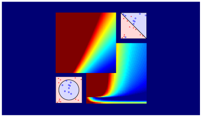
Machine Learning Foundations
(機器學習基石)



Lecture 6: Theory of Generalization

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Roadmap

1. When Can Machines Learn?
2. **Why** Can Machines Learn?

Lecture 5: Training versus Testing

effective price of choice in training: **(wishfully)** **growth function** $m_{\mathcal{H}}(N)$ with **a break point**

Lecture 6: Theory of Generalization

- Restriction of Break Point
- Bounding Function: Basic Cases
- Bounding Function: Inductive Cases
- A Pictorial Proof

3. How Can Machines Learn?
4. How Can Machines Learn Better?

The Four Break Points

growth function $m_{\mathcal{H}}(N)$: max number of dichotomies

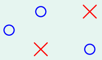
- positive rays: $m_{\mathcal{H}}(N) = N + 1$

  ○× $m_{\mathcal{H}}(2) = 3 < 2^2$: break point at 2

- positive intervals: $m_{\mathcal{H}}(N) = \frac{1}{2}N^2 + \frac{1}{2}N + 1$

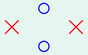
  ○×○ $m_{\mathcal{H}}(3) = 7 < 2^3$: break point at 3

- convex sets: $m_{\mathcal{H}}(N) = 2^N$

  $m_{\mathcal{H}}(N) = 2^N$ always: no break point

- 2D perceptrons: $m_{\mathcal{H}}(N) < 2^N$ **in some cases**

  $m_{\mathcal{H}}(4) = 14 < 2^4$: break point at 4

If k is a break point, then k+1, k+2, … are all break points.

break point $k \Longrightarrow$ break point $k + 1, \ldots$

what else?

Any two points cannot enumerate all possibles.

Restriction of Break Point (1/2)

what 'must be true' when **minimum** $\boxed{\textbf{break point } k = 2}$

No matter what N is, any k points can not be shattered.

- $N = 1$: every $m_{\mathcal{H}}(N) = 2$ by definition $\mathcal{H}$ can be shattered.
- $N = 2$: every $m_{\mathcal{H}}(N) < 4$ by definition $\mathcal{H}$ can't be shattered by 2 points. ($\because$ k = 2)
  (so **maximum possible** = 3)

maximum possible $m_{\mathcal{H}}(N)$ when $N = 3$ and $k = 2$?

1 dichotomy , shatter any two points? **no**

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
○	○	○

x_1	x_2
O	O
O	X
X	O
X	X

When we add the fourth dichotomy, it can be shattered by two points.
So it's illegal to add the fourth dichotomy.

Restriction of Break Point (1/2)

what 'must be true' when **minimum break point** $k = 2$

- $N = 1$: every $m_{\mathcal{H}}(N) = 2$ by definition
- $N = 2$: every $m_{\mathcal{H}}(N) < 4$ by definition
  (so **maximum possible =** 3)

maximum possible $m_{\mathcal{H}}(N)$ when $N = 3$ and $k = 2$?

2 dichotomies , shatter any two points? **no**

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
○	○	○
○	○	×

Restriction of Break Point (1/2)

what 'must be true' when **minimum break point $k = 2$**

- $N = 1$: every $m_{\mathcal{H}}(N) = 2$ by definition
- $N = 2$: every $m_{\mathcal{H}}(N) < 4$ by definition
  (so **maximum possible = 3**)

maximum possible $m_{\mathcal{H}}(N)$ when $N = 3$ and $k = 2$?

3 dichotomies , shatter any two points? **no**

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
○	○	○
○	○	×
○	×	○

Restriction of Break Point (1/2)

what 'must be true' when **minimum break point** $k = 2$

- $N = 1$: every $m_{\mathcal{H}}(N) = 2$ by definition
- $N = 2$: every $m_{\mathcal{H}}(N) < 4$ by definition (so **maximum possible =** 3)

maximum possible $m_{\mathcal{H}}(N)$ when $N = 3$ and $k = 2$?

