Language accepted by a 2-way DFA is regular

- Claim: The language accepted by a 2-way finite automaton is regular
- We will prove this using Myhill-Nerode theorem
- Recall that the Myhill-Nerode theorem states that a language is regular if and only if the canonical equivalence relation has finitely many equivalence classes

Language accepted by a 2-way DFA is regular

- Let $M = (Q, \Sigma, \vdash, \dashv, \delta, s, t, r)$ be a 2-way finite automaton
- Let $w = xz$ be a string in Σ^*
- As the automaton is 2-way, its read head can cross the boundary between x and z several times.
- Consider the function $T_x : (Q \cup \{\bullet\}) \rightarrow (Q \cup \{\perp\})$
- If the automaton goes to state q when it first crosses the boundary between x and z, define $T_x(\bullet) = q$
- If the read head never crosses the boundary between x and z, define $T_x(\bullet) = \perp$

Language accepted by a 2-way DFA is regular

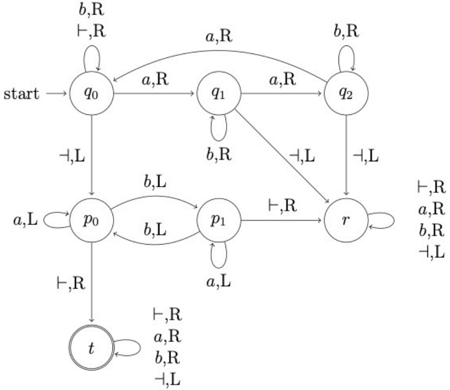
- Suppose the read head comes back from z to x and reaches state q
- Then it may either go back to z reaching state p, in which case define $T_x(q) = p$
- Else, it may never go back to z, in which case define $T_x(q) = \perp$
- Note that the function T_x is well-defined as the automaton is deterministic
- Also, T_x depends only on x and is independent of z
- And if y is another string in Σ^* such that $T_x = T_y$, then yz is accepted by the automaton if and only if xz is accepted by the automaton

Language accepted by a 2-way DFA is regular

- Hence, xRy if and only if $T_x = T_y$ becomes a canonical equivalence relation
- Since the number of unique functions is finite (at most $(k+1)^{k+1}$), the canonical equivalence relation has finitely many equivalence classes
- Hence the language accepted by the 2-way DFA is regular

Example

- Let us consider an example of a 2-way DFA accepting the language
  $L = \{x \in \{a, b\}^* | \#a(x)$ is a multiple of 3 and $\#b(x)$ is even$\}$



Example

- For any $x \in \{a, b\}^*$,
- If $\#a(x) = k \mod 3$, then $T_x(\bullet) = p_k$
- If $\#b(x) = 0 \mod 2$, then $T_x(p_0) = t$ and $T_x(p_1) = r$
- If $\#b(x) = 1 \mod 2$, then $T_x(p_0) = r$ and $T_x(p_1) = t$
- And $T_x(t) = t$, $T_x(r) = r$
- Hence, the canonical equivalence relation has 6 equivalence classes and the language is regular

Constructing DFA from 2-way DFA

- Once we have the canonical equivalence relation, we can easily construct a DFA.
- Let $M = (Q, \Sigma, \vdash, \dashv, \delta, s, t, r)$ be a 2-way finite automaton
- Let Q' be the set of all equivalence classes of the canonical equivalence relation
- $s' = T_\epsilon$
- $\delta'(T_x, a) = T_{xa}$
- $F' = \{T_x | x \in L(M)\}$
- DFA $M' = (Q', \delta', s', F')$ will accept the same language as the 2-way DFA M

Example

- The DFA accepting the same language as the 2-way DFA is as follows: