](right bot-tom cor-ner) is des-ti-na-tion. There are few cells which are blocked, means rat can-not enter into those cells. The rat can move only in two directions: forward and down.

Approach Towards the Solution :

Here we have to generate all paths from source to destination and one by one check if the generated path satisfies the constraints or not.

Let's the given maze is in the form of matrix whose entries are either 0 or 1. Entry 0 represent the block path from where the rat can't move. We have to print another matrix whose entries are either 0 or 1. In output matrix, entry 1 represents the the path of Rat from starting point to destination point.

Algorithm to generate all the Paths :

```
While there are untried paths
{
      Generate the next paths
      If this path has all blocks as 1
        {
              Print this path;
        }
}
```

Backtracking Algorithm :

If destination is reached

Print the solution

Else

{

a) Mark current cell in solution matrix as 1.

b) Move forward in horizontal direction and recursively check if this move leads to a solution.

c) If the move chosen in the above step doesn't lead to a solution then move down and check of this move leads to a solution.

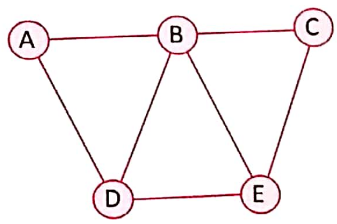
d) If none of the above solutions work then unmark this cell as 0 (Backtrack) and return false.

}

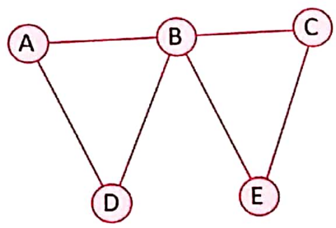
1. Finding Hamiltonian Cycle :

Hamiltonian Path in an undirected graph is a path that visits each vertex exactly once. A **Hamiltonian cycle (or Hamiltonian circuit)** is a Hamiltonian Path such that there is an edge (in graph) from the last vertex to the first vertex of the Hamiltonian Path.Determine whether a given graph contains Hamiltonian Cycle or not.

Example :



A Hamiltonian Cycle in the above graph is {A, B, C, E, D, A}. There are more Hamiltonian Cycles in the graph like {A, D, E, C, B, A}.



There is no any Hamiltonian Cycle in the above graph.

Approach towards the solution :

A naive algorithm is to generate all possible configurations of vertices and print a configuration that satisfies the given constraints. It will be n! (n factorial) configurations.

While there are untried configurations

{

```
Generate the next configuration

  If (there are edges between two consecutive vertices of this configuration and there is an edge
from the last vertex to the first )
    {
        Print this configuration;
        break;
    }

}
```

Backtracking Algorithm :

Create an empty path array and add vertex 0 to it. Add other vertices, starting from the vertex 1. Before adding a vertex, check for whether it is adjacent to the previously added vertex and not already added. If we find such a vertex, we add the vertex as part of the solution. If we do not find a vertex then we return false.

TIME TO TRY

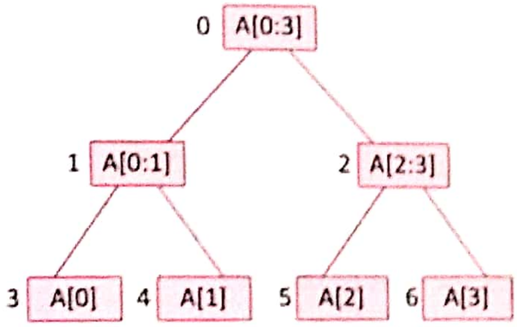
1. Knight's Tour
2. N-Queen
3. Sudoku Solver
4. Permutations
5. Rat in a Maze
6. T-shirts (Cochef)

7

What is a Segment Tree?

A Segment Tree is a full binary tree where each node represents an interval.

Generally a node would store one or more properties of an interval which can be queried later.



Segment Tree of an Array consisting of four elements

Why do we need Segment Tree?

Many problem require that we give results based on query over a range or segment of available data. This can be a tedious and slow process, especially if the number of queries is large.

A segment tree let's us process such queries efficiently in logarithmic order of time.

How do we make a Segment Tree?

Let the data be in array arr[]

1. The root of our tree will represent the entire interval of data we are interested in, i.e, arr[0...n-1].
2. Each leaf of the tree will represent a range consisting of just a single element. Thus the leaves represent arr[0], arr[1], ... , arr[n-1].
3. The internal nodes will represent the merged result of their child nodes.
4. Each of the children nodes could represent approximately half of the range represented by their parent.

Segment Tree generally contain three types of methods: **BUILD, QUERY, UPDATE.**

Let's Start with an Example

Problem : Range Minimum Query

You are given a list of N numbers and Q queries. There are two types of query:

1. 0 l r - find the minimum element in range [l,r].
2. 1 i a - update the element at ith position of array to val.

The problem states that we have to find minimum element in a given range [i,j], or update the element of the given array.

Basic Approach

For any range [i, j], we can answer the minimum element in O(n). But as we'll be having Q queries then we have to find minimum of a range, Q times.

So the overall complexity will be O(QxN).

So, to solve range based problems and to bring the complexity to logarithmic time we use Segment Tree.

We'll use tree[] to store the nodes of our Segment Tree(initialized to all zeros).

- The root of the tree is at index 1, i.e, tree[1] is the root of our tree.
- The children of tree[i] are stored at tree[i * 2] and tree[i * 2 +1].

```
#include <iostream>
using namespace std;
int tree[400005]; //will store our segment tree structure
int arr[100005]; //input array
```

BUILD Function :

Here, Build Function takes three parameter, **node, a, b.**

```
void build_tree(int node, int a, int b)
{ //BUILD function
// node represents the current node number
// a, b represents the current node range
// for leaf, node range will be single element
// so 'a' will be equal to 'b' for leaf node
if (a == b)
  { //checking if node is leaf
    //for single element, minimum will be the element itself
    tree[node] = arr[a]; //storing the answer in our tree structure
    return;
  }
```

```
    int mid = (a + b) >> 1; //middle element of our range
    build_tree(node * 2, a, mid); //recursively call for left half
    build_tree(node * 2 + 1, mid + 1, b); //call for right half
     //left child and right child are build
     // now build the parent node using its child nodes
   tree[node] = __min(tree[node*2], tree[node * 2 + 1]);
}
```

QUERY Function

```
int query_tree(int node, int a, int b, int i, int j)
{
// i, j represents the range to be queried
 if (a > b || a > j || b < i) return INT_MAX; //out of range
 if (a >= i && b <= j)
 { //current segment is totally within range[i,j]
   return tree[node];
 }
 int mid = (a + b) >> 1;
 //Query left child
 int q1 = query_tree(node * 2, a, mid, i, j);
 //Query right child
 int q2 = query_tree(node * 2 + 1, mid + 1, b, i, j);
 return __min(q1, q2); // return final result
}
```

