9

SEMIDEFINITE PROGRAMMING

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ABSTRACT

Semidefinite Programming is a rapidly emerging area of mathematical programming. It involves optimization over sets defined by semidefinite constraints. In this chapter, several facets of this problem are presented.

9.1 INTRODUCTION

Let $\mathcal{S}_n$ be the space of $n \times n$ real symmetric matrices, and for $A, B \in \mathcal{S}_n$, $A \bullet B$ denotes the inner product $\sum_{i,j} A_{ij}B_{ij}$, and we write $A \succeq B$ if $A - B$ is positive semidefinite. Suppose that $Q_0, \ldots, Q_m \in \mathcal{S}_n$ are given matrices, and $c \in \mathbf{R}^m$. Then the semidefinite program in **equality standard form** is defined to be the following optimization problem.

$$\begin{array}{rrcl} \inf: & U \bullet Q_0 & & \\ & U \bullet Q_i & = & c_i \; \forall\, i = 1, \ldots, m \\ & U & \succeq & 0. \end{array} \tag{SDP-E}$$

We also define the semidefinite program in **inequality standard form** to be:

$$\begin{array}{rrcl} \sup: & c^T x & & \\ & \sum_{i=1}^m x_i Q_i & \preceq & Q_0 \end{array} \tag{SDP-I}$$

The two problems SDP-E and SDP-I are equivalent in the sense that one can be transformed into the other with relative ease. Furthermore, as will be seen in sections

T. Terlaky (ed.), Interior Point Methods of Mathematical Programming 369–398.

to follow, these problems are the so-called *standard duals* of each other. The main motivation for starting out with both problems is that, the first form appears to be more suitable for algebraic purposes while the latter has a strong geometric flavor. Let f_E^*, f_I^* denote the optimal values of the problems SDP-E and SDP-I respectively. Both problems will be collectively referred to as SDP.

The main subject matter of **Semidefinite Programming (SDP)** can be broadly classified into the following three categories.

1. Geometric, algebraic and duality theoretic treatment of SDP.
2. Algorithmic, complexity theoretic and computational development.
3. Applications of SDP.

At the outset, it should be mentioned that two recent survey articles have already appeared on SDP, namely [3] and [92] (an earlier version of the latter is [91]). The main thrust of these two surveys had been interior point methodologies for SDP. In addition, in [3], applications to combinatorial optimization have been discussed, and in [92], applications to engineering problems and other optimization problem classes were presented. Keeping the above in mind, here we will dwell upon aspects that have received less attention in the abovementioned references. In particular, only sketchy attention will be paid to interior point methods, despite the stated title of the current volume. Several open problems will be stated with the hope that they will inspire further developments in this highly promising subject area.

9.2 GEOMETRY AND DUALITY

In this section, we will look at several geometric and duality theoretic aspects concerning SDP. Throughout this chapter, $Q(x)$ will denote the linear matrix map:

$$Q(x) = \sum_{i=1}^{m} x_i Q_i.$$

A **Spectrahedron** is defined to be a closed convex set of the following type,

$$G = \{x \mid Q(x) \preceq Q_0\},$$

where $Q(x)$ is a linear symmetric matrix map as defined above, and $Q_0 \in \mathcal{S}_n$. In other words, G is the feasible region of the semidefinite program SDP-I. It is not hard to see that the feasible region of SDP-E can be recast in the above inequality form, and hence spectrahedra are precisely the feasible regions of semidefinite programs. The name spectrahedron is chosen for the reason that their definition involves the spectrum (the eigenvalues) of matrices, and they bear a resemblance to, and are a generalization of polyhedra.

9.2.1 Analysis of Spectrahedra

We begin by first introducing some special classes of spectrahedra.

- If $P = \{x \mid Ax \leq b\}$ is a polyhedron, then $\{x | \mathrm{Diag}(b - Ax) \succeq 0\}$ is a spectrahedral representation of P. Thus every polyhedron is a spectrahedron.
- Let $S = \{x \mid x^T Q x + b^T x + c \leq 0\}$ be a *generic ellipsoid*, where Q is a PSD matrix. Then, it can easily be shown that S is a spectrahedron (see [92], [75]). Moreover, the intersection of finitely many spectrahedra is another spectrahedron, and hence the intersection of several generic ellipsoids is a spectrahedron. As an example, every Euclidean ball is a spectrahedron. It is also interesting that the unit ball in l_4 norm is the projection of a spectrahedron. To see this, let $m = 2$, and consider $S = \{(x_1, x_2) \mid x_1^4 + x_2^4 \leq 1\}$. Then consider $R = \{(x_1, x_2, y_1, y_2) \mid y_1^2 + y_2^2 \leq 1, x_1^2 \leq y_1, x_2^2 \leq y_2\}$. It follows that R is a spectrahedron, and S is the projection of R onto its first two coordinates.

Certain properties of spectrahedra have been studied in [79] and [68]. Some of these properties are:

1. Given a point $\bar{x}$ in a spectrahedron G as defined earlier, the smallest face of G containing $\bar{x}$ is given by

$$F_G(\bar{x}) = \{x \in G \mid \mathrm{Null}(Q_0 - Q(x)) \supseteq \mathrm{Null}(Q_0 - Q(\bar{x}))\}.$$

   Using this, one can characterize extreme points and extreme rays of spectrahedra. It is also known that every face of a spectrahedron is exposed (i.e., each face of G can be written as the intersection of a hyperplane with G; see [84] for examples of nonexposed faces of general convex sets).

2. Spectrahedra are closed under intersections, but are not closed under linear mappings, projections, polar operation or Minkowski sums ([79]).

3. Unlike polyhedra [8], the dimensions of the faces of a spectrahedron need not form a contiguous string. Take the PSD cone, for instance, which is a spectrahedron and it is well known that the dimensions of its faces are the triangular integers $k(k+1)/2$ for $k = 0, ..., n$ (see [9] and [21]).