4 dichotomies , shatter any two points? **yes**

contradict

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
○	○	○
○	○	×
○	×	○
~~○~~	~~×~~	~~×~~

Any other possible combination of dichotomies that is not shattered by any two points?

Restriction of Break Point (1/2)

what 'must be true' when **minimum break point** $k = 2$

- $N = 1$: every $m_{\mathcal{H}}(N) = 2$ by definition
- $N = 2$: every $m_{\mathcal{H}}(N) < 4$ by definition
  (so **maximum possible =** 3)

maximum possible $m_{\mathcal{H}}(N)$ when $N = 3$ and $k = 2$?

4 dichotomies , shatter any two points? **no**

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
○	○	○
○	○	×
○	×	○
×	○	○

Restriction of Break Point (1/2)

what 'must be true' when **minimum break point** $k = 2$

- $N = 1$: every $m_{\mathcal{H}}(N) = 2$ by definition
- $N = 2$: every $m_{\mathcal{H}}(N) < 4$ by definition
  (so **maximum possible =** 3)

maximum possible $m_{\mathcal{H}}(N)$ when $N = 3$ and $k = 2$?

5 dichotomies , shatter any two points? **yes**

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
○	○	○
○	○	×
○	×	○
×	○	○
~~×~~	~~○~~	~~×~~

Restriction of Break Point (1/2)

what 'must be true' when **minimum break point** $k = 2$

- $N = 1$: every $m_{\mathcal{H}}(N) = 2$ by definition
- $N = 2$: every $m_{\mathcal{H}}(N) < 4$ by definition
  (so **maximum possible =** 3)

maximum possible $m_{\mathcal{H}}(N)$ when $N = 3$ and $k = 2$?

5 dichotomies , shatter any two points? **yes**

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
○	○	○
○	○	×
○	×	○
×	○	○
~~×~~	~~×~~	~~○~~

Restriction of Break Point (1/2)

what 'must be true' when **minimum break point** $k = 2$

- $N = 1$: every $m_{\mathcal{H}}(N) = 2$ by definition
- $N = 2$: every $m_{\mathcal{H}}(N) < 4$ by definition
  (so **maximum possible =** 3)

maximum possible $m_{\mathcal{H}}(N)$ when $N = 3$ and $k = 2$?

5 dichotomies , shatter any two points? **yes**

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
○	○	○
○	○	×
○	×	○
×	○	○
~~×~~	~~×~~	~~×~~

Restriction of Break Point (1/2)

what 'must be true' when **minimum break point** $k = 2$

- $N = 1$: every $m_{\mathcal{H}}(N) = 2$ by definition
- $N = 2$: every $m_{\mathcal{H}}(N) < 4$ by definition
  (so **maximum possible =** 3)

maximum possible $m_{\mathcal{H}}(N)$ when $N = 3$ and $k = 2$?

maximum possible so far: **4 dichotomies** upper bond

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
○	○	○
○	○	×
○	×	○
×	○	○
:-(	:-(	:-(

Restriction of Break Point (2/2)

what 'must be true' when **minimum break point** $k = 2$

- $N = 1$: every $m_{\mathcal{H}}(N) = 2$ by definition
- $N = 2$: every $m_{\mathcal{H}}(N) < 4$ by definition N = 2: 4 → 3
  (so **maximum possible =** 3)
- $N = 3$: **maximum possible =** $4 \ll 2^3$ N = 3: 8 → 4

Why not "≥"?

—break point k **restricts maximum possible** $m_{\mathcal{H}}(N)$ **a lot** for $N > k$

When N = k, $m_{\mathcal{H}}(N) = 2^N - 1$
⇒ Not restrict a lot

idea:
$$
\begin{aligned}
& m_{\mathcal{H}}(N) \\
\leq\ & \text{maximum possible } m_{\mathcal{H}}(N) \text{ given } k \\
\leq\ & poly(N)
\end{aligned}
$$

Fun Time

When minimum break point $k = 1$, what is the maximum possible $m_{\mathcal{H}}(N)$ when $N = 3$?

1. 1
2. 2
3. 4
4. 8

Fun Time

When minimum break point $k = 1$, what is the maximum possible $m_{\mathcal{H}}(N)$ when $N = 3$?

1. 1
2. 2
3. 4
4. 8

Reference Answer: 1

Because $k = 1$, the hypothesis set cannot even shatter one point. Thus, every 'column' of the table cannot contain both $\circ$ and $\times$. Then, after including the first dichotomy, it is not possible to include any other different dichotomy. Thus, the maximum possible $m_{\mathcal{H}}(N)$ is 1.

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
$\circ$	$\times$	$\circ$
~~$\circ$~~	~~$\times$~~	~~$\times$~~

Bounding Function

bounding function $B(N, k)$: Don't care about $\mathcal{H}$.
maximum possible $m_{\mathcal{H}}(N)$ when break point = k

- combinatorial quantity:
  maximum number of length-N vectors with ($\circ$, $\times$)
  while **'no shatter' any length-k** subvectors
- ~~irrelevant of the details of $\mathcal{H}$~~
  e.g. $B(N, 3)$ bounds both
  - positive intervals ($k = 3$)
  - 1D perceptrons ($k = 3$)

new goal: $B(N, k) \leq poly(N)$**?**

Table of Bounding Function (1/4)

	$B(N,k)$	1	2	3	4	5	6	...
	1							
	2		**3**					
	3		**4**					
N	4							
	5							
	6							
	⋮							

(columns indexed by k)

Known

- $B(2,2) = 3$ (maximum < 4)
- $B(3,2) = 4$ ('pictorial' proof previously)

Table of Bounding Function (2/4)

	$B(N,k)$	k: 1	2	3	4	5	6	$\ldots$
	1	1						
	2	1	3					
	3	1	4					
N	4	1						
	5	1						
	6	1						
	$\vdots$	$\vdots$						

Known

- $B(N, 1) = 1$ (see previous quiz)

Table of Bounding Function (3/4)

	$B(N,k)$	k: 1	2	3	4	5	6	...
	1	1	**2**	**2**	**2**	**2**	**2**	...
	2	1	3	**4**	**4**	**4**	**4**	...
	3	1	4		**8**	**8**	**8**	...
N	4	1				**16**	**16**	...
	5	1					**32**	...
	6	1						...
	$\vdots$	$\vdots$						

Known

- $B(N,k) = 2^N$ for $\underline{N < k}$
  —including all dichotomies not violating 'breaking condition'

Table of Bounding Function (4/4)

	$B(N,k)$	k = 1	2	3	4	5	6	...
	1	**1**	2	2	2	2	2	...
	2	1	**3**	4	4	4	4	...
	3	1	4	**7**	8	8	8	...
N	4	1			**15**	16	16	...
	5	1				**31**	32	...
	6	1					**63**	...
	$\vdots$	$\vdots$						$\ddots$

Known

- $B(N,k) = 2^N - 1$ for $\underline{N = k}$
  —**removing a single dichotomy** satisfies 'breaking condition'

more than halfway done! :-)

Fun Time

For the 2D perceptrons, which of the following claim is true?

1. minimum break point $k = 2$
2. $m_{\mathcal{H}}(4) = 15$
3. $m_{\mathcal{H}}(N) < B(N, k)$ when $N = k =$ minimum break point
4. $m_{\mathcal{H}}(N) > B(N, k)$ when $N = k =$ minimum break point

Fun Time

For the 2D perceptrons, which of the following claim is true?

1. minimum break point $k = 2$
2. $m_{\mathcal{H}}(4) = 15$
3. $m_{\mathcal{H}}(N) < B(N, k)$ when $N = k =$ minimum break point
4. $m_{\mathcal{H}}(N) > B(N, k)$ when $N = k =$ minimum break point

Reference Answer: 3

As discussed previously, minimum break point for 2D perceptrons is 4, with $m_{\mathcal{H}}(4) = 14$. Also, note that $B(4, 4) = 15$. So bounding function upper bound $B(N, k)$ can be 'loose' in bounding $m_{\mathcal{H}}(N)$.

Estimating $B(4,3)$

	$B(N,k)$	1	2	3	4	5	6	...
	1	1	2	2	2	2	2	...
	2	1	3	4	4	4	4	...
	3	1	4	7	8	8	8	...
N	4	1		?	15	16	16	...
	5	1				31	32	...
	6	1					63	...
	⋮	⋮						⋱

Motivation

- $B(4,3)$ shall be related to $B(3,?)$
  —'adding' one point from $B(3,?)$

next: reduce $B(4,3)$ to $B(3,?)$

'Achieving' Dichotomies of $B(4,3)$

after checking all 2^{2^4} sets of dichotomies, **the winner is** ...

	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
01	○	○	○	○
02	×	○	○	○
03	○	×	○	○
04	○	○	×	○
05	○	○	○	×
06	×	×	○	×
07	×	○	×	○
08	×	○	○	×
09	○	×	×	○
10	○	×	○	×
11	○	○	×	×

	$B(N,k)$	k = 1	2	3	4	5	6
	1	1	2	2	2	2	2
	2	1	3	4	4	4	4
	3	1	4	7	8	8	8
N	4	1		**11**	15	16	16
	5	1				31	32
	6	1					63

Smell a rat?

how to reduce $B(4,3)$ to $B(3,?)$ cases?

Reorganized Dichotomies of $B(4, 3)$

after checking all 2^{2^4} sets of dichotomies, **the winner is** ...

	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
01	○	○	○	○
02	×	○	○	○
03	○	×	○	○
04	○	○	×	○
05	○	○	○	×
06	×	×	○	×
07	×	○	×	○
08	×	○	○	×
09	○	×	×	○
10	○	×	○	×
11	○	○	×	×

$\Rightarrow$

	$\mathbf{x}_1$	same $\mathbf{x}_2$	$\mathbf{x}_3$	different $\mathbf{x}_4$
01	○	○	○	○
05	○	○	○	×
02	×	○	○	○
08	×	○	○	×
03	○	×	○	○
10	○	×	○	×
04	○	○	×	○
11	○	○	×	×
06	×	×	○	×
07	×	○	×	○
09	○	×	×	○

orange: pair; **purple**: single

Estimating Part of $B(4,3)$ (1/2)

$$B(4,3) = 11 = 2\alpha + \beta$$

B(3,3)

	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
α	○	○	○
	×	○	○
	○	×	○
	○	○	×
β	×	×	○
	×	○	×
	○	×	×

- $\alpha + \beta$: dichotomies on $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$
- $B(4,3)$ 'no shatter' any 3 inputs
  $\Longrightarrow \alpha + \beta$ 'no shatter' any 3 including $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$

B(4,3)

	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
2α	○	○	○	○
	○	○	○	×
	×	○	○	○
	×	○	○	×
	○	×	○	○
	○	×	○	×
	○	○	×	○
	○	○	×	×
β	×	×	○	×
	×	○	×	○
	○	×	×	○

$$\alpha + \beta \le B(3,3)$$

Estimating Part of $B(4,3)$ (2/2)

$$B(4,3) = 11 = 2\alpha + \beta$$

B(3,2)

	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
α	○	○	○
	×	○	○
	○	×	○
	○	○	×

- α: dichotomies on $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ with $\mathbf{x}_4$ **paired**
- $B(4,3)$ 'no shatter' any 3 inputs $\Longrightarrow \alpha$ 'no shatter' any 2

If we chose (?, ?, x4) and we had already known that x4 are in pairs.