UPDATE Function

```
// updates the ith element to val
void update_tree(int node, int a, int b, int i, int val)
{
 if (a > b || a > i || b < i) return; //out of range
 if (a == b) { //leaf node
 tree[node] = val;
 return;
}
 int mid = (a + b) >> 1;
 update_tree(node * 2, a, mid, i, val); // updating left child
```

```
    int mid = (a + b) >> 1; //middle element of our range
    build_tree(node * 2, a, mid); //recursively call for left half
    build_tree(node * 2 + 1, mid + 1, b); //call for right half
     //left child and right child are build
     // now build the parent node using its child nodes
   tree[node] = __min(tree[node*2], tree[node * 2 + 1]);
}
```

QUERY Function

```
int query_tree(int node, int a, int b, int i, int j)
{
// i, j represents the range to be queried
  if (a > b || a > j || b < i) return INT_MAX; //out of range
  if (a >= i && b <= j)
  { //current segment is totally within range[i,j]
     return tree[node];
  }
  int mid = (a + b) >> 1;
  //Query left child
  int q1 = query_tree(node * 2, a, mid, i, j);
  //Query right child
  int q2 = query_tree(node * 2 + 1, mid + 1, b, i, j);
  return __min(q1, q2); // return final result
}
```

UPDATE Function

```
// updates the ith element to val
void update_tree(int node, int a, int b, int i, int val)
{
  if (a > b || a > i || b < i) return; //out of range
  if (a == b) { //leaf node
  tree[node] = val;
  return;
}
  int mid = (a + b) >> 1;
  update_tree(node * 2, a, mid, i, val); // updating left child
```

```
update_tree(node * 2 + 1, mid + 1, b, i, val);// updating right child
// updating node with min value
tree[node] = __min(tree[node * 2], tree[node * 2 + 1]);
}
```

Complexity Analysis

BUILD

We visit each leaf of the Segment Tree. That makes n leaves. Also there will be n-1 internal nodes. So we process about 2 * n nodes. This makes the Build process run in O(n) linear complexity.

UPDATE

The update process discards half of the range for every level of recursion to reach the appropriate leaf in the tree. This is similar to binary search and takes logarithmic time. After the leaf is updated, its direct ancestors at each level of the tree are updated. This takes time linear to height of the tree. So the complexity will be O(lg(n)).

QUERY

The query process traverses depth-first through the tree looking for node(s) that match exactly with the queried range. At best, we query for the entire range and get our result from the root of the segment tree itself. At worst, we query for a interval/range of size 1 (which corresponds to a single element), and we end up traversing through the height of the tree. This takes time linear to height of the tree. So the complexity will be O(lg(n)).

Problem : Range Sum

We have an array arr{0,.......n,– 1} and we have to perform two types of queries:

1. Find sum of elements from index l to r where 0 <= l <= r <= n
2. Increase the value of all elements from l to r by a given number.

As we can build the parent node using its two child nodes. So, we'll use the Segment Tree to answer the sum of elements from l to r.

Code :

```
int tree[400005]; //will store our segment tree structure
int arr[100005]; //input array
void build_tree(int node, int a, int b)
{ //BUILD function
if (a == b)
{ //checking if node is leaf
    //for single element, sum will be the element itself
```

```
    tree[node] = arr[a]; //storing the answer in our tree structure
    return;
}
int mid = (a + b) >> 1; //middle element of our range
build_tree(node * 2, a, mid); //recursively call for left half
build_tree(node * 2 + 1, mid + 1, b); //call for right half
//left child and right child are build
// now build the parent node using its child nodes
tree[node] = tree[node * 2] + tree[node * 2 + 1];
}
int query_tree(int node, int a, int b, int i, int j)
{
// i, j represents the range to be queried
if (a > b || a > j || b < i)
    return 0; //out of range
if (a >= i && b <= j)
{ //current segment is totally within range[i,j]
  return tree[node];
}
int mid = (a + b) >> 1;
int q1 = query_tree(node * 2, a, mid, i, j); //Query left child
int q2 = query_tree(node * 2 + 1, mid + 1, b, i, j);
//Query right child
return (q1 + q2); // return final result
}
void update_tree(int node, int a, int b, int i,int j, int val)
{
if (a > b || a > j || b < i)
    return; //out of range
if (a == b)
{ //leaf node
  tree[node] += val;
  return;
}
int mid = (a + b) >> 1;
```

```
update_tree(node * 2, a, mid, i, j, val);;// updating left child
update_tree(node * 2 + 1, mid + 1, b, i, j, val); // updating right child
tree[node] = tree[node * 2] + tree[node * 2 + 1];
}
```

The above approach will query the tree in O(lg(n)) time. But the problem will arise in the update function. As updating a single element takes O(lg(n)) time, now we are doing a range update. So, the overall complexity for update will be O(nlg(n)), which will get TLE as we have to answer for multiple queries.

Lazy Propagation

In the current version when we update a range, we branch its childs even if the segment is covered within range. In the lazy version we only mark its child that it needs to be updated and update it when needed.

In short, we try to postpone updating descendants of a node, until the descendants themselves need to be accessed.

We use another array lazy[] which is the same size as our segment tree array tree[] to represent a lazy node. lazy[i] holds the amount by which the node tree[i] needs to be incremented, when that node is finally accessed or queried. When lazy[i] is zero, it means that node tree[i] is not lazy and has no pending updates.

UPDATE function

```
void update_tree(int node, int a, int b, int i,int j, int val)
{
if (lazy[node] != 0)
{ // lazy node
 tree[node] += (b - a + 1)*lazy[node]; //normalize current node by removing
laziness
 if (a != b)
 { // update lazy[] for children nodes
 lazy[node * 2] += lazy[node];
 lazy[node * 2 + 1] += lazy[node];
 }
 lazy[node] = 0; // remove laziness from current node
}
if (a > b || a > j || b < i)
  return; //out of range
if (a >= i && b <= j)
{ // segment is fully within update range
```

```
tree[node] += (b - a + 1)*val;
if (a != b)
{ // update lazy[] for children nodes
  lazy[node * 2] += lazy[node];
  lazy[node * 2 + 1] += lazy[node];
}
return;
}
int mid = (a + b) >> 1;
update_tree(node * 2, a, mid, i, j, val);;// updatingleft child
update_tree(node * 2 + 1, mid + 1, b, i, j, val); //updating right child
tree[node] = tree[node * 2] + tree[node * 2 + 1];
}
```

QUERY function

```
int query_tree(int node, int a, int b, int i, int j)
{
// i, j represents the range to be queried
if (a > b || a > j || b < i)
 return INT_MAX; //out of range
if (lazy[node] != 0)
{ // lazy node
  tree[node] += (b - a + 1)*lazy[node]; //normalize current node by removing
laziness
 if (a != b)
 { // update lazy[] for children nodes
   lazy[node * 2] += lazy[node];
   lazy[node * 2 + 1] += lazy[node];
 }
 lazy[node] = 0; // remove laziness from current node
}
if (a >= i && b <= j)
{ //current segment is totally within range[i,j]
 return tree[node];
}
int mid = (a + b) >> 1;
int q1 = query_tree(node * 2, a, mid, i, j); //Query left child
```

```
int q2 = query_tree(node * 2 + 1, mid + 1, b, i, j);
//Query right child
return (q1 + q2); // return final result
}
```

Using Lazy Propagation the range update will be done in O(lg(n))time complexity.