In [50], the following subclass of spectrahedra, called *Elliptopes* were introduced:

$$\mathcal{E}_n := \{U \in \mathcal{S}_n \,|\, U_{ii} = 1 \; \forall \; i, U \succeq 0\}.$$

Such matrices are also known as correlation matrices, and they play a critical role in the approximation algorithm for the MAXCUT problem developed in [26]. More specifically, as we will see in more detail later, their method is a relaxation in which one optimizes a linear objective function over $\mathcal{E}_n$. In [50] and [51], this object has been investigated. In particular, their results include the following.

- Expressions for the normal cones.
- Proof that $\mathcal{E}_n$ has exactly 2^n vertices (points at which the normal cone is full dimensional), namely, matrices of the form vv^T where v is a binary (± 1) vector.
- Various results concerning regular points (points where the normal cone is one dimensional), tangent cones and faces of $\mathcal{E}_n$.

In [68], results concerning facial structure of spectrahedra are given. The following results are also derived.

1. Bounds on the ranks of the matrices (U for the SDP-E case and $Q_0 - Q(x)$ for the SDP-I problem), when the solutions are extreme points.
2. Bounds on the multiplicity of the eigenvalues of the matrices at extreme point optimal solutions ([69]).
3. In [70] and [49], the extreme points are treated as a generalization of the notion of *basic feasible solutions* from LP, and "simplex-type" methods for SDP has been proposed.

The polar of a convex set G containing the origin is defined by:

$$G^\circ = \{x \,|\, x^T y \leq 1 \; \forall \; y \in G\}.$$

When G is a spectrahedron of the form

$$G = \{x \mid Q(x) \preceq Q_0\},$$

clearly G contains the origin exactly when $Q_0 \succeq 0$. Supposing that this latter condition holds, it is not hard to derive (see [79]) the following expression for the polar:

$$G^\circ = \mathrm{Cl}(\{Q^*(U) \mid U \succeq 0, \;\; Q_0 \bullet U \leq 1\}),$$

where $Q^*(U)$ denotes the adjoint of the linear map $Q(x)$, and Cl(.) is the closure operation. When G is full dimensional, it is not necessary to take the closure in the above expression, thus yielding an algebraic description of the polar for this case. However, when full dimensionality is not satisfied, this fails to hold. In [76], by using an incremental argument, an expression for G° is derived for the most general situation. This in turn yields a polynomial size gapfree dual program for SDP which will be discussed in *S*9.2.2.

Since spectrahedra are a generalization of polyhedra, a seemingly interesting problem is that of characterizing when a spectrahedron is polyhedral. More generally, one can ask when a given projection of a spectrahedron is polyhedral. What is rather surprising is that, a satisfactory answer to this latter question will likely yield a good characterization of perfect graphs, as will be seen in *S*9.4.1.

On the Nonlinear Geometry of Spectrahedra

Much has been understood concerning the *linear geometry* (i.e., description of objects such as faces and polars) of spectrahedra. However, these objects do not seem to capture the inherently nonlinear nature of the surfaces of spectrahedra. To illustrate our point, we consider the following simple example.

Let G be a spectrahedron in $\mathbf{R}^3$ defined as the intersection of the unit ball $B = \{x \mid x^T x \leq 1\}$ and the ellipsoid

$$E = \{x \mid f(x) \leq 1\},$$

where, $f(x) = x_1^2 + (x_2 - 2)^2/4 + x_3^2/4$. Then every point on the boundary of G is an extreme point, and consequently, all faces are zero dimensional, except for the whole set itself which is 3 dimensional. However, the surface of G can be partitioned into three pieces; two smooth surfaces which are given by exactly one of the functions $x^T x - 1$ and $f(x) - 1$ being zero (and the other being negative), and one closed nonplanar curve which is the intersection of the two surfaces of B and E. This curve is parametrized as

$$\phi(t) := \left(\pm\sqrt{(4t-1)/3}, t, \pm\sqrt{(t+2)(2-3t)/3}\right),$$

where t is in the range $[1/4, 2/3]$.

Prompted by the above and other similar examples, we define the following nonlinear notion of faces, called *plates*. Let $G = \{x \in \mathbf{R}^m \,|\, Q(x) \preceq Q_0\}$ be a spectrahedron, where $Q(x)$ is a linear $n \times n$ matrix map. Then, for every $0 \leq k \leq n$, define the subset of G given by:

$$G_{[k]} := \{x \in G \,|\, \mathrm{rank}(Q_0 - Q(x)) = k\}.$$

Clearly, $G = \sum_{k=0}^{n} G_{[k]}$. Then, a **plate** of G of **order** k is defined to be the closure of a connected component of $G_{[k]}$. It is not hard to show the following:

1. The rank of $Q_0 - Q(x)$ is constant over the relative interior of an (ordinary) face (in fact, the null space is constant; see [79]). Hence the relative interior of a face on which $\mathrm{rank}\{Q_0 - Q(x)\} = k$ is contained in exactly one connected component of $G_{[k]}$.
2. Using the classical results of Whitney [94], it can be shown that every spectrahedron has at most finitely many plates.
3. If we have a polyhedron given by $P = \{x \,|\, Ax \leq b\}$, we can reexpress it as $P = \{x \,|\, \mathrm{Diag}(b - Ax) \succeq 0\}$. Then the above definition of plates reduces to the usual notion of polyhedral faces.

Of course, very little is understood concerning the plates of spectrahedra and their structure at this point. However, it appears that Algebraic Geometry techniques such as the Groebner bases ([10] and [16] are good introductory texts) are applicable here.

9.2.2 Duality in SDP

As mentioned earlier, the two formulations of SDP, namely SDP-E and SDP-I (of S9.1) have a certain duality correspondence. More specifically, they are Lagrangian duals (or *standard duals*) of each other. To show this, consider the following minmax reformulation of SDP-I, which is not hard to establish:

$$f_I^* = \sup_{x \in \mathbf{R}^m} \inf_{U \succeq 0} \{\, c^T x + U \bullet (Q_0 - Q(x)) \,\}.$$

One can reverse the minmax into maxmin and, it can be shown once again that

$$f_E^* = \inf_{U \succeq 0} \sup_{x \in \mathbf{R}^m} \{\, c^T x + U \bullet (Q_0 - Q(x)) \,\}.$$

This implies that $f_I^* \leq f_E^*$. There exist several examples for which equality fails to hold (see, for instance, [91], [76] or [22]). Let us define, for the pair of semidefinite programs SDP-E and SDP-I, the *standard duality gap (SDG)* to be the difference $f_E^* - f_I^*$. Listed below are some conditions under which SDG is zero (from [91]; see [59] for a thorough treatment).

