MATH 405/607E Numerical methods for differential equations

Project 3

1 Question Review

In this project, we want to solve the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

numerically under periodic boundary conditions by applying 2D Fourier transform. To generate the initial condition, I use the following picture (picture source: http://ultra8bit.com/wp-content/uploads/2013/05/iron_man_3_movie-wide.jpg)

Figure 1: initial condition

The final goal of this project is to get the smeared picture in the last instance and create a video recording the process of smearing.

2 Algorithm Discussion

We apply 2D Fourier transform F_xy to the heat equation in the part 1 and then transform the PDE to the ODE as follows:

$$\frac{\partial \mathcal{F}_{xy}[u]}{\partial t} = -(k_x^2 + k_y^2)\mathcal{F}_{xy}[u].$$

This is a quite elementary ode w.r.t. t so we can easily get the solution is

$$e^{-(k_x^2+k_y^2)t}\mathcal{F}_{xy}[u(x,y,0)].$$

Finally we apply inverse 2D Fourier transform to the above solution and then get the solution of original PDE as below:

$$u(x,y,t) \;=\; \mathcal{F}_{xy}^{-1}\left[e^{-(k_x^2+k_y^2)t}\mathcal{F}_{xy}[u(x,y,0)]\right].$$

The main question in this project is to decide what kx and ky is. First of all, as we know, fft command in Matlab is defined based on ordinary frequency, and the formula is given as:

$$X(k) = \sum_{j=1}^{N} x(j)\omega_N^{(j-1)(k-1)}$$

where

$$\omega_N = e^{(-2\pi i)/N}.$$

However, the Fourier transform we use in the project is based on angular frequency, which is calculated by the function:

$$\hat{f}(t) \;=\; \int_{-\infty}^{\infty} f(x)e^{-ixt}dx.$$

Thus there is a 2*pi scaling in kx and ky.

Next task is to discover the relationship between k in discrete Fourier transform given by fft and the continuous Fourier transform we use in the project. For a function define on the interval, we consider the Fourier series of it. Then we have the following relation:

$$\begin{aligned}
\hat{f}_n(t) \;&=\; \frac{1}{b-a}\int_a^b f(x)e^{-inx}dx \\
&\approx\; \frac{1}{b-a}\sum_{k=0}^{n-1} f(x_k)e^{-inx_k} * \Delta x \\
(\text{notice } \Delta x/(b-a) = N) \;&=\; \frac{1}{N}\sum_{k=0}^{n-1} f(x_k)e^{-inx_k} \\
&=\; \frac{1}{N}fft(\{x_0, x_1, \cdots, x_{N-1}\})(n).
\end{aligned}$$

However, we should notice that fft command in Matlab always transfers the frequency to the interval fs*(0,1), where fs is the sample frequency and L is the number of sample points, while the truncate Fourier series always range from [–N/2+1, N/2] when N is even. Thus we use the sequence fs/L*[0:L/2,-L/2+1:-1]. Noticing that in our project, it is a two dimension question, then kx and ky are (notice that fs = 1 in this question):

$$Kx = [0: Nx/2,-Nx/2+1:-1]/Nx,$$
$$Ky = [0: Ny/2,-Ny/2+1:-1]/Ny.$$

Here Nx and Ny is the length of x and y coordinate respectively.

3 Result

3.1 More detail about video

Here we set the video 100 frames per second. To avoid long and boring waiting of smearing photo, we choose the time interval of PDE to be [0,1e3] but compress the process into a 3 second video, thus we use linspace(0,1e3,3*100) as our sequence of time.

3.2 Smeared picture in the last instance