Note :

The following lines:

tree[node] += (b - a + 1)*lazy[node]; //normalize currentnode by removing laziness and

tree[node] += (b - a + 1)*val; // update segment and

tree[node] = tree[node * 2] + tree[node * 2 + 1]; //merge updates are specific to Range Sum Query problem. Different problems may have different updating and merging schemes.

Problem : GSS1 - Can you answer these queries

You are given a sequence A[1], A[2], ..., A[N] . ($|A[i]| \leq 15007$, $1 \leq N \leq 50000$). A query is defined as follows:

Query(x,y) = Max { a[i]+a[i+1]+...+a[j] ; $x \leq i \leq j \leq y$ }.

Given M queries, your program must output the results of these queries.

INPUT :

3

-1 -2 -3

1

1 2

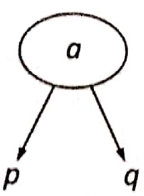
OUTPUT :

2

http://www.spoj.com/problems/GSS1/

Now we need to use a Segment Tree. But what data to store in each node, such that it is easy to compute the data associated with a given node if we already know the data associated with its two child nodes.

We need to find a maximum sum subarray in a given range.



Say we have a as a parent node and p and q as its child nodes. Now we need to build data for a from p and q such that node a can give the maximum sum subinterval for its range when queried.

So for this what do we need??

Maximum sum subarray in a can be equal to either:

1. Maximum sum subarray in p
2. Maximum sum subarray in q
3. Elements including from both p and q

So for each node we need to store:

1. Maximum Prefix Sum
2. Maximum Suffix Sum
3. Total Sum
4. Best Sum (Maximum Sum Subarray for a)

Maximum Prefix Sum can be calculated for a node as

a.prefix = max(p.prefix,p.total + q.prefix)

Maximum Suffix Sum can be calculated for a node as

a.suffix = max(p.suffix,q.total + p.suffix)

Total Sum

a.total = p.total + q.total

Best Sum

a.best = max(p.suffix + q.prefix , max(p.best,q.best))

Code :

```
long long int arr[50005]; //input array
struct Tree
{
 long long int prefix, suffix, total, best;
};
Tree tree[200005];
void build_tree(long long int node, long long int a, long long int b)
{
if (a == b)
{ //leaf node
  tree[node].prefix = arr[a]; // prefix sum
  tree[node].suffix = arr[a]; // suffix sum
  tree[node].total = arr[a]; // total sum
  tree[node].best = arr[a]; // best sum
  return;
 }
```

```
int mid = (a + b) >> 1;
build_tree(node * 2, a, mid);
build_tree(node * 2 + 1, mid + 1, b);
tree[node].prefix = max(tree[node * 2].prefix, tree[node*2].total + tree[node *
2 + 1].prefix); //prefix sum
tree[node].suffix = max(tree[node * 2 + 1].suffix, tree[node * 2].suffix +
tree[node * 2 + 1].total); //suffix sum
tree[node].total = tree[node * 2].total + tree[node *2 + 1].total; //total sum
tree[node].best = max(tree[node * 2].suffix + tree[node * 2 + 1].prefix,
max(tree[node * 2].best, tree[node *2 + 1].best)); //best sum
}
Tree query_tree(long long int node, long long int a, long long int b, long long int
i, long long int j)
{
Tree t;
if (a > b || a > j || b < i)
{
  t.prefix = t.suffix = t.best = INT_MIN ;
  t.total = 0;
  return t;
}
if (a >= i && b <= j)
{
  return tree[node];
}
long long int mid = (a + b) >> 1;
Tree q1 = query_tree(node * 2, a, mid, i, j);
Tree q2 = query_tree(node * 2 + 1, mid + 1, b, i, j);
t.prefix = max(q1.prefix, q1.total + q2.prefix);
t.suffix = max(q2.suffix, q1.suffix + q2.total);
t.total = q1.total + q2.total;
t.best = max(q1.suffix + q2.prefix, max(q1.best, q2.best));
return t;
}
```

Similar Problem: http://www.spoj.com/problems/GSS3/

The problem also includes an update operation, rest of the methods are same.

Problem : QSET (Queries on the String)

You have a string of N decimal digits, denoted by numbers A1,A2, ..., AN.

Now you are given M queries, each of whom is of following two types.

1. 1 X Y: Replace AX by Y.
2. 2 C D: Print the number of sub-strings divisible by 3 in range [C,D]

Formally, you have to print the number of pairs (i,j) such that the string Ai,Ai+1...Aj, (C ≤ i ≤ j ≤ D), when considered as a decimal number, is divisible by 3.

INPUT :

5 3
01245
2 1 3
1 4 5
2 3 5

OUTPUT :

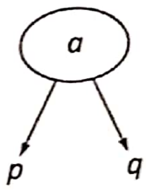
3
1

https://www.codechef.com/problems/QSET

A number is divisible by 3 if sum of its digits are divisible by 3.

Use segment trees. Store in node for each interval,

- the answer for that interval
- prefix[], where prefix[i] denotes number of prefixes of interval which modulo 3 give i.
- suffix[], where suffix[i] denotes number of suffixes of interval which modulo 3 give i.
- total sum of interval modulo 3.



Answer for interval denoted by a will be:

a.ans = p.ans + q.ans.

(Number of Suffixes in p which when taken with modulo 3 gives 1) + (Number of Prefixes in q which when taken with modulo 3 gives 2) —> also gives the substring divisible by 3 in a.

Similarly, for all possible combinations we can calculate the answer for an interval.

```
a.ans = p.ans + q.ans;
for (int i = 0; i < 3; ++i)
{
```

```
for (int j = 0; j < 3; ++j)
  {
    if ((i + j) % 3 == 0)
    {
      a.ans += p.suffix[i] * q.prefix[j];
    }
  }
}
```

How to build prefix and suffix array for a node?

There are p.prefix[i] prefixes of p which when taken modulo with 3 gives i, and there are some prefixes which are made by combining whole of p and some prefixes of q.

So, total prefixes in a which when taken modulo with 3 gives i will be:
(Same for suffix)

```
for (int i = 0; i < 3; ++i)
{
a.prefix[i] = p.prefix[i] + q.prefix[(3 - q1.total + i) % 3];
a.suffix[i] = q.suffix[i] + p.suffix[(3 - q2.total + i) % 3];
}
```

Code :

```
int arr[100005]; //input array
struct Tree
{
int ans;
int prefix[3], suffix[3];
int total;
};
Tree tree[400005];
void build_tree(int node, int a, int b)
{
if (a == b)
{
  tree[node].prefix[arr[a] % 3] = 1;
  tree[node].suffix[arr[a] % 3] = 1;
  tree[node].total = arr[a] % 3;
```

```
    tree[node].ans = (arr[a] % 3 == 0 ? 1 : 0);
    return;
}
int mid = (a + b) >> 1;
build_tree(node * 2, a, mid);
build_tree(node * 2 + 1, mid + 1, b);
for (int i = 0; i < 3; ++i)
{
    tree[node].prefix[i] = tree[node * 2].prefix[i] +
    tree[node * 2 + 1].prefix[(3 - tree[node * 2].total + i)% 3];
tree[node].suffix[i] = tree[node * 2 + 1].suffix[i] + tree[node * 2].suffix[(3 -
tree[node * 2 + 1].total+ i) % 3];
}
tree[node].total = (tree[node * 2].total + tree[node* 2 + 1].total) % 3;
tree[node].ans = tree[node * 2].ans + tree[node * 2 +1].ans;
for (int i = 0; i < 3; ++i)
```

PROBLEM : JTREE (JosephLand)

Nick lives in a country named JosephLand. JosephLand consists of **N** cities. **City 1** is the capital city. There are **N - 1** directed roads. It's guaranteed that it is possible to reach capital city from any city, and in fact there is a unique path from any city to the capital city.

Besides, you can't cross roads for free. To pass a road, you must have a ticket. There are total **M** tickets. You can not have more than one ticket at a time! Each ticket is represented by three integers:

v k w : you can buy a ticket with cost w in the city v. This ticket can be used at max k times. That means, after using this ticket for k roads ticket can't be used!

By the way, you can tear your ticket any time and buy a new one. But you are not allowed to buy a ticket if you are still having a ticket with you!

Nick's home is located in the capital city. He has Q friends, and he wants to invite all of them for dinner! So he is interested in knowing about how much each of his friends is going to spend in the journey!

His friends are quite smart and always choose a route to capital city that minimizes his/her spending! Nick has to prepare dinner, so he doesn't have time to figure out himself, Can you please help him?

Please note that **it's guaranteed that, one can reach the capital from any city using the tickets!**

INPUT :

7 7
3 1
2 1
7 6
6 3
5 3
4 3
7 2 3
7 1 1
2 3 5
3 6 2
4 2 4
5 3 10
6 1 20
3
5
6
7

OUTPUT :

10
22
5

https://www.codechef.com/problems/JTREE

The problem in simple words is:

Given a tree with edges directed towards root and nodes having some ticket information which allows you to travel k units towards the root with cost w. Answer Q queries i.e output the minimum cost for travelling from a node x to root.

Let's try to solve problem for a node X at depth H from root 1. Think of this path as a 1D vector V where we have all the H-1 nodes between root and X. The nodes are stored in vector in increasing order of depth.

Let dp[x] be the minimum cost to travel from X to the root.

So, for a given node, iterate over all the tickets present at node X and DP state will be like this:

```
for(int i=0;i<total_tickets;i++)
{
    K=current_ticket_jump_info;
    W=weight_ticket_info;
    for(int j=V.size()-1;j>= max(0,V.size()-1-k);j++) //loop 2
```

```
    {
      DP[X]=min( DP[X] , DP[V[j]] + W );
    }
  }
```

We will use DFS to find vector V for every node X.

Now the code will be :

```
void dfs(int u,int p)
{
V.push_back(u);
for(int l=0 ; l< adj[u].size() ; l++)
{
   if(adj[u][l]==p)
     continue;
   int x =adj[u][l];
   for(int i=0;i<total_tickets;i++) // loop 1
   {
     K=current_ticket_jump_info;
     W=weight_ticket_info;
     for(int j=V.size()-1;j>= max(0,V.size()-1-k);j++) //loop 2
     {
       DP[x]=min( DP[x] , DP[V[j]] + W );
     }
   }
   dfs(adj[u][l],u);
 }
V.pop_back();
}
```

The complexity will be **O(n + m * n).**

But If we look at the inner loop of the code which goes up to K times and we find the minimum of DP, this can be done using any data structure that supports **RMQ(Range Minimum QUery) + Point updates** in **O(logn)** time . Which will make the total complexity to be **O(N + MlogN).**

So we can build a segment tree that supports two operations.

First : find the minimum element in range L,R and

Second: update an element's value to VAL.

And we need to query for the minimum element between [H-K , H] so we will build the tree over height H.

Now just print DP[x] for every query.

Code :

```
void dfs(long long int cur, long long int h)
{
for (int j = 0; j < n_info[cur].size(); ++j)
{ //no. of tickets
   long long int k1 = n_info[cur][j].first;
   long long int w1 = n_info[cur][j].second;
   dp[cur] = min(dp[cur], query_tree(1, 0, N - 1, max(0LL, h - k1), h) + w1); //
find the min cost from cur to root
}
update_tree(1, 0, N - 1, h, dp[cur]); //update the
tree at posn h with its DP value
for(int i = 0; i < g_tree[cur].size(); ++i)
{
 long long int x = g_tree[cur][i];
 dfs(x, h + 1);
}
update_tree(1, 0, N - 1, h, INF_n); //update the tree at posn h with INF as we
are done with its subtree
}
int main()
{
for(int i = 0; i < 400003; ++i)
{
tree[i] = INF_n;
 dp[i / 4] = INF_n;
}
 cin >> N >> M;
for(int i = 0; i < N - 1; ++i)
{
 cin >> a >> b;
 g_tree[b].push_back(a);
}
for(int i = 0; i < M; ++i)
```

```
{
 cin >> v >> k >> w;
 n_info[v].push_back(make_pair(k, w));
}
dp[1] = 0;
update_tree(1, 0, N - 1, 0, dp[1]);
dfs(1, 0);
cin >> Q;
while (Q--)
{
  cin >> a;
  cout << dp[a] << "\n";
}
return 0;
}
```

PROBLEM : COUNZ (Counting Zeroes)

You are given an array of N integers(Indexed at 1)

For the given array you have to answer some queries given later. The queries are of 2 types:

1. **TYPE 1 -> 1 L R (where L and R are integers)**

   For this query you have to calculate the product of elements of the array in the range L to R (both inclusive) and print the number of zeros at the end of the result.

2. **TYPE 2 -> 0 L R V (where L,R,V are integers)**

   For this query you have to set the value of all the elements in the array ranging from L to R (both inclusive) to V.

INPUT :

5

1 3 5 8 9

3

1 2 5

0 1 4 10

1 1 5

OUTPUT :

1

4

https://www.codechef.com/problems/COUNZ

We'll use Segment Tree to answer the queries.

To determine the number of zeroes at the end of a number, we should know number of 2's and 5's in its prime factorization.

Number Of Zeroes = min(No. of 2's, No. of 5's)

Number of Zeroes in product = min(sum of 2's in elements in product, sum of 5's in elements in product).

Be careful of element having value 0 in product as number of zeroes will be 1.*

Since any array element can have value 0 at any point of time we store 3 values at a node in segment tree sum of number of twos in prime factorization of all the elements in range, sum of number of twos in prime factorization of all the elements in range and number of elements in range with value 0.

Now, using lazy propagation we can answer queries in **O(log2(MaxA[i])+logN) time.**

Total Complexity : O(Nlog2(Max A[i])+Q(log2(Max A[i])+logN))

Code :

```
void process_node(int node,int num)
{
tree[node][0] = tree[node][1] = tree[node][2] = 0;
if (num == 0)
    tree[node][0]++;
while (num % 2 == 0 && num != 0)
{
  num /= 2;
  tree[node][1]++; //no of 2 in PF of arr[a];
}
while (num % 5 == 0 && num != 0)
{
  num /= 5;
tree[node][2]++; //no of 5 in PF of arr[a];
}
return;
}
void build_tree(int node, int a, int b)
{
if (a == b)
{
int num = arr[a];
```

```
  process_node(node, num);
  return;
}
int mid = (a + b) >> 1;
build_tree(node * 2, a, mid);
build_tree(node * 2 + 1, mid + 1, b);
for (int i = 0; i < 3;++i) //build the parent node
 tree[node][i] = tree[node * 2][i] + tree[node * 2+ 1][i];
}
void update_tree(int node, int a, int b, int i, int j,int val)
{
if (lazy[node] != -1)
{ //if lazy
  process_node(node, lazy[node]);
  for (int i = 0; i < 3; ++i)
   tree[node][i] *= (b - a + 1);
  if (a != b)
  { //not a leaf node
   lazy[node * 2] = lazy[node * 2 + 1] = lazy[node]; //mark lazy
  }
 lazy[node] = -1; //unmark lazy
 }
 if (a > b || a > j || b < i)return;
 if (a >= i && b <= j)
 {
  process_node(node, val);
  for (int i = 0; i < 3;++i)
   tree[node][i] *= (b - a + 1);
  if (a != b)
 {
   lazy[node * 2] = lazy[node * 2 + 1] = val;
  }
  return;
 }
 int mid = (a + b) >> 1;
```

```
    update_tree(node * 2, a, mid, i, j, val);
    update_tree(node * 2 + 1, mid + 1, b, i, j, val);
    for (int i = 0; i < 3; ++i)
      tree[node][i] = tree[node * 2][i] + tree[node * 2 + 1][i];
    }

    void query_tree(int node, int a, int b, int i, int j)
    {
    if (lazy[node] != -1)
    { //if lazy
      process_node(node, lazy[node]);
       for (int i = 0; i < 3; ++i)
        tree[node][i] *= (b - a + 1); //multiply value by range times
      if (a != b)
      { //if not leaf
      lazy[node * 2] = lazy[node * 2 + 1] = lazy[node];//mark as lazy
      }
      lazy[node] = -1; //unmark lazy
    }
    if (a > b || a > j || b < i)
    {
      q_tree[node][0] = q_tree[node][1] = q_tree[node][2] = 0;
      return;
    }
    if (a >= i && b <= j)
    {
     for (int i = 0; i < 3;++i)
        q_tree[node][i] = tree[node][i];
     return;
    }
    int mid = (a + b) >> 1;
    query_tree(node * 2, a, mid, i, j);
    query_tree(node * 2 + 1, mid + 1, b, i, j);
    for (int i = 0; i < 3; ++i)
    q_tree[node][i] = q_tree[node * 2][i] + q_tree[node * 2 + 1][i];
    }
```

PROBLEM : DIVMAC (Dividing Machine)

Chef has created a special dividing machine that supports the below given operations on an array of positive integers.

There are two operations that Chef implemented on the machine.

```
Type 0 Operation
Update(L,R):
for i = L to R:
   a[i] = a[i] / LeastPrimeDivisor(a[i])
Type 1 Operation
Get(L,R):
result = 1
for i = L to R:
   result = max(result, LeastPrimeDivisor(a[i]))
return result;
```

The function **LeastPrimeDivisor(x)** finds the smallest prime divisor of a number. If the number does not have any prime divisors, then it returns 1.

Chef has provided you an array of size N, on which you have to apply M operations using the special machine. Each operation will be one of the above given two types. Your task is to implement the special dividing machine operations designed by Chef. Chef finds this task quite easy using his machine, do you too?

INPUT :

```
2
6 7
2 5 8 10 3 44
1 2 6
0 2 3
1 2 6
0 4 6
1 1 6
0 1 6
1 4 6
2 2
1 3
0 2 2
1 1 2
```

OUTPUT :

5 3 5 11

1

The problem is :

Given an array of numbers A, support the following two queries:

- for a given range [L, R], reduce each number in A[L...R] by its smallest prime factor
- for a given range [L, R], find the number in A[L...R] with largest smallest prime factor.

When we get the range modification entry, we modify each element in the range one by one, and hence perform (R - L + 1) single element update queries. This approach would be to slow, as it makes the complexity of range update query to **O ((R - L + 1) lg N)**

Note that, if the value of an element A[x] is 1, then update query on A[x] does not have any effect on the segment tree. We call such queries as degenerate queries, and can discard them. Also note that a number M can be written as a product of at most O (lg M) prime numbers, Hence, we can perform at most O (lg M) non-degenerate modification queries on A[x] = M all the latter queries will be degenerate. This means that if all numbers in the array are smaller than M, then a most O (N lg M) non-degenerate single element update queries would be performed. Each single element update query takes O (lg N) time, and hence all update queries can be performed in **O (N lg M lg N)** time.