1. There exists a primal feasible solution U that is positive definite, or less restrictively (see [79] for explanation), the primal feasible region is full dimensional.
2. The dual feasible region is full dimensional.
3. The primal optimal solution set is nonempty and bounded.
4. The dual optimal solution set is nonempty and bounded.

When none of the above conditions hold, one may have a nonzero duality gap. Therefore, it is a natural question to ask if there exists a polynomial size dual program for SDP which can be written down using the primal data and for which the duality gap is zero, without any assumptions. A first step in this direction was taken in [13], where it was shown that for any cone programming problem, restricting attention to the minimal cone will result in zero duality gap. Furthermore, a theoretical (and unimplementable) method for regularizing a cone program was given. While this approach to duality gives zero duality gap, resulting dual programs are not explicit polynomial size programs that depend only on the primal data. The derivation of such a dual was an open problem before it was resolved in [76]. The approach used there was to establish a description of polars of spectrahedra and use it to formulate the dual program (for SDP-I) called **Extended Lagrange-Slater Dual (ELSD)**. In the following, we will present the ELSD program and state the main duality theorem on ELSD. But first some notation is introduced.

- $Q(x) = \sum_{i=1}^m x_i Q_i$.
- $G := \{x \mid Q(x) \preceq Q_0\}$ is the feasible region of (P).
- $Q^* : \mathcal{M}_n \to \mathbf{R}^m$ is defined by $Q^*(U) = U \bullet Q_i, i = 1, ..., m$ (here, and in what follows, $\mathcal{M}_n$ denotes the space of $n \times n$ real matrices).
- $Q^\# : \mathcal{M}_n \to \mathbf{R}^{m+1}$ is defined to be

$$Q^\#(U) = \begin{pmatrix} Q_0 \bullet U \\ Q^*(U) \end{pmatrix}.$$

– If $y \in \mathbf{R}^{m+1}$ with indexing starting at zero,

$$\tilde{Q}(y) = \sum_{i=0}^{m} y_i Q_i.$$

The following is a gapfree dual semidefinite program, called the **Extended Lagrange-Slater Dual (ELSD)** for SDP-I.

$$\begin{array}{lrcl} \inf: & (U + W_m) \bullet Q_0 & & \\ \text{s.t.} & Q^*(U + W_m) & = & c \\ & Q^{\#}(U_i + W_{i-1}) & = & 0, i = 1, ..., m \\ & U_i & \succeq & W_i W_i^T, i = 1, ..., m \\ & U & \succeq & 0 \\ & W_0 & = & 0 \end{array} \qquad \text{(ELSD)}$$

Note that the constraint $U_i \succeq W_i W_i^T$ can alternately be written as

$$\begin{bmatrix} I & W_i^T \\ W_i & U_i \end{bmatrix} \succeq 0,$$

and consequently ELSD is a semidefinite program. The domains of different variables are given by: $U \in \mathcal{S}_n, U_i \in \mathcal{S}_n \ \forall\ i = 1, ..., m$ and $W_i \in \mathcal{M}_n \ \forall\ i = 1, ..., m$ (and we use an auxiliary matrix variable $W_0 = 0$ for notational convenience). The size of ELSD is easily seen to be polynomial in the size of the primal problem SDP-I.

The duality theorem for ELSD is given below, wherein (U, W) is said to be *dual feasible*, if these matrices, along with some $U_i, W_i, i = 1, ..., , m$, where $W_m = W$, satisfy the constraints of the dual program ELSD.

Theorem 9.2.1 (Duality Theorem) *The following hold for the primal problem SDP-I and the dual problem ELSD:*

1. *(**Weak Duality**) If x is primal feasible and (U, W) is dual feasible, then $c^T x \leq (U + W) \bullet Q_0$.*
2. *(**Primal Boundedness**) If the primal is feasible, then its optimal value is finite if and only if the dual ELSD is feasible.*
3. *(**Zero Gap**) If both the primal and the dual ELSD are feasible, then the optimal values of these two programs are equal.*
4. *(**Dual Attainment**) Whenever the common optimal value of the primal and ELSD is finite, the latter attains this value.*

In [83], connections between the minimal cone based approach and ELSD were discussed. Furthermore, the extended dual of the standard SDP in equality form, i.e., SDP-E was also given. In the recent work [78], the Lagrangian dual (or standard dual) of ELSD has been considered. After some reformulation, the standard dual of ELSD in variables are $z \in \mathbf{R}^m$ and $R_i \in \mathcal{S}_n$, $y(i) \in \mathbf{R}^{m+1}$, $i = 1, .., m$ takes the form given below.

$$\begin{array}{lrcl} \sup: & c^T z - \sum_{i=1}^m R_i \bullet I & & \\ & Q(z) & \preceq & Q_0 \\ & \begin{bmatrix} R_i & \tilde{Q}(y(i+1)) \\ \tilde{Q}(y(i+1)) & \tilde{Q}(y(i)) \end{bmatrix} & \succeq & 0 \ \forall\, i = 1, .., m-1 \\ & \begin{bmatrix} R_m & Q_0 - Q(z) \\ Q_0 - Q(z) & \tilde{Q}(y(m)) \end{bmatrix} & \succeq & 0 \end{array} \qquad \text{(P2)}$$

In any feasible solution of P2, the z part is also feasible for SDP-I, and every R_i is positive semidefinite. Therefore, it follows that the optimal value of P2 is at most that of SDP-I. In [78], it was shown that these are actually equal. Since the Lagrangian dual of P2 will be ELSD, it follows that the SDG (standard duality gap) of P2 is zero. Thus, starting wih any arbitrary SDP, one can obtain another (polynomial size) SDP with the same optimal value and whose SDG is zero. For this reason, we will call the problem P2, the **corrected primal** of the semidefinite program SDP-I. The corrected primal of SDP-E can be developed in a similar way. Now, in order to develop interior point methods (or other complexity bounded algorithms) for the most general SDPs, one may assume without loss of generality that the SDP at hand (which may be taken to be in either SDP-E form of SDP-I form) has zero standard duality gap. Note however that one can not still assume that Slater condition is satisfied, raising the possibility of developing infeasible interior point methods in this framework.