B(4,3)

	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
2α	○	○	○	○
	○	○	○	×
	×	○	○	○
	×	○	○	×
	○	×	○	○
	○	×	○	×
	○	○	×	○
	○	○	×	×
β	×	×	○	×
	×	○	×	○
	○	×	×	○

$$\alpha \le B(3,2)$$

Putting It All Together

$$
\begin{aligned}
B(4,3) &= 2\alpha + \beta \\
\alpha + \beta &\leq B(3,3) \\
\alpha &\leq B(3,2) \\
\Rightarrow B(4,3) &\leq B(3,3) + B(3,2)
\end{aligned}
$$

	$B(N,k)$	k = 1	2	3	4	5	6
	1	1	2	2	2	2	2
	2	1	3	4	4	4	4
	3	1	4	7	8	8	8
N	4	1	≤ 5	11	15	16	16
	5	1	≤ 6	≤ 16	≤ 26	31	32
	6	1	≤ 7	≤ 22	≤ 42	≤ 57	63

n+2 upper bound

now have **upper bound** of bounding function

Putting It All Together

$$
\begin{aligned}
B(N,k) &= 2\alpha + \beta \\
\alpha + \beta &\le B(N-1,k) \\
\alpha &\le B(N-1,k-1) \\
\Rightarrow B(N,k) &\le B(N-1,k) + B(N-1,k-1)
\end{aligned}
$$

	$B(N,k)$	k = 1	2	3	4	5	6
	1	1	2	2	2	2	2
	2	1	3	4	4	4	4
	3	1	4	7	8	8	8
N	4	1	≤ 5	11	15	16	16
	5	1	≤ 6	≤ 16	≤ 26	31	32
	6	1	≤ 7	≤ 22	≤ 42	≤ 57	63

now have **upper bound** of bounding function

Bounding Function: The Theorem

$$B(N,k) \le \underbrace{\sum_{i=0}^{k-1} \binom{N}{i}}_{\text{highest term } N^{k-1}} \quad \because$$

- simple induction using **boundary and inductive formula**
- for fixed k, $B(N,k)$ upper bounded by $poly(N)$ $\therefore$
  $\Longrightarrow$ **$m_{\mathcal{H}}(N)$ is $poly(N)$ if break point exists**

'$\le$' can be '$=$' actually,
go play and prove it if math lover! :-)

The Three Break Points

$$B(N,k) \leq \underbrace{\sum_{i=0}^{k-1} \binom{N}{i}}_{\text{highest term } N^{k-1}}$$

- positive rays: $m_{\mathcal{H}}(N) = N + 1 \leq N + 1$

  ○ × $m_{\mathcal{H}}(2) = 3 < 2^2$: break point at 2

- positive intervals: $m_{\mathcal{H}}(N) = \frac{1}{2}N^2 + \frac{1}{2}N + 1 \leq \frac{1}{2}N^2 + \frac{1}{2}N + 1$

  ○ × ○ $m_{\mathcal{H}}(3) = 7 < 2^3$: break point at 3

- 2D perceptrons: $m_{\mathcal{H}}(N) = ? \leq \frac{1}{6}N^3 + \frac{5}{6}N + 1$

  The upper bound of B(N, 4)

  × ○ ○ × $m_{\mathcal{H}}(4) = 14 < 2^4$: break point at 4

can bound $m_{\mathcal{H}}(N)$ by only **one break point**

Fun Time

For 1D perceptrons (positive and negative rays), we know that $m_{\mathcal{H}}(N) = 2N$. Let k be the minimum break point. Which of the following is not true?

1. $k = 3$
2. for some integers $N > 0$, $m_{\mathcal{H}}(N) = \sum_{i=0}^{k-1} \binom{N}{i}$
3. for all integers $N > 0$, $m_{\mathcal{H}}(N) = \sum_{i=0}^{k-1} \binom{N}{i}$
4. for all integers $N > 2$, $m_{\mathcal{H}}(N) < \sum_{i=0}^{k-1} \binom{N}{i}$

Fun Time

For 1D perceptrons (positive and negative rays), we know that $m_{\mathcal{H}}(N) = 2N$. Let k be the minimum break point. Which of the following is not true?