Code :

```
void modified_sieve()
{ //linear time sieve
lp[1] = 1;
for (int i = 2; i < maxn; ++i)
{
 if (lp[i] == 0)
 {
  lp[i] = i;
  prime.push_back(i);
 }
 for (int j = 0; j < prime.size() && i*prime[j] <maxn; ++j)
 {
  lp[i*prime[j]] = prime[j];
 }
}
}
void build_tree(int node, int a, int b)
{
 if (a == b)
 {
```

```
        tree[node] = lp[arr[a]];
        ones[node] = (arr[a] == 1 ? 1 : 0);
        return;
    }
    int mid = (a + b) >> 1;
    build_tree(node * 2, a, mid);
    build_tree(node * 2 + 1, mid + 1, b);
    ones[node] = ones[node * 2] + ones[node * 2 + 1];
    tree[node] = max(tree[node * 2], tree[node * 2 + 1]);
}
void update_tree(int node, int a, int b, int i, int j)
{
    if (a > b || a > j || b < i)
        return;
    if (a >= i && b <= j)
    {
        if (ones[node] == (b - a + 1))
        {
            return;
        }
        if (a == b)
        { //leaf node
            arr[a] /= lp[arr[a]];
            tree[node] = lp[arr[a]];
            ones[node] = (arr[a] == 1 ? 1 : 0);
            return;
        }
    }
    int mid = (a + b) >> 1;
    update_tree(node * 2, a, mid, i, j);
    update_tree(node * 2 + 1, mid + 1, b, i, j);
    ones[node] = ones[node * 2] + ones[node * 2 + 1];
    tree[node] = max(tree[node * 2], tree[node * 2 + 1]);
}
int query_tree(int node, int a, int b, int i, int j)
{
    if (a > b || a > j || b < i)
    {
        return INT_MIN;
    }
```

```
if (a >= i && b <= j)
{
  return tree[node];
}
int mid = (a + b) >> 1;
int q1 = query_tree(node * 2, a, mid, i, j);
int q2 = query_tree(node * 2 + 1, mid + 1, b, i, j);
return q1 > q2 ? q1 : q2;
}
```

8 Binary Indexed Tree

Binary Indexed Tree also called Fenwick Tree provides a way to represent an array of numbers in an array, allowing prefix sums to be calculated efficiently.

Calculating prefix sums efficiently is useful in various scenarios. Let's start with a simple problem.

We are given an array a[], and we want to be able to perform two types of operations on it.

1. Change the value stored at an index i. (This is called a point update operation)
2. Find the sum of a prefix of length k. (This is called a range sum query)

A straightforward implementation of the above would look like this.

```
int a[] = {2, 1, 4, 6, -1, 5, -32, 0, 1};
void update(int i, int v) //assigns value v to a[i]
{
    a[i] = v;
}
int prefixsum(int k) //calculate the sum of all a[i] s
uch that 0 <= i < k
{
    int sum = 0;
    for(int i = 0; i < k; i++)
        sum += a[i];
    return sum;
}
```

This is a perfect solution, but unfortunately the time required to calculate a prefix sum is proportional to the length of the array, so this will usually time out when large number of such operations are performed.

Can We Do Better Than This?

One efficient solution is to use segment tree that can perform both operation in O(logN) time.

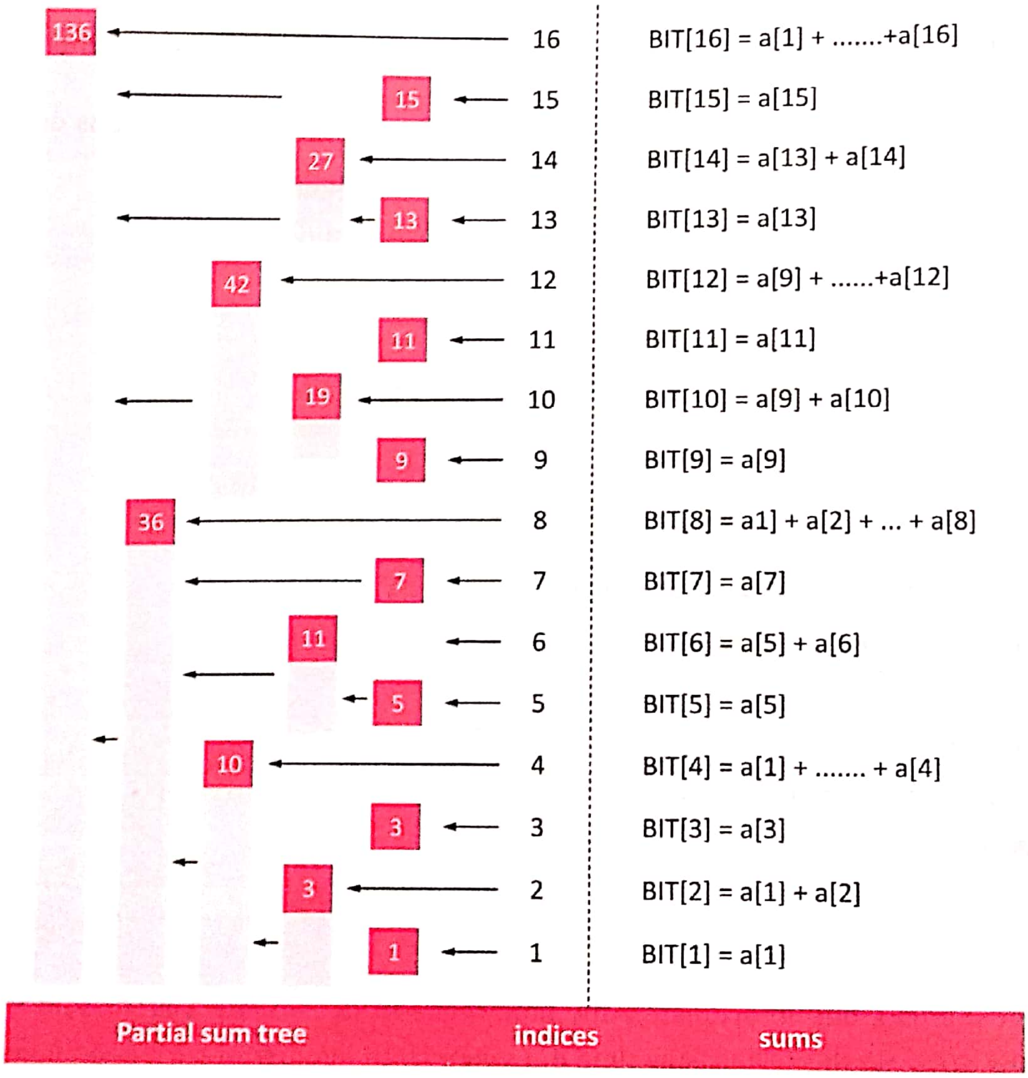
Using Binary Indexed Tree also, we can perform both the tasks in O(logN) time. But then why learn another data structure when segment tree can do the work for us. It's because binary indexed trees require less space and are very easy to implement during programming contests (the total code is not more than 8-10 lines).

Basic Idea of Binary Indexed Tree :

We know the fact that each integer can be represented as sum of powers of two. Similarly, for a given arra of size N, we can maintain an array BIT[] such that, at any index we can store sum of some numbers of th given array. This can also be called a partial sum tree.

Let's use an example to understand how BIT[] stores partial sums.

```
//for ease, we make sure our given array is 1-based indexed
int a[] = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16};
```



The value in the enclosed box represents BIT[index]

The above picture shows the binary indexed tree, each enclosed box of which denotes the value BIT[index] and each BIT[index] stores partial sum of some numbers.

Every index i in the BIT[] array stores the cumulative sum from the index i to i - (1<<r) + 1 (both inclusive), where r represents the last set bit in the index i.

Sum of first 12 numbers in array a[] = BIT[12] + BIT[8] = (a[12] + ... + a[9]) + (a[8] + ... + a[1])

Similarly, sum of first 6 elements = BIT[6] + BIT[4] = (a[6] + a[5]) + (a[4] + ... + a[1])

Sum of first 8 elements = BIT[8] = a[8] + ... + a[1]

Construct The BIT :

Let's see how to construct this tree and then we will come back to querying the tree for prefix sums.

BIT[] is an array of size = 1 + the size of the given array a[] on which we need to perform operations. Initially all values in BIT[] are equal to 0. Then we call update() operation for each element of given array to construct the Binary Indexed Tree. The update() operation is discussed below.

```
void update(int index,int val) {
    while (index <= maxn) {
        tree[index] += val;
        index += (index & (-index));
    }
}
```

Suppose we call update(13, 2).

Here we see from the above figure that indices 13, 14, 16 cover index 13 and thus we need to add 2 to them also.

Initially x is 13, we update BIT[13]

```
BIT[13] += 2;
```

Now isolate the last set bit of x = 13(1101) and add that to x , i.e. x += x&(-x)

Last bit is of x = 13(1101) is 1 which we add to x, then x = 13+1 = 14, we update BIT[14]

```
BIT[14] += 2;
```

Now 14 is 1110, isolate last bit and add to 14, x becomes 14+2 = 16(10000), we update BIT[16]

```
BIT[16] += 2;
```

In this way, when an update() operation is performed on index x we update all the indices of BIT[] which cover index **x** and maintain the BIT[].

If we look at the for loop in update() operation, we can see that the loop runs at most the number of bits in index x which is restricted to be less or equal to n (the size of the given array), so we can say that the update operation takes at most O(log2(n)) time.

Query in BIT :

```
int query(int index) {
    int sum = 0;
    while (index > 0) {
        sum += tree[index];
        index -= (index & (-index));
    }
    return sum;
}
```

The above function query() returns the sum of first x elements in given array. Let's see how it works.

Suppose we call query(14), initially sum = 0

x is 14(1110) we add BIT[14] to our sum variable, thus sum = BIT[14] = (a[14] + a[13])

Now we isolate the last set bit from x = 14(1110) and subtract it from x last set bit in 14(1110) is 2(10), thus x = 14 – 2 = 12

We add BIT[12] to our sum variable, thus sum = BIT[14] + BIT[12] = (a[14] + a[13]) + (a[12] + ... + a[9])

again we isolate last set bit from x = 12(1100) and subtract it from x last set bit in 12(1100) is 4(100), thus x = 12 – 4 = 8 we add BIT[8] to our sum variable, thus

sum = BIT[14] + BIT[2] + BIT[8] = (a[14] + a[13]) + (a[12] + ... + a[9]) + (a[8] + ... + a[1])

Once again we isolate last set bit from x = 8(1000) and subtract it from x last set bit in 8(1000) is 8(1000), thus x = 8 – 8 = 0 since x = 0, the for loop breaks and we return the prefix sum.

Talking about complexity, again we can see that the loop iterates at most the number of bits in x which will be at most n(the size of the given array). Thus the query operation takes O(log2(n)) time.

When To use BIT ?

Before going for Binary Indexed tree to perform operations over range, one must confirm that the operation or the function is:

Associative. i.e f(f(a, b), c) = f(a, f(b, c)) this is true even for seg-tree

Has an inverse. eg:

- Addition has inverse subtraction (this example we have discussed)
- Multiplication has inverse division
- gcd() has no inverse, so we can't use BIT to calculate range gcd's

PROBLEM : INVCNT (Inversion Count)

Let A[0...n - 1] be an array of n distinct positive integers. If i < j and A[i] > A[j] then the pair (i, j) is called an inversion of A. Given n and an array A your task is to find the number of inversions of A.

For example, the array a={2,3,1,5,4} has three inversions: (1,3), (2,3), (4,5), for the pairs of entries (2,1), (3,1), (5,4).

We'll use BIT to solve the problem of Inversion Count.

Replacing the Values of the Array with the Indexes :

Usually when we are implementing a BIT is necessarily to map the original values of the array to a new range with values between [1,N], where N is the size of the array. This is due to the following reasons:

1. The values in one or more A[i] entry are too high or too low.

   (e.g. 10 – 12 or 10^ – 12).

   For example imagine that we are given an array of 3 integers: {1,10^12,5}

   This means that if we want to construct a frequency table for our BIT data structure, we are going to need at least an array of 10^12 elements.

2. The values in one or more A[i] entry are negative.

   Because we are using arrays it's not possible to handle in our BIT frequency of negative values (e.g. we are not able to do freq[–12]).

A simple way to deal with this issues is to replace the original values of the target array for indexes that maintain its relative order.

For example, given the following array:

A	9	1	0	5	4

The first step is to make a copy of the original array **A** let's call it **B**.