Finally, certain analytical aspects of SDP have been studied in [52] and [85].

9.3 ALGORITHMS AND COMPLEXITY

9.3.1 An Overview of Known Complexity Results

Let $Q_i, i = 0, ..., m$ be given rational symmetric matrices, c is a rational vector, and let

$$G = \{x \mid Q(x) \preceq Q_0\}$$

be the feasible region of SDP-I.

By applying ellipsoid and interior point methods, one can deduce the following complexity results for SDP.The maximum of the bitlengths of the entries of the Q_i and the components of c will be denoted by L, and define for $\epsilon > 0$,

$$S(G, \epsilon) = G + B(0, \epsilon) \text{ and } S(G, -\epsilon) = \{x \mid B(x, \epsilon) \subseteq G\}.$$

- If a positive integer R is known a priori such that either $G = \emptyset$ or $G \cap B(0, R) \neq \emptyset$, then there is an algorithm that solves the "weak optimization" problem, i.e., for any rational $\epsilon > 0$, the algorithm either finds a point $y \in S(G, \epsilon)$ that satisfies $c^T x \leq c^T y + \epsilon \ \forall \ x \in S(G, -\epsilon)$, or asserts that $S(G, -\epsilon)$ is empty ([30]). The complexity of the algorithm is polynomial in n, m, L, and $\log(1/\epsilon)$.
- There are algorithms which, given any rational $\epsilon > 0$ and an x_0 such that $Q_0 - Q(x_0) \succ 0$, compute a rational vector $\overline{x}$ such that $Q_0 - Q(\overline{x}) \succ 0$, and $c^T \overline{x}$ is within an additive factor ϵ of the optimum value of SDP. The arithmetic complexity of these algorithms is polynomial in $n, m, L, \log(1/\epsilon), \log(R)$ and the bitlength of x_0, where R is an integer such that the feasible region of the SDP lies inside the ball of radius R around the origin ([3],[59]). However, it should be mentioned that a polynomial bound has not been established for the bitlengths of the intermediate numbers occurring in these algorithms.
- For any fixed m, there is a polynomial time algorithm (in n, L) that checks whether there exists an x such that $Q(x) \succ 0$, and if so, computes such a vector ([75]). For the nonstrict case of $Q(x) \succeq 0$, the feasibility can be verified in polynomial time for the fixed dimensional problem as shown in [72].

9.3.2 Interior Point Methods

The development of IPMs for SDP is currently an extremely active research area. The reader is referred to the surveys [3] and [92] for extensive details. Below, we will describe in a somewhat cursory fashion, some of the specific interior point algorithms developed.

At the outset, we emphasize the facts that these methods deal with the computation of approximate optimal solutions only and that no bitlength analysis has been carried out by any of the authors.

The main feature that enables one to extend LP interior point methods to SDP is the fact that the logarithm of the determinant function serves as a barrier function for SDP. Its self concordance was established and used by Nesterov and Nemirovskii [59] in developing barrier methods for SDP. In [1] and [3], a potential reduction algorithm was developed based on Ye's projective algorithm for LP [96]. Alizadeh ([1]) also pointed out the striking similarity between LP and SDP and suggested a mechanical way of extending results from LP to SDP. In [40], Jarre developed a barrier method. More potential reduction methods are given in [92]. In [35], a convergent and easily implementable method was given (a matlab code is available at the ftp site ftp://orion.uwaterloo.ca/pub/henry/software). A primal-dual method was presented in [4]. In [60] and [61], Nesterov and Todd discuss primal-dual methods for self-scaled cone problems and develop what has come to be known as the Nesterov-Todd (NT) direction. In a recent work [22], Freund discusses interior-point algorithms for SDPs in which no regularity (Slater-like) conditions are assumed. A self-dual skew-symmetric embedding method was presented in [46] for the initialization of interior point methods for SDP.

Recently, several papers have appeared on interior point methods for SDP, and these can be obtained from the interior point archive maintained at the Argonne National Laboratories (WWW URL is http://www.mcs.anl.gov/home/otc/InteriorPoint/index.html). To follow are some details on these results.

The primal-dual central path is defined as the set of solutions $(U(\mu), x(\mu), S(\mu))$ of the system

$$\begin{array}{rcll} U \bullet Q_i & = & c_i \ \ \forall\, i=1,\ldots,m & U \succeq 0 \\ \sum_{i=1}^m x_i Q_i + S & = & Q_0 & S \succeq 0 \\ US & = & \mu I. & \end{array} \tag{SDP-Path}$$

If we assume that the matices $Q_i \ \ \forall\, i = 1,\ldots,m$ are independent then for each $\mu > 0$ the solution $(U(\mu), x(\mu), S(\mu))$ is unique. If we have a solution with $\mu = 0$ then we have an optimal solution pair with duality gap zero. Given an interior primal dual-solution (U, x, S) that satisfy the first two of above requirements and satisfy approximately the last one ("approximately centered solution"), the search direction $(\Delta U, \Delta x, \Delta S)$ in the primal-dual methods is derived by solving the following Newton system;

$$\begin{array}{rcl} \Delta U \bullet Q_i & = & 0 \ \forall\, i=1,\ldots,m \\ \sum_{i=1}^m \Delta x_i Q_i + \Delta S & = & 0 \\ \Delta U S + U \Delta S & = & \mu I - US. \end{array} \tag{SDP-Newt}$$

The solution of the above system under the usual mild assumption is unique. The ΔS part is symmetric, while the ΔU part is not. Then one has to symmetrize this part and determine a step length α to obtain the new iterate $(U+\alpha\Delta U, x+\alpha\Delta x, S+\alpha\Delta S)$. Then the procedure is repeated while the parameter μ is driven to zero.