1. $k = 3$
2. for some integers $N > 0$, $m_{\mathcal{H}}(N) = \sum_{i=0}^{k-1} \binom{N}{i}$
3. for all integers $N > 0$, $m_{\mathcal{H}}(N) = \sum_{i=0}^{k-1} \binom{N}{i}$
4. for all integers $N > 2$, $m_{\mathcal{H}}(N) < \sum_{i=0}^{k-1} \binom{N}{i}$

Reference Answer: 3

The proof is generally trivial by listing the definitions. For 2, $N = 1$ or 2 gives the equality. One thing to notice is 4: the upper bound can be 'loose'.

BAD Bound for General $\mathcal{H}$

want:

$$\mathbb{P}\left[\exists h \in \mathcal{H} \text{ s.t. } \left|E_{\text{in}}(h) - E_{\text{out}}(h)\right| > \epsilon\right] \le 2 \quad m_{\mathcal{H}}(\ \ N) \cdot \exp\left(-2 \qquad \epsilon^2 N\right)$$

3 steps

actually, when N large enough,

$$\mathbb{P}\left[\exists h \in \mathcal{H} \text{ s.t. } \left|E_{\text{in}}(h) - E_{\text{out}}(h)\right| > \epsilon\right] \le 2 \cdot 2 m_{\mathcal{H}}(2N) \cdot \exp\left(-2 \cdot \frac{1}{16} \epsilon^2 N\right)$$

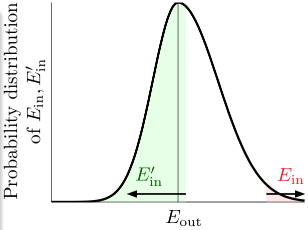
next: **sketch** of proof

Ignore the mathematical details of the proof.

Step 1: Replace E_{out} by E'_{in}

$$
\begin{aligned}
&\frac{1}{2}\mathbb{P}\Big[\exists h \in \mathcal{H} \text{ s.t. } \big|E_{\text{in}}(h) - E_{\text{out}}(h)\big| > \epsilon\Big] \\
\leq \quad & \mathbb{P}\Big[\exists h \in \mathcal{H} \text{ s.t. } \big|E_{\text{in}}(h) - E'_{\text{in}}(h)\big| > \frac{\epsilon}{2}\Big]
\end{aligned}
$$

- $E_{\text{in}}(h)$ finitely many, $E_{\text{out}}(h)$ infinitely many —replace the evil E_{out} first
- how? sample verification set $\mathcal{D}'$ of size N to calculate E'_{in}
- BAD h of $E_{\text{in}} - E_{\text{out}}$ $\overset{\text{probably}}{\Longrightarrow}$ BAD h of $E_{\text{in}} - E'_{\text{in}}$

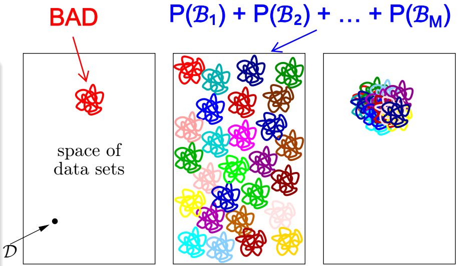


evil E_{out} removed by verification with '**ghost data**'

Step 2: Decompose $\mathcal{H}$ by Kind

$$\begin{aligned} \text{BAD} &\leq 2\mathbb{P}\left[\exists h \in \mathcal{H} \text{ s.t. } \left|E_{\text{in}}(h) - E'_{\text{in}}(h)\right| > \frac{\epsilon}{2}\right] \\ &\leq 2m_{\mathcal{H}}(2N)\mathbb{P}\left[\underline{\text{fixed } h} \text{ s.t. } \left|E_{\text{in}}(h) - E'_{\text{in}}(h)\right| > \frac{\epsilon}{2}\right] \end{aligned}$$

- E_{in} with $\underline{\mathcal{D}}$, E'_{in} with $\underline{\mathcal{D}'}$
  —now $m_{\mathcal{H}}$ comes to play
- how? infinite $\mathcal{H}$ becomes $|\mathcal{H}(\underline{\mathbf{x}_1, \ldots, \mathbf{x}_N}, \underline{\mathbf{x}'_1, \ldots, \mathbf{x}'_N})|$ kinds
- union bound on $m_{\mathcal{H}}(\underline{2N})$ kinds