Then we proceed to sort **B** in non-descending order as follow:

sorted copy of the array A

B	0	1	4	5	9

Using binary search over the array B we are going to seek for every element in the array A, and stored the resulting position indexes (1-based) in the new array A.

binary_search(B,9)=4 found at position 4 of the array B

binary_search(B,1)=1 found at position 1 of the array B

binary_search(B,0)=0 found at position 0 of the array B

binary_search(B,5)=3 found at position 3 of the array B

binary_search(B,4)=2 found at position 2 of the array B

The resulting array after increment each position by one is the following:

new array A

A	5	2	1	4	3

The following C++ code fragment illustrate the ideas previously explained:

```
for(int i = 0; i < N; ++i)
    B[i] = A[i]; // copy the content of array A to array B

sort(B, B + N); // sort array B

for(int i = 0; i < N; ++i) {

    int ind = int(lower_bound(B, B + N, A[i]) - B);
    A[i] = ind + 1;
}
```

Counting inversions with the accumulate frequency :

Initially the cumulative frequency table is empty, we start the process with the element 3, the last one in our array.

A					
	5	2	1	4	3

cumulative frequency array

0	0	0	0	0

how many numbers less than 3 have we seen so far

x=read(3"1) = 0

inv_counter=inv_counter+x

update the count of 3's so far

update(3,+1)

inv_counter=0

A					
	5	2	1	4	3

cumulative frequency array

0	0	1	1	1

The cumulative frequency of value 3 was increased in the previous step, this is why the read(4"1) count the inversion (4,3).

how many numbers less than 4 have we seen so far

x=read(4"1)=1

inv_counter=inv_counter+x

update the count of 4's so far

update(4,+1)

inv_counter=1

The term 1 is the lowest in our array, this is why there is no inversions beginning at 1.

A	5	2	1	4	3

cumulative frequency array

0	0	1	2	2

how many numbers less than 1 have we seen so far

x=read(1"1)=0

inv_counter=inv_counter+x

update the count of 1's so far

update(1,+1)

inv_counter=1

And so on for the other elements...

PROBLEM : BAADSHAH

You are provided an array consisting of 'n' integers and asked to do following operations:

1. Update the position at "x" of array to "y" ie A[x]=y.
2. Find if there is any prefix in the array whose sum equals to "S".If yes, then output "Found" and the last index of prefix, else output "Not Found".

INPUT :

4 4

1 2 3 4

2 6

1 4 10

2 16

2 5

OUTPUT :

Found 3

Found 4

Not Found

https://www.codechef.com/problems/BAADSHAH

Construct a BIT(Binary Indexed Tree) from the given array, where internal nodes represents range sum.

Use Point updates for query 1, and for query 2, using Binary search on range-query sums, where starting range is 1 to n, check if the prefix sum exists or not.

Code :

```
ll read(ll index) {
    ll sum = 0;
    while (index > 0) {
        sum += BIT[index];
        index -= (index & (-index));
    }
    return sum;
}

void update(ll index, ll val) {
    while (index < maxn) {
        BIT[index] += val;
        index += (index & (-index));
    }
}

int main() {
    cin >> N >> M;
    for (int i = 1; i <= N; ++i) {
        cin >> arr[i];
        update(i, arr[i]);
    }

    while (M--) {
        ll idx;
        cin >> type;
        if (type == 1) {
            cin >> a >> b;
            update(a, b - arr[a]);
            arr[a] = b;
    }
    else {
```

```
        bool flag = false;
        cin >> a;
        ll lo = 1, hi = N;
        while (lo <= hi) {
            int mid = (lo + hi) >> 1;
            ll ans = read(mid);
            if (ans < a) {
                lo = mid + 1;
            }
            else if (ans > a) {
                hi = mid - 1;
            }
            else {
                flag = true;
                idx = mid;
                break;
            }
        }
        if (flag)cout << "Found " << idx << "\n";
        else cout << "Not Found\n";
    }
}
return 0;
}
```

PROBLEM : CTRICK (CARD TRICK)

The magician shuffles a small pack of cards, holds it face down and performs the following procedure:

The top card is moved to the bottom of the pack. The new top card is dealt face up onto the table. It is the Ace of Spades.

Two cards are moved one at a time from the top to the bottom. The next card is dealt face up onto the table. It is the Two of Spades.

Three cards are moved one at a time...

This goes on until the nth and last card turns out to be the n of Spades.

This impressive trick works if the magician knows how to arrange the cards beforehand (and knows how to give a false shuffle). Your program has to determine the initial order of the cards for a given number of cards, 1 d" n d" 20000.

INPUT :

2

4

5

OUTPUT :

2 1 4 3

3 1 4 5 2

We are asked to place 1 in 2nd free cell of our answer, then to place 2 in 3rd free cell of our answer while starting counting from position where we had placed 1 (and starting from the beginning if we reached end of array), then to place 3 in 4th free cell, and so on.

Now add one more optimization. At every moment we know how many free cells are in our array now. Assume we are at position X now, need to move 20 free cells forward, and you know that at given moment

there are 2 free cells before X and 4 free cells after X. It means that we need to find 22nd free cell starting from beginning of array, which is same as going full circle 3 times and then taking 4th free cell. As we can see, we moved to a problem "find i-th zero in array".

Code :

```
int read(int ind) {
    int sum = 0;
    while (ind > 0) {
        sum += tree[ind];
        ind -= (ind & -ind);
    }
    return sum;
}

void update(int ind, int val) {
    while (ind < 20002) {
        tree[ind] += val;
        ind += (ind & -ind);
    }
}
```

```
int ans[20005];
int b_search(int val) {

    int l = 1, r = n;
    while (l < r) {

        int mid = (l + r) >> 1;
        int tt = read(mid);

        if (tt > val) {
        r = mid;
        }
        else if (tt < val) {
            l = mid + 1;
        }
        else {
            r = mid;
            if (ans[mid] == 0) return mid;
        }
    }
    return l;

}
int main() {

    ios_base::sync_with_stdio(false);
    cin.tie(NULL);

    cin >> t;
    while (t--) {

        cin >> n;
        memset(tree, 0, sizeof(tree));
        for (int i = 1; i <= n;i++) {
            update(i, 1);
        }
```

```
        int fcells = n;
        int cur = 1;
        for (int i = 1; i < n + 1;++i) {
            int free = i + 1;
            int freeb = read(cur - 1);
            int freef = read(n) - freeb;
            int yy = free + freeb;
            int cell = yy % fcells;
            if (cell == 0) cell = fcells;
            int ret_ind = b_search(cell);
            ans[ret_ind] = i;
            fcells–;
            update(ret_ind, -1);
            cur = ret_ind;
        }

        for (int i = 1; i < n + 1;++i) {
            cout << ans[i] << " ";
        }
        cout << "\n";
    }

    return 0;
}
```

TIME TO TRY

DQUERY

Given a sequence of n numbers a_1, a_2, ..., a_n and a number of d-queries. A d-query is a pair (i, j) (1 = i = j = n). For each d-query (i, j), you have to return the number of distinct elements in the subsequence a_i, a_{i+1}, ..., a_j.

http://www.spoj.com/problems/DQUERY/