If the candidate solution (U, x, S) does not satisfy the first two requirements of SDP-Path, then we enter the domain of infeasible interior point methods. In this case, the right hand sides of the first two Newton equations in SDP-Newt are not zero, but instead they equal the current primal and dual infeasibility, respectively. These methods simultaneously reduce the infeasibility and μ.

The papers [59, 92] deal with potential reduction methods. Much work has recently been done on primal–dual central path following algorithms. Detailed study of search directions can be found in [48, 56]. The properties of central trajectories are studied in detail in the papers [59, 27, 20, 86],

The so-called primal or dual logarithmic-barrier path-following methods are generalized by Faybusovich [20] (general analysis), de Klerk et al. [34] (full-step methods with local quadratic convergence) and Anstreicher and Fampa [7] (large update method). Primal-dual path following methods were independently developed by Sturm and Zhang [87] (a full-step primal-dual algorithm, predictor-corrector algorithm and the largest-step method) and Jiang [41] (long-step primal-dual logarithmic barrier algorithm of Jansen et al. [38]).

An infeasible interior point method for SDP was developed by Potra and Sheng [71]. This method is based on the Lagrange dual. It would be an interesting result to develop infeasible interior point methods based on the ELSD duality approach (see below). Interior point methods for monotone semidefinite complementarity problems have been developed by Shida and Shindoh [73]. They prove that the central trajectory converges to the analytic center of the optimal set. Further, they prove global convergence of an infeasible interior point algorithm for the monotone semidefinite complementarity problem.

With the exception of [22], most of the methods mentioned above make an explicit assumption that the primal and/or the dual have a strictly feasible solution. As mentioned in S9.2.2, it seems that infeasible interior point methods can be developed using the gapfree dual ELSD and the "corrected primal" problem P2. The suitability of infeasible IPMs for this situation can be justified as follows. Some difficulties with initialization can be circumvented using a corrected primal based infeasible IPM approach. Unlike in the case of LP, "Phase 1" type initialization can run into some difficulties for SDP. For instance, for the SDP-I problem, consider the "Phase 1" problem: $\inf\{z_0 | Q(x) \preceq Q_0 + z_0 I, z_0 \geq 0\}$. It may happen here that the infimum is zero without being attained. No satisfactory "Big-M" method has been devised for SDP (based on the examples of "ill-behaved SDPs" in [76], it is our conjecture that M will need to be exponentially large in bitlength here). Also, even if the initialization step is somehow carried out, there are instances of SDPs, for which all rational solutions are exponential in bitlength, and hence the whole process becomes inherently exponential, contradicting the initial objective of devising a polynomial time algorithm, even in an approximate sense.

Open Problem 9.1 *Develop an infeasible IPM for general semidefinite programs using ELSD and the corrected primal P2.*

Open Problem 9.2 *Perform a bitlength analysis of the interior point methods for SDP.*

We now turn our attention to affine scaling algorithms. The affine scaling linear programming algorithms have gained tremendous popularity owing to their charming simplicity. The global convergence properties of these methods (for LP) have been uncovered relatively recently (see [89] in this volume). In particular, Tsuchiya and Muramatsu ([90]) proved that when the step length taken is in $(0, 2/3]$, then both primal and dual iterates converge to optimal solutions for the respective problems. It is not hard to extend the LP affine scaling algorithm to semidefinite programming. For instance, for the problem SDP-I, let $\bar{x}$ be a strictly feasible solution. Let $P = Q_0 - \sum_{i=1}^{m} \bar{x}_i Q_i \succ 0$. Then consider the inequality

$$\operatorname{trace}((Q_0 - \sum_i x_i Q_i) P^{-2} (Q_0 - \sum_i x_i Q_i)) \leq 1.$$

It can be shown that every feasible solution to the above inequality is feasible for SDP-I. One can easily maximize $c^T x$ over the above ellipsoid and repeat as in the standard dual affine scaling method. It remains to be seen if the proofs of [90] can be extended to the above approach.

Open Problem 9.3 *Prove the global convergence of the above affine scaling method.*

The primal-dual affine scaling algorithms (both the Dikin-affine scaling of Jansen et al. [39] and the classical primal-dual affine scaling algorithm of Monteiro et al. [57]) have been generalized by de Klerk et al. [45]. The iteration complexity results are analogous to the LP case.

We will briefly mention about non-interior point methods for SDP. In [63], Overton discusses an active set type method. In [75] (see [81]), and later independently in [85], a notion of the convexity of a matrix map was introduced. Using this, one can define what may be called a "convex nonlinear SDP". In [81] a Newton-like method was developed for convex nonlinear semidefinite inequality systems, and in [85], certain sensitivity results have been derived. While attempts towards extending the LP simplex method to SDP have been made ([49, 70]), we consider that this problem remains unsolved. Also, since the SDP can be treated as a nondifferentiable convex optimization problem (NDO), most NDO algorithms can be applied to solve semidefinite programs. See [67] for interior point methods for global optimization problems, which solve some SDP relaxations in a disguised form.

Open Problem 9.4 *Develop an appropriate (globally convergent) extension of the simplex method to SDP.*

9.3.3 Feasibility and Complexity in SDP

In this section, we address the issue of *exact* complexity of Semidefinite Programming. As is well known, every rational linear inequality system that is feasible has a rational solution of polynomial bitlength. In sharp contrast, the following situations can occur for a feasible rational semidefinite inequality:

- it only has irrational solutions
- all its rational solutions have exponential bitlength.

Many such examples are discussed in [22] and [76]. Therefore, rigorously speaking, it is not a well stated problem to want to compute an exact optimal solution of an arbitrary rational SDP, since the output is not representable in the Turing machine model. Let us consider the feasibility problem defined below.

Definition 9.3.1 Semidefinite Feasibility Problem (SDFP) *Given rational symmetric matrices $Q_0, ..., Q_m$, determine if the semidefinite system*

$$\sum_{i=1}^{m} x_i Q_i \preceq Q_0$$

is feasible.

Note that the required output of this problem is a "Yes" or a "No" (decision problem). Therefore, it is reasonable to ask whether there is a polynomial time algorithm for the solution of SDFP. In our opinion, this is the most challenging and outstanding problem in semidefinite programming, at least in the context of complexity theory.