(a) Hoeffding Inequality (b) Union Bound (c) Now

no overlap

use $m_{\mathcal{H}}(2N)$ to calculate BAD-overlap properly

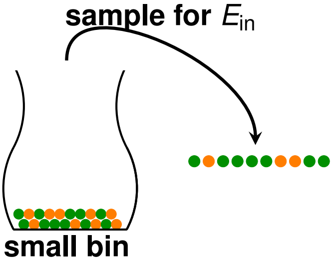
Step 3: Use Hoeffding without Replacement

$$\begin{aligned} \text{BAD} &\le 2m_{\mathcal{H}}(2N)\mathbb{P}\left[\text{fixed } h \text{ s.t. } \left|E_{\text{in}}(h) - E'_{\text{in}}(h)\right| > \frac{\epsilon}{2}\right] \\ &\le 2m_{\mathcal{H}}(2N) \cdot 2\exp\left(-2\left(\frac{\epsilon}{4}\right)^2 N\right) \end{aligned}$$

- consider bin of $2N$ examples, choose N for E_{in}, leave others for E'_{in}
  $|E_{\text{in}} - E'_{\text{in}}| > \frac{\epsilon}{2} \Leftrightarrow \left|E_{\text{in}} - \frac{E_{\text{in}}+E'_{\text{in}}}{2}\right| > \frac{\epsilon}{4}$
- so? just 'smaller bin', 'smaller ϵ', and Hoeffding without replacement

After taking out the balls, we don't put them back into the bin.

The result is similar with the original Hoeffding.



use Hoeffding after zooming to fixed h

That's All!

Vapnik-Chervonenkis (VC) bound:

$$\mathbb{P}\left[\exists h \in \mathcal{H} \text{ s.t. } \left|E_{\text{in}}(h) - E_{\text{out}}(h)\right| > \epsilon\right]$$
$$\leq \quad 4 m_{\mathcal{H}}(2N) \exp\left(-\frac{1}{8}\epsilon^2 N\right)$$

- replace E_{out} by E'_{in}
- decompose $\mathcal{H}$ by kind
- use Hoeffding without replacement

2D perceptrons:

- break point? 4
- $m_{\mathcal{H}}(N)$? $O(N^3)$

learning with 2D perceptrons feasible! :-)

Fun Time

For positive rays, $m_{\mathcal{H}}(N) = N + 1$. Plug it into the VC bound for $\epsilon = 0.1$ and $N = 10000$. What is VC bound of BAD events?

$$\mathbb{P}\Big[\exists h \in \mathcal{H} \text{ s.t. } \big|E_{\text{in}}(h) - E_{\text{out}}(h)\big| > \epsilon\Big] \quad \leq \quad 4 m_{\mathcal{H}}(2N) \exp\left(-\frac{1}{8}\epsilon^2 N\right)$$

1. 2.77×10^{-87}
2. 5.54×10^{-83}
3. 2.98×10^{-1}
4. 2.29×10^{2}

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1. 2.77×10^{-87}
2. 5.54×10^{-83}
3. 2.98×10^{-1}
4. 2.29×10^{2}

Reference Answer: 3

Simple calculation. Note that the BAD probability bound is <u>not very small</u> even with 10000 examples. We use a lot of approximation to get upper bound, so it won't be very accurate.

Summary

1. When Can Machines Learn?
2. **Why** Can Machines Learn?

Lecture 5: Training versus Testing

Lecture 6: Theory of Generalization

- Restriction of Break Point
  break point 'breaks' consequent points
- Bounding Function: Basic Cases
  $B(N, k)$ **bounds** $m_{\mathcal{H}}(N)$ **with break point** k
- Bounding Function: Inductive Cases
  $B(N, k)$ **is** $poly(N)$
- A Pictorial Proof
  $m_{\mathcal{H}}(N)$ **can replace** M **with a few changes**

- **next: how to 'use' the break point?**

3. How Can Machines Learn?
4. How Can Machines Learn Better?