Open Problem 9.5 *Determine whether the problem SDFP is NP-Hard, or else find a polynomial time algorithm for its solution.*

In [76], the following results concerning the exact complexity of SDP are established.

1. If SDFP$\in$NP, then SDFP$\in$Co-NP, and vice versa.
2. In the Turing Machine model [25], SDFP is not NP-Complete unless NP=Co-NP,
3. SDFP is in NP$\cap$Co-NP in the real number model of Blum, Shub and Smale [12].
4. There are polynomial time reductions from the following problems to SDFP:
   (a) Checking whether a feasible SDP is bounded (i.e., it has a finite optimal value).
   (b) Checking whether a feasible and bounded SDP attains the optimum.
   (c) Checking the optimality of a given feasible solution.

In [72], the authors discuss complexity results for fixed dimensional SDPs (both n-fixed and m-fixed cases), extending and strengthening certain results of [75].

9.4 APPLICATIONS

Applications of semidefinite programming can be broadly classified into three groups:

- SDP as a relaxation of nonconvex problems, in particular, mathematical programs involving quadratic functions.
- Combinatorial Optimization applications.
- Direct SDP models, arising in some engineering problems.

Also, as seen earlier, SDP generalizes linear and convex quadratic programming problems, more generally, convex quadratic programming with convex quadratic constraints. Since the latter has not been extensively studied by itself, most of its applications (which arise in certain facility location problems as studied in [18]) can also be considered to be applications of SDP.

Semidefinite Programming can be naturally arrived (see [75]) at by relaxing *Multiquadratic Programs (MQP)*, which are optimization problems of the type given below.

$$\begin{array}{lll} \min: & x^TQ_0x + 2b_0^Tx + c_0 & \\ \text{s.t.} & x^TQ_ix + 2b_i^Tx + c_i & = 0 \ \forall\, i = 1, ..., m. \end{array} \qquad \text{(MQP)}$$

By introducing a new matrix variable U and imposing an additional constraint $U = xx^T$, we can rewrite the above problem as

$$\begin{array}{lll} \min: & U \bullet Q_0 + 2b_0^Tx + c_0 & \\ \text{s.t.} & U \bullet Q_i + 2b_i^Tx + c_i & = 0 \ \forall\, i = 1, ..., m \\ & U - xx^T & = 0 \end{array} \qquad \text{(MQP2)}$$

Now, let us relax the condition $U - xx^T = 0$ to $U - xx^T \succeq 0$, to obtain

$$\begin{array}{lll} \min: & U \bullet Q_0 + 2b_0^Tx + c_0 & \\ \text{s.t.} & U \bullet Q_i + 2b_i^Tx + c_i & = 0 \ \forall\, i = 1, ..., m \\ & U - xx^T & \succeq 0. \end{array} \qquad \text{(RMQP)}$$

The condition $U - xx^T \succeq 0$ is equivalent to

$$\begin{bmatrix} U & x^T \\ x & 1 \end{bmatrix} \succeq 0.$$

Therefore, the relaxed MQP (RMQP) is a semidefinite program. This SDP relaxation of MQP will be referred to as the *convexification Relaxation* of MQP. The reason being that, if $f : \mathbf{R}^n \to \mathbf{R}^m$ is the quadratic map composed of the constraint functions of the MQP, then the feasibility of that problem can be restated as $0 \in f(\mathbf{R}^n)$. On the other hand, it can be shown (see [75]) that the semidefinite program RMQP is feasible if and only if 0 is in the convex hull of the image, i.e., $\text{Conv}(f(\mathbf{R}^n))$. This relaxation was originally introduced by Shor [74], although in a somewhat different form. It is also investigated in [24].

We will return to the connections between MQP and SDP after discussing some results on the application of SDP to combinatorial optimization.

9.4.1 Combinatorial Optimization

Stable Set Problems and Perfect Graphs

One of the early works in semidefinite programming emanated in the context of certain graph optimization problem such as the Maximum Stable Set (MSS) and related problems.

A clique (resp. stable set) in a graph $G = (V, E)$ is a subset S of V in which every pair of nodes is adjacent (resp. nonadjacent). The problem MSS is that of finding the largest stable set in G. Let STAB(G) denote the convex hull of the characteristic vectors of the stable sets of G. If u, v are the characteristic vectors of a clique and a stable set in G, we have the inequality $u^T v \leq 1$. This implies that the polyhedron

$$\text{QSTAB}(G) = \{x \geq 0 \mid x^T u \leq 1 \ \forall \ \text{characteristic vectors } u \text{ of cliques of } G\}$$

contains STAB(G). Now, note that the problem of finding a maximum stable set is equivalent to each of the following problems:

1. maximize $e^T x$ over STAB(G)
2. maximize $e^T x$ where x satisfies $x_i x_j = 0 \ \forall \ i, j \in E$ and $x_i \in \{0, 1\} \ \forall \ i \in V$.

Note that the second of these problems is a multiquadratic program, and hence we apply the convexification relaxation to it. Accordingly, we define the spectrahedron

$$S(G) := \{(U, x) \mid x \geq 0, U \succeq xx^T, U_{ii} = x_i \ \forall \ i \in V, U_{ij} = 0 \ \forall \ i, j \in E\}.$$

This spectrahedron can be projected on the x variables to get the following set defined in, for instance, [30]:

$$\text{TH}(G) = \{x \mid \exists U \text{ such that } (U, x) \in S(G)\}.$$

It is not hard to show ([30],[54]) that

$$\text{STAB}(G) \subseteq \text{TH}(G) \subseteq \text{QSTAB}(G),$$

and therefore, as a relaxation to MSS, one can maximize $e^T x$ over TH(G), which is an SDP in both variables x and U.

For general graphs, not much is known about the effectiveness of the above relaxation. However, for a class of graphs known as *perfect graphs*, the relaxation is exact. We will circumvent the usual combinatorial definition of perfect graphs as, for our purposes, it suffices to define these graphs as those for which STAB(G) = QSTAB(G). Clearly, in this case, all the three sets STAB(G), TH(G) and QSTAB(G) coincide. Thus, one can approximately maximize $e^T x$ over TH(G) by the use of a polynomial approximation algorithm for SDP. For techniques that extract discrete solutions from this approximation, the reader is referred to [30] and [1]. Furthermore, when G is perfect, the following additional problems can be solved using this methodology:

- Find the largest clique in G.

- Find the smallest number of colors required to color the vertices of G such that every pair of adjacent vertices receive different colors.

In [1], a sublinear time parallel algorithm was presented for solving the stable set and other problems for perfect graphs. The reader is also referred to the expository article [47] by Knuth on this approach.

Finally, we turn to the problem of characterization and recognition of perfect graphs. The definition of perfect graphs ($\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$) involves two polyhedra whose description involves the set of all maximal cliques and maximal stable sets of G. Since these may be exponential in number, the above definition does not seem to yield directly a recognition algorithm for perfect graphs. For that matter, even if the number of cliques and stable sets is polynomial, no polynomial time algorithm is known for solving this type of problems (see [55]). However, the following was shown in [30].

Proposition 9.4.1 *A graph G is perfect if and only if TH(G) is a polyhedron.*

This proposition may be useful in both addressing the complexity of perfect graph recognition, as well as settling what might be considered the most celebrated and yet unresolved conjecture in Graph Theory, which states that a graph G is perfect if and only neither G nor its complement $\bar{G}$ induce an odd cycle of size at least 5.

Open Problem 9.6 *Characterize the polyhedrality of TH(G).*

Since TH(G) of G is a projected spectrahedron, it is natural to ask about the complexity of verifying the polyhedrality of an arbitrary projected spectrahedron. Unfortunately, this general problem turns out to be NP-Hard as shown in an upcoming paper ([77]) by Ramana. There, it was also shown that, under an irredundancy assumption, the verification of polyhedrality of a spectrahedron can be done in randomized polynomial time. This latter result, however, does not seem to extend easily to projected spectrahedra such as TH(G).

The Maximum Cut Problem

Let us now turn our attention to another celebrated combinatorial optimization problem, called the *Maximum Cut Problem* (abbreviated MAXCUT): given a set of nonnegative weights $w_{ij}, 1 \leq i < j \leq n$, the problem is to determine a partition $S \cup \bar{S}$ of the set $N = \{1, ..., n\}$ that maximizes $\sum_{i \in S, j \in \bar{S}} w_{ij}$. This problem can be modeled as the quadratic integer program given below, where W is the matrix of weights, and J is the matrix of all ones.

$$\begin{array}{rll} \max: & W \bullet J - y^T W y & \\ & y_i & \in \{-1,+1\} \end{array} \qquad \text{(MAXCUT)}$$

Note that $y_i \in \{-1,+1\}$ is equivalently written as $y_i^2 = 1$. In [26], the following SDP relaxation of MAXCUT was considered.

$$\begin{array}{rll} \max: & W \bullet J - W \bullet U & \\ & U & \succeq 0 \\ & U_{ii} & \in \{-1,+1\}. \end{array} \qquad \text{(GWR)}$$

It is not hard to see that this is nothing but the convexification relaxation of the MQP form of MAXCUT. Let us call it the Goemans-Williamson Relaxation (GWR) of the maximum cut problem. The remarkable results of [26] are the following.

1. The optimal objective value of GWR is at most 1.14 times of that of MAXCUT.
2. From an optimal solution to GWR, a cut whose expected value is at least .878 times the optimal cut value can be obtained using randomization.

The underlying geometric reason behind item 1 above is best described using the following theorem formulated by Laurent [51]. First, let $\mathcal{C}_n$ denote the convex hull of all matrices of the form vv^T, where $v \in \{-1,+1\}^n$. It is clear that the MAXCUT problem amounts to maximizing a linear function over $\mathcal{C}_n$. Let us return to the convex set (called elliptope) $\mathcal{E}_n$ defined in $\mathcal{S}$9.2.1. The main geometrical result concerning these sets is given below. For a matrix A and a univariate function f, $f_\circ(A)$ denotes the matrix whose (i,j)th entry is $f(A_{ij})$.

Theorem 9.4.1 $\mathcal{C}_n \subseteq \mathcal{E}_n \subseteq \{\sin_\circ(\frac{\pi}{2}U) \,|\, U \in \mathcal{C}_n\}$.

Furthermore, the following nonlinear semidefinite program has the same objective function value as MAXCUT (see [26]):

$$\frac{1}{\pi}\max\{W \bullet (\arccos_\circ(U)) \,|\, U \in \mathcal{E}_n\}.$$

As mentioned above, to obtain an approximately optimal cut, a randomized rounding technique is applied, and then the entire algorithm is derandomized. However, it seems to be an interesting question to ask if there is a direct deterministic procedure that achieves the same. In particular, such a method might make use of semidefinite programming duality. We state this as an open problem.

Open Problem 9.7 *Find a deterministic procedure for obtaining an approximate maximum cut from an optimal solution to the semidefinite relaxation GWR.*

Other Combinatorial Optimization Problems

Several other applications of SDP to combinatorial optimization have recently been developed. The following is a list of some of these results.

1. In [26], the authors extend their analysis for MAX CUT to derive strong approximation results for the following problems: MAX SAT, MAX 2SAT, MAX DICUT (the first two problems are related to the Satisfiability Problem, and the last is a directed version of the MAX CUT problem).
2. In [43], an approximate graph coloring algorithm was developed.
3. Extensions of the Goemans-Williamson approach to the max-k-cut problem are given in [23].
4. Here are some results that are somewhat negative concerning the application of SDP to combinatorial optimization. Recently, Kleinberg and Goemans [44] have shown that certain SDP relaxations of the vertex cover problem have a worst case performance guarantee of only 2 (in the limit) coinciding with what the standard LP relaxation guarantees. A similar result for the independent set problem has been established by Alon and Kahale [6].

A topic that was studied well before SDP became popular was that of PSD completions of partially specified matrices. An early and well-written paper is [28]. These problems involve determinant maximization subject to semidefinite constraints and a recent reference to these is [93].

We would like to mention that the well known *Graph Isomorphism Problem* might perhaps be reducible to an SDP feasibility problem. Given two graphs G_1, G_2 with adjacency matrices A, B, respectively, the graphs are isomorphic if and only if there exists a permutation matrix X such that $A = X^T BX$. This can be written as the MQP

$$A = X^T BX, \; Xe = e, \; X^T e = e, \; X \circ X = X,$$

where e is the vector of all ones, and $\circ$ gives the entrywise (Hadamard) product of two matrices. In [82], it was shown that one can relax the condition that X is a permutation matrix to it being doubly stochastic. This gives the MQP:

$$A = X^T BX, \; Xe = e, \; X^T e = e, \; 0 \leq X \leq J,$$

where J is the matrix of all ones. Whether the convexification relaxation of either of the above systems is exact appears to be an intriguing question.

Open Problem 9.8 *Is Graph Isomorphism reducible to Semidefinite Feasibility?*

9.4.2 More on Multiquadratic Programming

We will return to the connections between multiquadratic programming and SDP. As we have seen, for every MQP, there is a natural SDP relaxation, called the convexification relaxation. It is natural to ask when this relaxation is exact. This is quite similar to the question of when LP relaxation is exact for Integer Programming problems (when the coefficient matrix is unimodular, for instance). We will present two slightly different ways of addressing this problem. The first of these comes from the analysis of [75], and the second is inspired by results on perfect graphs and the work of [54].

Let $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a constant-free (i.e., $f(0) = 0$) quadratic map, and consider the problem of verifying the feasibility of $f(x) = b$, or equivalently, $b \in f(\mathbf{R}^n)$. As mentioned earlier, the convexification relaxation $0 \in \text{Conv}(f(\mathbf{R}^n))$ reduces to a semidefinite program. The convexification relaxation is exact for *every* $b \in \mathbf{R}^m$ if and only if $f(\mathbf{R}^n)$, the image of f, is convex. This inspires the definition of ICON maps, which are maps that have convex images. In [80], quadratic ICON maps were characterized. Unfortunately, as shown in the same paper, the recognition of ICON maps is NP-Hard. However, the restriction to special classes of quadratic maps might yield polynomial time recognition results. For the optimization problem, $\min\{f_0(x) \mid f(x) = 0\}$, the convexification relaxation can be shown to be exact when the combined map $(f_0(x), f(x))$ is ICON.

The second formulation is very closely related to N_+ operator defined [54]. Let $f(x) = (f_1(x), ..., f_n(x))$, where $f_i(x) = x^T Q_i x + b_i^T x + c_i \; \forall \, i = 1, ..., m$, and consider the problem $\max\{c^T x \mid f(x) = 0\}$. Then, the convexification relaxation is

$$\max\{c^T x \mid Q_i \bullet U + b_i^T x + c_i = 0 \; \forall \; i, U \succeq xx^T\}.$$

Define the convex set (projection of a spectrahedron)

$$\text{TH}[f] = \{x \mid \exists U \text{ such that } Q_i \bullet U + b_i^T x + c_i = 0 \; \forall \; i, \; U \succeq xx^T\}.$$

Clearly, $\text{TH}[f]$ contains the convex hull of the MQP feasible region $Z = \{x \mid f(x) = 0\}$. Let us say that f is a *perfect quadratic* map if

$$\text{TH}[f] = \text{Conv}(\{x \mid f(x) = 0\}).$$

Now, let $G = (V, E)$ be a graph and f_G be the quadratic map composed both of $|E|$ components given by $x_i x_i \ \forall\ i,j \in E$ as well as $|V|$ components given by $x_i^2 - x_i \ \forall\ i \in V$. Then it is seen that $\mathrm{TH}[f_G]$ is nothing but the usual $\mathrm{TH}(G)$ defined for graphs earlier, and $\mathrm{Conv}(Z)$ is precisely $\mathrm{STAB}(G)$, and hence the perfectness of the graph G is the same as the perfectness of the quadratic map f_G.

Open Problem 9.9 *Characterize the perfectness of quadratic maps.*

9.4.3 Engineering and Other Applications

Listed below are certain applications that have been discussed for most part in [14] and [92].

- Logarithmic Chebychev Approximation ([92]).
- Structural Design problems, such as Truss design are found in [11].
- Pattern separation problems ([92]).
- Statistical applications such as minimum trace factor analysis ([92]).
- Control Theory applications (see [14]).

9.5 CONCLUDING REMARKS

Since Semidefinite Programming came to light four to five years ago, significant strides have been made in this subject. Theoretical as well as algorithmic advances continue to be made fairly rapidly. Several open problems and future research directions were presented in this chapter. We will conclude the chapter by mentioning a promising generalization of semidefinite programming which was unearthed by Güler [31].

A real homogeneous polynomial p is said to be ***hyperbolic*** with respect to a nonzero vector d, if $p(d) > 0$ and the univariate (in t) polynomial $p(x + td)$ has only real roots for every real vector x. Let $K(p, d)$ denote the component of $\{x \,|\, p(x) > 0\}$ that contains d. It is well known that this is an open convex cone and is called the *hyperbolicity cone* of p (in the direction d). Now let us define a *Hyperbolic Program (HP)* to be one of the type

$$\max\{c^T x \,|\, L(x) \in \mathrm{Cl}(K)\},$$

where $L(x)$ is an affine map of x. Hyperbolic programs generalize semidefinite programs. To see this, let $p(U)$ be the determinant polynomial of a symmetric matrix variable U. Then $p(U + tI) = 0$ gives precisely the negatives of the eigenvalues of the matrix U, which are real. Furthermore, $p(I) = 1 > 0$, and hence, in this case, the hyperbolicity cone $K(p, I)$ is simply the cone of positive definite matrices, whose closure is the cone of PSD matrices. Taking the affine map $L(x)$ to be $Q_0 - Q(x)$, we recover the semidefinite program SDP-I.

In [31], Güler discusses the existence of barrier functions for problems of this type. We strongly believe that many results that are known for SDP, such as interior point methods and duality theories (both standard and ELSD duals) can be extended to hyperbolic programs.

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